ELASTO-PLASTIC THERMAL STRESSES IN PLATES WITH CIRCULAR HOLES

H. McCLOSKEY
Engineering Science, University of Notre Dame,

T. ARIMAN
Department of Aerospace and Mechanical Engineering, University of Notre Dame, Notre Dame, Indiana 46556, U.S.A.

SUMMARY

Our energy conscious nuclear age with its reactors and hypersonic conveyances has brought numerous problems to structural mechanics and areas in which thermal stresses play an important role are increasing. One of the problems concerns the elasto-plastic analysis of plates. Several approaches have been taken to this problem. Among these is a method of successive elastic solutions with a finite difference technique to determine the plastic stresses; see e.g. A. Mendelson, "Solutions of Some Plane Thermal and Crack Problems for Strain-Hardening", University Microfilms, Inc., Ann Arbor, Michigan (1965); A. Mendelson, "Plasticity: Theory and Application", The MacMillan Company (1965).

Plates with holes are used extensively in nuclear reactors. These plates are subjected to thermal gradients in both the axial and radial directions. This paper, therefore, deals with the membrane elasto-plastic stress distribution in a finite rectangular plate with an insulated circular hole. Both circular hole and outer plate boundaries are assumed to be stress free. The plate is heated uniformly through the thickness, so there are no temperature gradients outside the plane. A method of successive elastic solutions with a finite difference technique is employed.

The first step in the solution of the problem is the selection of a suitable grid network. One is chosen that would guarantee points on the hole and the outer plate edges. Therefore, both Laplace's equation for the temperature field T ($\nabla^2 T = 0$), and the biharmonic equation for the stress function ϕ ($\nabla^4\phi = 0$), have to be derived for unequal grid point spacing. This derivation is carried out using Taylor's series.

Then, the particular solution of the stress function ($\nabla^2\phi_p = -E\alpha T$), and the complementary solution of the stress function ($\nabla^4\phi_c = 0$) are obtained for the elastic case to yield the complete solution ($\phi = \phi_p + \phi_c$). For plastic analysis, the biharmonic equation of plastic strains (ε_x^p, ε_y^p, and ε_{xy}^p),

$$\nabla^4\phi = -E\nabla^2(\alpha T) - E\left(\frac{\partial^2\varepsilon_x^p}{\partial y^2} + \frac{\partial^2\varepsilon_y^p}{\partial x^2} - 2\frac{\partial^2\varepsilon_{xy}^p}{\partial x\partial y}\right),$$

is solved by employing an experimentally determined stress-strain curve for aluminium 7075-T6.

The method of successive elastic solutions using the plastic-strain-total-strain relations is employed. The numerical solution is obtained using a Gauss-Seidel program for banded matrices and the University of Notre Dame's IBM 370 computer. Extensive numerical results are presented and convergence and error of the solution are discussed.

1. Introduction

In nuclear engineering there are numerous problems in which thermal stresses play an important and frequently even a primary role. One of the interesting problems is the analysis of temperature and stress distribution in rectangular plates with a circular hole subjected to heat flow. In the last decade a number of papers on membrane elastic thermal stresses in infinite and finite plates with holes or inclusions appeared in the literature [1-6]. However, the authors of this article have not seen a publication on the analysis of elastoplastic thermal stresses in finite rectangular plates with circular holes.

This paper attempts to determine the temperature distribution and the elastoplastic thermal stress solution for a finite rectangular plate with an insulated circular hole by means of the finite difference method. The plate is heated uniformly through the thickness, that is, the temperature varies only along the coordinates within the plane. The temperature distribution is determined as the solution of the steady state heat conduction equation. Afterwards, the elastoplastic membrane stresses due to the temperature variation are analyzed.

2. Temperature Field

A rectangular elastic thin plate of uniform thickness t is subjected to heat flow in the x-direction. The plate contains an insulated circular hole of radius R located at the center of the plate. The two edges of the plate $x = \pm a$ are kept at constant temperatures, T_1 and T_2 respectively ($T_1 > T_2$), and the other two edges, $y = \pm b$, are insulated.

The governing equation for the steady-state heat conduction in a thin plate with no sources or sinks is the Laplace equation:

$$\nabla^2 T = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2}$$

and T (x, y) is the temperature. The boundary conditions are given by:

$$T = T_1 \text{ at } x = a; \quad \frac{\partial T}{\partial y} = 0 \text{ at } y = \pm b$$

$$T = T_2 \text{ at } x = -a; \quad \frac{\partial T}{\partial n} = 0 \text{ on the hole}$$

where n is the coplanar normal to the hole. The temperature distribution is symmetrical about the x-axis. Therefore, only one half of the plate ($0 \le y \le b$) will be analyzed. The location of finite difference grid points is such that points can be guaranteed to be located on the hole and on all boundaries. This was accomplished by representing the Laplace equation in unequal intervals using the expanded Taylor's series [7]. Spacing of points on the hole was at equal arc lengths. Seven points were chosen to lie on the semicircle of the top half plate. The points started at $\theta = 0^\circ$ and were incremented by $\Delta\theta = 30^\circ$. With more extensive algebraic considerations, more points could have been placed on the hole. Therefore, with the equations in terms of unequal spacing, finite difference points could be placed nearly at will around the rest of the plate. The present program was written for a grid of 316 points in the half plate. Core storage, computing time, and shear ease in programming must be considered if a more extensive grid network is desired.

Figure 1 shows the non-dimensional temperature variation $\phi = (T-T_2)/(T_1-T_2)$ on the perimeter of the hole in a square plate. Curves are given for various sizes of the circular hole. It may be noted that as the size of the hole is increased, the temperatures at $\theta = 0^\circ$

and $\theta = 180^o$ approach the constant values at the outer edges. Figure 2 is the plot of the non-dimensional temperature distribution in a square plate along the line y = R/b= constant. The curves correspond to the various sizes of the circular hole. The temperature variation for the case of no hole, R/b = 0, is linear and becomes increasingly nonlinear as the value of R/b increases. The results are in excellent agreement with those in ref. [6] where the method of solution was a boundary point matching technique in the least squares sense.

3. Stress Field

An iteration technique called the method of successive elastic approximations is used in the determination of the stress field. This technique has been successfully applied to the analysis of a number of elastoplastic problems by Mendelson [8]. The governing partial differential equation for the stress function ϕ in the case of plane stress in a thin plate and for thermal elastoplastic deformation is given by [8]:

$$\nabla^2\nabla^2\phi = -E\,\nabla^2(\alpha T) - E\left(\frac{\partial^2\varepsilon_x^p}{\partial y^2} + \frac{\partial^2\varepsilon_y^p}{\partial x^2} - 2\frac{\partial^2\varepsilon_{xy}^p}{\partial x\partial y}\right) \tag{4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

E = Young's modulus

α = coefficient of linear thermal expansion

$$\varepsilon_x^p,\ \varepsilon_y^p,\ \varepsilon_{xy}^p = \text{plastic strain components} \tag{5}$$

and the stress function ϕ is related to the stress components by

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2},\quad \sigma_y = \frac{\partial^2\phi}{\partial x^2},\quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \tag{6}$$

Total stress-strain relations for plane stress are given by [8]

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y) + \alpha T + \varepsilon_x^p$$
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x) + \alpha T + \varepsilon_y^p \tag{7}$$

and

$$\varepsilon_{xy} = \frac{1+\mu}{E}\tau_{xy} + \varepsilon_{xy}^p$$

Then, by following the procedure of the successive elastic solutions, the increment to the governing biharmonic (eq. 4) is found from:

$$\Delta g = \frac{\partial^2(\Delta\varepsilon_x^p)}{\partial y^2} + \frac{\partial^2(\Delta\varepsilon_y^p)}{\partial x^2} - 2\frac{\partial^2(\Delta\varepsilon_{xy}^p)}{\partial x\partial y} \tag{8}$$

and from the modified total strain increments:

$$\varepsilon'_{x,i} = (\sigma_{x,i} - \mu\,\sigma_{y,i}) + \Delta\varepsilon_{x,i}^p$$

$$\varepsilon'_{y,i} = (\sigma_{y,i} - \mu\,\sigma_{x,i}) + \Delta\varepsilon_{y,i}^p \tag{9}$$

$$\varepsilon'_{xy,i} = (1+\mu)\,\tau_{xy,i} + \Delta\varepsilon_{xy,i}^p$$

Where i denotes the increment of load. We also have the equivalent total strain:

$$\varepsilon_{et} = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon'_x-\varepsilon'_y)^2 + (\varepsilon'_y-\varepsilon'_z)^2 + (\varepsilon'_z-\varepsilon'_x)^2 + 6\,(\varepsilon'_{xy})^2} \tag{10}$$

and the equivalent stress:

$$\sigma_{e,i-1} = \sqrt{\sigma^2_{x,i-1} + \sigma^2_{y,i-1} - \sigma_{x,i-1}\sigma_{y,i-1} + 3\tau^2_{xy,i-1}} \tag{11}$$

The equivalent plastic strain increments can be found from [8].

$$\Delta\varepsilon_p = (\varepsilon_{et} - (2/3)\frac{(1+\mu)}{E}\sigma_{e,i-1})/(1+(2/3)(1+\mu)(m/(1-m))) \tag{12}$$

where m is the strain hardening parameter.

The final set of equations necessary for the plastic calculations are:

$$\begin{aligned} \Delta\varepsilon^p_x &= (\Delta\varepsilon_{p,i}/3\varepsilon_{et,i})(2\varepsilon'_{x,i} - \varepsilon'_{y,i} - \varepsilon'_{z,i}) \\ \Delta\varepsilon^p_y &= (\Delta\varepsilon_{p,i}/3\varepsilon_{et,i})(2\varepsilon'_{y,i} - \varepsilon'_{z,i} - \varepsilon'_{x,i}) \\ \Delta\varepsilon^p_{xy,i} &= (\Delta\varepsilon_{p,i}/\varepsilon_{et,i})\varepsilon'_{xy,i} \end{aligned} \tag{13}$$

where $\Delta\varepsilon_p$ is the plastic strain increment. Of course, for the present case of plane stress, all components in the z-direction are assumed to be zero.

A typical finite difference equation for the Laplacian takes the form:

$$\frac{f_1}{\Delta_1(\Delta_1+\Delta_3)} + \frac{f_2}{\Delta_2(\Delta_2+\Delta_4)} + \frac{f_3}{\Delta_3(\Delta_1+\Delta_3)} + \frac{f_4}{\Delta_4(\Delta_2+\Delta_4)} - f_o\left(\frac{1}{\Delta_o\Delta_3} + \frac{1}{\Delta_2\Delta_4}\right) = 0 \tag{14}$$

Where f_1, f_2, f_3 and f_4 are nodal points of the finite difference scheme and f_o is the central point. f_1, f_2, f_3 and f_4 are located at $\theta = 0^\circ$, 90°, 180°, 270° counterclockwise, respectively.

The biharmonic undergoes a similar but more extensive transformation since the 13 point formula is used and each interval in an axis direction may be unique. First the stress function φ is broken up into a particular and complimentary part:

$$\phi = \phi_p + \phi_c \tag{15}$$

where

$$\nabla^2\phi_p = -E\alpha T_{x,y} \tag{16}$$

and

$$\nabla^4\phi_c = 0 \tag{17}$$

$\nabla^4\phi_c$ is subject to the conditions

$$\phi = \phi_p + \phi_c = 0 \quad \text{and} \quad \frac{\partial\phi}{\partial n} = \frac{\partial\phi_p}{\partial n} + \frac{\partial\phi_c}{\partial n} = 0 \tag{18}$$

on the whole plate boundaries and on the hole. The shear stress τ_{xy} and the displacement v are taken as zero on the x-axis due to the symmetry of the problem.

The particular solution is found, independent of boundary conditions, by using the Laplacian in the usual five point finite difference manner, except on boundaries. On those boundaries, however, it is necessary to use forward or backward differences to approximate the second derivative. Since forward and backward differences are intrinsically less accurate than central differences, the second term of the series expansion for the non-central second differences was also used to improve the accuracy of the finite difference second order partial derivative approximation. For instance, the second forward difference is:

$$\begin{aligned} \left(\frac{\partial^2 f}{\partial x^2}\right)_o &= f_o\left(\frac{1}{h_1^2} + \frac{1}{h_1^3}\right) - f_1\left(\frac{1}{h_1^3} + \frac{(h_1+h_2)}{h_1^2 h_2} + \frac{(h_1+h_2)}{h_1^2 h_2^2}\right) \\ &+ f_2\left(\frac{1}{h_1h_2h_3} + \frac{(h_1+h_2)}{h_1^2 h_2^2} + \frac{1}{h_1h_2}\right) - \frac{f_3}{h_1h_2h_3} \end{aligned} \tag{19}$$

where f_0, f_1, f_2, and f_3 are consecutive points from the left edge toward the right and h_1, h_2, h_3 are the three consecutive intervals between the four points.

The next step in the solution is the complimentary solution. This entails solving eq. (17) subject to the previous boundary conditions and the restrictions of the particular solution. The general thirteen point finite difference form of the biharmonic is written for each one of the grid points. Points near the boundaries must be handled with care since some of the nodal points may be outside of the physical boundaries. These "imaginary" points are found in terms of known points and values by applying the boundary conditions. These points are then computed in with the other known points and placed in the coefficient matrix of the simultaneous equations for the biharmonic. After these equations have been solved, the total stress function is obtained from the combined particular and complimentary stress functions. The stress may then be calculated using eq. (6).

With these values, the successive approximation method [8, 9] for the plastic strains is, in outline form, as follows:

(a) Assume small plastic strain increments in the x and y directions. (eq. (13))

(b) Calculate the increment Δg (eq. (8)) to eq. (1), the nonhomogeneous biharmonic.

(c) Solve eq. (4) for the total stress function ϕ.

(d) Calculate the stresses from eq. (6).

(e) Calculate the modified total strains due to mechanical loading using eq. (9). At this point the stresses have been nondimensionalized.

(f) Find the equivalent total strain from eq. (10).

(g) Solve for the plastic strain increment, (eq. 12), where m is the strain hardening parameter and $\sigma_{e,i-1}$ is from eq. (11).

(h) Calculate the new plastic strain increments from eq. (13).

(i) Repeat steps b through h until convergence.

(j) Increment the load and go back to step a. This involves taking the next increment along the actual stress-strain curve of the plate material(an aluminum alloy 7075-T6) [10]. The previous method, according to Mendelson [7, 8], is the most rapidly convergent. He discusses other methods in [9].

3. Computer Programming

The programming was done on an IBM 370/155 computer using complete double precision for all calculations. The matrices of coefficients were stored in banded form and solved by a Gauss-Jordan elimination solver written especially for these matrices. The solution was checked with a Gauss-Seidel banded matrix solver which was to be the main solver, but proved to be too slowly convergent to be of use. However, using the Gauss-Jordan solution as the initial approximation to the Gauss-Seidel, the convergence was immediate. This is encouraging since the roundoff error in solving large systems by Gauss elimination can become significant. The test with the Gauss-Seidel proved it to be minimal, at least, for double precision arithmetic. At the time of preparation of this paper, debug operations were not completed on the elastoplastic stress analysis part of the investigation. It is expected that the numerical results will be presented at the Conference.

4. References

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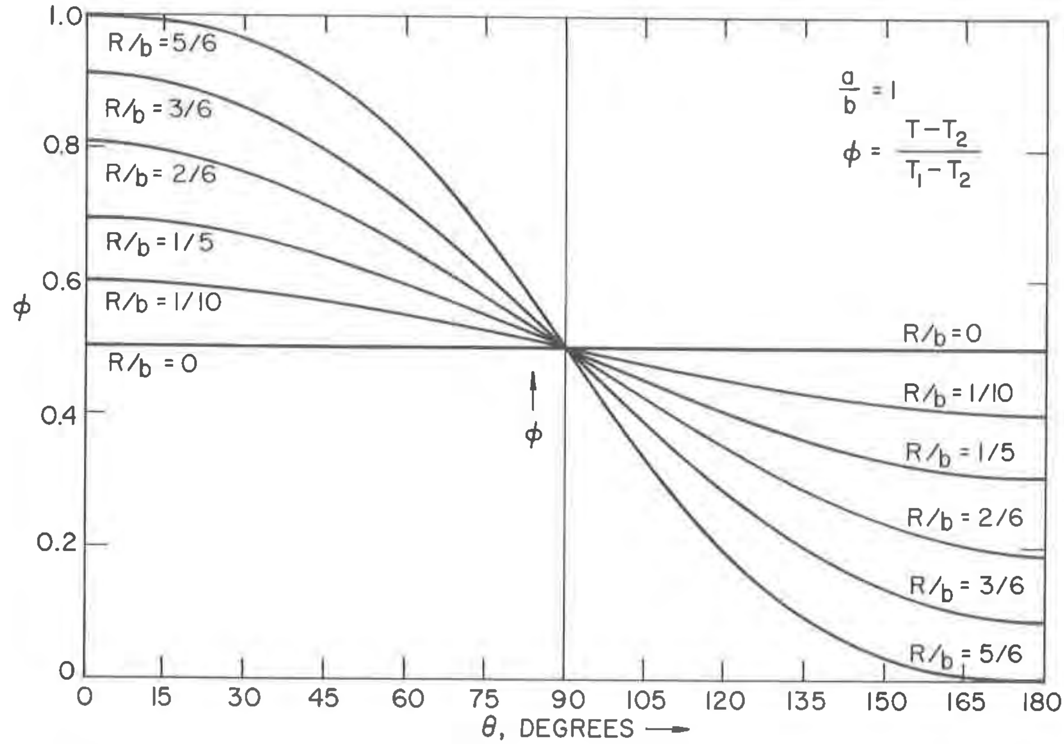

Figure 1. Normalized temperature distribution in a square plate

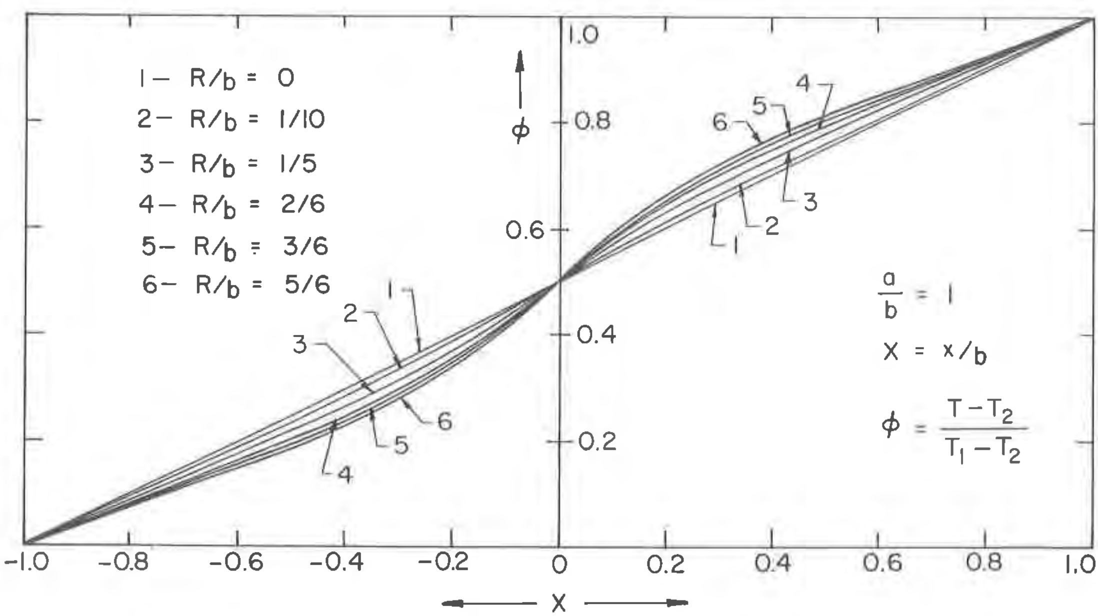

Figure 2. Normalized temperature distribution along the line y = R/b in a square plate.