A WORK-HARDENING RULE FOR FINITE ELASTIC-PLASTIC DEFORMATION OF METALS AT ELEVATED TEMPERATURES

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SUMMARY

The paper is concerned with an extension of Prager-Ziegler's kinematic work-hardening rule for infinitesimal elastic-plastic deformation to a work-hardening rule for finite elastic-plastic deformation of a polycrystalline metal. It is shown that the finite work-hardening rule, which accounts for the Bauschinger and temperature effects within certain pressure and temperature ranges, satisfies certain invariant, continuity and thermodynamic requirements.

A description of the kinematics of an elastic-plastic body is employed with reference to three separate configurations: initial, current and an intermediate configuration. The intermediate configuration is a conceptual, local configuration obtained by removing the stress and temperature changes in the neighborhood of an element. A rigid body rotation of the intermediate configuration is allowed. Piola-Kirchhoff stresses and Green deformation tensors referred to the initial and intermediate configurations are employed as stress and strain measures.

The plastic deformation has been associated with the motion and production of dislocations. It has been observed that the motion of mobile dislocations usually occur in the narrow slip bands in each grain, leaving the basic lattice structure practically intact, so that the macroscopic elastic properties of the material are essentially independent of plastic deformation. Employing this fact and the thermodynamic laws, a simplified elastic stress-strain relationship of the plastically deformed material, which agrees with the results of Naghdi and Trapp, is obtained.

At finite deformation, an elastic state may be represented by a point in the Piola-Kirchhoff stress ($S_{\alpha\beta}$) space referred to the intermediate configuration. The plastic parts of the kinematic changes are embedded in the configuration reference. The yield surface which bounds the elastic states may be described in the same stress space. Available experimental results have shown that a temperature change causes essentially a change only in size of the yield surface of a polycrystalline metal. Following Prager and Ziegler's approach, it is assumed that the yield surface may change in size but not in shape and may undergo a translation in the $S_{\alpha\beta}$ stress space depending on the states of stress, strain, temperature and history of plastic deformation. The translation of the center of the yield surface and a flow rule on plastic strain increments measured with respect to the initial configuration are determined by Ziegler's assumptions, the continuity and thermodynamic requirements and the condition of positive work in an open or closed path of plastic deformation. By appropriate transformations, the constitutive relationships are expressed in terms of stress and strain measures in the initial configuration. The work-hardening rule is applicable to all loading programs, including that of simple tension and compression. Thus, a proportionality scalar function occurred in the flow rule may be determined from experimentally obtained nonlinear, uniaxial stress-strain curves at various temperatures. The work-hardening rule may serve as a simplified approach for practical problems of finite thermoplasticity.

1. Introduction

This paper is concerned with an extension of Prager-Ziegler's kinematic work-hardening rule [1,2] for infinitesimal elastic-plastic deformation to a work-hardening rule for finite elastic-plastic deformation of polycrystalline metals. It will be shown that the finite work-hardening rule, which accounts for the Bauschinger and temperature effects within certain pressure and temperature ranges, satisfies certain invariant, continuity and thermodynamic requirements.

From a continuum view point, the mechanical properties of a metal would be described in terms of macro-variables such as stress, strain, temperature and their histories. From a physical view point, the properties would be described in terms of micro-structural changes. A polycrystalline metal has grains of various sizes, each of which has its own orientations of atomic lattice structure and defects known as dislocations. The plastic deformation has been associated with the motion and production of dislocations. The work-hardening effect is attributed to impediments to mobile dislocations, such as grain boundaries, dislocation densities or other obstacles. However, it is difficult to determine the strength of impediments or stress fields around an arbitrary arrangement of dislocations and boundaries, especially when micro-structural changes are coupled with finite deformations. To establish a sound foundation for the field of finite plasticity, relationships between continuum variables and micro-structural changes should be analyzed. Such attempts in various forms have been made [3-12]. The concept of internal or hidden state variables [13-18] has also been employed to account for the effect of micro-structural changes on constitutive relationships.

In some sense, the effects of micro-structural changes have been indirectly and approximately accounted for by the classical concept of a yield (or loading) function. A yield stress, even that obtained from a simple tension stress-strain curve, is an idealized value. It is known that motions of some dislocations and consequently irreversible geometric changes may occur at a stress level lower than that of a "yield stress". A yield function is a mathematical simplification employed to characterize the micro-structural changes or the irreversible behavior of elastic-plastic solids, especially the notable (plastic) loading and (elastic) unloading phenomenon.

To describe a yield function of a metal at finite deformation, proper stress and deformation measures are necessary [9]. Lee and Liu [19,20] have introduced a description of the kinematics of an elastic-plastic body with reference to three separate configurations: initial, current and an intermediate configuration. The plastic deformation has been separated from the total deformation through the decomposition of elastic and plastic deformation gradients [21-24]. The concept of a plastic strain as a state variable and as a kinematic measure has been developed by Green and Naghdi [25-27]. Employing these measures, various forms of yield function and associated flow rule have been formulated. However, experimental verification and correlation of the formulations with micro-structural changes are still lacking. The work-hardening rule proposed in this paper which concerns the yield function, may serve as a simplified approach for solving practical problems of finite plasticity.

2. Deformation and Stress Measures

Consider a body of a continuum occupying a region R_o in its initial, natural state with a homogeneous temperature T_o at time t_o. Let the initial position X_K of a material point be referred to a fixed system of rectangular cartesian coordinates. The history of deformation of the body under influences of external forces is defined by the position x_m in a region R to each X_K at each instant of time t, i.e.

$$x_m = x_m(X_K, t), \quad m = 1,2,3 \quad K = 1,2,3 \tag{1}$$

The vector function is assumed to be single-valued and continuously differentiable within the domain of the body. In general, lower case Latin indices associate with x_m and upper case Latin indices associated with X_K.

Consider a line element dX_K of an initial length dS in R_o. The element is deformed to dx_m in R and a length ds at time t. If at time t, the stress and temperature changes in the neighborhood are removed, the length of the element becomes $\overline{ds}$, which may be referred to an intermediate configuration $\overline{R}$ defined by the particle position

$$Y_\alpha = Y_\alpha(X_K, t), \quad \alpha = 1,2,3 \quad K = 1,2,3 \tag{2}$$

Greek indices are for quantities associated with the intermediate configuration $\overline{R}$. It is assumed that the functions Y_α are single-valued and continuously differentiable only in the neighborhood of the element. In other words, the intermediate configuration is a conceptual, local configuration embedded in the memory of the material. Thus, a rigid body rotation of the intermediate configuration may be allowed.

The deformation of the body may be described in terms of Lagrangian strain tensor, E_{KL}, defined by

$$ds^2 - dS^2 = 2E_{KL}\, dX_K dX_L \tag{3}$$

where

$$E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL}) \tag{4}$$

and C_{KL} is Green's deformation tensor given by

$$C_{KL} = x_{k,K} x_{k,L} \tag{5}$$

Here, a partial differentiation of a variable with respect to X_K is designated as $(\,)_{,K}$. δ_{KL} is Kronecker symbol and repetition of an index in a term indicates summation. Lagrangian strain tensor may be divided into elastic, E_{KL}', and plastic, E_{KL}'', parts by the relationship

$$ds^2 - dS^2 = (ds^2 - \overline{ds}^2) + (\overline{ds}^2 - dS^2) = 2(E_{KL}' + E_{KL}'')dX_K dX_L \tag{6}$$

The elastic strain, referring to R_o, is given by

$$E_{KL}' = \overline{E}_{\alpha\beta}\, Y_{\alpha,K}\, Y_{\beta,L} \tag{7}$$

where $\overline{E}_{\alpha\beta}$ is the elastic strain referred to $\overline{R}$, or

$$\overline{E}_{\alpha\beta} = \frac{1}{2}(\overline{C}_{\alpha\beta} - \delta_{\alpha\beta}) \tag{8}$$

where

$$\overline{C}_{\alpha\beta} = x_{k,\alpha} x_{k,\beta} \tag{9}$$

The plastic strain is given by

$$E_{KL}'' = \frac{1}{2}(C_{KL}'' - \delta_{KL}) \tag{10}$$

where

$$C_{KL}'' = Y_{\alpha,K} Y_{\alpha,L}\,. \tag{11}$$

Equations (3) and (6) show that Lagrangian strain can be expressed as

$$E_{KL} = E_{KL}' + E_{KL}'' \tag{12}$$

Furthermore, it may be shown that the values of E_{KL}, E_{KL}' and C_{KL}'' do not change by a rigid-body rotation of the intermediate or current configuration.

The true stress or Cauchy stress tensor, σ_{ij}, referred to R, is a natural physical concept. However, Cauchy stress rate, $\dot{\sigma}_{ij}$, is not an objective quantity and cannot occur directly in a description of material properties. Instead, the second Piola-Kirchhoff stress tensor, S_{KL}, referred to R_o, is frequently employed. By definition,

$$S_{KL} = j\, X_{K,i} X_{L,j} \sigma_{ij} \tag{13}$$

where

$$j = |x_{m,M}| \tag{14}$$

is the Jacobian of the deformation gradient. For later reference, symmetric Piola-Kirchhoff stress, $\overline{S}_{\alpha\beta}$, referred to the intermediate configuration $\overline{R}$ is defined by

$$\overline{S}_{\alpha\beta} = j^e Y_{\alpha,i} Y_{\beta,j} \sigma_{ij} \tag{15}$$

or

$$S_{\alpha\beta} = (j^P)^{-1} Y_{\alpha,K} Y_{\beta,L} S_{KL} \tag{16}$$

where

$$j^e = |x_{i,\alpha}| \quad \text{and} \quad j^P = |Y_{\alpha,K}| \tag{17}$$

3. Thermodynamic Considerations

The first and second laws of thermodynamics may be expressed respectively as

$$\rho_o r - \rho_o(\dot{A} + \dot{T}S + T\dot{S}) - Q_{K,K} + S_{KL}\dot{E}_{KL} = 0 \tag{18}$$

and

$$-\rho_o(\dot{A} + S\dot{T}) + S_{KL}\dot{E}_{KL} - \frac{Q_K T_{,K}}{T} \geq 0 \tag{19}$$

where ρ_o is the initial mass density of the body, r the specific heat supply, S the specific entropy, T the temperature and Q_K the heat flux across the surface X_K = constant, all the specific quantities being for unit mass. A is the specific Helmholtz free energy function defined by

$$A = U - TS \tag{20}$$

where U is the specific internal energy.

For an elastic-plastic continuum, the following constitutive equations may be assumed

$$A = A(E_{KL}', E_{KL}'', \varkappa, T) \tag{21}$$

$$S = S(E_{KL}', E_{KL}'', \varkappa, T) \tag{22}$$

$$S_{KL} = S_{KL}(E_{MN}', E_{MN}'', \varkappa, T) \tag{23}$$

where $\varkappa$ is a work-hardening parameter which depends on the whole history of plastic deformation and accounts indirectly for the effects of micro-structural changes. It is further assumed that the material admits a yield function (or yield surface) which defines a range of stress such that for stresses within this range, deformations are wholly elastic, while plastic deformations may occur for stresses on the boundary of this range. The yield function may be described as

$$F(S_{KL}, E_{KL}'', \varkappa, T) = 0 \tag{24}$$

where F is a regular (continuously differentiable) function of its variables. With the consideration of the time rate of change of the yield function, i.e.

$$\dot{F} = \frac{\partial F}{\partial S_{KL}} \dot{S}_{KL} + \frac{\partial F}{\partial T} \dot{T} + \frac{\partial F}{\partial E_{KL}''} \dot{E}_{KL}'' + \frac{\partial F}{\partial \varkappa} \dot{\varkappa} = 0, \tag{25}$$

the state $F = 0$ and $\dot{F} < 0$ at the next instant of time, is called the state of elastic unloading. It is required that in an unloading process, no additional plastic strain occurs, $\dot{E}_{KL}'' = 0$, and by its very nature, the rate of change of the strain hardening parameter must also vanish. Hence, the different loading conditions described by

$$\frac{\partial F}{\partial S_{KL}} \dot{S}_{KL} + \frac{\partial F}{\partial T} \dot{T} < 0, \ = 0 \ \text{and} \ > 0 \ \text{on} \ F = 0 \tag{26}$$

and called the unloading, neutral loading and loading conditions, respectively. Furthermore, the plastic strain rate may be written as

$$\dot{E}_{KL}'' = e_{KL} (S_{MN}, \dot{S}_{MN}, E_{MN}'', \varkappa, \dot{\varkappa}, T, \dot{T})$$

when

$$F = 0, \ \dot{\varkappa} \neq 0 \ \text{and} \ \frac{\partial F}{\partial S_{KL}} \dot{S}_{KL} + \frac{\partial F}{\partial T} \dot{T} > 0 \tag{27}$$

and

$$\dot{E}_{KL}'' = 0$$

when

$$F = 0, \ \dot{\varkappa} = 0 \ \text{and} \ \frac{\partial F}{\partial S_{KL}} \dot{S}_{KL} + \frac{\partial F}{\partial T} \dot{T} \leq 0$$

or

when

$$F < 0 \ \text{with} \ \dot{\varkappa} = 0 \, . \tag{28}$$

By employing the constitutive equations, (21) (22) and (23), the thermodynamic laws may be written as

$$\rho_o r - \rho_o (S + \frac{\partial A}{\partial T}) \dot{T} + (S_{KL} - \rho_o \frac{\partial A}{\partial E_{KL}'}) \dot{E}_{KL}'$$

$$+ (S_{KL} - \rho_o \frac{\partial A}{\partial E_{KL}''}) \dot{E}_{KL}'' - \rho_o \frac{\partial A}{\partial \varkappa} \dot{\varkappa} - \rho_o \dot{S} T - Q_{K,K} = 0 \tag{29}$$

and

$$-\rho_o (S + \frac{\partial A}{\partial T}) \dot{T} + (S_{KL} - \rho_o \frac{\partial A}{\partial E_{KL}'}) \dot{E}_{KL}'$$

$$+ (S_{KL} - \rho_o \frac{\partial A}{\partial E_{KL}''}) \dot{E}_{KL}'' - \rho_o \frac{\partial A}{\partial \varkappa} \dot{\varkappa} - \frac{Q_K T_{,K}}{T} \geq 0 \tag{30}$$

both of which must hold during loading as well as unloading. For the case of unloading, $\dot{E}_{KL}'' = 0$ and $\dot{\varkappa} = 0$, the inequality (30), known as Clausius-Duhem inequality, becomes

$$-\rho_o (S + \frac{\partial A}{\partial T}) \dot{T} + (S_{KL} - \rho_o \frac{\partial A}{\partial E_{KL}'}) \dot{E}_{KL}' - \frac{Q_K T_{,K}}{T} \geq 0 \tag{31}$$

which holds for all arbitrary values of $\dot{T}$, $\dot{E}_{KL}'$ and temperature distribution. Hence,

$$-Q_K T_{,K} \geq 0 \tag{32}$$

$$S = -\frac{\partial A}{\partial T} \tag{33}$$

and

$$S_{KL} = \rho_o \frac{\partial A}{\partial E_{KL}'}, \tag{34}$$

Equation (7) shows that E_{KL}' may be expressed in terms of $\overline{E}_{\alpha\beta}$ and $Y_{\alpha,K}$. During unloading, the intermediate configuration $\overline{R}$, or $Y_{\alpha,K}$, remains unchanged. Therefore, eq. (34) may be written as

$$\begin{aligned} S_{KL} &= \rho_o \frac{\partial \overline{A}}{\partial \overline{E}_{\alpha\beta}} \frac{\partial \overline{E}_{\alpha\beta}}{\partial E_{KL}'}, \\ &= \rho_o X_{K,\alpha} X_{L,\beta} \frac{\partial \overline{A}}{\partial \overline{E}_{\alpha\beta}} \end{aligned} \tag{35}$$

where, by eqs. (7), (10) and (11),

$$\overline{A}[\overline{E}_{\alpha\beta}, E_{KL}'', \kappa, T] = A[E_{KL}'(\overline{E}_{\alpha\beta}, Y_{\alpha,K}), E_{KL}'', \kappa, T] \tag{36}$$

By employing eq. (16), eq. (35) may also be expressed as

$$S_{\alpha\beta} = \rho_o (j^p)^{-1} \frac{\partial \overline{A}}{\partial \overline{E}_{\alpha\beta}} \tag{37}$$

The free energy $\overline{A}$ may be approximated by a polynomial expansion of the form

$$\overline{A} = A'' + a_{\alpha\beta} \overline{E}_{\alpha\beta} + \frac{1}{2} a_{\alpha\beta\gamma\delta} \overline{E}_{\alpha\beta} \overline{E}_{\gamma\delta} + \cdots \tag{38}$$

where A'', $a_{\alpha\beta}$, $a\alpha_{\beta\gamma\delta}$ etc. may be functions of plastic strain E_{KL}'', history of plastic deformation κ and temperature T. Without loss in generality, it is assumed that

$$\begin{aligned} a_{\alpha\beta} &= a_{\beta\alpha} \\ a_{\alpha\beta\gamma\delta} &= a_{\gamma\delta\alpha\beta} = a_{\beta\alpha\gamma\delta} = a_{\alpha\beta\delta\gamma} \end{aligned} \tag{39}$$

since $\overline{S}_{\alpha\beta}$ and $\overline{E}_{\alpha\beta}$ are both symmetric. Furthermore, $a_{\alpha\beta} = 0$, because the intermediate configuration $\overline{R}$ is free of stress. It has been observed [28] that the motion of mobile dislocations usually occur in the narrow slip bands in each grain, leaving the basic lattice structure practically intact, so that the macroscopic elastic properties of a polycrystalline metal are essentially independent of plastic deformation even if plastic deformation is quite large. Therefore, it is assumed that $a_{\alpha\beta\gamma\delta}$ etc. may be functions of temperature T only.

For incorporating with plastic strain-rate equations, it is more convenient to express the elastic stress-strain relationship in terms of E_{KL}' and S_{KL}. As an illustration, the elastic stress-strain relationship of a material which is homogeneous, isotropic, thermal-mechanically uncoupled, and has no volume change in plastic deformation and relatively small elastic strains, may be written as

$$\overline{E}_{\alpha\beta} = \frac{1}{E}[(1+\nu)\overline{S}_{\alpha\beta} - \nu S_{\gamma\gamma}\delta_{\alpha\beta}] + \alpha\delta_{\alpha\beta}(T - T_o) \tag{40}$$

where E is Young's modulus of elasticity, ν and α may be functions of temperature. Employing eqs. (7), (11) and (16), eq. (40) may be transformed to

$$E_{MN}' = \frac{1}{E}[(1+\nu) C_{MK}'' C_{NL}'' - \nu C_{MN}'' C_{KL}''] S_{KL} + \alpha(T - T_o) C_{MN}'' \tag{41}$$

It is to be noted that eq. (41) is form invariant with respect to any rotation of the intermediate configuration and it agrees with the results of Naghdi and Trapp [27].

4. A Work-Hardening Rule

In the classical theory of plasticity for infinitesimal deformation, a yield surface is treated as a closed surface in the only stress space. With infinitesimal kinematic changes, a point in the stress space is adequate to represent an elastic state. Mises' yield function has been found experimentally and analytically to describe well the initial yielding of polycrystalline metals. The shape and location of the subsequent yield surface may be influenced by micro-structural and temperature changes. At finite deformation, the additional effects of kinematic changes on the yield surface must also be considered.

At finite deformation, an elastic state of a continuum may be represented by a point in the $\bar{S}_{\alpha\beta}$ stress space. The kinematic changes associated with the plastic deformation are embedded in the configuration reference. Available experimental results [29] have shown that the temperature change causes essentially a change only in size of the yield surface. Following Prager and Ziegler's approach, it is assumed that the yield surface may change in size but not in shape and may undergo a translation depending on the history of plastic deformation. In other words, it is assumed that the yield function in eq. (24) may be simplified to the form

$$F(S_{KL},\ E_{KL}'',\ \varkappa,\ T) = \bar{f}(\bar{S}_{\alpha\beta} - \bar{\theta}_{\alpha\beta}) - k^2(T) = 0 \tag{42}$$

where the tensor $\bar{\theta}_{\alpha\beta}$ indicates the position or total translation of the "center" (initially coincides with the null stress state) of the yield surface in the $\bar{S}_{\alpha\beta}$ stress space. It is further assumed that the transformation of $\bar{\theta}_{\alpha\beta}$ in $\bar{R}$ to θ_{KL} in R_o follows that of $\bar{S}_{\alpha\beta}$ to S_{KL} in eq. (16). Furthermore, the yield surface when expressed in terms of S_{KL} and θ_{KL} must be form invariant for any rigid rotation of the intermediate configuration. Therefore, the yield surface may be expressed as

$$F = f(S_{KL} - \theta_{KL},\ E_{KL}'') - k^2(T) = 0 \tag{43}$$

The tensorial function θ_{KL} is a function of the stress, plastic strain, temperature and history of plastic deformation, or

$$\theta_{KL} = \theta_{KL}(S_{MN},\ E_{MN}'',\ \varkappa,\ T) \tag{44}$$

Furthermore, $\dot{\theta}_{KL} \neq 0$ when plastic deformation occurs and $\dot{\theta}_{KL} = 0$ otherwise.

During a loading process, the time rate of change of the yield function in eq. (43) may be expressed as

$$\dot{F} = \frac{\partial f}{\partial S_{KL}}(\dot{S}_{KL} - \dot{\theta}_{KL}) + \frac{\partial f}{\partial E_{KL}}\dot{E}_{KL}'' - 2k\frac{\partial k}{\partial T}\dot{T} = 0\ , \tag{45}$$

in which the convective rates of change of $\bar{S}_{\alpha\beta}$ are included implicitly. By omitting higher order terms, $\dot{E}_{KL}''$ may be expressed as a linear function of $\dot{S}_{KL}$ and $\dot{T}$ such as

$$\dot{E}_{KL}'' = \beta_{KLMN}\dot{S}_{MN} + \beta_{KL}\dot{T} \tag{46}$$

where β_{KLMN} and β_{KL} are tensor functions of stress, plastic strain, temperature and their histories. During neutral loading, it is required that $\dot{E}_{KL}'' = 0$ and $\dot{\theta}_{KL} = 0$ and that, by eqs. (45) and (46),

$$\beta_{KLMN}\dot{S}_{MN} + \beta_{KL}\dot{T} = 0 \tag{47}$$

and

$$\frac{\partial f}{\partial S_{MN}}\dot{S}_{MN} - 2k\frac{\partial k}{\partial T}\dot{T} = 0 \tag{48}$$

Hence, the continuity of the various loading processes requires that

$$(\beta_{KLMN} - \lambda\gamma_{KL}\frac{\partial f}{\partial S_{MN}})\dot{S}_{MN} + (\beta_{KL} + 2\lambda\gamma_{KL}k\frac{\partial k}{\partial T})\dot{T} = 0 \tag{49}$$

where γ_{KL} is a symmetric tensor function and λ is a positive scalar function of S_{MN}, E_{MN}'', T and their histories. For arbitrary stress and temperature rates, it is necessary that

$$\beta_{KL} = -2\lambda \gamma_{KL} k \frac{\partial k}{\partial T} \tag{50}$$

and

$$\beta_{KLMN} = \lambda \gamma_{KL} \frac{\partial f}{\partial S_{MN}} \tag{51}$$

Thus, the plastic strain rate in eq. (46) may be written as

$$\dot{E}_{KL}'' = \lambda \gamma_{KL} \left(\frac{\partial f}{\partial S_{MN}} \dot{S}_{MN} - 2k \frac{\partial k}{\partial T} \dot{T}\right)$$

$$\text{when } F = 0 \text{ and } \frac{\partial f}{\partial S_{MN}} \dot{S}_{MN} - 2k \frac{\partial k}{\partial T} \dot{T} > 0 \tag{52}$$

and $\dot{E}_{KL}'' = 0$

$$\text{when } F < 0, \text{ or } F = 0 \text{ and } \frac{\partial f}{\partial S_{MN}} \dot{S}_{MN} - 2k \frac{\partial k}{\partial T} \dot{T} \le 0 \tag{53}$$

The tensor functions γ_{KL} must satisfy the two thermodynamic laws. By employing eqs. (33) and (34), the thermodynamic laws, eqs. (29) and (30), may be reduced to

$$\rho_o r + \left(S_{KL} - \rho_o \frac{\partial A''}{\partial E_{KL}''}\right) \dot{E}_{KL}'' - \rho_o \frac{\partial A''}{\partial \varkappa} \dot{\varkappa} - \rho_o \dot{S} T - Q_{K,K} = 0 \tag{54}$$

and

$$\left(S_{KL} - \rho_o \frac{\partial A''}{\partial E_{KL}''}\right) \dot{E}_{KL}'' - \rho_o \frac{\partial A''}{\partial \varkappa} \dot{\varkappa} - \frac{Q_K T_{,K}}{T} \ge 0 \tag{55}$$

To account for the irreversible nature of plastic deformation, the work-hardening parameter $\varkappa$ may take the form [30] such as

$$\dot{\varkappa} = (\dot{E}_{KL}'' \dot{E}_{KL}'')^{1/2} \tag{56}$$

By substituting eq. (52) into eq. (55) and using an arbitrary homogeneous temperature distribution for which $T_{,K} = 0$, it is found that

$$\gamma_{KL}\left(S_{KL} - \rho_o \frac{\partial A''}{\partial E_{KL}''}\right) - \rho_o \frac{\partial A''}{\partial \varkappa} \left(\gamma_{KL} \gamma_{KL}\right)^{1/2} \ge 0 \tag{57}$$

for any admissible states of stress and strain and history of plastic formation.

It is known that the internal energy increases with increasing dislocation density caused by plastic deformation within a certain temperature range. The work-hardening parameter $\varkappa$ may be associated with the microstructural change or dislocation density, therefore,

$$\partial A''/\partial \varkappa \ge 0 \tag{58}$$

In inequality (57), the term $\rho_o \frac{\partial A''}{\partial E_{KL}''}$, which is associated with the configuration change, may be regarded as a point in the S_{KL} stress space. Inequality (57) may be interpreted as that the angle between the nine dimensional strain vector γ_{KL} and stress vector $(S_{KL} - \rho_o[\partial A''/\partial E_{KL}''])$ be less than a certain acute angle or $\pi/2$. Since the function A'' is not readily available, other constitutive postulates may be needed to simplify the relationships. Therefore, the postulate of Ilyushin [31] is employed. For arbitrary stress cycles Ilyushin [31] proposed that the net work should be positive if plastic deformation occurs:

$$\oint S_{KL}\, dE_{KL} \ge 0 \tag{59}$$

Referring to a state of stress S_{KL} on the yield surface, the postulate may be further reduced to the form [9]

$$(S_{KL} - S_{KL}^{*})\dot{E}_{KL}'' > 0 \tag{60}$$

where S_{KL}^{*} may be any state of stress within the elastic domain. Inequality (60) infers that the yield surface is at least locally convex [9] and the normality rule

$$\gamma_{KL} = \partial f/\partial S_{KL} \, . \tag{61}$$

The normality rule (61) and inequality (57) require that the value of $\rho_o[\partial A''/\partial E_{KL}'']$ be in the vicinity of that of the center of the yield surface.

The translation of the center of the yield surface may be determined by an assumption similar to Ziegler's assumption [2], i.e.

$$\dot{\theta}_{KL} = (S_{KL} - \theta_{KL})\dot{\mu} \tag{62}$$

in which $\dot{\mu}$ as a scalar function. Ziegler's assumption is in general accordance with experimentally observed Bauschinger effects. The value of $\dot{\mu}$ may be obtained by substitution of eqs. (52), (61) and (62) into (45), which yields

$$\dot{\mu} = \left(1+\lambda \frac{\partial f}{\partial S_{KL}} \frac{\partial f}{\partial E_{KL}''}\right)\left(\frac{\partial f}{\partial S_{MN}}\dot{S}_{MN} - 2k\frac{\partial k}{\partial T}\dot{T}\right)\Big/\left[\frac{\partial f}{\partial S_{KL}}(S_{KL} - \theta_{KL})\right] \tag{63}$$

The foregoing work-hardening rule is applicable to all loading programs, including that of simple tension and compression. Therefore, to complete the hardening rule (62), (63) and the flow rule (52) the quantity λ may be determined from experimentally obtained, uniaxial Piola-Kirchhoff stress versus Lagrangian strain curves at various temperatures. This may be illustrated by the following example.

5. An Example

Mises' yield function and its translation in the $\bar{S}_{\alpha\beta}$ stress space may be expressed by eq. (42), as

$$F = \tfrac{1}{2}[(\bar{S}_{\alpha\beta} - \bar{\theta}_{\alpha\beta}) - \tfrac{1}{3}\delta_{\alpha\beta}(\bar{S}_{\gamma\gamma} - \bar{\theta}_{\gamma\gamma})] \cdot [(\bar{S}_{\alpha\beta} - \bar{\theta}_{\alpha\beta}) - \tfrac{1}{3}\delta_{\alpha\beta}(\bar{S}_{\gamma\gamma} - \bar{\theta}_{\gamma\gamma})] - k^2(T) = 0 \tag{64}$$

By employing equations (11) and (16) and assuming no volume change in plastic deformation, the yield surface may be transformed to

$$F = \tfrac{1}{2}(C_{KM}'' C_{LN}'' - \tfrac{1}{3}C_{KL}'' C_{MN}'')(S_{KL} - \theta_{KL})(S_{MN} - \theta_{MN}) - k^2(T) = 0 \tag{65}$$

For an uniaxial state of stress, eq. (65) becomes

$$F = \tfrac{1}{3}C''^{\,2}_{11}(S_{11} - \theta_{11})^2 - k^2(T) = 0 \tag{66}$$

It may next be shown by (52), that

$$\dot{E}''_{11} = \lambda \frac{4k^2}{3}[C''^{\,2}_{11}\dot{S}_{11} - \sqrt{3}\frac{\partial k}{\partial T}C''_{11}\dot{T}]$$

$$\text{when } \frac{2}{3}C''^{\,2}_{11}(S_{11} - \theta_{11})\dot{S}_{11} - 2k\frac{\partial k}{\partial T}\dot{T} > 0 \tag{67}$$

Therefore,

$$\lambda = [3/4k^2][1/(\xi_1 + \xi_2)] \, , \tag{68}$$

where

$$\xi_1 = C''^{\,2}_{11}\dot{S}_{11}/\dot{E}_{11} \tag{69}$$

and

$$\xi_2 = -\sqrt{3}\,C_{11}''\frac{\partial k}{\partial T}\frac{\dot{T}}{E_{11}''} \tag{70}$$

may be called stiffness coefficients which may be obtained from experimentally determined S_{11} versus E_{11} curves at various temperatures. ξ_1 and ξ_2 are functions of temperature and history of plastic deformation, $\varkappa$, or the irreversible dissipative work, W^P,

$$W^P = \int_o^t S_{KL}\dot{E}_{KL}{}'' \, dt \tag{71}$$

Therefore, it is assmed that $\xi_1(W^P)$ and $\xi_2(W^P)$ of a general state of stress are identical to that of an uniaxial state of stress.

By employing eqs. (52) and (65), it may be shown that the plastic strain rates satisfy the condition of no volume change in plastic deformation, i.e.

$$j^P = 1 = (1 + 2I_E{}'' + 4II_E{}'' + 8III_E{}'')^{1/2}$$

or $$\dot{I}_E{}'' + 2\dot{II}_E{}'' + 4\dot{III}_E{}'' = 0 \tag{72}$$

where $I_E{}''$, $II_E{}''$ and $III_E{}''$ are the invariants of the plastic strain tensor $E_{KL}{}''$.

6. Remarks

It is to be noted that the present flow and hardening rules may be applicable to a polycrystalline metal only where temperature and pressure are within certain ranges such that there are negligible viscous effects and no structural phase change from a solid state to another state. The present flow rule may be reduced to Ziegler's rule [2] for infinitesimal deformation and that of classical, simple flow theory for initial yielding. It is also to be noted that about 90% of the plastic work, W^P, is converted to heat while the rest of it is locked in the dislocation field associated with the breakdown of the crystal structure [20].

7. Acknowledgement

This research was sponsored by the National Science Foundation, U.S.A., under Engineering Mechanics Program and Grant GK-37053 to the University of Notre Dame.

References

[1] PRAGER, W., "A New Method of Analyzing Stresses and Strains in Work-Hardening Plastic Solids," J. Appl. Mech., 23, 493 (1956).

[2] ZIEGLER, H., "A Modification of Prager's Hardening Rules," Q. Appl. Math., 17, 55 (1959).

[3] BUDIANSKY, B. and WU, T.T., "Theoretical Predictions of Plastic Strains of Polycrystals," Proc. 4th U.S. Natl. Congr. Appl. Mech., ASME, 1175 (1962).

[4] HILL, R., "The Essential Structure of Constitutive Laws for Metal Composites and Polycrystals," J. Mech. Phys. Solids, 15, 79 (1967).

[5] LIN, T.H. and ITO, M., "Theoretical Plastic Distortion of Polycrystalline Aggregate Under Combined and Reverse Stresses," J. Mech. Phys. Solids, 13, 103 (1965).

[6] JOHNSON, J.N., "Constitutive Relation for Rate-Dependent Plastic Flow in Polycrystalline Metals," J. Appl. Phys., 40, 2287 (1969).

[7] RICE, J.R., "On the Structure of Stress-Strain Relations for Time-Dependent Plastic Deformation in Metals," J. Appl. Mech., 37, 728 (1970).

[8] HUTCHINSON, J.W., "Elastic-Plastic Behavior of Polycrystalline Metals and Composites," Proc. Royal Soc. London A. 319, 27 (1970).

[9] HILL, R., "On Constitutive Inequalities for Simple Materials-II," J. Mech. Phys. Solids, 16, 315 (1968).

[10] HILL, R., and RICE, J.R., "Constitutive Analysis of Elastic-Plastic Crystals at Arbitrary Strain," J. Mech. Phys. Solids, 20, 401 (1972).

[11] MANDEL, J., "Relations de Comportement des Milieux Elastiques-Plastiques et Elastiques-Viscoplastiques. Notion de Reper Directeur," Foundations of Plasticity, (edited by Sawczyk, A.) 1, Noordhoff, 387 (1973).

[12] HAVNER, K.S., "On the Mechanics of Crystalline Solids," J. Mech. Phys. Solids, 21, 383 (1973).

[13] RICE, J.R., "Inelastic Constitutive Relations for Solids: An Internal-Variable Theory and Its Application to Metal Plasticity," J. Mech. Phys. Solids, 19, 433 (1971).

[14] COLEMAN, B.D. and GURTIN, M.E., "Thermodynamics with Internal State Variables," J. Chem. Phys., 47, 597 (1967).

[15] PERZYNA, P. and WONJO, W., "Thermodynamics of a Rate Sensitive Plastic Material," Arch. Mech., 20, 501 (1968).

[16] KRATOCHVIL, J., "On a Finite Strain Theory of Elastic-Inelastic Materials," Acta Mech., 16, 127 (1973).

[17] ODEN, J.T. and BHANDARI, D.R., "Thermo-Plastic Materials with Memory," J. Engng. Mech. Div. ASCE, 99, 131 (1973).

[18] VALANIS, K.C., "A Theory of Viscoplasticity Without a Yield Surface," Arch. Mech., 23, 517 (1971).

[19] LEE, E.H. and LIU, D.T., "Finite Strain Elastic-Plastic Theory with Application to Plane Wave Analysis," J. Appl. Phys., 38, 19 (1967).

[20] LEE, E.H., "Elastic-Plastic Deformation at Finite Strains," J. Appl. Mech., 36, 1 (1969).

[21] FREUND, L.B., "Constitutive Equations for Elastic-Plastic Materials at Finite Strain," Int. J. Solids Str., 6, 1193 (1970).

[22] HOLSAPPLE, K.A., "Elastic-Plastic Materials as Simple Materials," Zi. Ang. Math. Mech., 53, 261 (1973).

[23] WANG, Y.S., "A Simplified Theory of the Constitutive Equation of Metal Plasticity at Finite Deformation," J. Appl. Mech., 40, 941 (1973).

[24] HAHN, H.T., "A Finite-Deformation Theory of Plasticity," Int. J. Solids and Str., 10, 111 (1974).

[25] GREEN, A.E. and NAGHDI, P.M., "A General Theory of an Elastic-Plastic Continuum," Arch. Rat. Mech. Analy., 18, 251 (1965).

[26] GREEN, A.E. and NAGHDI, P.M., "Some Remarks on Elastic-Plastic Deformation at Finite Strain," Int. J. of Engng. Sci., 9, 1219 (1971).

[27] NAGHDI, P.M. and TRAPP, J.A., "On Finite Elastic Plastic Deformation of Metals," J. Appl. Mech., 41, 254 (1974).

[28] HILL, R., The Mathematical Theory of Plasticity, Oxford Univ. Press, (1950).

[29] PHILLIPS, A. and KASPER, R., "On the Foundations of Thermoplasticity-An Experimental Investigation," J. Appl. Mech., 40, 891 (1973).

[30] EISENBERG, M.A. and PHILLIPS, A., "On Nonlinear Kinematic Hardening," Acta Mech., 5, 1 (1968).

[31] ILYUSHIN, A.A., "On the Postulate of Plasticity," Prik. Mat. Mekh., 25, 503 (1961).