A NONLINEAR ENDOCHRONIC THEORY OF CYCLIC PLASTIC HARDENING AND SOFTENING

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Abstract

Using the internal state variable approach a nonlinear finite endochronic theory of plasticity consisted of a nonlinear finite constitutive equation, a nonlinear evolution equation, and an intrinsic time equation is constructed and applied to problems of cyclic plasticity.

The constitutive and evolution equations are more general than the original Valanis' ones and satisfy the principle of material objectivity. For an isotropic material we retain in the free energy function terms up to third order in principal values of a tensorial state variable and the right Cauchy-Green strain tensor. The operational definition requires the internal variable to coincide with the plastic strain at unloaded /stress free/ states. The assumption of the plastic incompressibility is made.

The evolution equation is expressed in polynomial form. It is shown that the incompressibility condition and the vanishing of the rate of the internal variable at unloaded state lead to the nonlinear evolution equation which has only one material constant. Intrinsic time measure is expressed in terms of invariants of strain rate tensor. For the problems of cyclic plasticity a time scale with state variables is proposed.

The theory with explicit form of time scales describes experimentally observed phenomena for large cyclic number such as quick increasing of the peak stress for annealed metals /cyclic hardening/ or its rapid decreasing for cold-worked metals /cyclic softening/. Numerical examples are calculated and compared with experimental data.

Former explanation by Valanis using linear theory with a special type of the intrinsic time was limited. The present theory can explain both cyclic hardening and softening in which peak stresses saturate with certain values. This property is rather common among many metals in whole cyclic process.

1. Introduction

We propose herein a new nonlinear endochronic theory of finite inelastic strain which consistes of three constitutive relations, that is (a) a nonlinear second order finite constitutive equation, (b) a nonlinear evolution equation having the proper physical meaning of internal variable, and (c) an intrinsic time equation. New types of time measures and time scales depending on state variables which are especially appropriate for cyclic plasticity are formulated. The aim of the paper is to formulate explicit constitutive relations of endochronic theory of plasticity and to show new results and phenomena which could not be predicted nor described by the original Valanis linear theory [1,2] for infinitesimal strains. It is also our intention to describe consistently the changes of both the peak stress and the shape of hysteresis loop with respect to cyclic number in transient and steady states.

2. Framework of the proposed theory. Definitions and assumptions

We restrict our investigations to one tensorial internal state variable $\underset{\sim}{q}$. We claim that the symmetric tensor $\underset{\sim}{q}$ and the right Cauchy-Green strain tensor $\underset{\sim}{C}$ are independent variables treated as state variables. For convenience we mainly use the following variables $\underset{\sim}{E} = \frac{1}{2}(\underset{\sim}{C}-\underset{\sim}{1})$, $\hat{\underset{\sim}{q}} = \frac{1}{2}(\underset{\sim}{q}-\underset{\sim}{1})$ where $\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$ and $\underset{\sim}{F}$ is deformation gradient, $\underset{\sim}{1}$ is unit tensor.

The second Piola-Kirchhoff stress tensor $\tilde{\underset{\sim}{T}}$ under isothermal conditions is a function of these tensors $\underset{\sim}{C}$ and $\underset{\sim}{q}$, $\tilde{\underset{\sim}{T}} = \tilde{\underset{\sim}{T}}(\underset{\sim}{C},\underset{\sim}{q})$.

To describe inelastic properties of the material we need an evolution equation which is an initial value problem

$$\frac{d\underset{\sim}{q}}{dz} = \hat{A}(\underset{\sim}{C}(z), \underset{\sim}{q}(z)), \qquad \underset{\sim}{q}(0) = \underset{\sim}{q}_0, \qquad z \in [0,\infty) \tag{1}$$

where z is an intrinsic time not the natural time. We make precise definition of $\underset{\sim}{q}$ through following relations

$$\tilde{\underset{\sim}{T}}(\underset{\sim}{C},\underset{\sim}{q})\Big|_{\underset{\sim}{C}=\underset{\sim}{q}} = \underset{\sim}{0}, \qquad \frac{d\underset{\sim}{q}}{dz}\Big|_{\underset{\sim}{C}=\underset{\sim}{q}} = \underset{\sim}{0} \tag{2}$$

It means that at unloaded(stress-free)state $\tilde{\underset{\sim}{T}} = \underset{\sim}{0}$, $\underset{\sim}{C}$ coincides with $\underset{\sim}{q}$ ($\underset{\sim}{C} = \underset{\sim}{q}$) and at unloaded state $\underset{\sim}{q}$ measures plastic (permanent) deformation. Furthermore eq.(2) means that at unloaded state plastic material cannot relaxe at zero stress. We make one more important assumption

$$\det(\underset{\sim}{q}) = 1 \tag{3}$$

i.e., in all deformation processes inelastic deformation is isochoric.

We assume that there exists a free energy function ψ of $\underset{\sim}{C}$ and $\underset{\sim}{q}$ for the material under consideration. Representation theorem for an isotropic function of two tensors $\underset{\sim}{C}$ and $\underset{\sim}{q}$ tells us that there are only ten independent variables. But because of the incompressibility condition we may use a specific set of 9 invariants for ψ

$$\begin{gathered}\psi = \hat{\psi}(J_1, J_2, J_3, J_4, J_5', J_7, J_8, J_9, J_{10}) \\ J_1 = 2I_E,\ J_2 = 4II_E,\ J_3 = 8III_E,\ J_4 = 2I_{\hat{q}},\ J_5' = 4II_{\hat{q}} - 2I_{\hat{q}} \\ J_7 = 4\underset{\sim}{E}\cdot\hat{\underset{\sim}{q}},\ J_8 = 8\underset{\sim}{E}^2\cdot\hat{\underset{\sim}{q}},\ J_9 = 8\underset{\sim}{E}\cdot\hat{\underset{\sim}{q}}^2,\ J_{10} = 16\underset{\sim}{E}^2\cdot\hat{\underset{\sim}{q}}^2\end{gathered} \tag{4}$$

where $I_{( )}$, $II_{( )}$, $III_{( )}$ are the first, second and third invariant and '·' denotes the inner product. Since $\det(\underline{q}) = III_{q} = 1$ we use the invariant J_5' defined above. Note that the order of $J_4 = 2\mathrm{tr}(\hat{q})$ is two and the order of J_5' is three. We assume that free energy function is a polynomial function of these nine invariants.

3. Constitutive equation

We make an approximation of ψ and retain all terms up to and including third order. The complete form is

$$\psi = a_0 + a_1 J_1 + a_2 J_2 + a_3 J_3 + a_4 J_4 + a_5 J_5' + a_7 J_7 + a_8 J_8 + a_9 J_9 + \\ + a_{10} J_1^2 + a_{11} J_1 J_2 + a_{12} J_1^3 + a_{13} J_1 J_4 + a_{17} J_1 J_7 \tag{5}$$

where a's are material constants. The existence of the elasto-plastic coupling is manifested by terms $a_7 J_7$, $a_8 J_8$, $a_{13} J_1 J_4$ and $a_{17} J_1 J_7$. Let ψ be 0 at natural state, then $a_0 = 0$.

Using established relation [1] that $\tilde{\underline{T}} = 2\rho_0(\partial\psi/\partial\underline{C})$ we can get the constitutive equation

$$\frac{1}{2\rho_0}\tilde{\underline{T}} = \left[2(a_2+2a_{10})I_{\underline{E}} + 4(a_{11}+3a_{12})I_{\underline{E}}^2 + 4(a_3+a_{11})II_{\underline{E}}\right]\underline{1} + \left[2a_{13} I_{\hat{q}} + 4a_{17}\underline{E}\cdot\hat{\underline{q}}\right]\underline{1} + \\ - \left[a_2 + 2(a_3+a_{11})I_{\underline{E}}\right]\underline{E} + 4a_3\underline{E}^2 + 2a_7\hat{\underline{q}} + 2a_{17}I_{\underline{E}}\hat{\underline{q}} + 4a_8(\underline{E}\hat{\underline{q}} + \hat{\underline{q}}\underline{E}) + 4a_9\hat{\underline{q}}^2 \tag{6}$$

We let $\tilde{\underline{T}}$ at natural state be $\underline{0}$, then $a_1 = 0$. Natural state is given by the condition that both $\underline{C}$ and $\underline{q}$ are $\underline{1}$. We assume that if both $\underline{C}$ and $\underline{q}$ tend to $\underline{1}$, the linear part of the initial response from the natural state is given by the classical Lame constants λ and μ that is

$$\left.\frac{\partial \tilde{T}_{12}}{\partial C_{12}}\right|_{\underline{C}=\underline{q}=\underline{1}} = -2\rho_0 a_2 \equiv \mu, \quad \left.\frac{\partial \tilde{T}_{11}}{\partial C_{11}}\right|_{\underline{C}=\underline{q}=\underline{1}} = 2\rho_0\left[(a_2+2a_{10}) - a_2\right] \equiv \lambda + 2\mu \tag{7}$$

then

$$a_2 = -\frac{1}{2\rho_0}\mu, \quad (a_2+2a_{10}) = \frac{1}{4\rho_0}\lambda$$

Noting that at any unloaded state $\underline{q} = \underline{C}$, $\tilde{\underline{T}} = \underline{0}$ we have the following identities $J_4 = J_1$, $J_5 = J_2$, $J_7 = J_1^2 - 2J_2$. Substituting these identities into eq.(6) gives us the set of homogeneous equations for the material constants a's which should be satisfied for all unimodular tensor $\underline{C}$ (because $\det(\underline{q}) = 1$). These equations together with eq.(6) enable us to reduce material constants and to write the final form of the constitutive equation.

$$\tilde{\underline{T}} = \lambda\,\mathrm{tr}(\underline{E}-\hat{\underline{q}})\underline{1} + 2\mu(\underline{E}-\hat{\underline{q}}) + 8\rho_0\left[a_3\underline{E}^2 + a_8(\underline{E}\hat{\underline{q}} + \hat{\underline{q}}\underline{E}) + a_9\hat{\underline{q}}^2\right] \tag{8}$$

It has only two unknown material constants for $a_3 + 2a_8 + a_9 = 0$. The first two linear terms include Valanis' theory. The second nonlinear part describes second order properties and elasto-plastic coupling.

4. Evolution equation

In the following way we can specify the evolution equation using two restrictions (2) and (3). The restriction $\det(\underline{q}) = 1$ will be satisfied if

$$\frac{d}{dz}(\det(\underline{q})) = \mathrm{tr}\left(\underline{q}^{-1}\frac{d\underline{q}}{dz}\right) = \mathrm{tr}\left(\frac{d\underline{q}}{dz}\underline{q}^{-1}\right) = 0 \qquad \text{and} \quad \det(\underline{q}(z))\big|_{z=0} = 1 \tag{9}$$

in all deformation processes. It means that the product $\underline{q}^{-1}\dot{\underline{q}}$ (or $\dot{\underline{q}}\,\underline{q}^{-1}$) should be the deviatoric part of some tensor $\underline{e}$ treated as a function of $\underline{C}$ and $\underline{q}$ i.e.

$$\dot{q}q^{-1} = \operatorname{dev} e \quad (q^{-1}\dot{q} = \operatorname{dev} e), \quad e = \tilde{e}(C, q) \tag{10}$$

where dev e means the deviatoric part of e. Since $\frac{d}{dz}(q) = \frac{1}{2}[q(\operatorname{dev} e) + (\operatorname{dev} e)q]$ the function $e = \tilde{e}(C, q)$ should be linear in C and q in order to be consistent with the assumption of the second order approximation:

$$\tilde{e}(C, q) = 2b_0 C + 2b_1 q + 2b_2(\operatorname{tr} C)1, \quad b_0, b_1, b_2: \text{consts.} \tag{11}$$

The condition (2) results in $b_1 = -b_0$, $b_2 = 0$. Hence the final form of the evolution equation is following

$$\frac{1}{b_0}\frac{d\hat{q}}{dz} = (E - \hat{q}) - \frac{1}{3}\operatorname{tr}(E - \hat{q})1 + (E - \hat{q})\hat{q} + \hat{q}(E - \hat{q}) - \frac{2}{3}(\operatorname{tr}(E - \hat{q}))\hat{q} \tag{12}$$

The first two terms are linear. The nonlinear terms manifest elasto-plastic coupling. For small deformations, the incompressibility condition reduces to tr $d\hat{q}/dz = 0$, and two linear terms are sufficient. But at finite deformations the incompressibility condition is no longer tr$(dq/dz) = 0$ but $d/dz(\det q) = 0$ so, the second group must be included to satisfy it. Note that it does not exist a linear evolution equation that satisfies the condition $\det(q) = 1$. Obtained above evolution equation is deformation history dependent and it has only one material constant b_0.

5. Intrinsic time equation

We propose the following form of the intrinsic time measure

$$\left(\frac{d\varsigma}{dt}\right)^2 = k_1^2 |I_{\dot{E}}|^2 + k_2^2 |II_{\dot{E}}| + k_3^2 |III_{\dot{E}}|^{2/3}, \quad \dot{E} = \frac{d}{dt}(E(t)) \tag{13}$$

where 't' is natural time; $I_{\dot{E}}$, $II_{\dot{E}}$ and $III_{\dot{E}}$ are invariants of $\dot{E}$; k_1, k_2, k_3 are material constants or in general functions and may depend on $\dot{E}$ and E.

Valanis used time scale of the form

$$dz/d\varsigma = (1 + \beta\varsigma)^{\alpha}, \quad \alpha = -1, \quad \beta = \text{const.} \tag{14}$$

In the problems of cyclic plasticity in which the cyclic number is very big the time scale which reflects the deformation history, must cover the limit case $\varsigma \to \infty$. In this limit case Valanis' time scale does not make any sense, for $dz/d\varsigma$ tends to zero very rapidly. If we put $dz/d\varsigma$ = const., it can merely describe elasto-perfectly plastic material.

For the description of cyclic properties and periodic response of material the time scale itself should have cyclic and periodic properties. Two possibilities are conceivable. The first one is to introduce a periodic function (e.g. sin x) into the time scale. But this form restricts considerations to special loading conditions and is not applicapable to different non-periodic loading conditions. Another possibility is to introduce state variables in the time scale as Bažant and his co-workers did [4]. To this end we propose the following time scale

$$\frac{dz}{d\varsigma} = \hat{f}(E, \hat{q}, \varsigma) = \hat{f}_1(E, \hat{q})\hat{f}_2(\varsigma) \tag{15}$$

there we have a function of the time measure and of state variables E and $\hat{q}$ as well. This form is different from Bažant's one for it includes the internal variables. We have found out that the following form is suitable for problems of cyclic plasticity

$$\hat{f}_1(\underline{E}, \underline{\hat{q}}) = |\mathrm{II}(\underline{E} - \underline{\hat{q}})|^{c_2} \tag{17}$$

where c_2 is a material constant. For $\hat{f}_2(\varsigma)$ the following two simple forms can be used

i) $\hat{f}_2(\varsigma) = (c_3 + c_4 e^{-c_5 \varsigma})$ for cyclic hardening (18)

ii) $\hat{f}_2(\varsigma) = (c_6 + c_7(1 - e^{-c_8 \varsigma}))$ for cyclic softening (19)

where c_3, c_4, c_5, c_6, c_7 and c_8 are material constants.

6. Description of cyclic hardening and cyclic softening

When metallic materials are under strain-controlled large cyclic straining accompanying plastic strains then they (a) cyclically harden, (b) soften or (c) harden at first and soften later, depending upon the initial internal structure of material. The cyclic hardening and cyclic softening can be characterized by two factors, that is the peak stress and the shape of hysteresis loop at each cyclic number. Both phenomena cyclic hardening and cyclic softening have transient and steady states. Our objective is to describe consistently the peak stress and the hysteresis loop, and both the phenomena using our endochronic theory proposed above.

6.1. Description of torsional cyclic hardening and comparison with experimental data

When we apply only shear stress $\tilde{T}_{12}$ we can assume that $\underline{E}$ and $\underline{\hat{q}}$ have the following forms

$$\underline{E} = \begin{bmatrix} E_1 & E_{12} & 0 \\ E_{12} & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \qquad \underline{\hat{q}} = \begin{bmatrix} \hat{q}_1 & \hat{q}_{12} & 0 \\ \hat{q}_{12} & \hat{q}_2 & 0 \\ 0 & 0 & \hat{q}_3 \end{bmatrix} \tag{20}$$

in some Cartesian coordinate system. We can not write in general $E_1 = E_2 = E_3 = \hat{q}_1 = \hat{q}_2 = \hat{q}_3 = 0$ in this shear stress loading condition because this theory is not of infinitesimal strain. The linear part of constitutive equation has the following form

$$\begin{aligned} \tilde{T}_{12} &= 2\mu(E_{12} - \hat{q}_{12}) \\ 0 &= (\lambda + 2\mu)(E_1 - \hat{q}_1) + \lambda(E_2 - \hat{q}_2) + \lambda(E_3 - \hat{q}_3) \\ 0 &= \lambda(E_1 - \hat{q}_1) + (\lambda + 2\mu)(E_2 - \hat{q}_2) + \lambda(E_3 - \hat{q}_3) \\ 0 &= \lambda(E_1 - \hat{q}_1) + \lambda(E_2 - \hat{q}_2) + (\lambda + 2\mu)(E_3 - \hat{q}_3) \end{aligned} \tag{21}$$

The last three equations give the relation $E_1 - \hat{q}_1 = E_2 - \hat{q}_2 = E_3 - \hat{q}_3 = 0$. But it does not mean that $E_1 = E_2 = E_3$. The evolution equation under cyclic torsion is

$$\begin{aligned} &\frac{d\hat{q}_1}{dz} = \frac{d\hat{q}_2}{dz} = 2b_0(E_{12} - \hat{q}_{12})\hat{q}_{12}, \quad \frac{d\hat{q}_3}{dz} = \frac{d\hat{q}_{13}}{dz} = \frac{d\hat{q}_{23}}{dz} = 0 \\ &\frac{d\hat{q}_{12}}{dz} = b_0(E_{12} - \hat{q}_{12}) + b_0(E_{12} - \hat{q}_{12})(\hat{q}_1 + \hat{q}_2) = b_0(E_{12} - \hat{q}_{12})(1 + 2\hat{q}_1) \end{aligned} \tag{22}$$

The relations $d\hat{q}_1/dz = d\hat{q}_2/dz$, $d\hat{q}_3/dz = d\hat{q}_{13}/dz = d\hat{q}_{23}/dz = 0$ mean that the tensor $d\underline{\hat{q}}/dz$ can be expressed by two components $d\hat{q}_1/dz$ and $d\hat{q}_{12}/dz$ in this case. Note that $\hat{q}_1 = \hat{q}_2$, $\hat{q}_3 = 0$, $E_1 = \hat{q}_1 = E_2 = \hat{q}_2$. The invariants of $\dot{\underline{E}}$ are

$$I_{\dot{\underline{E}}} = 2\frac{d\hat{q}_1}{dz}, \quad II_{\dot{\underline{E}}} = \left(\frac{d\hat{q}}{dz}\right)^2 - \left(\frac{dE_{12}}{dz}\right)^2, \quad III_{\dot{\underline{E}}} = 0 \tag{23}$$

then time measure is

$$d\varsigma = k_1|2d\hat{q}_1| + k_2\left|(d\hat{q}_1)^2 - (dE_{12})^2\right|^{1/2} \tag{24}$$

For the function $\hat{f}$, eqs.(17) and (21) give $|II\ (E-q)|^{c_2} = |E_{12} - \hat{q}_{12}|^{2c_2}$; then the time scale is given by

$$\frac{dz}{d\varsigma} = |E_{12} - \hat{q}_{12}|^{2c_2}(c_3 + c_4 e^{-c_5\varsigma}) \tag{25}$$

Numerical calculation was made using eqs.(21), (22), (24), (25) with material constants of annealed copper $E_0 = 16700$ ksi, $\mu = 5600$ ksi, $\nu = 0.33$ and

$$b_0 = 14270, \quad k_1 = k_2 = k_3 = 1, \quad c_2 = 0.125, \quad c_3 = 0.4, \quad c_4 = 0.6, \quad c_5 = 20.0 \tag{26}$$

and compared with the experimental data by Lamba [4].
Fig.1 is Lamba's figure of the torsional cyclic straining of amplitude $\gamma = 1.1\%$ (engineering strain) of annealed copper. Fig.2 shows theoretical description using above material constants of the same material under the same loading condition. Fig.3 shows the change of peak stress with respect to the cyclic number n, where solid line is the theoretical prediction. Fig.2 and 3 are in good accordance with the experiment. It can also describe properties in both the transient state and the steady state.

6.2. Simulation of torsional cyclic softening

Since we do not have at hand systematic experimental data of cyclic hardening and softening under various loading conditions for the same material but in different initial structure (or state) we can make only a numerical simulation of cyclic softening of material in imaginary initially fully cold-worked (or quenched) state. For this simulation we use the same constitutive relations which we used for analysis of cyclic hardening except for the time scale. The material constants for copper were used except material constants with some changes as follows: $c_6 = 1.0$, $c_7 = 0.2$, $c_8 = 20.0$, but the same $c_2 = 0.125$. Fig.4 shows the simulation of cyclic softening under cyclic shear of strain amplitude $\gamma = 1.1\%$ (engineering strain). One can see that the figure can describe both changes of peak stress and hysteresis loop in both transient and steady states.

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4 LAMBA H.S., "Nonproportional cyclic plasticity", UILU-ENG 76, T.A.M.Report NO.413, University of Illinois, Urbana, Illinois, 1976

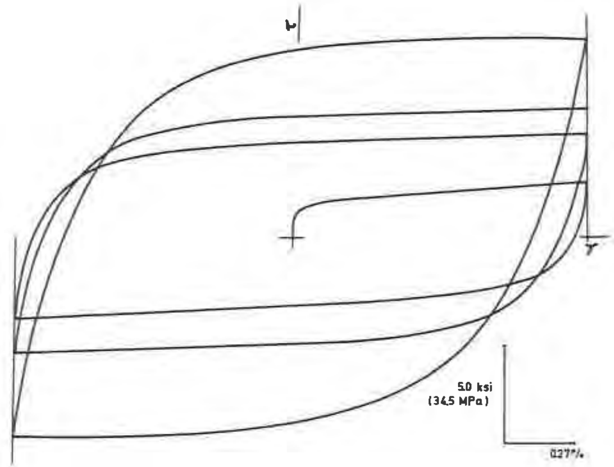

Fig. 1 Recording of torsional cyclic hardening by Lamba.

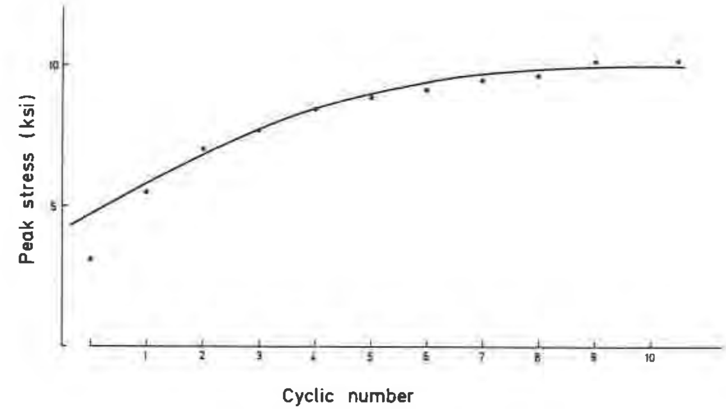

Fig. 3 Change of peak stress.

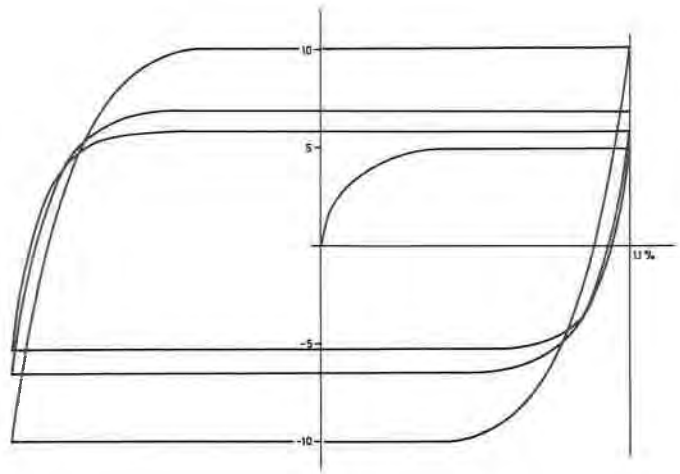

Fig. 2 Calculation of torsional cyclic hardening.

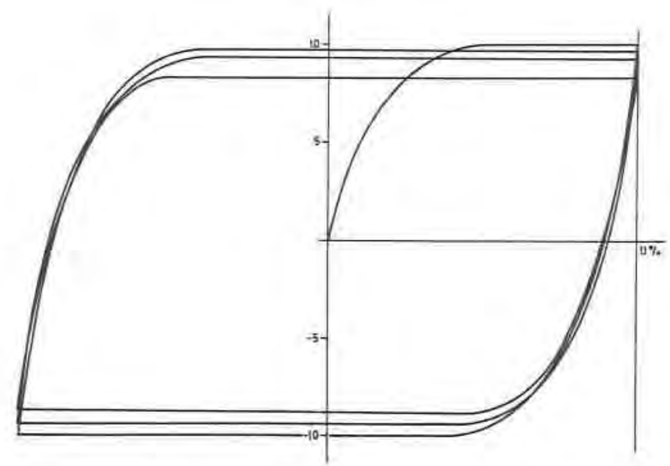

Fig. 4 Simulation of torsional cyclic softening.