A Note on Families of Fragility Curves
Is the Composite Curve Equivalent to the Mean Curve?

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1. INTRODUCTION

In the quantitative assessment of seismic risk, uncertainty in the fragility of a structural component is usually expressed by putting forth a family of fragility curves, with probability serving as the parameter of the family (Figure 1). Commonly, a lognormal shape is used both for the individual curves and for the expression of uncertainty over the family. A so-called "composite" single curve can also be drawn and used for purposes of approximation (Kennedy, et al., 1980, and Kaplan, et al., 1983). This composite curve is often regarded as equivalent to the "mean curve" of the family. The equality seems intuitively reasonable, but to our knowledge, has never been proven.

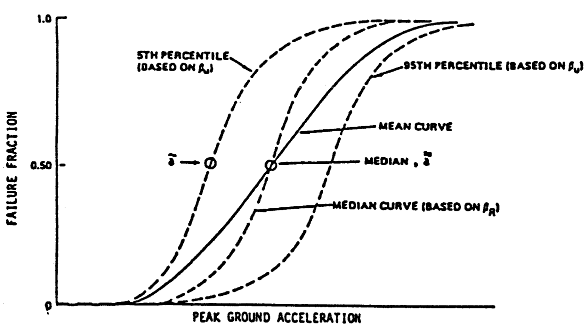

Figure 1. Typical Fragility Family Showing Composite Curves

The present paper proves this equivalence hypothesis mathematically. Moreover, we show that this equivalence hypothesis between fragility curves is itself equivalent to an identity property of the standard normal probability curve. Thus, in the course of proving the fragility curve hypothesis, we have also proved a rather obscure, but interesting and perhaps previously unrecognized, property of the standard normal curve.

2. FORMULATION OF THE FRAGILITY FAMILY

An individual fragility curve (i.e., one member of the fragility family) can be written in parametric form as:

$$F(a \mid \tilde{a}) = N(z)$$
$$a = \tilde{a} e^{\beta_R z} \tag{1}$$

where

a = the peak ground acceleration (the abscissa in Figure 1).

$F(a|\tilde{a})$ = the fragility of the structure at acceleration a (the ordinate in Figure 1). This number represents the fraction of the population of earthquakes having peak ground acceleration a that will cause the structure to fail.

$N(z)$ = the area under the standard normal probability curve up to the point z; i.e.,

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}x^2} dx$$

$\tilde{a}$ = the median acceleration for the particular curve chosen; that is, the value that a takes on when $F(a|\tilde{a}) = \frac{1}{2}$.

β_R = the lognormal standard deviation that expresses the degree of randomness or variability in the underlying earthquake population. β_R governs the degree of steepness of the individual fragility curves in the family.

Note that $\tilde{a}$ also serves as a parameter of the family of fragility curves. This family of curves expresses the fact that we have uncertainty about which member curve of the fragility family is the "true" fragility curve. This uncertainty can be quantified by associating a probability $\underline{P}(\tilde{a})$ to each value $\tilde{a}$, according to the relationship

$$\begin{aligned} \underline{P}(\tilde{a}) &= N(\zeta) \\ \tilde{a} &= \tilde{\tilde{a}} e^{\beta_U \zeta} \end{aligned} \tag{2}$$

where

$\underline{P}(\tilde{a})$ = the probability (i.e., our degree of confidence) that the true fragility curve has a median acceleration value less than or equal to $\tilde{a}$.

$\tilde{\tilde{a}}$ = the "median median" of the fragility family; i.e., the median of the individual curve medians $\tilde{a}$. Thus, the true value of $\tilde{a}$ is equally likely to be above $\tilde{\tilde{a}}$ as below $\tilde{\tilde{a}}$.

ζ = a standard normal variate serving as the intermediate parameter in Equation (2) describing the relationship between $\underline{P}$ and $\tilde{a}$.

β_U = the lognormal standard deviation for the uncertainty. This parameter governs the width of the family of fragility curves.

3. THE COMPOSITE CURVE

The composite curve, $\hat{F}(a)$, is defined, again in parametric form, by

$$\begin{aligned} \hat{F}(a) &= N(\xi) \\ a &= \tilde{\tilde{a}} e^{\beta_C \xi} \end{aligned} \tag{3}$$

with

$$\beta_C = \sqrt{\beta_R^2 + \beta_U^2} \tag{4}$$

Here, ξ is the intermediate parameter, and β_C is the composite standard deviation.

4. FORMULATION OF THE MEAN CURVE OF THE FAMILY

The mean curve of the family has an ordinate, $\bar{F}(a)$, at the acceleration level a, given by

$$\bar{F}(a) = \int_0^{\infty} F(a\,|\,\tilde{a}) \frac{d\underline{P}(\tilde{a})}{d\tilde{a}}\, d\tilde{a} \tag{5}$$

Our task, then, is to show that $\bar{F}(a)$, as defined by Equation (5), is equal to $\hat{F}(a)$, as defined by Equation (3).

For this purpose, we begin by casting Equation (5) into other forms. To do this, let us change the dummy variable, $\tilde{a}$, in Equation (5) to the variable z, according to Equation (1); i.e.,

$$\tilde{a} = a\ e^{-\beta_R z} \tag{6}$$

With $\tilde{a}$ and z related in this way, then for fixed a, we have by Equation (1) the relationship

$$F(a\,|\,\tilde{a}) = N(z) \tag{7}$$

Similarly, $d\underline{P}(\tilde{a})/d\tilde{a}$ is a function of $\tilde{a}$ that we wish, through Equation (6), to change to a function of z. For this purpose, note from Equation (2) that

$$\frac{d\underline{P}(\tilde{a})}{d\tilde{a}} = \frac{dN(\zeta)}{d\zeta}\ \frac{d\zeta}{d\tilde{a}} \tag{8a}$$

$$\frac{d\underline{P}(\tilde{a})}{d\tilde{a}} = N'(\zeta)\ \frac{d\zeta}{d\tilde{a}} \tag{8b}$$

$$\frac{d\underline{P}(\tilde{a})}{d\tilde{a}} = N'\left(\frac{1}{\beta_U}\ \ln\frac{\tilde{a}}{\approx{a}}\right)\frac{1}{\beta_U\,\tilde{a}} \tag{8c}$$

Now, using Equation (6), we have

$$\frac{d\underline{P}(\tilde{a})}{d\tilde{a}} = N'\left(\frac{1}{\beta_U}\ \ln\frac{a}{\approx{a}} - \frac{\beta_R}{\beta_U}\ z\right)\ \frac{1}{\beta_U\, a}\ e^{\beta_R z} \tag{9}$$

and also

$$\begin{aligned} d\tilde{a} &= \frac{d\tilde{a}}{dz}\ dz = -\beta_R\ \tilde{a}\, dz \\ &= -\beta_R\left(a\ e^{-\beta_R z}\right) dz \end{aligned} \tag{10}$$

Thus, combining Equations (9) and (10), we obtain

$$\frac{d\underline{P}(\tilde{a})}{d\tilde{a}}\, d\tilde{a} = -\frac{\beta_R}{\beta_U}\, N'\left(\frac{1}{\beta_U}\ln\frac{a}{\approx\atop a} - \frac{\beta_R}{\beta_U}\, z\right) dz \tag{11}$$

Using Equations (7) and (11), Equation (5) now becomes

$$\overline{F}(a) = \left(\frac{\beta_R}{\beta_U}\right)\int_{-\infty}^{\infty} N(z)\, N'\left(\frac{1}{\beta_U}\ln\frac{a}{\approx\atop a} - \frac{\beta_R}{\beta_U}\, z\right) dz \tag{12}$$

or, equivalently, by changing the variable of integration from z to

$$x = \frac{1}{\beta_U}\ln\frac{a}{\approx\atop a} - \frac{\beta_R}{\beta_U}\, z \tag{13}$$

Equation (12) can be rewritten as

$$\overline{F}(a) = \int_{-\infty}^{\infty} N\left(\frac{1}{\beta_R}\ln\frac{a}{\approx\atop a} - \frac{\beta_U}{\beta_R}\, x\right) N'(x)\, dx \tag{14}$$

5. OUR PROBLEM

Summarizing Equations (3), (12), and (14), we have the following results:

$$\hat{F}(a) = N\left(\frac{1}{\beta_C}\ln\frac{a}{\approx\atop a}\right) \tag{15}$$

$$\overline{F}(a) = \frac{\beta_R}{\beta_U}\int_{-\infty}^{\infty} N(z)\, N'\left(\frac{1}{\beta_U}\ln\frac{a}{\approx\atop a} - \frac{\beta_R}{\beta_U}\, z\right) dz \tag{16}$$

and

$$\overline{F}(a) = \int_{-\infty}^{\infty} N\left(\frac{1}{\beta_R}\ln\frac{a}{\approx\atop a} - \frac{\beta_U}{\beta_R}\, z\right) N'(z)\, dz \tag{17}$$

Our problem is to prove that

$$\hat{F}(a) = \overline{F}(a) \tag{18}$$

where $\hat{F}(a)$ is given by Equation (15) and $\overline{F}(a)$ by Equation (16), or equivalently by Equation (17).

6. IDENTITIES OF THE NORMAL CURVE

Notice that Equation (18) can be rewritten as

$$\left(\frac{\beta_R}{\beta_U}\right)\int_{-\infty}^{\infty} N(z)\, N'\left(\frac{1}{\beta_U}\ln\frac{a}{\approx\atop a} - \frac{\beta_R}{\beta_U}\, z\right) dz = N\left(\frac{1}{\beta_C}\ln\frac{a}{\approx\atop a}\right) \tag{19}$$

This equation, if true, expresses a property of the normal probability curve N(z). In other words, it is an identity for N. The question of whether Equation (18) is true thus boils down to the question: Does the normal curve N have the (somewhat obscure) property of Equation (19)? It turns out that it does, as we shall prove shortly.

7. PROOF OF THE GENERAL CASE

Consider now the identity of Equation (19), which can be written as

$$\left(\frac{\beta_R}{\beta_U}\right)\int_{-\infty}^{\infty} N(z)\, N'\left(\frac{\gamma}{\beta_U} - \frac{\beta_R}{\beta_U} z\right) dz = N\left(\frac{\gamma}{\beta_C}\right) \tag{20}$$

or equivalently

$$\int_{-\infty}^{\infty} N\left(\frac{\gamma}{\beta_R} - \frac{\beta_U}{\beta_R} x\right) N'(x)\, dx = N\left(\frac{\gamma}{\beta_C}\right) \tag{21}$$

where

$$\gamma = \ln \frac{a}{\approx a} \tag{22}$$

We first note that the left-hand side of Equation (21) can be written as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\frac{\gamma}{\beta_R} - \frac{\beta_U}{\beta_R} x} N'(z)\, N'(x)\, dz\, dx \tag{23}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{\gamma}{\beta_R} - \frac{\beta_U}{\beta_R} x} e^{-\frac{1}{2}(x^2 + z^2)}\, dz\, dx$$

Now, we can make the transformation

$$u = \frac{\beta_U}{\beta_R} x + z \tag{24a}$$

$$v = x - \frac{\beta_U}{\beta_R} z \tag{24b}$$

The inverse of this transformation is

$$\begin{bmatrix} x \\ z \end{bmatrix} = \frac{1}{\left[1 + \left(\frac{\beta_U}{\beta_R}\right)^2\right]} \begin{bmatrix} \frac{\beta_U}{\beta_R} & 1 \\ 1 & -\frac{\beta_U}{\beta_R} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{25}$$

with Jacobian

$$\frac{\partial xy}{\partial uv} = \frac{1}{1+\left(\frac{\beta_U}{\beta_R}\right)^2} \tag{26}$$

Therefore, Equation (23) becomes

$$\frac{1}{2\pi\left[1+\left(\frac{\beta_U}{\beta_R}\right)^2\right]}\int_{-\infty}^{\infty}\int_{-\infty}^{\frac{\gamma}{\beta_R}} e^{-\frac{1}{2}\left\{\left[\frac{\frac{\beta_U}{\beta_R}u+v}{1+\left(\frac{\beta_U}{\beta_R}\right)^2}\right]^2+\left[\frac{u-\frac{\beta_U}{\beta_R}v}{1+\left(\frac{\beta_U}{\beta_R}\right)^2}\right]^2\right\}} du\,dv$$

$$=\frac{1}{2\pi}\left[1+\left(\frac{\beta_U}{\beta_R}\right)^2\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\frac{\gamma}{\beta_R}} e^{-\frac{1}{2}\left\{\left[\frac{u}{\sqrt{1+\left(\frac{\beta_U}{\beta_R}\right)^2}}\right]^2+\left[\frac{v}{\sqrt{1+\left(\frac{\beta_U}{\beta_R}\right)^2}}\right]^2\right\}} du\,dv$$

$$=\frac{1}{2\pi\left[1+\left(\frac{\beta_U}{\beta_R}\right)^2\right]}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{v}{\sqrt{1+\left(\frac{\beta_U}{\beta_R}\right)^2}}\right]^2} dv\int_{-\infty}^{\frac{\gamma}{\beta_R}} e^{-\frac{1}{2}\left[\frac{u}{\sqrt{1+\left(\frac{\beta_U}{\beta_R}\right)^2}}\right]^2} du \tag{27}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx\int_{-\infty}^{\frac{\gamma}{\beta_R\sqrt{1+(\quad)^2}}} e^{-\frac{1}{2}z^2}\,dz$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\gamma}{\beta_C}} e^{-\frac{1}{2}z^2}\,dz$$

$$=N\left(\frac{\gamma}{\beta_C}\right)$$

Thus, by following the chain backwards, we can see that Equation (27) is equal to Equation (23), which in turn is equal to the left-hand sides of Equations (20) and (21). Hence, Equations (20) and (21) are proven as a general property of the normal curve N. Also, since Equation (20) is equivalent to Equation (18), in proving Equation (20) we have proved the equivalence of the mean and composite fragility curves.

8. REFERENCES

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