BRANCHING PROCESSES IN STOCHASTIC ENVIRONMENTS

by

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TABLE OF CONTENTS

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ABSTRACT

This paper is concerned with two simple models for branching processes in stochastic environments. The models are identical to the model for the classical Galton-Watson branching process in all respects but one. In the family-tree language commonly used to describe branching processes, that difference is that the probability distribution for the number of offspring of an object changes stochastically from one generation to the next, and is the same for all members of the same generation. That is, the probability distribution of the number of offspring is a function of the "environment." The two models considered herein have a random environment and a Markov environment. In the classical Galton-Watson process, it is assumed that the families of distinct objects in a given generation develop independently of one another. This independence, which renders the Galton-Watson process so tractable mathematically, is lacking in the stochastic environment models. The object of study is the probability distribution of the number of objects Z_n in the n<u>th</u> generation. Of particular interest is the determination of conditions under which the family has probability one of dying out. In the random environment model, the asymptotic behavior of the probability generating function for Z_n (and, consequently, the question of extinction probability) is studied by analyzing the asymptotic behavior of a closely related Markov process on the unit interval. Necessary and sufficient conditions for extinction with probability one are obtained in the case of a finite number of environmental states; for a denumerably infinite number of environmental states, an additional condition, which has not been shown to be necessary, is required, and this precludes

obtaining necessary and sufficient conditions for almost certain extinction. In some special cases, procedures are given for approximating extinction probabilities when the population has a non-zero chance of surviving indefinitely. The asymptotic behavior of the probability generating function for family size in the Markov environment model is studied by relating it to a probability generating function of the type obtained in the random environment model.

CHAPTER I

INTRODUCTION AND PRELIMINARY RESULTS

1. Introduction and description of the process

In the preface to The Theory of Branching Processes, Harris (1963) defines a branching process as "a mathematical representation of the development of a population whose members reproduce and die, according to laws of chance. The objects may be of different types, depending on their age, energy, position, or other factors. However, they must not interfere with one another." This assumption, that different objects reproduce independently, unifies the mathematical theory and characterizes virtually all of the branching process models in the literature. While this assumption allows the definition to encompass a large number of models, it also limits the application of the models of branching processes, since the natural processes of multiplication are often affected by interaction among objects or other factors which introduce dependencies.

The model with which we shall be concerned in the first three chapters may be described mathematically as follows. Let $\{p_r\}$ be a finite or denumerably infinite sequence of nonnegative real numbers satisfying $\sum_r p_r = 1$, and let $\{\phi_r(s)\}$ be a corresponding sequence of probability generating functions. Define a matrix (P_{ij}) with elements

$$P_{ij} = \text{coefficient of } s^j \text{ in } \sum_r p_r[\phi_r(s)]^i, |s| \leq 1, \qquad i,j = 0,1,\cdots. \tag{1.1}$$

Clearly $P_{ij} \geq 0$ for all i and j, and since

$$\sum_{j=0}^{\infty} P_{ij}s^j = \sum_r p_r[\phi_r(s)]^i, \tag{1.2}$$

it follows that $\sum_j P_{ij} = 1$ for all i.

We can define a temporally homogeneous Markov chain $\{Z_n\}$ on the nonnegative integers by choosing initial probabilities

$$P(Z_0 = i) = \delta_{i,\kappa} = \begin{cases} 1, & i = \kappa \\ 0, & i \neq \kappa, \end{cases}$$

for some positive integer κ,

and defining

$$P(Z_0 = a_0, \cdots, Z_n = a_n) = P(Z_0 = a_0)P_{a_0 a_1} \cdots P_{a_{n-1} a_n} .$$

If $P(Z_n = i) > 0$, then P_{ij} is the transition probability

$$P(Z_{n+1} = j | Z_n = i).$$

While all our results follow from the mathematical description of the model given above, we may interpret the process $\{Z_n\}$ as a branching process[1] developing in an environment which changes stochastically in time and which affects the reproductive behavior of the population. For example, the development of an animal population is often affected by such environmental factors as weather conditions, food supply, and so forth. In the physical interpretation of the model considered in the first three chapters, what we have in mind is that the environmental factors which affect reproductive behavior can be classified into a countable number of "states," and that these states of the environment

[1] We shall refer to the stochastic process $\{Z_n\}$ as a branching process, even though the objects reproduce independently only in a conditional sense.

are sampled randomly from one generation to the next.

As in the classical Galton-Watson process [cf. Harris (1963), Chapter I], we consider objects that can generate additional objects of the same kind. The initial set of objects, called the zeroth generation, has offspring that constitute the first generation; their offspring constitute the second generation, and so on. Since we are interested only in the sizes of the successive generations, and not the number of offspring of individual objects, we shall let Z_n, $n = 0,1,\cdots$, denote the size of the n<u>th</u> generation, and shall always assume that $Z_0 = 1$, unless stated otherwise.

In the Galton-Watson process, it is assumed that the number of offspring of different objects are independent, identically distributed random variables with probability generating function $\phi(s)$, say. In our model, the probability generating function $\phi(s)$ is replaced by one of a countable number of probability generating functions $\phi_r(s)$, $r = 1,2,\cdots$, depending on the "state" of the environment at the time. If $\phi_r(s) = \sum_{j=1}^{\infty} p_{rj}s^j$, then p_{rj} is interpreted as the probability that an object existing in the n<u>th</u> generation and environmental state r has j offspring in the (n+1)<u>st</u> generation.

That is, we assume that the environment passes through a sequence of states governed by a process $\{V_n\}$ of independent, identically distributed random variables with

$$P(V_n = r) = p_r, \quad n = 0,1,\cdots, r = 1,2,\cdots; \Sigma\, p_r = 1,$$

independent of n. Then, given that $V_n = r$, the number of offspring of different objects in the n<u>th</u> generation are independent, identically distributed random variables with probability generating function $\phi_r(s)$.

Without loss of generality, we shall assume that $p_r > 0$ for all r. We shall further assume throughout that, unless stated otherwise, the following basic assumptions are satisfied:

a) $Z_0 = 1$ (but V_0 is random);

b) $m_r = \phi_r'(1)$ is finite for all r;

c) $p_{r0} < 1$ for all r, and $p_{r0} + p_{r1} < 1$ for some r (i.e. at least one $\phi_r(s)$ is strictly convex on the unit interval).

2. The generating function and moments of Z_n

Let $\Pi_n(s)$ designate the probability generating function of Z_n, $n = 0,1,\cdots$.

Theorem 2.1 The generating function of Z_n is given by

$$\Pi_n(s) = \sum_r p_r \Pi_{n-1}(\phi_r(s)), \quad n = 1,2,\cdots, \tag{2.1}$$

where, consistent with Assumption (a), $\Pi_0(s) = s$.

Proof.

$$\Pi_n(s) = E(s^{Z_n})$$

$$= \sum_{i=0}^{\infty} P(Z_{n-1} = i) E(s^{Z_n} | Z_{n-1} = i)$$

$$= \sum_{i=0}^{\infty} P(Z_{n-1} = i) \{\sum_r p_r [\phi_r(s)]^i\}$$

from (1.2). Rearranging the double series, we obtain

$$\Pi_n(s) = \sum_r p_r \{\sum_{i=0}^{\infty} P(Z_{n-1} = i) [\phi_r(s)]^i\}$$

$$= \sum_r p_r \Pi_{n-1}(\phi_r(s)),$$

and the proof is complete.

By repeated application of (2.1), we obtain the representation

$$\Pi_n(s) = \sum_{r_0,\cdots,r_{n-1}} p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \phi_{r_0}(\phi_{r_1}(\cdots\phi_{r_{n-1}}(s)\cdots)), \tag{2.2}$$

which we will use in Chapter II.

Let $m = EZ_1 (= \sum_r p_r m_r)$ and $\gamma = \sum_r p_r m_r^2$. If $m < \infty$, we can differentiate (2.1) at $s = 1$ and obtain

$$\Pi_n'(1) = \sum_r p_r \Pi_{n-1}'(1)\phi_r'(1)$$

$$= m\Pi_{n-1}'(1),$$

so by induction, $\Pi_n'(1) = m^n$, $n = 0,1,\cdots$. If $\Pi_1''(1) < \infty$, we can differentiate (2.1) again, obtaining

$$\Pi_n''(1) = \sum_r p_r\{\Pi_{n-1}''(1)[\phi_r'(1)]^2 + \Pi_{n-1}'(1)\phi_r''(1)\}$$

$$= \gamma\Pi_{n-1}''(1) + \Pi_1''(1)m^{n-1}.$$

We thus obtain

$$\Pi_n''(1) = \Pi_1''(1) \sum_{i=0}^{n-1} \gamma^i m^{n-1-i}, \quad n = 1,2,\cdots$$

and

$$\text{Var } Z_n = \Pi_1''(1) \sum_{i=0}^{n-1} \gamma^i m^{n-1-i} + m^n(1 - m^n), \quad n = 1,2,\cdots. \tag{2.3}$$

Hence we have the following result.

Theorem 2.2. If $EZ_1 < \infty$, then $EZ_n = m^n$, $n = 0,1,\cdots$, and if $\text{Var } Z_1 < \infty$, $\text{Var } Z_n$ is given by (2.3).

3. Instability of Z_n and convergence of Π_n

The state space of the Markov chain Z_n, $n = 0,1,\cdots$, is the set S of nonnegative integers. The state $z \varepsilon S$ is said to be transient if

$$P(Z_{n+t} = z \text{ for some } t = 1,2,\cdots | Z_n = z) < 1.$$

Theorem 3.1 *If z is a positive integer, then*

$$P(Z_{n+t} = z \text{ for some } t = 1,2,\cdots|Z_n = z) < 1. \tag{3.1}$$

It follows that for an arbitrary positive integer N,

$$P(0 < Z_n < N) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Proof. If $\phi_r(0) > 0$ for some r, then

$$P(Z_{n+1} = 0|Z_n = z) > 0.$$

Hence

$$P(Z_{n+t} = z \text{ for some } t = 1,2,\cdots|Z_n = z) \leq 1 - P(Z_{n+1} = 0|Z_n = z) < 1,$$

so the state z is transient.

If $\phi_r(0) = 0$ for all r, then since at least one ϕ_r is strictly convex,

$$P(Z_{n+1} > z|Z_n = z) > 0.$$

But $\phi_r(0) = 0$ for all r implies that Z_n is nondecreasing, so that

$$P(Z_{n+t} = z \text{ for some } t = 1,2,\cdots|Z_n = z) \leq 1 - P(Z_{n+1} > z|Z_n = z) < 1,$$

and the proof of (3.1) is complete.

If i is a positive integer, we have just shown that the state i is transient. It follows [Feller (1957), p. 353] that

$$\lim_{n\to\infty} P(Z_n = i) = 0.$$

Hence

$$P(0 < Z_n < N) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Theorem 3.2 *For* $s\varepsilon[0,1)$,

$$\lim_{n\to\infty} \Pi_n(s) = c \leq 1.$$

<u>Proof</u>. $\Pi_n(0)$ is a nondecreasing function of n, and so tends to a limit c, say $(0 \leq c \leq 1)$, as n tends to infinity.

If N is a positive integer,

$$\Pi_n(s) = P(Z_n = 0) + \sum_{i=1}^{N} P(Z_n = i)s^i + \sum_{i=N+1}^{\infty} P(Z_n = i)s^i.$$

Given s, $0 < s < 1$, and an arbitrarily small $\varepsilon > 0$, choose N sufficiently large that $s^{N+1}/(1 - s) < \varepsilon/2$. Then

$$\sum_{i=N+1}^{\infty} P(Z_n = i)s^i < \varepsilon/2.$$

By the latter half of Theorem 3.1, we can choose n sufficiently large that

$$\sum_{i=1}^{N} P(Z_n = i) < \varepsilon/2.$$

Thus for n sufficiently large,

$$\Pi_n(s) \leq \Pi_n(0) + \varepsilon,$$

that is,

$$\overline{\lim}\, \Pi_n(s) \leq c + \varepsilon.$$

Since ε is arbitrary, we have $\overline{\lim}\, \Pi_n(s) \leq c$. But $\Pi_n(0) \leq \Pi_n(s)$, so $c \leq \underline{\lim}\, \Pi_n(s)$, and the proof is complete.

4. <u>Martingales</u>

In this section, we shall assume that $m = \sum_r m_r$ is finite. Let $\xi_n = Z_n/m^n$, $n = 0,1,\cdots$.

From the definition of Z_n, we have with probability one, $E(Z_{n+1}|Z_n) = mZ_n$, $n = 0,1,\cdots$. Hence, with probability one,

$$E(Z_{n+k}|Z_n) = E[E(Z_{n+k}|Z_{n+k-1},\cdots,Z_n)|Z_n]$$

$$= E[E(Z_{n+k}|Z_{n+k-1})|Z_n]$$

$$= E[mZ_{n+k-1} | Z_n]$$

$$= \cdots = m^k Z_n, \quad k,n = 0,1,\cdots.$$

Dividing both sides by m^{n+k}, we obtain

$$E(\xi_{n+k} | \xi_n) = E(\xi_{n+k} | \xi_{n+k-1},\cdots,\xi_n) = \xi_n, \tag{4.1}$$

the first equality holding because ξ_0, $\xi_1,\cdots$ is a Markov chain. The second equality in (4.1) reveals that ξ_0, $\xi_1,\cdots$ is a martingale.

Since $E\xi_n = 1$, $n = 0,1,\cdots$, it follows from a theorem of Doob (1953, Theorem 4.1, p. 319) that $\lim_{n\to\infty} \xi_n = \xi$ exists with probability one, and $E\xi \leq 1$.

In the very special case where m_r is independent of r ($m_r = m$), we can obtain a stronger result.

Theorem 4.1 *If $m_r = m$ for all r, $m > 1$ and $EZ_1^2 < \infty$, then the random variables ξ_n converge with probability one and in mean square to a random variable ξ with*

$$E\xi = 1, \text{ Var } \xi = \text{Var } Z_1/(m^2 - m). \tag{4.2}$$

Proof. By the same theorem of Doob, we have only to prove that $\lim_{n\to\infty} E\xi_n^2 < \infty$ to obtain mean square convergence. From (2.3),

$$E\xi_n^2 = m^{-2n}\ \Pi_1''(1) \sum_{i=0}^{n-1} \gamma^i m^{n-1-i} + m^{-n}.$$

But $\gamma = \sum_r p_r m_r^2 = m^2$, and $\Pi_1''(1) = \text{Var } Z_1 + m(m-1)$, so we have

$$E\xi_n^2 = [\text{Var } Z_1 + m(m-1)]m^{-2n} \sum_{i=0}^{n-1} m^{2i+n-1-i} + m^{-n}$$

$$= [\text{Var } Z_1 + m(m-1)]m^{-(n+1)}\ (m^n - 1)/(m-1) + m^{-n}$$

$$= \frac{\text{Var } Z_1}{m(m-1)}\ (1 - m^{-n}) + 1\ .$$

Hence

$$\lim_{n\to\infty} E\xi_n^2 = 1 + \frac{\text{Var } Z_1}{m^2 - m} < \infty.$$

Since mean square convergence of ξ_n to ξ implies $\lim E\xi_n = E\xi$ and $\lim E\xi_n^2 = E\xi^2$, we obtain

$$E\xi = 1, \text{ Var } \xi = \frac{\text{Var } Z_1}{m^2 - m},$$

and the proof is complete.

We shall see later that ξ_n does not generally converge in mean square if the m_r are unequal.

CHAPTER II

CONDITIONS FOR ALMOST CERTAIN EXTINCTION

5. Introduction

Extinction is the event that the population eventually dies out, or, more accurately, the event that the random sequence $\{Z_n\}$ consists of zeros for all but a finite number of values of n. Since $P(Z_{n+k} = 0 | Z_n = 0) = 1$ for $k = 1,2,\cdots$, extinction is the event that $Z_n = 0$ for some $n = 1,2,\cdots$. We thus obtain for the probability of extinction

$$\begin{aligned} P(Z_n = 0 \text{ for some } n) &= P\{(Z_1 = 0) \cup (Z_2 = 0) \cup \cdots\} \\ &= \lim_{n\to\infty} P\{(Z_1 = 0) \cup \cdots \cup (Z_n = 0)\} \\ &= \lim_{n\to\infty} P(Z_n = 0) \\ &= \lim_{n\to\infty} \Pi_n(0). \end{aligned}$$

Hence if q is the probability of extinction,

$$q = \lim_{n\to\infty} \Pi_n(0). \tag{5.1}$$

In the classical Galton-Watson process (this model with $p_j = 1$ for some j and $\phi_j(s) = f(s)$), the probability generating function $f_n(s)$ for Z_n satisfies

$$\begin{aligned} f_{n+1}(s) &= f_n(f_1(s)) \\ &= f_1(f_n(s)), \quad n = 0,1,\cdots, \end{aligned}$$

where $f_0(s) = s$ and $f_1(s) = f(s)$.

The probability of extinction q is then easily seen to be the smallest nonnegative solution of the equation

$$s = f(s),$$

from which it follows that $q = 1$ if and only if $f'(1) \leq 1$.

In our model, $m = \sum_r m_r \leq 1$ is a sufficient condition for extinction with probability one, as the following theorem shows; however, we shall see later that it is not a necessary condition.

Theorem 5.1 If $m \leq 1$, then

$$\lim_{n\to\infty} \Pi_n(0) = 1.$$

Proof. We recall that $EZ_n = m^n$. Hence if $m \leq 1$ and N is an arbitrary positive integer,

$$P(Z_n \geq N) \leq (1/N)EZ_n \leq 1/N.$$

Given $\varepsilon > 0$, choose N sufficiently large that

$$P(Z_n \geq N) < \varepsilon/2 \tag{5.2}$$

for all n, and then choose n sufficiently large that

$$P(0 < Z_n < N) < \varepsilon/2 \tag{5.3}$$

(by the latter half of Theorem 3.1). It follows from (5.2) and (5.3) that for n sufficiently large,

$$P(Z_n = 0) > 1 - \varepsilon.$$

Hence

$$\lim_{n\to\infty} \Pi_n(0) \geq 1 - \varepsilon,$$

and since ε is arbitrary, $\lim \Pi_n(0) = 1$.

6. The dual process

The solution to the problem of almost certain extinction (i.e., extinction with probability one) can apparently not be found by analysis involving the generating functions $\Pi_n(s)$ alone. The solution lies, instead, in the behavior of random walks on the unit interval and the nonnegative real axis, the first of which is defined below.

Consider the random walk $\{X_n\}$ on the unit interval defined as follows: for arbitrary but fixed $s_0 \varepsilon [0,1)$, let $X_0 = s_0$, and define

$$X_{n+1} = \phi_{V_n}(X_n), \quad n = 0,1,\cdots,$$

where $\{V_n\}$ is the environmental process defined in Section 1 and $\{\phi_r(s)\}$ is the sequence of probability generating functions corresponding to the possible states of the environment. The stochastic process $\{X_n\}$ will be called the dual process associated with the branching process $\{Z_n\}$.

For the expected location of the dual process after n steps, we obtain[1]

$$\begin{aligned} E(X_n | X_0 = s_0) &= \sum_{r_0,\ldots,r_{n-1}} p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \phi_{r_{n-1}}(\phi_{r_{n-2}}(\cdots \phi_{r_0}(s_0)\cdots)) \\ &= \sum_{r_0,\ldots,r_{n-1}} p_{r_{n-1}} p_{r_{n-2}} \cdots p_{r_0} \phi_{r_{n-1}}(\phi_{r_{n-2}}(\cdots \phi_{r_0}(s_0)\cdots)) \\ &= \sum_{r_0,\ldots,r_{n-1}} p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \phi_{r_0}(\phi_{r_1}(\cdots \phi_{r_{n-1}}(s_0)\cdots)). \end{aligned}$$

Comparing this expression with (2.2), we have the rather surprising result that

$$E(X_n | X_0 = s_0) = \Pi_n(s_0). \tag{6.1}$$

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[1] $E(X_n | X_0 = s_0)$ is not a conditional expectation; we write it this way, however, to emphasize the initial assumption.

7. Some special cases

Insight into the nature of the problem and its solution can be obtained by considering some special cases when there are only two possible environmental states, so that (2.1) becomes

$$\Pi_{n+1}(s) = p_1\Pi_n(\phi_1(s)) + p_2\Pi_n(\phi_2(s)), \quad p_1 + p_2 = 1. \tag{7.1}$$

If $m_1 \leq 1$ and $m_2 \leq 1$, then $m = p_1m_1 + p_2m_2 \leq 1$, so by Theorem 5.1, extinction occurs with probability one.

If $m_1 > 1$ and $m_2 > 1$, then there exist q_1 and q_2, $q_1 < 1$, $q_2 < 1$, satisfying

$$q_1 = \phi_1(q_1); \quad q_2 = \phi_2(q_2).$$

Suppose that $q_1 \leq q_2$. Since $\phi_1(q_2) \leq q_2$ and $\Pi_n(s)$ is increasing in s, we have from (7.1) that

$$\Pi_{n+1}(q_2) = p_1\Pi_n(\phi_1(q_2)) + p_2\Pi_n(\phi_2(q_2)) \leq p_1\Pi_n(q_2) + p_2\Pi_n(q_2) = \Pi_n(q_2),$$

$$n = 0,1,\cdots,$$

from which it follows that $\Pi_n(q_2) \leq q_2$, $n = 1,2,\cdots$. Therefore

$$\lim_{n\to\infty} \Pi_n(0) = \lim_{n\to\infty} \Pi_n(q_2) \leq q_2,$$

so $q \leq q_2 < 1$, and extinction is not almost certain.

The difficulty arises when, for example, $m_1 < 1$ and $m_2 > 1$ (and $p_1m_1 + p_2m_2 > 1$). We shall consider two special cases of this kind.

Case I. Suppose $m_1 = \theta$ and $m_2 = 1/\theta$, where θ is a real number in (0,1). Let X_n, $n = 0,1,\cdots$, be the dual process defined in Section 6.

Define piecewise linear functions ψ_1 and ψ_2 on the unit interval by

$$\psi_1(s) = (1 - m_1) + m_1s$$

and

$$\psi_2(s) = \begin{cases} 0 & , s < 1 - 1/m_2 \\ (1 - m_2) + m_2 s, & s \geq 1 - 1/m_2. \end{cases}$$

Note that $\psi_i(1) = 1$, $\psi_i'(1) = \phi_i'(1)$, and for $s\varepsilon[0,1]$, $\psi_i(s) \leq \phi_i(s)$, $i = 1,2$. (Figure 1).

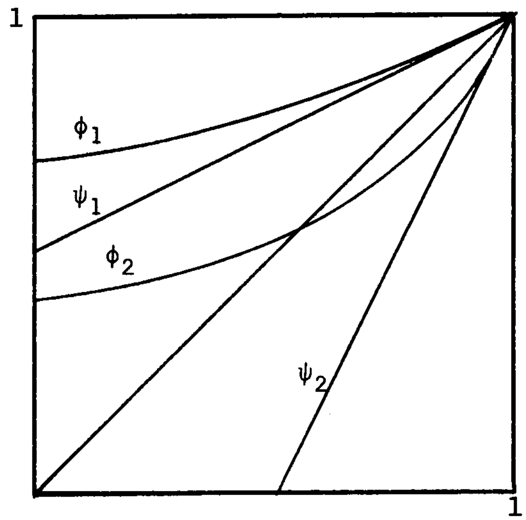

Figure 1.

Since $m_1 = \theta$ and $m_2 = 1/\theta$, ψ_1 and ψ_2 become

$$\psi_1(s) = (1 - \theta) + \theta s$$

and

$$\psi_2(s) = \begin{cases} 0 & , s < 1 - \theta \\ (1 - \theta^{-1}) + \theta^{-1}s, & s \geq 1 - \theta. \end{cases}$$

Define a new random walk $\{Y_n\}$ on the unit interval as follows: let $Y_0 = s_0$, and

$$Y_{n+1} = \psi_{V_n}(Y_n), \quad n = 0,1,\cdots.$$

We then obtain

$$E(Y_n|Y_0 = s_0) = \sum_{r_0,\cdots,r_{n-1}=1}^{2} p_{r_0}p_{r_1}\cdots p_{r_{n-1}}\psi_{r_{n-1}}(\psi_{r_{n-2}}(\cdots\psi_{r_0}(s_0)\cdots))$$

$$\leq \sum p_{r_0}p_{r_1}\cdots p_{r_{n-1}}\psi_{r_{n-1}}(\psi_{r_{n-2}}(\cdots\psi_{r_1}(\phi_{r_0}(s_0))\cdots))$$

$$\leq \cdots$$

$$\leq \sum p_{r_0}p_{r_1}\cdots p_{r_{n-1}}\phi_{r_{n-1}}(\phi_{r_{n-2}}(\cdots\phi_{r_0}(s_0)\cdots))$$

$$= E(X_n|X_0 = s_0), \qquad n = 0,1,\cdots.$$

Consequently, if $\lim_n EY_n = 1$, it follows that $\lim_n EX_n = 1$, and thus that $\lim_n \Pi_n(s_0) = 1$. Since the limit of Π_n is independent of s for $s\varepsilon[0,1)$, we would have, in particular, $\lim_n \Pi_n(0) = 1$ (i.e. $q = 1$). We thus want to determine conditions under which $\lim_n EY_n = 1$.

Returning to the random walk $\{Y_n\}$, let us assume that $s_0 = 1 - \theta^i$ for an arbitrary nonnegative integer i. Since

$$\psi_1(1 - \theta^j) = 1 - \theta^{j+1}, \quad j = 0,1,\cdots$$

and (7.2)

$$\psi_2(1 - \theta^j) = \begin{cases} 0 & , \quad j = 0 \\ 1 - \theta^{j-1}, & j = 1,2,\cdots, \end{cases}$$

the state space of the random walk $\{Y_n\}$ is the set of all real numbers of the form $1 - \theta^j$, $j = 0,1,\cdots$.

Define a process W_n, $n = 0,1,\cdots$, as follows: let

$$W_n = \log_\theta(1 - Y_n), \quad n = 0,1,\cdots.$$

Hence we have

$$P(W_n = j) = P(Y_n = 1 - \theta^j), \quad n,j = 0,1,\cdots.$$

From the definition of the Y_n - process and (7.2), it follows that $\{W_n\}$ is a random walk on the nonnegative integers with matrix of transition probabilities

$$\begin{bmatrix} p_2 & p_1 & 0 & 0 & 0 & \cdots \\ p_2 & 0 & p_1 & 0 & 0 & \cdots \\ 0 & p_2 & 0 & p_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

This is, of course, immediately recognized as the classical random walk on the nonnegative integers with a reflecting barrier at zero.

It is well known that the states of the Markov chain $\{W_n\}$ are positive recurrent if and only if $p_2 > p_1$. If $p_1 \geq p_2$, it follows that for an arbitrary positive integer N,

$$P(W_n \in \{0,1,\cdots,N\}) \to 0 \quad \text{as } n \to \infty.$$

Consequently, in the Y_n-process,

$$P(Y_n \leq 1 - \theta^N) \to 0 \quad \text{as } n \to \infty.$$

Thus $Y_n \to 1$ in probability, and, by bounded convergence, $EY_n \to 1$. It follows from the remarks above that $q = 1$ for $p_1 \geq 1/2$.

We next want to investigate the nature of the arithmetic mean

$$m = p_1 m_1 + p_2 m_2$$

and the geometric mean

$$m_1^{p_1} m_2^{p_2}$$

in this example under the condition that $p_1 \geq 1/2$.

For the geometric mean, we have

$$m_1^{p_1}\, m_2^{p_2} = \theta^{p_1}\, \theta^{p_1-1} = \theta^{2p_1-1},$$

so $p_1 \geq 1/2$ if and only if $m_1^{p_1}\, m_2^{p_2} \leq 1$. For the arithmetic mean, we have

$$p_1 m_1 + p_2 m_2 = p_1\theta + (1 - p_1)\theta^{-1},$$

so $m > 1$ if and only if $p_1\theta^2 + (1 - p_1) > \theta$, that is, if and only if $p_1(1 - \theta^2) < 1 - \theta$. Hence for $0 < \theta < 1$, $m > 1$ if and only if $p_1 < 1/(1 + \theta)$. But $1/(1 + \theta) > 1/2$ for $0 < \theta < 1$, so for p_1 in the non-empty interval $[1/2, 1/(1 + \theta)]$, the arithmetic mean is > 1 while the geometric mean is ≤ 1. Hence for $p_1 \varepsilon [1/2, 1/(1 + \theta)]$, $\Pi_n(0) \longrightarrow 1$ although $\Pi_n'(1) = m^n \longrightarrow \infty$ as $n \longrightarrow \infty$.

We can now justify the statement that Theorem 4.1 cannot hold in general for the m_r unequal and $m > 1$, for if, while $m > 1$, the geometric mean is ≤ 1, $\xi_n = Z_n/m^n \longrightarrow 0$ with probability one, but $E\xi_n = 1$, so the martingale $\{\xi_n\}$ cannot converge in mean square, or even in the mean (order one).

<u>Case II</u>. Suppose there exists a real number θ, $0 < \theta < 1$, satisfying the conditions

$$m_1 > \theta;\ m_2 > \theta^{-1};\ \theta^{p_1}\, \theta^{-p_2} > 1.$$

The last condition is, of course, equivalent to $p_1 < p_2$, or $p_1 < 1/2$.

In this situation, we want to construct piecewise linear functions with desired properties which bound ϕ_1 and ϕ_2 above. To obtain the required functions, choose an integer N_1 sufficiently large that

$$(1 - \theta) + \theta s > \phi_1(s), \text{ for } s \geq 1 - \theta^{N_1},$$

and an integer N_2 sufficiently large that

$$(1 - \theta^{-1}) + \theta^{-1}s > \phi_2(s), \text{ for } s \geq 1 - \theta^{N_2}.$$

Then choose N_1 or N_2 still larger, if necessary, so that $N_1 + 1 = N_2 - 1 = N$, say.

Now define piecewise linear functions ψ_1 and ψ_2 on the unit interval by

$$\psi_1(s) = \begin{cases} 1 - \theta^N, & s < 1 - \theta^{N-1} \\ (1 - \theta) + \theta s, & s \geq 1 - \theta^{N-1} \end{cases}$$

and

$$\psi_2(s) = \begin{cases} 1 - \theta^N, & s < 1 - \theta^{N+1} \\ (1 - \theta^{-1}) + \theta^{-1}s, & s \geq 1 - \theta^{N+1} \end{cases}$$

The relationship of ψ_1 and ψ_2 to ϕ_1 and ϕ_2 is of the form described in Figure 2.

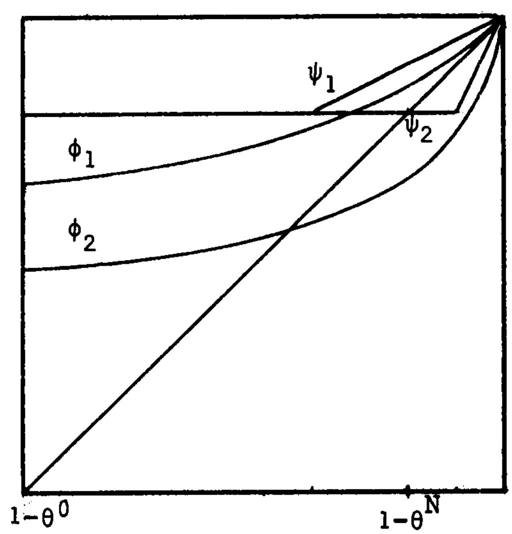

Figure 2.

Define the dual process $\{X_n\}$ and the Y_n-process determined by ψ_1 and ψ_2 as before. In this case it is again easily seen that

$EX_n \leq EY_n$, $n = 1,2,\cdots$. The Y_n-process is again a random walk on the space of all points of the form $1 - \theta^j$, $j = 0,1,\cdots$, provided that the initial point s_0 is of this form. If, however, we assume that

$$s_0 = 1 - \theta^i \text{ for some } i \geq N,$$

then the state space of the Y_n-process is the set of all real numbers $1 - \theta^j$, $j \geq N$.

We define

$$W_n = \log_\theta(1 - Y_n) - N, \quad n = 0,1,\cdots,$$

so that

$$P(W_n = j) = P[\log_\theta(1 - Y_n) - N = j]$$

$$= P(Y_n = 1 - \theta^{N+j}), \quad n,j = 0,1,\cdots.$$

The W_n-process is again the classical random walk on the nonnegative integers with a reflecting barrier at zero.

Since $p_2 > p_1$ (by assumption), the states of the W_n-process are positive recurrent, and therefore a stationary distribution $\{u_j\}$ exists. In particular

$$P(Y_n = 1 - \theta^N | Y_0 = 1 - \theta^i) = P(W_n = 0 | W_0 = i - N) \longrightarrow u_0 > 0$$

as $n \longrightarrow \infty$. Hence for all $n \geq n_0$, say,

$$P(Y_n = 1 - \theta^N | Y_0 = 1 - \theta^i) > u_0/2,$$

from which it follows that

$$E(Y_n | Y_0 = 1 - \theta^i) \leq (1 - \theta^N)u_0/2 + (1 - u_0/2) = 1 - \theta^N u_0/2$$

for $n \geq n_0$. Thus

$$\lim_{n\to\infty} E(Y_n | Y_0 = 1 - \theta^i) < 1,$$

and

$$\lim_{n\to\infty} \Pi_n(s_0) = \lim_{n\to\infty} E(X_n|X_0 = s_0) \le \lim_{n\to\infty} E(Y_n|Y_0 = s_0) < 1.$$

It follows that $\lim \Pi_n(s) = c$, say, for $c < 1$ and $s\epsilon[0,1)$; in particular, $\lim \Pi_n(0) < 1$, so extinction is not almost certain.

To sum up the results of this section, we have seen that: (1) $m = p_1m_1 + p_2m_2 \le 1$ implies $q = 1$, (2) $m_1 > 1$ and $m_2 > 1$ implies $q < 1$, (3) $m_1 = \theta$, $m_2 = \theta^{-1}$ for $\theta\epsilon(0,1)$ and $\theta^{p_1}\ \theta^{-p_2} \le 1$ $(p_1 \ge 1/2)$ implies $q = 1$, and (4) $m_1 > \theta$, $m_2 > \theta^{-1}$ for $\theta\epsilon(0,1)$ and $\theta^{p_1}\ \theta^{-p_2} > 1$ $(p_1 < 1/2)$ implies $q < 1$. The assumptions in (1) - (4) are not, of course, exhaustive, but consideration of these special cases has shown that the arithmetic mean m is not a critical parameter in determining whether extinction occurs with probability one. These cases suggest, however, that the geometric mean ($m_1^{p_1}\ m_2^{p_2}$ for two environmental states) is a critical parameter, as indeed we shall prove in Section 9.

8. The Lindley process

In Section 9, the W_n-process is much more general than the classical random walk of the special cases in Section 7; this section is devoted to a discussion of the more general random walk which we will encounter in Section 9. The results of this section are due to Lindley (1952), who introduced the process in connection with a waiting-time problem in queueing theory. He also indicated that the methods employed could be applied to random walks of the kind encountered in the next section.

Let $U_1, U_2, \cdots$ be a sequence of independent, identically distributed random variables with $E|U_1|$ finite.

Define a sequence of random variables W_n, $n = 1,2,\cdots$, as follows: $W_0 = 0$, and

$$W_{n+1} = \begin{cases} W_n + U_{n+1}, & \text{if } W_n + U_{n+1} > 0, \\ 0, & \text{if } W_n + U_{n+1} \leq 0, \end{cases} \qquad n = 0,1,\cdots. \tag{8.1}$$

The sequence $\{W_n\}$ of nonnegative random variables will be called the <u>Lindley process</u>. The process also has the representation

$$W_n = \max(0,\ U_n,\ U_{n-1} + U_n, \cdots,\ U_1 + U_2 + \cdots + U_n), \qquad n = 1,2,\cdots. \tag{8.2}$$

For clearly $W_1 = \max(0, U_1)$. Suppose (8.2) holds for $n = r$. Then if $W_r + U_{r+1} > 0$,

$$\begin{aligned} W_{r+1} &= W_r + U_{r+1} \\ &= \max(0,\ U_r,\ U_{r-1} + U_r, \cdots,\ U_1 + U_2 + \cdots + U_r) + U_{r+1} \\ &= \max(U_{r+1},\ U_r + U_{r+1}, \cdots,\ U_1 + U_2 + \cdots + U_{r+1}), \end{aligned}$$

and if $W_r + U_{r+1} \leq 0$, $W_{r+1} = 0$. Combining these two possibilities in the single expression

$$W_{r+1} = \max(0,\ U_{r+1},\ U_r + U_{r+1}, \cdots,\ U_1 + U_2 + \cdots + U_{r+1}),$$

it follows, by induction, that (8.2) holds for all n.

For $x \geq 0$, we have

$$P(W_n \leq x) = P(U_n \leq x,\ U_{n-1} + U_n \leq x, \cdots,\ U_1 + U_2 + \cdots + U_n \leq x). \tag{8.3}$$

Since the U_n's are independent and identically distributed, we may renumber them without affecting the right hand member of (8.3); in particular, if we replace U_r by U_{n+1-r}, $r = 1,\cdots,n$, we obtain

$$P(W_n \leq x) = P(U_1 \leq x,\ U_1 + U_2 \leq x, \cdots,\ U_1 + U_2 + \cdots + U_n \leq x).$$

If we write $S_r = \sum_{s=1}^{r} U_s$, this becomes

$$P(W_n \leq x) = P(S_r \leq x \text{ for all } r \leq n). \tag{8.4}$$

We now want to establish the existence of a limit for $F_n(x) = P(W_n \leq x)$ as n tends to infinity, for any x.

If $E_n(x)$ is the event

$$E_n(x): \quad S_r \leq x \text{ for all } r \leq n,$$

the sequence of events $\{E_n(x)\}$, $n = 1,2,\cdots$, for fixed x, is decreasing and tending to the limit event

$$E(x): \quad S_r \leq x \text{ for all } r \geq 1.$$

Hence by a property of probability measures,

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} P(E_n(x)) = P(E(x))$$

exists. If we denote this limit by F(x), then since

$$F(x) = P(S_r \leq x \text{ for all } r \geq 1),$$

it follows that F(x) is a nonnegative, nondecreasing function with $F(x) = 0$ for $x < 0$ (since $F_n(x) = P(W_n \leq x) = 0$ for $x < 0$).

The question immediately poses itself: under what conditions is F(x) a distribution function. There are three cases to consider according to the value of EU_1.

<u>(i) $EU_1 > 0$</u>. By the strong law of large numbers,

$$\lim_{n\to\infty} S_n/n = EU_1$$

with probability one. Hence with probability one, for any sequence $\{U_1, U_2,\cdots\}$, there exists n_0 such that $S_n > nEU_1/2$ for $n \geq n_0$. It follows that for $x > 0$, there exists a random $n_0(x)$ such that

$$S_n > x \text{ for } n \geq n_0(x)$$

with probability one. Thus $F(x) = 0$ for all x; that is

$$P(W_n \leq x) \longrightarrow 0, \text{ as } n \longrightarrow \infty,$$

for all x.

<u>(ii) $EU_1 < 0$</u>. Again by the strong law of large numbers,

$$\lim_{n\to\infty} S_n/n = EU_1$$

with probability one. Given $\varepsilon > 0$, it follows by Egoroff's theorem that there exists an integer n_0 such that

$$P(S_n \leq 0 \text{ for all } n \geq n_0) > 1 - \varepsilon/2. \tag{8.5}$$

Further, considering the joint distribution of $S_1, S_2, \cdots, S_{n_0}$, we can choose x sufficiently large that

$$P(S_n \leq x \text{ for all } n \leq n_0) > 1 - \varepsilon/2. \tag{8.6}$$

From (8.5) and (8.6), we obtain

$$P(S_n > x \text{ for some } n \geq 1) < \varepsilon,$$

that is,

$$F(x) = P(S_n \leq x \text{ for all } n \geq 1) > 1 - \varepsilon$$

for sufficiently large x. Since $\lim_{x\to\infty} F(x) \leq 1$, it follows that $\lim_{x\to\infty} F(x) = 1$. Thus in this case the existence of a nondefective limiting distribution function has been established.

<u>(iii) $EU_1 = 0$</u>. Chung and Fuchs (1951) show that if $E|U_1| < \infty$ and $EU_1 = 0$, then for arbitrary $\varepsilon > 0$, and any real number x,

$$P(|S_n - x| < \varepsilon \text{ infinitely often}) = 1, \tag{8.7}$$

unless all values assumed by U_1 are of the form $i\lambda$ $(i = 0, \pm 1, \pm 2, \cdots)$ for λ a real number, in which case (8.7) holds for every x of this form.

The case where $U_1 = 0$ with probability one is excluded. Since

$$F(x) = P(S_n \leq x \text{ for all } n \geq 1),$$

it follows immediately that $F(x) = 0$ for all x since any value x will be exceeded by S_n for some n with probability one.

Hence we have the following result.

Theorem 8.1 (Lindley). Suppose $E|U_1|$ is finite and U_1 is not zero with probability one. Then a necessary and sufficient condition that the distribution function $F_n(x) = P(W_n \leq x)$ tends to a nondefective limit as $n \to \infty$ is that $EU_1 < 0$. If $EU_1 \geq 0$, $P(W_n \leq x)$ tends to zero for any x.

9. Conditions for almost certain extinction

Theorem 9.1 Suppose $\sum_r p_r |\log m_r| < \infty$.

(a) If $\sum_r p_r \log m_r \leq 0$, then $\Pi_n(0) \to 1$ as $n \to \infty$.

(b) If $\sum_r p_r \log m_r > 0$, and

$$\sum_r p_r \log(1 - \phi_r(0)) \text{ converges,} \tag{9.1}$$

then $\Pi_n(0) \to q < 1$ as $n \to \infty$.

We have not determined whether the condition (9.1) is necessary, but have been unable to obtain the result without it. We can, however, remove all conditions if there are only a finite number of environmental states (Corollary 9.1).

Proof of Theorem 9.1. Let $\{V_n\}$ be the environmental process consisting of independent, identically distributed random variables with probability mass function $\{p_r\}$, and let $\{\phi_r(s)\}$ be the corresponding sequence of probability generating functions with $\phi_r'(1) = m_r$, $r = 1,2,\cdots$. Let

$\{X_n\}$ be the dual process of Section 6, for which

$$E(X_n|X_0 = s_0) = \Pi_n(s_0), \tag{9.2}$$

for $s_0 \varepsilon [0,1)$.

(a) $\underline{\sum_r p_r \log m_r \leq 0}$. If $m_r = 1$ for all r, then m = 1 and by Theorem 5.1, $\Pi_n(0) \to 1$. Thus we shall henceforth assume that $m_r \neq 1$ for some r.

Define two subsets of the positive integers by $M_1 = \{r: m_r \leq 1\}$ and $M_2 = \{r: m_r > 1\}$.

We then define piecewise linear functions ψ_r, $r = 1,2,\cdots$, on the unit interval by

$$\psi_r(s) = (1 - m_r) + m_r s \qquad , r\varepsilon M_1$$

and

$$\psi_r(s) = \begin{cases} 0 & , s < 1 - 1/m_r, \\ (1 - m_r) + m_r s, & s \geq 1 - 1/m_r, \end{cases} \quad r\varepsilon M_2. \tag{9.3}$$

Then $\psi_r(1) = 1$, $\psi_r'(1) = \phi_r'(1)$, and, for $s\varepsilon[0,1]$, $\psi_r(s) \leq \phi_r(s)$, $r = 1,2,\cdots$.

Based on these functions ψ_r, define a process Y_n, $n = 0,1,\cdots$, on the unit interval as follows: $Y_0 = 0$ and

$$Y_{n+1} = \psi_{V_n}(Y_n), \quad n = 0,1,\cdots.$$

Then

$$E(Y_n|Y_0 = 0) = \sum_{r_0,\ldots,r_{n-1}} p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \psi_{r_{n-1}}(\psi_{r_{n-2}}(\cdots\psi_{r_0}(0)\cdots))$$

$$\leq \sum p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \psi_{r_{n-1}}(\psi_{r_{n-2}}(\cdots\phi_{r_0}(0)\cdots))$$

$$\leq \cdots$$

$$\leq \Sigma\, p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \phi_{r_{n-1}}(\phi_{r_{n-2}}(\cdots\phi_{r_0}(0)\cdots))$$

$$= E(X_n | X_0 = 0) \tag{9.4}$$

If we write $s = 1 - e^{-x}$, $x \geq 0$, the equations (9.3) become

$$\psi_r(1 - e^{-x}) = \begin{cases} 1 - e^{0} & , x - \log m_r < 0 \\ 1 - e^{-(x - \log m_r)} & , x - \log m_r \geq 0, \end{cases} \tag{9.5}$$

for $r \epsilon M_1 \cup M_2$, $x \geq 0$.

Define

$$W_n = -\log(1 - Y_n), \quad n = 0,1,\cdots,$$

from which it follows that

$$P(W_n \leq x) = P(Y_n \leq 1 - e^{-x}), \; x \geq 0, \quad n = 0,1,\cdots.$$

Define a sequence $U_1, U_2, \cdots$ of independent, identically distributed, random variables by

$$U_n = -\log m_{V_{n-1}}, \quad n = 1,2,\cdots.$$

Then

$$Y_{n+1} = \psi_{V_n}(Y_n)$$

$$= \psi_{V_n}(1 - e^{-W_n})$$

$$= \begin{cases} 0 & , W_n + U_{n+1} < 0 \\ 1 - e^{-(W_n + U_{n+1})} & , W_n + U_{n+1} \geq 0, \end{cases}$$

by (9.5). Hence

$$W_{n+1} = -\log(1 - Y_{n+1}) = \begin{cases} 0 & , W_n + U_{n+1} < 0 \\ W_n + U_{n+1} & , W_n + U_{n+1} \geq 0, \end{cases}$$

that is, $\{W_n\}$ is a Lindley process.

Since

$$EU_1 = E(-\log m_{V_0})$$

$$= -\sum_r p_r \log m_r \geq 0,$$

it follows from Theorem 8.1 that

$$P(W_n \leq x) \longrightarrow 0 \text{ for all } x \geq 0, \text{ as } n \longrightarrow \infty.$$

Hence

$$P(Y_n \leq 1 - e^{-x} | Y_0 = 0) \longrightarrow 0 \text{ for all } x \geq 0, \text{ as } n \longrightarrow \infty,$$

that is, $Y_n \longrightarrow 1$ in probability, given $Y_0 = 0$. By bounded convergence, $E(Y_n | Y_0 = 0) \longrightarrow 1$, and thus by (9.4), $E(X_n | X_0 = 0) \longrightarrow 1$. Finally, by (9.2), $\Pi_n(0) \longrightarrow 1$ as $n \longrightarrow \infty$, so extinction occurs with probability one.

(b) $\sum_r p_r \log m_r > 0$ and (9.1) holds. Suppose $\sum_r p_r \log m_r = a > 0$. Choose n_0 sufficiently large that

$$\sum_{r=1}^{n} p_r \log m_r > a/2 \text{ for } n \geq n_0.$$

Since $\sum_r p_r \log(1 - \phi_r(0))$ converges, we can choose $N \geq n_0$ sufficiently large that

$$\left| \sum_{r>N} p_r \log(1 - \phi_r(0)) \right| < a/2.$$

It follows that

$$\sum_{r=1}^{N} p_r \log m_r + \sum_{r>N} p_r \log(1 - \phi_r(0)) > 0.$$

There exists ε, $0 < \varepsilon < 1$, such that

$$\sum_{r=1}^{N} p_r \log m_r + \sum_{r>N} p_r \log(1 - \phi_r(0)) + (\log \varepsilon) \sum_{r=1}^{N} p_r > 0,$$

that is,

$$\sum_{r=1}^{N} p_r \log(\varepsilon m_r) + \sum_{r>N} p_r \log(1 - \phi_r(0)) > 0. \tag{9.6}$$

Let $m_r' = \varepsilon m_r$, $r = 1,\cdots,N$; $m_r' = 1 - \phi_r(0)$, $r = N+1,\cdots$, so that (9.6) becomes

$$\sum_r p_r \log m_r' > 0. \tag{9.7}$$

We note that

$$\phi_r(s) = \sum_{j=0}^{\infty} p_{rj}\, s^j = \phi_r(0) + \sum_{j=1}^{\infty} p_{rj}\, s^j$$

$$\leq \phi_r(0) + (1 - \phi_r(0))s.$$

Therefore, $m_r > 1 - \phi_r(0)$ for all r, and, in particular, for $r > N$. Thus since $m_r' < m_r$ for all r, $\phi_r(s) \leq (1 - m_r') + m_r's$ for $s = s(r)$ sufficiently close to 1. For each $r = 1,\cdots,N$, let $t_r < 1$ be the smallest nonnegative real number such that

$$(1 - m_r') + m_r's \geq \phi_r(s) \text{ for } s \geq t_r,$$

and let $t = \max\{(1 - m_r') + m_r't_r,\ r = 1,\cdots,N\}$. Clearly $0 \leq t < 1$.

Define piecewise linear functions $\psi_1, \psi_2,\cdots$ on the unit interval by

$$\psi_r(s) = \begin{cases} t & , s < [m_r' - (1 - t)]/m_r', \\ (1 - m_r') + m_r's, & s \geq [m_r' - (1 - t)]/m_r', \end{cases} \quad r = 1,2,\cdots. \tag{9.8}$$

Thus $\phi_r(s) \leq \psi_r(s)$ for $0 \leq s \leq 1$ and $r = 1,2,\cdots$.

Define a process Y_n, $n = 0,1,\cdots$, on the unit interval as follows: $Y_0 = 1 - e^{-|\log(1-t)|} = 1 - e^{-\alpha}$, say, and

$$Y_{n+1} = \psi_{V_n}(Y_n), \quad n = 0,1,\cdots.$$

In this case, we have $E(X_n|X_0 = 1 - e^{-\alpha}) \leq E(Y_n|Y_0 = 1 - e^{-\alpha})$.

With the representation $1 - e^{-x}$, $x \geq 0$, for real numbers in the semi-closed interval $[0,1)$, the equations (9.8) become

$$\psi_r(1 - e^{-x}) = \begin{cases} 1 - e^{-\alpha} & , x - \log m'_r < \alpha, \\ 1 - e^{-(x - \log m'_r)} & , x - \log m'_r \geq \alpha, \end{cases} \quad r = 1,2,\cdots. \qquad (9.9)$$

Define

$$W_n = -\log(1 - Y_n) - \alpha, \quad n = 0,1,\cdots,$$

from which it follows that

$$P(W_n \leq x) = P(Y_n \leq 1 - e^{-(x+\alpha)}), \; x \geq 0, \quad n = 0,1,\cdots.$$

Define a sequence $U_1, U_2, \cdots$ of independent, identically distributed random variables by

$$U_n = -\log m'_{V_{n-1}}, \quad n = 1,2,\cdots.$$

Then

$$\begin{aligned} Y_{n+1} &= \psi_{V_n}(Y_n) \\ &= \psi_{V_n}(1 - e^{-(W_n+\alpha)}) \\ &= \begin{cases} 1 - e^{-\alpha} & , W_n + U_{n+1} < 0 \\ 1 - e^{-(W_n+U_{n+1}+\alpha)} & , W_n + U_{n+1} \geq 0, \end{cases} \end{aligned}$$

by (9.9). Hence

$$W_{n+1} = -\log(1 - Y_{n+1}) - \alpha = \begin{cases} 0 & , W_n + U_{n+1} < 0 \\ W_n + U_{n+1}, & W_n + U_{n+1} \geq 0, \end{cases}$$

that is, $\{W_n\}$ is a Lindley process.

Since

$$EU_1 = E(-\log m'_{V_0})$$

$$= -\sum_r p_r \log m'_r < 0$$

by (9.7), it follows from Theorem 8.1 that the W_n-process has a non-defective limiting distribution. That is,

$$\lim_{x\to\infty} \lim_{n\to\infty} P(W_n \le x) = 1. \qquad (9.10)$$

For some ε, $0 < \varepsilon < 1$, choose x_0 sufficiently large that

$$\lim_{n\to\infty} P(W_n \le x_0) > \varepsilon,$$

and then choose n_0 sufficiently large that

$$P(W_n \le x_0) > \varepsilon/2 \text{ for } n \ge n_0.$$

Thus we have

$$P(Y_n \le 1 - e^{-(x_0+\alpha)} | Y_0 = 1 - e^{-\alpha}) > \varepsilon/2 \text{ for } n \ge n_0.$$

It follows that

$$E(Y_n | Y_0 = 1 - e^{-\alpha}) \le (\varepsilon/2)[1 - e^{-(x_0+\alpha)}] + (1 - \varepsilon/2)$$

$$= 1 - (\varepsilon/2)e^{-(x_0+\alpha)}, \text{ for } n \ge n_0.$$

Since the right side of this inequality is independent of n,

$$\lim_{n\to\infty} E(Y_n | Y_0 = 1 - e^{-\alpha}) < 1.$$

We therefore obtain

$$\lim_{n\to\infty} \Pi_n(1 - e^{-\alpha}) = \lim_{n\to\infty} E(X_n | X_0 = 1 - e^{-\alpha})$$

$$\le \lim_{n\to\infty} E(Y_n | Y_0 = 1 - e^{-\alpha})$$

$$< 1 .$$

It follows that for all $s\varepsilon[0,1)$, and in particular $s = 0$, $\lim \Pi_n(s) < 1$ and extinction does not occur with probability one. The proof of the theorem is complete.

The following results follow immediately from Theorem 9.1.

Corollary 9.1. If, for some positive integer N, $\sum_{r=1}^{N} p_r = 1$, then a necessary and sufficient condition for extinction with probability one is that $\sum_r p_r \log m_r \leq 0$.

Corollary 9.2. If $\sum_r p_r |\log m_r| < \infty$, $\sum_r p_r \log m_r > 0$, and there exists $c < 1$ such that $\phi_r(0) \leq c$ for all r, then $\Pi_n(0) \longrightarrow q < 1$ as $n \longrightarrow \infty$.

10. Comparison with related processes

It is interesting to compare the branching process in a random environment with two related branching processes which may be classified as multitype Galton-Watson processes, the latter being generalizations of the simple Galton-Watson process to processes involving several different types of objects. Before introducing these two processes, the multitype Galton-Watson process will be defined and those properties of interest here will be stated. For a detailed discussion of the multitype Galton-Watson process, the reader is referred to Chapter II of Harris (1963).

We will consider a multitype process consisting of k types ($k < \infty$). Associated with type i is the probability generating function

$$f^i(s_1,\cdots,s_k) = \sum_{r_1,\ldots,r_k=0}^{\infty} p^i(r_1,\cdots,r_k)s_1^{r_1}\cdots s_k^{r_k}, \quad (10.1)$$

$$|s_1|,\cdots,|s_k| \leq 1, \quad i = 1,\cdots,k,$$

where $p^i(r_1,\cdots,r_k)$ is the probability that an object of type i has r_1 offspring of type 1, r_2 of type 2,$\cdots$, r_k of type k.

Let $\underline{Z}_n = (Z_n^1,\cdots,Z_n^k)'$ represent the population size, by type, in the $n\underline{th}$ generation. If $\underline{Z}_n = (r_1,\cdots,r_k)'$, then $\underline{Z}_{n+1}$ is the sum of $r_1 +\cdots+ r_k$ independent random vectors, r_1 of them having the generating function f^1, r_2 of them having the generating function $f^2,\cdots$, r_k of them having the generating function f^k. If $\underline{Z}_n = \underline{0}$, then $\underline{Z}_{n+1} = \underline{0}$. If $f_n^i(s_1,\cdots,s_k) = f_n^i(\underline{s})$ represents the generating function of $\underline{Z}_n$, given a single object of type i in the zeroth generation, then

$$\begin{aligned} f_{n+1}^i(\underline{s}) &= f^i[f_n^1(\underline{s}),\cdots,f_n^k(\underline{s})], \quad && n = 0,1,\cdots \\ f_0^i(\underline{s}) &= s_i, && i = 1,\cdots,k, \end{aligned} \tag{10.2}$$

or, in vector form,

$$\underline{f}_{n+1}(\underline{s}) = \underline{f}[\underline{f}_n(\underline{s})], \tag{10.3}$$

where $\underline{f}(\underline{s}) = [f^1(\underline{s}),\cdots,f^k(\underline{s})]$.

Let $M = (m_{ij})$ be the matrix of first moments

$$m_{ij} = E(Z_1^j|\underline{Z}_0 = \underline{e}_i) = \frac{\partial f^i(1,\cdots,1)}{\partial s_j}. \quad i,j = 1,\cdots,k, \tag{10.4}$$

where $\underline{e}_i$ denotes the column vector whose $i\underline{th}$ component is 1 and whose other components are 0. Then

$$E(\underline{Z}_{n+m}|\underline{Z}_n) = \underline{Z}_n' M^m, \quad n,m = 0,1,\cdots,$$

with probability one, and, in particular,

$$E(\underline{Z}_n|\underline{Z}_0 = \underline{e}_i) = \underline{e}_i' M^n, \tag{10.5}$$

the $i\underline{th}$ row of M^n.

We shall call a vector $\underline{v}$ or a matrix A positive ($\underline{v} > \underline{0}$, $A > 0$) or nonnegative ($\underline{v} \geq \underline{0}$, $A \geq 0$) if all its components have these properties. If $\underline{u}$ and $\underline{v}$ are vectors, then $\underline{u} > \underline{v}$ ($\underline{u} \geq \underline{v}$) means that $\underline{u} - \underline{v} > \underline{0}$ ($\underline{u} - \underline{v} \geq \underline{0}$).

The basic theorem concerning extinction with probability one can now be stated. We shall assume that M^N is positive for some positive integer N. The process is then said to be positively regular. Such a matrix has a positive characteristic root ρ which is simple and greater in absolute value than any other characteristic root.[2] We shall also assume that the process is not singular. (The process is said to be singular if the generating functions $f^1(s_1, \cdots, s_k), \cdots, f^k(s_1, \cdots, s_k)$ are all linear in $s_1, \cdots, s_k$ with no constant term.) Let q^i be the probability of extinction if initially there is one object of type i; that is,

$$q^i = P(\underline{Z}_n = \underline{0} \text{ for some } n \mid \underline{Z}_0 = \underline{e}_i).$$

The vector $(q^1, \cdots, q^k)'$ is denoted by $\underline{q}$. Let $\underline{1}$ denote the vector $(1, 1, \cdots, 1)'$.

Theorem 10.1 (Harris). Suppose the process is positively regular and not singular. If $\rho \leq 1$, then $\underline{q} = \underline{1}$. If $\rho > 1$, then $\underline{0} \leq \underline{q} < \underline{1}$, and $\underline{q}$ is the unique solution of the equation

$$\underline{s} = \underline{f}(\underline{s}) \tag{10.6}$$

satisfying $\underline{0} \leq \underline{s} < \underline{1}$.

---

[2] If $M^N > 0$, the matrix M must be irreducible, since every power of a reducible matrix is reducible. The result now follows from the Frobenius Theorem [Gantmacher (1959), p. 65].

We now want to define two multitype Galton-Watson processes of a special type. Let $\{p_i\}$ be a probability mass function with $\sum_{i=1}^{k} p_i = 1$ for some positive integer k and let $\{\phi_i(s)\}$ be a corresponding sequence of probability generating functions satisfying the assumptions of Section 1; $\phi_i(s) = \Sigma\, p_{ij} s^j$ is the probability generating function for the number of offspring of an i-type object.

A-process. In this process, we shall assume that all offspring of an object of type i are of the same type, type j with probability p_j, $j = 1,\cdots,k$. That is, in the notation of (10.1),

$$\begin{aligned} p^i(0,\cdots, 0) &= p_{i0} \\ p^i(0,\cdots, 0, r_j, 0,\cdots, 0) &= p_j p_{ir_j} \qquad j = 1,\cdots,k; \quad r_j \geq 1 \\ p^i(r_1,\cdots, r_k) &= 0, \text{ otherwise.} \end{aligned}$$

Then

$$f^i(s_1,\cdots, s_k) = p_1\phi_i(s_1) +\cdots+ p_k\phi_i(s_k)$$

and, by (10.2),

$$f^i_{n+1}(\underline{s}) = p_1\phi_i[f^1_n(\underline{s})] +\cdots+ p_k\phi_i[f^k_n(\underline{s})]. \tag{10.7}$$

The matrix M of first moments is given by

$$M = \begin{bmatrix} p_1m_1 & p_2m_1 & \cdots & p_km_1 \\ p_1m_2 & p_2m_2 & \cdots & p_km_2 \\ \cdots & \cdots & \cdots & \cdots \\ p_1m_k & p_2m_k & \cdots & p_km_k \end{bmatrix}$$

B-process. In this process, we shall assume that each offspring of an i-type object is of type j with probability p_j, independently of the type of other offspring (and of the type of the parent object). If $r = \sum_{j=1}^{k} r_j$, then, in the notation of (10.1),

$$p^i(r_1,\cdots, r_k) = p_{ir}\binom{r}{r_1\ r_2 \cdots r_k} p_1^{r_1}\cdots p_k^{r_k}.$$

Thus

$$f^i(s_1,\cdots,s_k) = \sum_{r_1,\cdots,r_k=0}^{\infty} p_{ir}\binom{r}{r_1 \cdots r_k}(p_1s_1)^{r_1}\cdots(p_ks_k)^{r_k}$$

$$= \sum_{r=0}^{\infty} p_{ir}\Big\{\sum_{\substack{r_1,\cdots,r_k \\ \ni r_1+\cdots+r_k=r}} \binom{r}{r_1\cdots r_k}(p_1s_1)^{r_1}\cdots(p_ks_k)^{r_k}\Big\}$$

$$= \sum_{r=0}^{\infty} p_{ir}(p_1s_1 +\cdots+ p_ks_k)^r$$

$$= \phi_i(p_1s_1 +\cdots+ p_ks_k),$$

and

$$f^i_{n+1}(\underline{s}) = \phi_i(p_1f^1_n(\underline{s}) +\cdots+ p_kf^k_n(\underline{s})). \tag{10.8}$$

The matrix M is the same as in the A-process.

Since these two processes have the same moment matrix M, they both become extinct with probability one if and only if $\rho \leq 1$. Now it is a well-known result of matrix theory that if C is a nonsingular matrix, then M and $C^{-1}MC$ have the same characteristic roots. Therefore, if we let

$$C = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & m_k \end{bmatrix},$$

then

$$C^{-1}MC = \begin{bmatrix} p_1m_1 & p_2m_2 & \cdots & p_km_k \\ p_1m_1 & p_2m_2 & \cdots & p_km_k \\ \cdots & \cdots & \cdots & \cdots \\ p_1m_1 & p_2m_2 & \cdots & p_km_k \end{bmatrix}.$$

Hence $\rho = p_1m_1 + p_2m_2 + \cdots + p_km_k$, since the dominant root of a nonnegative matrix cannot be greater than the maximum row sum nor less than the minimum row sum [Gantmacher (1959), p. 82].

Thus, if $\rho > 1$, we have by (10.6) that the probabilities of extinction satisfy

$$q^i = p_1\phi_i(q^1) + p_2\phi_i(q^2) + \cdots + p_k\phi_i(q^k), \quad i = 1,\cdots,k, \tag{10.9}$$

for the A-process, and

$$q^i = \phi_i(p_1q^1 + \cdots + p_kq^k), \quad i = 1,\cdots,k, \tag{10.10}$$

for the B-process.

Using an induction argument, it is easily seen that ${}_Bf_n^i(\underline{0}) \leq {}_Af_n^i(\underline{0})$ for $i = 1,\cdots,k$, from which we obtain $\underline{q}_B \leq \underline{q}_A$ (i.e., $q_B^i \leq q_A^i$ for $i = 1,\cdots,k$).

We recall that $\underline{Z}_0$ is nonrandom in the A- and B-processes. Suppose we let $\underline{Z}_0 = \underline{e}_i$ with probability p_i, $i = 1,\cdots,k$, so that these processes are more analogous to the random environment process defined by $\{p_i\}$ and $\{\phi_i(s)\}$. We still obtain extinction with probability one if and only if $\rho \leq 1$, and in case $\rho > 1$, the probability of extinction is given by

$$q_A = p_1\, q_A^1 + \cdots + p_k\, q_A^k$$

for the A-process, and

$$q_B = p_1 q_B^1 + \cdots + p_k q_B^k$$

for the B-process, with $q_B \leq q_A$.

Let $f_n(s)$ be the probability generating function for the size of the nth generation, without regard to type, in the A-process with $\underline{Z}_0$ random (i.e. $\underline{Z}_0 = \underline{e}_i$ with probability p_i, $i = 1, \cdots, k$). Then

$$f_n(s) = \sum_{i=1}^{k} p_i f_n^i(s, \cdots, s). \qquad (10.11)$$

From (10.7) we obtain

$$f_n^i(\underline{s}) = \sum_{i_1=1}^{k} p_{i_1} \phi_i(f_{n-1}^{i_1}(\underline{s}))$$

$$\leq \sum_{i_1=1}^{k} \sum_{i_2=1}^{k} p_{i_1} p_{i_2} \phi_i(\phi_{i_1}(f_{n-2}^{i_2}(\underline{s})))$$

$$\leq \cdots$$

$$\leq \sum_{i_1, \cdots, i_{n-1}=1}^{k} p_{i_1} p_{i_2} \cdots p_{i_{n-1}} \phi_i(\phi_{i_1}(\cdots \phi_{i_{n-2}}(f_1^{i_{n-1}}(\underline{s}))\cdots)).$$

Therefore, from (10.11),

$$f_n(s) \leq \sum_{i, i_1, \cdots, i_{n-1}=1}^{k} p_i p_{i_1} \cdots p_{i_{n-1}} \phi_i(\phi_{i_1}(\cdots \phi_{i_{n-2}}(\phi_{i_{n-1}}(s))\cdots))$$

$$= \Pi_n(s)$$

by (2.2). In particular, $f_n(0) \leq \Pi_n(0)$, so $q_A \leq q$ where q is the probability of extinction in the random environment process.

We note that if $p_1 m_1 + \cdots + p_k m_k > 1$ while $m_1^{p_1} m_2^{p_2} \cdots m_k^{p_k} \leq 1$, the random environment process becomes extinct with probability one, while the A- and B-processes (with $\underline{Z}_0$ random or non-random) have a positive probability of surviving indefinitely.

The results of this section may be summarized as follows.

Theorem 10.1 (a) Suppose $\underline{Z}_0 = \underline{e}_i$ (non-random).

If $\rho = p_1 m_1 + \cdots + p_k m_k \le 1$, then $\underline{q}_A = \underline{q}_B = \underline{1}$. If $\rho > 1$, then $\underline{0} \le \underline{q}_B \le \underline{q}_A < \underline{1}$; $\underline{q}_A$ satisfies the equations

$$q_A^i = p_1 \phi_i(q_A^1) + \cdots + p_k \phi_i(q_A^k), \quad i = 1,\cdots,k, \qquad (10.12)$$

and $\underline{q}_B$ satisfies the equations

$$q_B^i = \phi_i(p_1 q_B^1 + \cdots + p_k q_B^k), \quad i = 1,\cdots,k. \qquad (10.13)$$

(b) Suppose $\underline{Z}_0 = \underline{e}_i$ with probability p_i, $i = 1,\cdots,k$.

If $\rho \le 1$, then $q_A = q_B = q = 1$ (q is the probability of extinction in the random environment process). If $\rho > 1$, then $0 \le q_B \le q_A < 1$ and $q_A \le q \le 1$, where

$$q_A = p_1 q_A^1 + \cdots + p_k q_A^k$$

and

$$q_B = p_1 q_B^1 + \cdots + p_k q_B^k,$$

with q_A^i and q_B^i ($i = 1,\cdots,k$) given by (10.12) and (10.13) respectively.

CHAPTER III

EXTINCTION PROBABILITIES

11. Introduction

In the classical types of branching processes, in which distinct objects reproduce independently, the iterative nature of the probability generating functions for the sizes of successive generations often leads to equations such as $s = f(s)$ or $\underline{s} = \underline{f}(\underline{s})$, which are satisfied by the extinction probabilities. Further, given the probability of extinction q_1, say, with one initial object, it follows from the assumption of independence between objects that the probability of extinction with j initial objects, say q_j, is simply $q_j = q_1^j$. Such a relationship does not exist in the random environment process. In this chapter, we determine bounds for these probabilities.

We shall occasionally require the conditions in Theorem 9.1, and therefore state them here for reference:

$$\sum_r p_r |\log m_r| < \infty, \ \sum_r p_r \log m_r \leq 0; \tag{11.1}$$

$$\sum_r p_r |\log m_r| < \infty, \ \sum_r p_r \log m_r > 0, \ \sum_r p_r \log(1 - \phi_r(0)) \text{ converges.} \tag{11.2}$$

12. Moments of the dual process

Let $\Pi_n^{(k)}(s)$ designate the generating function of Z_n, given that $Z_0 = k$, $n = 0,1,\cdots$, $k = 1,2,\cdots$. For $\Pi_n^{(1)}(s)$ we shall simply write $\Pi_n(s)$.

<u>Theorem 12.1</u> <u>The generating function of Z_n, given that $Z_0 = k$, is given by</u>

$$\Pi_n^{(k)}(s) = \sum_r p_r \, \Pi_{n-1}^{(k)}(\phi_r(s)), \quad n = 1,2,\cdots; \; k = 1,2,\cdots, \tag{12.1}$$

<u>where $\Pi_0^{(k)}(s) = s^k$.</u>

The proof is analogous to the proof of Theorem 2.1 and will be omitted. From (12.1) we obtain

$$\Pi_n^{(k)\prime}(1) = m\Pi_{n-1}^{(k)\prime}(1) = m^n \, \Pi_0^{(k)\prime}(1) = km^n.$$

By repeated application of (12.1), we obtain the representation

$$\Pi_n^{(k)}(s) = \sum_{r_0,\cdots,r_{n-1}} p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \, \phi_{r_0}^k(\phi_{r_1}(\cdots \phi_{r_{n-1}}(s)\cdots)). \tag{12.2}$$

We shall also state, without proof, the following theorem, which is analogous in statement and proof to Theorem 3.2.

<u>Theorem 12.2</u> <u>For $s \in [0,1)$</u>,

$$\lim_{n\to\infty} \Pi_n^{(k)}(s) = q_k \le 1,$$

<u>where q_k is the probability of extinction, given $Z_0 = k$.</u>

We next want to establish a relationship between $\Pi_n^{(k)}(s)$ and the dual process $\{X_n\}$. One obtains for the <u>kth</u> moment of X_n, given $X_0 = s_0 \in [0,1)$,

$$E(X_n^k | X_0 = s_0) = \sum_{r_0,\cdots,r_{n-1}} p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \, \phi_{r_{n-1}}^k(\phi_{r_{n-2}}(\cdots \phi_{r_0}(s_0)\cdots))$$

$$= \sum p_{r_{n-1}} p_{r_{n-2}} \cdots p_{r_0} \, \phi_{r_{n-1}}^k(\phi_{r_{n-2}}(\cdots \phi_{r_0}(s_0)\cdots))$$

$$= \sum_{r_0,\cdots,r_{n-1}} p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \, \phi_{r_0}^k(\phi_{r_1}(\cdots \phi_{r_{n-1}}(s_0)\cdots)),$$

so from (12.2) we have

$$E(X_n^k | X_0 = s_0) = \Pi_n^{(k)}(s_0). \tag{12.3}$$

From (12.3) and Theorem 12.2, we obtain

Theorem 12.3 *As n tends to infinity, the kth moment of the dual process (at the nth step) tends to a limit, which is the probability of extinction for the random environment process (the Z_n-process), given k objects in the zeroth generation, k = 1,2,⋯. That is,*

$$\lim_{n\to\infty} E(X_n^k | X_0 = s_0) = q_k, \tag{12.4}$$

for $s_0 \varepsilon [0,1)$ and k = 1,2,⋯.

It follows [Feller (1966), p. 244] that there exists a random variable X such that X_n tends to X in distribution and $q_k = EX^k$ is the kth moment of X. We shall refer to this limiting distribution as the ergodic distribution of the dual process.

Under the assumptions of Theorem 9.1(b), and with $\{Y_n\}$ and $\{W_n\}$ defined as in its proof, it follows that

$$P(Y_n \le z | Y_0 = s_0) \le P(X_n \le z | X_0 = s_0).$$

Hence

$$\lim_{x\to\infty} \lim_{n\to\infty} P(X_n \le 1 - e^{-(x+\alpha)} | X_0 = 1 - e^{-\alpha})$$

$$\ge \lim_{x\to\infty} \lim_{n\to\infty} P(Y_n \le 1 - e^{-(x+\alpha)} | Y_0 = 1 - e^{-\alpha})$$

$$= \lim_{x\to\infty} \lim_{n\to\infty} P(W_n \le x | W_0 = 0) = 1,$$

by (9.10). Given $\varepsilon > 0$, choose x_0 sufficiently large that

$$\lim_{n\to\infty} P(X_n \le 1 - e^{-(x_0+\alpha)} | X_0 = 1 - e^{-\alpha}) > 1 - \varepsilon. \tag{12.5}$$

Let G be the distribution function of X on [0,1]; that is, G is the ergodic distribution of the dual process. By (12.5) it follows that $G(1^-) = 1$. Hence by bounded convergence, $EX^k \to 0$ as $k \to \infty$; that is, $q_k \to 0$ as $k \to \infty$.

We may summarize these results as follows.

Theorem 12.4 *To any branching process $\{Z_n\}$ in a random environment (as defined in Section 1), there corresponds, on the unit interval, a dual Markov process $\{X_n\}$ (as defined in Section 6), possessing an ergodic distribution G, the kth moment of which is the probability q_k that the branching process becomes extinct, given that $Z_0 = k$.*

Furthermore, if (11.1) holds,

$$q_k = \int_0^1 x^k \, dG(x) = 1 \quad \text{for all } k = 1,2,\cdots,$$

while if (11.2) holds,

$$q_k = \int_0^1 x^k \, dG(x) \downarrow 0 \quad \text{as } k \to \infty.$$

13. Equations for extinction probabilities

We shall assume, in this section, that the conditions (11.2) are satisfied. Let

$$p_i^{(k)} = P(Z_1 = i | Z_0 = k), \quad i = 0,1,\cdots, \; k = 1,2,\cdots,$$

so $\Pi_1^{(k)}(s)$ can be written in the form

$$\Pi_1^{(k)}(s) = \sum_{i=0}^{\infty} p_i^{(k)} s^i, \; |s| \leq 1, \quad k = 1,2,\cdots. \tag{13.1}$$

We then have the system of equations

$$
\begin{aligned}
q_1 &= p_0^{(1)} + p_1^{(1)} q_1 + p_2^{(1)} q_2 + \cdots + p_n^{(1)} q_n + \cdots \\
q_2 &= p_0^{(2)} + p_1^{(2)} q_1 + p_2^{(2)} q_2 + \cdots + p_n^{(2)} q_n + \cdots \\
&\cdots \\
q_n &= p_0^{(n)} + p_1^{(n)} q_1 + p_2^{(n)} q_2 + \cdots + p_n^{(n)} q_n + \cdots \\
&\cdots
\end{aligned}
\tag{13.2}
$$

We can write the first n of equation (13.2) as

$$
\begin{aligned}
q_1 &= p_0^{(1)} + p_1^{(1)} q_1 + \cdots + p_n^{(1)} q_n + C_n^{(1)} \\
q_2 &= p_0^{(2)} + p_1^{(2)} q_1 + \cdots + p_n^{(2)} q_n + C_n^{(2)} \\
&\cdots \\
q_n &= p_0^{(n)} + p_1^{(n)} q_1 + \cdots + p_n^{(n)} q_n + C_n^{(n)}
\end{aligned}
\tag{13.3}
$$

Let γ_j, $j = 1,2,\cdots$, be a decreasing sequence of real numbers tending to zero, and such that

$$q_j = EX^j \leq \gamma_j, \quad j = 1,2,\cdots.$$

It then follows that

$$C_n^{(j)} \leq \gamma_{n+1}(1 - \sum_{r=0}^{n} p_r^{(j)}). \tag{13.4}$$

Let

$$A_n = \begin{bmatrix} p_1^{(1)} & p_2^{(1)} & \cdots & p_n^{(1)} \\ p_1^{(2)} & p_2^{(2)} & \cdots & p_n^{(2)} \\ & \cdots & & \\ p_1^{(n)} & p_2^{(n)} & \cdots & p_n^{(n)} \end{bmatrix}$$

If $\phi_j(0) = 0$ for all j, then $q_1 = q_2 = \cdots = 0$, and the problem is trivial. If $\phi_j(0) > 0$ for some j, then $p_0^{(k)} > 0$ for all $k = 1,2,\cdots$, and it follows that A_n is a nonnegative matrix whose row sums are all less than 1 (for any n).

From the theory of nonnegative matrices [Gantmacher (1959), Chapter III], it follows that A_n has a nonnegative characteristic root λ_n such that no characteristic root of A_n has modulus exceeding λ_n. Further, if s_n and S_n denote respectively the minimum and maximum row sums of A_n, then

$$s_n \leq \lambda_n \leq S_n,$$

and, since $S_n < 1$, we have $\lambda_n < 1$, whatever the positive integer n.

Let $\underline{q}_n = (q_1, q_2, \cdots, q_n)'$, $\underline{P}_n = (p_0^{(1)}, p_0^{(2)}, \cdots, p_0^{(n)})'$, and $\underline{C}_n' = (C_n^{(1)}, C_n^{(2)}, \cdots, C_n^{(n)})'$. Equations (13.3) thus become

$$(E - A_n)\underline{q}_n = \underline{P}_n + \underline{C}_n, \tag{13.5}$$

where $E = E(n \times n)$ is the identity matrix.

Since $\lambda_n < 1$, $|E - A_n| \neq 0$. Hence $(E - A_n)^{-1}$ exists for all n, and we have further [Gantmacher (1959), Chapter III] that

$$(\lambda E - A_n)^{-1} \geq 0 \quad \text{for } \lambda > \lambda_n.$$

In particular, therefore, $(E - A_n)^{-1} \geq 0$. From (13.5), we thus obtain

$$\underline{q}_n = (E - A_n)^{-1}[\underline{P}_n + \underline{C}_n].$$

Let $R_n^{(j)}$, $j = 1, \cdots, n$, denote the row sums of $(E - A_n)$. Then, from (13.4),

$$C_n^{(j)} \leq \gamma_{n+1}(1 - \sum_{r=0}^{n} p_r^{(j)})$$

$$< \gamma_{n+1}(1 - \sum_{r=1}^{n} p_r^{(j)})$$

$$= \gamma_{n+1} R_n^{(j)}. \tag{13.6}$$

Since $(E - A_n)^{-1}$ is a nonnegative matrix,

$$(E - A_n)^{-1}\underline{P}_n \le \underline{q}_n \le (E - A_n)^{-1}[\underline{P}_n + \gamma_{n+1} \underline{R}_n]. \tag{13.7}$$

where $\underline{R}_n = (R_n^{(1)}, R_n^{(2)}, \cdots, R_n^{(n)})'$.

But $\underline{R}_n = (E - A_n)\underline{1}$, where $\underline{1} = (1,1,\cdots,1)'$, so $(E - A_n)^{-1}\underline{R}_n = \underline{1}$, and (13.7) becomes

$$(E - A_n)^{-1}\underline{P}_n \le \underline{q}_n \le (E - A_n)^{-1}\underline{P}_n + \gamma_{n+1} \underline{1}. \tag{13.8}$$

Hence we have the following result.

<u>Theorem 13.1</u> <u>Suppose the conditions (11.2) are satisfied. Let γ_j, $j = 1,2,\cdots$, be a decreasing sequence of real numbers tending to zero and such that $q_j = EX^j \le \gamma_j$, $j = 1,2,\cdots$. Then (13.8) holds for all $n = 1,2,\cdots$.</u>

Theorem 13.1 shows that we can obtain any number of the q_j's to any desired accuracy given a suitable sequence $\{\gamma_j\}$. We note that (13.8) holds provided only that $q_j = EX^j \le \gamma_j$ and that $\{\gamma_j\}$ is a decreasing sequence. However, the conditions (11.2) imply the existence of such a sequence with $\gamma_j \downarrow 0$.

By using the first of inequalities (13.6), we can obtain upper bounds on the q_j's which are better than those given by (13.8). That is, if $T_n^{(j)} = (1 - \sum_{r=0}^{n} p_r^{(j)})$, then by the first inequality in (13.6),

$$C_n^{(j)} \le \gamma_{n+1} T_n^{(j)}.$$

It follows from (13.5), with $\underline{T}_n = (T_n^{(1)}, T_n^{(2)}, \cdots, T_n^{(n)})'$, that

$$(E - A_n)^{-1}\underline{P}_n \leq \underline{q}_n \leq (E - A_n)^{-1}\underline{P}_n + \gamma_{n+1}(E - A_n)^{-1}\underline{T}_n. \qquad (13.9)$$

In one special situation, we can find a simple sequence $\{\gamma_j\}$ as the following theorem shows.

Theorem 13.2 Suppose the probability generating functions $\phi_r(s)$, $r = 1,2,\cdots$, are all such that for some $\gamma < 1$,

$$\phi_r(s) \leq s, \quad \text{for } s \geq \gamma. \qquad (13.10)$$

Then

$$(E - A_n)^{-1}\underline{P}_n \leq \underline{q}_n \leq (E - A_n)^{-1}\underline{P}_n + \gamma^{n+1}\,\underline{1}, \qquad (13.11)$$

for all $n = 1,2,\cdots$.

Note that (13.10) implies that $m_r > 1$ for all r, and if there are only a finite number of environmental states, then $m_r > 1$ for all r implies (13.10).

Proof.

$$\begin{aligned} E(X_n^j | X_0 = \gamma) &= \sum_{r_0,\cdots,r_{n-1}} p_{r_0} p_{r_1} \cdots p_{r_{n-1}} \phi^j_{r_{n-1}}(\phi_{r_{n-2}}(\cdots\phi_{r_0}(\gamma)\cdots)) \\ &\leq \sum_{r_1,\cdots,r_{n-1}} p_{r_1} \cdots p_{r_{n-1}} \phi^j_{r_{n-1}}(\phi_{r_{n-2}}(\cdots\phi_{r_1}(\gamma)\cdots)) \\ &\leq \cdots \\ &\leq \gamma^j. \end{aligned}$$

Hence

$$\lim_{n\to\infty} E(X_n^j | X_0 = \gamma) = EX^j \leq \gamma^j,$$

so we may put $\gamma_j = \gamma^j$, $j = 1,2,\cdots$, in (13.8).

Example. Suppose there are just two environmental states. Let $\phi_1(s) = \frac{1}{4} + \frac{3}{4}s^2$, $\phi_2(s) = \frac{1}{3}s + \frac{2}{3}s^2$, and $p_1 = p_2 = \frac{1}{2}$. In this example, γ in (13.10) is the solution less than 1 of the equation

$$\phi_1(s) = s;$$

that is, $\gamma = \frac{1}{3}$. The equations (13.1) are given by

$$\Pi_1^{(1)}(s) = .12500 + .16667s + .70834s^2$$

$$\Pi_1^{(2)}(s) = .03125 + .24306s^2 + .22222s^3 + .50347s^4$$

$$\Pi_1^{(3)}(s) = .00782 + .07032s^2 + .01852s^3 + .32205s^4 + .22222s^5 + .35909s^6$$

$$\Pi_1^{(4)}(s) = .00196 + .02344s^2 + .11165s^4 + .04939s^5 + .35909s^6 + .19753s^7 + .25697s^8.$$

. . .

With n = 4, we have

$$A_4 = \begin{bmatrix} .16667 & .70834 & 0 & 0 \\ 0 & .24306 & .22222 & .50347 \\ 0 & .07032 & .01852 & .32205 \\ 0 & .02344 & 0 & .11165 \end{bmatrix}$$

and

$$(E - A_4)^{-1} = \begin{bmatrix} 1.20000 & 1.17112 & 0.26517 & 0.75986 \\ 0 & 1.37777 & 0.31195 & 0.89392 \\ 0 & 0.11065 & 1.04392 & 0.44101 \\ 0 & 0.03635 & 0.00823 & 1.14925 \end{bmatrix}$$

$\underline{P}_4' = (0.12500, 0.03125, 0.00782, 0.00196)$, and $\gamma^5 = 0.00412$. From (13.11) we obtain

$$\begin{bmatrix} 0.19016 \\ 0.04725 \\ 0.01249 \\ 0.00345 \end{bmatrix} \leq \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \leq \begin{bmatrix} 0.19428 \\ 0.05137 \\ 0.01661 \\ 0.00757 \end{bmatrix}$$

Suppose we let $n = 8$ in (13.11) in order to obtain better bounds for $q_1, \cdots, q_4$. The same procedure leads to the bounds

$$\begin{bmatrix} .19040 \\ .04753 \\ .01294 \\ .00368 \end{bmatrix} \leq \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \leq \begin{bmatrix} .19045 \\ .04758 \\ .01299 \\ .00373 \end{bmatrix}$$

In taking n larger to improve the bounds on $q_1, \cdots, q_4$, we also obtain the lower bounds .00107, .00032, .00009, .00003 for $q_5, \cdots, q_8$, respectively, each differing from the real value by less than $\gamma^9 \doteqdot 5 \times 10^{-5}$.

14. The case of two environmental states

In this section we shall assume that there are just two environmental states, and that $m_1^{p_1} m_2^{p_2} > 1$ (*i.e.*, extinction is not certain). In Section 13, we obtained bounds on the q_j's in the case when all the m_r's are greater than 1. Hence we shall further assume here that $m_1 \leq 1$ and $m_2 > 1$. In this case, we obtain explicit bounds for the moments of the ergodic distribution of the dual process and consequently bounds on the sequence $\{q_j\}$ of extinction probabilities. This will be accomplished by "bounding" the dual process by a process of a very special kind for which an equilibrium distribution exists and can be obtained.

We shall need the following result.

<u>Lemma 14.1</u> <u>If $m_1 \leq 1$, $m_2 > 1$, and $m_1^{p_1} m_2^{p_2} > 1$, $p_1 + p_2 = 1$, then there exist relatively prime positive integers a and b and a real number θ, $0 < \theta < 1$, such that $\theta^a < m_1$, $\theta^{-b} < m_2$, and $ap_1 - bp_2 < 0$.</u>

<u>Proof</u>. Since $x^{p_1} y^{p_2}$ is a continuous function of x and y, there exists a spherical neighborhood N of (m_1, m_2) such that $(x, y)\varepsilon N$ implies $x^{p_1} y^{p_2} > 1$. Let $R = \{(x, y) | 0 < x < m_1, 1 < y < m_2\}$. There exists $(x, y)\varepsilon N \cap R$ such that

$$\frac{\log x}{\log(1/y)} = r > 0,$$

where r is a rational number. Let $r = a/b$, for a and b relatively prime positive integers. Then

$$\frac{1}{a} \log x = \frac{1}{b} \log(1/y),$$

that is,

$$x^{1/a} = y^{-1/b}.$$

Let $\theta = x^{1/a} > 0$ (<u>i</u>.<u>e</u>. θ is the positive real a<u>th</u> root of x), so $x = \theta^a$ and $y = \theta^{-b}$. Thus $\theta^a < m_1$, $\theta^{-b} < m_2$, $0 < \theta < 1$, and $\theta^{ap_1} \cdot \theta^{-bp_2} > 1$ (<u>i</u>.<u>e</u>., $ap_1 - bp_2 < 0$).

Note that this is only an existence theorem and does not describe a procedure for finding appropriate values of a, b, and θ. However, in practice, these can generally be found by trial and error without too much effort.

We now proceed as in Case II of Section 7 to construct piecewise linear functions which bound ϕ_1 and ϕ_2 above. By Lemma 14.1, there exist relatively prime positive integers a and b, and a real number θ, $0 < \theta < 1$, such that $\theta^a < m_1$, $\theta^{-b} < m_2$, and $ap_1 - bp_2 < 0$.

To obtain the required functions, choose an integer N_1 sufficiently large that

$$(1 - \theta^a) + \theta^a s \geq \phi_1(s) \quad \text{for } s \geq 1 - \theta^{N_1},$$

and an integer N_2 sufficiently large that

$$(1 - \theta^{-b}) + \theta^{-b} s \geq \phi_2(s) \quad \text{for } s \geq 1 - \theta^{N_2}.$$

Then choose N_1 or N_2 still larger, if necessary, so that $N_1 + a = N_2 - b = N$, say. Define piecewise linear functions ψ_1 and ψ_2 on the unit interval by

$$\psi_1(s) = \begin{cases} 1 - \theta^N & , s < 1 - \theta^{N-a} \\ (1 - \theta^a) + \theta^a s, & s \geq 1 - \theta^{N-a} \end{cases}$$

and

$$\psi_2(s) = \begin{cases} 1 - \theta^N & , s < 1 - \theta^{N+b} \\ (1 - \theta^{-b}) + \theta^{-b} s, & s \geq 1 - \theta^{N+b}. \end{cases}$$

Let $\{X_n\}$ be the dual process with $X_0 = s_0 \varepsilon [0, 1)$, and define the Y_n-process as usual with $Y_0 = s_0$ and $Y_{n+1} = \psi_{V_n}(Y_n)$, $n = 0,1,\cdots$. Then it easily follows that

$$E\, X_n^k \leq E\, Y_n^k, \quad n,k = 1,2,\cdots. \tag{14.1}$$

We want to show that $\lim_{n\to\infty} E\, Y_n^k$ exists for all k and tends monotonically to zero as $k \to \infty$. Since $\lim_{n\to\infty} E\, X_n^k = q_k$, we shall have

$$q_k \le \gamma_k, \quad k = 1,2,\cdots,$$

where $\gamma_k = \lim_{n\to\infty} E\, Y_n^k$.

If we let $s_0 = 1 - \theta^N$, then the state space of the Y_n-process is the set of all real numbers $1 - \theta^{N+j}$, $j = 0,1,\cdots$.

The Y_n-process is a Markov chain with transition probabilities

$$p_{ij} = P(Y_{n+1} = 1 - \theta^{N+j} | Y_n = 1 - \theta^{N+i}), \quad i,j,n = 0,1,\cdots,$$

given by

$$\begin{aligned} p_{0,0} &= p_2,\ p_{0,a} = p_1 \\ p_{1,0} &= p_2,\ p_{1,a+1} = p_1 \\ &\ \vdots \\ p_{b,0} &= p_2,\ p_{b,a+b} = p_1 \\ p_{i,i-b} &= p_2,\ p_{i,i+a} = p_1, \quad i = b,b+1,\cdots \\ p_{ij} &= 0, \quad \text{otherwise.} \end{aligned} \tag{14.2}$$

<u>Lemma 14.2</u>. <u>The Markov chain $\{Y_n\}$ is irreducible and aperiodic.</u>

<u>Proof</u>. Once we have shown the chain to be irreducible, it follows immediately that it is aperiodic, since $p_{00} > 0$.

To show that the chain is irreducible, we must show that every state is accessible from any given state. (We shall refer to the real number $1 - \theta^{N+i}$ as state i, $i = 0,1,\cdots$.) Clearly 0 is accessible from every state. If the state 1 is accessible from the state 0, then every state is accessible from 0 (and thus from every other state), so we have only to show that 1 is accessible from 0. That is, we must prove that there exist positive integers n and m such that

$$na - mb = 1.$$

The argument used is based on arguments in Hardy and Wright (1960) employed in obtaining more general results in number theory.

Let S be the set of all integers of the form na - mb, where n and m are integers (a and b are the given relatively prime positive integers). If $s_1, s_2 \varepsilon S$, $s_1 \pm s_2 = (n_1 \pm n_2)a - (m_1 \pm m_2)b \varepsilon S$, so S contains the sum and difference of any two of its members. Clearly S contains positive integers; let d be the smallest positive number in S. If s is any positive number in S, then $s-zd \varepsilon S$ for every integer z. Now if c is the remainder when s is divided by d, and

$$s = zd + c,$$

then $c \varepsilon S$ and $0 \leq c < d$. Since d is the smallest positive number in S, c = 0, and s = zd. It follows that S consists of integral multiples of the number d. Since d divides every number in S, it divides a and b, and therefore $d \leq 1$ (since a and b are relatively prime). That is, d = 1, and hence S consists of all the integers. Let n_0, m_0 be a solution of na - mb = 1. Let r be a positive integer sufficiently large that $n_1 = n_0 + rb$ and $m_1 = m_0 + ra$ are both positive. Then

$$n_1 a - m_1 b = n_0 a - m_0 b = 1,$$

so (n_1, m_1) is a solution with positive integers. It follows that the state 1 is accessible from the state 0, and the proof of the lemma is complete.

An irreducible aperiodic Markov chain is positive recurrent if the equations

$$\sum_{j=0}^{\infty} u_j \, p_{jk} = u_k, \quad k = 0,1,\cdots$$

have a solution $\{u_k\}$ (u_k not all zero) satisfying $\Sigma|u_k| < \infty$. Furthermore, such a solution is a scalar multiple of the unique stationary distribution $\{\omega_k\}$ for the chain. Thus, in order to show that the states of the Markov chain $\{Y_n\}$ are positive recurrent and find the stationary distribution, we must obtain an absolutely convergent solution to the system of equations

$$\begin{aligned} u_0 &= p_2(u_0 + u_1 + \cdots + u_b) \\ u_1 &= p_2\, u_{b+1} \\ &\;\vdots \\ u_{a-1} &= p_2\, u_{a+b-1} \\ u_n &= p_1\, u_{n-a} + p_2\, u_{n+b}, \quad n = a, a+1, \cdots. \end{aligned} \tag{14.3}$$

Consider the auxiliary equation of the difference equation $u_n = p_1\, u_{n-a} + p_2\, u_{n+b}$:

$$h(z) = p_2\, z^{a+b} - z^a + p_1. \tag{14.4}$$

Now $h(0) = p_1$, $h(1) = 0$, and $h'(1) = p_2 b - p_1 a > 0$. Hence there exists λ_1, $0 < \lambda_1 < 1$, such that $h(\lambda_1) = 0$. By Descartes' rule of signs, $h(z)$ has at most two positive roots. Hence for $r \varepsilon (\lambda_1, 1)$, $h(r) < 0$, and for some $r_0 \varepsilon (\lambda_1, 1)$, $h'(r_0) = 0$.

Let Γ_r be the circle with center the origin and radius r for $r \varepsilon (\lambda_1, 1)$. Define

$$f(z) = -z^a$$

and

$$g(z) = p_2\, z^{a+b} + p_1.$$

On Γ_r, $|f(z)| = r^a$ and $|g(z)| \le p_2 r^{a+b} + p_1$. Since $h(r) < 0$, $p_2 r^{a+b} + p_1 < r^a$, and thus $|g(z)| < |f(z)|$ on Γ_r. By Rouché's theorem, it follows that $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside Γ_r; hence $h(z)$ has a zeros $\lambda_1, \lambda_2, \cdots, \lambda_a$ inside the unit circle and $|\lambda_i| \le \lambda_1$ for $i = 2, \cdots, a$.

Furthermore,

$$h'(z) = z^{a-1}[p_2(a+b)z^b - a],$$

so $h'(z)$ has a-1 zeros at 0, and the b remaining zeros are the b<u>th</u> roots of $a/p_2(a+b)$. But $h'(r_0) = 0$ for $r_0 \varepsilon (\lambda_1, 1)$, so these b roots have modulus greater than λ_1. Therefore, the a zeros of $h(z)$ in the unit circle are all distinct.

If we let

$$u_k = C_1(1 - \lambda_1)\lambda_1^k + \cdots + C_a(1 - \lambda_a)\lambda_a^k, \quad k = 0,1,\cdots, \tag{14.5}$$

then $\Sigma|u_k| < \infty$ and the stationary distribution for the Markov chain $\{Y_n\}$ is a scalar multiple of $\{u_k\}$ provided the C's in (14.5) can be chosen to satisfy the boundary conditions (the first a of equations (14.3)). The matrix of coefficients of the C's for the boundary conditions is given by

$$D = \begin{bmatrix} (1 - \lambda_1) - p_2(1 - \lambda_1^{b+1}) & \cdots & (1 - \lambda_a) - p_2(1 - \lambda_a^{b+1}) \\ \lambda_1(1 - \lambda_1)(1 - p_2\lambda_1^b) & \cdots & \lambda_a(1 - \lambda_a)(1 - p_2\lambda_a^b) \\ \cdots\cdots & \cdots & \cdots\cdots \\ \lambda_1^{a-1}(1 - \lambda_1)(1 - p_2\lambda_1^b) & \cdots & \lambda_a^{a-1}(1 - \lambda_a)(1 - p_2\lambda_a^b) \end{bmatrix},$$

and in order for the homogeneous system

$$D\underline{C} = \underline{0} \tag{14.6}$$

to have a non-null solution $\underline{C} = (C_1, \cdots, C_a)'$, it is necessary and sufficient that $|D| = 0$. Subtracting the sum of the last a - 1 rows from the first row (and using the fact that $p_2\lambda_i^{a+b} - \lambda_i^a + p_1 = 0$), the first row consists of zeros, and hence $|D| = 0$. Thus a non-null solution exists.

It follows that a stationary distribution exists and is determined to within a multiplicative constant (chosen so that $\Sigma u_k = 1$, i.e., $\Sigma C_i = 1$) by the solution of (14.6). Therefore the stationary distribution is given by (14.5), where the C's are the unique solution of the non-homogeneous system

$$B\underline{C} = \underline{\delta},$$

where $\underline{\delta}' = (1,0,\cdots,0)$, and

$$B = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1(1-\lambda_1)(1-p_2\lambda_1^b) & \cdots & \lambda_a(1-\lambda_a)(1-p_2\lambda_a^b) \\ \cdots & \cdots & \cdots \\ \lambda_1^{a-1}(1-\lambda_1)(1-p_2\lambda_1^b) & \cdots & \lambda_a^{a-1}(1-\lambda_a)(1-p_2\lambda_a^b) \end{bmatrix}. \tag{14.7}$$

That is,

$$\underline{C} = B^{-1}\,\underline{\delta},$$

so the vector $\underline{C}$ is the first column of B^{-1}. We therefore have the following result.

Theorem 14.1. Let $\{Y_n\}$ be a Markov chain with transition probabilities given by (14.2) with $ap_1 - bp_2 < 0$ (a and b are relatively prime and $p_1 + p_2 = 1$). The Markov chain is irreducible and aperiodic and possesses a unique stationary distribution $\{\omega_k\}$ given by

$$\omega_k = C_1(1 - \lambda_1)\lambda_1^k + \cdots + C_a(1 - \lambda_a)\lambda_a^k, \quad k = 0,1,\cdots, \tag{14.8}$$

where $\lambda_1,\cdots,\lambda_a$ are the roots of modulus less than 1 of the polynomial $p_2 z^{a+b} - z^a + p_1$, and $C_1,\cdots,C_a$ are the elements of the first row of B^{-1}, for B given by (14.7).

If $p_{ij}^{(n)}$ denotes the n-step transition probability from state i to state j of the Markov chain, it follows from the ergodicity of $\{Y_n\}$ that

$$\lim_{n\to\infty} p_{ij}^{(n)} = \omega_j > 0$$

for all i and j. Since $s_0 = 1 - \theta^N$, $\{Y_n\}$ is initially in state 0, so

$$EY_n^k = \sum_{j=0}^{\infty} (1 - \theta^{N+j})^k \, p_{0j}^{(n)},$$

and we want to show that

$$\lim_{n\to\infty} E\, Y_n^k = \sum_{j=0}^{\infty} (1 - \theta^{N+j})^k \, \omega_j, \quad k = 1,2,\cdots. \tag{14.9}$$

For an arbitrary positive integer m,

$$E\, Y_n^k = \sum_{j=0}^{m} (1 - \theta^{N+j})^k \, p_{0j}^{(n)} + \sum_{j=m+1}^{\infty} (1 - \theta^{N+j})^k \, p_{0j}^{(n)}.$$

Given $\varepsilon > 0$, there exists m_0 sufficiently large that $\sum_{j=0}^{m_0} \omega_j > 1 - \varepsilon$. Then there exists n_0 such that

$$|p_{0j}^{(n)} - \omega_j| < \varepsilon/m_0$$

for $j = 0,1,\cdots, m_0$, and all $n \geq n_0$.

Thus for $n \geq n_0$,

$$\sum_{j=0}^{m_0} p_{0j}^{(n)} \geq \sum_{j=0}^{m_0} (\omega_j - |\omega_j - p_{0j}^{(n)}|)$$

$$\geq 1 - 2\varepsilon.$$

Hence

$$\sum_{j=m+1}^{\infty} (1 - \theta^{N+j})^k \, p_{0j}^{(n)} \leq 2\varepsilon,$$

for $n \geq n_0$. It follows, since ε is arbitrary, that

$$\overline{\lim_{n\to\infty}} \, EY_n^k \leq \sum_{j=0}^{\infty} (1 - \theta^{N+j})^k \, \omega_j.$$

Further,

$$\underline{\lim}_{n\to\infty} \, EY_n^k \geq \sum_{j=0}^{m} (1 - \theta^{N+j})^k \, \omega_j$$

for every positive integer m. Hence

$$\underline{\lim}_{n\to\infty} \, EY_n^k \geq \sum_{j=0}^{\infty} (1 - \theta^{N+j})^k \, \omega_j,$$

and (14.9) holds. Clearly $\lim_n EY_n^k = \sum_j (1 - \theta^{N+j})^k \, \omega_j$ is a decreasing function of k, and, by dominated convergence, tends to zero as $k \to \infty$.

Thus we have shown that $\lim_n EY_n^k$ exists and tends monotonically to zero as $k \to \infty$. Since $\lim_n EX_n^k = q_k$, it follows from (14.1) that

$$q_k \leq \lim_{n\to\infty} EY_n^k = \sum_{j=0}^{\infty} (1 - \theta^{N+j})^k \, \omega_j.$$

From (14.8), we obtain

$$\sum_{j=0}^{\infty} (1 - \theta^{N+j})^k \omega_j$$

$$= \sum_{j=0}^{\infty} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \theta^{(N+j)r} [C_1(1 - \lambda_1)\lambda_1^j + \cdots + C_a(1 - \lambda_a)\lambda_a^j]$$

$$= \sum_{r=0}^{k} (-1)^r \binom{k}{r} \theta^{Nr} \left[\frac{C_1(1 - \lambda_1)}{1 - \lambda_1\theta^r} + \cdots + \frac{C_a(1 - \lambda_a)}{1 - \lambda_a\theta^r}\right],$$

the change in order of summation being justified by absolute convergence. Hence if we let $\gamma_k = \sum_j (1 - \theta^{N+j})^k \omega_j$ and let $f(s)$ be the generating function of the stationary distribution $\{\omega_k\}$, we have

$$\gamma_k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \theta^{Nr} f(\theta^r), \tag{14.10}$$

and $\gamma_k \downarrow 0$ as $k \longrightarrow \infty$.

We may summarize these results as follows.

Theorem 14.2 *Suppose there are just two environmental states. If $m_1 \leq 1$, $m_2 > 1$, and $m_1^{p_1} m_2^{p_2} > 1$, then there exist a real number θ, $0 < \theta < 1$ (Lemma 14.1), a positive integer N (see p. 50) and a probability generating function*

$$f(s) = \sum_{j=0}^{\infty} \omega_j s^j$$

(defined by (14.8)), such that

$$q_k \leq \gamma_k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \theta^{Nr} f(\theta^r),$$

and $\gamma_k \downarrow 0$ as $k \longrightarrow \infty$.

With the γ's thus obtained, (13.8) provides upper and lower bounds on any finite number of the extinction probabilities, with the width of the bound intervals tending to zero as n (in (13.8)) tends to infinity.

CHAPTER IV

MARKOV ENVIRONMENT

15. Description of the process

In this chapter, we shall introduce a more elaborate model of a branching process in a stochastic environment, and we shall show that the results of Chapter II can be applied to the problem of determining conditions for extinction with probability one. Whereas the model introduced in Chapter I assumes that the environmental states are sampled randomly from one generation to the next, we shall now assume that the environmental states constitute a Markov chain. That is, we shall consider a branching process in a Markov environment.

While we shall give a purely mathematical description of the process from which all our results follow, the arguments will be intuitively more appealing if we rely mainly on the physical interpretation of the model as a branching process.

Let $\mathbb{P} = (p_{ij})$ be a stochastic matrix of order k, and let $\phi_1(s),\cdots, \phi_k(s)$ be probability generating functions. We shall assume that the environment passes through a sequence of states governed by a Markov chain $\{V_n\}$ on the positive integers $1,2,\cdots,k$, defined as follows:

$$P(V_0 = i) = \begin{cases} 1, & i = \eta \\ 0, & i \neq \eta \end{cases}$$

for some fixed η, $1 \leq \eta \leq k$, and

$$P(V_0 = i_0, V_1 = i_1, \cdots, V_n = i_n) = P(V_0 = i_0) p_{i_0 i_1} \, p_{i_1 i_2} \cdots p_{i_{n-1} i_n}.$$

Then, given that $V_n = i$, the number of offspring of different objects in the $n\underline{th}$ generation are independent, identically distributed random variables with probability generating function $\phi_i(s)$, $1 \leq i \leq k$. If $\phi_r(s) = \sum_{j=1}^{\infty} a_{rj} s^j$, then a_{rj} is interpreted as the probability that an object existing in the $n\underline{th}$ generation and environmental state r has j offspring in the (n + 1)st generation. The initial population size is assumed to be nonrandom.

We shall say that objects in the $n\underline{th}$ generation are of type i if $V_n = i$; thus let $\underline{Z}_n = (Z_n^1, Z_n^2, \cdots, Z_n^k)'$ represent the population size, by type, in the $n\underline{th}$ generation. By definition of the process, at most one Z_n^r is nonzero for each n. Hence if we define

$$||Z_n|| = [(Z_n^1)^2 + (Z_n^2)^2 + \cdots + (Z_n^k)^2]^{1/2},$$

it follows that $||\underline{Z}_n||$ is the size of the $n\underline{th}$ generation.

Given the stochastic matrix $\mathbb{P}$ and the probability generating functions $\phi_1(s), \cdots, \phi_k(s)$, the process $\{\underline{Z}_n\}$, described above, may be defined mathematically, without reference to branching processes, as follows. Let $\underline{e}_i$, $1 \leq i \leq k$, denote the k-dimensional vector whose $i\underline{th}$ component is 1 and whose other components are 0. Let T denote the set of all k-dimensional vectors of the form $r\underline{e}_i$, $1 \leq i \leq k$, $r = 0, 1, \cdots$. We want to define a Markov chain on the vectors in T. To this end, define

$$Q(r\underline{e}_i, \underline{0}) = [\phi_i(0)]^r, \quad 1 \leq i \leq k; \quad r = 0, 1, \cdots,$$

$$Q(\underline{0}; t\underline{e}_j) = 0 \quad , 1 \leq j \leq k, \quad t = 1, 2, \cdots,$$

and

$$Q(r\underline{e}_i, t\underline{e}_j) = \text{coefficient of } s^t \text{ in } p_{ij}[\phi_i(s)]^r,$$

$$1 \leq i,j \leq k; \quad r,t = 1,2,\cdots.$$

Clearly, $Q(r\underline{e}_i, t\underline{e}_j) \geq 0$ for all i, j, r and t, and since

$$Q(r\underline{e}_i, \underline{0}) + \sum_{t=1}^{\infty} \sum_{j=1}^{k} Q(r\underline{e}_i, t\underline{e}_j)s^j$$

$$= [\phi_i(0)]^r + \{\sum_j p_{ij}[\phi_i(s)]^r - [\phi_i(0)]^r\}$$

$$= [\phi_i(s)]^r, \quad 1 \leq i \leq k, \quad r = 1,2,\cdots,$$

it follows that

$$Q(r\underline{e}_i, \underline{0}) + \sum_{t=1}^{\infty} \sum_{j=1}^{k} Q(r\underline{e}_i, t\underline{e}_j) = 1, \quad 1 \leq i \leq k; \quad r = 1,2,\cdots.$$

If r = 0, we have

$$Q(\underline{0}, \underline{0}) + \sum_{t=1}^{\infty} \sum_{j=1}^{k} Q(\underline{0}, t\underline{e}_j) = 1.$$

Define a temporally homogeneous, vector-valued Markov chain $\underline{Z}_n = (Z_n^1, Z_n^2, \cdots, Z_n^k)'$ on the set T by choosing initial probabilities

$$P(\underline{Z}_0 = \underline{a}_0) = \begin{cases} 1, \text{ if } \underline{a}_0 = \kappa\underline{e}_\eta, \\ 0, \text{ otherwise,} \end{cases}$$

for some positive integers κ and $\eta (1 \leq \eta \leq k)$, and defining

$$P(\underline{Z}_0 = \underline{a}_0, \underline{Z}_1 = \underline{a}_1, \cdots, \underline{Z}_n = \underline{a}_n)$$

$$= P(\underline{Z}_0 = \underline{a}_0)Q(\underline{a}_0, \underline{a}_1)\cdots Q(\underline{a}_{n-1}, \underline{a}_n), \ \underline{a}_i \varepsilon T, \quad i = 0,1,\cdots n.$$

If $P(\underline{Z}_n = r\underline{e}_i) > 0$, then $Q(r\underline{e}_i, t\underline{e}_j)$ is the transition probability

$$P(\underline{Z}_{n+1} = t\underline{e}_j | \underline{Z}_n = r\underline{e}_i).$$

We shall make the following basic assumptions:

a) $\underline{Z}_0 = \underline{e}_i$ for some fixed i, $1 \leq i \leq k$;

b) $m_r = \phi_r'(1)$ is finite for $r = 1,2,\cdots,k$;

c) $a_{r0} < 1$ for all r, and $a_{r0} + a_{r1} < 1$ for some r;

d) the stochastic matrix $\mathbb{P}$ is irreducible and aperiodic.[1]

16. First and second moments of $\underline{Z}_n$

With the notation of Section 10, let $M = (m_{ij})$ be the matrix of first moments

$$m_{ij} = E(Z_1^j|\underline{Z}_0 = \underline{e}_i) = p_{ij}m_i, \quad i,j = 1,\cdots,k.$$

It follows as in Section 10 that

$$E(\underline{Z}_n|\underline{Z}_0 = \underline{e}_i) = \underline{e}_i' M^n,$$

the ith row of M^n.

Since $Z_n^i Z_n^j = 0$ for $i \neq j$, the second moments of interest are $E(Z_n^i)^2$, $i = 1,\cdots,k$. It also follows that $||\underline{Z}_n||^2 = \underline{Z}_n' \underline{Z}_n$.

By considering conditional expectations, and assuming that $\phi_i''(1) < \infty$, $i = 1,2,\cdots,k$, we obtain the second moments as follows:

$$\begin{aligned} E[(Z_{n+1}^i)^2|\underline{Z}_n] &= \sum_{r=1}^k Z_n^r \, E[(Z_1^i)^2|\underline{Z}_0 = \underline{e}_r] \\ &\quad + \sum_{r=1}^k [(Z_n^r)^2 - Z_n^r] p_{ri} \, m_r^2 \\ &= \sum_{r=1}^k (Z_n^r)^2 \, p_{ri} \, m_r^2 \\ &\quad + \sum_{r=1}^k Z_n^r \, p_{ri}[\phi_r''(1) + m_r - m_r^2] \\ &= \sum_{r=1}^k (Z_n^r)^2 \, n_{ri} + \sum_{r=1}^k Z_n^r \, v_{ri}, \end{aligned}$$

---

[1]Strictly speaking, it is the Markov chain $\{V_n\}$ with transition matrix $\mathbb{P}$ which is assumed to be aperiodic.

where $n_{ri} = p_{ri}\, m_r^2$ and $v_{ri} = p_{ri}[\phi_r''(1) + m_r - m_r^2]$.

Taking expectations, we obtain

$$E[(Z_{n+1}^i)^2] = \sum_{r=1}^{k} E(Z_n^r)^2\, n_{ri} + \sum_{r=1}^{k} E(Z_n^r)\, v_{ri}.$$

Let $\underline{C}_n = [E(Z_n^1)^2, E(Z_n^2)^2, \cdots, E(Z_n^k)^2]'$, $N = (n_{ij})$, $V = (v_{ij})$, and we have

$$\underline{C}_{n+1}' = \underline{C}_n'\, N + \underline{Z}_0'\, M^n V,$$

from which we obtain

$$\underline{C}_{n+1}' = \underline{Z}_0'\, N^{n+1} + \underline{Z}_0' \sum_{i=0}^{n} M^i V N^{n-i}.$$

Since $E||\underline{Z}_n||^2 = E\underline{Z}_n'\,\underline{Z}_n = \underline{C}_n'\,\underline{1}$, where $\underline{1} = (1, 1, \cdots, 1)'$, we have the following result (the basic assumptions are not required except as stated in the theorem).

Theorem 16.1 If $\phi_r'(1) < \infty$, $r = 1, \cdots, k$, and initially there is one object of type i, then

$$E(||\underline{Z}_n|| \mid \underline{Z}_0 = \underline{e}_i) = \underline{e}_i'\, M^n \underline{1};$$

if also $\phi_r''(1) < \infty$, $r = 1, \cdots, k$, then

$$E(||\underline{Z}_n||^2 \mid \underline{Z}_0 = \underline{e}_i) = \underline{e}_i'\, N^n \underline{1} + \underline{e}_i' \sum_{j=0}^{n-1} M^j V N^{n-1-j}\, \underline{1}.$$

17. The generating function of population size at epochs.

Defn. Let $\Pi_n^i(s)$ designate the generating function of $||\underline{Z}_n||$, $n = 0, 1, \cdots$, given that $\underline{Z}_0 = \underline{e}_i$.

A representation of $\Pi_n^i(s)$ analogous to (2.2) is easily obtained, for under the condition that $(V_0, V_1, \cdots, V_n) = (i, r_1, \cdots, r_n)$,

$1 \le r_j \le k$, $j = 1,\cdots,n$, the generating function of $||\underline{Z}_{n+1}||$ is

$$\phi_i(\phi_{r_1}(\cdots\phi_{r_n}(s)\cdots)).$$

Hence

$$\Pi^i_{n+1}(s) = \sum_{r_1,\cdots,r_n=1}^{k} P(V_1 = r_1,\cdots, V_n = r_n | V_0 = i)\phi_i(\phi_{r_1}(\cdots\phi_{r_n}(s)\cdots))$$

$$= \sum_{r_1,\cdots,r_n=1}^{k} p_{ir_1}p_{r_1r_2}\cdots p_{r_{n-1}r_n} \phi_i(\phi_{r_1}(\cdots\phi_{r_n}(s)\cdots)). \qquad (17.1)$$

The right-hand side of (17.1) is not easily related to a random walk, as was (2.2), so this representation doesn't seem very helpful.

However, suppose we consider the generating function of $||\underline{Z}_n||$ only at those points in time (which we shall call epochs) when the environmental process returns to its initial state. Since the Markov chain (of environmental states) is finite and irreducible, the realized number of epochs tends to infinity with probability one as the number of generations tends to infinity. Thus, to determine necessary and sufficient conditions for almost certain extinction, it suffices to study the limiting behavior of the generating function of population size at these epochs.

To this end, let $\{s_j^{(i)},\ j = 1,2,\cdots\}$ be an ordering of all finite paths from state i to state i (in the environmental process). If $s_j^{(i)} = i, j_1, j_2,\cdots, j_m, i$ $(j_\ell \ne i)$, let $p_j^{(i)} = p_{ij_1}\, p_{j_1j_2}\cdots p_{j_mi}$, and let $\Phi_j^{(i)}(s) = \phi_i(\phi_{j_1}(\cdots\phi_{j_m}(s)\cdots))$. The generating function $\hat{\Pi}^i_\nu(s)$ of population size at the νth epoch is then given by

$$\hat{\Pi}^i_\nu(s) = \sum_{i_1,\cdots,i_\nu=1}^{\infty} p_{i_1}^{(i)}\, p_{i_2}^{(i)}\cdots p_{i_\nu}^{(i)}\, \Phi_{i_1}^{(i)}(\Phi_{i_2}^{(i)}(\cdots\Phi_{i_\nu}^{(i)}(s)\cdots)),$$

$$\nu = 1,2,\cdots, \qquad (17.2)$$

with $\hat{\Pi}_0^i(s) = s$. Since $p_j^{(i)} \geq 0$, $\sum_{j=1}^{\infty} p_j^{(i)} = 1$, and each $\Phi_j^{(i)}(s)$ is a probability generating function, $\hat{\Pi}_\nu^i(s)$ is a generating function of the type studied in Chapter II.

18. Extinction probabilities

From (17.2) we have that

$$\hat{\Pi}_1^i(s) = \sum_{r=1}^{\infty} p_r^{(i)} \Phi_r^{(i)}(s), \quad i = 1,\cdots,k. \tag{18.1}$$

Let $\mu_j^{(i)} = \Phi_j^{(i)\prime}(1)$, $i = 1,\cdots k;\ j = 1,2,\cdots$.

If

$$\sum_{r=1}^{\infty} p_r^{(i)} |\log \mu_r^{(i)}| < \infty \tag{18.2}$$

then it follows from Theorem 9.1 that: a)

$$\sum_{r=1}^{\infty} p_r^{(i)} \log \mu_r^{(i)} \leq 0 \tag{18.3}$$

implies $\hat{\Pi}_\nu^i(0) \longrightarrow 1$ as $\nu \longrightarrow \infty$ (extinction occurs with probability one, given $\underline{Z}_0 = \underline{e}_i$), and b)

$$\sum_{r=1}^{\infty} p_r^{(i)} \log \mu_r^{(i)} > 0 \tag{18.4}$$

and

$$\sum_{r=1}^{\infty} p_r^{(i)} \log(1 - \Phi_r^{(i)}(0)) \text{ converges} \tag{18.5}$$

imply $\hat{\Pi}_\nu^i(0) \longrightarrow q_i < 1$ as $\nu \longrightarrow \infty$.

We shall first show that (18.2) holds so that the sums in (18.3) and (18.4) are unambiguously defined (since the ordering of the paths is arbitrary), and then express (18.3) and (18.4) in terms of parameters of the Markov chain $\{V_n\}$ and the means $m_1, m_2,\cdots, m_k$.

Let $n_j^{(i)}(r)$ be the number of times j occurs in the sequence $s_r^{(i)}$. Then

$$\sum_{r=1}^{\infty} p_r^{(i)} \; n_j^{(i)}(r)$$

is the expected number of visits to state j between successive visits to state i. But if $\{\omega_\ell, \ \ell = 1,\cdots,k\}$ is the stationary distribution for the chain $\{V_n\}$, then ω_j/ω_i is also the expected number of visits to state j between successive visits to state i [see e.g. Harris (1952); a more direct proof is given by Hodgson (1965)]. Hence we have

$$\omega_j/\omega_i = \sum_{r=1}^{\infty} p_r^{(i)} \; n_j^{(i)}(r), \quad i,j = 1,\cdots,k.$$

Thus

$$\begin{aligned}
&(\omega_1/\omega_i)|\log m_1| + \cdots + (\omega_k/\omega_i)|\log m_k| \\
&= \sum_{r=1}^{\infty} p_r^{(i)} \{|\log m_1^{n_1^{(i)}(r)}| + \cdots + |\log m_k^{n_k^{(i)}(r)}|\} \\
&\geq \sum_{r=1}^{\infty} p_r^{(i)} |\log(m_1^{n_1^{(i)}(r)} \, m_2^{n_2^{(i)}(r)} \cdots m_k^{n_k^{(i)}(r)})| \\
&= \sum_{r=1}^{\infty} p_r^{(i)} |\log \mu_r^{(i)}|.
\end{aligned}$$

Hence $\sum_{r=1}^{\infty} p_r^{(i)} |\log \mu_r^{(i)}| < \infty$ for $i = 1,\cdots,k$, so the condition (18.2) is satisfied whatever the initial state of the environment. If the absolute value signs are removed above, the inequality becomes an equality, and we have

$$\sum_{r=1}^{\infty} p_r^{(i)} \log \mu_r^{(i)} = \sum_{j=1}^{k} (\omega_j/\omega_i) \log m_j. \tag{18.6}$$

Thus we have the following result.

Theorem 18.1 *If*

$$\sum_{j=1}^{k} \omega_j \log m_j \le 0, \tag{18.7}$$

then extinction occurs with probability one, whatever the initial state of the environment.

We have been unable to determine whether (18.5) always holds or whether (9.1) can be weakened to a condition which always holds for the Markov environment case. Thus we cannot prove much beyond Theorem 18.1 for an arbitrary finite number of environmental states. (We can, however, obtain necessary and sufficient conditions for almost certain extinction if there are just two environmental states, as we shall see below.) We note that Corollary 9.2 does not help here, for if $m_j \le 1$, $\Phi_r^{(i)}(0)$ is not bounded away from 1 as $r \to \infty$ for any initial state i, since the sequence $\{\Phi_r^{(i)}(0)\}$ must contain the terms

$$\{\phi_i(\underbrace{\phi_j(\phi_j(\cdots\phi_j(0)\cdots))}_{\ell \text{ times}}), \qquad \ell = 1,2,\cdots\},$$

which, of course, tends to 1 as $\ell \to \infty$.

In the case of only two environmental states, this difficulty can be avoided by using the fact, suggested by (18.7), that extinction cannot be almost certain for some initial states and not for others.

Hence consider the case of two environmental states with transition probabilities given by

$$\mathbb{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

and probability generating functions $\phi_1(s)$ and $\phi_2(s)$, with means $m_1 = \phi_1'(1)$ and $m_2 = \phi_2'(1)$. Let q^i be the probability of extinction given $\underline{Z}_0 = \underline{e}_i$, $i = 1,2$.

If $\omega_1 \log m_1 + \omega_2 \log m_2 \le 0$, then $q^1 = q^2 = 1$ by (18.7). Since $m_1 \le 1$ and $m_2 \le 1$ implies $\omega_1 \log m_1 + \omega_2 \log m_2 \le 0$, we need only consider the case when at least one m_i is greater than 1; suppose $m_2 > 1$. Let $s_1^{(1)} = 1,1$ and $s_r^{(1)} = 1, \underbrace{2,\cdots,2}_{(r-1)\text{ times}}, 1$, $r = 2,3,\cdots$.

Then $p_1^{(1)} = p_{11}$, $\Phi_1^{(1)}(s) = \phi_1(s)$, and $p_r^{(1)} = p_{12}\, p_{22}^{r-2}\, p_{21}$, $\Phi_r^{(1)}(s) = \phi_1(\underbrace{\phi_2(\cdots\phi_2(s)}_{(r-1)\text{ times}}\cdots))$, $r = 2,3,\cdots$. If s_2 is the unique solution less than 1 of $\phi_2(s) = s$, it follows that $\Phi_r^{(1)}(0) \le \phi_1(s_2) < 1$ for all r. Thus by Corollary 9.2, $\omega_1 \log m_1 + \omega_2 \log m_2 > 0$ implies $q^1 < 1$.

Hence $q^1 = 1$ if and only if

$$\omega_1 \log m_1 + \omega_2 \log m_2 \le 0. \tag{18.8}$$

Note that since $\omega_1 = p_{21}/(p_{12} + p_{21})$ and $\omega_2 = p_{12}/(p_{12} + p_{21})$, (18.8) is equivalent to

$$p_{21} \log m_1 + p_{12} \log m_2 \le 0.$$

Let q_r^i be the probability of extinction given that $\underline{Z}_0 = r\underline{e}_i$, $i = 1,2$; $r = 1,2,\cdots$ $(q^i = q_1^i)$, and recall that $\phi_i(s) = \sum_j a_{ij} s^j$, $i = 1,2$. If $q^1 = 1$, then

$$1 = p_{11}(a_{10} + a_{11}\, q_1^1 + a_{12}\, q_2^1 + \cdots)$$
$$+ p_{12}(a_{10} + a_{11}\, q_1^2 + a_{12}\, q_2^2 + \cdots).$$

Since $\sum_j a_{ij} = 1$ and $0 \le q_r^i \le 1$, we must have $q_r^i = 1$, $i = 1,2$; $r = 1,2,\cdots$. In particular, therefore, $q^2 = 1$. This argument immediately generalizes to any finite number of environmental states. Hence it is impossible that extinction occurs with probability one for some initial states of the environment and not for others.

We have thus proved the following result.

Theorem 18.2 For two environmental states and $\underline{Z}_0 = \underline{e}_i$, $i = 1,2$, extinction occurs with probability one if and only if

$$p_{21} \log m_1 + p_{12} \log m_2 \le 0.$$

The following question remains unanswered. For an arbitrary finite number k of environmental states in a branching process with a Markov environment, does extinction occur with probability one if and only if

$$\sum_{j=1}^{k} \omega_j \log m_j \le 0?$$

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