ON THE CHOICE OF OPTIMALITY CRITERIA IN COMPARING STATISTICAL DESIGNS

by

K. R. Shah and B. K. Sinha

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K. R. Shah
University of Waterloo
Waterloo, Ontario, Canada

B. K. Sinha*
North Carolina State University
Raleigh, NC 27695-8203 USA

ABSTRACT

We develop in a reasonable sense a class of optimality functionals for comparing feasible statistical designs available in a given set-up. It is desired that the optimality functionals reflect *symmetric measures* of the *lack of information* contained in the designs being compared. In this sense, Kiefer's (1975) *Universal Optimality* criteria rests on stringent conditions some of which can be relaxed and, yet, optimality (in a reasonable sense) of the so-called *balanced* designs may be reinforced.

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*On leave from Indian Statistical Institute (Calcutta), India.

We address the problem of seeking a class of suitable criteria for comparison of experiments involving linear models. Specifically, denote by $\mathcal{C}$ the class of available C-matrices in a given traditional block-design set-up or in a row-column design setting where the interest lies primarily in the so-called fixed varietal effects. Denote by Φ the class of optimality functionals ϕ defined on the members of $\mathcal{C}$. Let g denote a permutation on the variety symbols and let G_g denote the corresponding permutation matrix i.e., the matrix obtained by applying g to the columns of the identity matrix.

We list below reasonable requirements to be satisfied by the optimality funtionals ϕ:

(i) $\phi(C) = \phi(G_g'CG_g)$ for every member g of the symmetric group of permutations.

(ii) $C_1 \geq C_2$ (in the sense of $C_1 - C_2$ being nnd) $\Rightarrow \phi(C_1) \leq \phi(C_2)$.

(iii) $\phi(C_1) \geq \phi(C_2) \Leftrightarrow \phi(bC_1) \geq \phi(bC_2)$ for all positive integers $b \geq 1$.

(iv) $\phi(\Sigma b_g G_g'CG_g) \leq \phi((\Sigma b_g)C)$ for all non-negative integers b_g's, at least one b_g being positive.

In particular, (iv) implies

(iv)' $\phi(\Sigma(G_g'CG_g)) \leq \phi((v!)C)$, assuming C is of order (vxv).

The optimality functionals are aimed at providing *symmetric measures* of the *lack of information* contained in the C-matrices (which represent information matrices for the fixed varietal effects) and, hence, it would be reasonable to impose invariance wrt the symmetric group of permutations (of the rows and/or columns of C). This justifies (i).

The requirement (ii) exhibits another desirable property of ϕ in the sense that the lack of information in C_1 is less than that in C_2 whenever $C_1 \geq C_2$. This is quite reasonable in every sense.

The requirement (iii) simply means that the design producing C_2 is

Φ-better than that producing C_1 if and only if so are b copies of the former compared to b copies of the latter, for all positive integer values of b. This justifies the understanding of repeated use of better designs.

In (iv), we essentially compare two *derived* designs. Given any design d to start with, we construct d_g as a variation of d by permuting the varieties according to the permutation g. Clearly, $G_g'CG_g$ is the resulting C-matrix of d_g. Then $\Sigma\, b_g\, G_g'CG_g$ represents the C-matrix underlying the design formed by considering b_g copies of d_g, simultaneously for every $b_g > 0$. On the other hand, $(\Sigma b_g)C$ corresponds to the C-matrix of another *derived* design which is simply formed of (Σb_g) copies of the original design d. Since the optimality functionals are permutation-invariant (the requirement (i)), one would expect that a combination of various forms of d would do rather better than an exclusive use of d itself.

The usual convexity requirement on the part of Φ states that

$$\Phi(\Sigma_g C_g/v!) \le \frac{1}{v!}\ \Sigma_g\ \Phi(C_g)\ ,\quad C_g = G_g'CG_g$$

wherein the component matrices as well as their convex combination need <u>not</u> be feasible C-matrices. Kiefer's (1975) notion of *Universal Optimality* considered only the criteria which satisfy this definition of convexity. By contrast, the requirements (i) - (iv) above need be satisfied only by feasible C-matrices i.e., by C-matrices which correspond to actual designs.

The fact that (iv)' is weaker than (iv) can be seen through the following example:

d_1

1	2
1	3

d_2

2	1
2	3

$$2\,C_1 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad 2C_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad C_2 = G_g'\,C_1 G_g \text{ with } g=(1 \longleftrightarrow 2)$$

Define $$\phi_d = \Delta_d + \left(\otimes_{\max d} - \otimes_{\min d}\right)^{\delta} \left(\frac{A}{\otimes_{\max d}} + \frac{1}{\otimes_{\min d}}\right) .$$

Here Δ_d is a function of the design parameters and, is the same for all variants of the design d. One possible choice could be $\Delta_d = \bar{r}_d - \bar{\lambda}_d$ where $\bar{r}_d = \Sigma r_{d_{ii}}/v$ and $\bar{\lambda}_d = \Sigma_{i<j}\, \Sigma\, \lambda_{d_{ij}}/\binom{v}{2}$. In the above $\otimes_d$'s represent positive eigenvalues of C_d. For some (δ, A), for example $\delta = 1$ and $A = 10$, it follows that $\phi(C_{d1} + C_{d2}) > \phi(2C_{d1})$ while it is easy to verify that $\phi(\Sigma\, G_g' C_{d1} G_g) < \phi(6\, C_{d1})$ for any choice of $\delta > 0$, $A > 0$.

We may refer to (iv)' as a *symmetry* requirement on the optimality functionals ϕ. The requirement (iv) may be referred to as the property of *weak convexity*. This is motivated by the fact that (i) together with the notion of convexity in the usual sense implies (iv). In fact, (iv) is satisfied by all functionals of the form $\Phi(C) = g(f(C))$ where g is monotone increasing, and f is convex in the usual sense satisfying (i). Certainly plenty of examples of such forms of ϕ can be constructed where ϕ as such is *not* convex. The following is one of them.

$$\phi(C) = \sqrt[4]{tr(C^2)} \ .$$

Taking $C_1 = 9(I-J/v)$ and $C_2 = 16(I-J/v)$, it is seen that

$$\phi(\frac{1}{2}(C_1 + C_2)) > \frac{1}{2}\left[\phi(C_1) + \phi(C_2)\right]$$

thereby violating convexity in respect of ϕ . However, since ϕ is a monotone increasing function of $tr(C^2)$ which is a convex function of C, weak convexity holds. When we assume (i), the following are seen to be progressively weaker:

(1) ϕ is convex in the usual sense

(2) ϕ is a monotone increasing function of a convex function

(3) ϕ is weakly convex in the sense of (iv)

(4) ϕ satisfies the symmetry requirement (iv)' .

Kiefer's proof of universal optimality of *balanced* designs requires only the last of the above conditions (on ϕ) which is the weakest one. Further, convexity in this context does *not* appear to have statistical meaning or interpretation whereas condition (iv)' (even condition (iv) of which (iv)' is a special case) is appealing in a reasonable statistical sense. We note that Kiefer's notion of universal optimality is based on (i), a version of (ii), and convexity is the usual sense. The proof uses nonfeasible C-matrices of the type $\Sigma\, G_g' C G_g / v!$. However, consideration of (i), (ii), (iii) and (iv)' leads to a neat proof of the following Proposition 1 of Kiefer (1975):

Proposition 1 (Restated) If there is a feasible C-matrix which is completely symmetric and has maximum trace, then it is universally optimal (in the sense of minimizing ϕ for all functionals ϕ in Φ satisfying (i), (ii), (iii) and (iv)').

It may be mentioned that various optimality functionals discussed in the literature do satisfy these requirements. See Hedayat (1981) for an excellent review of the existing optimality criteria.

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