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FEM & BIEM - A NEW INFINITE HYBRID FINITE ELEMENT

P. Drakaliev

1. INTRODUCTION

The finite element method (MFE) and the boundary integral equation method (BIEM) are general approximation procedures applicable to a wide variety of engineering problems. Each of them has many variants and each possesses certain merits and limitations of its own. The FEM may be easier to apply in domains with anisotropic or nonlinear behaviour. On the other hand the BEM is more attractive for unbounded domains or regions of high stress concentration. Therefore, the idea of combining both numerical techniques is of great interest in many practical problems, especially in solid and fluid mechanics, such as soil-structure and structure-fluid interaction problems. An extensive number of papers now exist on the coupling between boundary and finite elements, some of them are [2,4,5].

In the developments to follow an energy approach for symmetrizing the indirect BIEM is being used to obtain the stiffness matrix for the infinite or semi-infinite elastic medium. Thus the subdomain is considered as an infinite super element with an arbitrary shaped boundary and can be easily implemented into existing finite element codes.

2. VARIATIONAL FORMULATION FOR A FINITE SUBREGION

Throughout the paper the Cartesian coordinate system x_i (i=1,2,3) and the standard tensor notation are used. The subscripts following commas indicate the differentiations with respect to the coordinate variables and the repeated indices imply the summation convention.

An elastic body occupying the domain $V \subset R^3$ with a boundary S is considered. The body is deformed under the action of prescribed body forces $\bar{f}_i(x)$, boundary tractions $\bar{t}_i(x)$ on the regular part S_σ of S, and displacements $\bar{u}_i(x)$ on S_u. The stress-strain state of the body is characterized by the displacements $u_i(x)$ and the corresponding stresses $\sigma_{ij}(x)$ and strains $\varepsilon_{ij}(x)$.

Let the body consist of a finite number of homogeneous subdomains $V^{(m)} \subseteq V$ with arbitrary shaped boundaries $S^{(m)}$. The subdomains might also be multiconnected.

The stress-strain state of the body can be described by a modified functional of the complementary energy, proposed by Pian [1]:

$$\Pi(\sigma,\tilde{u}) = \sum_m \left[-\frac{1}{2} \int_{V^{(m)}} C_{ijkl}\, \sigma_{ij}\, \sigma_{kl}\, dV + \int_{S^{(m)}} t_i\, \tilde{u}_i\, dS - \int_{S_\sigma^{(m)}} \bar{t}_i\, \tilde{u}_i\, dS \right] \tag{1}$$

under the additional conditions:
- equilibrium equations

$$\sigma_{ij,j}(x) + \bar{f}_i(x) = 0\,, \qquad x \in V^{(m)}\,; \tag{2}$$

- constitutive relationships

$$\sigma_{ij}(x) = \lambda\,\delta_{ij}\,u_{k,k} + \mu(u_{i,j} + u_{j,i})\,, \qquad x \in V^{(m)},\ S^{(m)}\,; \tag{3}$$

- prescribed boundary displacements

$$\tilde{u}_i(x) = \bar{u}_i(x)\,, \qquad x \in S_u^{(m)}\,, \tag{4}$$

where C_{ijkl} is the elastic compliance tensor, λ and μ are the Lame constants, δ_{ij} - Kronecker's delta, and $\tilde{u}_i(x)$ - the displacements over the subdomain boundary $S^{(m)}$.

The independent variables subject to variation in Eq.(1) are the stresses $\sigma_{ij}(x)$ and the boundary displacements $\tilde{u}_i(x)$ which are common for two adjacent subdomains and satisfy the prescribed boundary conditions (4).

It is easy to show that

$$\delta\Pi = \sum_m \Big[-\int_{S^{(m)}} (u_i - \tilde{u}_i)\,\delta t_i\,dS + \int_{S_\sigma^{(m)}} (t_i - \bar{t}_i)\,\delta\tilde{u}_i\,dS \ \Big]\,. \tag{5}$$

The vanishing $\delta\Pi$ leads to the Euler equations:

$$u_i(x) = \tilde{u}_i(x)\,, \qquad x \in S^{(m)}\,; \tag{6}$$

$$t_i(x) = \sigma_{ij}(x)\,n_j(x) = \bar{t}_i(x)\,, \qquad x \in S_\sigma^{(m)}\,, \tag{7}$$

where $n_i(x)$ is the unit outward normal at a point $x \in S^{(m)}$.

Equation (6) expresses the compatibility conditions

$$u_i^{(m)}(x) = u_i^{(k)}(x)\,, \qquad x \in S_{int}^{(m)} \tag{8}$$

over the interface between subdomains $V^{(m)}$ and $V^{(k)}$ because $\tilde{u}_i(x)$ are common for these two adjacent subregions. Eq. (7) gives the prescribed boundary traction conditions. Functional (1) does not satisfy the interface equilibrium conditions

$$t_i^{(m)}(x) = -t_i^{(k)}(x)\,, \qquad x \in S_{int}^{(m)}\,. \tag{9}$$

Integrating the first integral in Eq.(1) by parts, we have

$$\int_{V^{(m)}} C_{ijkl}\,\sigma_{ij}\,\sigma_{kl}\,dV = \int_{S^{(m)}} u_i\,t_i\,dS + \int_{V^{(m)}} u_i\bar{f}_i\,dV\,, \tag{10}$$

so that the functional (1) becomes

$$\Pi(\sigma,\tilde{u}) = \sum_m \Big[-\frac{1}{2}\int_{S^{(m)}} t_i\,u_i\,dS + \int_{S^{(m)}} t_i\,\tilde{u}_i\,dS - \int_{S_\sigma^{(m)}} \bar{t}_i\,\tilde{u}_i\,dS - \frac{1}{2}\int_{V^{(m)}} u_i\bar{f}_i\,dV \ \Big]\,. \tag{11}$$

This functional will be used to obtain the stiffness matrices for subdomains which are considered as hybrid finite or infinite super elements.

3. FINITE ELEMENT MODEL

To compute the displacements, the strains and the stresses within the subdomain $V^{(m)}$, the wellknown from the indirect boundary integral equation method identities are used. The boundary integral equations for the displacements and the tractions are obtained from them by carrying out a limiting process for $x \in S^{(m)}$. Then for a smooth boundary

$$u_i(x) = \int_{S^{(m)}} U_{ij}(x,y)\,\varphi_j(y)\,dS(y) + \int_{V^{(m)}} U_{ij}(x,z)\bar{f}_j(z)\,dV(z)\,; \tag{12}$$

$$t_i(x) = \frac{1}{2}\delta_{ij}\varphi_j(x) + \int_{S^{(m)}} T_{ij}(x,y)\,\varphi_j(y)\,dS(y) + \int_{V^{(m)}} T_{ij}(x,z)\bar{f}_j(z)\,dV(z)\,; \tag{13}$$

$$x,y \in S^{(m)};\ z \in V^{(m)},$$

where $U_{ij}(x,y)$ and $T_{ij}(x,y)$ are the fundamental solutions, corresponding to a concentrated force, acting at a point y in the infinite elastic space; $\varphi_i(y)$ are the fictitious tractions acting on the subdomain boundary. The first integral in Eq.(13) is to be interpreted in the sense of Cauchy principal value and the remaining integrals in Eqs.(12) and (13) present no special singularities and can be interpreted in the normal sense of integrations.

Subject to certain conditions of functional and surface smoothness. the solution of the integral equations exists, and varies smoothly over the boundary. However, there is no guarantee that in general the proposed fictitious force distribution exits if the boundary has corners and edges. It is likely to be unbounded and difficult to represent numerically. In that case one can use partially discontinuous boundary elements [6] which have both continuous and discontinuous edges. They perform satisfactorily and realise the desired optimal compromise. Then the boundary is always smooth and Eq.(13) is valid.

Both Eqs.(12) and (13) satisfy the additional conditions (2) to (4) and can be used as trial functions for the functional (11). The boundary $S^{(m)}$ is represented by boundary elements and the functions $\varphi_i(x)$ and $\tilde{u}_i(x)$ are given in terms of nodal values b_i^n and q_i^n

$$\varphi_i(x) = M^n(x)\,b_i^n, \qquad x \in S^{(m)}; \tag{14}$$

$$\tilde{u}_i(x) = N^n(x)\,q_i^n, \qquad x \in S^{(m)}; \tag{15}$$

Substituting Eq.(14) into Eqs.(12) and (13) one obtains (in matrix notations)

$$\begin{aligned} \boldsymbol{u}(x) &= \boldsymbol{U}(x)\,\boldsymbol{b} + \boldsymbol{F}_1(x)\,, \\ \boldsymbol{t}(x) &= \frac{1}{2}\varphi + \boldsymbol{T}(x)\,\boldsymbol{b} + \boldsymbol{F}_2(x), \qquad x \in S^{(m)} \end{aligned} \tag{16}$$

where

$$\begin{aligned} \boldsymbol{U}(x) &= \int_{S^{(m)}} \boldsymbol{U}(x,y)\,\boldsymbol{M}(y)\,dS(y)\,, \\ \boldsymbol{T}(x) &= \int_{S^{(m)}} \boldsymbol{T}(x,y)\,\boldsymbol{M}(y)\,dS(y)\,, \end{aligned} \tag{17}$$

$$F_1(x) = \int_{V^{(m)}} U(x,z)\bar{f}(z)\, dV(z) ,$$
$$F_2(x) = \int_{V^{(m)}} T(x,z)\bar{f}(z)\, dV(z) .$$

A substitution of Eqs.(15) and (16) into the functional expression (11) yields

$$\Pi = \sum_m [-b^T H b + b^T R q + b^T B + Q q + \text{const}] \tag{18}$$

where

$$\begin{aligned} H &= \frac{1}{2}\int_{S^{(m)}} [\frac{1}{2} M(x) + T(x)]^T U(x)\, dS(x) , \\ R &= \int_{S^{(m)}} [\frac{1}{2} M(x) + T(x)]^T N(x)\, dS(x) , \\ B &= -\frac{1}{2}\int_{S^{(m)}} [\frac{1}{2} M(x) + T(x)]^T F_1(x)\, dS(x) - \\ &\quad - \frac{1}{2}\int_{S^{(m)}} U^T(x)\, F_2(x)\, dS(x) - \frac{1}{2}\int_{V^{(m)}} U^T(z)\bar{f}(z)\, dV(z) , \\ Q &= \int_{S^{(m)}} F_2(x)\, N(x)\, dS(x) - \int_{S_\sigma^{(m)}} \bar{t}^T(x)\, N(x)\, dS(x) . \end{aligned} \tag{19}$$

The variation of Π with respect to the quantities b gives

$$-(H + H^T)\, b + Rq + B = 0 . \tag{20}$$

From this equation b can be solved in terms of q and then substituted into Eq.(18), thus obtaining

$$\begin{aligned} \Pi^{(m)} = & \; q^T [-R^T(H + H^T)^{-1} H (H + H^T)^{-1} R + R^T(H + H^T)^{-1}R]\, q + \\ & \; q^T [R^T(H + H^T)^{-1} B + Q^T] + const \end{aligned} \tag{21}$$

The variation of $\Pi^{(m)}$ with respect to the quantities q gives

$$K^{(m)} q + F^{(m)} = 0 , \tag{22}$$

where

$$\begin{aligned} K^{(m)} &= R^T(H + H^T)^{-1}R , \\ F^{(m)} &= R^T(H + H^T)^{-1}B + Q^T . \end{aligned} \tag{23}$$

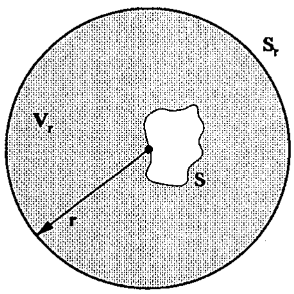
$K^{(m)}$ is the stiffness matrix for the subdomain considered as a hybrid finite super element. $F^{(m)}$ is the generalized force vector. By the standard procedure, we can determine the stiffness matrix of the entire body and solve for q.

4. INFINITE MODELS

Although the functional (11) and the boundary integral equations (12) and (13) have been derived considering that the domain $V^{(m)}$ is bounded, all concepts presented thus far are also valid for infinite regular regions in the sense defined by Kellog, i.e., regions bounded by a regular surface and containing all sufficiently distant points. However, for this extension to be valid, certain regular conditions concerning the behaviour of the functions on a surface which is infinitely remote from the origin must be fulfilled.

Let r be the radius of spherical surface S_r and centred at x, which encloses the cavity of the external problem. Eqs. (12) and (13) can be written for the region V_r as follows

$$u_i(x) = \int_S U_{ij}(x,y)\,\varphi_j(y)\,dS(y) + \int_{S_r} U_{ij}(x,y)\,\varphi_j(y)\,dS(y)\;;$$
$$t_i(x) = \frac{1}{2}\delta_{ij}\,\varphi_j(x) + \int_S T_{ij}(x,y)\,\varphi_j(y)\,dS(y) + \int_{S_r} T_{ij}(x,y)\,\varphi_j(y)\,dS(y)\,. \tag{24}$$



Clearly if the limiting case $r \to \infty$ is considered, Eqs. (12) and (13) can be expressed in terms of boundary integrals over S alone if

$$\lim_{r\to\infty} \int_{S_r} U_{ij}(x,y)\,\varphi_j(y)\,dS(y) = 0\,,$$
$$\lim_{r\to\infty} \int_{S_r} T_{ij}(x,y)\,\varphi_j(y)\,dS(y) = 0\,. \tag{25}$$

For three-dimensional problems, one has

$$dS(y) = |J|\,d\xi_1 d\xi_2 \quad \text{with } |J| = O(r^2)\,,$$
$$U_{ij}(x,y) = O(r^{-1})\,, \tag{26}$$
$$T_{ij}(x,y) = O(r^{-2})\,.$$

where O( ) represents the asymptotic behaviour as $r \to \infty$. Therefore, if at most $\varphi_i(y)$ have the behaviour r^{-2} at infinity, the regular conditions (25) are satisfied. Thus

$$u_i(x) = O(r^{-1})\,,$$
$$t_i(x) = O(r^{-2})\,. \tag{27}$$

Carrying out the same limiting process for the functional (11) one has

$$\lim_{r\to\infty}\left[-\frac{1}{2}\int_{S_r} t_i(y)\,u_i(y)\,dS(y) + \int_{S_r} t_i(y)\,\tilde{u}_i(y)\,dS(y) \right] = 0\,. \tag{28}$$

Since $u_i(x) = O(r^{-1})$ we suppose $\tilde{u}_i(x) = O(r^{-1})$ and the regular condition (28) is also satisfied. The functional (11) can be expressed in terms of boundary integrals over S. The above discussion strongly suggest that the regularity conditions are always satisfied if $u_i(x)$ and $t_i(x)$ behave at most like the corresponding fundamental solution at infinity.

This statement is also verified for semi-infinite problems where the half-space fundamental solutions dictate the corresponding conditions. To compute the surface integrals the discretization has to be curtailed at some arbitrary distance and beyond it the infinite boundary elements are used [3].

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