ON THE DISTRIBUTION OF THE LAST OCCURRENCE TIME IN AN INTERVAL FOR A REGENERATIVE PHENOMENON[1.]

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1. Introduction and Summary

Let $(\Omega, \mathcal{B}, Pr)$ be a probability space on which a regenerative phenomenon $\mathcal{E}$ is defined. That is, $\mathcal{E}$ is a family $\{E(t), t > o\}$ of subsets of Ω, each belonging to $\mathcal{B}$, and having the property that whenever

$$o < t_1 < t_2 < \dots < t_k,$$

then

$$Pr\{E(t_1), E(t_2), \dots, E(t_k)\} = Pr\{E(t_1)\} Pr\{E(t_2 - t_1), \dots, E(t_k - t_1)\}.$$

Let $\mathcal{E}$ be such that

$$p(t) = Pr\{E(t)\} \to 1$$

as $t \to o$. That is, in the terminology of KINGMAN [4], $\mathcal{E}$ is standard. We write $Z(t, \omega)$ for the indicator process of $\mathcal{E}$ defined by

$$Z(t, \omega) = \begin{cases} 1 & \text{if } \omega \in E(t) \\ o & \text{if } \omega \notin E(t), \end{cases}$$

and we shall suppose, as we may do without essential loss of generality, that this process is separable ([4], Section 13). Take $Z(o, \omega) = 1$ for convenience. Then, associated with $\mathcal{E}$ there is a stochastic process T_t defined by

$$T_t = \sup\{u : o \leq u \leq t;\ Z(u, \omega) = 1\}.$$

That is, T_t is the time of last occurrence of $\mathcal{E}$ in the interval $[o, t]$.

This paper is concerned with a study of the process T_t. We shall obtain various representation results for T_t and examine aspects of its limit behaviour as $t \to \infty$.

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2. Representation Results

Theorem 1. For $\theta > o$, define

$$r(\theta) = \int_o^\infty e^{-\theta t} p(t)dt.$$

Then, for $s > o$, $z > o$,

$$z\int_o^\infty E(e^{-sT_t})e^{-zt}dt.$$

$$(1) \qquad = \exp\left\{-\int_o^\infty t^2 ET_t e^{-zt}\left[(1 + tz)-(1 + t(z + s)e^{-st})\right]dt\right\}$$

$$= r(z + s)\left[r(z)\right]^{-1}.$$

Proof. For integer valued $m \geq 1$, consider the family $\{E(2^{-m}n), n = o, 1, 2, \ldots\}$ of subsets of Ω. This discrete skeleton forms a recurrent event in the sense of FELLER ([1], I Chapter 13). Now, define the sequences $\{u_n, n \geq o\}$, $\{f_n, n \geq 1\}$ by

$$u_n = Pr\{E(2^{-m}n)\}, \; n \geq o,$$

$$f_n = Pr\{\overline{E(2^{-m})}, \overline{E(2^{-(m-1)})}, \ldots, \overline{E(2^{-m}(n-1))}, E(2^{-m}n)\}, \; n \geq 1,$$

where $\overline{E(2^{-m}k)}$ denotes the complement of $E(2^{-m}k)$. These sequences are related by the well-known power series identity

$$U(t) = \left[1 - F(t)\right]^{-1}, \; o \leq t < 1,$$

where

$$U(t) = \sum_{n=o}^{\infty} u_n t^n, \qquad F(t) = \sum_{n=1}^{\infty} f_n t^n, \; 0 \leq t < 1.$$

Let $2^{-m}T_n^{(m)}$ denote the time of last occurrence of $\mathcal{E}$ in the set $\{o, 2^{-m}, 2^{-(m-1)}, \ldots, 2^{-m}n\}$. Then, the representations

$$u_n = Pr(T_n^{(m)} = n), \; q_n = \sum_{r = n+1}^{\infty} f_r = Pr(T_n^{(m)} = o), \; n \geq o,$$

are obvious, together with the extremal factorization property

$$Pr(T_n^{(m)} = k) = Pr(T_k^{(m)} = k)\, Pr(T_{n-k}^{(m)} = o)$$

$$= u_k q_{n-k},$$

$o \leq k \leq n$. Then, upon taking generating functions we readily find that for $s > o$, $z > o$,

(2)
$$(1 - e^{-2^{-m}z}) \sum_{n=0}^{\infty} E(e^{-2^{-m}sT_n^{(m)}})e^{-2^{-m}zn} = U(e^{-2^{-m}(z+s)}) \left[U(e^{-2^{-m}z})\right]^{-1}.$$

The next step is to take the limit as $m \to \infty$ in (2) and in order to proceed with this we need the following lemma.

<u>Lemma</u> <u>Write</u>

$$\phi_m(\lambda, t) = E(e^{-2^{-m}\lambda T^{(m)}_{[2^m t]}}), \quad \phi(\lambda, t) = E(e^{-\lambda T_t}),$$

$[2^m t]$ <u>denoting the integer part of</u> $2^m t$. <u>Then,</u>

$$\lim_{m\to\infty} \phi_m(\lambda, t) = \phi(\lambda, t)$$

<u>uniformly in t in any finite interval</u>.

<u>Proof</u>. Firstly, we note that $2^{-m}T^{(m)}_{[2^m t]}$ is monotone non-decreasing in m and hence that $\phi_m(\lambda, t)$ is monotone non-increasing in m. Consequently, it will be sufficient to show that

$$\lim_{m\to\infty} \phi_m(\lambda, t) = \phi(\lambda, t)$$

for each fixed t and that $\phi(\lambda, t)$ is a continuous function of t.

Now, the indicator process of $\mathcal{E}$ is separable so that for any $x \geq 0$ and any countably dense subset $\{t_1, t_2, \ldots\}$ of $[0, t]$,

$$\Pr(T_t \leq x) = \Pr(Z(u, \omega) = 0 \text{ for } x < u \leq t) = \Pr(T_t^* \leq x)$$

where $T_t^* = \sup\left\{t_j : Z(t_j, \omega) = 1\right\}$. It follows then, using the Helly-Bray Theorem, that

$$\lim_{m\to\infty} \phi_m(\lambda, t) = \phi(\lambda, t)$$

for fixed t. Also, if

$$\tau(t) = \tau(t, \omega) = \int_0^t Z(u, \omega)du,$$

then

$$\Pr(T_{t+\Delta t} > T_t) \leq \Pr(\tau(t + \Delta t) > \tau(t)) \to 0$$

as $\Delta t \to 0$ so that T_t is continious in probability. Finally, since T_t is monotone non-decreasing in t,

(3)
$$\phi(\lambda, t + \Delta t) \leq \phi(\lambda, t),$$

while, letting δ be arbitrarily small and positive and using integration by parts,

$$\phi(\lambda,\ t+\Delta t) = 1 - \lambda\int_0^\infty e^{-\lambda x}\Pr(T_{t+\Delta t} \geq x)dx$$

$$= 1 - \lambda\int_0^\infty e^{-\lambda x}\Pr(T_{t+\Delta t} - T_t + T_t \geq x)dx$$

$$\geq 1 - \lambda\int_0^\infty e^{-\lambda x}\left[\Pr(T_{t+\Delta t} - T_t \geq \delta) + \Pr(T_t \geq x - \delta)\right]dx$$

$$= 1 - \Pr(T_{t+\Delta t} - T_t \geq \delta) - \lambda e^{-\lambda\delta}\int_{-\delta}^\infty e^{-\lambda y}\Pr(T_t \geq y)\,dy$$

$$(4)\qquad = 1 - \Pr(T_{t+\Delta t} - T_t \geq \delta) - e^{-\lambda\delta}\left[1 - \phi(\lambda,\ t)\right]$$

$$-\lambda e^{-\lambda\delta}\int_{-\delta}^{0} e^{-\lambda y}\Pr(T_t \geq y)dy,$$

so that, from (3) and (4) and since T_t is continuous in probability,

$$\lim_{\Delta t\to 0} \phi(\lambda,\ t+\Delta t) = \phi(\lambda,\ t).$$

This completes the proof of the lemma and we resume the proof of the theorem.

We can write,

$$(5)\qquad z\int_0^\infty E(e^{-sT_t})e^{-zt}dt = \lim_{m\to\infty}(1 - e^{-2^{-m}z})\sum_{n=0}^{\infty}\phi(s,\ 2^{-m}n)e^{-2^{-m}nz},$$

and we shall show that it is possible to replace $\phi(s,\ 2^{-m}n)$ by $\phi_m(s,\ 2^{-m}n)$ on the right hand side of (5). To show this, let $\epsilon > 0$ be arbitrarily small and let N be a positive integer so large that for fixed z, $e^{-2^N z} < (\frac{1}{4})\epsilon$. Then, from the lemma, we have for sufficiently large m,

$$|\phi_m(s,\ 2^{-m}k) - \phi(s,\ 2^{-m}k)| \leq (1/2)\epsilon,\ k = 0, 1, 2, \ldots, 2^{m+N}.$$

Therefore, for sufficiently large m,

$$\left|(1 - e^{-2^{-m}z})\sum_{n=0}^{\infty}\left[\phi_m(s,\ 2^{-m}n) - \phi(s,\ 2^{-m}n)\right]e^{-2^{-m}nz}\right|$$

$$\leq (1 - e^{-2^{-m}z})\sum_{n=0}^{2^{m+N}}|\phi_m(s,\ 2^{-m}n) - \phi(s,\ 2^{-m}n)|e^{-2^{-m}nz}$$

$$(6)\qquad + 2(1 - e^{-2^{-m}z})\sum_{n=2^{m+N}}^{\infty}e^{-2^{-m}nz}$$

$$\leq \frac{\epsilon}{2} + 2\cdot\frac{\epsilon}{4} = \epsilon,$$

and from (5) and (6) we obtain

$$z\int_0^\infty E(e^{-sT_t})e^{-zt}dt = \lim_{m\to\infty}(1 - e^{-2^{-m}z})\sum_{n=0}^\infty \phi_m(s,\ 2^{-m}n)e^{-2^{-m}nz},$$

which deals with the left hand side of (2).

For the right hand side of (2), we firstly make use of a result of DWASS (see PORT [7]) which allows us to express $U(t)$ in the form

$$U(t) = \exp\left\{\sum_{k=1}^\infty t^k\Delta_{km}k^{-1}\right\},\quad 0 \leq t < 1,$$

where

$$\Delta_{km} = 2^{-m}E\left[T_k^{(m)} - T_{k-1}^{(m)}\right],\ k \geq 1.$$

Then,

$$U(e^{-2^{-m}(z+s)})\left[U(e^{-2^{-m}z})\right]^{-1}$$

(7) $$= \exp\left\{-\sum_{k=1}^\infty k^{-1}e^{-2^{-m}kz}(1 - e^{-2^{-m}ks})\Delta_{km}\right\}.$$

Now, using summation by parts,

$$\sum_{k=1}^\infty k^{-1}e^{-2^{-m}kz}(1 - e^{-2^{-m}ks})\Delta_{km}$$

(8) $$= \sum_{k=1}^\infty 2^{-m}ET_k^{(m)}\left\{\frac{1}{k}e^{-2^{-m}ks}) - \frac{1}{(k+1)}e^{-2^{-m}(k+1)z}(1 - e^{-2^{-m}(k+1)s}\right\},$$

while, again making use of the lemma, we obtain without difficulty that

$$\lim_{m\to\infty}\sum_{k=1}^\infty 2^{-m}ET_k^{(m)}\left\{\frac{1}{k}e^{-2^{-m}kz}(1 - e^{-2^{-m}ks}) - \frac{1}{(k+1)}e^{-2^{-m}(k+1)z}(1 - e^{-2^{-m}(k+1)s}\right\}$$

$$= -\int_0^\infty ET_t\,d_t\left\{\frac{1}{t}e^{-tz}(1 - e^{-ts})\right\}$$

(9) $$= \int_0^\infty t^{-2}ET_te^{-zt}\left[(1 + tz) - (1 + t(z+s)e^{-st})\right]dt.$$

The second part of (1) follows immediately from (7), (8) and (9).

Finally, for $\theta > 0$,

$$2^{-m}U(e^{-2^{-m}\theta}) = 2^{-m}\sum_{n=0}^\infty \Pr(T_n^{(m)} = n)e^{-2^{-m}\theta}$$

$$= 2^{-m}\sum_{n=0}^{\infty} p(2^{-m}n)e^{-2^{-m}\theta}$$

$$\to r(\theta) = \int_0^{\infty} e^{-\theta t}p(t)dt$$

as $m\to\infty$ and, consequently, the right hand side of (2) also converges to $r(z+s)\left[r(z)\right]^{-1}$ as $m\to\infty$. This provides the third part of (1) and thus completes the proof of the theorem.

Corollary 1. For $z > 0$,

$$zr(z) = \exp\left\{-\int_0^{\infty} t^{-1}e^{-tz}d_t(t - ET_t)\right\}. \tag{10}$$

Proof. Using Theorem 1, we have with the aid of easy calculations that

$$\frac{zr(z)}{(z+s)r(z+s)} = \exp\left\{\int_0^{\infty} t^{-2}ET_te^{-zt}\left[(1+tz) - (1 + t(z+s)e^{-st})\right]dt - \log(1 + z^{-1}s)\right\}$$

$$= \exp\left\{\int_0^{\infty}\left[t^{-1}ET_t - 1\right]t^{-1}e^{-zt}\left[(1 + tz) - (1 + t(z+s)e^{-st})\right]dt\right\}$$

$$= \exp\left\{-\int_0^{\infty}\left[ET_t - t\right]d_t\left[t^{-1}e^{-zt}(1 - e^{-st})\right]\right\}$$

$$= \exp\left\{\int_0^{\infty} t^{-1}e^{-zt}(1 - e^{-st})d_t(t - ET_t)\right\}. \tag{11}$$

Furthermore, in Theorem 3 of [4] it is shown that there exists a unique positive measure μ on $(0,\infty]$ with

$$\int_{(0,\infty]} (1 - e^{-x})\mu(dx) < \infty\,,$$

such that for $\theta > 0$,

$$r(\theta) = \left[\theta + \int_{(0,\infty]} (1 - e^{-\theta x})\mu(dx)\right]^{-1},$$

and from this represtation it follows immediately that $\theta r(\theta)\to 1$ as $\theta\to\infty$. The result (10) is then obtained by letting $s\to\infty$ in (11).

Theorem 2. Let μ be the canonical measure of the regenerative phenomenon $\mathcal{E}$. That is, μ is a positive measure on $(0,\infty]$ for which

$$\int_{(0,\infty]} (1 - e^{-x})\mu(dx) < \infty, \tag{12}$$

and such that for $\theta > o$,

$$(13) \qquad r(\theta) = \int_o^\infty e^{-\theta t} p(t)dt = \left[\theta + \int_{(o,\infty]} (1 - e^{-\theta x})\mu(dx)\right]^{-1}.$$

Then, the distribution of T_t *is given by*

$$(14) \qquad \begin{aligned} Pr(T_t < u) &= \int_o^u \mu(t - v, \infty]\, p(v)dv, \quad o \leq u < t, \\ Pr(T_t = t) &= p(t). \end{aligned}$$

Proof. In view of (12), $\mu(x,\infty]$ is bounded in $(a, \infty]$ and integrable over (o, a) for each $a > o$. Furthermore, its Laplace transform $\int_o^\infty e^{-\theta x}\mu(x,\infty]\, dx$ exists for $\theta > o$ and

$$\int_o^\infty e^{-\theta x}\mu(x, \infty]dx = \theta^{-1}\int_{(o,\infty]} (1 - e^{-\theta x})\mu(dx).$$

Consequently, from (13),

$$\left[\theta r(\theta)\right]^{-1} = 1 + \int_o^\infty e^{-\theta x}\mu(x, \infty]dx.$$

It then follows from Theorem 1 that

$$\begin{aligned} \int_o^\infty E(e^{-sT_t})e^{-zt}dt &= r(z + s) \left[1 + \int_o^\infty e^{-zx}\mu(x, \infty]dx\right] \\ &= \int_o^\infty e^{-(z + s)t} p(t)dt + \int_o^\infty e^{-zt}\left\{\int_o^t e^{-sv} p(v)\mu(t - v,\infty]dv\right\} dt, \end{aligned}$$

so that

$$E(e^{-sT_t}) = e^{-st}p(t) + \int_o^t e^{-sv}p(v)\mu(t - v, \infty]dv,$$

and a further inversion yields (14). We note that

$$Pr(T_t \leq t) = 1 = p(t) + \int_o^t p(t - v)\mu(v, \infty]dv,$$

which is the Volterra integral equation obtained in Proposition 7 of [4]. Theorem 2 shows us clearly that the function $p(t)$ uniquely determines the distribution of T_t for all $t > o$ and vice-versa. We note also the following result which is obtained by differentiating in (14),

$$\mu(t - u, \infty] = \frac{\frac{d}{du} Pr(T_t < u)}{Pr(T_u = u)}, \quad o < u < t.$$

3. Limit behavior

In [4], it is shown that there are three possibilities for the ergodic behavior of a standard regenerative phenomenon $\mathcal{E}$:

$$\begin{array}{lll} \text{(I)} & \mu\{\infty\} > 0 & \text{(transient)}, \\ \text{(II)} & \mu\{\infty\} = 0, \int_{(0,\infty)} x\mu(dx) = \infty & \text{(null)}, \\ \text{(III)} & \mu\{\infty\} = 0, \int_{(0,\infty)} x\mu(dx) < \infty & \text{(positive)}. \end{array}$$

<u>Theorem 3.</u> <u>If $\mathcal{E}$ is transient, then</u>

$$\lim_{t\to\infty} \Pr(T_t < u) = \left[\mu\{\infty\}\right]^{-1} \int_0^u p(x)dx. \tag{15}$$

<u>If $\mathcal{E}$ is positive, then</u>

$$\lim_{t\to\infty} \Pr(t - T_t < u) = \frac{1 + \int_0^u \mu(x, \infty]dx}{1 + \int_0^\infty \mu(x, \infty]dx}.$$

<u>Proof.</u> Suppose firstly that $\mathcal{E}$ is transient. Then, from Proposition 8 of [4],

$$\left[\mu\{\infty\}\right]^{-1} = \int_0^\infty p(t)dt < \infty.$$

The result (15) follows immediately upon proceeding to the limit in (14).

Next, suppose that $\mathcal{E}$ is positive. From Theorem 6 of [4],

$$p(t) \to \frac{1}{1 + \int_0^\infty \mu(x,\infty]dx}$$

as $t \to \infty$. Then, from Theorem 2,

$$\begin{aligned} \Pr(t - T_t < u) &= 1 - \int_0^{t-u} \mu(t - v, \infty]p(v)dv \\ &= p(t) + \int_t^{t-u} \mu(t - v, \infty]p(v)dv \\ &= p(t) + \int_0^u \mu(x, \infty]p(t - x)dx \\ &\to \frac{1 + \int_0^u \mu(x, \infty]dx}{1 + \int_0^\infty \mu(x, \infty]dx} \end{aligned}$$

as $t \to \infty$. This completes the proof of the theorem.

<u>Definition.</u> A regenerative phenomenon $\mathcal{E}$ will be called <u>β- regular</u> if $\lim_{t \to \infty} t^{-1}ET_t = \beta$ (obviously $0 \leq \beta \leq 1$).

The concept of β-regularity is important by virtue of the following theorem which is the regenerative phenomenon analogue of Theorem 3.2 of LAMPERTI [6] for the recurrent event context.

<u>Theorem 4.</u> The limiting distribution

(17) $$\lim_{t \to \infty} \Pr(t^{-1}T_t < x) = F(x)$$

exists if and only if $\mathcal{E}$ is β-regular and then $F(x)$ is related to β by

$$F(x) = F_\beta(x) = \frac{\sin \pi\beta}{\pi} \int_0^x v^{-(1-\beta)}(1-v)^{-\beta} dv, \quad 0 < \beta < 1,\ 0 \leq x \leq 1,$$

(18) $$F_0(x) = 0 \text{ if } x < 0,\ 1 \text{ if } x \geq 0,$$
$$F_1(x) = 0 \text{ if } x < 1,\ 1 \text{ if } x \geq 1.$$

<u>Proof.</u> We shall first establish that the condition of β-regularity is sufficient for the existence of the limiting distribution (17). In order to do this, we show firstly that under the condition of β-regularity and when $0 < \lambda < 1$,

(19) $$\lim_{z \to 0} \int_0^\infty t^{-2}ET_t e^{-zt}\left[(1 + tz) - (1 + tz(1+\lambda))e^{-\lambda zt}\right]dt = \beta \log(1 + \lambda).$$

Write

$$C(z,\ t) = t^{-1}e^{-zt}\left[(1 + tz) - (1 + tz(1+\lambda)e^{-\lambda zt}\right],$$

and note that $C(z,\ t) \geq 0$ and $\lim_{z \to 0} C(z,\ t) = 0$. Thus, for $z > 0$,

$$\int_0^\infty t^{-1}ET_t C(z,\ t)dt = \int_0^\infty \left[t^{-1}ET_t - \beta\right] C(z,\ t)dt + \beta \int_0^\infty C(z,\ t)dt$$
$$= \int_0^\infty \left[t^{-1}ET_t - \beta\right] C(z,\ t)dt + \beta \log(1 + \lambda),$$

upon performing a simple integration. Now, in view of the β-regularity condition we can, given $\epsilon > 0$ arbitrarily small, choose T so large that $|t^{-1}ET_t - \beta| < \epsilon$ for $t \geq T$ and then

$$\left| \int_0^\infty \left[t^{-1}ET_t - \beta \right] C(z,\ t)dt \right| \leq \int_0^T |t^{-1}ET_t - \beta| C(z,\ t)dt + \epsilon \log(1 + \lambda)$$

$$\to \epsilon \log(1 + \lambda)$$

as $z \to o$ since $\lim_{z \to o} C(z,\ t) = o$. The result (19) follows immediately. Then, putting $s = \lambda z$ where $o < \lambda < 1$ in the result of Theorem 1 and making use of (19), we obtain

(20)
$$\lim_{z \to o} z \int_0^\infty e^{-zt} E(e^{-\lambda\, zT_t})dt = (1 + \lambda)^{-\beta}.$$

Now,

$$z\int_0^\infty e^{-zt} E(e^{-\lambda z T_t})dt = z\int_0^\infty e^{-zt} \sum_{k=o}^{\infty} \frac{(-\lambda\, z)^k ET_t^k}{k!}\, dt$$

$$= \sum_{k=o}^{\infty} \lambda^k A_k(z),$$

where

$$A_k(z) = -\frac{(-z)^{k+1}}{k!} \int_0^\infty e^{-zt} ET_t^k dt,$$

so that from (20),

$$\lim_{z \to o} \sum_{k=o}^{\infty} \lambda^k A_k(z) = (1 + \lambda)^{-\beta} = \sum_{k=o}^{\infty} \lambda^k \binom{-\beta}{k},$$

and consequently,

(21)
$$\lim_{z \to o} A_k(z) = \binom{-\beta}{k},\ k \geq o.$$

But, ET_t^k is monotone in t so, using Theorem 4, 423, Vol. II of [1], it follows from (21) that as $t \to \infty$,

(22)
$$E(t^{-1}T_t)^k \to (-1)^k \binom{-\beta}{k},\ k \geq o.$$

Furthermore, it is easy to verify that

$$(-1)^k \binom{-\beta}{k} = \int_0^1 x^k F_\beta(dx),$$

where $F_\beta(x)$ is given by (18) and, since the moment problem in this case has a unique solution, the proof of the sufficiency part of the theorem is complete.

Finally, suppose that $t^{-1}T_t$ has a proper limiting distribution. Then, necessarily, $t^{-1}ET_t \to \beta$ for some $0 \leq \beta \leq 1$ so that $\mathcal{E}$ is β-regular. This completes the proof of the theorem.

In view of the importance of the β-regularity concept, we shall next give some equivalent forms which may provide more useful criteria under certain circumstances.

Theorem 5. *A regenerative phenomenon $\mathcal{E}$ is β-regular if and only if*

$$r(z) \sim z^{-\beta} L(z^{-1}) \tag{23}$$

as $z \to o$ or equivalently,

$$\int_o^t p(u)du \sim \frac{1}{\Gamma(1 + \beta)} t^{\beta} L(t) \tag{24}$$

as $t \to \infty$, $L(x)$ being a slowly varying function as $x \to \infty$. In the particular case where $p(t)$ is ultimately monotone and $\beta > o$, the conditions (23) and (24) are also equivalent to

$$p(t) \sim \frac{1}{\Gamma(\beta)} t^{-(1 - \beta)} L(t) \tag{25}$$

as $t \to \infty$.

Proof. We shall deal firstly with the condition (23). Suppose that $\mathcal{E}$ is β-regular. Then, making use of Theorem 1 and equation (20),

$$\lim_{z \to o} \frac{r((1 + \lambda)z)}{r(z)} = (1 + \lambda)^{-\beta}$$

for $o < \lambda < 1$ and (23) follows by use of the Theorem, 270 Vol. II of [1].

Conversely, suppose that the condition (23) holds. Then, again making use of Theorem 1, we have for $o < \lambda < 1$,

$$\lim_{z \to o} \int_o^{\infty} e^{-zt} E(e^{-sT_t})dt = \lim_{z \to o} \frac{r((1 + \lambda)z)}{r(z)} = (1 + \lambda)^{-\beta},$$

which is equation (20). Following the proof of Theorem 4, we then deduce from (22) the required β-regularity condition. This completes the proof that (23) is a necessary and sufficient condition for β-regularity. The remainder of the proof is then immediately completed by appeal to Theorem 2, 421 Vol. II of [1] for condition (24) and Theorem 4, 423, Vol. II of [1] for condition (25).

For $s > o$, $t > o$, write

$$P(s, t) = Pr\left(\sup_{s \leq u \leq s + t} Z(u, w) = 1 \right).$$

That is, $P(s, t)$ is the probability that $\mathcal{E}$ will occur in the time interval $[s, s + t]$.

Theorem 6. *Suppose $\mathcal{E}$ is β-regular. Then, for any $\alpha > o$,*

$$\text{(26)} \quad \lim_{t\to\infty} P(t, \alpha t) = \begin{cases} \dfrac{\sin \pi\beta}{\pi} \displaystyle\int_{(1+\alpha)^{-1}}^{1} x^{-(1-\beta)}(1-x)^{-\beta}dx, & 0 < \beta < 1 \\ o & , \beta = o, \\ 1 & , \beta = 1. \end{cases}$$

If $\mathcal{E}$ is not β-regular for some β, $o \leq \beta \leq 1$, then the limit of $P(t, \alpha t)$ as $t\to\infty$ does not exist. In the particular case $\beta = 1/2$, (26) yields

$$\lim_{t\to\infty} P(t, \alpha t) = 1 - 2\pi^{-1} \arcsin \left[(1+\alpha)^{-\frac{1}{2}}\right],$$

so that

$$\text{(27)} \qquad \lim_{t\to\infty}\left[P(t, \alpha t) + P(\alpha t, t)\right] = 1.$$

Proof. We have

$$P(s, t) = \Pr(T_{s+t} \geq s),$$

so that

$$\lim_{t\to\infty} P(t, \alpha t) = \lim_{t\to\infty} \Pr(T_{t(1+\alpha)} \geq t) = \lim_{t\to\infty} \Pr(t^{-1}T_t \geq (1+\alpha)^{-1}),$$

and the result (26) follows from Theorem 4. It also follows from Theorem 4 that the limit as $t\to\infty$ of $P(t, \alpha t)$ only exists in the case of β-regularity.

In the case $\beta = \frac{1}{2}$, we have

$$\lim_{t\to\infty} P(t, \alpha t) = \pi^{-1}\int_{(1+\alpha)^{-1}}^{1} x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\, dx$$

$$= 1 - \pi^{-1}\int_{0}^{(1+\alpha)^{-1}} x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx = 1 - 2\pi^{-1} \arcsin\left[(1+\alpha)^{-\frac{1}{2}}\right]$$

by a simple transformation. The result (27) follows as

$$\arcsin\left[(1+\alpha)^{-\frac{1}{2}}\right] + \arcsin\left[\alpha^{\frac{1}{2}}(1-\alpha)^{-\frac{1}{2}}\right] = \tfrac{1}{2}\pi.$$

This completes the proof of the theorem.

Theorem 6 is the counterpart of Theorem 2 of HEYDE [2] for recurrent events. Unlike the recurrent event case, however, it cannot happen that

$$\text{(28)} \qquad P(s, t) + P(t, s) = 1$$

for all $s > o$, $t > o$. In order to see this, we note that (28) is precisely the condition that T_t and $t - T_t$ should have the same distribution for each $t > o$. This is clearly impossible by Theorem 2.

4. Examples

Let $\xi(t)$, $t \geq o$, $\xi(o) = o$ be a separable stochastic process with stationary independent increments whose sample functions are continuous on the left. Write $\bar{\xi}(t) = \sup_{o \leq s \leq t} \xi(s)$ and $T_t = \min[u : \xi(u) = \bar{\xi}(t)]$. We shall show that $\mathcal{E}$, where $E(t)$ is the event $\{T_t = t\}$, is a proper regenerative phenomenon if and only if

$$\int_o^1 t^{-1} \Pr(\xi(t) < o)dt < \infty.$$

This is in contrast with the corresponding discrete case where $\mathcal{E}$ is always a recurrent event. A discussion of the condition

$$\int_o^1 t^{-1} \Pr(\xi(t) < o)dt < \infty$$

can be found in ROGOZIN [8]. It is satisfied, for example, if $\xi(t)$ is a process with negative jumps and positive drift.

Write $\eta(t) = -\xi(t)$ and $\bar{\eta}(t) = \sup_{o \leq s \leq t} \eta(s)$. Then, it can readily be verified that

$$p(t) = \Pr\{E(t)\} = \Pr(\bar{\eta}(t) = o).$$

Furthermore, from equation (1) of ROGOZIN [8], we have for $\mathrm{Re}\lambda \leq o$,

$$u\int_o^\infty e^{-ut}\left\{\int_o^\infty e^{\lambda x} d_x \Pr(\bar{\eta}(t) < x)\right\}dt = \exp\left\{-\int_o^\infty (e^{\lambda x} - 1)d_x \int_o^\infty t^{-1}\Pr(\eta(t) > x)e^{-ut}dt\right\},$$

and upon letting $\lambda \to -\infty$ we obtain

$$\text{(29)} \quad ur(u) = u\int_o^\infty e^{-ut}\Pr(\bar{\eta}(t) = o)dt = \exp\left\{-\int_o^\infty t^{-1}\Pr(\eta(t) > o)e^{-ut}dt\right\}$$

when the integral

$$\text{(30)} \quad \int_o^1 t^{-1}\Pr(\xi(t) < o)dt = \int_o^1 t^{-1}\Pr(\eta(t) > o)dt$$

converges and

$$\text{(31)} \quad u\int_o^\infty e^{-ut}\Pr(\bar{\eta}(t) = o)dt = o$$

when (30) diverges. The equation (31) of course implies that $p(t) = o$ for $t > o$ and the regenerative phenomenon definition breaks down. In the case where (30) converges we let $u \to \infty$ in (29) and obtain $\lim_{t \to o} p(t) = 1$. It is easily checked in this case that whenever

$$o < t_1 < t_2 < \dots < t_k,$$

then

$$\Pr\left\{E(t_1), E(t_2), \dots, E(t_k)\right\} = \Pr\left\{E(t_1)\right\} \Pr\left\{E(t_2 - t_1), \dots, E(t_k - t_1)\right\}.$$

Equation (29) can, of course, be indentified with equation (10) (Corollary 1) so we see that

$$ET_t = \int_o^t \Pr(\xi(u) \geq o)du,$$

and therefore $\mathcal{E}$ is β-regular if and only if

$$\lim_{t \to \infty} t^{-1} \int_o^t \Pr(\xi(u) \geq o)du = \beta.$$

The use of Theorem 4 in this context provides a result which is a special case of that of HEYDE [3].

As a final example, we mention a regenerative phenomenon which occurs in connection with the queueing system $M|G|1$. Customers arrive at a single server in a Poisson process of rate λ and the service time distribution is of arbitrary type. Then, if the server is initially idle, the event of the server being idle forms a regenerative phenomenon E (KINGMAN [4], [5]). We have

$$p(t) = \Pr\{\text{queue empty at time } t\},$$

and the random variable $t - T_t$ represents the time for which the current busy period has been in progress. It is not very difficult to show, making use of the results of Section 3.5 of [5] and of Theorem 6 and Section 16 of [4], that $\mathcal{E}$ is 1-regular if and only if the mean service time is less than λ^{-1}, o-regular if and only if the mean service time is greater than λ^{-1}, and β-regular, $\frac{1}{2} \leq \beta < 1$, if and only if the mean service time is equal to λ^{-1} and its distribution belongs to the domain of attraction of a stable law of index β^{-1}. $\mathcal{E}$ cannot be β-regular for $o < \beta < \frac{1}{2}$.

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13. ABSTRACT

Let $\{E(t), \ t > 0\}$ be a family of regenerative events, as originally introduced by KINGMAN, and let $Z(t, \omega) = 1$ if $\omega \in E(t)$ and be zero otherwise. Define

$$T_t = \sup\{u : o \leq u \leq t;\ Z(u, \omega) = 1\}$$

This paper obtains various theorems about T_t and studies its limiting behavior as $t \to \infty$.

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