ESTIMATING VARIANCE COMPONENTS FOR TWO-WAY DISPROPORTIONATE DATA BY THE METHOD OF UNWEIGHTED MEANS

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1. INTRODUCTION

Crossed classifications with unequal numbers of observations in the cells pose difficult computational problems for the method of fitting constants, Yates [1934]. To avoid these computational difficulties, Yates [1934], has suggested an alternative procedure, the method of unweighted means, which is a simple analysis based on the cell means. This analysis is limited to cases for which all of the cells have one or more observations. To avoid this limitation, the primary purpose of this dissertation is to study the method of unweighted means with regard to estimating the variance components for a two-way classification with unequal numbers of observations in the cells and with one or more cells having no observations. The model used is the variance component Model II given by Eisenhart [1947].

In particular, two estimating procedures are examined which are based upon the method of unweighted means. The first (denoted by Procedure M) estimates the variance components in the following manner:

1. a missing value estimation procedure based on cell means is used to estimate an observation (observations) for the missing cell (cells);
2. the mean squares and their expectations are then computed for the method of unweighted means and
3. the variance components are estimated by equating the expected mean squares to the computed mean squares in the analysis of variance table and solving the resulting set of equations for the estimates of the variance components.

Procedure M gives unbiased estimates for the variance components.

The second procedure (denoted by Procedure $\hat{M}$) is the same as Procedure M except that approximate expected mean squares are used which are essentially those that would be obtained by using the method of unweighted means for a design where there is no missing cell (cells). Procedure $\hat{M}$ gives biased estimates for the variance components.

In order to evaluate Procedures M and $\hat{M}$ the variances and mean square errors of the estimated variance components are computed for both procedures for several experimental designs with missing cells and a number of parameter combinations (true values of the variance components). These variances and mean square errors are then compared with corresponding variances obtained by a widely used estimating procedure based on the method of fitting constants. The variances for all three procedures were computed by using a computer program written for the IBM/360 model 75. The computer program uses a computational method given by Bush [1962] to compute the variances for the procedure based on the method of fitting constants.

In addition to the two-way classification with unequal numbers of observations in the cells, the method of unweighted means is also studied for estimating the variance components for a two-way classification with two levels of nesting within each of the cells (in this case no missing cells were allowed). Here the estimating procedure is again based on computing mean squares and their expectations by the method of unweighted means; but due to the two levels of nesting, the estimating procedure is examined for two different estimates of the cell means: (a) the simple arithmetic means within cells and (b) the unweighted means of class means within cells.

The format of this dissertation is the following: a review of the pertinent literature is given in Chapter 2; for two-way disporportionate designs with one missing cell the (a) analytic formulas for the expected mean squares using the method of unweighted means after having estimated the missing cell and (b) the estimates of the variance components for Procedures M and $\hat{M}$ are given in Chapter 3; the results given in Chapter 3 are extended to designs with more than one missing cell in Chapter 4; for a number of experimental designs and parameter values the computer results for (a) the variances and mean square errors of the estimated variance components using Procedures M, $\hat{M}$ and the procedure based on the method of fitting constants, (b) the numerical values of the expected mean squares for the method of unweighted means and (c) the biases in the estimated variance components obtained by Procedure $\hat{M}$ are presented in Chapter 5; the method of unweighted means is examined for estimating variance components for a two-way design with two levels of nesting within each cell in Chapter 6; a summary and suggestions for future research are given in Chapter 7.

2. REVIEW OF LITERATURE

There have been several articles written about the estimation of variance components in a two-way classification with unequal numbers of observations in the cells for the completely random model. In most of these articles the estimation procedure involves calculating mean squares, computing the expectations of these mean squares under the variance component model and then equating calculated mean squares to their expected values and solving the set of equations for the variance components. This procedure (usually referred to as the analysis of variance procedure) gives unbiased estimators for the variance components under the completely random model. Using the analysis of variance procedure Henderson [1953] describes and gives some examples of estimating the variance components (a) by computing mean squares by the method of fitting constants (i.e. mean squares for rows adjusted for columns ignoring iteractions; mean squares for columns adjusted for rows ignoring interactions; mean squares for interactions adjusted for rows and columns) and (b) by using unadjusted mean squares computed by the usual formulas for the analysis of orthogonal data (these two methods are called Method 3 and Method 1, respectively). Searle [1956, 1958, 1961] studies Henderson's Method 1 and develops matrix methods by which the variances of the estimated variance components for this method can be obtained. Searle found the variances of the estimated variance components by using the fact that if $Z_1 = \underline{y}'G_1\underline{y}$ and $Z_2 = \underline{y}'G_2\underline{y}$ are functions of variables $\underline{y}' = (y_1, y_2, \dots y_n)$ having a multinormal distribution with means zero and variance-covariance matrix V and G_1 and G_2 are square symmetric matrices, then $\text{Var}(Z_1) = 2\text{tr}(V\cdot G_1)^2$ and $\text{Cov}(Z_1, Z_2) = 2\text{tr}(VG_1\cdot VG_2)$.

Searle [1968] reviews and reformulates the methods given by Henderson [1953]. Searle's work on the variances of the Method 1 estimated variance components is extended by Mahamunulu [1963] to the three-level nested classification and by Blischke [1968] to the r-way crossed classification. Low [1964] gives algebraic expressions similar to Searle's for the variances of the estimators obtained by Henderson's Method 3 when there is no interaction in the two-way classification model. Harville [1969] extends this work by giving formulas for the variances of Method 3 estimators for the two-way classification with interaction and indicates how these formulas may be compared with those of Searle for the variances of the Method 1 estimators.

Bush and Anderson [1963] compare three analysis of variance procedures for estimating variance components by developing an analytic procedure for obtaining the variances of estimates of the variance components and then numerically computing these variances for each procedure for a number of experimental designs. The three procedures studied are a procedure based on the method of weighted squares of means, Yates [1934]; and Henderson's Method 1 and Method 3. Based on their computations of the variances the authors reach the conclusion that of the three estimating procedures studied the procedure based on the method of fitting constants (Method 3) should be used. Hirotsu [1966] examines estimating the variance components by computing mean squares by the method of unweighted means when there are no missing cells in the design. This method has the advantage that it requires a minimum of computation. Hirotsu gives analytic formulas for the variances of his estimates and compares these variances for several designs to those for the three procedures studied by Bush and Anderson [1963]. He finds that his procedure compared quite favorably with the other three procedures. The estimating procedure used by Hirotsu is examined in detail in Chapter 3 of this dissertation.

In many of the articles which examine the analysis of variance procedure for estimating variance components for unbalanced designs the authors find that the expectations of the computed mean squares and the variances of the estimated variance components are difficult to obtain. This is especially true of Henderson's Method 3. Accordingly, suitable numerical techniques which can be used on a computer for any unbalanced design can be quite helpful. Besides the work of Bush and Anderson [1963] in this area, Hartley [1967] develops a computer procedure called 'synthesis' which can be used to find the numerical values of the coefficients in the formulas for expected mean squares for any analysis of variance. Hartley's procedure can also be used to find numerical values of the variance and covariances of the expected mean squares. Similarly, Gaylor _et al_. [1970] give a procedure which uses the forward solution of the Abbreviated Doolittle Method to obtain the numerical values of the coefficients in the expected mean squares for the method of fitting constants.

In addition to the analysis of variance procedure, several other methods have been examined for estimating variance components. Koch [1967, 1968] develops a procedure which does not require an analysis of variance table as most other procedures do. The method uses the fact that the expected value of the squared difference of two observations is a function of the variance components. For example, for the usual two-level nested design

$$y_{ij} = \mu + a_i + e_{ij} \qquad \begin{array}{l} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n_{ij} \end{array} \tag{2.1}$$

where μ is a constant and a_i and e_{ij} are normally independently distributed random variables with means zero and variances σ_a^2 and σ_e^2 , respectively. Then

$$E\left[(y_{ij} - y_{k\ell})^2\right] = 2\sigma_e^2 \qquad \text{if} \quad i = k,\ j \neq \ell \tag{2.2}$$

$$= 2(\sigma_a^2 + \sigma_e^2) \quad \text{if} \quad i \neq k \ .$$

Koch forms sums of all the unbiased estimators given in (2.2) and equates these sums to their expected values and solves the resulting equations for the estimates of the variance components. In his articles Koch estimates variance components by the above method for nested and/or crossed designs. Koch's estimates are unbiased and consistent and in the case of balanced designs his estimators coincide with those obtained from the analysis of variance procedure.

Hartley and Rao [1967] develop an iterative maximum-likelihood procedure for estimating the variance components from unbalanced designs. Their procedure which requires the use of a computer leads to consistent and asymptotically efficient estimates and avoids the occurrence of negative estimates of variance components. In addition, the authors also give small sample confidence regions for the parameters in their model. Hartley and Rao point out that their technique will permit Monte Carlo evaluations of small sample variances of their estimators as was done by Bush and Anderson [1963]. Anderson and Bancroft [1952] and Anderson [1961] also discuss the use and underlying complexities of maximum likelihood or modified maximum likelihood for estimating variance components. Searle [1970] obtains the large sample variance-covariance matrix for the maximum likelihood estimators of variance components for unbalanced data from a general mixed model. Searle points out that even though the maximum likelihood estimators themselves cannot be obtained the large sample variances of these estimators are relatively easy to compute. Thus, these large sample variances may be compared with the variances from other methods of estimation to obtain a measure of asymptotic efficiency of these other methods.

Anderson [1961] gives an estimation procedure which lists certain mean squares and their corresponding expectations and performs iterative weighted least squares to estimate the variance components using prior estimates of the variance components to determine the weights. This procedure uses the computed mean squares as the dependent variables and the variance components as regression coefficients.

The method of unweighted means given by Yates [1934] has been studied by many authors in addition to Hirotsu [1966]; for example, Anderson and Bancroft [1952], Gosslee [1956], Gosslee and Lucas [1965], Webster [1968] and Hirotsu [1968]. However, these publications do not deal primarily with the estimation of variance components. Instead, most of these authors study the method of unweighted means from the standpoint of testing hypothesis.

The method of unweighted means is appealing because of its computational simplicity, but it has been limited to designs in which observations exist in all cells. In this dissertation the method of unweighted means is extended to designs which contain empty cells. After supplying estimates of the means for the missing cells, the method of unweighted means is applied to provide estimates of the variance components. Properties of these estimates are studied.

3. ANALYTICAL RESULTS FOR ONE MISSING CELL

3.1 Introduction

The purpose of this chapter is to examine the method of unweighted means with regard to estimating the variance components when there is one cell with no observations in a two-way classification with unequal numbers of observations in the cells. The mathematical model that will be used is the variance components Model II given by Eisenhart [1947]. It can be written as

$$y_{ijk} = \mu + r_i + c_j + rc_{ij} + e_{ijk} \qquad (3.1)$$

$$i = 1,2,\ldots,a; \quad j = 1,2,\ldots,b; \quad k = 1,2,\ldots,n_{ij}$$

where y_{ijk} is the observed value for the k^{th} observation in the i^{th} row and the j^{th} column. In the model, μ is a constant; r_i, c_j, rc_{ij} and e_{ijk} are normally independently distributed random variables with means zero and variances σ_r^2, σ_c^2, σ_{rc}^2 and σ_e^2 respectively; e.g. $e_{ijk} \sim NID\ (0, \sigma_e^2)$.

Totals and means are denoted by dropping subscripts, i.e.,

$$\sum_k y_{ijk} = y_{ij}, \quad \sum_j \sum_k y_{ijk} = y_i, \quad \sum_i \sum_k y_{ijk} = y_j,$$

$$\sum_i \sum_j \sum_k y_{ijk} = y, \quad \sum_j n_{ij} = n_i, \quad \sum_i n_{ij} = n_j,$$

$$\sum_i \sum_j n_{ij} = n, \quad \frac{1}{n_{ij}} \sum_k y_{ijk} = \bar{y}_{ij}, \quad \frac{1}{n_i} \sum_j \sum_k y_{ijk} = \bar{y}_i,$$

$$\frac{1}{n_j} \sum_i \sum_k y_{ijk} = \bar{y}_j \quad \text{and} \quad \frac{1}{n} \sum_i \sum_j \sum_k y_{ijk} = \bar{y}.$$

A missing cell implies that one of the $n_{ij} = 0$, *i.e.* cell (ij) has no observations in it.

In the next section the method of unweighted means and the work of Hirotsu [1966] will be reviewed for the case of all $n_{ij} > 0$.

3.2 The Method of Unweighted Means

The method of unweighted means for the two-way classification is simply to compute the mean for each of the ab cells of the design and then to run the usual analysis of variance on these means as if they were the original observations. That is, compute

$$\bar{y}_{ij} = \frac{1}{n_{ij}} \sum_k y_{ijk} \qquad \text{for all i and j} \tag{3.2}$$

and compute the usual sum of squares for rows, columns and interaction using the $\bar{y}_{ij}$. Using the notation given by Hirotsu [1966] this implies that the mean square for rows is

$$MSR = \sum_i \sum_j (\bar{y}_i - \bar{y})^2/(a - 1) = \underline{y}' \, G_R \underline{y} \tag{3.3}$$

where $\underline{y}'$ is a vector of cell means $= (\bar{y}_{11}, \bar{y}_{12}, \ldots, \bar{y}_{1b}, \ldots, \bar{y}_{a1}, \ldots, \bar{y}_{ab})$ and

$$G_R = \{P/Q\}_a \, , \quad P = \frac{1}{ab} J_{b\cdot b} \, , \quad Q = \frac{-1}{ab(a-1)} J_{b\cdot b} \, .$$

Here $\{P/Q\}_a$ denotes a matrix whose diagonal elements are all equal to P and whose off diagonal elements are all equal to Q. Note P and Q are b x b matrices; and therefore, G_R is a ab x ab matrix. $J_{b\cdot b}$ denotes a b x b matrix whose every element is one. Likewise, the mean square for columns is

$$MSC = \sum_i \sum_j (\bar{y}_j - \bar{y})^2 / (b-1) = \underline{y}' G_C \underline{y}$$

where

$$G_C = \{R/R\}_a, \quad R = \frac{1}{a(b-1)} [I_b - \frac{1}{b} J_{b \cdot b}]$$

and the mean square for interaction is

$$MSRC = \sum_i \sum_j (\bar{y}_{ij} - \bar{y}_i - \bar{y}_j + \bar{y})^2 / (a-1)(b-1) \tag{3.4}$$

$$= \underline{y}' G_{RC} \underline{y}$$

where

$$G_{RC} = \frac{1}{(a-1)(b-1)} [I_{ab} - \frac{1}{ab} J_{ab \cdot ab} - (a-1) G_R - (b-1) G_C]$$

assuming no $n_{ij} = 0$. The mean square for error is the usual one, namely

$$MSE = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij})^2 / (n-ab) .$$

Now $\underline{y}'$ has a multivariate normal distribution with variance-covariance matrix V which can be written as

$$V = \{D/C\}_a + \text{diag} \left(\frac{\sigma_e^2}{n_{ij}} \right) \tag{3.5}$$

where $D = \{\sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2/\sigma_r^2\}_b$ and $C = \sigma_c^2 I_b$.

Thus, the results of Lancaster [1954] and Searle [1956] can be used which state that if $Z = \underline{y}' G \underline{y}$ is a function of variables $\underline{y}$, having a multinormal distribution with variance-covariance matrix V, then[1]

$$E(Z) = \mathrm{tr}\,(V\cdot G) \tag{3.6.1}$$

and

$$\mathrm{Var}(Z) = 2\,\mathrm{tr}\,(V\cdot G)^2 . \tag{3.6.2}$$

Using this Hirotsu showed for the case of all $n_{ij} > 0$

$$E(MSR) = E(\underline{y}'G_R\underline{y}) = \mathrm{tr}(V\cdot G_R) = \frac{1}{ab}\sum_i\sum_j \frac{1}{n_{ij}}\,\sigma_e^2 + \sigma_{rc}^2 + b\sigma_r^2$$

$$E(MSC) = E(\underline{y}'G_C\underline{y}) = \mathrm{tr}(V\cdot G_C) = \frac{1}{ab}\sum_i\sum_j \frac{1}{n_{ij}}\,\sigma_e^2 + \sigma_{rc}^2 + a\sigma_c^2$$

and

$$E(MSRC) = E(\underline{y}'G_{RC}\underline{y}) = \mathrm{tr}(V\cdot G_{RC}) = \frac{1}{ab}\sum_i\sum_j \frac{1}{n_{ij}}\,\sigma_e^2 + \sigma_{rc}^2 . \tag{3.7}$$

With these results and the fact that

$$E(MSE) = \sigma_e^2$$

it is then possible to estimate the variance components σ_r^2, σ_c^2, σ_{rc}^2 and σ_e^2 by setting the expected mean squares in (3.7) equal to the mean

---

[1] To be more precise

$$E(Z) = \mathrm{tr}(V\cdot G) + \underline{\mu}'\,G\,\underline{\mu} \quad \text{and}$$

$$\mathrm{Var}(Z) = 2\mathrm{tr}(V\cdot G)^2 + 4\underline{\mu}'\,G\,V\,G\,\underline{\mu}$$

where $\underline{\mu}'$ is a vector of μ's. However as pointed out by Searle [1970] analysis of variance mean squares are such that row sums of G are zero; i.e., $G\,\underline{1} = 0$ where $\underline{1}$ is a vector of 1's. Therefore, $G\underline{\mu} = G\,\underline{1}\,\mu = 0$ which implies

$$\mathrm{tr}(V\cdot G) + \underline{\mu}'\,G\,\underline{\mu} = \mathrm{tr}(V\cdot G) \quad \text{and}$$

$$2\mathrm{tr}(V\cdot G)^2 + 4\underline{\mu}'\,G\,V\,G\,\underline{\mu} = 2\mathrm{tr}(V\cdot G)^2 .$$

squares in the analysis of variance table (this method of estimation is referred to as the analysis of variance procedure). The results are

$$
\begin{aligned}
\hat{\sigma}_r^2 &= \frac{1}{b}\, \underline{y}'\, (G_R - G_{RC})\, \underline{y} \\
\hat{\sigma}_c^2 &= \frac{1}{a}\, \underline{y}'\, (G_C - G_{RC})\, \underline{y} \\
\hat{\sigma}_{rc}^2 &= \underline{y}'\, G_{RC}\, \underline{y} - \frac{1}{ab} \sum_i \sum_j \frac{1}{n_{ij}}\, (\mathrm{MSE}) \\
\hat{\sigma}_e^2 &= \mathrm{MSE} \quad .
\end{aligned}
\tag{3.8}
$$

Hirotsu then showed by using (3.6.2) and (8) that

$$
\begin{aligned}
\mathrm{Var}(\hat{\sigma}_r^2) &= 2\mathrm{tr}\, \left(V \cdot \frac{1}{b}\, (G_R - G_{RC})\right)^2 \\
&= \frac{2}{b^2} \left[\frac{b^2}{a-1}\, \sigma_r^4 + \frac{2b}{a-1}\, \sigma_r^2\, \sigma_{rc}^2 + \frac{b\, \sigma_{rc}^4}{(a-1)(b-1)} \right. \\
&\quad + 2\sigma_e^2 \left(\sigma_r^2 + \frac{\sigma_{rc}^2}{b-1} \right) \frac{H1}{a(a-1)} \\
&\quad \left. + \frac{\sigma_e^4}{a^2(b-1)^2} \left((H3 - H2) + \frac{(H5 + H2 - H3 - H4)}{(a-1)^2} \right) \right]
\end{aligned}
\tag{3.9}
$$

where

$$
H1 = \sum_i \sum_j \frac{1}{n_{ij}}\,, \quad H2 = \sum_i \sum_j \frac{1}{n_{ij}^2}\,, \quad H3 = \sum_i \sum_j \sum_\ell \frac{1}{n_{ij}}\, \frac{1}{n_{i\ell}}\,,
$$

$$
H4 = \sum_i \sum_j \sum_k \frac{1}{n_{ij} n_{kj}} \quad \text{and} \quad H5 = \sum_i \sum_j \sum_k \sum_\ell \frac{1}{n_{ij} n_{k\ell}} \quad .
$$

$\mathrm{Var}(\hat{\sigma}_c^2) = \mathrm{Var}(\hat{\sigma}_r^2)$ with a and b interchanged, σ_r^2 replaced by σ_c^2 and H4 and H3 interchanged.

$$\mathrm{Var}(\hat{\sigma}^2_{rc}) = 2\mathrm{tr}(V\cdot G_{RC})^2 + \left(\frac{1}{ab}\sum_i\sum_j \frac{1}{n_{ij}}\right)^2 \mathrm{Var}(\hat{\sigma}^2_e)$$

$$= \frac{2}{(a-1)(b-1)}\left[\sigma^4_{rc} + \frac{2}{ab}(H1)\ \sigma^2_{rc}\sigma^2_e + \sigma^4_e\left(\frac{ab(a-2)(b-2)H2 + a(a-2)H3 + b(b-2)H4 + H5}{a^2b^2(a-1)(b-1)}\right)\right]$$

$$+\left(\frac{H1}{ab}\right)^2 \mathrm{Var}(\hat{\sigma}^2_e)\ .$$

$$\mathrm{Var}(\hat{\sigma}^2_e) = \frac{2\sigma^4_e}{n-ab}\ .$$

3.3 Expected Mean Squares with One Missing Cell

In order to extend the results of Hirotsu which were given above to the case when one $n_{ij} = 0$ (denote the missing cell by $(i_1 j_1)$) it is necessary to estimate an observation for the missing cell. This is due to the fact that the expected mean squares for the method of unweighted means, see (3.7), include the term $\sum_i \sum_j 1/n_{ij}$. The formula that is used to estimate this missing cell is the same one given by Yates [1933] for the least squares estimate, X, of the missing value for a randomized complete block (R.C.B.) design, namely

$$X = (a\ R_{i_1} + b\ C_{j_1} - G)/(a-1)(b-1) \tag{3.10}$$

where R_{i_1} is the total of the observations in row i_1, C_{j_1} is the total of the observations in column j_1 and G is the grand total of the existing observations. This formula is used because of the fact that the method of unweighted means takes a two-way disproportionate design and converts it to a R.C.B. design by analyzing cell means. In terms of the model given in (3.1), (3.10) becomes

$$X = \frac{a \sum\limits_{j \neq j_1} \bar{y}_{i_1 j} + b \sum\limits_{i \neq i_1} \bar{y}_{i j_1} - \sum\limits_{i} \sum\limits_{j} \bar{y}_{ij}}{(a-1)(b-1)} \quad \text{with } \textstyle\sum\sum \text{ over } \neq (i_1 j_1) . \tag{3.11}$$

Equation (3.11) can be derived directly by minimizing the least squares equation from (3.1) for the sum of squares for interaction after having replaced the observations in each cell by their cell means (see (3.2)). That is, the formula given for the missing cell in (3.11) is that value which minimizes the sum of squares for interaction for the method of unweighted means with respect to X. Section 4.2 below shows this minimization procedure in detail for the case of two missing cells. By replacing $\bar{y}_{i_1 j}$, $\bar{y}_{i j_1}$ and $\bar{y}_{ij}$ with

$$\bar{y}_{i_1 j} = \mu + r_{i_1} + c_j + rc_{i_1 j} + \frac{\sum\limits_k e_{i_1 jk}}{n_{i_1 j}} \qquad \text{etc.} \tag{3.12}$$

in (3.11) we can then obtain X in terms of the model parameters, i.e.,

$$X = \mu + r_{i_1} + c_{j_1} + \frac{\sum\limits_{i \neq i_1} rc_{ij_1}}{(a-1)} + \frac{\sum\limits_{j \neq j_1} rc_{i_1 j}}{(b-1)} - \frac{\sum\limits_{i \neq i_1} \sum\limits_{j \neq j_1} rc_{ij}}{(a-1)(b-1)} \tag{3.13}$$

$$+ \sum_{i \neq i_1} \frac{\sum\limits_k e_{ij_1 k}}{n_{ij_1}(a-1)} + \sum_{j \neq j_1} \frac{\sum\limits_k e_{i_1 jk}}{n_{i_1 j}(b-1)} - \sum_{i \neq i_1} \sum_{j \neq j_1} \frac{\sum\limits_k e_{ijk}}{n_{ij}(a-1)(b-1)} .$$

Having estimated a value for the missing cell $(i_1 j_1)$ it is then possible based upon the method of unweighted means to estimate the variance components σ_r^2, σ_c^2, σ_{rc}^2 and σ_e^2 using the same procedure employed by Hirotsu. That is, equation (3.6.1) is used to compute the expected mean squares for rows, columns, and interaction. Then the estimates of the variance components are obtained by equating the expected mean squares to the computed mean squares in the analysis of variance table. However, this procedure is complicated somewhat by the fact that the vector of cell means $\underline{y}'$ (assuming for example that cell (1, 1) is missing) now is equal to $(X, \bar{y}_{12}, \ldots, \bar{y}_{1b}, \ldots, \bar{y}_{a1}, \ldots, \bar{y}_{ab})$. Thus, the variance-covariance matrix, call it V_1, of $\underline{y}'$ no longer has the simple form shown in (3.5) but will have in its first row and column the variance and covariances of X. That is,

$$V_1 = \left[\begin{array}{c|c|c|c|c}
\mathrm{Var}(X) & \mathrm{Cov}(X,\bar{y}_{12}) & \mathrm{Cov}(X,\bar{y}_{13}) & \cdots\cdots & \mathrm{Cov}(X,\bar{y}_{ab}) \\
\hline
\mathrm{Cov}(X,\bar{y}_{12}) & \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_e^2/n_{12} & \sigma_r^2 & \cdots\cdots & 0 \\
\hline
\mathrm{Cov}(X,\bar{y}_{13}) & \sigma_r^2 & \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_e^2/n_{13} & \cdots\cdots & 0 \\
\hline
\vdots & \vdots & \vdots & & \vdots \\
\hline
\mathrm{Cov}(X,\bar{y}_{ab}) & 0 & 0 & \cdots\cdots & \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_e^2/n_{ab}
\end{array}\right] \tag{3.14.1}$$

$$
= \begin{bmatrix}
V_{11} & V_{12} & V_{13} & V_{14} & \cdots & V_{1a} \\
V'_{12} & V_{22} & V_{23} & V_{23} & \cdots & V_{23} \\
V'_{13} & V'_{23} & V_{33} & V_{23} & \cdots & V_{23} \\
V'_{14} & V'_{23} & V'_{23} & V_{44} & \cdots & V_{23} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
V'_{1a} & V'_{23} & V'_{23} & V'_{23} & \cdots & V_{aa}
\end{bmatrix}
$$

where V_{11}, V_{22} ... V_{aa}; V_{12}, V_{13} ... V_{1a}; and V_{23} are all b x b matrices which are defined as

$$
V_{11} = \left[\begin{array}{c|c|c|c}
\mathrm{Var}(X) & \mathrm{Cov}(X,\bar{y}_{12}) & \cdots & \mathrm{Cov}(X,\bar{y}_{1b}) \\
\hline
\mathrm{Cov}(X,\bar{y}_{12} & \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_e^2/n_{12} & \cdots & \sigma_r^2 \\
\hline
\vdots & \vdots & & \vdots \\
\hline
\mathrm{Cov}(X,\bar{y}_{1b}) & \sigma_r^2 & \cdots & \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_e^2/n_{1b}
\end{array}\right] \tag{3.14.2}
$$

$$
V_{22} = \{\sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2/\sigma_r^2\}_b + \mathrm{diag}\left(\frac{\sigma_e^2}{n_{2j}}\right)
$$

etc. for V_{33}, ..., V_{aa} (Recall the notation $\{\ \ \}_b$ was defined in (3.3) above.)

$$V_{12} = \left[\begin{array}{c|c|c|c} Cov(X,\bar{y}_{21}) & Cov(X,\bar{y}_{22}) & \cdots & Cov(X,\bar{y}_{2b}) \\ \hline 0 & \sigma_c^2 & \cdots & 0 \\ \hline \vdots & \vdots & & \vdots \\ \hline 0 & 0 & \cdots & \sigma_c^2 \end{array}\right] \quad \text{etc. for } V_{13}, V_{14}, \ldots, V_{1a}$$

(3.14.3)

and

$$V_{23} = \sigma_c^2 \, I_b \; .$$

It is easy to show using (3.13) that

$$Var(X) = \sigma_r^2 + \sigma_c^2 + \frac{(a+b-1)}{(a-1)(b-1)} \sigma_{rc}^2 + \sigma_e^2 \left[\frac{1}{(a-1)^2} \sum_{i \neq i_1} \frac{1}{n_{ij_1}} + \frac{1}{(b-1)^2} \sum_{j \neq j_1} \frac{1}{n_{i_1 j}} + \frac{1}{(a-1)^2 (b-1)^2} \sum_{i \neq i_1} \sum_{j \neq j_1} \frac{1}{n_{ij}} \right],$$

(3.15)

$$Cov(X,\bar{y}_{ij_1}) = \sigma_c^2 + \frac{\sigma_{rc}^2}{a-1} + \frac{\sigma_e^2}{(a-1) n_{ij_1}} \qquad i \neq i_1 \quad ,$$

$$Cov(X,\bar{y}_{i_1 j}) = \sigma_r^2 + \frac{\sigma_{rc}^2}{b-1} + \frac{\sigma_e^2}{(b-1) n_{i_1 j}} \qquad j \neq j_1 \quad ,$$

and

$$Cov(X,\bar{y}_{ij}) = \frac{-\sigma_{rc}^2}{(a-1)(b-1)} + \frac{-\ \sigma_e^2}{(a-1)(b-1) n_{ij}} \qquad \begin{array}{l} i \neq i_1 \\ j \neq j_1 \end{array} \quad .$$

To illustrate the above notation, suppose that a 3 x 3 design with cell (1, 1) missing is to be used to estimate the variance components, *i.e.*, the n_{ij} arrangement is given by

X	n_{12}	n_{13}
n_{21}	n_{22}	n_{23}
n_{31}	n_{32}	n_{33}

where an X indicates that the cell has no observations. Here, using (3.13) we have

$$
\begin{aligned}
X &= \mu + r_1 + c_1 + \frac{\sum_{i\neq 1} rc_{i1}}{2} + \frac{\sum_{j\neq 1} rc_{1j}}{2} - \frac{\sum_{i\neq 1}\sum_{j\neq 1} rc_{ij}}{4} \\
&\quad + \sum_{i\neq 1} \frac{\sum_k e_{i1k}}{2n_{i1}} + \sum_{j\neq 1} \frac{\sum_k e_{1jk}}{2n_{1j}} + \sum_{i\neq 1}\sum_{j\neq 1} \frac{\sum_k e_{ijk}}{4n_{ij}}
\end{aligned} \tag{3.16}
$$

and the vector of cell means is

$$\underline{y}' = (X, \bar{y}_{12}, \bar{y}_{13}, \bar{y}_{21}, \bar{y}_{22}, \bar{y}_{23}, \bar{y}_{31}, \bar{y}_{32}, \bar{y}_{33}) .$$

Then using (3.14.1) and (3.15) the variance-covariance matrix of $\underline{y}'$ is given by

$$V_1 = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
\sigma^2_{r+c} + \frac{5}{4}\sigma^2_{rc} + \sigma^2_e\,[1] & \sigma^2_r + \frac{\sigma^2_{12}}{2} & \sigma^2_r + \frac{\sigma^2_{13}}{2} & \sigma^2_c + \frac{\sigma^2_{21}}{2} & -\frac{\sigma^2_{22}}{4} & -\frac{\sigma^2_{23}}{4} & \sigma^2_c + \frac{\sigma^2_{31}}{2} & -\frac{\sigma^2_{32}}{4} & -\frac{\sigma^2_{33}}{4} \\
\hline
\sigma^2_r + \frac{\sigma^2_{12}}{2} & \sigma^2_{r+c} + \sigma^2_{12} & \sigma^2_r & 0 & \sigma^2_c & 0 & 0 & \sigma^2_c & 0 \\
\hline
\sigma^2_r + \frac{\sigma^2_{13}}{2} & \sigma^2_r & \sigma^2_{r+c} + \sigma^2_{13} & 0 & 0 & \sigma^2_c & 0 & 0 & \sigma^2_c \\
\hline
\multicolumn{3}{c|}{\text{Sym.}} & \multicolumn{6}{c}{\{D/C\}_2 + \mathrm{diag}\left(\frac{\sigma^2_e}{n_{ij}}\right)}
\end{array}\right]$$

where

$$D = \{\sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2/\sigma_r^2\}_3 \quad \text{and} \quad C = \sigma_c^2\, I_3 \quad \text{as in (3.5)},$$

$$\sigma_{ij}^2 = \sigma_{rc}^2 + \frac{\sigma_e^2}{n_{ij}}, \quad \sigma_{r+c}^2 = \sigma_r^2 + \sigma_c^2 \quad \text{and}$$

$$[1] = \frac{1}{(2)^2} \sum_{i\neq 1} \frac{1}{n_{i1}} + \frac{1}{(2)^2} \sum_{j\neq 1} \frac{1}{n_{1j}} + \frac{1}{(2)^2(2)^2} \sum_{i\neq 1} \sum_{j\neq 1} \frac{1}{n_{ij}} .$$

$$= \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}' & V_{22} & V_{23} \\ V_{13}' & V_{23}' & V_{33} \end{bmatrix} .$$

Having set up V_1, the expected mean square for rows in the general a x b design (assuming cell (1,1) is the missing cell) is calculated as follows:

$$E(MSR) = tr(V_1 \cdot G_R)$$

$$= tr \begin{bmatrix} V_{11} & V_{12} & V_{13} & \cdots & V_{1a} \\ V_{12}' & V_{22} & V_{23} & \cdots & V_{23} \\ V_{13}' & V_{23}' & V_{33} & \cdots & V_{23} \\ \vdots & \vdots & \vdots & & \vdots \\ V_{1a}' & V_{23}' & V_{23}' & \cdots & V_{aa} \end{bmatrix} \cdot \begin{bmatrix} P & Q & Q & \cdots & Q \\ Q & P & Q & \cdots & Q \\ Q & Q & P & \cdots & Q \\ \vdots & \vdots & \vdots & & \vdots \\ Q & Q & Q & \cdots & P \end{bmatrix}$$

$$= tr[(V_{11} + V_{22} + V_{33} + \ldots V_{aa}) \cdot P]$$

$$+ tr[(2(V_{12} + V_{13} + \ldots V_{1a}) + (a-1)(a-2)\ V_{23}) \cdot Q] \qquad (3.17)$$

where P and Q were defined in (3.3). Now from (3.14.2) and (3.14.3)

$V_{11} + V_{22} + \dots + V_{aa} =$

$$\begin{bmatrix}
a\,\sigma^2_{r+c} + \sigma^2_{rc}\,[1] + \sigma^2_e\,[2] & a\,\sigma^2_r + \frac{\sigma^2_{12}}{(b-1)} & a\,\sigma^2_r + \frac{\sigma^2_{13}}{(b-1)} & \cdots & a\,\sigma^2_r + \frac{\sigma^2_{1b}}{(b-1)} \\
a\,\sigma^2_r + \frac{\sigma^2_{12}}{(b-1)} & a\,\sigma^2_{r+c} + a\,\sigma^2_{rc} + \sigma^2_e \sum_i \frac{1}{n_{i2}} & a\,\sigma^2_r & \cdots & a\,\sigma^2_r \\
a\,\sigma^2_r + \frac{\sigma^2_{13}}{(b-1)} & a\,\sigma^2_r & a\,\sigma^2_{r+c} + a\,\sigma^2_{rc} + \sigma^2_e \sum_i \frac{1}{n_{i3}} & \cdots & a\,\sigma^2_r \\
\vdots & \vdots & \vdots & & \vdots \\
a\,\sigma^2_r + \frac{\sigma^2_{1b}}{(b-1)} & a\,\sigma^2_r & a\,\sigma^2_r & & a\,\sigma^2_{r+c} + a\,\sigma^2_{rc} + \sigma^2_e \sum_i \frac{1}{n_{ib}}
\end{bmatrix} \tag{3.18}$$

where $[1] = \frac{a + b - 1}{(a-1)(b-1)} + (a-1)$,

$$[2] = \frac{a^2 - 2a + 2}{(a-1)^2} \sum_{i \neq 1} \frac{1}{n_{i1}} + \frac{1}{(b-1)^2} \sum_{j \neq 1} \frac{1}{n_{1j}} + \frac{1}{(a-1)^2 (b-1)^2} \sum_{i \neq 1} \sum_{j \neq 1} \frac{1}{n_{ij}},$$

$$\sigma^2_{ij} = \sigma^2_{rc} + \frac{\sigma^2_e}{n_{ij}} \quad \text{and} \quad \sigma^2_{r+c} = \sigma^2_r + \sigma^2_c \ ,$$

$V_{12} + V_{13} + \dots V_{1a} =$

$$\left[\begin{array}{c|c|c|c}
(a-1)\sigma_c^2 + \sigma_{rc}^2 + \dfrac{\sigma_e^2}{(a-1)} \sum\limits_{i \neq 1} \dfrac{1}{n_{i1}} & - \dfrac{\left[(a-1)\sigma_{rc}^2 + \sigma_e^2 \sum\limits_{i \neq 1} \dfrac{1}{n_{i2}}\right]}{(a-1)(b-1)} & \cdots & - \dfrac{\left[(a-1)\sigma_{rc}^2 + \sigma_e^2 \sum\limits_{i \neq 1} \dfrac{1}{n_{ib}}\right]}{(a-1)(b-1)} \\
\hline
0 & (a-1)\sigma_c^2 & \cdots & 0 \\
\hline
\vdots & \vdots & & \vdots \\
\hline
0 & 0 & \cdots & (a-1)\sigma_c^2
\end{array}\right]$$

and $V_{23} = \sigma_c^2 I_b$. Therefore,

$$\mathrm{tr}\left[(V_{11} + V_{22} + \ldots + V_{aa}) \cdot P \right]$$

$$= \frac{1}{ab}\left[ab^2\sigma_r^2 + ab\ \sigma_c^2 + \sigma_{rc}^2 \left(\frac{a+b-1}{(a-1)(b-1)} + a-1 + \frac{2(b-1)}{b-1} + a(b-1) \right) \right.$$

$$\left. + \sigma_e^2 \left([2] + \sum_i \sum_{j\neq 1} \frac{1}{n_{ij}} + \frac{2}{b-1} \sum_{j\neq 1} \frac{1}{n_{1j}} \right) \right] ,$$

$$\mathrm{tr}\left[2(V_{12} + V_{13} + \ldots + V_{1a}) \cdot Q \right]$$

$$= \frac{-2}{ab(a-1)}\left[(a-1)b\sigma_c^2 + \sigma_e^2 \left(\frac{1}{(a-1)} \sum_{i\neq 1} \frac{1}{n_{i1}} \right.\right.$$

$$\left.\left. - \frac{1}{(a-1)(b-1)} \sum_{i\neq 1} \sum_{j\neq 1} \frac{1}{n_{ij}} \right) \right]$$

and

$$\mathrm{tr}\ [(a-1)(a-2)\ V_{23} \cdot Q] = \frac{-(a-1)(a-2)\ b}{ab(a-1)} \sigma_c^2 .$$

Accordingly,

$$\mathrm{tr}(V_1 \cdot G_R) = b\sigma_r^2 + \sigma_c^2(1-1) + \frac{\sigma_{rc}^2}{ab} \left(\frac{a+b-1}{(a-1)(b-1)} + ab + 1 \right)$$

$$+ \frac{\sigma_e^2}{ab} \left(\frac{1}{(a-1)^2} \sum_{i\neq 1} \frac{1}{n_{i1}} + \frac{1}{(b-1)^2} \sum_{j\neq 1} \frac{1}{n_{1j}} \right.$$

$$+ \frac{1}{(a-1)^2(b-1)^2} \sum_{i\neq 1} \sum_{j\neq 1} \frac{1}{n_{ij}}$$

$$+ \sum_{i\neq 1} \frac{1}{n_{i1}} + \sum_{i\neq 1} \sum_{j\neq 1} \frac{1}{n_{ij}} + \sum_{j\neq 1} \frac{1}{n_{1j}}$$

$$+ \frac{2}{(b-1)} \sum_{j\neq 1} \frac{1}{n_{1j}} - \frac{2}{(a-1)^2} \sum_{i\neq 1} \frac{1}{n_{i1}}$$

$$+ \frac{2}{(a-1)^2(b-1)} \sum_{i\neq 1} \sum_{j\neq 1} \frac{1}{n_{ij}} \Bigg)$$

$$= b\sigma_r^2 + \sigma_{rc}^2 \left(1 + \frac{1}{(a-1)(b-1)}\right) \tag{3.19}$$

$$+ \frac{\sigma_e^2}{ab} \Bigg[\sum_{i\neq 1} \sum_{j\neq 1} \frac{1}{n_{ij}} \left(1 + \frac{2b-1}{(a-1)^2(b-1)^2}\right)$$

$$+ \sum_{i\neq 1} \frac{1}{n_{i1}} \left(1 - \frac{1}{(a-1)^2}\right) + \sum_{j\neq 1} \frac{1}{n_{1j}} \left(1 + \frac{2b-1}{(b-1)^2}\right) \Bigg] .$$

Similarly, the expected mean square for columns is

$$E(MSC) = tr(V_1 \cdot G_C)$$

$$= tr[(V_{11} + V_{22} + \cdots + V_{aa}) \cdot R]$$

$$+ tr[(2(V_{12} + V_{13} + \ldots + V_{1a}) + (a-1)(a-2)V_{23}) \cdot R]$$

where R was defined in (3.4). Here

$$tr[(V_{11} + V_{22} + \ldots + V_{aa}) \cdot R]$$

$$= \frac{1}{a(b-1)} \Bigg[\sigma_r^2 \Big(a(b-1) - a(b-1) \Big) + a(b-1)\ \sigma_c^2$$

$$+ \sigma_{rc}^2 \left(\frac{a+b-1}{(a-1)b} + \frac{(a-1)(b-1)}{b} - \frac{2(b-1)}{b(b-1)} + \frac{a(b-1)^2}{b} \right)$$

$$+ \sigma_e^2 \left(\frac{(b-1)}{b} [2] + \frac{b-1}{b} \sum_i \sum_{j\neq 1} \frac{1}{n_{ij}} - \frac{2}{b(b-1)} \sum_{j\neq 1} \frac{1}{n_{1j}} \right) \Bigg]$$

$$\mathrm{tr}[2(V_{12} + V_{13} + \ldots + V_{1a})\cdot R]$$

$$= \frac{2}{a(b-1)}\left[(a-1)(b-1)\sigma_c^2 + \sigma_{rc}^2\left(\frac{b-1}{b} + \frac{1}{b}\right)\right.$$

$$\left. + \sigma_e^2\left(\frac{b-1}{(a-1)b}\sum_{i\neq 1}\frac{1}{n_{i1}} + \frac{1}{(a-1)b(b-1)}\sum_{i\neq 1}\sum_{j\neq 1}\frac{1}{n_{ij}}\right)\right]$$

and

$$\mathrm{tr}[(a-1)(a-2)\ V_{23}\cdot R] = \frac{(a-1)(a-2)}{a}\sigma_c^2 .$$

Thus, the expected mean square for columns can be shown to be

$$E(MSC) = a\sigma_c^2 + \sigma_{rc}^2\left(1 + \frac{1}{(a-1)(b-1)}\right) + \frac{\sigma_e^2}{ab}\left[\sum_{i\neq 1}\sum_{j\neq 1}\frac{1}{n_{ij}}\left(1\right.\right.$$

$$\left. + \frac{2a-1}{(a-1)^2(b-1)^2}\right) + \sum_{i\neq 1}\frac{1}{n_{i1}}\left(1 + \frac{2a-1}{(a-1)^2}\right)$$

$$\left. + \sum_{j\neq 1}\frac{1}{n_{1j}}\left(1 - \frac{1}{(b-1)^2}\right)\right] . \qquad (3.20)$$

Finally, the expected mean square for interaction is

$$E(MSRC) = \mathrm{tr}(V_1\cdot G_{RC1})$$

where $G_{RC1} = \frac{(a-1)(b-1)}{(a-1)(b-1)-1}G_{RC}$. G_{RC1} must be used in the present case instead of G_{RC} (defined in (3.4)) because the missing cell takes one degree of freedom away from interaction. Proceeding in the same manner as for the expected mean squares of rows and columns it can be shown that

$$E(MSRC) = \frac{(a-1)(b-1)}{ab[(a-1)(b-1)-1]}\left[\sum_{i\neq 1}\sum_{j\neq 1}\frac{1}{n_{ij}}\left(1-\frac{1}{(a-1)^2(b-1)^2}\right)\right. \tag{3.21}$$

$$\left. + \sum_{i\neq 1}\frac{1}{n_{i1}}\left(1-\frac{1}{(a-1)^2}\right) + \sum_{j\neq 1}\frac{1}{n_{1j}}\left(1-\frac{1}{(b-1)^2}\right)\right]\sigma_e^2 + \sigma_{rc}^2 .$$

Table 3.1 gives a summary of the preceding work on expected mean squares for the case of one missing cell. Examination of Table 3.1 shows that as a and b become large

$$K1 \doteq K3 \doteq K5 \doteq \frac{1}{ab}\left[\Sigma_1 + \Sigma_2 + \Sigma_3\right] = \frac{1}{ab}\sum_{\substack{i\ j \\ \neq(i_1 j_1)}}\sum \frac{1}{n_{ij}}$$

and K2 $\doteq$ K4 $\doteq$ 1, i.e. the coefficients of the expected mean squares approach those shown in (3.7) above for the case of no missing cells. Also, if the number of observations in each cell of the design are equal (i.e. all $n_{ij} = n$) then

$$\Sigma_1 = \frac{(a-1)(b-1)}{n}, \quad \Sigma_2 = \frac{(a-1)}{n} \quad \text{and} \quad \Sigma_3 = \frac{b-1}{n} .$$

In this case the expected mean squares in Table 3.1 reduce to

$$E(MSR) = \frac{1}{n}\left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_{rc}^2 + b\sigma_r^2$$

$$E(MSC) = \frac{1}{n}\left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_{rc}^2 + a\sigma_c^2$$

$$E(MSRC) = \frac{1}{n}\sigma_e^2 + \sigma_{rc}^2$$

$$E(MSE) = \sigma_e^2 .$$

Table 3.1 Expected mean squares for a two-way disproportionate design with one missing cell using the method of unweighted means

Source	Coefficients of Variance Components in E(MS) σ_e^2	σ_{rc}^2	σ_c^2	σ_r^2
Rows	K1 $= \frac{1}{ab}\left[\left(1+\frac{2b-1}{(a-1)^2(b-1)^2}\right)\cdot\Sigma_1 + \left(1-\frac{1}{(a-1)^2}\right)\cdot\Sigma_2 + \left(1+\frac{2b-1}{(b-1)^2}\right)\cdot\Sigma_3\right]$	K2 $= 1+\frac{1}{(a-1)(b-1)}$		b
Columns	K3 $= \frac{1}{ab}\left[\left(1+\frac{2a-1}{(a-1)^2(b-1)^2}\right)\cdot\Sigma_1 + \left(1+\frac{2a-1}{(a-1)^2}\right)\cdot\Sigma_2 + \left(1-\frac{1}{(b-1)^2}\right)\cdot\Sigma_3\right]$	K4 $= 1+\frac{1}{(a-1)(b-1)}$	a	
Interaction	K5 $= \frac{C_1}{ab}\left[\left(1-\frac{1}{(a-1)^2(b-1)^2}\right)\cdot\Sigma_1 + \left(1-\frac{1}{(a-1)^2}\right)\cdot\Sigma_2 + \left(1-\frac{1}{(b-1)^2}\right)\cdot\Sigma_3\right]$	1		
Error	1			

where the missing cell is $(i_1 j_1)$ and

$$\Sigma_1 = \sum_{i\neq i_1}\sum_{j\neq j_1}\frac{1}{n_{ij}},\ \Sigma_2 = \sum_{i\neq i_1}\frac{1}{n_{ij_1}},\ \Sigma_3 = \sum_{j\neq j_1}\frac{1}{n_{i_1 j}} \text{ and } C_1 = \frac{(a-1)(b-1)}{(a-1)(b-1)-1}.$$

As mentioned previously, the results given in Table 3.1 can be used to estimate the variance components by setting the expected mean squares equal to the computed mean squares. This gives

$$\hat{\sigma}_r^2 = \frac{1}{b}\,[\text{MSR} - \text{K2}\cdot\text{MSRC} + (\text{K2}\cdot\text{K5} - \text{K1})\ \text{MSE}]$$

$$= \frac{1}{b}\,[\underline{y}'\ G_R\underline{y} - \text{K2}\cdot\underline{y}'\ G_{RC1}\underline{y}] + \frac{(\text{K2}\cdot\text{K5} - \text{K1})}{b}\cdot\text{MSE}$$

$$= \frac{1}{b}\,[\underline{y}'\ (G_R - \text{K2}\cdot G_{RC1})\ \underline{y}] + \frac{(\text{K2}\cdot\text{K5} - \text{K1})}{b}\cdot\text{MSE} \tag{3.22}$$

$$\hat{\sigma}_c^2 = \frac{1}{a}\,[\underline{y}'\ (G_C - \text{K4}\cdot G_{RC1})\ \underline{y}] + \frac{(\text{K4}\cdot\text{K5} - \text{K3})}{a}\cdot\text{MSE}$$

$$\hat{\sigma}_{rc}^2 = \underline{y}'\ G_{RC1}\underline{y} - \text{K5}\cdot\text{MSE} \qquad \text{and}$$

$$\hat{\sigma}_e^2 = \text{MSE}\ .$$

In the case of all $n_{ij} = n$ the equations in (3.22) will reduce to

$$\hat{\sigma}_r^2 = \frac{\text{MSR} - \left[1 + \frac{1}{(a-1)(b-1)}\right]\text{MSRC}}{b}$$

$$\hat{\sigma}_c^2 = \frac{\text{MSC} - \left[1 + \frac{1}{(a-1)(b-1)}\right]\text{MSRC}}{a}$$

$$\hat{\sigma}_{rc}^2 = \text{MSRC} - \frac{\text{MSE}}{n} \qquad \text{and}$$

$$\hat{\sigma}_e^2 = \text{MSE}\ .$$

As a notational convenience, the estimating procedure used to obtain the estimates in (3.22) will be called Procedure M.

The estimates of the variance components given in (3.22) have the property that they are unbiased estimates. Another set of estimates for the variance components which are biased but have relatively simple computational formulas can be obtained by essentially ignoring the missing cell and using the following approximate expected mean squares:

$$\widehat{E(MSR)} = \left(\frac{1}{s} \sum_{i} \sum_{\substack{j \\ \neq (i_1 j_1)}} \frac{1}{n_{ij}} \right) \sigma_e^2 + \sigma_{rc}^2 + b\sigma_r^2$$

$$\widehat{E(MSC)} = \left(\frac{1}{s} \sum_{i} \sum_{\substack{j \\ \neq (i_1 j_1)}} \frac{1}{n_{ij}} \right) \sigma_e^2 + \sigma_{rc}^2 + a\sigma_c^2 \tag{3.23}$$

$$\widehat{E(MSRC)} = \left(\frac{1}{s} \sum_{i} \sum_{\substack{j \\ \neq (i_1 j_1)}} \frac{1}{n_{ij}} \right) \sigma_e^2 + \sigma_{rc}^2$$

$$E(MSE) = \sigma_e^2$$

where s is the number of occupied cells in the design. These approximate expected mean squares are the same as those given in (3.7) for the case of no missing cells except that in the present case (harmonic mean)$^{-1}$, $\frac{1}{ab} \sum_i \sum_j \frac{1}{n_{ij}}$, is replaced by $\frac{1}{s} \sum_{i} \sum_{\substack{j \\ \neq (i_1 j_1)}} \frac{1}{n_{ij}}$ (denoted by $\hat{K}$).

The term $\hat{K}$ is used because it gives a better approximation to K5, the correct coefficient for σ_e^2 in E(MSRC), than does (harmonic mean)$^{-1}$ (see Chapter 5); and therefore, reduces the bias in the estimate of σ_{rc}^2 (see (3.25) below). That $\hat{K}$ is a better approximation to K5 than (harmonic mean)$^{-1}$ is easy to see in the case of all $n_{ij} = n$ where

$$\frac{1}{ab}\sum_i\sum_j \frac{1}{n_{ij}} = \frac{1}{ab}\left[\frac{1}{1} + \sum_{\substack{i\ j \\ \neq(i_1 j_1)}}\sum \frac{1}{n_{ij}}\right] = \frac{1}{ab}\left[\frac{1}{1} + \frac{ab-1}{n}\right]$$

$$= \frac{1}{n} + \frac{1}{ab}\left(1 - \frac{1}{n}\right) \qquad \text{and}$$

$$\hat{K} = \frac{1}{s}\sum_{\substack{i\ j \\ \neq(i_1 j_1)}}\sum \frac{1}{n_{ij}} = \frac{1}{ab-1}\left(\frac{ab-1}{n}\right) = \frac{1}{n} = K5 \ .$$

Setting the approximate expected mean squares in (3.23) equal to the computed mean squares gives the following biased estimates of the variance components (the estimating procedure used to obtain these estimates will be called Procedure $\hat{M}$):

$$\hat{\hat{\sigma}}_r^2 = \frac{MSR - MSRC}{b}$$

$$\hat{\hat{\sigma}}_c^2 = \frac{MSC - MSRC}{a}$$

$$\hat{\hat{\sigma}}_{rc}^2 = MSRC - \left(\frac{1}{s}\sum_{\substack{i\ j \\ \neq(i_1 j_1)}}\sum \frac{1}{n_{ij}}\right) MSE \qquad \text{and} \tag{3.24}$$

$$\hat{\hat{\sigma}}_e^2 = \hat{\sigma}_e^2 = MSE \ .$$

The biases for the Procedure $\hat{M}$ estimates given in (3.24) are found by taking expectations and using the results given in Table 3.1, i.e.,

$$E(\hat{\hat{\sigma}}_r^2) - \sigma_r^2 = \frac{1}{b}\left[K1\cdot\sigma_e^2 + K2\cdot\sigma_{rc}^2 + b\sigma_r^2 - K5\cdot\sigma_e^2 - \sigma_{rc}^2\right] - \sigma_r^2$$

$$= \left(\frac{K2-1}{b}\right)\sigma_{rc}^2 + \left(\frac{K1-K5}{b}\right)\sigma_e^2$$

$$E(\hat{\sigma}_c^2) - \sigma_c^2 = \frac{1}{a}\left[K3\cdot\sigma_e^2 + K4\cdot\sigma_{rc}^2 + a\sigma_c^2 - K5\cdot\sigma_e^2 - \sigma_{rc}^2 \right] - \sigma_c^2$$

$$= \left(\frac{K4-1}{a}\right)\sigma_{rc}^2 + \left(\frac{K3-K5}{a}\right)\sigma_e^2 \qquad \text{and} \tag{3.25}$$

$$E(\hat{\sigma}_{rc}^2) - \sigma_{rc}^2 = K5\cdot\sigma_e^2 + \sigma_{rc}^2 - \hat{K}\cdot\sigma_e^2 - \sigma_{rc}^2$$

$$= (K5 - \hat{K})\ \sigma_e^2 \ .$$

In general it does not appear possible to state that the biases given in (3.25) are always positive or negative. However, if $\sigma_{rc}^2 \gg \sigma_e^2$ then the biases in $\hat{\sigma}_r^2$ and $\hat{\sigma}_c^2$ will be positive since $(K2-1)/b$ and $(K4-1)/a$ are always greater than zero. Also, since $\hat{K}$ is usually very close to K5 the bias in $\hat{\sigma}_{rc}^2$ will be quite small. In Chapter 5 a detailed numerical study is made of the biases in (3.25) for several different designs.

3.4 Variances of Estimated Variance Components with One Missing Cell

The variances of the estimated variance components for Procedure M given in (3.22) can be computed by using equation (3.6.2). This computation is simplified by the fact that the error sum of squares is independent of the sum of squares for rows, columns and interaction. That is, the within cells sum of squares is independent of all between cells sum of squares. Also, the sum of squares for error divided by σ_e^2 (i.e. SSE/σ_e^2) has a χ^2 distribution with N-s degrees of freedom (where again s is equal to the number of occupied cells in the design). Thus, the variance of the error mean square is $2\sigma_e^4/(n-s)$. Accordingly,

$$\mathrm{Var}(\hat{\sigma}_r^2) = \frac{2}{b^2}\,\mathrm{tr}\left[V_1\cdot(G_R - K2\cdot G_{RC1})\right]^2 + \frac{(K2\cdot K5 - K1)^2}{b^2}\cdot\frac{2\sigma_e^4}{n-s}$$

$$\mathrm{Var}(\hat{\sigma}_c^2) = \frac{2}{a^2}\,\mathrm{tr}\left[V_1\cdot(G_C - K4\cdot G_{RC1})\right]^2 + \frac{(K4\cdot K5 - K3)^2}{a^2}\cdot\frac{2\sigma_e^4}{n-s}$$

$$\mathrm{Var}(\hat{\sigma}_{rc}^2) = 2\mathrm{tr}\left[V_1\cdot G_{RC1}\right]^2 + (K5)^2\cdot\frac{2\sigma_e^4}{n-s}\,, \qquad (3.26)$$

In attempting to reduce the equations given in (3.26) to an analytic form such as that obtained by Hirotsu (see (3.9)) the variance of $\hat{\sigma}_{rc}^2$ was looked at initially since it had the easiest computational formula. However, the matrix algebra involved in reducing this equation was quite tedious and only the result of the computation will be given here, namely:

$$\begin{aligned}\mathrm{Var}(\hat{\sigma}_{rc}^2) = C_1^2 \Big(& 2\sigma_{rc}^4\left[(b-1)T1 + (a-1)T2 + (a-1)(b-1)T3\right] \\ & + 4\sigma_{rc}^2\sigma_e^2\left[T1\cdot\Sigma_3 + T2\cdot\Sigma_2 + T3\cdot\Sigma_1\right] \\ & + 2\sigma_e^4\left[T1\cdot\Sigma_6 + T2\cdot\Sigma_5 + T3\cdot\Sigma_4\right]\Big) \\ & + (K5)^2\,\frac{2\sigma_e^4}{n-s}\end{aligned} \qquad (3.27)$$

where

$$T_1 = \frac{(b-2)}{a(a-1)(b-1)^3}\,, \qquad T2 = \frac{(a-2)}{b(b-1)(a-1)^3}\,, \qquad \text{and}$$

$$T3 = \frac{(b-2)}{a^2(a-1)^2(b-1)^3} + \frac{(a-2)}{b^2(a-1)^3(b-1)^2}$$

$$+ \frac{1}{a^2b^2}\left[\left(1 - \frac{1}{(a-1)^2(b-1)^2}\right)^2 + \frac{a-2}{(a-1)^2}\left(1 + \frac{1}{(a-1)(b-1)^2}\right)^2\right.$$

$$+ \frac{(b-2)}{(b-1)^2}\left(1 + \frac{1}{(a-1)^2(b-1)}\right)^2$$

$$\left. + \frac{(a-2)(b-2)}{(a-1)^2(b-1)^2}\left(1 - \frac{1}{(a-1)(b-1)}\right)^2\right] ;$$

Σ_1 , Σ_2 , Σ_3 are defined in Table 3.1; and

$$\Sigma_4 = \sum_{i \neq i_1} \sum_{j \neq j_1} \frac{1}{n_{ij}^2} , \qquad \Sigma_5 = \sum_{i \neq i_1} \frac{1}{n_{ij_1}^2} , \qquad \text{and}$$

$$\Sigma_6 = \sum_{j \neq j_1} \frac{1}{n_{i_1 j}^2} , \qquad \text{and} \qquad C_1 = \frac{(a-1)(b-1)}{(a-1)(b-1) - 1} .$$

Unfortunately, the matrix algebra that would have been involved in solving analytically for $\text{Var}(\hat{\sigma}_r^2)$ and $\text{Var}(\hat{\sigma}_c^2)$ in (3.26) was enormous and it was not felt that it was worthwhile to carry it out in view of the complex formula obtained in (3.27) for $\text{Var}(\hat{\sigma}_{rc}^2)$. Instead, it was decided to write a computer program that would solve the equations given in (3.26) numerically for the desired variances given a particular design (i.e. a particular n_{ij} pattern). Using this computer program it would then be possible to calculate the variances of estimated variance components obtained by Procedure M for any desired design with one missing cell. In addition, the computer program could also be used to obtain the variances of estimated variance components for designs which had more than one missing cell (see Chapter 4). Chapter 5 below describes the computer program (called VC) that was written to estimate these variances and the numerical results obtained by running VC with several different design patterns.

In addition to deriving the variance equations in (3.26), equation (3.6.2) can also be used to obtain the variances of the estimated variance components for Procedure $\hat{M}$ given in (3.24). The result of this computation is

$$\mathrm{Var}(\hat{\hat{\sigma}}_r^2) = \frac{2}{b^2}\,\mathrm{tr}\left[V_1\cdot(G_R - G_{RC1})\right]^2$$

$$\mathrm{Var}(\hat{\hat{\sigma}}_c^2) = \frac{2}{a^2}\,\mathrm{tr}\left[V_1\cdot(G_C - G_{RC1})\right]^2 \quad \text{and} \tag{3.28}$$

$$\mathrm{Var}(\hat{\hat{\sigma}}_{rc}^2) = 2\mathrm{tr}\left[V_1\cdot G_{RC1}\right]^2 + (\hat{K})^2\,\frac{2\sigma_e^4}{n-s} \quad .$$

Here the $\mathrm{Var}(\hat{\hat{\sigma}}_{rc}^2)$ can be reduced to the analytic form given for $\mathrm{Var}(\hat{\sigma}_{rc}^2)$ in (3.27) with K5 replaced with $\hat{K}$. As was the case in (3.26) the $\mathrm{Var}(\hat{\hat{\sigma}}_r^2)$ and $\mathrm{Var}(\hat{\hat{\sigma}}_c^2)$ were not reduced due to the matrix algebra involved in these calculations.

Using (3.25), (3.26) and (3.28) the differences between the variances of the unbiased estimates for Procedure M given in (3.22) and the mean square errors[2] of the biased estimates for Procedure $\hat{M}$ in

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[2]The mean square error (M.S.E.) of an estimator $\hat{\sigma}^2$ of the parameter σ^2 is defined as

$$\mathrm{M.S.E.}(\hat{\sigma}^2) = E(\hat{\sigma}^2 - \sigma^2)^2 = \mathrm{Var}(\hat{\sigma}^2) + \mathrm{bias}^2(\hat{\sigma}^2) \quad .$$

(3.24) can be shown to be[3]

$$\mathrm{Var}(\hat{\sigma}_r^2) - \mathrm{M.S.E.}(\hat{\hat{\sigma}}_r^2)$$

$$= \frac{1}{b^2}\Big[(K2^2-1)\ \mathrm{Var}(MSRC) + (K2\cdot K5-K1)^2\ \mathrm{Var}(MSE)$$

$$- 2(K2-1)\ \mathrm{Cov}(MSR, MSRC) - (K2-1)^2\ \sigma_{rc}^4$$

$$- (K1-K5)^2\ \sigma_e^4 - 2(K2-1)(K1-K5)\ \sigma_{rc}^2\sigma_e^2 \Big] \tag{3.29}$$

$$= \frac{1}{b^2}\Big[2(K2^2-1)\ \mathrm{tr}(V_1\cdot G_{RC1})^2 - 4(K2-1)\ \mathrm{tr}(V_1\cdot G_R)\cdot(V_1\cdot G_{RC1})$$

$$- (K2-1)^2\ \sigma_{rc}^4 - 2(K2-1)(K1-K5)\ \sigma_{rc}^2\ \sigma_e^2$$

$$+\left(\frac{2(K2\cdot K5-K1)^2}{n-s} - (K1-K5)^2 \right) \sigma_e^4 \Big]$$

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[3]It should be noted here that in general

$$\mathrm{Cov}(MSR, MSRC) = 2\mathrm{tr}(V_1\cdot G_R)\cdot(V_1\cdot G_{RC1}) + 4\underline{\mu}'G_RV_1G_{RC1}\underline{\mu} \quad .$$

However, as for the case of $E(Z) = E(\underline{y}'G\underline{y})$ given in (3.6.1) $G_R\underline{\mu} = G_{RC1}\underline{\mu} = 0$ which implies

$$\mathrm{Cov}(MSR, MSRC) = 2\mathrm{tr}(V_1\cdot G_R)\cdot(V_1\cdot G_{RC1}) \quad .$$

A similar result holds for Cov(MSC, MSRC) .

$$\mathrm{Var}(\hat{\sigma}_c^2) - \mathrm{M.S.E.}(\hat{\hat{\sigma}}_c^2)$$

$$= \frac{1}{a^2}\Big[(K4^2-1)\ \mathrm{Var(MSRC)} + (K4.K5-K3)^2\ \mathrm{Var(MSE)}$$

$$- 2(K4-1)\ \mathrm{Cov(MSC,\ MSRC)} - (K4-1)^2\ \sigma_{rc}^4$$

$$- (K3-K5)^2\ \sigma_e^4 - 2(K4-1)(K3-K5)\ \sigma_{rc}^2\ \sigma_e^2 \Big]$$

$$= \frac{1}{a^2}\Big[2(K4^2-1)\ \mathrm{tr}(V_1 \cdot G_{RC1})^2 - 4(K4-1)\ \mathrm{tr}(V_1 \cdot G_C)\cdot(V_1 \cdot G_{RC1})$$

$$- (K4-1)^2\ \sigma_{rc}^4 - 2(K4-1)(K3-K5)\ \sigma_{rc}^2\ \sigma_e^2$$

$$+ \left(\frac{2(K4\cdot K5-K3)^2}{n-s} - (K3-K5)^2 \right) \sigma_e^4 \Big] \qquad \text{and}$$

$$\mathrm{Var}(\hat{\sigma}_{rc}^2) - \mathrm{M.S.E.}(\hat{\hat{\sigma}}_{rc}^2) = (K5^2-\hat{K}^2)\ \frac{2\sigma_e^4}{n-s} - (K5-\hat{K})^2\ \sigma_e^4 \ .$$

Again, the matrix algebra involved in reducing the first two of these equations was not considered worthwhile. Instead, the computer program VC was modified to solve the equations in (3.28) numerically for a particular design. It was then possible to compare the mean square errors of the estimates for Procedure $\hat{M}$ with the variances of the estimates for Procedure M for a given design. The results of this comparison for several different designs are also discussed in Chapter 5.

4. ANALYTICAL RESULTS FOR MORE THAN ONE MISSING CELL

4.1 Introduction

In this chapter the results of Chapter 3 for computing expected mean squares and estimating variance components based upon the method of unweighted means will be extended to the case of two or more cells missing from a two-way disproportionate design. The method of analysis is essentially the same as that used in Chapter 3 but the results obtained are more complex especially those for missing cells in different rows and columns. No attempt is made in this chapter to determine analytic formulas for the variances of the estimated variance components due to their complexity; instead, Chapter 5 will examine these variances numerically. Initially, in Section 4.2 designs with missing cells in the same row (or column) will be studied. Then in Section 4.3 designs with missing cells in different rows and columns are examined.

4.2 Two or More Missing Cells in the Same Row (or Column)

First, it will be assumed that we have a design with two missing cells in the same row. Denote the two missing cells by (i_1j_1) and (i_1j_2). As discussed in Chapter 3 a missing value formula must first be used to estimate an observation for each of these missing cells before the method of unweighted means can be applied. Burrows [1966] gives formulas for obtaining one or more missing values. The formulas for missing values can be derived by minimizing the sum of squares for interaction with respect to the missing values. That is, assume the usual model for a randomized complete block design,

$$y_{ij} = \mu + t_i + b_j + tb_{ij}$$

$$i = 1, 2, \ldots, a; \quad j = 1, 2, \ldots, b$$

with the usual definitions for μ, t_i, b_j and tb_{ij}. Let two cells in the same row be missing from this design and denote the missing values for these cells by $X_{i_1j_1}$ and $X_{i_1j_2}$. Then the sum of squares for interaction is given by

$$SS(TB) = \sum_{ij} y_{ij}^2 - \frac{\sum_i T_i^2}{b} - \frac{\sum_j B_j^2}{a} + \frac{G^2}{ab} \tag{4.1}$$

where

$$T_i = \sum_j y_{ij} \qquad i \neq i_1$$

$$B_j = \sum_i y_{ij} \qquad j \neq j_1, j_2$$

$$T_{i_1} = \sum_{j \neq j_1, j_2} y_{i_1j} + X_{i_1j_1} + X_{i_1j_2} = T_{i_10} + X_{i_1j_1} + X_{i_1j_2}$$

$$B_{j_1} = \sum_{i \neq i_1} y_{ij_1} + X_{i_1j_1} = B_{j_10} + X_{i_1j_1}$$

$$B_{j_2} = \sum_{i \neq i_1} y_{ij_2} + X_{i_1j_2} = B_{j_20} + X_{i_1j_2}$$

and

$$G = \sum_{\substack{i \\ \neq(i_1j_1)}} \sum_{\substack{j \\ (i_1j_2)}} y_{ij} + X_{i_1j_1} + X_{i_1j_2} = G_0 + X_{i_1j_1} + X_{i_1j_2} \ .$$

Taking the partial derivatives of (4.1) with respect to $X_{i_1j_1}$ and $X_{i_1j_2}$ gives

$$\frac{\partial(SSTB)}{\partial X_{i_1j_1}} = 2\left[X_{i_1j_1} - \frac{T_{i_10} + X_{i_1j_1} + X_{i_1j_2}}{b} - \frac{B_{j_10} + X_{i_1j_1}}{a} + \frac{G_0 + X_{i_1j_1} + X_{i_1j_2}}{ab}\right] \tag{4.2}$$

and

$$\frac{\partial(SSTB)}{\partial X_{i_1j_2}} = 2\left[X_{i_1j_2} - \frac{T_{i_10} + X_{i_1j_1} + X_{i_1j_2}}{b} - \frac{B_{j_20} + X_{i_1j_2}}{a} + \frac{G_0 + X_{i_1j_1} + X_{i_1j_2}}{ab}\right] . \tag{4.3}$$

Note, that the second derivatives of (4.1) with respect to $X_{i_1j_1}$ and $X_{i_1j_2}$ are positive. Then setting (4.2) and (4.3) equal to zero and solving for $X_{i_1j_1}$ and $X_{i_1j_2}$ we obtain the formulas for the missing values given by Burrows for minimizing SSTB

$$X_{i_1j_1} = \frac{a\ T_{i_10} + (b-1)\ B_{j_10} + B_{j_20} - G_0}{(a-1)(b-2)} \tag{4.4}$$

and

$$X_{i_1j_2} = X_{i_1j_1} + \frac{B_{j_20} - B_{j_10}}{a-1} .$$

Now (4.4) can also be used to estimate two missing values in the same row for a two-way disproportionate design after having replaced the observations in each cell by the cell mean (equation (3.2)). This gives

$$X_{i_1j_1} = \frac{a \sum\limits_{j \neq j_1, j_2} \bar{y}_{i_1j} + (b-1) \sum\limits_{i \neq i_1} \bar{y}_{ij_1} + \sum\limits_{i \neq i_1} \bar{y}_{ij_2} - \sum\limits_{i} \sum\limits_{\substack{j \\ \neq (i_1j_1)(i_1j_2)}} \bar{y}_{ij}}{(a-1)(b-2)}$$

and

$$X_{i_1j_2} = X_{i_1j_1} + \frac{\sum\limits_{i \neq i_1} \bar{y}_{ij_2} - \sum\limits_{i \neq i_1} \bar{y}_{ij_1}}{(a-1)} \quad . \tag{4.5}$$

Thus, (4.5) gives those values that minimize the sum of squares for interaction, using the method of unweighted means, with respect to $X_{i_1j_1}$ and $X_{i_1j_2}$ for a two-way disproportionate design.

By substituting (3.12) into (4.5) we obtain $X_{i_1j_1}$ and $X_{i_1j_2}$ in terms of the model parameters, i.e.

$$X_{i_1j_1} = \mu + r_{i_1} + c_{j_1} + \frac{\sum\limits_{i \neq i_1} rc_{ij_1}}{a-1} + \frac{\sum\limits_{j \neq j_1j_2} rc_{i_1j}}{b-2}$$

$$- \frac{\sum\limits_{i \neq i_1} \sum\limits_{j \neq j_1j_2} rc_{ij}}{(a-1)(b-2)} + \sum_{i \neq i_1} \frac{\sum\limits_{k} e_{ij_1k}}{n_{ij_1}(a-1)} + \sum_{j \neq j_1j_2} \frac{\sum\limits_{k} e_{i_1jk}}{n_{i_1j}(b-2)}$$

$$- \sum_{i \neq i_1} \sum_{j \neq j_1j_2} \frac{\sum\limits_{k} e_{ijk}}{n_{ij}(a-1)(b-2)} \ , \tag{4.6}$$

$X_{i_1j_2}$ has the same formula with j_1 replaced by j_2.

With the above estimates for the two missing cells Procedures M and $\hat{M}$ can again be used to estimate the variance components as in

Chapter 3. First, for Procedure M the expected mean squares for rows, columns and interaction are derived by using (3.6.1) which implies that the variance-covariance matrix of $\underline{y}'$ (the vector of cell means including the missing cells) must be computed. The elements of this matrix are the same as those given in (3.14.1) except for the variances and covariances of $X_{i_1j_1}$ and $X_{i_1j_2}$. Using (4.6) we obtain for $X_{i_1j_1}$

$$\begin{aligned} \mathrm{Var}(X_{i_1j_1}) &= \sigma_r^2 + \sigma_c^2 + \frac{a+b-2}{(a-1)(b-2)}\,\sigma_{rc}^2 \\ &\quad + \sigma_e^2\left[\frac{1}{(a-1)^2}\sum_{i\neq i_1}\frac{1}{n_{ij_1}} + \frac{1}{(b-2)^2}\sum_{j\neq j_1j_2}\frac{1}{n_{i_1j}}\right. \\ &\quad \left. + \frac{1}{(a-1)^2(b-2)^2}\sum_{i\neq i_1}\sum_{j\neq j_1j_2}\frac{1}{n_{ij}}\right] \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(X_{i_1j_1}, X_{i_1j_2}) &= \sigma_r^2 + \frac{a\sigma_{rc}^2}{(a-1)(b-2)} \\ &\quad + \sigma_e^2\left[\frac{1}{(b-2)^2}\sum_{j\neq j_1j_2}\frac{1}{n_{ij}}\right. \\ &\quad \left. + \frac{1}{(a-1)^2(b-2)^2}\sum_{i\neq i_1}\sum_{j\neq j_1j_2}\frac{1}{n_{ij}}\right] \end{aligned}$$

$$\mathrm{Cov}(X_{i_1j_1}, \bar{y}_{ij_1}) = \sigma_c^2 + \frac{\sigma_{rc}^2}{a-1} + \frac{\sigma_e^2}{(a-1)n_{ij_1}} \qquad i\neq i_1 \tag{4.7}$$

$$\mathrm{Cov}(X_{i_1j_1}, \bar{y}_{ij_2}) = 0 \qquad i\neq i_1$$

$$\mathrm{Cov}(X_{i_1j_1}, \bar{y}_{i_1j}) = \sigma_r^2 + \frac{\sigma_{rc}^2}{b-2} + \frac{\sigma_e^2}{(b-2)n_{i_1j}} \qquad j \neq j_1, j_2$$

$$\mathrm{Cov}(X_{i_1j_1}, \bar{y}_{ij}) = \frac{\sigma_{rc}^2}{(a-1)(b-2)} - \frac{\sigma_e^2}{(a-1)(b-2)n_{ij}} \qquad \begin{matrix} i \neq i_1 \\ j \neq j_1, j_2 \end{matrix} .$$

The variances and covariances for $X_{i_1j_2}$ are the same as those in (4.7) with j_1 replaced by j_2.

By using (4.7) the variance-covariance matrix for $\underline{y}'$, call it V_2, can then be written down. For example, for a 3 x 3 design with cells (1, 1) and (1, 3) missing, *i.e.*

X	n_{12}	X
n_{21}	n_{22}	n_{23}
n_{31}	n_{32}	n_{33}

the variance-covariance matrix of $\underline{y}' = (X_{11}, \bar{y}_{12}, X_{13}, \bar{y}_{21}, \bar{y}_{22}, \bar{y}_{23}, \bar{y}_{31}, \bar{y}_{32}, \bar{y}_{33})$ is given by

$$V_2 = \left[\begin{array}{ccc|cccccc}
\sigma^2_{r+c} + 2\sigma^2_{rc} + \sigma^2_e[1] & \sigma^2_r + \sigma^2_{12} & \sigma^2_r + \frac{3}{2}\sigma^2_{rc} + \sigma^2_e[2] & \sigma^2_c + \frac{\sigma^2_{21}}{2} & -\frac{\sigma^2_{22}}{2} & 0 & \sigma^2_c + \frac{\sigma^2_{31}}{2} & -\frac{\sigma^2_{32}}{2} & 0 \\
\hline
\sigma^2_r + \sigma^2_{12} & \sigma^2_{r+c} + \sigma^2_{12} & \sigma^2_r + \sigma^2_{12} & 0 & \sigma^2_c & 0 & 0 & \sigma^2_c & 0 \\
\hline
\sigma^2_r + \frac{3}{2}\sigma^2_{rc} + \sigma^2_e[2] & \sigma^2_r + \sigma^2_{12} & \sigma^2_{r+c} + 2\sigma^2_{rc} + \sigma^2_e[3] & 0 & -\frac{\sigma^2_{22}}{2} & \sigma^2_c + \frac{\sigma^2_{23}}{2} & 0 & -\frac{\sigma^2_{32}}{2} & \sigma^2_c + \frac{\sigma^2_{33}}{2} \\
\hline
 & \text{Sym.} & & \multicolumn{6}{c}{\{D/C\}_2 + \mathrm{diag}\left(\frac{\sigma^2_e}{n_{ij}}\right)}
\end{array}\right] \qquad (4.8)$$

$$= \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V'_{12} & V_{22} & V_{23} \\ V'_{13} & V'_{23} & V_{33} \end{bmatrix}$$

where $D = \{\sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2/\sigma_r^2\}_3$ and $C = \sigma_c^2 I_3$ as in (3.5),

$$[1] = \left[\frac{1}{2^2} \sum_{i\neq 1} \frac{1}{n_{i1}} + \frac{1}{1^2} \sum_{j\neq 1,3} \frac{1}{n_{1j}} + \frac{1}{(2)^2(1^2)} \sum_{i\neq 1} \sum_{j\neq 1,3} \frac{1}{n_{ij}} \right],$$

$$[2] = \left[\frac{1}{1^2} \sum_{j\neq 1,3} \frac{1}{n_{1j}} + \frac{1}{(2^2)(1^2)} \sum_{i\neq 1} \sum_{j\neq 1,3} \frac{1}{n_{ij}} \right],$$

$[3] = [1]$ with $j_1 = 1$ replaced by $j_2 = 3$,

$$\sigma_{ij}^2 = \sigma_{rc}^2 + \frac{\sigma_e^2}{n_{ij}} \quad \text{and} \quad \sigma_{r+c}^2 = \sigma_r^2 + \sigma_c^2 \quad .$$

Given V_2 the expected mean square for rows in the general a x b disproportionate design (assuming that cells (1, 1) and (1, 3) are missing) is given by

$$E(MSR) = tr(V_2 \cdot G_R) = tr[(V_{11} + V_{22} + \ldots + V_{aa}) \cdot P]$$

$$+ tr[(2\cdot(V_{12} + V_{13} + \ldots + V_{1a}) + (a-1)(a-2)V_{23}) \cdot Q]$$

which is the same as equation (3.17) for the case of one missing cell except that in the present case the equations in (4.7) are used to define the elements of V_{11}, V_{22}, etc. instead of the equations in (3.15). Thus, in the present case it can be shown that

$$V_{11} + V_{22} + \cdots + V_{aa} =$$

$$\left[\begin{array}{c|c|c|c|c}
a\sigma^2_{r+c} + \sigma^2_{rc}[1] + \sigma^2_e[2] & a\sigma^2_r + \frac{\sigma^2_{12}}{(b-2)} & a\sigma^2_r + \frac{a\sigma^2_{rc}}{(a-1)(b-2)} + \sigma^2_e[3] & \cdots & a\sigma^2_r + \frac{\sigma^2_{1b}}{(b-2)} \\
\hline
a\sigma^2_r + \frac{\sigma^2_{12}}{(b-2)} & a\sigma^2_{r+c} + a\sigma^2_{rc} + \sigma^2_e \sum_i \frac{1}{n_{i2}} & a\sigma^2_r + \frac{\sigma^2_{12}}{(b-2)} & \cdots & a\sigma^2_r \\
\hline
a\sigma^2_r + \frac{a\sigma^2_{rc}}{(a-1)(b-2)} + \sigma^2_e[3] & a\sigma^2_r + \frac{\sigma^2_{12}}{(b-2)} & a\sigma^2_{r+c} + \sigma^2_{rc}[1] + \sigma^2_e[4] & \cdots & a\sigma^2_r + \frac{\sigma^2_{1b}}{(b-2)} \\
\hline
\vdots & \vdots & \vdots & & \vdots \\
\hline
a\sigma^2_r + \frac{\sigma^2_{1b}}{(b-2)} & a\sigma^2_r & a\sigma^2_r + \frac{\sigma^2_{1b}}{(b-2)} & \cdots & a\sigma^2_{r+c} + a\sigma^2_{rc} + \sigma^2_e \sum_i \frac{1}{n_{ib}}
\end{array}\right]$$

where

$$[1] = \frac{a+b-2}{(a-1)(b-2)} + (a-1) \quad ,$$

$$[2] = \frac{a^2 - 2a + 2}{(a-1)^2} \sum_{i \neq 1} \frac{1}{n_{i1}} + \frac{1}{(b-2)^2} \sum_{j \neq 1,3} \frac{1}{n_{1j}}$$

$$+ \frac{1}{(a-1)^2 (b-2)^2} \sum_{i \neq 1} \sum_{j \neq 1,3} \frac{1}{n_{ij}} \quad ,$$

$$[3] = \frac{1}{(b-2)^2} \sum_{j \neq 1,3} \frac{1}{n_{1j}} + \frac{1}{(a-1)^2 (b-2)^2} \sum_{i \neq 1} \sum_{j \neq 1,3} \frac{1}{n_{ij}} \quad ,$$

$[4]$ = $[1]$ with $j_1 = 1$ replaced by $j_2 = 3$,

$$\sigma^2_{ij} = \sigma^2_{rc} + \frac{\sigma^2_e}{n_{ij}} \quad \text{and} \quad \sigma^2_{r+c} = \sigma^2_r + \sigma^2_c \quad .$$

$$V_{12} + V_{13} + \cdots + V_{1a} =$$

$$\left[\begin{array}{c|c|c|c|c}
(a-1)\sigma_c^2 + \sigma_{rc}^2 + \dfrac{\sigma_e^2}{a-1} \displaystyle\sum_{i\neq 1} \frac{1}{n_{i1}} & -\dfrac{\left[(a-1)\sigma_{rc}^2 + \sigma_e^2 \displaystyle\sum_{i\neq 1} \frac{1}{n_{i2}}\right]}{(a-1)(b-2)} & 0 & \cdots & -\dfrac{\left[(a-1)\sigma_{rc}^2 + \sigma_e^2 \displaystyle\sum_{i\neq 1} \frac{1}{n_{ib}}\right]}{(a-1)(b-2)} \\
\hline
0 & (a-1)\sigma_c^2 & 0 & \cdots & 0 \\
\hline
0 & -\dfrac{\left[(a-1)\sigma_{rc}^2 + \sigma_e^2 \displaystyle\sum_{i\neq 1} \frac{1}{n_{i2}}\right]}{(a-1)(b-2)} & (a-1)\sigma_c^2 + \sigma_{rc}^2 + \dfrac{\sigma_e^2}{a-1} \displaystyle\sum_{i\neq 1} \frac{1}{n_{i3}} & \cdots & -\dfrac{\left[(a-1)\sigma_{rc}^2 + \sigma_e^2 \displaystyle\sum_{i\neq 1} \frac{1}{n_{ib}}\right]}{(a-1)(b-2)} \\
\hline
\vdots & \vdots & \vdots & & \vdots \\
\hline
0 & 0 & 0 & \cdots & (a-1)\sigma_c^2
\end{array}\right]$$

Straightforward matrix algebra can then be used to show that the expected mean square for rows reduces to

$$E(MSR) = \frac{\sigma_e^2}{ab}\left[\sum_{i\neq 1} \sum_{j\neq 1,3} \frac{1}{n_{ij}} \left(1 + \frac{2(2b-2)}{(a-1)^2(b-2)^2}\right) \right.$$

$$+ \sum_{i\neq 1} \frac{1}{n_{i1}} \left(1 - \frac{1}{(a-1)^2}\right) + \sum_{i\neq 1} \frac{1}{n_{i3}} \left(1 - \frac{1}{(a-1)^2}\right)$$

$$\left. + \sum_{j\neq 1,3} \frac{1}{n_{1j}} \left(1 + \frac{2(2b-2)}{(b-2)^2}\right) \right]$$

$$+ \sigma_{rc}^2 \left(1 + \frac{2}{(a-1)(b-2)}\right) + b\sigma_r^2 \quad .$$

In the same manner

$$E(MSC) = tr(V_2 \cdot G_C) \qquad \text{and}$$

$$E(MSRC) = tr(V_2 \cdot G_{RC2})$$

where

$$G_{RC2} = \frac{(a-1)(b-1)}{(a-1)(b-1)-2} \; G_{RC}$$

can be used to derive the expected mean squares for columns and interaction. Table 4.1 summarizes the expected mean squares for the case of two missing cells in the same row.

Table 4.1 Expected mean squares for a two-way disproportionate design with two missing cells in the same row using the method of unweighted means

Source	Coefficients of Variance Components in E(MS) σ_e^2	σ_{rc}^2	σ_c^2	σ_r^2
Rows	$K1 = \frac{1}{ab}\left[\left(1 + \frac{2(2b-2)}{(a-1)^2(b-2)^2}\right)\cdot\Sigma_1 + \left(1 - \frac{1}{(a-1)^2}\right)\cdot\Sigma_2 + \left(1 + \frac{2(2b-2)}{(b-2)^2}\right)\cdot\Sigma_3\right]$	$K2 = 1 + \frac{2}{(a-1)(b-2)}$		b
Columns	$K3 = \frac{1}{ab}\left[\left(1 + \frac{2(2a-1)}{(a-1)^2(b-1)(b-2)}\right)\cdot\Sigma_1 + \left(1 + \frac{2a-1}{(a-1)^2}\right)\cdot\Sigma_2 + \left(1 - \frac{2}{(b-1)(b-2)}\right)\cdot\Sigma_3\right]$	$K4 = 1 + \frac{2}{(a-1)(b-1)}$	a	
Interaction	$K5 = \frac{C_2}{ab}\left[\left(1 - \frac{2}{(a-1)^2(b-1)(b-2)}\right)\cdot\Sigma_1 + \left(1 - \frac{1}{(a-1)^2}\right)\cdot\Sigma_2 + \left(1 - \frac{2}{(b-1)(b-2)}\right)\cdot\Sigma_3\right]$	1		
Error	1			

where the missing cells are (i_1j_1) and (i_1j_2) and

$$\Sigma_1 = \sum_{i\neq i_1}\sum_{j\neq j_1j_2}\frac{1}{n_{ij}}, \quad \Sigma_2 = \sum_{i\neq i_1}\frac{1}{n_{ij_1}} + \sum_{i\neq i_1}\frac{1}{n_{ij_2}},$$

$$\Sigma_3 = \sum_{j\neq j_1j_2}\frac{1}{n_{i_1j}} \quad \text{and} \quad C_2 = \frac{(a-1)(b-1)}{(a-1)(b-1)-2}.$$

As expected, Table 4.1 shows that as a and b become large

$$K1 \hat{=} K3 \hat{=} K5 \hat{=} \frac{1}{ab} \sum_{\substack{i \\ \neq(i_1 j_1)}} \sum_{\substack{j \\ (i_1 j_2)}} \frac{1}{n_{ij}}$$

and $K2 \hat{=} K4 \hat{=} 1$.

Also, for the special case of equal observations in the cells ($n_{ij} = n$) the expected mean squares in Table 4.1 reduce to

$$E(MSR) = \frac{1}{n}\left[1 + \frac{2}{(a-1)(b-2)}\right]\sigma_e^2 + \left[1 + \frac{2}{(a-1)(b-2)}\right]\sigma_{rc}^2 + b\sigma_r^2$$

$$E(MSC) = \frac{1}{n}\left[1 + \frac{2}{(a-1)(b-1)}\right]\sigma_e^2 + \left[1 + \frac{2}{(a-1)(b-1)}\right]\sigma_{rc}^2 + a\sigma_c^2$$

$$E(MSRC) = \frac{1}{n}\sigma_e^2 + \sigma_{rc}^2 \qquad \text{and}$$

$$E(MSE) = \sigma_e^2 \quad .$$

The preceding results for the expected mean squares for two missing cells in the same row can be generalized to the case of M missing cells (denoted by i_1j_1, i_1j_2, ... i_1j_m) in the same row. The procedure for obtaining the expected mean squares in the M missing cell case is exactly the same as that just described for two missing cells. That is: (a) A value is estimated for each of the M missing cells by minimizing the sum of squares for interaction with respect to each of the missing cells after having replaced the observations in each cell by their cell means. For example, the missing value formula for $X_{i_1j_1}$ is given by

$$X_{i_1j_1} = \frac{a \sum_{j \neq j_1 \ldots j_m} \bar{y}_{i_1 j} + (b-M+1) \sum_{i \neq i_1} \bar{y}_{i j_1} + \sum_{j=j_2 \ldots j_m} \sum_{i \neq i_1} \bar{y}_{ij}}{(a-1)(b-M)}$$

$$- \frac{\sum_{i} \sum_{j} \bar{y}_{ij} \quad \neq (i_1 j_1) \ldots (i_1 j_m)}{(a-1)(b-M)} \quad . \tag{4.9}$$

The missing value formulas for the other missing cells are exactly the same except that j_1 is replaced by the appropriate subscript.

(b) The variance-covariance matrix, V_M, of the vector of cell means, $\underline{y}'$, is set up by using formulas similar to (4.7) for $Var(X_{i_1j_1})$, $Cov(X_{i_1j_1}, X_{i_1j_2})$ etc. This matrix has essentially the same form as the matrix given in (4.8) for the case of two missing cells. (c) V_M is substituted into (3.6.1) for the row, column and interaction mean squares. Straightforward matrix algebra is then used to obtain the expected mean squares (see Table 4.2) which are similar to those given in Table 4.1.

Table 4.2 Expected mean squares for a two-way disproportionate design with M missing cells in the same row using the method of unweighted means

Source	Coefficients of Variance Components in E(MS): σ_e^2	σ_{rc}^2	σ_c^2	σ_r^2
Rows	K1 $= \frac{1}{ab}\left[\left(1 + \frac{M(2b-M)}{(a-1)^2(b-M)^2}\right)\cdot\Sigma_1 + \left(1 - \frac{1}{(a-1)^2}\right)\cdot\Sigma_2 + \left(1 + \frac{M(2b-M)}{(b-M)^2}\right)\cdot\Sigma_3\right]$	K2 $= 1 + \frac{M}{(a-1)(b-M)}$		b
Columns	K3 $= \frac{1}{ab}\left[\left(1 + \frac{M(2a-1)}{(a-1)^2(b-1)(b-M)}\right)\cdot\Sigma_1 + \left(1 + \frac{2a-1}{(a-1)^2}\right)\cdot\Sigma_2 + \left(1 - \frac{M}{(b-1)(b-M)}\right)\cdot\Sigma_3\right]$	K4 $= 1 + \frac{M}{(a-1)(b-1)}$	a	
Interaction	K5 $= \frac{C_M}{ab}\left[\left(1 - \frac{M}{(a-1)^2(b-1)(b-M)}\right)\cdot\Sigma_1 + \left(1 - \frac{1}{(a-1)^2}\right)\cdot\Sigma_2 + \left(1 - \frac{M}{(b-1)(b-M)}\right)\cdot\Sigma_3\right]$	1		
Error	1			

where the M missing cells are i_1j_1, i_1j_2, i_1j_3, ..., i_1j_m and

$$\Sigma_1 = \sum_{i\neq i_1}\ \sum_{j\neq j_1j_2\cdots j_m} \frac{1}{n_{ij}}, \quad \Sigma_2 = \sum_{i\neq i_1}\frac{1}{n_{ij_1}} + \sum_{i\neq i_1}\frac{1}{n_{ij_2}} + \cdots$$

$$+ \sum_{i\neq i_1}\frac{1}{n_{ij_m}}, \quad \Sigma_3 = \sum_{j\neq j_1j_2\cdots j_m}\frac{1}{n_{i_1j}} \quad \text{and} \quad C_M = \frac{(a-1)(b-1)}{(a-1)(b-1)-M}.$$

(Note that if M = 2 the coefficients in Table 4.2 reduce to those given in Table 4.1 and if M = 1 they reduce to those given in Table 3.1.)

The results obtained above for two or M missing cells in the same row extend directly to two or M missing cells in the same column. For example, if there are M missing cells in the same column then the expected mean square for columns is given by

$$\begin{aligned} E(MSC) &= a\sigma_c^2 + \sigma_{rc}^2 \left(1 + \frac{M}{(a-M)(b-1)}\right) \\ &\quad + \sigma_e^2 \left[\frac{1}{ab}\left(1 + \frac{M(2a-M)}{(a-M)^2(b-1)^2}\right)\cdot\Sigma_1 + \left(1 + \frac{M(2a-M)}{(a-M)^2}\right)\cdot\Sigma_2 \right. \\ &\quad \left. + \left(1 - \frac{1}{(b-1)^2}\right)\cdot\Sigma_3 \right] \end{aligned} \tag{4.10}$$

where the M missing cells are i_1j_1, i_2j_1, i_3j_1, ..., i_mj_1, and

$$\Sigma_1 = \sum_{i\neq i_1\ldots i_m}\ \sum_{j\neq j_1} \frac{1}{n_{ij}}, \quad \Sigma_2 = \sum_{i\neq i_1\ldots i_m} \frac{1}{n_{ij_1}} \quad \text{and}$$

$$\Sigma_3 = \sum_{j\neq j_1} \frac{1}{n_{i_1j}} + \sum_{j\neq j_1} \frac{1}{n_{i_2j}} + \ldots + \sum_{j\neq j_1} \frac{1}{n_{i_mj}} .$$

Similar results hold for E(MSR) and E(MSRC).

The expected mean squares given in Tables 4.1 and 4.2 can be used to obtain unbiased estimates of the variance components σ_r^2, σ_c^2, σ_{rc}^2, and σ_e^2 for the cases of two or M missing cells in the same row by using the formulas given in (3.22) above (Procedure M estimates) with G_{RC1} replaced by G_{RC2} or G_{RCM}. In addition, biased estimates of these variance components may be obtained by using (3.24) (Procedure $\hat{M}$

estimates) with

$$\frac{1}{s} \underset{\neq(i_1 j_1)}{\sum_i \sum_j} \frac{1}{n_{ij}} \quad \text{replaced with} \quad \frac{1}{s} \underset{\neq(\text{missing cells})}{\sum_i \sum_j} \frac{1}{n_{ij}} .$$

The biases of the Procedure $\hat{M}$ estimated variance components, given in (3.25), are examined in Chapter 5 for several designs. The variances of the Procedure M and $\hat{M}$ estimated variance components can again be derived analytically by using (3.26) and (3.28) with V_1 replaced by V_2 or V_M and G_{RC1} replaced with G_{RC2} or G_{RCM}. However, as in the case of one missing cell (Chapter 3) these variances were not derived analytically due to the complexity of the matrix algebra involved in solving (3.26) and (3.28). Instead, Chapter 5 gives numerical results obtained by using the computer program VC to estimate the variances of the Procedure M and $\hat{M}$ estimates of the variance components for several designs with two and three missing cells in the same row.

4.3 Two or More Missing Cells in Different Rows and Columns

In this section the estimation of variance components for two-way disproportionate designs with missing cells in different rows and columns based upon the method of unweighted means is examined. Although the procedures for estimating the variance components are exactly the same as described above for the case of two or M missing cells in the same row, the matrix algebra involved in the present case is much more tedious and the results are quite complex.

First, consider a design which has two missing cells which are in different rows and columns (denote the two cells by $(i_1 j_1)$ and

(i_2j_2)). For example, the design might have the following n_{ij} pattern:

X	n_{12}	n_{13}
n_{21}	X	n_{23}
n_{31}	n_{32}	n_{33}

(4.11)

The missing value formulas for these two missing cells can be derived as in Section 4.2 by minimizing the interaction sum of squares with respect to the missing cells ($X_{i_1j_1}$ and $X_{i_2j_2}$). These formulas are given by Burrows [1966] as

$$X_{i_1j_1} = \frac{(a-1)(b-1)\left[a \sum_{j\neq j_1} \bar{y}_{i_1j} + b \sum_{i\neq i_1} \bar{y}_{ij_1} - G\right] - \left[a \sum_{j\neq j_2} \bar{y}_{i_2j} + b \sum_{i\neq i_2} \bar{y}_{ij_2} - G\right]}{(a-1)^2(b-1)^2-1} \tag{4.12}$$

and

$$X_{i_2j_2} = \frac{a \sum_{j\neq j_2} \bar{y}_{i_2j} + b \sum_{i\neq i_2} \bar{y}_{ij_2} - (G + X_{i_1j_1})}{(a-1)(b-1)}$$

where

$$G = \underset{\neq(i_1j_1)}{\sum_i} \underset{(i_2j_2)}{\sum_j} \bar{y}_{ij} .$$

By again using (3.12), $X_{i_1j_1}$ and $X_{i_2j_2}$ (in (4.12)) can be written in terms of the model parameters. Unfortunately, this model representation is quite complicated, namely

$$X_{i_1j_1} = \mu + r_{i_1} + c_{j_1} + \frac{f_1}{F}(g_{i_1j_2}) + \frac{f_2}{F}\left[\sum_{j\neq j_1j_2} g_{i_1j}\right]$$
$$+ \frac{f_3}{F}(g_{i_2j_1}) + \frac{f_5}{F}\left[\sum_{j\neq j_1j_2} g_{i_2j}\right] + \frac{f_6}{F}\left[\sum_{i\neq i_1i_2} g_{ij_1}\right] \quad (4.13)$$
$$+ \frac{f_7}{F}\left[\sum_{i\neq i_1i_2} g_{ij_2}\right] + \frac{f_8}{F}\left[\sum_{i\neq i_1i_2}\sum_{j\neq j_1j_2} g_{ij}\right]$$

where

$$g_{i_\ell j_m} = rc_{i_\ell j_m} + \frac{\sum_k e_{i_\ell j_m k}}{n_{i_\ell j_m}} ,$$

$$f_1 = a(a-2)(b-1) ,$$

$$f_2 = (a-1)^2(b-1) + 1 ,$$

$$f_3 = (a-1)b(b-2) ,$$

$$f_5 = -(a-1)b ,$$

$$f_6 = (a-1)(b-1)^2 + 1 ,$$

$$f_7 = -a(b-1) ,$$

$$f_8 = -(a-1)(b-1) + 1 \quad \text{and}$$

$$F = (a-1)^2(b-1)^2 - 1 .$$

The model representation for $X_{i_2j_2}$ is exactly the same as (4.13) with i_2 replacing i_1, j_2 replacing j_1 and vice-versa.

Using (4.13) the variance of $X_{i_1j_1}$ and the covariances of $X_{i_1j_1}$ with the other cell means can be shown to be

$$\begin{aligned}
\mathrm{Var}(X_{i_1j_1}) &= \sigma_r^2 + \sigma_c^2 + \frac{\sigma_{rc}^2}{F^2}\Big[f_1^2 + (b-2)f_2^2 + f_3^2 + (b-2)f_5^2 \\
&\qquad + (a-2)(f_6^2 + f_7^2) + (a-2)(b-2)f_8^2 \Big] \\
&\qquad + \frac{\sigma_e^2}{F^2}\Big[\frac{f_1^2}{n_{i_1j_2}} + f_2^2 \sum_{j\neq j_1j_2} \frac{1}{n_{i_1j}} + \frac{f_3^2}{n_{i_2j_1}} \\
&\qquad + f_5^2 \sum_{j\neq j_1j_2} \frac{1}{n_{i_2j}} + f_6^2 \sum_{i\neq i_1i_2} \frac{1}{n_{ij_1}} \\
&\qquad + f_7^2 \sum_{i\neq i_1i_2} \frac{1}{n_{ij_2}} + f_8^2 \sum_{i\neq i_1i_2} \sum_{j\neq j_1j_2} \frac{1}{n_{ij}} \Big] \\
&= \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 \cdot v_{rc}(X_{i_1j_1}) + \sigma_e^2 \cdot v_e(X_{i_1j_1}) \ ;
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(X_{i_1j_1}, X_{i_2j_2}) &= \frac{2\sigma_{rc}^2}{F^2}\Big[f_1f_3 + (b-2)f_2f_5 + (a-2)f_6f_7 \\
&\qquad + \frac{(a-2)(b-2)}{2} f_8^2 \Big] \\
&\qquad + \frac{\sigma_e^2}{F^2}\Big[f_1f_3\cdot\Sigma_1 + f_2f_5\cdot\Sigma_2 + f_6f_7\cdot\Sigma_3 + f_8^2\cdot\Sigma_4 \Big] \\
&= \sigma_{rc}^2 \cdot c_{rc}(X_{i_1j_1}, X_{i_2j_2}) + \sigma_e^2 \cdot c_e(X_{i_1j_1}, X_{i_2j_2})
\end{aligned}$$

where

$$\Sigma_1 = \sum_{i=i_1i_2} \sum_{\substack{j=j_1j_2 \\ \neq i}} \frac{1}{n_{ij}} \ ; \qquad \Sigma_2 = \sum_{i=i_1i_2} \sum_{j\neq j_1j_2} \frac{1}{n_{ij}} \ ;$$

$$\Sigma_3 = \sum_{i \neq i_1 i_2} \sum_{j = j_1 j_2} \frac{1}{n_{ij}} \quad \text{and} \quad \Sigma_4 = \sum_{i \neq i_1 i_2} \sum_{j \neq j_1 j_2} \frac{1}{n_{ij}} ;$$

$$\mathrm{Cov}(X_{i_1 j_1}, \bar{y}_{i_1 j_2}) = \sigma_r^2 + \frac{f_1}{F}\left(\sigma_{rc}^2 + \frac{\sigma_e^2}{n_{i_1 j_2}}\right) ;$$

$$\mathrm{Cov}(X_{i_1 j_1}, \bar{y}_{i_1 j}) = \sigma_r^2 + \frac{f_2}{F}\left(\sigma_{rc}^2 + \frac{\sigma_e^2}{n_{i_1 j}}\right) \quad \text{for} \quad j \neq j_1 j_2 ;$$

$$\mathrm{Cov}(X_{i_1 j_1}, \bar{y}_{i_2 j_1}) = \sigma_c^2 + \frac{f_3}{F}\left(\sigma_{rc}^2 + \frac{\sigma_e^2}{n_{i_2 j_1}}\right) ; \tag{4.14}$$

$$\mathrm{Cov}(X_{i_1 j_1}, \bar{y}_{i_2 j}) = \frac{f_5}{F}\left(\sigma_{rc}^2 + \frac{\sigma_e^2}{n_{i_2 j}}\right) \quad \text{for} \quad j \neq j_1 j_2 ;$$

$$\mathrm{Cov}(X_{i_1 j_1}, \bar{y}_{i j_1}) = \sigma_c^2 + \frac{f_6}{F}\left(\sigma_{rc}^2 + \frac{\sigma_e^2}{n_{i j_1}}\right) \quad \text{for} \quad i \neq i_1 i_2 ;$$

$$\mathrm{Cov}(X_{i_1 j_1}, \bar{y}_{i j_2}) = \frac{f_7}{F}\left(\sigma_{rc}^2 + \frac{\sigma_e^2}{n_{i j_2}}\right) \quad \text{for} \quad i \neq i_1 i_2 \quad \text{and}$$

$$\mathrm{Cov}(X_{i_1 j_1}, \bar{y}_{ij}) = \frac{f_8}{F}\left(\sigma_{rc}^2 + \frac{\sigma_e^2}{n_{ij}}\right) \quad \text{for} \quad \begin{array}{l} i \neq i_1 i_2 \\ j \neq j_1 j_2 \end{array} .$$

The above formulas also give the variance of $X_{i_2 j_2}$ and its covariances with the other cell means when $(i_2 j_2)$ and $(i_1 j_1)$ are interchanged. A comparison of the equations given in (4.14) with those in (4.7) indicates why the results for missing cells in different rows and columns are much more complex than those for missing cells all in the same row. Using (4.14) the variance-covariance matrix, V_2, of the vector of cell means for the case of two missing cells in different rows and columns can then be written. For example, for the design given in (4.11) the variance-covariance matrix of the vector of cell means, $\underset{\sim}{y}' = (X_{11}, \bar{y}_{12}, \bar{y}_{13}, \bar{y}_{21}, X_{22}, \bar{y}_{23}, \bar{y}_{31}, \bar{y}_{32}, \bar{y}_{33})$, is given by

$$V_2 = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
\begin{matrix}\sigma^2_{r+c} \\ + (1.4)\sigma^2_{rc} \\ + \sigma^2_e \cdot v_e(X_{11})\end{matrix} & \begin{matrix}\sigma^2_r \\ + .4\sigma^2_{12}\end{matrix} & \begin{matrix}\sigma^2_r \\ + .6\sigma^2_{13}\end{matrix} & \begin{matrix}\sigma^2_c \\ + .4\sigma^2_{21}\end{matrix} & \begin{matrix}- .6\sigma^2_{rc} \\ + \sigma^2_e \cdot c_e(X_{11},X_{21})\end{matrix} & - .4\sigma^2_{23} & \begin{matrix}\sigma^2_c \\ + .6\sigma^2_{31}\end{matrix} & - .4\sigma^2_{32} & - .2\sigma^2_{33} \\
\hline
 & \begin{matrix}\sigma^2_{r+c} \\ + \sigma^2_{12}\end{matrix} & \sigma^2_r & 0 & \begin{matrix}\sigma^2_c \\ + .4\sigma^2_{12}\end{matrix} & 0 & 0 & \sigma^2_c & 0 \\
\hline
 & & \begin{matrix}\sigma^2_{r+c} \\ + \sigma^2_{13}\end{matrix} & 0 & - .4\sigma^2_{13} & \sigma^2_c & 0 & 0 & \sigma^2_c \\
\hline
 & & & \begin{matrix}\sigma^2_{r+c} \\ + \sigma^2_{21}\end{matrix} & \begin{matrix}\sigma^2_r \\ + .4\sigma^2_{21}\end{matrix} & \sigma^2_r & \sigma^2_c & 0 & 0 \\
\hline
 & & & & \begin{matrix}\sigma^2_{r+c} + (1.4)\sigma^2_{rc} \\ + \sigma^2_e \cdot v_e(X_{22})\end{matrix} & \begin{matrix}\sigma^2_r \\ + .6\sigma^2_{23}\end{matrix} & - .6\sigma^2_{31} & \begin{matrix}\sigma^2_c \\ + .6\sigma^2_{32}\end{matrix} & - .2\sigma^2_{33} \\
\hline
 & \text{Sym.} & & & & \begin{matrix}\sigma^2_{r+c} \\ + \sigma^2_{23}\end{matrix} & 0 & 0 & \sigma^2_c \\
\hline
 & & & & & & \multicolumn{3}{l}{\{D/C\}_1 + \operatorname{diag}\left(\dfrac{\sigma^2_e}{n_{ij}}\right)}
\end{array}\right]$$

where

$$v_e(X_{11}) = \frac{1}{(15)^2}\left[\frac{(6)^2}{n_{12}} + (9)^2 \sum_{j\neq 1,2} \frac{1}{n_{1j}} + \frac{(6)^2}{n_{21}} + (-6)^2 \sum_{j\neq 1,2} \frac{1}{n_{2j}}\right.$$

$$\left. + (9)^2 \sum_{i\neq 1,2} \frac{1}{n_{i1}} + (-6)^2 \sum_{i\neq 1,2} \frac{1}{n_{i2}} + (-3)^2 \sum_{i\neq 1,2} \sum_{j\neq 1,2} \frac{1}{n_{ij}}\right],$$

$$c_e(X_{11},X_{22}) = \frac{1}{(15)^2}\left[(6)(6) \sum_{i=1,2} \sum_{j=1,2} \frac{1}{n_{ij}} + (9)(-6) \sum_{i=1,2} \sum_{j\neq 1,2} \frac{1}{n_{ij}}\right.$$

$$+ (9)(-6) \sum_{i\neq 1,2} \sum_{j=1,2} \frac{1}{n_{ij}}$$

$$\left. + (-3)^2 \sum_{i\neq 1,2} \sum_{j\neq 1,2} \frac{1}{n_{ij}}\right],$$

$v_e(X_{22}) = v_e(X_{11})$ with $i_1 = 1$ replaced with $i_2 = 2$ and $j_1 = 1$ replaced with $j_2 = 2$, D and C are defined in (4.8),

$\sigma^2_{ij} = \sigma^2_{rc} + \sigma^2_e/n_{ij}$ and $\sigma^2_{r+c} = \sigma^2_r + \sigma^2_c$.

Once V_2 is determined, equation (3.6.1) can again be used to obtain the expected mean squares for rows, columns and interaction. The result for the expected mean square for rows is given by

$$E(MSR) = tr(V_2 \cdot G_R)$$

$$= b\sigma_r^2 + \sigma_{rc}^2 \left[1 + \frac{v_{rc}(X_{i_1j_1}) + v_{rc}(X_{i_2j_2}) - \dfrac{2c_{rc}(X_{i_1j_1}, X_{i_2j_2})}{(a-1)} + M}{ab} \right]$$

$$+ \frac{\sigma_e^2}{ab}\Bigg[\sum_{i} \sum_{j}_{\neq(i_1j_1)(i_2j_2)} \frac{1}{n_{ij}} + v_e(X_{i_1j_1}) + v_e(X_{i_2j_2})$$

$$- \frac{2c_e(X_{i_1j_1}, X_{i_2j_2})}{(a-1)} \tag{4.15}$$

$$+ \frac{2}{F}\Big(f_1 \cdot \Sigma_1 + f_2 \cdot \Sigma_2 - \frac{1}{(a-1)}(f_3 \cdot \Sigma_1 + f_5 \cdot \Sigma_2 + f_6 \cdot \Sigma_3$$

$$+ f_7 \cdot \Sigma_3 + M \cdot f_8 \cdot \Sigma_4)\Big)\Bigg]$$

where M = number of missing cells (i.e., 2); $v_{rc}(X_{i_1j_1})$, $v_{rc}(X_{i_2j_2})$, $v_e(X_{i_1j_1})$, $v_e(X_{i_2j_2})$, $c_{rc}(X_{i_1j_1}, X_{i_2j_2})$, $c_e(X_{i_1j_1}, X_{i_2j_2})$ and $\Sigma_1 \cdots \Sigma_4$ were defined in (4.14); and $f_1, \cdots, f_8, F$ were defined in (4.13). Further simplification of (4.15) to a less complex form does not appear to be possible.

Note here that (4.15) reduces to the formula given in Table 3.1 for the case of 1 missing cell (i_1j_1) when the proper substitutions are made. That is, if (i_1j_1) is the only missing cell then in (4.15) replace $v_{rc}(X_{i_1j_1})$ with $\frac{a + b - 1}{(a-1)(b-1)}$ and $v_e(X_{i_1j_1})$ with $\frac{1}{(a-1)^2} \sum_{i \neq i_1} \frac{1}{n_{ij_1}}$

$+ \frac{1}{(b-1)^2} \sum_{j \neq j_1} \frac{1}{n_{i_1j}} + \frac{1}{(a-1)^2(b-1)^2} \sum_{i \neq i_1} \sum_{j \neq j_1} \frac{1}{n_{ij}}$ as given in (3.15);

$Var(X_{i_2j_2})$ and $Cov(X_{i_1j_1}, X_{i_2j_2})$ with 0 since $X_{i_2j_2}$ is no longer a missing cell;

M with 1 ;

$\sum\limits_{i}\sum\limits_{j}_{\neq(i_1j_1)(i_2j_2)}$ with $\sum\limits_{i}\sum\limits_{j}_{\neq(i_1j_1)} \frac{1}{n_{ij}}$;

$$\frac{f_1}{F} = \frac{f_3}{F} = \frac{f_5}{F} = \frac{f_7}{F} = 0 \quad ;$$

$$\frac{f_2}{F} = \frac{1}{b-1}, \quad \frac{f_6}{F} = \frac{1}{a-1} \quad \text{and} \quad \frac{f_8}{F} = -\frac{1}{(a-1)(b-1)} \; ; \quad \text{and}$$

Σ_2 , Σ_3 and Σ_4 with $\sum\limits_{j\neq j_1} \frac{1}{n_{i_1j}}$,

$\sum\limits_{i\neq i_1} \frac{1}{n_{ij_1}}$ and $\sum\limits_{i\neq i_1}\sum\limits_{j\neq j_1} \frac{1}{n_{ij}}$ respectively.

The resulting formula for E(MSR) will be that given in Table 3.1.

The results of using (3.6.1) to derive the expected mean squares for rows, columns and interaction for the case of two missing cells in different rows and columns are summarized in Table 4.3, where the missing cells are (i_1j_1) and (i_2j_2); M is the number of missing cells (i.e., two);

$$SO = \sum_{i}\sum_{j}{}_{\neq(\text{missing cells})} \frac{1}{n_{ij}} \; ; \quad v_{rc}(X_{i_1j_1}),\ v_{rc}(X_{i_2j_2}),\ c_{rc}(X_{i_1j_1},X_{i_2j_2}),$$

$v_e(X_{i_1j_1})$, $v_e(X_{i_2j_2})$, $c_e(X_{i_1j_1},X_{i_2j_2})$, Σ_1, Σ_2, Σ_3 and Σ_4

are the same as defined in (4.14) above; $f_1, \ldots, f_8$, F are defined in (4.13);

Table 4.3 Expected mean squares for a two-way disproportionate design with two missing cells in different rows and columns using the method of unweighted means

Source	Coefficients of Variance Components in E(MS)			
	σ^2_e	σ^2_{rc}	σ^2_c	σ^2_r
Rows	$K1 = \frac{1}{ab}\Big[SO + v_e(X_{i_1j_1}) + v_e(X_{i_2j_2})$ $- 2c_e(X_{i_1j_1}, X_{i_2j_2}) / (a-1)$ $+ \frac{2}{F}\Big(f_1 \cdot \Sigma_1 + f_2 \cdot \Sigma_2 - (f_3 \cdot \Sigma_1 + f_5 \cdot \Sigma_2$ $+ [f_6+f_7] \cdot \Sigma_3 + f_8 \cdot M \cdot \Sigma_4)/(a-1)\Big)\Big]$	$K2 = 1 + \frac{v_{rc}(X_{i_1j_1}) + v_{rc}(X_{i_2j_2})}{ab} - \frac{\frac{2c_{rc}(X_{i_1j_1}, X_{i_2j_2})}{(a-1)} - M}{ab}$		b
Columns	$K3 = \frac{1}{ab}\Big[SO + v_e(X_{i_1j_1}) + v_e(X_{i_2j_2})$ $- 2c_e(X_{i_1j_1}, X_{i_2j_2}) / (b-1)$ $+ \frac{2}{F}\Big(f_3 \cdot \Sigma_1 + f_6 \cdot \Sigma_3 - (f_1 \cdot \Sigma_1 + f_7 \cdot \Sigma_3$ $+ [f_2+f_5] \cdot \Sigma_2 + f_8 \cdot M \cdot \Sigma_4)/(b-1)\Big)\Big]$	$K4 = 1 + \frac{v_{rc}(X_{i_1j_1}) + v_{rc}(X_{i_2j_2})}{ab} - \frac{\frac{2c_{rc}(X_{i_1j_1}, X_{i_2j_2})}{(b-1)} - M}{ab}$	a	

continued

Table 4.3 (continued)

Source	Coefficients of Variance Components in E(MS)			
	σ_e^2	σ_{rc}^2	σ_c^2	σ_r^2
Interaction	K5 = $\frac{C_2}{ab}\left[SO + v_e(X_{i_1j_1}) + v_e(X_{i_2j_2}) + \frac{2c_e(X_{i_1j_1}, X_{i_2j_2})}{(a-1)(b-1)} - FI\right]$	1		
Error	1			

$$FI = \frac{2}{F}\left(\frac{f_1\cdot\Sigma_1}{(b-1)} + \frac{f_2\cdot\Sigma_2}{(b-1)} + \frac{f_3\cdot\Sigma_1}{(a-1)} - \frac{f_5\cdot\Sigma_2}{(a-1)(b-1)} + \frac{f_6\cdot\Sigma_3}{(a-1)}\right.$$

$$\left. - \frac{f_7\cdot\Sigma_3}{(a-1)(b-1)} - \frac{f_8\cdot M\cdot\Sigma_4}{(a-1)(b-1)}\right) \qquad \text{and}$$

$$C_2 = \frac{(a-1)(b-1)}{(a-1)(b-1)-2} \quad .$$

As in previous cases, the formulas given in Table 4.3 show that as a and b become large

$$K1 \hat{=} K3 \hat{=} K5 \hat{=} \frac{1}{ab} \sum_{i} \sum_{\substack{j \\ \neq (\text{missing cells})}} \frac{1}{n_{ij}}$$

and $K2 \hat{=} K4 \hat{=} 1$.

The extension of the results in Table 4.3 to the case of M missing cells in different rows and columns is straightforward but as might be expected the matrix algebra involved is quite complicated. A brief summary of the calculations needed to obtain the expected mean squares in this general M missing cell case will be given here. First, the missing value formulas for the M missing cells (denote the missing values by $X_{i_1j_1}$, $X_{i_2j_2}$, $\cdots$, $X_{i_mj_m}$) are derived as in previous cases by minimizing the interaction sum of squares. The resulting formulas for these missing values are given in general by

$$X_{i_1j_1} = \frac{[(a-1)(b-1) + M - 2][a \sum_{j\neq j_1} \bar{y}_{i_1j} + b \sum_{i\neq i_1} \bar{y}_{ij_1} - G]}{[(a-1)(b-1) - 1][(a-1)(b-1) + M - 1]}$$

$$- \frac{[a \sum_{i=i_2i_3\ldots i_m} \sum_{j\neq j_i} \bar{y}_{ij} + b \sum_{j=j_2j_3\ldots j_m} \sum_{i\neq i_j} \bar{y}_{ij} - (M-1)G]}{[(a-1)(b-1) - 1][(a-1)(b-1) + M - 1]}$$

$$\text{where } G = \sum_{i} \sum_{j} \bar{y}_{ij} \, , \qquad (4.16)$$
$$\neq \binom{\text{missing}}{\text{cells}}$$

$$X_{i_2j_2} = \frac{[(a-1)(b-1) + M - 2][a \sum_{j\neq j_2} \bar{y}_{i_2j} + b \sum_{i\neq i_2} \bar{y}_{ij_2} - G]}{[(a-1)(b-1) - 1][(a-1)(b-1) + M - 1]}$$

$$- \frac{[a \sum_{i=i_1i_3\ldots i_m} \sum_{j\neq j_i} \bar{y}_{ij} + b \sum_{j=j_1j_3\ldots j_m} \sum_{i\neq i_j} \bar{y}_{ij} - (M-1)G]}{[(a-1)(b-1) - 1][(a-1)(b-1) + M - 1]}$$

etc. for $X_{i_3j_3} \ldots X_{i_mj_m}$.

In terms of the model parameters (4.16) becomes

$$
\begin{aligned}
X_{i_1 j_1} = \mu + r_{i_1} + c_{j_1} &+ \frac{f_1}{F} \sum_{j=j_2 j_3 \cdots j_m} g_{i_1 j} \\
&+ \frac{f_2}{F} \sum_{j \neq j_1 j_2 \cdots j_m} g_{i_1 j} + \frac{f_3}{F} \sum_{i=i_2 i_3 \cdots i_m} g_{i j_1} \\
&+ \frac{f_4}{F} \sum_{i \neq i_2 i_3 \cdots i_m} \; \sum_{j \neq j_2 j_3 \cdots j_m} g_{ij} \\
&\qquad\qquad \neq \binom{\text{missing}}{\text{cells}} \\
&+ \frac{f_5}{F} \sum_{i=i_2 i_3 \cdots i_m} \; \sum_{j \neq j_1 j_2 \cdots j_m} g_{ij} \\
&+ \frac{f_6}{F} \sum_{i \neq i_1 i_2 \cdots i_m} g_{i j_1} \\
&+ \frac{f_7}{F} \sum_{i \neq i_1 i_2 \cdots i_m} \; \sum_{j=j_2 j_3 \cdots j_m} g_{ij} \\
&+ \frac{f_8}{F} \sum_{i \neq i_1 i_2 \cdots i_m} \; \sum_{j \neq j_1 j_2 \cdots j_m} g_{ij}
\end{aligned} \tag{4.17}
$$

where

$$f_1 = (a-1)[(a-1)(b-1) + M - 2] - b + M - 1 \;,$$

$$f_2 = (a-1)[(a-1)(b-1) + M - 2] + M - 1 \;,$$

$$f_3 = (b-1)[(a-1)(b-1) + M - 2] - a + M - 1 \;,$$

$$f_4 = -[(a-1)(b-1) + M - 2] - a - b + M - 1 = -ab \;,$$

$$f_5 = -[(a-1)(b-1) + M - 2] - a + M - 1 = -b(a-1) \;,$$

$$f_6 = (b-1)[(a-1)(b-1) + M - 2] + M - 1 \;,$$

$$f_7 = -[(a-1)(b-1) + M - 2] - b + M - 1 = -a(b-1) \;,$$

$$f_8 = -[(a-1)(b-1) + M - 2] + M - 1 = -(a-1)(b-1) + 1 \;,$$

$$F = [(a-1)(b-1) - 1][(a-1)(b-1) + M - 1]$$

and $g_{i_\ell j_m}$ was defined in (4.13).

Similar formulas hold for $X_{i_2 j_2}$, $X_{i_3 j_3}$, $\cdots$, $X_{i_m j_m}$. The complicated formulas for $X_{i_1 j_1}$, $X_{i_2 j_2}$, $\cdots$ $X_{i_m j_m}$ in (4.17) can be visualized more readily for a particular design by using the following notation:

Suppose we have a 4 x 5 design with missing cells (1,1), (2,2) and (3,3). Then (4.17) can be written as

$$X_{11} = \mu + r_1 + c_1 + \frac{1}{F}\Big[f_1 \sum_{j=2,3} g_{1j} + f_2 \sum_{j\neq 1,2,3} g_{1j} + \cdots + f_8 \sum_{i\neq 1,2,3}\;\sum_{j\neq 1,2,3} g_{ij}\Big] \tag{4.18}$$

$$= \mu + r_1 + c_1 + \frac{J_4'}{F} \cdot \begin{bmatrix} 0 & f_1 \cdot g_{12} & f_1 \cdot g_{13} & f_2 \cdot g_{14} & f_2 \cdot g_{15} \\ f_3 \cdot g_{21} & 0 & f_4 \cdot g_{23} & f_5 \cdot g_{24} & f_5 \cdot g_{25} \\ f_3 \cdot g_{31} & f_4 \cdot g_{32} & 0 & f_5 \cdot g_{34} & f_5 \cdot g_{35} \\ f_6 \cdot g_{41} & f_7 \cdot g_{42} & f_7 \cdot g_{43} & f_8 \cdot g_{44} & f_8 \cdot g_{45} \end{bmatrix} \cdot J_5 \tag{4.19}$$

where J_4 and J_5 are (4 x 1) and (5 x 1) column vectors of one's.

Similarly,

$$X_{22} = \mu + r_2 + c_2 + \frac{1}{F}[f_1 \sum_{j=1,3} g_{2j} + f_2 \sum_{j\neq 1,2,3} g_{2j} + \cdots$$

$$+ f_8 \sum_{i\neq 1,2,3} \sum_{j\neq 1,2,3} g_{ij}]$$

$$= \mu + r_2 + c_2 + \frac{J_4'}{F} \cdot \begin{bmatrix} 0 & f_3\cdot g_{12} & f_4\cdot g_{13} & f_5\cdot g_{14} & f_5\cdot g_{15} \\ f_1\cdot g_{21} & 0 & f_1\cdot g_{23} & f_2\cdot g_{24} & f_2\cdot g_{25} \\ f_4\cdot g_{31} & f_3\cdot g_{32} & 0 & f_5\cdot g_{34} & f_5\cdot g_{35} \\ f_7\cdot g_{41} & f_6\cdot g_{42} & f_7\cdot g_{43} & f_8\cdot g_{44} & f_8\cdot g_{45} \end{bmatrix} \cdot J_5$$

and

$$X_{33} = \mu + r_3 + c_3 + \frac{1}{F}[f_1 \sum_{j=1,2} g_{3j} + f_2 \sum_{j\neq 1,2,3} g_{3j} + \cdots$$

$$+ f_8 \sum_{i\neq 1,2,3} \sum_{j\neq 1,2,3} g_{ij}]$$

$$= \mu + r_3 + c_3 + \frac{J_4'}{F} \cdot \begin{bmatrix} 0 & f_4\cdot g_{12} & f_3\cdot g_{13} & f_5\cdot g_{14} & f_5\cdot g_{15} \\ f_4\cdot g_{21} & 0 & f_3\cdot g_{23} & f_5\cdot g_{24} & f_5\cdot g_{25} \\ f_1\cdot g_{31} & f_1\cdot g_{32} & 0 & f_2\cdot g_{34} & f_2\cdot g_{35} \\ f_7\cdot g_{41} & f_7\cdot g_{42} & f_6\cdot g_{43} & f_8\cdot g_{44} & f_8\cdot g_{45} \end{bmatrix} \cdot J_5 .$$

The author has found that the notation given in (4.19) made the calculations of the expected mean squares for a particular design much easier as opposed to the equations given in (4.17).

Having set up (4.17) the variance-covariance matrix, V_M, of the vector of cell means can then be set up and used in equation (3.6.1) to obtain the desired expected mean squares for rows, columns and interaction. The results of this calculation are given in Table 4.4, where the missing cells are $(i_1 j_1) \cdots (i_m j_m)$; SO is defined in Table 4.3; $v_{rc}(X_{ij})$, $c_{rc}(X_{ij}, X_{k\ell})$, $v_e(X_{ij})$ and $c_e(X_{ij}, X_{k\ell})$ are derived directly from equation (4.17) (as was done previously in the case of two missing cells using (4.13));

$$\Sigma_1 = \sum_{i=i_1 i_2 \cdots i_m} \sum_{\substack{j=j_1 j_2 \cdots j_m \\ \neq i}} \frac{1}{n_{ij}} ,$$

$$\Sigma_2 = \sum_{i=i_1 i_2 \cdots i_m} \sum_{j \neq j_1 j_2 \cdots j_m} \frac{1}{n_{ij}} ,$$

$$\Sigma_3 = \sum_{i \neq i_1 i_2 \cdots i_m} \sum_{j=j_1 j_2 \cdots j_m} \frac{1}{n_{ij}} ,$$

$$\Sigma_4 = \sum_{i \neq i_1 i_2 \cdots i_m} \sum_{j \neq j_1 j_2 \cdots j_m} \frac{1}{n_{ij}} ;$$

$f_1, \cdots, f_8$, F are defined in (4.17);

$$FI = \frac{2}{F} \left(\frac{f_1 \cdot \Sigma_1}{(b-1)} + \frac{f_2 \cdot \Sigma_2}{(b-1)} + \frac{f_3 \cdot \Sigma_1}{(a-1)} - \frac{f_4 \cdot \Sigma_1}{(a-1)(b-1)} - \frac{f_5 \cdot \Sigma_2}{(a-1)(b-1)} + \frac{f_6 \cdot \Sigma_3}{(a-1)} \right.$$

$$\left. - \frac{f_7 \cdot \Sigma_3}{(a-1)(b-1)} - \frac{f_8 \cdot M \cdot \Sigma_4}{(a-1)(b-1)} \right) \qquad \text{and}$$

$$C_M = \frac{(a-1)(b-1)}{(a-1)(b-1) - M} .$$

Table 4.4 Expected mean squares for a two-way disproportionate design with M missing cells in different rows and columns using the method of unweighted means

Source	Coefficients of Variance Components in E(MS)			
	σ_e^2	σ_{rc}^2	σ_c^2	σ_r^2
Rows	$K1 = \frac{1}{ab}\Big[SO + \sum_{ij=i_1j_1}^{i_mj_m} v_e(X_{ij}) - 2\sum_{ij=i_1j_1}^{i_mj_m}\sum_{k\ell>ij}^{i_mj_m} \frac{c_e(X_{ij},X_{k\ell})}{(a-1)} + \frac{2}{F}\Big\{f_1\cdot\Sigma_1+f_2\cdot\Sigma_2-([f_3+f_4]\cdot\Sigma_1+(M-1)f_5\cdot\Sigma_2 + (f_6+(M-1)f_7)\cdot\Sigma_3+f_8\cdot M\cdot\Sigma_4)/(a-1)\Big\}\Big]$	$K2 = 1 + \sum_{ij=i_1j_1}^{i_mj_m} v_{rc}(X_{ij})/ab - 2\sum_{ij=i_1j_1}^{i_mj_m}\sum_{k\ell>ij}^{i_mj_m} \frac{c_{rc}(X_{ij},X_{k\ell})}{ab(a-1)} + \frac{M}{ab}$		b

continued

Table 4.4 (continued)

Source	Coefficients of Variance Components in E(MS)			
	σ_e^2	σ_{rc}^2	σ_c^2	σ_r^2
Columns	K3 = $\frac{1}{ab}\Big[SO + \sum_{ij=i_1j_1}^{i_mj_m} v_e(X_{ij})$ $- 2\sum_{ij=i_1j_1}^{i_mj_m}\sum_{k\ell>ij}^{i_mj_m}\frac{c_e(X_{ij},X_{k\ell})}{(b-1)}$ $+ \frac{2}{F}\Big(f_3\cdot\Sigma_1+f_6\cdot\Sigma_3-((f_1+f_4)\cdot\Sigma_1+(M-1)f_7\cdot\Sigma_3$ $+ (f_2+(M-1)f_5)\cdot\Sigma_2+f_8\cdot M\cdot\Sigma_4)/(b-1)\Big)\Big]$	K4 = $1 + \sum_{ij=i_1j_1}^{i_mj_m} v_{rc}(X_{ij})/ab$ $- 2\sum_{ij=i_1j_1}^{i_mj_m}\sum_{k\ell>ij}^{i_mj_m}\frac{c_{rc}(X_{ij},X_{k\ell})}{ab(b-1)}$ $+ \frac{M}{ab}$	a	

continued

Table 4.4 (continued)

Source	Coefficients of Variance Components in E(MS)			
	σ^2_e	σ^2_{rc}	σ^2_c	σ^2_r
Interaction	K5 = $\frac{C_M}{ab}\left[SO + \sum_{ij=i_1j_1}^{i_mj_m} v_e(X_{ij}) + 2\sum_{ij=i_1j_1}^{i_mj_m}\sum_{k\ell>ij}^{i_mj_m} \frac{c_e(X_{ij},X_{k\ell})}{(a-1)(b-1)} - FI\right]$	1		
Error	1			

Note that for M equal to two the results in Table 4.4 reduce to those given in Table 4.3.

As an example of the expected mean squares given in Table 4.4 consider the following 3 x 3 design with missing cells (1,1), (2,2) and (3,3):

X	n_{12}	n_{13}
n_{21}	X	n_{23}
n_{31}	n_{32}	X

For this design equation (4.19) becomes

$$X_{11} = \mu + r_1 + c_1 + \frac{J_3'}{F} \cdot \begin{bmatrix} 0 & f_1 \cdot g_{12} & f_1 \cdot g_{13} \\ f_3 \cdot g_{21} & 0 & f_4 \cdot g_{23} \\ f_3 \cdot g_{31} & f_4 \cdot g_{32} & 0 \end{bmatrix} \cdot J_3 \quad (4.20)$$

where $J_3' = [1,1,1]$,

$f_1 = a[(a-2)(b-1) + 1]$,

$f_3 = b[(a-1)(b-2) + 1]$,

$f_4 = -ab$, $F = [(a-1)(b-1) - 1][(a-1)(b-1) + 2]$ and

g_{ij} was defined in (4.13) .

Note for this particular design that f_2, f_5, f_6, f_7 and f_8 given in (4.17) do not exist since there is a missing cell in every row and column. Similarly,

$$X_{22} = \mu + r_2 + c_2 + \frac{J_3'}{F} \cdot \begin{array}{|c|c|c|} \hline 0 & f_3 \cdot g_{12} & f_4 \cdot g_{13} \\ \hline f_1 \cdot g_{21} & 0 & f_1 \cdot g_{23} \\ \hline f_4 \cdot g_{31} & f_3 \cdot g_{32} & 0 \\ \hline \end{array} \cdot J_3$$

and

$$X_{33} = \mu + r_3 + c_3 + \frac{J_3'}{F} \cdot \begin{array}{|c|c|c|} \hline 0 & f_4 \cdot g_{12} & f_3 \cdot g_{13} \\ \hline f_4 \cdot g_{21} & 0 & f_3 \cdot g_{23} \\ \hline f_1 \cdot g_{31} & f_1 \cdot g_{32} & 0 \\ \hline \end{array} \cdot J_3 \quad .$$

Using the results given in Table 4.4 the coefficients of the expected mean squares for this 3 x 3 example can then be shown to be

$$K1 = \frac{1}{ab} \Bigg[SO + v_e(X_{11}) + v_e(X_{22}) + v_e(X_{33}) \frac{-2c_e(X_{11},X_{22}) - 2c_e(X_{11}X_{33}) - 2c_e(X_{22},X_{33})}{(a-1)} + \frac{2 \cdot \Sigma_1}{F} \left(f_1 - \frac{(f_3 + f_4)}{(a-1)}\right) \Bigg] \quad ,$$

$$K2 = 1 + \frac{1}{ab} \Bigg[3 + v_{rc}(X_{11}) + v_{rc}(X_{22}) + v_{rc}(X_{33}) - \frac{2}{(a-1)} \left(c_{rc}(X_{11},X_{22}) + c_{rc}(X_{11},X_{33}) + c_{rc}(X_{22},X_{33})\right) \Bigg],$$

$$K3 = \frac{1}{ab}\left[SO + v_e(X_{11}) + v_e(X_{22}) + v_e(X_{33})\right.$$

$$\frac{-2c_e(X_{11},X_{22}) - 2c_e(X_{11},X_{33}) - 2c_e(X_{22},X_{33})}{(b-1)}$$

$$\left. + \frac{2\cdot\Sigma_1}{F}\left(f_3 - \frac{(f_1 + f_4)}{(b-1)}\right)\right] ,$$

$$K4 = 1 + \frac{1}{ab}\left[3 + v_{rc}(X_{11}) + v_{rc}(X_{22}) + v_{rc}(X_{33})\right.$$

$$\left. - \frac{2}{(b-1)}\left(c_{rc}(X_{11},X_{22}) + c_{rc}(X_{11},X_{33}) + c_{rc}(X_{22},X_{33})\right)\right]$$

and $$K_5 = \frac{C_3}{ab}\left[SO + v_e(X_{11}) + v_e(X_{22}) + v_e(X_{33})\right.$$

$$+ \frac{2c_e(X_{11},X_{22}) + 2c_e(X_{11},X_{33}) + 2c_e(X_{22},X_{33})}{(a-1)(b-1)}$$

$$\left. - \frac{2\cdot\Sigma_1}{F}\left(\frac{f_1}{b-1} + \frac{f_3}{a-1} - \frac{f_4}{(a-1)(b-1)}\right)\right]$$

where $$SO = \Sigma_1 = \sum_{i} \sum_{j} \frac{1}{n_{ij}} \quad (\neq \text{missing cells}), \qquad C_3 = \frac{(a-1)(b-1)}{(a-1)(b-1) - 3} ,$$

$$v_{rc}(X_{11}) = \frac{2}{F^2}\left[f_1^2 + f_3^2 + f_4^2\right] ,$$

$$v_e\,(X_{11}) = \frac{1}{F^2}\left[f_1^2\left(\frac{1}{n_{12}} + \frac{1}{n_{13}}\right) + f_3^2\left(\frac{1}{n_{21}} + \frac{1}{n_{31}}\right) + f_4^2\left(\frac{1}{n_{23}} + \frac{1}{n_{33}}\right)\right] ,$$

$$c_{rc}(X_{11},X_{22}) = \frac{2}{F^2}[f_1f_3 + f_1f_4 + f_3f_4] ,\qquad \text{and}$$

$$c_e\ (X_{11},X_{22}) = \frac{1}{F^2}[f_1f_3\ (\frac{1}{n_{12}} + \frac{1}{n_{21}}) + f_1f_4(\frac{1}{n_{13}} + \frac{1}{n_{23}})$$

$$+ f_3f_4(\frac{1}{n_{31}} + \frac{1}{n_{32}})\ ] \ .$$

Similar formulas can be derived for $v_{rc}(X_{22})$, $v_{rc}(X_{33})$, $v_e(X_{22})$, $v_e(X_{33})$, $c_{rc}(X_{11}, X_{33})$, $c_{rc}(X_{22}, X_{33})$, $c_e(X_{11}, X_{33})$ and $c_e(X_{22}, X_{33})$ by using (4.20).

The results given in Tables 4.3 and 4.4 for the expected mean squares for designs with missing cells in different rows and columns can be used along with equation (3.22) to obtain unbiased estimates of the variance components σ_r^2, σ_c^2, σ_{rc}^2 and σ_e^2 (Procedure M estimates). These variance components can also be estimated by again using the biased estimating equations given in (3.24) (Procedure $\hat{M}$ estimates). The variances of the Procedure M and $\hat{M}$ estimated variance components can then be derived analytically by using (3.26) and (3.28). However, as in Chapter 3 and Section 4.2 above, the variances of the estimated variance components are so complex that it did not appear worthwhile to derive them analytically; and therefore, they are examined numerically in Chapter 5 by using the computer program VC for both Procedure M and $\hat{M}$ estimates for several different design patterns.

The results presented for expected mean squares in this section have produced, as the reader has no doubt noticed, complex formulas which do not appear to reduce to forms similar to the relatively simple formulas derived for designs with one missing cell, Table 3.1, or missing cells in the same row (or column), Tables 4.1 and 4.2. Accordingly, it seems worthwhile to examine these complex formulas numerically for various designs to determine if there is any possibility of simplifying them. For example, for certain designs can they be approximated well by the equations given in (3.23)? Chapter 5 presents the numerical values of the coefficients K1, $\cdots$, K5 in Tables 4.3 and 4.4 for several different designs and then examines the possibility of simplifying or approximating these coefficients.

In this chapter we have examined the expected mean squares for the method of unweighted means for designs with missing cells in the same row and in different rows and columns. The procedures employed in these cases can also be used to obtain the expected mean squares for designs that have a combination of these two cases. That is, for designs that have some missing cells in the same row (or column) and some missing cells in different rows and columns. For example, a design with the following n_{ij} pattern:

$$\begin{array}{|c|c|c|} \hline X & n_{12} & n_{13} \\ \hline n_{21} & n_{22} & X \\ \hline n_{31} & n_{32} & X \\ \hline \end{array} \quad . \qquad (4.21)$$

Due to the wide variety of design patterns for these types of designs no general formula for their expected mean squares has been derived but it seems obvious that these formulas will be a compromise between the cases of M cells missing in one row (or column) and M cells missing in different rows and columns. Of course, for any particular design pattern, such as that given in (4.21), the expected mean squares can be derived by using the procedures described in detail in the present chapter. In Chapter 5 numerical results are obtained for the expected mean squares and the variances of the resulting estimated variance components (using Procedures M and $\hat{M}$) for several designs which have some missing cells in the same row and some missing cells in different rows and columns.

5. COMPUTER STUDY

5.1 Introduction

The purpose of this chapter is to describe the results obtained by using a computer program, VC, which computes and prints out the following quantities for a given two-way disproportionate design (with missing cells) and a given parameter set (e.g., $\sigma_r^2 = 16$, $\sigma_c^2 = 4$, $\sigma_{rc}^2 = 1$ and $\sigma_e^2 = 1$):

1. the coefficients K1, K2, K3, K4 and K5 of the expected mean squares for the method of unweighted means (see Tables 3.1 and 4.1 through 4.4, these coefficients are used in Procedure M);
2. the variances of the unbiased estimated variance components obtained by Procedure M (recall that Procedure M estimates the variance components by using (3.22));
3. the variances of the biased estimated variance components obtained by Procedure $\hat{M}$ (Procedure $\hat{M}$ estimates the variance components by using (3.24));
4. the variances of the unbiased estimated variance components obtained by a procedure based upon the method of fitting constants (This procedure computes the mean squares for rows adjusted for columns, for columns adjusted for rows, and for interaction adjusted for rows and columns. The analysis of variance procedure is then used to estimate the variance components. This procedure will be called Procedure A. It has been examined in detail by Bush and Anderson [1963]).

The above output from the computer program VC gives the information necessary to (a) study the coefficients K1 ⋯ K5 to determine if they can be approximated by simpler formulas, (b) study the biases of the estimated variance components obtained by Procedure $\hat{M}$, (c) study the magnitudes of the variances and mean square errors of the estimated variance components obtained by Procedures M and $\hat{M}$ relative to each other and to the corresponding variances obtained by Procedure A and (d) study for Procedures M, $\hat{M}$ and A how the variances and mean square errors of the estimated variance components vary over different designs (n_{ij} patterns) and over different parameter sets.

A brief description of VC will be given first.

5.2 General Description of the Computer Program VC

In general, for a particular design (i.e., a given n_{ij} pattern) and a particular parameter set VC computes and prints out the quantities given above in 1 through 4. The program then checks to see if another parameter set has been inputed for the design under study and if so recomputes 1 through 4 for this new set. If a new parameter set is not given then the program reads in another design and parameter set etc. The main inputs to VC for each design under study besides the n_{ij} pattern of the design are (a) parameter sets for σ_r^2, σ_c^2, σ_{rc}^2 and σ_e^2, (b) the component matrices (see (5.2)) of the variance-covariance matrix of the cell means and (c) matrices needed in the computations of the variances of the estimated variance components for Procedure A (see Bush and Anderson [1963]). VC was written in Fortran IV for the IBM System/360 model 75.

In particular, the procedure used by VC to compute K1 ··· K5 for the method of unweighted means is the following:

Recall from Chapter 3 that for any mean square (MS = $\underline{y}'G\underline{y}$) computed by unweighted means the formula for computing its expected value for a particular design is

$$E(MS) = E(\underline{y}'G\underline{y}) = tr(V\cdot G) = k_1\sigma_e^2 + k_2\sigma_{rc}^2 + k_3\sigma_c^2 + k_4\sigma_r^2 \tag{5.1}$$

where V is the variance-covariance matrix of the cell means, $\underline{y}$, and the k_i's are coefficients of the variance components which depend on the dimensions and n_{ij} pattern of the design. In order to use (5.1) to compute the individual coefficients k_1, k_2, k_3 and k_4 which multiply σ_e^2, σ_{rc}^2, σ_c^2 and σ_r^2 respectively it is necessary to partition the V matrix into component matrices that contain only terms that involve σ_e^2, σ_{rc}^2, ···. For example, suppose we have a 2 x 2 design; *i.e.*, the n_{ij} arrangement is given by

n_{11}	n_{12}
n_{21}	n_{22}

.

Then the variance-covariance matrix of the cell means for this design has the form

$$V = \left[\begin{array}{c|c|c|c} \sigma^2_{r+c} + \sigma^2_{11} & \sigma^2_r & \sigma^2_c & 0 \\ \hline \sigma^2_r & \sigma^2_{r+c} + \sigma^2_{12} & 0 & \sigma^2_c \\ \hline \sigma^2_c & 0 & \sigma^2_{r+c} + \sigma^2_{21} & \sigma^2_r \\ \hline 0 & \sigma^2_c & \sigma^2_r & \sigma^2_{r+c} + \sigma^2_{22} \end{array}\right]$$

where $\sigma^2_{r+c} = \sigma^2_r + \sigma^2_c$ and $\sigma^2_{ij} = \sigma^2_{rc} + \dfrac{\sigma^2_e}{n_{ij}}$.

This variance-covariance matrix can be partitioned into

$$V = \left[\begin{array}{cc|cc} \sigma^2_r & \sigma^2_r & 0 & 0 \\ \sigma^2_r & \sigma^2_r & 0 & 0 \\ \hline 0 & 0 & \sigma^2_r & \sigma^2_r \\ 0 & 0 & \sigma^2_r & \sigma^2_r \end{array}\right] + \left[\begin{array}{cc|cc} \sigma^2_c & 0 & \sigma^2_c & 0 \\ 0 & \sigma^2_c & 0 & \sigma^2_c \\ \hline \sigma^2_c & 0 & \sigma^2_c & 0 \\ 0 & \sigma^2_c & 0 & \sigma^2_c \end{array}\right] + \left[\begin{array}{cc|cc} \sigma^2_{rc} & 0 & 0 & 0 \\ 0 & \sigma^2_{rc} & 0 & 0 \\ \hline 0 & 0 & \sigma^2_{rc} & 0 \\ 0 & 0 & 0 & \sigma^2_{rc} \end{array}\right]$$

$$+ \left[\begin{array}{cc|cc} \dfrac{\sigma^2_e}{n_{11}} & 0 & 0 & 0 \\ 0 & \dfrac{\sigma^2_e}{n_{12}} & 0 & 0 \\ \hline 0 & 0 & \dfrac{\sigma^2_e}{n_{21}} & 0 \\ 0 & 0 & 0 & \dfrac{\sigma^2_e}{n_{22}} \end{array}\right] = V_r + V_c + V_{rc} + V_e \quad . \qquad (5.2)$$

Using (5.2) we can write (5.1) as

$$E(MS) = tr(V \cdot G) = tr[(V_e + V_{rc} + V_c + V_r) \cdot G]$$

$$= tr(V_e \cdot G) + tr(V_{rc} \cdot G) + tr(V_c \cdot G) + tr(V_r \cdot G) \qquad (5.3)$$

$$= k_1\sigma_e^2 + k_2\sigma_{rc}^2 + k_3\sigma_c^2 + k_4\sigma_r^2$$

where $\quad tr(V_e \cdot G) = k_1\sigma_e^2 \quad$ etc.

Thus, the matrices V_e, V_{rc}, V_c and V_r are used as input to VC and then (5.3) is used to calculate K1 ··· K5 for the method of unweighted means.

Having computed K1 ··· K5, VC then computes the variances of the estimated variance components obtained by Procedure M by using equation (3.26). The variances of the estimated variance components for Procedure $\hat{M}$ are calculated by VC by using (3.28) and the corresponding variances for Procedure A are obtained by the computational method described in detail in Bush and Anderson [1963].

5.3 Designs Investigated

The designs investigated by using the computer program VC consisted of 3 x 3, 4 x 4 and 6 x 6 designs with one or more missing cells. The 3 x 3 designs investigated were derived from those studied by Bush and Anderson [1963] and Hirotsu [1966] by deleting one, two or three cells from the full (i.e. no missing cells)3 x 3 designs that these authors studied. These 3 x 3 designs are given in Table 5.1 along with the base designs (i.e., the full 3 x 3 designs) from which they were derived. The 4 x 4 and 6 x 6 designs examined by VC and the base designs from which they were derived are given in Tables 5.2 and 5.3, respectively. The notation used to denote the various designs in

Table 5.1 The n_{ij} arrangements for the 3 x 3 designs investigated and the base designs from which they were derived[a]

Design	Base Design					
	18	16-1	14-2	12-3		
	2 2 2	X 2 2	X 2 2	X 2 2		
A.	2 2 2	2 2 2	2 X 2	2 X 2		
	2 2 2	2 2 2	2 2 2	2 2 X		
	18	17-1	15-1	15-2	13-2	12-3
	1 1 1	X 1 1	1 1 1	X 1 1	1 1 1	X 1 1
B.	2 2 2	2 2 2	2 2 2	2 X 2	2 X 2	2 X 2
	3 3 3	3 3 3	3 3 X	3 3 3	3 3 X	3 3 X
	18	17-1	16-1	14-2	13-2	12-3
	1 2 3	X 2 3	1 2 3	X 2 3	1 2 3	X 2 3
C.	2 3 1	2 3 1	2 3 1	2 X 1	2 X 1	2 X 1
	3 1 2	3 1 2	3 1 X	3 1 2	3 1 X	3 1 X
		15-1	12-2	9-3		
		1 2 X	1 2 X	1 2 X		
		2 3 1	2 X 1	2 X 1		
		3 1 2	3 1 2	X 1 2		
	18	17-1	13-2	12-3	14-1	10-2
	1 1 4	X 1 4	X 1 4	X 1 4	1 1 X	1 1 X
D.	1 4 1	1 4 1	1 X 1	1 X 1	1 4 1	1 X 1
	4 1 1	4 1 1	4 1 1	4 1 X	4 1 1	4 1 1
	18	17-1	13-1	15-2	11-2	10-3
	1 1 1	X 1 1	1 1 1	X 1 1	1 1 1	X 1 1
E.	1 2 3	1 2 3	1 2 3	1 X 3	1 X 3	1 X 3
	1 3 5	1 3 5	1 3 X	1 3 5	1 3 X	1 3 X

[a]An X in a cell of the design means there are no observations in this cell.

Table 5.2 The n_{ij} arrangements for the 4 x 4 designs investigated and the base designs from which they were derived[a]

Design	Base Design				
A.	32 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	30-1 X 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	28-2-1 X 2 X 2 2 2 2 2 2 2 2 2 2 2 2 2	28-2-2 X 2 2 2 2 X 2 2 2 2 2 2 2 2 2 2	26-3 X X 2 2 X 2 2 2 2 2 2 2 2 2 2 2
		24-4-1 X X 2 2 X X 2 2 2 2 2 2 2 2 2 2	24-4-2 X 2 2 2 2 X 2 2 2 2 X 2 2 2 2 X		
B.	32 1 1 1 1 1 2 2 2 1 2 3 3 1 2 3 6	31-1 X 1 1 1 1 2 2 2 1 2 3 3 1 2 3 6	30-2 X 1 X 1 1 2 2 2 1 2 3 3 1 2 3 6	24-2 1 1 1 1 1 2 2 2 1 2 3 3 1 X 3 X	29-2 X 1 1 1 1 X 2 2 1 2 3 3 1 2 3 6
		23-2 1 1 1 1 1 2 2 2 1 2 X 3 1 2 3 X	29-3 X X 1 1 X 2 2 2 1 2 3 3 1 2 3 6	20-3 1 1 1 1 1 2 2 2 1 2 3 X 1 2 X X	27-4 X X 1 1 X X 2 2 1 2 3 3 1 2 3 6
		20-4 X 1 1 1 1 X 2 2 1 2 X 3 1 2 3 X	17-4 1 1 1 1 1 2 2 2 1 2 X X 1 2 X X		

[a]An X in a cell of the design means there are no observations in this cell.

Table 5.3 The n_{ij} arrangements for the 6 x 6 designs investigated and the base designs from which they were derived[a]

Design | Base Design

A.

48 (Base Design)

1	1	1	1	1	1
1	1	1	1	2	2
1	1	1	1	2	2
2	2	1	1	1	1
2	2	1	1	2	2
1	1	1	1	2	2

46-2

X	1	1	1	1	1
1	X	1	1	2	2
1	1	1	1	2	2
2	2	1	1	1	1
2	2	1	1	2	2
1	1	1	1	2	2

44-3

1	1	1	1	1	1
1	1	1	1	2	2
1	1	1	1	2	2
2	2	1	1	1	1
2	2	1	1	2	2
1	X	1	X	2	X

42-4

1	1	1	1	1	1
1	1	1	1	2	2
1	1	X	1	2	2
2	2	1	X	1	1
2	2	1	1	X	2
1	1	1	1	2	X

40-4

1	1	1	1	1	1
1	1	1	1	2	2
1	1	1	1	2	2
2	2	1	1	1	1
2	2	1	1	X	X
1	1	1	1	X	X

B.

58 (Base Design)

1	1	1	1	1	1
1	1	2	2	2	2
1	2	3	3	3	3
1	2	1	1	1	1
1	1	1	2	2	2
1	1	1	1	1	6

56-2

X	1	1	1	1	1
1	X	2	2	2	2
1	2	3	3	3	3
1	2	1	1	1	1
1	1	1	2	2	2
1	1	1	1	1	6

50-3

1	1	1	1	1	1
1	1	2	2	2	2
1	2	3	3	3	3
1	2	1	1	1	1
1	1	1	2	2	2
1	X	1	X	1	X

46-4

1	1	1	1	1	1
1	1	2	2	2	2
1	2	X	3	3	3
1	2	1	X	1	1
1	1	1	2	X	2
1	1	1	1	1	X

47-4

1	1	1	1	1	1
1	1	2	2	2	2
1	2	3	3	3	3
1	2	1	1	1	1
1	1	1	2	X	X
1	1	1	1	X	X

continued

Table 5.3 (continued)

Design	Base Design 76						72-2						68-3					
	2	2	1	1	1	1	X	2	1	1	1	1	2	2	1	1	1	1
	2	2	1	1	3	3	2	X	1	1	3	3	2	2	1	1	3	3
C.	1	1	1	1	3	3	1	1	1	1	3	3	1	1	1	1	3	3
	3	3	1	1	1	1	3	3	1	1	1	1	3	3	1	1	1	1
	3	3	1	1	6	6	3	3	1	1	6	6	3	3	1	1	6	6
	1	1	1	1	6	6	1	1	1	1	6	6	1	X	1	X	6	X

62-4						52-4					
2	2	1	1	1	1	2	2	1	1	1	1
2	2	1	1	3	3	2	2	1	1	3	3
1	1	X	1	3	3	1	1	1	1	3	3
3	3	1	X	1	1	3	3	1	1	1	1
3	3	1	1	X	6	3	3	1	1	X	X
1	1	1	1	6	X	1	1	1	1	X	X

[a]An X in a cell of the design means there are no observations in this cell.

Tables 5.1 through 5.3 is as follows: within the 3 x 3, 4 x 4 and 6 x 6 designs, the designs that come from the same base design are labeled A or B or etc. Thus, for the 3 x 3 designs there are the 5 base design patterns A, B, C, D and E studied by Bush and Anderson [1963]. Then within designs that come from the same base design the designs are first given a number which represents the total number of observations in the design, and then a number representing the number of empty cells in the design. For example, for the 3 x 3 - A designs given in Table 5.1 the first design investigated is labeled 16-1. This means that there are a total of 16 observations and one missing cell in the design. For some of the 4 x 4 - A designs (Table 5.2) a third index number had to be added to distinguish between designs with the same number of observations and missing cells. In these cases the third number is nothing more than an index which labels the designs in order of appearance.

For each of the 3 x 3 and 4 x 4 designs given in Tables 5.1 and 5.2 the following 13 sets of parameter values were used by VC for computing the variances of the estimated variance components for the three estimation procedures under study:

σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2
1	1	1/4	1
16	1	1/4	1
4	4	1/4	1
16	16	1/4	1
1	1	1	1
4	1	1	1
16	1	1	1

σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2
16	4	1	1
16	16	1	1
1	1	4	1
4	1	4	1
16	1	4	1
4	4	4	1

For each of the 6 x 6 designs given in Table 5.3 the parameters were allowed to take the following 10 sets of values:

σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2
1	1	1/4	1
16	1	1/4	1
4	4	1/4	1
1	1	1	1
16	1	1	1
4	4	1	1
16	16	1	1
1	1	4	1
4	4	4	1
16	4	4	1

Note that the above parameter combinations include the cases of $\sigma_{rc}^2 \geq$ or $< \sigma_e^2$ and σ_r^2 (σ_c^2) $\geq$ or $< \sigma_{rc}^2$. Note also that all three estimation procedures have the properties that (a) the variance of the estimated row variance component, Var ($\hat{\sigma}_r^2$), remains invariant under changes in σ_c^2, (b) Var ($\hat{\sigma}_c^2$) remains invariant under changes in σ_r^2 and (c) Var ($\hat{\sigma}_{rc}^2$) remains invariant under changes in both σ_r^2 and σ_c^2. Therefore, the variances of the estimated variance components computed by VC for the above parameter combinations imply additional combinations of interest. For example, the results for the parameter sets $\sigma_r^2 = 16$, $\sigma_c^2 = 1$, $\sigma_{rc}^2 = 4$, $\sigma_e^2 = 1$ and $\sigma_r^2 = 4$, $\sigma_c^2 = 4$, $\sigma_{rc}^2 = 4$ and $\sigma_e^2 = 1$ imply the results for the parameter set $\sigma_r^2 = 16$, $\sigma_c^2 = 4$, $\sigma_{rc}^2 = 4$, $\sigma_e^2 = 1$. All three estimation procedures also have the property that for a fixed σ_r^2 (σ_c^2), an increasing σ_{rc}^2 increases Var ($\hat{\sigma}_r^2$) (Var($\hat{\sigma}_c^2$)).

5.4 Results

In this section the results of running VC for the designs and parameter sets given in the previous section will be examined in detail. These results are summarized in Appendix Tables 9.1 through

9.10 (which give mean square errors for all three estimation procedures under study), Table 5.5 (expected mean squares for Procedure M) and Tables 5.6, 5.7 and 5.8 (biases of the estimated variance components for Procedure $\hat{M}$). Note in the following discussion that for Procedures A and M the mean square error of an estimated variance component is the same as its variance (which was computed by VC) since both of these methods are unbiased estimating procedures. On the other hand, for Procedure $\hat{M}$ the mean square error of an estimated variance component is not the same as its variance since $\hat{M}$ is a biased estimation procedure. For Procedure $\hat{M}$ the biases of the estimated variance components were computed on a desk calculator by using (3.25). The mean square errors of the estimated variance components were then computed using the square of these biases added to the variances computed by VC.

5.4.1 Mean Square Errors of the Estimated Variance Components

The results given in Tables 9.1 through 9.10 for the mean square errors of the estimated variance components for the three estimation procedures under study indicate the following general conclusions (here $\hat{\sigma}_r^2$, $\hat{\sigma}_c^2$ and $\hat{\sigma}_{rc}^2$ are used to denote estimates of the variance components, the ^ does not imply an estimate for a particular estimating procedure):

1. For all the designs and parameter values studied Procedure $\hat{M}$ compares favorably with Procedure M. In fact, Procedures M and $\hat{M}$ have mean square errors which are relatively close together especially for (a) $\hat{\sigma}_{rc}^2$ and for (b) $\hat{\sigma}_r^2$ ($\hat{\sigma}_c^2$) in the case of σ_r^2 (σ_c^2) larger than σ_{rc}^2. Moreover, both Procedures M and $\hat{M}$ compare quite favorably with Procedure A.

2. For the case of an equal number of observations in each cell (non-missing) of the design (i.e. designs 3 x 3 - A and 4 x 4 - A) the three estimation procedures give mean square errors which are identical for $\hat{\sigma}^2_{rc}$ and relatively close for $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$. The one noticeable exception to this conclusion is when there are a considerable percentage of missing cells (3 x 3 - A design 12-3) and $\sigma^2_r \leq \sigma^2_{rc}$ or σ^2_e. In this case, Procedure $\hat{M}$ is better for estimating σ^2_r and σ^2_c.

3. For designs with an unequal number of observations in the cells, if σ^2_r (σ^2_c) is much larger than σ^2_{rc} then Procedures M and $\hat{M}$ have mean square errors for $\hat{\sigma}^2_r$ $(\hat{\sigma}^2_c)$ which are usually smaller than the mean square errors for Procedure A (here "much larger" may vary somewhat depending upon the design but usually if σ^2_r $(\sigma^2_c) \geq 4\sigma^2_{rc}$ this result will hold). Also, if σ^2_{rc} is greater than σ^2_e then Procedures M and $\hat{M}$ give smaller mean square errors for $\hat{\sigma}^2_{rc}$ than does Procedure A. However, when σ^2_{rc} is less than or equal to σ^2_e then Procedure A will give smaller mean square errors for $\hat{\sigma}^2_{rc}$, particularly if there are not many missing cells in the design (e.g. 3 x 3 - B designs 17-1 and 15-1). For the case of σ^2_r (σ^2_c) less than or equal to σ^2_{rc} no one method gives in general a smaller mean square error for $\hat{\sigma}^2_r$ $(\hat{\sigma}^2_c)$ although $\hat{M}$ is quite often the best procedure; instead, the design pattern determines which method is better. However, it does appear in general that when σ^2_r (σ^2_c) and σ^2_{rc} are equal but larger than σ^2_e that Procedure $\hat{M}$ will give the smallest mean square errors for $\hat{\sigma}^2_r$ $(\hat{\sigma}^2_c)$. For the case of a relatively high percentage of

missing cells in a design (e.g. 3 x 3 designs with three missing cells) Procedure $\hat{M}$ gives mean square errors for $\hat{\sigma}_r^2$ and $\hat{\sigma}_c^2$ which are quite often much smaller than those given by Procedures A and M. This result is especially noticeable where σ_r^2 (σ_c^2) $\leqq \sigma_{rc}^2$ or σ_e^2. Table 5.4 summarizes these results for designs with an unequal number of observations in the cells.

4. The mean square errors of $\hat{\sigma}_r^2$ $(\hat{\sigma}_c^2)$ for Procedure A decrease as the number of observations in each row (column) of the design (i.e. $n_i = \sum_j n_{ij}$, $i = 1,2,\cdots,a$) become closer together. In fact, the mean square errors of $\hat{\sigma}_r^2$ $(\hat{\sigma}_c^2)$ may actually be reduced by decreasing the number of observations in the design so that the number of observations in each row (column) are nearly equal. This property of Procedure A is especially noticeable when σ_r^2 (σ_c^2) is much larger than σ_{rc}^2. For example, consider the 3 x 3 - E designs 17-1 and 10-3 for the parameter combination $\sigma_r^2 = 16$, $\sigma_c^2 = 16$, $\sigma_{rc}^2 = 1$ and $\sigma_e^2 = 1$. In this case the mean square error using Procedure A for $\hat{\sigma}_r^2$ is 338 for the 17-1 design and 302 for the 10-3 design (Table 9.5). Thus, here is a case where one design (17-1) has nearly double the number of observations of another design (10-3) and yet the mean square error of $\hat{\sigma}_r^2$ for 17-1 is larger than for 10-3. The reason for this apparent contradiction is that in 17-1 the number of observations in the three rows of the design (n_i, $i = 1,2,3$) are 2, 6 and 9 respectively while the corresponding totals for the 10-3

Table 5.4 General results using mean square errors to determine the best estimating procedure

Parameter Combinations			Best Estimating Procedure		
σ^2_r	σ^2_c	σ^2_{rc}	σ^2_r	σ^2_c	σ^2_{rc}
$> \sigma^2_{rc}$			M or $\hat{M}$		
	$> \sigma^2_{rc}$			M or $\hat{M}$	
$\leq \sigma^2_{rc}$		$\leq \sigma^2_e$	Varies with design[a]		
	$\leq \sigma^2_{rc}$	$\leq \sigma^2_e$		Varies with design[a]	
		$> \sigma^2_e$			M or $\hat{M}$
		$\leq \sigma^2_e$			A
$= \sigma^2_{rc}$		$> \sigma^2_e$	$\hat{M}$		
	$= \sigma^2_{rc}$	$> \sigma^2_e$		$\hat{M}$	

[a]Procedure $\hat{M}$ is best for estimating σ^2_r and σ^2_c if σ^2_r (σ^2_c) $\leq \sigma^2_{rc}$ or σ^2_e and there is a high percentage of missing cells in the design (e.g. 3 x 3 - E design 10-3).

design are 2, 4 and 4. Thus, the n_i's for the 10-3 design are much closer together than for the 17-1 design and this results in a smaller mean square error for $\hat{\sigma}_r^2$ using Procedure A[1,2].

This property of Procedure A indicates that this estimating procedure is not very satisfactory for estimating variance components for designs where the number of observations in the rows (columns) have wide imbalance, especially when σ_r^2 (σ_c^2) is much larger than σ_{rc}^2. Instead, Procedure M or $\hat{M}$ is recommended in these cases since these procedures do not appear to be as sensitive to the design pattern as is Procedure A. That is, the mean square errors of $\hat{\sigma}_r^2$ ($\hat{\sigma}_c^2$) using Procedures M and $\hat{M}$ usually increase as the number of observations in a design are decreased. For the 3 x 3 - E design example given above the mean square errors of $\hat{\sigma}_r^2$ using Procedures M and $\hat{M}$ are both 279 for design 17-1 and 298 and 296 respectively for design 10-3.

5. For the 6 x 6 designs investigated (Tables 9.8, 9.9 and 9.10) all three estimating procedures have mean square errors for the estimated variance components which are relatively close

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[1] Searle [1967] gives examples for a one-way classification where increasing the total number of observations increases the variances of the estimated between-group variance components using Procedure A. As in the present case, this result is due to imbalance in the number of observations in each group of his design.

[2] Gaylor [1960] shows that the optimal designs for estimating variance components in two-way classifications are as close to balanced designs as is possible under certain restrictions.

together. In fact, Procedures M and $\hat{M}$ are nearly identical. In addition, Procedures M and $\hat{M}$ have mean square errors for $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ which are approximately equal for a given parameter combination for all the 6 x 6 designs studied. For example, for the parameter combination $\sigma^2_r = 16$, $\sigma^2_c = \sigma^2_{rc} = \sigma^2_e = 1$ the mean square errors of $\hat{\sigma}^2_r$ for M and $\hat{M}$ for all the 6 x 6 - A and B designs are 107 while the corresponding values for the 6 x 6 - C designs are 106 or 107. On the other hand, the mean square errors of $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ for Procedure A vary somewhat from design to design for a given parameter combination due to the imbalances in the number of observations in the rows and columns of the 6 x 6 designs. This property of Procedure A was discussed in 4 above. The mean square errors of $\hat{\sigma}^2_{rc}$ for the 6 x 6 designs for all three estimating procedures depend mainly on the number of observations in a design; and therefore, the 6 x 6 - C designs generally have the smallest mean square errors for $\hat{\sigma}^2_{rc}$.

6. For Procedures M and $\hat{M}$, as the number of missing cells in a design increase the mean square errors of $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ also increase. However, this result does not always hold for $\hat{\sigma}^2_{rc}$. In estimating this variance component by Procedures M and $\hat{M}$ it appears that if $\sigma^2_e < \sigma^2_{rc}$ then the mean square error of $\hat{\sigma}^2_{rc}$ also increases with the number of missing cells. On the other hand, if $\sigma^2_e \geq \sigma^2_{rc}$, then the total number of observations in the design appear to be a much more important factor in determining the magnitude of the mean square error than

does the number of missing cells. For example, consider the 4×4 - B designs 20-3 and 27-4. If $\sigma^2_{rc} = .25$ and $\sigma^2_e = 1$ then Table 9.7 shows that the mean square errors of $\hat{\sigma}^2_{rc}$ for Procedures M and $\hat{M}$ are .52 and .51 respectively for design 20-3 and .33 and .34 for design 27-4. Similarly for $\sigma^2_{rc} = \sigma^2_e = 1$, the mean square errors are both 1.21 for design 20-3 and 1.05 for design 27-4. However, when $\sigma^2_{rc} = 4$ and $\sigma^2_e = 1$, this pattern is reversed. In this case the mean square errors of $\hat{\sigma}^2_{rc}$ for 20-3 are 7.74 and 7.73 respectively for M and $\hat{M}$ while for 27-4 the corresponding values are 8.40 and 8.41.

7. For Procedures M and $\hat{M}$, if σ^2_r (σ^2_c) is much larger than σ^2_{rc} then the design pattern has relatively little effect on the mean square errors of $\hat{\sigma}^2_r$ ($\hat{\sigma}^2_c$). That is, several designs can have a wide range of total number of observations and number of missing cells and still the mean square errors of the estimated variance components for rows (columns) will not change appreciably. For example, for the ten 4×4 - B designs studied by VC and for the parameter combination $\sigma^2_r = 16$, $\sigma^2_c = 16$, $\sigma^2_{rc} = .25$ and $\sigma^2_e = 1$ the range of the mean square errors for $\hat{\sigma}^2_r$ ($\hat{\sigma}^2_c$) for Procedures M and $\hat{M}$ is only 176 to 180[3] (Table 9.7).

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[3]It is interesting to note here how close the mean square errors 176 to 180 are to $2\sigma^4_r/(a-1)$ (= 171) which would be the variance of $\hat{\sigma}^2_r$ ($\hat{\sigma}^2_c$) if this estimated variance component were distributed as $\sigma^2_r \cdot X^2(a-1)/(a-1)$.

8. For designs with more cells missing in the same row than are missing in the same column (e.g. 4 x 4 - B designs 30-2 and 24-2) the mean square error of $\hat{\sigma}_r^2$ is greater than the mean square error of $\hat{\sigma}_c^2$ for all three estimating procedures. Of course, the reverse will be true for designs with more cells missing in the same column than are missing in the same row.
9. As would be expected the 3 x 3 designs had mean square errors for all three estimating procedures which were much more sensitive to the number of missing cells than were the 4 x 4 and 6 x 6 designs studied. This is due to the fact that it is not just the number of missing cells but the number of missing cells relative to the total number of cells in the design that determines the effect on the mean square errors of the estimated variance components. In fact, for the 6 x 6 designs studied where the percentage of missing cells was quite small and the number of observations large there was relatively little effect on the mean square errors as the number of missing cells was varied.

To summarize, for the majority of the designs and parameter combinations that VC computed Procedures M and $\hat{M}$ appear to be as good as or better than Procedure A for estimating variance components with respect to the mean square errors of the estimates. In fact, Procedures M and $\hat{M}$ are always better than Procedure A for estimating σ_r^2 (σ_c^2) when σ_r^2 (σ_c^2) is much larger than σ_{rc}^2. The only case where Procedure A is consistently better than Procedures M and $\hat{M}$ is for estimating σ_{rc}^2 when $\sigma_{rc}^2 \leq \sigma_e^2$ and there are a small proportion of missing cells in a design. Procedure $\hat{M}$ usually has mean square errors that are relatively close to those of Procedure M and in some cases (notably for the case of estimating σ_r^2 (σ_c^2) when (a) there are a large proportion of missing cells in a design and (b) $\sigma_r^2 < \sigma_{rc}^2$) significantly less than those of M. Furthermore, Procedure $\hat{M}$ is computationally much easier than Procedures M and A. In fact, Procedure $\hat{M}$ can easily be carried out on a desk calculator while Procedures M and A although they can be done on a desk calculator quite often require the use of an electronic computer. Accordingly, of the three estimation procedures studied for estimating variance components Procedure $\hat{M}$ is recommended. However, if one is interested only in unbiased estimates of the variance components then Procedure M is recommended over Procedure A because (a) its mean square errors are usually as small as or smaller than those of Procedure A and (b) it is computationally easier than Procedure A.

5.4.2 Expected Mean Squares for the Method of Unweighted Means

As discussed in Chapter 4, the coefficients K1, K2, K3, K4 and K5 of the expected mean squares for the method of unweighted means have rather complicated analytical formulas especially for designs with missing cells in different rows and columns. Therefore, VC was set up to calculate and print out these coefficients so that they could be examined numerically to determine if they could be approximated by simpler formulas. In particular, for the 3 x 3, 4 x 4 and 6 x 6 designs with missing cells studied it was of interest to determine how well K1 ··· K5 could be approximated by the coefficients for designs with no missing cells given in (3.7); namely, K2 = K4 = 1.0 and K1 = K3 = K5 = (harmonic mean)$^{-1}$. Recall also from Tables 3.1, and

4.1 through 4.4 that as the number of rows and columns of a design become large that K1 $\cdots$ K5 converge to the coefficients given in (3.7). In addition, examination of the analytical formulas for K5, K3 and K1 (Tables 4.3 and 4.4) indicated that perhaps these coefficients could be approximated by $\hat{K}$ (see equation (3.23)), $\hat{K}\cdot$K4 and $\hat{K}\cdot$K2 respectively. Thus, Table 5.5 presents the numerical values for K1 $\cdots$ K5, $\hat{K}$, (harmonic mean)$^{-1}$, $\hat{K}\cdot$K4 and $\hat{K}\cdot$K2 for all the designs given in Tables 5.1, 5.2 and 5.3.

Examination of Table 5.5 shows the following:

1. For all of the designs studied K5 is quite close to $\hat{K}$. However, (harmonic mean)$^{-1}$ does not seem to be a good approximation to K5. In fact, for all the cases given in Table 5.5 $\hat{K}$ is closer in magnitude to K5 than is (harmonic mean)$^{-1}$. It is interesting to note that if in the expected mean square for interaction for the method of unweighted means, $\hat{K}$ is used to estimate the variance component for interaction (recall Procedure $\hat{M}$ approximates K5 with $\hat{K}$) the bias in this estimate of σ^2_{rc} is negligible (see Table 5.6).
2. For most of the designs studied K1 and K3 are relatively close to $\hat{K}\cdot$K2 and $\hat{K}\cdot$K4, respectively. These approximations improve as the number of observations in the cells of a design become more nearly equal and for the equal sampling designs (_i.e._ $n_{ij} = n$; designs 3 x 3 - A and 4 x 4 - A) K1 equals $\hat{K}\cdot$K2 and K3 equals $\hat{K}\cdot$K4 as shown previously in Chapters 3 and 4. If $\hat{K}\cdot$K2, $\hat{K}\cdot$K4 and $\hat{K}$ are used to approximate K1, K3 and K5 and the resulting expected mean squares are then used to estimate

Table 5.5 Coefficients of the variance components in the expected mean squares for the method of unweighted means for several two-way disproportionate designs with missing cells

Analysis of Variance Table for the Method of Unweighted Means

Source	Coefficients of Variance Components in E(MS)			
	σ_e^2	σ_{rc}^2	σ_c^2	σ_r^2
Rows	K1	K2		b
Columns	K3	K4	a	
Interaction	K5	1		
Error	1			

Design	Coefficients[a]					Approximations to Coefficients			
	K2	K4	K1	K3	K5	$\hat{K}\cdot K2$	$\hat{K}\cdot K4$	$\hat{K}^b$	HM^{-1} c
3 x 3									
A.									
16-1	1.25	1.25	.625	.625	.500	.625	.625	.500	.556
14-2	1.60	1.60	.800	.800	.500	.800	.800	.500	.611
12-3	2.00	2.00	1.00	1.00	.500	1.00	1.00	.500	.667
B.									
17-1	1.25	1.25	.811	.617	.545	.704	.704	.563	.611
15-1	1.25	1.25	.729	.867	.656	.806	.806	.645	.685
15-2	1.60	1.60	.988	.821	.553	.838	.838	.524	.667
13-2	1.60	1.60	.969	1.18	702	1.07	1.07	.667	.741
12-3	2.00	2.00	1.22	1.22	.608	1.22	1.22	.611	.741
C.									
17-1	1.25	1.25	.690	.690	.577	.704	.704	.563	.611
16-1	1.25	1.25	.784	.784	.619	.791	.791	.625	.667
15-1	1.25	1.25	.815	.815	.633	.806	.806	.646	.685
14-2	1.60	1.60	.981	.981	.588	.952	.952	.595	.685
13-2	1.60	1.60	.961	.961	.688	1.07	1.07	.667	.741
12-2	1.60	1.60	1.13	1.13	.666	1.10	1.10	.690	.759
12-3	2.00	2.00	1.22	1.22	.608	1.22	1.22	.611	.741
9-3	2.00	2.00	1.50	1.50	.750	1.50	1.50	.750	.833

continued

Table 5.5 (continued)

Design	Coefficients[a]					Approximations to Coefficients			
	K2	K4	K1	K3	K5	$\hat{K}\cdot K2$	$\hat{K}\cdot K4$	$\hat{K}$[b]	HM^{-1}[c]
3 x 3									
D.									
17-1	1.25	1.25	.891	.891	.729	.899	.899	.719	.750
14-1	1.25	1.25	1.03	1.03	.792	1.02	1.02	.813	.833
13-2	1.60	1.60	1.19	1.19	.800	1.26	1.26	.786	.833
10-2	1.60	1.60	1.48	1.48	.850	1.43	1.43	.893	.917
12-3	2.00	2.00	1.50	1.50	.750	1.50	1.50	.750	.833
E.									
17-1	1.25	1.25	.865	.865	.633	.839	.839	.671	.707
13-1	1.25	1.25	.954	.954	.781	.964	.964	.771	.796
15-2	1.60	1.60	1.11	1.11	.660	1.11	1.11	.695	.763
11-2	1.60	1.60	1.37	1.37	.821	1.30	1.30	.810	.852
10-3	2.00	2.00	1.56	1.56	.774	1.56	1.56	.778	.852
4 x 4									
A.									
30-1	1.11	1.11	.556	.556	.500	.556	.556	.500	.531
28-2-1	1.33	1.22	.666	.611	.500	.666	.611	.500	.563
28-2-2	1.23	1.23	.617	.617	.500	.617	.617	.500	.563
26-3	1.44	1.44	.722	.722	.500	.722	.722	.500	.594
24-4-1	1.67	1.67	.833	.833	.500	.833	.833	.500	.625
24-4-2	1.49	1.49	.747	.747	.500	.747	.747	.500	.625
B.									
31-1	1.11	1.11	.748	.748	.630	.715	.715	.644	.667
30-2	1.33	1.22	.940	.741	.598	.823	.755	.619	.667
29-2	1.23	1.23	.840	.840	.637	.806	.806	.655	.698
24-2	1.33	1.22	.949	.861	.719	.950	.871	.714	.750
23-2	1.23	1.23	.882	.882	.736	.893	.893	.726	.760
29-3	1.44	1.44	.935	.935	.551	.850	.850	.590	.667
20-3	1.44	1.44	1.09	1.09	.763	1.09	1.09	.756	.802
27-4	1.67	1.67	1.02	1.02	.566	.997	.997	.597	.698
20-4	1.49	1.49	1.08	1.08	.725	1.08	1.08	.722	.792
17-4	1.67	1.67	1.31	1.31	.800	1.32	1.32	.792	.844
6 x 6									
A.									
46-2	1.08	1.08	.899	.886	.823	.890	.890	.824	.833
44-3	1.20	1.12	1.00	.933	.833	1.00	.933	.833	.847
42-4	1.17	1.17	.971	.985	.845	.987	.987	.844	.861
40-4	1.20	1.20	1.06	1.02	.877	1.05	1.05	.875	.889

continued

Table 5.5 (continued)

Design	Coefficients[a]					Approximations to Coefficients			
	K2	K4	K1	K3	K5	$\hat{K}\cdot K2$	$\hat{K}\cdot K4$	$\hat{K}$[b]	HM^{-1}[c]
6 x 6									
B.									
56-2	1.08	1.08	.827	.840	.763	.826	.826	.765	.778
50-3	1.20	1.12	.996	.861	.780	.940	.877	.783	.801
46-4	1.17	1.17	.945	.931	.814	.951	.951	.813	.833
47-4	1.20	1.20	1.01	.938	.806	.968	.968	.807	.829
C.									
72-2	1.08	1.08	.781	.763	.715	.773	.773	.716	.731
68-3	1.20	1.12	.847	.785	.701	.842	.786	.702	.727
62-4	1.17	1.17	.834	.853	.716	.841	.841	.719	.750
52-4	1.20	1.20	.945	.895	.771	.925	.925	.771	.796

[a] These coefficients are used in Procedure M.

[b] $\hat{K} = \frac{1}{s} \sum_{i} \sum_{j} \frac{1}{n_{ij}}$, $\ne$ (missing cells), where s is the number of occupied cells in the design.

[c] $HM^{-1} = \frac{1}{ab} \left[M + \sum_{i} \sum_{j} \frac{1}{n_{ij}} \right]$, $\ne$ (missing cells), where M is the number of missing cells in the design.

the variance components for rows and columns by the analysis of variance procedure the biases in these estimates are relatively small. That is, if we approximate the expected mean squares for rows, columns and interaction by

$$\widehat{E(MSR)} = (\hat{K}\cdot K2)\ \sigma_e^2 + K2\ \sigma_{rc}^2 + b\ \sigma_r^2$$

$$\widehat{E(MSC)} = (\hat{K}\cdot K4)\ \sigma_e^2 + K4\ \sigma_{rc}^2 + a\ \sigma_c^2$$

$$\widehat{E(MSRC)} = \hat{K}\ \sigma_e^2 \quad + \quad \sigma_{rc}^2$$

and estimate σ_r^2 and σ_c^2 from these approximate expected mean squares; then, the biases in the resulting estimates, namely

$$\text{Bias}\ (\hat{\sigma}_r^2) = \frac{(K1 - K2\cdot K5)}{b} \cdot \sigma_e^2 \qquad \text{and}$$

$$\text{Bias}\ (\hat{\sigma}_c^2) = \frac{(K3 - K4\cdot K5)}{a} \cdot \sigma_e^2$$

are quite small. For example, for the 4 x 4 - B designs studied these biases are the following when $\sigma_e^2 = 1$:

Design 4 x 4 - B	Bias in Estimated Row Component When K1 Approximated by $\hat{K}\cdot K2$	Bias in Estimated Column Component When K3 Approximated by $\hat{K}\cdot K4$
31-1	.012	.012
30-2	.036	.003
29-2	.014	.014
24-2	-.002	-.004
23-2	-.006	-.006
29-3	.035	.035
20-3	-.002	-.002
27-4	.019	.019
20-4	.000	.000
17-4	-.007	-.007

Thus, it would appear that although $\hat{K}\cdot K2$ and $\hat{K}\cdot K4$ are far from perfect approximations to K1 and K3 they are quite satisfactory in many cases when the analytical formulas for K1 and K3 are relatively complicated to compute.

3. For the designs studied neither $\hat{K}$ nor $(\text{harmonic mean})^{-1}$ are good approximations to K1 and K3 even though these quantities converge to one another as the number of rows and columns of a design become large. Even the 6 x 6 designs studied had relatively large differences between K1 and K3 versus $\hat{K}$ and $(\text{harmonic mean})^{-1}$. In addition, as the number of missing cells of a design increases so does the magnitude of the differences between these quantities.

4. For the designs studied K2 and K4 are not relatively close to 1.0, although as expected these coefficients become closer and closer to 1.0 as the number of rows and columns of the designs increases. Only for the 6 x 6 designs with two missing cells (K2 = K4 = 1.08) is the magnitude of these coefficients relatively close to 1.0. However, it should be pointed out here that K2 and K4 are independent of the n_{ij} pattern of a design. Therefore, once the values of these coefficients are computed for a particular design these values apply to all designs with the same dimensions (number of rows and columns) and number and pattern of missing cells. For example, the values of K2 and K4 for the 3 x 3 - B design 15-2 also apply to the 3 x 3 - C designs 14-2, 13-2 and 12-2. Thus, the results given in Table 5.5 give the values of K2 and K4 for most cases of interest for 3 x 3 and 4 x 4 designs with missing cells. The biases in the estimated variance components for Procedure $\hat{M}$ which sets K2 and K4 equal to 1.0 and K1, K3 and K5 equal to $\hat{K}$ are studied in Section 5.4.3.

In summary, in terms of the relative magnitude of the coefficients and the resulting biases introduced in estimated variance components it appears for the cases studied that $\hat{K}$ is a good approximation for K5 and $\hat{K} \cdot K2$ and $\hat{K} \cdot K4$ are reasonable approximations for K1 and K3 respectively. K2 and K4 cannot be approximated well by 1.0 for the 3 x 3 and 4 x 4 designs but for the 6 x 6 designs this approximation is fairly good if the design does not have a large number of missing cells. Of course, in practice how well K1 ... K5 are approximated by simpler formulas is a function of how the resulting expected mean squares are to be used. In terms of using expected mean squares to estimate variance components, Section 5.4.1 indicated that Procedure $\hat{M}$ (which approximates K2 and K4 with 1.0 and K1, K3 and K5 with $\hat{K}$) is usually as good as or better (in terms of mean square errors of the estimates) than Procedure M which uses the exact analytical formulas K1 ... K5. The effect of using the approximations studied in this section for K1 ... K5 to set up approximate tests of hypothesis deserves some study.

In addition, it is important to remember here that designs with one missing cell or missing cells in only one row (or column) have exact analytical formulas (Chapters 3 and 4) which are relatively easy to compute; and therefore, in these cases the approximations studied here will not save much computational time.

5.4.3 Biases of the Estimated Variance Components for Procedure $\hat{M}$

The biases of the estimated variance components obtained by using Procedure $\hat{M}$ were calculated for all the designs in Tables 5.1, 5.2 and 5.3 by using equation (3.25) and the results given in Table 5.5.

These biases are given in Tables 5.6, 5.7 and 5.8 for the parameter combinations $\sigma^2_{rc} = .25$, $\sigma^2_e = 1$; $\sigma^2_{rc} = 1$, $\sigma^2_e = 1$ and $\sigma^2_{rc} = 4$, $\sigma^2_e = 1$ respectively. The most striking result in these tables is that the biases of the estimated variance component for interaction are quite small for all cases studied. Thus, when estimating σ^2_{rc} it appears that very little bias is introduced by using Procedure $\hat{M}$ as opposed to using an unbiased estimating procedure. For the estimated row and column variance components ($\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$) the tables show that the biases (for a given σ^2_{rc}, σ^2_e combination) increase as (a) the number of missing cells in a design increases and (b) the number of rows and columns of the designs decreases (i.e. the 3 x 3 designs have larger biases than do the 4 x 4 designs while the biases for the 6 x 6 designs studied are relatively small even for the case of $\sigma^2_{rc} = 4$ and $\sigma^2_e = 1$). In general it is difficult to determine from Tables 5.6-5.8 the relative importance of the biases for $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ since this will depend upon the true values of σ^2_r and σ^2_c. For example, if $\sigma^2_r = 16$ and $\sigma^2_{rc} = 1$, $\sigma^2_e = 1$ then Table 5.7 shows that the biases in $\hat{\sigma}^2_r$ for all the designs studied are less than 4 per cent of 16 (= σ^2_r). On the other hand, if $\sigma^2_r = 1$ and $\sigma^2_{rc} = 4$, $\sigma^2_e = 1$ then Table 5.8 shows that in several cases the biases in $\hat{\sigma}^2_r$ are greater than 1 (= σ^2_r). One method that is commonly used to study the relative importance of the biases of an estimation procedure is to calculate the contribution of these biases to the corresponding mean square errors of the estimators. This in effect indicates how large the bias squared is relative to the mean square error of the estimators. Accordingly, the percentage contribution of the biases to the mean square errors of $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ for

Procedure $\hat{M}$ were computed for all the designs and parameter combinations given in Section 5.2. The result of these calculations is that the maximum percentage contribution of the biases to the mean square errors is 10 per cent and in most cases the contribution is much less than 10 per cent. For example, Table 5.9 gives the results for the 3 x 3 - B designs studied. This table shows that in most cases the percentage contribution is less than 1 per cent.

Thus, it would appear for Procedure $\hat{M}$ that the biases of the estimated variance components are relatively unimportant when compared with the corresponding mean square errors.

It is also worth noting here that the biases of $\hat{\sigma}_r^2$ and $\hat{\sigma}_c^2$ in Tables 5.6-5.8 are always positive. This indicates that the Procedure $\hat{M}$ estimated row and column variance components usually have a positive bias.

Table 5.6 Biases[a] of estimated variance components obtained by Procedure $\hat{M}$ when $\sigma^2_{rc} = .25$ and $\sigma^2_e = 1$

Design	Rows	Columns	Interaction	Design	Rows	Columns	Interaction
3 x 3				4 x 4			
A.				A.			
16-1	.063	.063	.000	30-1	.021	.021	.000
14-2	.150	.150	.000	28-2-1	.061	.042	.000
12-3	.250	.250	.000	28-2-2	.044	.044	.000
B.				26-3	.083	.083	.000
17-1	.110	.045	-.018	24-4-1	.125	.125	.000
15-1	.045	.091	.011	24-4-2	.092	.092	.000
15-2	.195	.139	.029	B.			
13-2	.139	.203	.035	31-1	.036	.036	-.014
12-3	.288	.288	-.003	30-2	.106	.050	-.021
C.				29-2	.065	.065	-.018
17-1	.059	.059	.014	24-2	.078	.049	.005
16-1	.076	.076	-.006	23-2	.051	.051	.010
15-1	.082	.082	-.013	29-3	.124	.124	-.039
14-2	.181	.181	-.007	20-3	.109	.109	.007
13-2	.141	.141	.021	27-4	.155	.155	-.031
12-2	.205	.205	-.024	20-4	.119	.119	.003
12-3	.287	.287	-.003	17-4	.169	.169	.008
9-3	.333	.333	.000	6 x 6			
D.				A.			
17-1	.075	.075	.010	46-2	.016	.014	-.001
14-1	.100	.100	-.021	44-3	.036	.022	.000
13-2	.180	.180	.014	42-4	.028	.030	.001
10-2	.260	.260	-.043	40-4	.039	.032	.002
12-3	.333	.333	.000	B.			
E.				56-2	.014	.016	-.002
17-1	.098	.098	-.038	50-3	.044	.019	-.003
13-1	.079	.079	.010	46-4	.029	.027	.001
15-2	.199	.199	-.035	47-4	.042	.030	-.001
11-2	.232	.232	.011	C.			
10-3	.344	.344	-.004	72-2	.014	.011	-.001
				68-3	.033	.019	-.001
				62-4	.027	.030	-.003
				52-4	.037	.029	.000

[a]The biases were computed by using equation (3.25). Note that the biases for $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ depend only upon the values of σ^2_{rc} and σ^2_e and are independent of the size of σ^2_r and σ^2_c.

Table 5.7 Biases[a] of estimated variance components obtained by Procedure $\hat{M}$ when $\sigma^2_{rc} = 1$ and $\sigma^2_e = 1$

Design	Rows	Columns	Interaction[b]
3 x 3			
A.			
16-1	.125	.125	
14-2	.300	.300	
12-3	.500	.500	
B.			
17-1	.172	.107	
15-1	.108	.154	
15-2	.345	.289	
13-2	.289	.353	
12-3	.538	.538	
C.			
17-1	.121	.121	
16-1	.138	.138	
15-1	.144	.144	
14-2	.331	.331	
13-2	.291	.291	
12-2	.355	.355	
12-3	.537	.537	
9-3	.583	.583	
D.			
17-1	.137	.137	
14-1	.163	.163	
13-2	.330	.330	
10-2	.410	.410	
12-3	.583	.583	
E.			
17-1	.161	.161	
13-1	.141	.141	
15-2	.349	.349	
11-2	.382	.382	
10-3	.594	.594	

Design	Rows	Columns
4 x 4		
A.		
30-1	.042	.042
28-2-1	.123	.083
28-2-2	.087	.087
26-3	.166	.166
24-4-1	.251	.251
24-4-2	.184	.184
B.		
31-1	.057	.057
30-2	.168	.091
29-2	.108	.108
24-2	.140	.091
23-2	.094	.094
29-3	.206	.206
20-3	.192	.192
27-4	.281	.281
20-4	.211	.211
17-4	.295	.295
6 x 6		
A.		
46-2	.026	.024
44-3	.061	.037
42-4	.049	.052
40-4	.064	.057
B.		
56-2	.024	.026
50-3	.069	.034
46-4	.050	.048
47-4	.067	.055
C		
72-2	.024	.021
68-3	.058	.034
62-4	.048	.051
52-4	.062	.054

[a]The biases were computed by using equation (3.25). Note that the biases for $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ depend only upon the values of σ^2_{rc} and σ^2_e and are independent of the size of σ^2_r and σ^2_c.

[b]The biases for $\hat{\sigma}^2_{rc}$ depend only upon the value of σ^2_e, and therefore, the biases for $\hat{\sigma}^2_{rc}$ in the present case are the same as those given in Table 5.6.

Table 5.8 Biases[a] of estimated variance components obtained by Procedure $\hat{M}$ when $\sigma^2_{rc} = 4$ and $\sigma^2_e = 1$

Design	Rows	Columns	Interaction[b]
3 x 3			
A.			
16-1	.375	.375	
14-2	.900	.900	
12-3	1.50	1.50	
B.			
17-1	.422	.357	
15-1	.358	.404	
15-2	.945	.889	
13-2	.889	.953	
12-3	1.54	1.54	
C.			
17-1	.371	.371	
16-1	.388	.388	
15-1	.394	.394	
14-2	.931	.931	
13-2	.891	.891	
12-2	.955	.955	
12-3	1.54	1.54	
9-3	1.58	1.58	
D.			
17-1	.387	.387	
14-1	.413	.413	
13-2	.930	.930	
10-2	1.01	1.01	
12-3	1.58	1.58	
E.			
17-1	.411	.411	
13-1	.391	.391	
15-2	.949	.949	
11-2	.982	.982	
10-3	1.59	1.59	

Design	Rows	Columns
4 x 4		
A.		
30-1	.124	.124
28-2-1	.370	.248
28-2-2	.259	.259
26-3	.496	.496
24-4-1	.753	.753
24-4-2	.552	.552
B.		
31-1	.140	.140
30-2	.416	.256
29-2	.281	.281
24-2	.388	.256
23-2	.267	.267
29-3	.536	.536
20-3	.522	.522
27-4	.784	.784
20-4	.579	.579
17-4	.798	.798
6 x 6		
A.		
46-2	.066	.064
44-3	.161	.097
42-4	.134	.137
40-4	.164	.157
B.		
56-2	.064	.066
50-3	.169	.094
46-4	.135	.133
47-4	.167	.155
C.		
72-2	.064	.061
68-3	.158	.094
62-4	.133	.136
52-4	.162	.154

[a]The biases were computed by using equation (3.25). Note that the biases for $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ depend only upon the values of σ^2_{rc} and σ^2_e and are independent of the size of σ^2_r and σ^2_c.

[b]The biases for $\hat{\sigma}^2_{rc}$ depends only upon the value of σ^2_e, and therefore, the biases for $\hat{\sigma}^2_{rc}$ in the present case are the same as those given in Table 5.6.

Table 5.9 Percentage contribution[a] of the bias of row and column variance components estimated by Procedure $\hat{M}$ to the mean square error, 3 x 3 - B designs

σ^2_{rc}	σ^2_e	Design	σ^2_r	Rows		σ^2_c	Columns
		3 x 3 - B		%			%
.25	1	17-1	1	< 1		1	< 1
		15-1		< 1			< 1
		15-2		2			< 1
		13-2		< 1			2
		12-3		3			3
		17-1	16	< 1		16	< 1
		15-1		< 1			< 1
		15-2		< 1			< 1
		13-2		< 1			< 1
		12-3		< 1			< 1
1	1	17-1	1	< 1		1	< 1
		15-1		< 1			< 1
		15-2		3			2
		13-2		2			3
		12-3		6			6
		17-1	4	< 1		4	< 1
		15-1		< 1			< 1
		15-2		< 1			< 1
		13-2		< 1			< 1
		12-3		1			1
		17-1	16	< 1		16	< 1
		15-1		< 1			< 1
		15-2		< 1			< 1
		13-2		< 1			< 1
		12-3		< 1			< 1
4	1	17-1	1	2		1	1
		15-1		1			1
		15-2		6			5
		13-2		5			6
		12-3		10			10
		17-1	4	< 1		4	< 1
		15-1		< 1			< 1
		15-2		2			2
		13-2		2			2
		12-3		4			4

continued

Table 5.9 (continued)

σ^2_{rc}	σ^2_e	Design	σ^2_r	Rows
4	1	17-1	16	< 1
		15-1		< 1
		15-2		< 1
		13-2		< 1
		12-3		< 1

[a]The percentage contribution was computed by

$$\frac{[\text{bias (estimator)}]^2}{[\text{bias (estimator)}]^2 + \text{Var (estimator)}} \times (100) .$$

6. UNWEIGHTED MEANS FOR DESIGNS WITH TWO LEVELS OF NESTING

6.1 Introduction

In this chapter the method of unweighted means will be examined for estimating variance components from a two-way classification with s_{ij} classes within each (ij) cell of the design and an unequal number of observations within each of the s_{ij} classes. The mathematical model for this design can be written as

$$y_{ijk\ell} = \mu + r_i + c_j + rc_{ij} + s_{k(ij)} + e_{ijk\ell} \tag{6.1}$$

$$i = 1,2,\cdots, a;\ j = 1,2,\cdots, b;$$

$$k = 1,2,\cdots,s_{ij};\ \ell = 1,2,\cdots,n_{ijk}$$

where $y_{ijk\ell}$ is the observed value for the ℓ^{th} observation in the s_k class in the i^{th} row and j^{th} column. In the model, μ is a constant; r_i, c_j, rc_{ij}, $s_{k(ij)}$ and $e_{ijk\ell}$ are normally independently distributed random variables with means zero and variances σ_r^2, σ_c^2, σ_{rc}^2, σ_s^2 and σ_e^2, respectively. As in Chapter 3 totals and means are denoted by dropping subscripts, i.e.,

$$\sum_k n_{ijk} = n_{ij}\ , \quad \sum_j n_{ij} = n_i\ , \quad \sum_i n_{ij} = n_j\ ,$$

$$\sum_i \sum_j \sum_k n_{ijk} = n\ , \quad \frac{1}{n_{ijk}} \sum_\ell y_{ijk\ell} = \bar{y}_{ijk}\ ,$$

$$\frac{1}{n_{ij}} \sum_k \sum_\ell y_{ijk\ell} = \bar{y}_{ij}\ , \quad \frac{1}{n_i} \sum_j \sum_k \sum_\ell y_{ijk\ell} = \bar{y}_i\ ,$$

$$\frac{1}{n_j} \sum_i \sum_k \sum_\ell y_{ijk\ell} = \bar{y}_j \quad \text{and} \quad \frac{1}{n} \sum_i \sum_j \sum_k \sum_\ell y_{ijk\ell} = \bar{y}\ .$$

The model given in (6.1) will be referred to as a model with two levels of nesting. An example of a design with two levels of nesting is the following:

Columns

Rows	1	2	3
1	$s_{1(11)}$ $\boxed{n_{111}}$ $s_{2(11)}$ $\boxed{n_{112}}$	$s_{1(12)}$ $\boxed{n_{121}}$ $s_{2(12)}$ $\boxed{n_{122}}$ $s_{3(12)}$ $\boxed{n_{123}}$	$s_{1(13)}$ $\boxed{n_{131}}$ $s_{2(13)}$ $\boxed{n_{132}}$
2	$s_{1(21)}$ $\boxed{n_{211}}$	$s_{1(22)}$ $\boxed{n_{221}}$ $s_{2(22)}$ $\boxed{n_{222}}$	$s_{1(23)}$ $\boxed{n_{231}}$ $s_{2(23)}$ $\boxed{n_{232}}$
3	$s_{1(31)}$ $\boxed{n_{311}}$	$s_{1(32)}$ $\boxed{n_{321}}$ $s_{2(32)}$ $\boxed{n_{322}}$	$s_{1(33)}$ $\boxed{n_{331}}$ $s_{2(33)}$ $\boxed{n_{332}}$ $s_{3(33)}$ $\boxed{n_{333}}$

Here $a = 3$, $b = 3$, $s_{11} = s_{13} = s_{22} = s_{23} = s_{32} = 2$, $s_{12} = s_{33} = 3$ and $s_{21} = s_{31} = 1$.

In order to apply the method of unweighted means to a design with model (6.1) it is first necessary as described in Chapter 3 to compute the mean for each of the (ij) cells of the design. Now in Chapter 3 the method of unweighted means was used for designs which had model (3.1), i.e., for designs with only one level of nesting within a cell. In this case the estimation of the cell means is straightforward because the arithmetic mean of the (ij) cell is known to be the best linear unbiased estimate of the cell mean. In the present case (two levels of nesting within a (ij) cell) the best linear unbiased estimate of the (ij) cell mean is to weight the class means, $\bar{y}_{ijk}$, by the reciprocals of their variances, i.e.

$$(\text{cell mean})_{ij} = \sum_{k=1}^{s_{ij}} \frac{\bar{y}_{ijk}}{(\sigma_s^2 + \frac{\sigma_e^2}{n_{ijk}})} \Bigg/ \sum_{k=1}^{s_{ij}} \frac{1}{(\sigma_s^2 + \frac{\sigma_e^2}{n_{ijk}})} \quad . \qquad (6.2)$$

Of course, the relative values of σ_e^2 and σ_s^2 are in general unknown, hence the best linear unbiased estimate of the cell mean cannot be obtained for model (6.1). However if $\sigma_s^2 = 0$, then from (6.2) the best linear unbiased estimate of the cell mean is

$$\bar{y}_{ij} = \frac{1}{n_{ij}} \sum_{k=1}^{s_{ij}} n_{ijk}\, \bar{y}_{ijk} = \frac{1}{n_{ij}} \sum_{k=1}^{s_{ij}} \sum_{\ell=1}^{n_{ijk}} y_{ijk\ell} \qquad (6.3.1)$$

the simple arithmetic mean of all observations in the (ij) cell which weights each class mean within the (ij) cell by the number of observations in the class means (n_{ijk}). On the other hand if $\sigma_e^2 = 0$, then the best linear unbiased estimate of the cell mean is

$$\bar{y}^{*}_{ij} = \frac{1}{s_{ij}} \sum_{k=1}^{s_{ij}} \bar{y}_{ijk} = \frac{1}{s_{ij}} \sum_{k=1}^{s_{ij}} \sum_{\ell=1}^{n_{ijk}} \frac{y_{ijk\ell}}{n_{ijk}} \tag{6.3.2}$$

which gives equal weight to each class mean within the (ij) cell.

Accordingly, Section 6.2 examines the expected mean squares for designs with model (6.1) using the method of unweighted means with cell means computed by the limiting cases given in (6.3.1) and (6.3.2). Based on these two sets of expected mean squares the variance components are estimated by again using the analysis of variance procedure. In Section 6.3 the variances of the estimated variance components are studied. Throughout this chapter it is assumed that the designs under study have $n_{ijk} > 0$ for all i, j and k.

6.2 Expected Mean Squares and Estimated Variance Components

In order to estimate the variance components in model (6.1) by the analysis of variance procedure based upon the method of unweighted means using either $\bar{y}_{ij}$ or $\bar{y}^{*}_{ij}$ as an estimate of the (ij) cell mean, essentially the same procedure is followed as was described in Chapter 3. In particular, the (ij) cell means are computed by (6.3.1) or (6.3.2) and then the formulas given in (3.3) and (3.4) are used to compute the mean squares for rows, columns and interaction. Then (3.6.1) is used to calculate the expected values of these computed mean squares. The mean squares for classes nested within the (ij) cells can be computed by the following weighted formulas:

$$MSS_w \text{ (using } \bar{y}_{ij}) = MSS_w = \sum_i \sum_j \sum_k \sum_\ell (\bar{y}_{ijk} - \bar{y}_{ij})^2 / \sum_i \sum_j s_{ij} - ab$$

and (6.4.1)

$$MSS_w \text{ (using } \bar{y}^*_{ij}) = MSS^*_w = \sum_i \sum_j \sum_k \sum_\ell (\bar{y}_{ijk} - \bar{y}^*_{ij})^2 / \sum_i \sum_j s_{ij} - ab$$

[*i.e.*, the (class mean-cell mean)^2 terms are weighted by the number of observations in the class means (n_{ijk}) in (6.4.1)]. The following unweighted formulas may also be used to compute the mean squares for classes:[1]

$$MSS_{\bar{w}} \text{ (using } \bar{y}_{ij}) = MSS_{\bar{w}} = \sum_i \sum_j \sum_k (\bar{y}_{ijk} - \bar{y}_{ij})^2 / \sum_i \sum_j s_{ij} - ab$$

and (6.4.2)

$$MSS_{\bar{w}} \text{ (using } \bar{y}^*_{ij}) = MSS^*_{\bar{w}} = \sum_i \sum_j \sum_k (\bar{y}_{ijk} - \bar{y}^*_{ij})^2 / \sum_i \sum_j s_{ij} - ab$$

[*i.e.*, the (class mean-cell mean)^2 terms are given equal weight in (6.4.2)]. The expected values of the mean squares in (6.4.1) and (6.4.2) can be computed directly using the definitions in (6.1). The mean square for subclasses nested within the $s_{k(ij)}$ classes is given by

$$MSE = \sum_i \sum_j \sum_k \sum_\ell (y_{ijk\ell} - \bar{y}_{ijk})^2 / (n - \sum_i \sum_j s_{ij}) .$$

Once the expected values of the mean squares are computed the estimates of the variance components in (6.1) are obtained by equating these expected mean squares to the computed mean squares. The resulting estimates of the variance components are unbiased.

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[1] Crump [1954] discusses for various parameter combinations the use of weighted versus unweighted mean squares for a design with two levels of nesting.

As in Chapter 3 the first step in using (3.6.1) to calculate the expected mean squares for rows, columns and interaction using the method of unweighted means for a design with model (6.1) is to determine the variance-covariance matrix of the (ij) cell means $\bar{y}_{ij}$ or $\bar{y}^*_{ij}$. Now in terms of the model parameters (6.3.1) becomes

$$\bar{y}_{ij} = \mu + r_i + c_j + rc_{ij} + \sum_k \frac{n_{ijk}\, s_{k(ij)}}{n_{ij}} + \sum_k \sum_\ell \frac{e_{ijk\ell}}{n_{ij}}$$

and (6.3.2) is (6.5)

$$\bar{y}^*_{ij} = \mu + r_i + c_j + rc_{ij} + \sum_k \frac{s_{k(ij)}}{s_{ij}} + \sum_k \sum_\ell \frac{e_{ijk\ell}}{s_{ij}(n_{ijk})} \quad .$$

Thus, the variances and covariances of the cell means $\bar{y}_{ij}$ and $\bar{y}_{m\ell}$ are given by

$$\begin{aligned}
\text{Cov}\,(\bar{y}_{ij}, \bar{y}_{m\ell}) &= \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_s^2 \frac{\sum_k n_{ijk}^2}{n_{ij}^2} + \frac{\sigma_e^2}{n_{ij}} && \text{if } i = m,\ j = \ell\,, \\
&= \sigma_r^2 && \text{if } i = m,\ j \neq \ell\,, \\
&= \sigma_c^2 && \text{if } i \neq m,\ j = \ell\,,
\end{aligned}$$

and

$$= 0 \qquad \text{if } i \neq m,\ j \neq \ell\,.$$

Using the notation given in (3.5), the variance-covariance matrix, V, of the vector of cell means $\underline{y}' = (\bar{y}_{11}, \bar{y}_{12}, \cdots, \bar{y}_{ab})$ can then be written as

$$V = \left\{ D/C \right\}_a + \text{diag}\left(\frac{\sigma_s^2 \sum_k n_{ijk}^2}{n_{ij}^2} + \frac{\sigma_e^2}{n_{ij}} \right) \tag{6.6}$$

where

$$D = \left\{ \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 \,/\, \sigma_r^2 \right\}_b \quad \text{and} \quad C = \sigma_c^2 \, I_b \, .$$

Similarly, the variances and covariances of $\bar{y}^*_{ij}$ and $\bar{y}^*_{m\ell}$ are

$$\begin{aligned}
\text{Cov}\,(\bar{y}^*_{ij}, \bar{y}^*_{m\ell}) &= \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \frac{\sigma_s^2}{s_{ij}} + \sigma_e^2 \sum_k \frac{1}{(s_{ij}^2)\, n_{ijk}} && \text{if } i = m,\ j = \ell, \\
&= \sigma_r^2 && \text{if } i = m,\ j \neq \ell, \\
&= \sigma_c^2 && \text{if } i \neq m,\ j = \ell
\end{aligned}$$

and

$$= 0 \qquad \text{if } i \neq m,\ j \neq \ell.$$

Thus, the variance-covariance matrix, V^*, of the vector of cell means $\underline{y}'^* = (\bar{y}^*_{11}, \bar{y}^*_{12}, \cdots, \bar{y}^*_{ab})$ is given by

$$V^* = \left\{ D/C \right\}_a + \text{diag}\left(\frac{\sigma_s^2}{s_{ij}} + \sigma_e^2 \sum_k \frac{1}{(s_{ij})^2\, n_{ijk}} \right) \quad . \tag{6.7}$$

Using (6.6), (6.7), (3.6.1) and the formula for G_R given in (3.3) it can then be shown for the case of all $n_{ijk} > 0$ that the expected mean squares for rows using the method of unweighted means are given by

$$E(MSR/\text{using } \bar{y}_{ij}) = E(MSR) = E(\underline{y}' G_R \underline{y}) = tr(V \cdot G_R)$$

$$= a\ tr(D \cdot P) + a(a-1)\ tr(C \cdot Q) + \left(\frac{1}{ab} \sum_i \sum_j \sum_k \frac{n_{ijk}^2}{n_{ij}^2}\right) \sigma_s^2$$

$$+ \left(\frac{1}{ab} \sum_i \sum_j \frac{1}{n_{ij}}\right) \sigma_e^2$$

$$= b\ \sigma_r^2 + \sigma_{rc}^2 + \left(\frac{1}{ab} \sum_i \sum_j \sum_k \frac{n_{ijk}^2}{n_{ij}^2}\right) \sigma_s^2$$

$$+ \left(\frac{1}{ab} \sum_i \sum_j \frac{1}{n_{ij}}\right) \sigma_e^2$$

and (6.8)

$$E(MSR/\text{using } \bar{y}_{ij}^*) = E(MSR^*) = E(\underline{y}'^* G_R \underline{y}^*) = tr(V^* \cdot G_R)$$

$$= b\ \sigma_r^2 + \sigma_{rc}^2 + \left(\frac{1}{ab} \sum_i \sum_j \frac{1}{s_{ij}}\right) \sigma_s^2$$

$$+ \left(\frac{1}{ab} \sum_i \sum_j \sum_k \frac{1}{(s_{ij}^2)(n_{ijk})}\right) \sigma_e^2 .$$

In the same manner the expected mean squares for columns and interaction may be derived by using

$$E(MSC) = tr(V\cdot G_C), \quad E(MSC^*) = tr(V^*\cdot G_C) \ ,$$

$$E(MSRC) = tr(V\cdot G_{RC}) \quad \text{and} \quad E(MSRC^*) = tr(V^*\cdot G_{RC})$$

where G_C and G_{RC} are defined in (3.3).

The expected mean squares for the $s_{k(ij)}$ classes within the (ij) cells can be derived directly from (6.4.1) and (6.4.2). The results of these calculations for the weighted mean squares for classes are

$$E(MSS_w) = \sigma_e^2 + \left[\frac{(n - \sum_i \sum_j \sum_k \frac{n_{ijk}^2}{n_{ij}})}{\sum_i \sum_j s_{ij} - ab} \right] \sigma_s^2 \qquad \text{and}$$

$$E(MSS_w^*) = \sigma_e^2 + \left[\frac{(n - \sum_i \sum_j \frac{n_{ij}}{s_{ij}})}{\sum_i \sum_j s_{ij} - ab} \right] \sigma_s^2 \quad .$$

The corresponding expected values for the unweighted mean squares for classes are

$$E(MSS_{\bar{w}}) = \left[\frac{\sum_i \sum_j \sum_k (1 - \frac{n_{ijk}}{n_{ij}}) \frac{1}{n_{ijk}}}{\sum_i \sum_j s_{ij} - ab} \right] \sigma_e^2$$

$$+ \left[1 - \frac{(ab - \sum_i \sum_j \sum_k \frac{s_{ij}\, n_{ijk}^2}{n_{ij}^2})}{\sum_i \sum_j s_{ij} - ab} \right] \sigma_s^2 \qquad \text{and}$$

$$E(MSS^*_{\bar{w}}) = \left[\frac{\sum\limits_i \sum\limits_j \sum\limits_k (1 - \frac{1}{s_{ij}}) \frac{1}{n_{ijk}}}{\sum\limits_i \sum\limits_j s_{ij} - ab} \right] \sigma_e^2 + \sigma_s^2 .$$

Tables 6.1 and 6.2 give a summary of the expected mean squares for designs with two levels of nesting using the method of unweighted means with cell means $\bar{y}_{ij}$ and $\bar{y}^*_{ij}$, respectively. Examination of Tables 6.1 and 6.2 shows that if the number of observations in each (ij) cell of the design are equal (i.e. $n_{ijk} = m$ and $s_{ij} = s$) then the expected mean squares in both tables reduce to

$$E(MSR) = \frac{1}{sm} \sigma_e^2 + \frac{1}{s} \sigma_s^2 + \sigma_{rc}^2 + b\, \sigma_r^2$$

$$E(MSC) = \frac{1}{sm} \sigma_e^2 + \frac{1}{s} \sigma_s^2 + \sigma_{rc}^2 + a\, \sigma_c^2$$

$$E(MSRC) = \frac{1}{sm} \sigma_e^2 + \frac{1}{s} \sigma_s^2 + \sigma_{rc}^2$$

$$E(MSS_w) = \sigma_e^2 + m\, \sigma_s^2$$

$$E(MSS_{\bar{w}}) = \frac{\sigma_e^2}{m} + \sigma_s^2$$

$$E(MSE) = \sigma_e^2 .$$

As mentioned previously, the results given in Tables 6.1 and 6.2 can be used to estimate the variance components in (6.1) by setting the expected mean squares equal to the computed mean squares. The results

Table 6.1 Expected mean squares for a two-way disproportionate design with two levels of nesting within each cell using the method of unweighted means with cell means $\bar{y}_{ij}$

Source	Coefficients of Variance Components in E(MS) σ_e^2	σ_s^2	σ_{rc}^2	σ_c^2	σ_r^2
Rows	$K1 = \frac{1}{ab}\left(\sum_i \sum_j \frac{1}{n_{ij}}\right)$	$K2 = \frac{1}{ab}\left(\sum_i \sum_j \sum_k \frac{n_{ijk}^2}{n_{ij}^2}\right)$	1		b
Columns	$\frac{1}{ab}\left(\sum_i \sum_j \frac{1}{n_{ij}}\right)$	$\frac{1}{ab}\left(\sum_i \sum_j \sum_k \frac{n_{ijk}^2}{n_{ij}^2}\right)$	1	a	
Interaction	$\frac{1}{ab}\left(\sum_i \sum_j \frac{1}{n_{ij}}\right)$	$\frac{1}{ab}\left(\sum_i \sum_j \sum_k \frac{n_{ijk}^2}{n_{ij}^2}\right)$	1		
Classes[a] (using MSS_w)	1	$K3 = \frac{\left(n - \sum_i \sum_j \sum_k \frac{n_{ijk}^2}{n_{ij}}\right)}{\sum_i \sum_j s_{ij} - ab}$			
Classes (using $MSS_{\bar{w}}$)	$\frac{\sum_i \sum_j \sum_k \left(1 - \frac{n_{ijk}}{n_{ij}}\right) \frac{1}{n_{ijk}}}{\sum_i \sum_j s_{ij} - ab}$	$1 - \frac{\left(ab - \sum_i \sum_j \sum_k \frac{s_{ij} n_{ijk}^2}{n_{ij}^2}\right)}{\sum_i \sum_j s_{ij} - ab}$			
Error	1				

[a]The mean square for classes nested within the (ij) cells may be computed in two different ways as shown in (6.4.1) and (6.4.2).

Table 6.2 Expected mean squares for a two-way disproportionate design with two levels of nesting within each cell using the method of unweighted means with cell means $\bar{y}^*_{ij}$

Source	Coefficients of Variance Components in E(MS) σ^2_e	σ^2_s	σ^2_{rc}	σ^2_c	σ^2_r
Rows	$K1^* = \frac{1}{ab}\left(\sum_i \sum_j \sum_k \frac{1}{(s^2_{ij})(n_{ijk})}\right)$	$K2^* = \frac{1}{ab}\left(\sum_i \sum_j \frac{1}{s_{ij}}\right)$	1		b
Columns	$\frac{1}{ab}\left(\sum_i \sum_j \sum_k \frac{1}{(s^2_{ij})(n_{ijk})}\right)$	$\frac{1}{ab}\left(\sum_i \sum_j \frac{1}{s_{ij}}\right)$	1	a	
Interaction	$\frac{1}{ab}\left(\sum_i \sum_j \sum_k \frac{1}{(s^2_{ij})(n_{ijk})}\right)$	$\frac{1}{ab}\left(\sum_i \sum_j \frac{1}{s_{ij}}\right)$	1		
Classes[a] (using MSS^*_w)	1	$K3^* = \dfrac{\left(n - \sum_i \sum_j \frac{n_{ij}}{s_{ij}}\right)}{\sum_i \sum_j s_{ij} - ab}$			
Classes (using $MSS^*_{\bar{w}}$)	$\dfrac{\sum_i \sum_j \sum_k \left(1 - \frac{1}{s_{ij}}\right) \frac{1}{n_{ijk}}}{\sum_i \sum_j s_{ij} - ab}$	1			
Error	1				

[a]The mean squares for classes nested within the (ij) cells may be computed in two different ways as shown in (6.4.1) and (6.4.2).

of this calculation from Table 6.1[2] are

$$\hat{\sigma}_r^2 = \frac{MSR - MSRC}{b} = \frac{1}{b}\,\underline{y}'(G_R - G_{RC})\,\underline{y}$$

$$\hat{\sigma}_c^2 = \frac{MSC - MSRC}{a} = \frac{1}{a}\,\underline{y}'(G_C - G_{RC})\,\underline{y}$$

$$\hat{\sigma}_{rc}^2 = MSRC - \frac{K2}{K3}\,(MSS_w) + (\frac{K2}{K3} - K1)\ (MSE) \qquad (6.9)$$

$$= \underline{y}'G_{RC}\,\underline{y} - \frac{K2}{K3}\,(MSS_w) + (\frac{K2}{K3} - K1)\ (MSE)$$

$$\hat{\sigma}_s^2 = \frac{MSS_w - MSE}{K3}$$

$$\hat{\sigma}_e^2 = MSE\ .$$

Similarly, the results from Table 6.2 are

$$(\hat{\sigma}_r^2)^* = \frac{1}{b}\,\underline{y}'^*\,(G_R - G_{RC})\,\underline{y}^*$$

$$(\hat{\sigma}_c^2)^* = \frac{1}{a}\,\underline{y}'^*\,(G_C - G_{RC})\,\underline{y}^*$$

$$(\hat{\sigma}_{rc}^2)^* = \underline{y}'^*\,G_{RC}\,\underline{y}^* - \frac{K2^*}{K3^*}\,(MSS_w^*) + (\frac{K2^*}{K3^*} - K1^*)\ MSE \qquad (6.10)$$

$$(\hat{\sigma}_s^2)^* = \frac{MSS_w^* - MSE}{K3^*}$$

and $\hat{\sigma}_e^2 = MSE$.

---

[2]In (6.9) and (6.10) MSS_w and MSS_w^* respectively are used to estimate the variance components $\hat{\sigma}_{rc}^2$ and $\hat{\sigma}_s^2$. These variance components can also be estimated in a straightforward manner from Tables 6.1 and 6.2 using the unweighted mean squares $MSS_{\bar{w}}$ and $MSS_{\bar{w}}^*$.

6.3 Variances of the Estimated Variance Components

Examination of (6.9) and (6.10) shows that the formulas for the estimated row and column variance components for a design with two levels of nesting have exactly the same form as the corresponding estimates given in (3.8) for the two-way classification with one level of nesting. Therefore, the results of Hirotsu for the variances of $\hat{\sigma}_r^2$ and $\hat{\sigma}_c^2$ given in (3.9) can be used (after replacing the variance-covariance matrix of the cell means, V, given in (3.5) by V given in (6.6) or V^* given in (6.7)) to obtain the variances of the estimated row and column variance components for a design with two levels of nesting. Thus, the following variances can be shown to hold for the estimated row and column variance components in (6.9) and (6.10):

$$\mathrm{Var}\,(\hat{\sigma}_r^2) = 2\,\mathrm{tr}(V \cdot \frac{1}{b}(G_R - G_{RC}))^2$$

$$\begin{aligned}
&= \frac{2}{b^2}\Bigg\{ \frac{b^2}{a-1}\sigma_r^4 + \frac{2b}{a-1}\sigma_r^2\sigma_{rc}^2 + \frac{b\,\sigma_{rc}^4}{(a-1)(b-1)} + \frac{2\sigma_e^2}{a(a-1)}(\sigma_r^2 + \frac{\sigma_{rc}^2}{b-1})\ \mathrm{H1} \\
&\quad + \frac{2\sigma_s^2}{a(a-1)}(\sigma_r^2 + \frac{\sigma_{rc}^2}{b-1})\,\mathrm{H1S} \\
&\quad + \frac{\sigma_s^4}{a^2(b-1)^2}\left[(\mathrm{H3S} - \mathrm{H2S}) + \frac{(\mathrm{H5S} + \mathrm{H2S} - \mathrm{H3S} - \mathrm{H4S})}{(a-1)^2}\right] \qquad (6.11) \\
&\quad + \frac{\sigma_e^4}{a^2(b-1)^2}\left[(\mathrm{H3} - \mathrm{H2}) + \frac{(\mathrm{H5} + \mathrm{H2} - \mathrm{H3} - \mathrm{H4})}{(a-1)^2}\right] \\
&\quad + \frac{2\sigma_s^2\,\sigma_e^2}{a^2(b-1)^2}\Bigg[\sum_i \sum_j \sum_{\ell \neq j} (\sum_k \frac{n_{ijk}^2}{n_{ij}^2})(\frac{1}{n_{i\ell}}) \\
&\qquad + \frac{1}{(a-1)^2}\sum_i \sum_j \sum_{m \neq i} \sum_{\ell \neq j} (\sum_k \frac{n_{ijk}^2}{n_{ij}^2})(\frac{1}{n_{m\ell}})\Bigg]\Bigg\}
\end{aligned}$$

where $H1 = \sum_i \sum_j \frac{1}{n_{ij}}$, $\quad H1S = \sum_i \sum_j \left(\sum_k \frac{n_{ijk}^2}{n_{ij}^2} \right)$,

$$H2 = \sum_i \sum_j \frac{1}{n_{ij}^2}, \qquad H2S = \sum_i \sum_j \left(\sum_k \frac{n_{ijk}^2}{n_{ij}^2} \right)^2,$$

$$H3 = \sum_i \sum_j \sum_\ell \frac{1}{n_{ij}} \cdot \frac{1}{n_{i\ell}},$$

$$H3S = \sum_i \sum_j \sum_\ell \left(\sum_k \frac{n_{ijk}^2}{n_{ij}^2} \right)\left(\sum_k \frac{n_{ijk}^2}{n_{i\ell}^2} \right), \qquad H4 = \sum_i \sum_j \sum_m \frac{1}{n_{ij}} \cdot \frac{1}{n_{mj}},$$

$$H4S = \sum_i \sum_j \sum_m \left(\sum_k \frac{n_{ijk}^2}{n_{ij}^2} \right)\left(\sum_k \frac{n_{mjk}^2}{n_{mj}^2} \right), \qquad H5 = \sum_i \sum_j \sum_m \sum_\ell \frac{1}{n_{ij}} \cdot \frac{1}{n_{m\ell}}$$

and $H5S = \sum_i \sum_j \sum_m \sum_\ell \left(\sum_k \frac{n_{ijk}^2}{n_{ij}^2} \right)\left(\sum_k \frac{n_{m\ell k}^2}{n_{m\ell}^2} \right)$.

$\mathrm{Var}(\hat{\sigma}_c^2) = \mathrm{Var}(\hat{\sigma}_r^2)$ with a and b interchanged, σ_r^2 replaced by σ_c^2, H3 and H4 interchanged and H3S and H4S interchanged.

$$\mathrm{Var}[(\hat{\sigma}_r^2)^*] = 2\ \mathrm{tr}(V^* \cdot \frac{1}{b}(G_R - G_{RC}))^2$$

$$= \frac{2}{b^2}\Big\{ \frac{b^2}{a-1}\sigma_r^4 + \frac{2b}{a-1}\sigma_r^2\sigma_{rc}^2 + \frac{b\,\sigma_{rc}^4}{(a-1)(b-1)} + \frac{2\sigma_e^2}{a(a-1)}(\sigma_r^2 + \frac{\sigma_{rc}^2}{b-1})\,H1^*$$

$$+ \frac{2\sigma_s^2}{a(a-1)}(\sigma_r^2 + \frac{\sigma_{rc}^2}{b-1})\ H1S^*$$

$$+ \frac{\sigma_s^4}{a^2(b-1)^2}\Big[(H3S^* - H2S^*) + \frac{(H5S^* + H2S^* - H3S^* - H4S^*)}{(a-1)^2}\Big]$$

$$+ \frac{\sigma_e^4}{a^2(b-1)^2}\Big[(H3^* - H2^*) + \frac{(H5^* + H2^* - H3^* - H4^*)}{(a-1)^2}\Big]$$

$$+ \frac{2\sigma_s^2\,\sigma_e^2}{a^2(b-1)^2}\Big[\sum_i\sum_j\sum_{\ell\neq j}(\frac{1}{s_{ij}})(\sum_k \frac{1}{(s_{i\ell}^2)n_{i\ell k}}) \qquad (6.12)$$

$$+ \frac{1}{(a-1)^2}\sum_i\sum_j\sum_{m\neq i}\sum_{\ell\neq j}(\frac{1}{s_{ij}})(\sum_k \frac{1}{(s_{m\ell}^2)\ n_{m\ell k}})\Big]\Big\}$$

where $H1^* = \sum_i\sum_j(\sum_k \frac{1}{(s_{ij}^2)\ n_{ijk}})$, $\quad H1S^* = \sum_i\sum_j \frac{1}{s_{ij}}$,

$H2^* = \sum_i\sum_j(\sum_k \frac{1}{(s_{ij}^2)\ n_{ijk}})^2$, $\quad H2S^* = \sum_i\sum_j \frac{1}{s_{ij}^2}$,

$H3^* = \sum_i\sum_j\sum_\ell(\sum_k \frac{1}{(s_{ij}^2)\ n_{ijk}})(\sum_k \frac{1}{(s_{i\ell}^2)\ n_{i\ell k}})$,

$H3S^* = \sum_i\sum_j\sum_\ell \frac{1}{s_{ij}} \cdot \frac{1}{s_{i\ell}}$,

$$H4^* = \sum_i \sum_j \sum_m \left(\sum_k \frac{1}{(s_{ij})^2 \, n_{ijk}} \right) \left(\sum_k \frac{1}{(s_{mj}^2) \, n_{mjk}} \right) \quad ,$$

$$H4S^* = \sum_i \sum_j \sum_m \frac{1}{s_{ij}} \cdot \frac{1}{s_{mj}} \quad ,$$

$$H5^* = \sum_i \sum_j \sum_m \sum_\ell \left(\sum_k \frac{1}{(s_{ij}^2) \, n_{ijk}} \right) \left(\sum_k \frac{1}{(s_{m\ell}^2) \, n_{m\ell k}} \right) \quad ,$$

$$H5S^* = \sum_i \sum_j \sum_m \sum_\ell \frac{1}{s_{ij}} \cdot \frac{1}{s_{m\ell}} \quad .$$

$\mathrm{Var}[(\hat{\sigma}_c^2)^*] = \mathrm{Var}[(\hat{\sigma}_r^2)^*]$ with a and b interchanged, σ_r^2 replaced by σ_c^2, $H3^*$ and $H4^*$ interchanged and $H3S^*$ and $H4S^*$ interchanged.

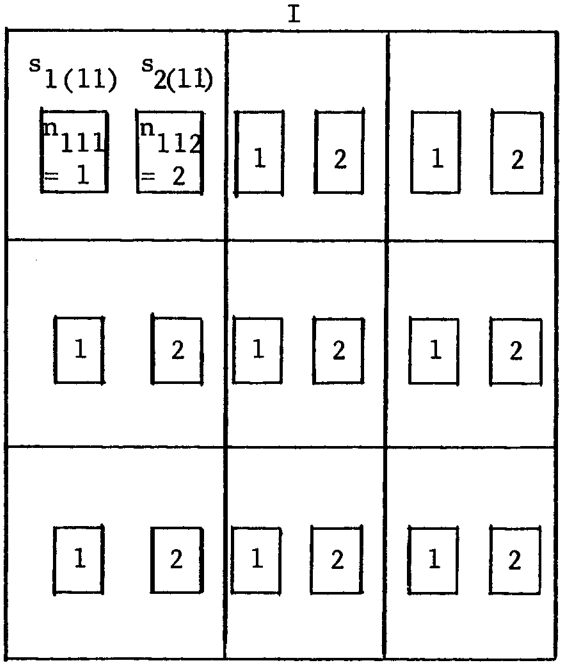
Unfortunately, the variances of the estimated row by column interaction variance components in (6.9) and (6.10) cannot be obtained from (3.9) due to the additional term in their estimating equations involving MSS_w and MSS_w^*. In fact, due to the unequal sampling within the (ij) cells of the design it can be shown that MSS_w (MSS_w^*) is not independent of MSRC (MSRC*); and in addition, that SSS_w (SSS_w^*) over its variance is not distributed as a chi-squared variate. Thus, the variances of the estimated interaction variance components in (6.9) and (6.10) are quite complicated and they were not derived.

The analytic formulas for the variances of the estimated row and column variance components given in (6.11) and (6.12) can be used to examine for what parameter combinations (i.e. σ_r^2, σ_c^2, σ_{rc}^2, σ_s^2 and σ_e^2) the variances of $\hat{\sigma}_r^2$ and $\hat{\sigma}_c^2$ are smaller (or larger) than the variances of $(\hat{\sigma}_r^2)^*$ or $(\hat{\sigma}_c^2)^*$. That is, for which parameter values is $\bar{y}_{ij}$ better (or worse) than $\bar{y}_{ij}^*$ in terms of minimizing the variances of the estimated row and column variance components obtained by using the analysis of variance procedure based upon the method of unweighted means?

Accordingly, the numerical values of the variances of $\hat{\sigma}_r^2$ and $(\hat{\sigma}_r^2)^*$ were computed from (6.11) and (6.12) for 40 parameter combinations using four designs with two levels of nesting. The designs had the following s_{ij} and n_{ijk} arrangements:

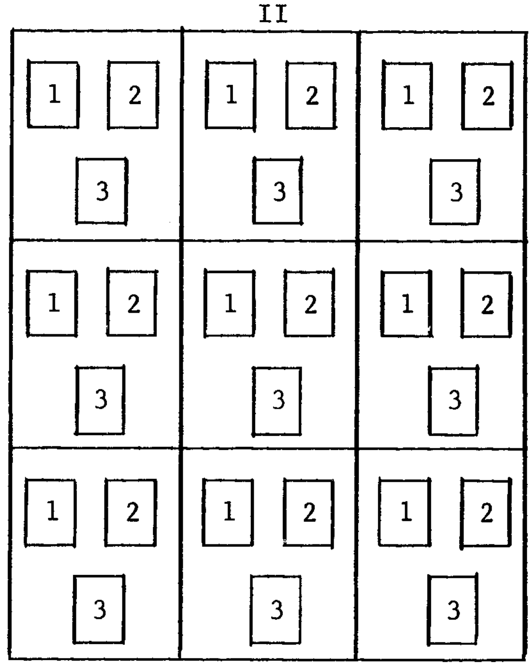
<u>Design</u>

I



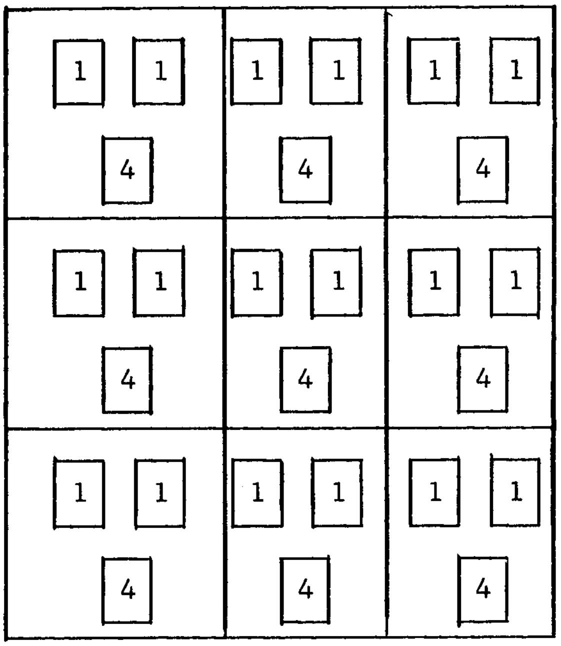
Here $a = b = 3$; $s_{ij} = 2$; $n_{ij1} = 1$ and $n_{ij2} = 2$ for all (ij) cells.

II



Here $a = b = 3$; $s_{ij} = 3$; $n_{ij1} = 1$, $n_{ij2} = 2$ and $n_{ij3} = 3$ for all (ij) cells.

III

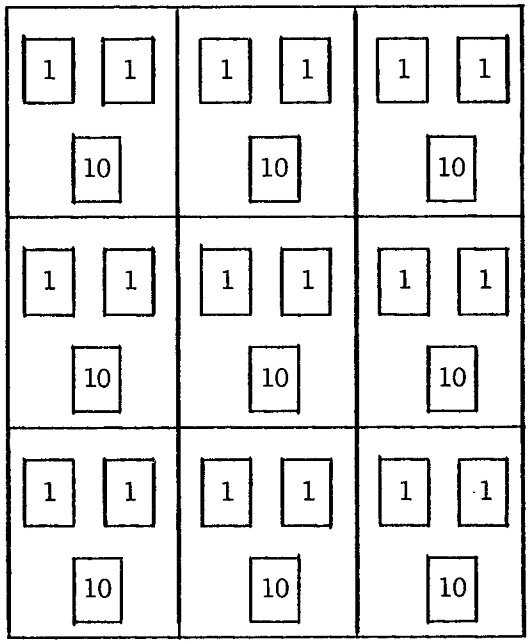


Here $a = b = 3$; $s_{ij} = 3$;

$n_{ij1} = 1$, $n_{ij2} = 1$ and

$n_{ij3} = 4$ for all (ij) cells.

IV



Here $a = b = 3$; $s_{ij} = 3$;

$n_{ij1} = 1$, $n_{ij2} = 1$ and

$n_{ij3} = 10$ for all (ij) cells.

The ratios of the variance of $\hat{\sigma}_r^2$ to the variance of $(\hat{\sigma}_r^2)^*$ were then computed for all designs and parameter combinations and these ratios are given in Table 6.3.

The results in Table 6.3 indicate the following conclusions: (a) The variances of $\hat{\sigma}_r^2$ and $(\hat{\sigma}_r^2)^*$ are quite close to each other for most of the cases examined. The only noticeable exception to this conclusion is when σ_s^2 is large compared to σ_e^2, σ_{rc}^2 and σ_r^2 (e.g. $\sigma_e^2 = 1$, $\sigma_s^2 = 8$, $\sigma_{rc}^2 = 1$ and $\sigma_r^2 = 2$). In this case, the ratios are relatively large implying that the variances of $(\hat{\sigma}_r^2)^*$ are somewhat smaller than the variances of $\hat{\sigma}_r^2$. (b) For the case of $\sigma_s^2 \ll \sigma_e^2$ (i.e. $\sigma_e^2 = 1$ and $\sigma_s = \frac{1}{8}$) and σ_{rc}^2 and σ_r^2 not more than twice σ_e^2 the

Table 6.3 Ratios of the variances of the estimated row variance components obtained by using the analysis of variance procedure based upon the method of unweighted means with cell means $\bar{y}_{ij}$ and $\bar{y}^*_{ij}$

Parameters				Design			
				I	II	III	IV
σ^2_e	σ^2_s	σ^2_{rc}	σ^2_r	$\mathrm{Var}(\hat{\sigma}^2_r) / \mathrm{Var}(\hat{\sigma}^2_r)^*$	$\mathrm{Var}(\hat{\sigma}^2_r) / \mathrm{Var}(\hat{\sigma}^2_r)^*$	$\mathrm{Var}(\hat{\sigma}^2_r) / \mathrm{Var}(\hat{\sigma}^2_r)^*$	$\mathrm{Var}(\hat{\sigma}^2_r) / \mathrm{Var}(\hat{\sigma}^2_r)^*$
1	1/8	1	2	.99	.99	.98	.97
1	1/2	1	2	1.00	1.00	1.00	1.01
1	1	1	2	1.00	1.01	1.02	1.07
1	2	1	2	1.02	1.02	1.07	1.17
1	8	1	2	1.08	1.09	1.30	1.75
1	1/8	2	2	.99	.99	.98	.97
1	1/2	2	2	1.00	1.00	1.00	1.01
1	1	2	2	1.00	1.00	1.02	1.06
1	2	2	2	1.02	1.02	1.06	1.16
1	8	2	2	1.07	1.09	1.27	1.67
1	1/8	1	8	1.00	1.00	.99	.99
1	1/2	1	8	1.00	1.00	1.00	1.00
1	1	1	8	1.00	1.00	1.01	1.02
1	2	1	8	1.01	1.01	1.02	1.05
1	8	1	8	1.03	1.03	1.10	1.23
1	1/8	2	8	1.00	1.00	1.00	.99
1	1/2	2	8	1.00	1.00	1.00	1.00
1	1	2	8	1.00	1.00	1.01	1.02
1	2	2	8	1.01	1.01	1.02	1.05
1	8	2	8	1.03	1.03	1.10	1.23
1	1/8	8	8	1.00	1.00	1.00	.99
1	1/2	8	8	1.00	1.00	1.00	1.00
1	1	8	8	1.00	1.00	1.01	1.02
1	2	8	8	1.00	1.00	1.02	1.04
1	8	8	8	1.02	1.03	1.08	1.19
1	1/8	1	16	1.00	1.00	1.00	1.00
1	1/2	1	16	1.00	1.00	1.00	1.00
1	1	1	16	1.00	1.00	1.00	1.01
1	2	1	16	1.00	1.00	1.01	1.02
1	8	1	16	1.02	1.02	1.05	1.12

continued

Table 6.3 (continued)

Parameters				Design			
				I	II	III	IV
σ_e^2	σ_s^2	σ_{rc}^2	σ_r^2	$\mathrm{Var}(\hat{\sigma}_r^2)$ / $\mathrm{Var}(\hat{\sigma}_r^2)^*$	$\mathrm{Var}(\hat{\sigma}_r^2)$ / $\mathrm{Var}(\hat{\sigma}_r^2)^*$	$\mathrm{Var}(\hat{\sigma}_r^2)$ / $\mathrm{Var}(\hat{\sigma}_r^2)^*$	$\mathrm{Var}(\hat{\sigma}_r^2)$ / $\mathrm{Var}(\hat{\sigma}_r^2)^*$
1	1/8	2	16	1.00	1.00	1.00	1.00
1	1/2	2	16	1.00	1.00	1.00	1.00
1	1	2	16	1.00	1.00	1.00	1.01
1	2	2	16	1.00	1.00	1.01	1.02
1	8	2	16	1.02	1.02	1.05	1.12
1	1/8	8	16	1.00	1.00	1.00	1.00
1	1/2	8	16	1.00	1.00	1.00	1.00
1	1	8	16	1.00	1.00	1.00	1.01
1	2	8	16	1.00	1.00	1.01	1.02
1	8	8	16	1.01	1.01	1.05	1.11

variance of $\hat{\sigma}_r^2$ is smaller than that of $(\hat{\sigma}_r^2)^*$. On the other hand, when $\sigma_s^2 \gg \sigma_e^2$ (e.g. $\sigma_e^2 = 1$, $\sigma_s^2 = 8$) and σ_{rc}^2 and σ_r^2 are not more than 8 times σ_e^2 the variance of $(\hat{\sigma}_r^2)^*$ is noticeably smaller than that of $\hat{\sigma}_r^2$. (c) Of the four designs studied, Design IV (the design with the most imbalance in the n_{ijk}'s) has ratios which are the most sensitive to varying parameter values. In fact, when $\sigma_s^2 = 8$ the ratios for Design IV are relatively large ranging from 1.75 ($\sigma_{rc}^2 = 1$, $\sigma_r^2 = 2$) to 1.11 ($\sigma_{rc}^2 = 8$, $\sigma_r^2 = 16$).

Thus, the ratios in Table 6.3 suggest the following procedure for estimating the row variance component in model (6.1) based upon the method of unweighted means if the relative magnitude of the variance components are known: If $\sigma_s^2 \ll \sigma_e^2$ then the (ij) cell mean should be estimated by $\bar{y}_{ij}$ while if $\sigma_s^2 \gg \sigma_e^2$ then this mean should be estimated by $\bar{y}_{ij}^*$. However, if (a) σ_e^2 is approximately equal to σ_s^2 (i.e. $\frac{\sigma_s^2}{2} \leq \sigma_e^2 \leq 2\sigma_s^2$) or (b) σ_{rc}^2 and/or $\sigma_r^2 \gg \sigma_s^2$ then the variances of $\hat{\sigma}_r^2$ and $(\hat{\sigma}_r^2)^*$ are approximately equal and it makes very little difference whether $\bar{y}_{ij}$ or $\bar{y}_{ij}^*$ is used to estimate the (ij) cell mean. By symmetry the same procedure is recommended when estimating the column variance component.

On the other hand, if no information is available concerning the relative magnitudes of the variance components then $\bar{y}_{ij}^*$ is recommended over $\bar{y}_{ij}$ for estimating σ_r^2 (σ_c^2) since the inappropriate choice of $\bar{y}_{ij}^*$ (i.e. when $\sigma_s^2 \ll \sigma_e^2$) still results in relative efficiencies near one and since $\bar{y}_{ij}^*$ provides better estimates of σ_r^2 (σ_c^2) when σ_s^2 is relatively large.

7. SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

7.1 Summary

The method of unweighted means which gives a relatively simple analysis for crossed classifications with unequal numbers of observations in the cells has been limited to designs where every cell has at least one observation. The principle objective of this dissertation was to examine the method of unweighted means with regard to estimating the variance components for a two-way classification with unequal numbers of observations in the cells and with one or more cells having no observations. To do this two procedures based upon the method of unweighted means were developed to estimate the variance components for a design with missing cells. These two procedures were then compared with a procedure based on the method of fitting constants which has been studied extensively by many authors and found to produce relatively good estimates of the variance components (Bush and Anderson [1963]). The estimation procedure for all three methods of estimation is the analysis of variance procedure which involves calculating mean squares, computing the expectations of these mean squares under the variance component model and then equating calculated mean squares to expected mean squares and solving the set of equations for the variance components. Specifically, the mean square errors of the estimated variance components for each of the three estimation procedures were compared numerically for several parameter combinations and two-way designs with missing cells.

The first of the two estimating procedures based upon the method of unweighted means, Procedure M, estimates the variance components

(σ_r^2, σ_c^2 and σ_{rc}^2) by first estimating observations for the missing cells in the design. These observations are estimated in a manner such that the row by column interaction sum of squares is minimized after having replaced the observations in each cell by their cell means. Having estimated values for the missing cells, Procedure M computes the mean squares by the method of unweighted means and then equates these mean squares to their expectations and solves these equations for the variance components. The second procedure, Procedure $\hat{M}$, is the same as Procedure M except that approximate expected mean squares are used which are essentially those that would be obtained by using the method of unweighted means for a design where there are no missing cells. Procedure M gives unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

The estimates of the variance components for Procedures M and $\hat{M}$ are given in Chapters 3 and 4 for designs with different patterns of missing cells. The expectations of the mean squares for these procedures were derived analytically by expressing the mean squares in matrix notation; e.g. the mean square for rows for the method of unweighted means is given by MSR = $\underline{y}'G_R\underline{y}$ where $\underline{y}'$ is a vector of cell means and G_R is a square symmetric matrix. Then, the results of Lancaster [1954] and Searle [1956] were used which state that if the quadratic form Z = $\underline{y}'G\underline{y}$ is a function of variables $\underline{y}$, having a multinormal distribution with means $\underline{\mu}$ and variance-covariance matrix V, then

$$E(Z) = tr(V \cdot G) + \underline{\mu}'G\underline{\mu} \ .$$

The analytical results in Chapters 3 and 4 show that the formulas for the expected mean squares for the method of unweighted means used in Procedure M (*i.e.*, the expected mean squares which account for the fact that the design has missing cells) are relatively simple for designs with one missing cell (Table 3.1) or missing cells in the same row or column (Tables 4.1 and 4.2). However, for designs with missing cells in different rows and columns these expected mean squares are quite complex (Tables 4.3 and 4.4). The approximate expected mean squares used in Procedure $\hat{M}$ (*i.e.*, the expected mean squares which ignore the fact that the design has missing cells) also have formulas which are relatively simple (equation (3.23)).

The variances of the estimated variance components for Procedures M and $\hat{M}$ can be derived analytically by expressing the estimated variance components in matrix notation (*e.g.*, the estimated row variance component for Procedure $\hat{M}$ is given by $\hat{\hat{\sigma}}_r^2 = \underline{y}'(G_R - G_{RC1})\underline{y}'/b)$ and again using the results of Lancaster and Searle which state for $Z = \underline{y}'G\underline{y}$ as defined above that

$$\text{Var}\ (Z) \quad = \quad 2\ \text{tr}(V \cdot G)^2 + 4\underline{\mu}'GVG\underline{\mu} \quad . \qquad (7.1)$$

However, only the variance of the estimated row by column interaction component ($\hat{\sigma}_{rc}^2$) for a design with one missing cell was derived analytically due to the enormous amount of matrix algebra involved in solving (7.1) and the resulting complexity of the variance formula for $\hat{\sigma}_{rc}^2$ (equation (3.27)). Instead, it was decided that for comparison purposes it was better to write a computer program to calculate the variances of the estimated variance components obtained by Procedures M and $\hat{M}$ rather than perform tedious matrix algebra which would lead to complex analytical formulas (of questionable value) for these variances. In addition, the computer program was set up to (a) compute the corresponding variances for the procedure based on the method of fitting constants (denoted by Procedure A) and (b) compute the coefficients of the expected mean squares for the method of unweighted means used in Procedure M. The computer program, written for the IBM/360 model 75, used (7.1) to compute the variances for Procedures M and $\hat{M}$ and a computational method given by Bush [1962] to compute the variances for Procedure A.

The results of running the computer program for twenty-six 3 x 3 designs, sixteen 4 x 4 designs, twelve 6 x 6 designs and 13 sets of parameter values are given in Chapter 5. A summary of these results indicated that for the majority of the designs and parameter combinations studied Procedures M and $\hat{M}$ appear to be as good as or better than Procedure A for estimating variance components with respect to the mean square errors of the estimates. In fact, Procedures M and $\hat{M}$ are always better than Procedure A for estimating σ_r^2 (σ_c^2) when σ_r^2 (σ_c^2) is much larger than σ_{rc}^2. The only case where Procedure A is consistently better than Procedures M and $\hat{M}$ is for estimating σ_{rc}^2 when $\sigma_{rc}^2 \leq \sigma_e^2$ and there is a small proportion of missing cells in the design. The mean square errors of Procedure $\hat{M}$ are usually relatively close to those of Procedure M. However in some cases, notably when there are a large proportion of missing cells in a design and σ_r^2 is dominated by σ_{rc}^2, Procedure $\hat{M}$ is significantly better than Procedure M for estimating σ_r^2 and σ_c^2. Based on these results, and the fact that Procedure $\hat{M}$ is computationally much easier than Procedures M and A (Procedure $\hat{M}$ can easily be carried out on a desk calculator) it is recommended that of the three procedures investigated Procedure $\hat{M}$ be used to estimate the variance components for a two-way classification with unequal numbers and missing cells. However, if only unbiased estimates of the variance components are desired then Procedure M is recommended over Procedure A because its mean square errors are usually as small as or smaller than those of Procedure A and it is computationally easier than Procedure A.

In addition to the comparison of Procedures M, $\hat{M}$ and A, the results from the computer runs also indicated that (a) the complicated analytical formulas for the expected mean squares for the method of unweighted means used in Procedure M can be approximated satisfactorily in many cases by simpler formulas and (b) the biases in the estimated variance components for Procedure $\hat{M}$ are relatively unimportant when compared with the corresponding mean square errors of the estimates.

In Chapter 6 the method of unweighted means was examined for estimating the variance components for a two-way classification with two levels of nesting within each of the cells of the design (classes within cells and observations within classes). For this type of design it was assumed that every cell had at least one observation. The estimating procedure was again based upon equating computed mean squares to their expected values; but due to the two levels of nesting two different cell means were considered for use in the method of unweighted means, namely (a) the mean of class means weighted by the numbers of observations per class ; i.e., the simple arithmetic mean of the observations in a cell (denoted by $\bar{y}_{ij}$) and (b) the unweighted mean of class means within cells (denoted by $\bar{y}^*_{ij}$). Accordingly, based upon the method of unweighted means two sets of estimates of the variance components were obtained using $\bar{y}_{ij}$ and $\bar{y}^*_{ij}$ as estimates of the cell means. The resulting variances of the estimated row and column variance components were obtained analytically using the results of Hirotsu [1966]. A numerical comparison was then carried out using these analytic variance formulas for $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ to determine for various parameter combinations which of the two cell means leads to the estimated variance components with the smallest variances. The numerical comparison evaluated the variances for four different 3 x 3 designs and 40 parameter combinations. The results showed that if (a) σ^2_r (σ^2_c) and/or σ^2_{rc} dominate the between classes variance component, σ^2_s, or if (b) σ^2_e and σ^2_s are approximately equal then the variances of the estimated row (column) variance components are nearly equal for both cell means. On the other hand, if σ^2_r (σ^2_c) and σ^2_{rc} are not more than twice σ^2_s then (a) if σ^2_s is dominated by σ^2_e, $\bar{y}_{ij}$ gives variances which are a little smaller than those given by $\bar{y}^*_{ij}$ and (b) if σ^2_s dominates σ^2_e, $\bar{y}^*_{ij}$ gives variances which are smaller than those given by $\bar{y}_{ij}$.

Based on these results, it is suggested if no prior knowledge of the relative magnitudes of the variance components is known, that $\bar{y}^*_{ij}$ (the unweighted cell mean) be used in estimating σ^2_r and σ^2_c for the two levels of nesting design. The reason for this recommendation is that the inappropriate choice of $\bar{y}^*_{ij}$ (i.e. when σ^2_s is dominated by σ^2_e) still results in variances for $\hat{\sigma}^2_r$ and $\hat{\sigma}^2_c$ that are approximately equal to those given by $\bar{y}_{ij}$ while if σ_s^2 is relatively large $\bar{y}_{ij}^*$ provides better estimates of σ_r^2 and σ_c^2 than does $\bar{y}_{ij}$.

7.2 Suggested Future Research

1. Extend the two-way model to three-way and higher order cross classifications with missing cells to see if similar results are obtained.
2. Study the level of significance and power of approximate tests of hypotheses of the variance components for designs with missing cells based upon the expected mean squares used in Procedures M and $\hat{M}$, the properties of these tests should be compared to previous work on the properties of tests for designs with no missing cells.
3. Study the possibility of approximating the variances of the variance component estimators by simpler formulas which do not require the use of a computer.
4. Investigate the use of other estimation procedures for the variance components from unbalanced data which do not necessarily produce unbiased estimates but are easy to compute and have other attractive properties such as relatively small mean square errors.
5. Compare Procedures M and $\hat{M}$ with other estimation procedures besides Procedure A, e.g. Henderson's Method 1, Koch's symmetric sums, maximum likelihood, etc.
6. Examine the possibility of using Procedures M and $\hat{M}$ to estimate the variance components for a two-way classification with missing cells and two levels of nesting within each of the occupied cells.

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9. APPENDIX. MEAN SQUARE ERRORS OF ESTIMATES OF VARIANCE COMPONENTS

The mean square errors of the estimates of the variance components for each design and parameter combination given in Section 5.3 are shown in Tables 9.1 through 9.10 for the estimation procedures A, M and $\hat{M}$. Note that since Procedures A and M are unbiased procedures, their mean square errors are the same as their variances.

Table 9.1 Mean square errors of estimates of variance components for 3 x 3 - A designs

Parameters				Design	Rows			Columns			Interaction		
					Procedure			Procedure			Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				18[b]							.34		
				16-1	1.79	1.79	1.77	1.79	1.79	1.77	.44	.44	.44
1	1	.25	1	14-2	2.09	2.13	2.05	2.09	2.13	2.05	.63	.63	.63
				12-3	2.75	2.75	2.44	2.75	2.75	2.44	1.21	1.21	1.21
				18	264.			c					
				16-1	276.	266.	266.						
16	1	.25	1	14-2	284.	269.	269.						
				12-3	273.	273.	272.						
				16-1	19.2	18.7	18.6	19.2	18.7	18.6			
4	4	.25	1	14-2	20.3	19.5	19.4	20.3	19.5	19.4			
				12-3	20.8	20.8	20.5	20.8	20.8	20.5			
				16-1				276.	266.	266.			
16	16	.25	1	14-2				284.	269.	269.			
				12-3				273.	273.	272.			
				18	2.38			2.38			1.18		
				16-1	2.84	2.92	2.83	2.84	2.92	2.83	1.56	1.56	1.56
1	1	1	1	14-2	3.69	3.92	3.62	3.69	3.92	3.62	2.32	2.32	2.32
				12-3	6.00	6.00	4.75	6.00	6.00	4.75	4.58	4.58	4.58
				18	20.4								
				16-1	22.0	21.7	21.6						
4	1	1	1	14-2	24.1	23.7	23.4						
				12-3	27.0	27.0	25.8						
				18	272.								
				16-1	286.	277.	277.						
16	1	1	1	14-2	297.	283.	283.						
				12-3	291.	291.	290.						
				18				20.4					
				16-1				22.0	21.7	21.6			
16	4	1	1	14-2				24.1	23.7	23.4			
				12-3				27.0	27.0	25.8			
				18				272.					
				16-1				286.	277.	277.			
16	16	1	1	14-2				297.	283.	283.			
				12-3				291.	291.	290.			

continued

Table 9.1 (continued)

Parameters				Design	Rows			Columns			Interaction		
					Procedure			Procedure			Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
1	1	4	1	18	7.38						10.2		
				16-1	10.0	10.8	10.1	10.0	10.8	10.1	13.6	13.6	13.6
				14-2	15.7	17.7	15.0	15.7	17.7	15.0	20.3	20.3	20.3
				12-3	34.0	34.0	22.8	34.0	34.0	22.8	40.6	40.6	40.6
4	1	4	1	16-1	36.4	37.0	36.3						
				14-2	45.1	47.1	44.4						
				12-3	67.0	67.0	55.8						
16	1	4	1	18	307.								
				16-1	329.	322.	321.						
				14-2	354.	345.	342.						
				12-3	379.	379.	368.						
4	4	4	1	18				31.4					
				16-1				36.4	37.0	36.3			
				14-2				45.1	47.1	44.4			
				12-3				67.0	67.0	55.8			

[a]Procedure A is based upon the method of fitting constants, and Procedures M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b]The mean square errors for the base design (i.e. design 18) are given only for those cases computed by Bush [1962] for Procedure A.

[c]Procedures A, M and $\hat{M}$ have the properties that M.S.E.($\hat{\sigma}_r^2$) remains invariant over any change in σ_c^2; M.S.E.($\hat{\sigma}_c^2$) remains invariant over any change in σ_r^2; and M.S.E.($\hat{\sigma}_{rc}^2$) remains invariant over any change in σ_r^2 and σ_c^2. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.

Table 9.2 Mean square errors of estimates of variance components for 3 x 3 - B designs

σ^2_r	σ^2_c	σ^2_{rc}	σ^2_e	Design	Rows A[a]	Rows M	Rows $\hat{M}$	Columns A	Columns M	Columns $\hat{M}$	Interaction A	Interaction M	Interaction $\hat{M}$
1	1	.25	1	18[b]							.39		
				17-1	2.01	1.99	1.97	1.76	1.79	1.76	.43	.51	.52
				15-1	1.93	1.92	1.88	2.00	2.04	2.01	.59	.69	.69
				15-2	2.28	2.35	2.28	2.09	2.15	2.06	.66	.74	.73
				13-2	2.35	2.37	2.24	2.49	2.57	2.47	1.01	1.05	1.04
				12-3	3.16	3.12	2.72	3.18	3.08	2.70	1.61	1.60	1.60
16	1	.25	1	18	292.			c					
				17-1	328.	268.	268.						
				15-1	278.	267.	267.						
				15-2	316.	271.	271.						
				13-2	272.	271.	271.						
				12-3	284.	275.	275.						
4	4	.25	1	17-1	22.5	19.2	19.2	18.6	18.6	18.6			
				15-1	19.6	19.0	19.0	20.7	19.4	19.4			
				15-2	22.4	20.1	20.0	19.8	19.6	19.5			
				13-2	20.1	20.1	20.0	22.3	20.8	20.6			
				12-3	22.1	21.7	21.2	22.6	21.5	21.2			
16	16	.25	1	17-1				268.	266.	266.			
				15-1				296.	269.	269.			
				15-2				278.	269.	269.			
				13-2				307.	273.	273.			
				12-3				295.	275.	275.			
1	1	1	1	18	2.63			2.59			1.31		
				17-1	3.09	3.17	3.10	2.95	2.91	2.82	1.59	1.68	1.69
				15-1	3.15	3.08	2.98	3.27	3.24	3.15	1.88	1.97	1.97
				15-2	3.95	4.23	3.93	3.97	3.96	3.63	2.44	2.51	2.50
				13-2	4.28	4.28	3.86	4.54	4.55	4.19	2.98	3.02	3.01
				12-3	6.79	6.59	5.16	7.04	6.55	5.15	5.32	5.31	5.31
4	1	1	1	18	22.5								
				17-1	25.5	22.3	22.2						
				15-1	22.8	22.0	21.9						
				15-2	26.5	24.4	24.1						
				13-2	24.5	24.4	24.0						
				12-3	28.9	28.0	26.6						

continued

Table 9.2 (continued)

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				18[b]	302.								
				17-1	339.	279.	279.						
				15-1	289.	278.	278.						
16	1	1	1	15-2	330.	285.	285.						
				13-2	286.	285.	285.						
				12-3	304.	294.	292.						
				18				20.9					
				17-1				21.8	21.6	21.6			
				15-1				24.1	22.5	22.4			
16	4	1	1	15-2				24.3	23.8	23.5			
				13-2				27.1	25.1	24.7			
				12-3				30.0	28.0	26.6			
				18				274.					
				17-1				279.	277.	277.			
				15-1				307.	279.	279.			
16	16	1	1	15-2				292.	283.	283.			
				13-2				323.	287.	287.			
				12-3				317.	294.	292.			
				18	8.15						11.3		
				17-1	10.3	11.2	10.6	11.4	10.8	10.0	14.3	13.9	13.9
				15-1	11.7	11.1	10.3	11.9	11.4	10.7	15.4	14.6	14.6
1	1	4	1	15-2	16.4	18.3	15.6	18.5	17.8	15.0	21.5	20.8	20.8
				13-2	18.7	18.5	15.5	19.2	19.1	16.2	22.3	22.2	22.1
				12-3	36.8	35.5	23.7	38.7	35.4	23.7	42.7	42.6	42.6
				17-1	40.6	37.9	37.2						
				15-1	39.4	37.5	36.7						
4	1	4	1	15-2	48.7	48.1	45.4						
				13-2	48.7	48.2	45.2						
				12-3	71.8	68.9	57.1						
				18	340.								
				17-1	385.	324.	324.						
				15-1	338.	323.	323.						
16	1	4	1	15-2	391.	347.	345.						
				13-2	350.	347.	344.						
				12-3	398.	383.	371.						

continued

Table 9.2 (continued)

Parameters				Design	Rows			Columns			Interaction		
					Procedure			Procedure			Procedure		
σ^2_r	σ^2_c	σ^2_{rc}	σ^2_e		A	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				18				34.1					
				17-1				38.3	37.0	36.2			
				15-1				41.2	38.1	37.4			
4	4	4	1	15-2				49.0	47.3	44.5			
				13-2				52.8	49.3	46.4			
				12-3				75.6	68.9	57.2			

[a]Procedure A is based upon the method of fitting constants, and Procedures M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b]The mean square errors for the base design (i.e. design 18) are given only for those cases computed by Bush [1962] for Procedure A.

[c]Procedures A, M and $\hat{M}$ have the properties that M.S.E.($\hat{\sigma}^2_r$) remains invariant over any change in σ^2_c; M.S.E.($\hat{\sigma}^2_c$) remains invariant over any change in σ^2_r; and M.S.E.($\hat{\sigma}^2_{rc}$) remains invariant over any change in σ^2_r and σ^2_c. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.

Table 9.3 Mean square errors of estimates of variance components for 3 x 3 - C designs

Parameters σ^2_r σ^2_c σ^2_{rc} σ^2_e	Design	Rows Procedure A[a]	M	$\hat{M}$	Columns Procedure A	M	$\hat{M}$	Interaction Procedure A	M	$\hat{M}$
1 1 .25 1	18[b]							.42		
	17-1	1.83	1.86	1.83	1.83	1.86	1.83	.49	.54	.54
	16-1	1.93	1.95	1.93	1.93	1.95	1.93	.57	.61	.61
	15-1	1.96	1.99	1.97	1.96	1.99	1.97	.60	.64	.65
	14-2	2.27	2.33	2.25	2.27	2.33	2.25	.80	.80	.80
	13-2	2.35	2.36	2.23	2.35	2.36	2.23	1.02	1.04	1.03
	12-2	2.45	2.54	2.44	2.45	2.54	2.44	.98	1.03	1.04
	12-3	3.16	3.12	2.72	3.16	3.12	2.72	1.60	1.60	1.60
	9-3	3.67	3.67	3.11	3.67	3.67	3.11	2.38	2.38	2.38
16 1 .25 1	18	265.			c					
	17-1	271.	267.	267.						
	16-1	282.	268.	268.						
	15-1	286.	268.	268.						
	14-2	292.	271.	271.						
	13-2	271.	271.	271.						
	12-2	306.	273.	273.						
	12-3	284.	275.	275.						
	9-3	279.	279.	279.						
4 4 .25 1	17-1	19.0	18.9	18.8	19.0	18.9	18.8			
	16-1	19.9	19.2	19.1	19.9	19.2	19.1			
	15-1	20.1	19.2	19.2	20.1	19.2	19.2			
	14-2	21.1	20.1	20.0	21.1	20.1	20.0			
	13-2	20.0	20.1	19.9	20.0	20.1	19.9			
	12-2	22.2	20.6	20.5	22.2	20.6	20.5			
	12-3	22.1	21.6	21.2	22.1	21.6	21.2			
	9-3	22.7	22.7	22.1	22.7	22.7	22.1			
16 16 .25 1	17-1				271.	267.	267.			
	16-1				282.	268.	268.			
	15-1				286.	268.	268.			
	14-2				292.	271.	271.			
	13-2				271.	271.	271.			
	12-2				306.	273.	273.			
	12-3				284.	275.	275.			
	9-3				279.	279.	279.			

continued

Table 9.3 (continued)

Parameters σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	Rows Procedure A^a	M	$\hat{M}$	Columns Procedure A	M	$\hat{M}$	Interaction Procedure A	M	$\hat{M}$
				18	2.57			2.57			1.35		
				17-1	3.01	3.01	2.91	3.01	3.01	2.91	1.70	1.74	1.74
				16-1	3.13	3.14	3.05	3.13	3.14	3.05	1.81	1.85	1.86
				15-1	3.17	3.18	3.09	3.17	3.18	3.09	1.86	1.90	1.91
1	1	1	1	14-2	4.07	4.21	3.89	4.07	4.21	3.89	2.62	2.62	2.62
				13-2	4.28	4.27	3.85	4.28	4.27	3.85	2.99	3.01	3.00
				12-2	4.26	4.50	4.15	4.26	4.50	4.15	2.91	2.96	2.98
				12-3	6.78	6.59	5.16	6.78	6.59	5.16	5.31	5.31	5.31
				9-3	7.42	7.42	5.72	7.42	7.42	5.72	6.50	6.50	6.50
				18	20.9								
				17-1	22.1	21.9	21.8						
				16-1	23.0	22.2	22.1						
				15-1	23.3	22.3	22.2						
4	1	1	1	14-2	25.3	24.4	24.1						
				13-2	24.4	24.4	24.0						
				12-2	26.5	25.0	24.6						
				12-3	28.9	28.0	26.6						
				9-3	29.4	29.4	27.7						
				18	274.								
				17-1	282.	277.	277.						
				16-1	293.	279.	278.						
				15-1	297.	279.	279.						
16	1	1	1	14-2	306.	285.	285.						
				13-2	285.	285.	284.						
				12-2	320.	287.	286.						
				12-3	304.	294.	292.						
				9-3	297.	297.	295.						
				18				20.9					
				17-1				22.1	21.9	21.8			
				16-1				23.0	22.2	22.1			
				15-1				23.3	22.3	22.2			
16	4	1	1	14-2				25.3	24.4	24.1			
				13-2				24.4	24.4	24.0			
				12-2				26.5	25.0	24.6			
				12-3				28.9	28.0	26.6			
				9-3				29.4	29.4	27.7			

continued

Table 9.3 (continued)

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				18				274.					
				17-1				282.	277.	277.			
				16-1				293.	279.	278.			
				15-1				297.	279.	279.			
16	16	1	1	14-2				306.	285.	285.			
				13-2				285.	285.	284.			
				12-2				320.	287.	286.			
				12-3				304.	294.	292.			
				9-3				297.	297.	295.			
				18	8.36						11.1		
				17-1	11.2	10.9	10.2	11.2	10.9	10.2	14.5	14.1	14.1
				16-1	11.3	11.2	10.5	11.3	11.2	10.5	14.6	14.3	14.3
				15-1	11.3	11.3	10.6	11.3	11.3	10.6	14.7	14.4	14.4
1	1	4	1	14-2	17.3	18.3	15.6	17.3	18.3	15.6	21.2	21.2	21.2
				13-2	18.7	18.5	15.5	18.7	18.5	15.5	22.3	22.1	22.1
				12-2	17.3	19.0	16.1	17.3	19.0	16.1	22.2	22.0	22.0
				12-3	36.8	35.5	23.7	36.8	35.5	23.7	42.6	42.6	42.6
				9-3	37.4	37.4	24.9	37.4	37.4	24.9	45.5	45.5	45.5
				17-1	38.2	37.3	36.6						
				16-1	39.1	37.8	37.1						
				15-1	39.5	37.9	37.2						
4	1	4	1	14-2	48.4	48.1	45.4						
				13-2	48.6	48.2	45.2						
				12-2	49.4	49.0	46.1						
				12-3	71.8	68.9	57.2						
				9-3	71.4	71.4	58.9						
				18	313.								
				17-1	329.	323.	322.						
				16-1	341.	324.	323.						
				15-1	345.	324.	324.						
16	1	4	1	14-2	369.	347.	345.						
				13-2	349.	347.	344.						
				12-2	382.	349.	347.						
				12-3	398.	383.	371.						
				9-3	387.	387.	375.						

continued

Table 9.3 (continued)

Parameters				Design	Rows			Columns			Interaction		
					Procedure			Procedure			Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				18				33.3					
				17-1				38.2	37.3	36.6			
				16-1				39.1	37.8	37.1			
				15-1				39.5	37.9	37.2			
4	4	4	1	14-2				48.4	48.1	45.4			
				13-2				48.6	48.2	45.2			
				12-2				49.4	49.0	46.1			
				12-3				71.8	68.9	57.2			
				9-3				71.4	71.4	58.9			

[a]Procedure A is based upon the method of fitting constants, and Procedures M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b]The mean square errors for the base design (*i.e.* design 18) are given only for those cases computed by Bush [1962] for Procedure A.

[c]Procedures A, M and $\hat{M}$ have the properties that M.S.E.$(\hat{\sigma}_r^2)$ remains invariant over any change in σ_c^2; M.S.E.$(\hat{\sigma}_c^2)$ remains invariant over any change in σ_r^2; and M.S.E.$(\hat{\sigma}_{rc}^2)$ remains invariant over any change in σ_r^2 and σ_c^2. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.

Table 9.4 Mean square errors of estimates of variance components for 3 x 3 - D designs

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
1	1	.25	1	18[b]							.57		
				17-1	2.03	2.07	2.04	2.03	2.07	2.03	.71	.77	.77
				14-1	2.16	2.23	2.19	2.16	2.23	2.19	.88	.95	.96
				13-2	2.59	2.64	2.49	2.59	2.64	2.49	1.31	1.32	1.31
				10-2	2.82	3.03	2.89	2.82	3.03	2.89	1.61	1.71	1.76
				12-3	3.67	3.65	3.10	3.67	3.65	3.10	2.19	2.19	2.19
16	1	.25	1	18	266.			c					
				17-1	277.	269.	269.						
				14-1	290.	271.	271.						
				13-2	279.	274.	273.						
				10-2	315.	277.	277.						
				12-3	283.	279.	278.						
4	4	.25	1	17-1	19.8	19.5	19.4	19.8	19.5	19.4			
				14-1	20.8	19.9	19.9	20.8	19.9	19.9			
				13-2	20.9	20.8	20.6	20.9	20.8	20.6			
				10-2	23.5	21.8	21.7	23.5	21.8	21.7			
				12-3	22.8	22.7	22.1	22.9	22.7	22.1			
16	16	.25	1	17-1				277.	269.	269.			
				14-1				290.	271.	271.			
				13-2				279.	274.	273.			
				10-2				315.	277.	277.			
				12-3				283.	279.	278.			
1	1	1	1	18	2.73			2.73			1.60		
				17-1	3.27	3.29	3.19	3.27	3.29	3.19	2.06	2.13	2.12
				14-1	3.45	3.49	3.39	3.45	3.49	3.39	2.29	2.37	2.38
				13-2	4.61	4.66	4.22	4.61	4.66	4.22	3.44	3.46	3.45
				10-2	4.78	5.17	4.75	4.78	5.17	4.75	3.81	3.93	3.98
				12-3	7.52	7.40	5.71	7.52	7.40	5.71	6.31	6.31	6.31
4	1	1	1	18	21.3								
				17-1	23.0	22.6	22.5						
				14-1	24.1	23.1	22.9						
				13-2	25.4	25.2	24.8						
				10-2	27.9	26.3	25.9						
				12-3	29.9	29.4	27.7						

continued

Table 9.4 (continued)

Parameters σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	Rows Procedure A^a	M	$\hat{M}$	Columns Procedure A	M	$\hat{M}$	Interaction Procedure A	M	$\hat{M}$
16	1	1	1	18	276.								
				17-1	288.	280.	280.						
				14-1	301.	281.	281.						
				13-2	293.	288.	287.						
				10-2	329.	291.	290.						
				12-3	303.	297.	295.						
16	4	1	1	18				21.3					
				17-1				23.0	22.6	22.5			
				14-1				24.1	23.1	22.9			
				13-2				25.4	25.2	24.8			
				10-2				27.9	26.3	25.9			
				12-3				29.9	29.4	27.7			
16	16	1	1	18				276.					
				17-1				288.	280.	280.			
				14-1				301.	281.	281.			
				13-2				293.	288.	287.			
				10-2				329.	291.	290.			
				12-3				303.	297.	295.			
1	1	4	1	18	8.69						11.7		
				17-1	11.7	11.5	10.7	11.7	11.5	10.7	15.3	15.0	15.0
				14-1	11.9	11.9	11.1	11.9	11.9	11.1	15.7	15.5	15.5
				13-2	19.1	19.3	16.3	19.1	19.3	16.3	23.3	23.3	23.3
				10-2	18.3	20.3	17.3	18.3	20.3	17.3	24.2	24.0	24.1
				12-3	38.2	37.4	24.9	38.2	37.4	24.9	45.3	45.3	45.3
4	1	4	1	17-1	39.2	38.3	37.5						
				14-1	40.6	39.0	38.2						
				13-2	49.8	49.5	46.4						
				10-2	51.6	51.1	48.0						
				12-3	73.1	71.4	58.9						
16	1	4	1	18	315.								
				17-1	335.	325.	325.						
				14-1	350.	327.	326.						
				13-2	357.	350.	347.						
				10-2	393.	354.	351.						
				12-3	396.	387.	375.						

continued

Table 9.4 (continued)

Parameters σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	Rows Procedure A^a	M	$\hat{M}$	Columns Procedure A	M	$\hat{M}$	Interaction Procedure A	M	$\hat{M}$
				18				33.9					
				17-1				39.2	38.3	37.5			
				14-1				40.6	39.0	38.2			
4	4	4	1	13-2				49.8	49.5	46.4			
				10-2				51.6	51.1	48.0			
				12-3				73.1	71.4	58.9			

[a] Procedure A is based upon the method of fitting constants, and Procedures M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b] The mean square errors for the base design (i.e. design 18) are given only for those cases computed by Bush [1962] for Procedure A.

[c] Procedures A, M and $\hat{M}$ have the properties that M.S.E.$(\hat{\sigma}_r^2)$ remains invariant over any change in σ_c^2; M.S.E.$(\hat{\sigma}_c^2)$ remains invariant over any change in σ_r^2; and M.S.E.$(\hat{\sigma}_{rc}^2)$ remains invariant over any change in σ_r^2 and σ_c^2. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.

Table 9.5 Mean square errors of estimates of variance components for 3 x 3 - E designs

Parameters					Rows			Columns			Interaction		
					Procedure			Procedure			Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				18[b]							.45		
				17-1	2.07	2.04	2.02	2.07	2.04	2.02	.49	.64	.65
				13-1	2.11	2.14	2.11	2.11	2.14	2.11	.92	.96	.96
1	1	.25	1	15-2	2.48	2.49	2.39	2.48	2.49	2.39	.90	.95	.96
				11-2	2.72	2.87	2.72	2.72	2.87	2.72	1.48	1.49	1.48
				10-3	3.78	3.76	3.19	3.78	3.76	3.19	2.42	2.41	2.41
				18	295.			c					
				17-1	325.	269.	269.						
				13-1	275.	270.	270.						
16	1	.25	1	15-2	315.	273.	272.						
				11-2	297.	276.	275.						
				10-3	283.	279.	279.						
				17-1	22.5	19.4	19.4	22.5	19.4	19.4			
				13-1	19.8	19.7	19.6	19.8	19.7	19.6			
4	4	.25	1	15-2	22.7	20.5	20.4	22.7	20.5	20.4			
				11-2	22.3	21.4	21.3	22.3	21.4	21.3			
				10-3	23.1	22.9	22.3	23.1	22.9	22.3			
				17-1				325.	269.	269.			
				13-1				275.	270.	270.			
16	16	.25	1	15-2				315.	273.	272.			
				11-2				297.	276.	275.			
				10-3				283.	279.	279.			
				18	2.91			2.91			1.48		
				17-1	3.39	3.25	3.17	3.39	3.25	3.17	1.78	1.90	1.91
				13-1	3.44	3.38	3.27	3.44	3.38	3.27	2.33	2.37	2.36
1	1	1	1	15-2	4.63	4.44	4.09	4.63	4.44	4.09	2.83	2.88	2.89
				11-2	4.61	4.96	4.54	4.61	4.96	4.54	3.65	3.66	3.65
				10-3	7.67	7.56	5.82	7.67	7.56	5.82	6.63	6.61	6.62
				18	23.4								
				17-1	26.1	22.5	22.4						
				13-1	23.2	22.8	22.7						
4	1	1	1	15-2	27.8	24.9	24.5						
				11-2	26.5	25.9	25.4						
				10-3	30.0	29.7	28.0						

continued

Table 9.5 (continued)

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ^2_r	σ^2_c	σ^2_{rc}	σ^2_e		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				18	306.								
				17-1	338.	279.	279.						
				13-1	286.	280.	280.						
16	1	1	1	15-2	332.	287.	286.						
				11-2	310.	290.	289.						
				10-3	302.	298.	296.						
				18				23.4					
				17-1				26.1	22.5	22.4			
				13-1				23.2	22.8	22.7			
16	4	1	1	15-2				27.8	24.9	24.5			
				11-2				26.5	25.9	25.4			
				10-3				30.0	29.7	28.0			
				18				306.					
				17-1				338.	279.	279.			
				13-1				286.	280.	280.			
16	16	1	1	15-2				332.	287.	286.			
				11-2				310.	290.	289.			
				10-3				302.	298.	296.			
				18	9.89						12.4		
				17-1	12.4	11.4	10.7	12.4	11.4	10.7	15.9	14.4	14.4
				13-1	12.5	11.7	10.9	12.5	11.7	10.9	15.6	15.5	15.5
1	1	4	1	15-2	20.2	18.8	16.0	20.2	18.8	16.0	22.1	21.9	21.9
				11-2	17.8	19.9	16.9	17.8	19.9	16.9	23.6	23.6	23.6
				10-3	38.4	37.8	25.1	38.4	37.8	25.1	46.0	45.9	45.9
				17-1	44.4	38.1	37.4						
				13-1	40.4	38.6	37.9						
4	1	4	1	15-2	55.4	48.9	46.0						
				11-2	49.1	50.5	47.4						
				10-3	73.2	71.9	59.2						
				18	353.								
				17-1	393.	325.	324.						
				13-1	336.	326.	325.						
16	1	4	1	15-2	408.	349.	346.						
				11-2	370.	353.	350.						
				10-3	395.	388.	376.						

continued

Table 9.5 (continued)

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				18				38.3					
				17-1				44.4	38.1	37.4			
				13-1				40.4	38.6	37.9			
4	4	4	1	15-2				55.4	48.9	46.0			
				11-2				49.1	50.5	47.4			
				10-3				73.2	71.9	59.2			

[a]Procedure A is based upon the method of fitting constants, and Procedures M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b]The mean square errors for the base design (i.e. design 18) are given only for those cases computed by Bush [1962] for Procedure A.

[c]Procedures A, M and $\hat{M}$ have the properties that M.S.E.$(\hat{\sigma}_r^2)$ remains invariant over any change in σ_c^2; M.S.E.$(\hat{\sigma}_c^2)$ remains invariant over any change in σ_r^2; and M.S.E.$(\hat{\sigma}_{rc}^2)$ remains invariant over any change in σ_r^2 and σ_c^2. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.

Table 9.6 Mean square errors of estimates of variance components for 4 x 4 - A designs

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
1	1	.25	1	30-1	.99	.98	.98	.99	.98	.98	.17	.17	.17
				28-2-1	1.07	1.07	1.06	1.03	1.02	1.02	.20	.20	.20
				28-2-2	1.03	1.03	1.02	1.03	1.03	1.02	.20	.20	.20
				26-3	1.11	1.11	1.10	1.11	1.11	1.10	.23	.23	.23
				24-4-1	1.21	1.19	1.19	1.21	1.19	1.19	.27	.27	.27
				24-4-2	1.13	1.12	1.12	1.13	1.12	1.12	.27	.27	.27
16	1	.25	1	30-1	178.	175.	175.	b					
				28-2-1	189.	176.	176.						
				28-2-2	180.	176.	176.						
				26-3	189.	177.	177.						
				24-4-1	198.	177.	177.						
				24-4-2	177.	177.	177.						
4	4	.25	1	30-1	12.0	11.8	11.8	12.0	11.8	11.8			
				28-2-1	12.8	12.1	12.1	12.1	11.9	11.9			
				28-2-2	12.2	12.0	12.0	12.2	12.0	12.0			
				26-3	12.9	12.2	12.2	12.9	12.2	12.2			
				24-4-1	13.6	12.4	12.4	13.6	12.4	12.4			
				24-4-2	12.3	12.2	12.2	12.3	12.2	12.2			
16	16	.25	1	30-1				178.	175.	175.			
				28-2-1				179.	176.	176.			
				28-2-2				180.	176.	176.			
				26-3				189.	177.	177.			
				24-4-1				198.	177.	177.			
				24-4-2				177.	177.	177.			
1	1	1	1	30-1	1.38	1.38	1.38	1.38	1.38	1.38	.60	.60	.60
				28-2-1	1.51	1.59	1.58	1.47	1.48	1.47	.68	.68	.68
				28-2-2	1.48	1.49	1.48	1.48	1.49	1.48	.68	.68	.68
				26-3	1.63	1.70	1.68	1.63	1.70	1.68	.79	.79	.79
				24-4-1	1.84	1.94	1.90	1.84	1.94	1.90	.94	.94	.94
				24-4-2	1.75	1.75	1.71	1.75	1.75	1.71	.94	.94	.94
4	1	1	1	30-1	13.2	13.1	13.0						
				28-2-1	14.1	13.6	13.6						
				28-2-2	13.5	13.3	13.3						
				26-3	14.4	13.9	13.8						
				24-4-1	15.3	14.4	14.4						
				24-4-2	14.0	14.0	13.9						

continued

Table 9.6 (continued)

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				30-1	182.	180.	180.						
				28-2-1	194.	182.	182.						
				28-2-2	185.	181.	181.						
16	1	1	1	26-3	194.	183.	183.						
				24-4-1	204.	184.	184.						
				24-4-2	183.	183.	183.						
				30-1				13.2	13.1	13.0			
				28-2-1				13.5	13.3	13.3			
				28-2-2				13.5	13.3	13.3			
16	4	1	1	26-3				14.4	13.9	13.8			
				24-4-1				15.3	14.4	14.4			
				24-4-2				14.0	14.0	13.9			
				30-1				182.	180.	180.			
				28-2-1				184.	181.	181.			
				28-2-2				185.	181.	181.			
16	16	1	1	26-3				194.	183.	183.			
				24-4-1				204.	184.	184.			
				24-4-2				183.	183.	183.			
				30-1	3.70	3.78	3.73	3.70	3.78	3.73	5.10	5.10	5.10
				28-2-1	4.26	5.00	4.86	4.22	4.31	4.20	5.82	5.82	5.82
				28-2-2	4.22	4.39	4.27	4.22	4.39	4.27	5.82	5.82	5.82
1	1	4	1	26-3	4.97	5.62	5.41	4.97	5.62	5.41	6.79	6.79	6.79
				24-4-1	6.04	7.10	6.77	6.04	7.10	6.77	8.14	8.14	8.14
				24-4-2	5.95	5.91	5.59	5.95	5.91	5.59	8.14	8.15	8.15
				30-1	18.8	18.8	18.7						
				28-2-1	20.5	21.0	20.8						
				28-2-2	19.9	19.9	19.9						
4	1	4	1	26-3	21.7	22.1	21.9						
				24-4-1	24.0	24.6	24.3						
				24-4-2	22.7	22.6	22.3						
				30-1	201.	199.	199.						
				28-2-1	215.	205.	205.						
				28-2-2	206.	202.	202.						
16	1	4	1	26-3	218.	208.	208.						
				24-4-1	231.	215.	215.						
				24-4-2	210.	210.	209.						

continued

Table 9.6 (continued)

Parameters σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	Rows Procedure A^a	M	$\hat{M}$	Columns Procedure A	M	$\hat{M}$	Interaction Procedure A	M	$\hat{M}$
				30-1				18.8	18.8	18.7			
				28-2-1				19.8	19.8	19.7			
				28-2-2				19.9	19.9	19.9			
4	4	4	1	26-3				21.7	22.1	21.9			
				24-4-1				24.0	24.6	24.3			
				24-4-2				22.7	22.6	22.3			

[a]Procedure A is based upon the method of fitting constants, and Procedure M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b]Procedures A, M and $\hat{M}$ have the properties that M.S.E.$(\hat{\sigma}_r^2)$ remains invariant over any change in σ_c^2; M.S.E.$(\hat{\sigma}_c^2)$ remains invariant over any change in σ_r^2; and M.S.E.$(\hat{\sigma}_{rc}^2)$ remains invariant over any change in σ_r^2 and σ_c^2. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.

Table 9.7 Mean square errors of estimates of variance components for 4 x 4 - B designs

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
				31-1	1.11	1.07	1.07	1.11	1.07	1.07	.19	.26	.26
				30-2	1.20	1.20	1.20	1.13	1.08	1.08	.20	.27	.27
				29-2	1.18	1.13	1.12	1.18	1.13	1.12	.22	.29	.30
				24-2	1.18	1.20	1.20	1.13	1.14	1.13	.32	.38	.38
1	1	.25	1	23-2	1.14	1.15	1.14	1.14	1.15	1.14	.35	.41	.40
				29-3	1.23	1.21	1.22	1.23	1.21	1.22	.20	.27	.27
				20-3	1.26	1.29	1.27	1.26	1.29	1.27	.46	.52	.51
				27-4	1.35	1.29	1.28	1.35	1.29	1.28	.24	.33	.34
				20-4	1.28	1.29	1.27	1.28	1.29	1.27	.46	.52	.52
				17-4	1.41	1.45	1.43	1.41	1.45	1.43	.68	.70	.70
				31-1	201.	176.	176.	b					
				30-2	216.	178.	178.						
				29-2	209.	177.	177.						
				24-2	196.	178.	178.						
16	1	.25	1	23-2	184.	177.	177.						
				29-3	219.	178.	178.						
				20-3	191.	179.	179.						
				27-4	232.	179.	178.						
				20-4	183.	179.	179.						
				17-4	195.	180.	180.						
				31-1	13.5	12.1	12.1	13.5	12.1	12.1			
				30-2	14.5	12.5	12.5	13.5	12.1	12.1			
				29-2	14.1	12.3	12.3	14.1	12.3	12.3			
				24-2	13.4	12.5	12.5	12.6	12.3	12.3			
4	4	.25	1	23-2	12.7	12.3	12.3	12.7	12.3	12.3			
				29-3	14.7	12.5	12.5	14.7	12.5	12.5			
				20-3	13.3	12.7	12.7	13.3	12.7	12.7			
				27-4	15.7	12.7	12.7	15.7	12.7	12.7			
				20-4	12.9	12.7	12.7	12.9	12.7	12.7			
				17-4	13.9	13.2	13.1	13.9	13.2	13.1			
				31-1				201.	176.	176.			
				30-2				200.	176.	176.			
				29-2				209.	177.	177.			
				24-2				183.	178.	177.			
16	16	.25	1	23-2				184.	177.	177.			
				29-3				219.	178.	178.			
				20-3				191.	179.	179.			
				27-4				232.	179.	178.			
				20-4				183.	179.	179.			
				17-4				195.	180.	180.			

continued

Table 9.7 (continued)

Parameters σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	Rows Procedure A^a	Rows M	Rows $\hat{M}$	Columns Procedure A	Columns M	Columns $\hat{M}$	Interaction Procedure A	Interaction M	Interaction $\hat{M}$
1	1	1	1	31-1	1.60	1.49	1.48	1.60	1.49	1.48	.67	.73	.73
				30-2	1.74	1.76	1.76	1.67	1.55	1.54	.73	.79	.79
				29-2	1.75	1.62	1.60	1.75	1.62	1.60	.77	.84	.84
				24-2	1.74	1.76	1.74	1.68	1.62	1.60	.90	.96	.96
				23-2	1.68	1.64	1.63	1.68	1.64	1.63	.93	.99	.99
				29-3	1.83	1.84	1.82	1.83	1.84	1.82	.80	.86	.86
				20-3	1.89	1.93	1.90	1.89	1.93	1.90	1.16	1.21	1.21
				27-4	2.08	2.07	2.04	2.08	2.07	2.04	.97	1.05	1.05
				20-4	2.05	1.96	1.91	2.05	1.96	1.91	1.27	1.33	1.33
				17-4	2.17	2.28	2.23	2.17	2.28	2.23	1.53	1.56	1.55
4	1	1	1	31-1	15.0	13.4	13.3						
				30-2	16.2	14.0	14.0						
				29-2	15.8	13.7	13.7						
				24-2	15.1	14.1	14.0						
				23-2	14.2	13.8	13.7						
				29-3	16.6	14.2	14.2						
				20-3	15.1	14.5	14.4						
				27-4	17.9	14.8	14.7						
				20-4	15.0	14.5	14.5						
				17-4	15.9	15.3	15.2						
16	1	1	1	31-1	207.	181.	181.						
				30-2	222.	183.	183.						
				29-2	215.	182.	182.						
				24-2	202.	183.	183.						
				23-2	189.	182.	182.						
				29-3	225.	184.	184.						
				20-3	198.	185.	185.						
				27-4	240.	186.	185.						
				20-4	190.	185.	185.						
				17-4	202.	187.	187.						
16	4	1	1	31-1				15.0	13.4	13.3			
				30-2				15.2	13.5	13.5			
				29-2				15.8	13.7	13.7			
				24-2				14.2	13.7	13.7			
				23-2				14.2	13.8	13.7			
				29-3				16.6	14.2	14.2			
				20-3				15.1	14.5	14.4			
				27-4				17.9	14.8	14.7			
				20-4				15.0	14.5	14.5			
				17-4				15.9	15.3	15.2			

continued

Table 9.7 (continued)

Parameters σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	Rows Procedure A^a	M	$\hat{M}$	Columns Procedure A	M	$\hat{M}$	Interaction Procedure A	M	$\hat{M}$
				31-1				207.	181.	181.			
				30-2				206.	181.	181.			
				29-2				215.	182.	182.			
				24-2				189.	182.	182.			
16	16	1	1	23-2				189.	182.	182.			
				29-3				225.	184.	184.			
				20-3				198.	185.	185.			
				27-4				240.	186.	185.			
				20-4				190.	185.	185.			
				17-4				202.	187.	187.			
				31-1	4.53	3.96	3.89	4.53	3.96	3.89	5.95	5.42	5.42
				30-2	5.04	5.32	5.20	5.07	4.44	4.33	6.64	6.10	6.10
				29-2	5.26	4.61	4.49	5.26	4.61	4.49	6.88	6.21	6.22
				24-2	5.29	5.32	5.16	5.09	4.56	4.44	6.78	6.48	6.47
1	1	4	1	23-2	5.05	4.66	4.52	5.05	4.66	4.52	6.82	6.54	6.53
				29-3	5.65	5.88	5.70	5.65	5.88	5.70	7.49	6.96	6.96
				20-3	5.93	6.06	5.82	5.93	6.06	5.82	7.98	7.74	7.73
				27-4	6.96	7.35	7.04	6.96	7.35	7.04	9.16	8.40	8.41
				20-4	7.13	6.33	5.98	7.13	6.33	5.98	9.38	9.06	9.06
				17-4	7.20	7.78	7.40	7.20	7.78	7.40	9.51	9.48	9.47
				31-1	22.1	19.2	19.1						
				30-2	24.0	21.6	21.5						
				29-2	23.9	20.4	20.3						
				24-2	22.9	21.6	21.5						
4	1	4	1	23-2	21.7	20.5	20.4						
				29-3	25.2	22.6	22.4						
				20-3	23.6	22.9	22.7						
				27-4	28.3	25.0	24.7						
				20-4	25.2	23.4	23.0						
				17-4	25.8	25.8	25.3						
				31-1	230.	200.	200.						
				30-2	248.	207.	206.						
				29-2	241.	204.	203.						
				24-2	227.	207.	206.						
16	1	4	1	23-2	213.	204.	204.						
				29-3	253.	209.	209.						
				20-3	224.	210.	210.						
				27-4	272.	216.	216.						
				20-4	220.	212.	211.						
				17-4	232.	218.	218.						

continued

Table 9.7 (continued)

Parameters σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	Rows Procedure A^a	M	$\hat{M}$	Columns Procedure A	M	$\hat{M}$	Interaction Procedure A	M	$\hat{M}$
				31-1				22.1	19.2	19.1			
				30-2				23.0	20.1	20.0			
				29-2				23.9	20.4	20.3			
				24-2				21.8	20.3	20.2			
4	4	4	1	23-2				21.7	20.5	20.4			
				29-3				25.2	22.6	22.4			
				20-3				23.6	22.9	22.7			
				27-4				28.3	25.0	24.7			
				20-4				25.2	23.4	23.0			
				17-4				25.8	25.8	25.3			

[a]Procedure A is based upon the method of fitting constants, and Procedures M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b]Procedures A, M and $\hat{M}$ have the properties that M.S.E.$(\hat{\sigma}_r^2)$ remains invariant over any change in σ_c^2; M.S.E.$(\hat{\sigma}_c^2)$ remains invariant over any change in σ_r^2; and M.S.E.$(\hat{\sigma}_{rc}^2)$ remains invariant over any change in σ_r^2 and σ_c^2. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.

Table 9.8 Mean square errors of estimates of variance components for 6 x 6 - A designs

Parameters				Design	Rows			Columns			Interaction		
					Procedure			Procedure			Procedure		
σ^2_r	σ^2_c	σ^2_{rc}	σ^2_e		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
1	1	.25	1	46-2	.58	.57	.57	.58	.57	.57	.19	.22	.22
				44-3	.60	.60	.60	.59	.58	.58	.21	.24	.24
				42-4	.59	.59	.59	.60	.59	.59	.23	.26	.26
				40-4	.61	.61	.61	.60	.60	.60	.29	.32	.32
16	1	.25	1	46-2	108.	105.	105.	b					
				44-3	112.	105.	105.						
				42-4	106.	105.	105.						
				40-4	111.	105.	105.						
4	4	.25	1	46-2	7.23	7.04	7.04	7.26	7.04	7.04			
				44-3	7.47	7.12	7.12	7.26	7.07	7.07			
				42-4	7.15	7.10	7.09	7.29	7.10	7.10			
				40-4	7.44	7.15	7.15	7.24	7.13	7.13			
1	1	1	1	46-2	.73	.72	.72	.73	.71	.71	.38	.41	.41
				44-3	.76	.77	.76	.75	.73	.73	.41	.44	.44
				42-4	.76	.75	.74	.77	.75	.75	.44	.47	.47
				40-4	.78	.77	.77	.77	.77	.76	.50	.53	.53
16	1	1	1	46-2	110.	107.	107.						
				44-3	114.	107.	107.						
				42-4	108.	107.	107.						
				40-4	113.	107.	107.						
4	4	1	1	46-2	7.74	7.51	7.51	7.77	7.50	7.50			
				44-3	8.00	7.65	7.64	7.78	7.55	7.55			
				42-4	7.70	7.60	7.60	7.84	7.61	7.61			
				40-4	7.99	7.68	7.67	7.79	7.65	7.65			
16	16	1	1	46-2				111.	107.	107.			
				44-3				110.	107.	107.			
				42-4				111.	107.	107.			
				40-4				110.	107.	107.			
1	1	4	1	46-2	1.56	1.47	1.46	1.55	1.46	1.46	2.19	2.14	2.14
				44-3	1.64	1.67	1.68	1.61	1.52	1.52	2.31	2.25	2.25
				42-4	1.69	1.59	1.59	1.68	1.59	1.59	2.43	2.38	2.38
				40-4	1.70	1.67	1.67	1.70	1.66	1.65	2.50	2.46	2.46

continued

Table 9.8 (continued)

Parameters σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	Rows Procedure A^a	M	$\hat{M}$	Columns Procedure A	M	$\hat{M}$	Interaction Procedure A	M	$\hat{M}$
4	4	4	1	46-2	9.99	9.56	9.55	10.0	9.55	9.54			
				44-3	10.4	9.99	10.0	10.1	9.69	9.69			
				42-4	10.2	9.84	9.84	10.3	9.85	9.85			
				40-4	10.4	10.0	10.0	10.2	9.99	9.98			
16	4	4	1	46-2	118.	114.	114.						
				44-3	122.	115.	115.						
				42-4	117.	115.	115.						
				40-4	121.	115.	115.						

[a]Procedure A is based upon the method of fitting constants, and Procedures M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b]Procedures A, M and $\hat{M}$ have the properties that M.S.E.$(\hat{\sigma}_r^2)$ remains invariant over any change in σ_c^2; M.S.E.$(\hat{\sigma}_c^2)$ remains invariant over any change in σ_r^2; and M.S.E.$(\hat{\sigma}_{rc}^2)$ remains invariant over any change in σ_r^2 and σ_c^2. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.

Table 9.9 Mean square errors of estimates of variance components for 6 x 6 - B designs

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
1	1	.25	1	56-2	.59	.56	.56	.58	.57	.56	.11	.15	.15
				50-3	.63	.60	.60	.57	.57	.57	.14	.17	.17
				46-4	.60	.59	.59	.58	.58	.58	.17	.21	.21
				47-4	.64	.60	.60	.58	.59	.59	.16	.20	.20
16	1	.25	1	56-2	114.	104.	105.	b					
				50-3	123.	105.	105.						
				46-4	113.	105.	105.						
				47-4	122.	105.	105.						
4	4	.25	1	56-2	7.54	7.00	7.00	7.42	7.01	7.01			
				50-3	8.11	7.12	7.12	7.16	7.03	7.03			
				46-4	7.52	7.08	7.08	7.18	7.07	7.07			
				47-4	8.13	7.12	7.12	7.21	7.08	7.08			
1	1	1	1	56-2	.76	.70	.70	.75	.71	.71	.30	.33	.33
				50-3	.80	.77	.76	.75	.72	.72	.33	.36	.37
				46-4	.78	.74	.74	.77	.74	.74	.38	.41	.41
				47-4	.82	.76	.76	.77	.75	.75	.37	.40	.40
16	1	1	1	56-2	116.	107.	107.						
				50-3	125.	107.	107.						
				46-4	115.	107.	107.						
				47-4	125.	107.	107.						
4	4	1	1	56-2	8.12	7.47	7.47	7.98	7.48	7.48			
				50-3	8.66	7.65	7.64	7.74	7.51	7.51			
				46-4	8.09	7.59	7.58	7.78	7.58	7.58			
				47-4	8.71	7.65	7.64	7.82	7.61	7.60			
16	16	1	1	56-2				114.	107.	107.			
				50-3				110.	107.	107.			
				46-4				110.	107.	107.			
				47-4				110.	107.	107.			
1	1	4	1	56-2	1.71	1.45	1.44	1.66	1.45	1.44	2.18	2.03	2.03
				50-3	1.68	1.68	1.68	1.70	1.50	1.50	2.28	2.15	2.15
				46-4	1.74	1.58	1.58	1.77	1.58	1.58	2.42	2.31	2.31
				47-4	1.76	1.66	1.66	1.82	1.63	1.63	2.42	2.29	2.29

continued

Table 9.9 (continued)

Parameters σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2	Design	Rows Procedure A^a	M	$\hat{M}$	Columns Procedure A	M	$\hat{M}$	Interaction Procedure A	M	$\hat{M}$
				56-2	10.7	9.51	9.50	10.5	9.52	9.51			
4	4	4	1	50-3	11.1	10.0	10.0	10.3	9.64	9.64			
				46-4	10.7	9.82	9.82	10.4	9.81	9.81			
				47-4	11.3	9.98	9.98	10.6	9.93	9.92			
				56-2	125.	114.	114.						
16	4	4	1	50-3	133.	115.	115.						
				46-4	124.	115.	115.						
				47-4	134.	115.	115.						

[a] Procedure A is based upon the method of fitting constants, and Procedures M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b] Procedures A, M and $\hat{M}$ have the properties that M.S.E.($\hat{\sigma}_r^2$) remains invariant over any change in σ_c^2; M.S.E.($\hat{\sigma}_c^2$) remains invariant over any change in σ_r^2; and M.S.E.($\hat{\sigma}_{rc}^2$) remains invariant over any change in σ_r^2 and σ_c^2. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.

Table 9.10 Mean square errors of estimates of variance components for 6 x 6 - C designs

Parameters				Design	Rows			Columns			Interaction		
					Procedure			Procedure			Procedure		
σ^2_r	σ^2_c	σ^2_{rc}	σ^2_e		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
1	1	.25	1	72-2	.59	.56	.55	.62	.55	.55	.07	.12	.12
				68-3	.59	.57	.57	.59	.56	.56	.08	.12	.12
				62-4	.57	.57	.57	.60	.57	.57	.09	.13	.13
				52-4	.60	.59	.59	.59	.58	.58	.13	.16	.16
16	1	.25	1	72-2	118.	105.	105.	b					
				68-3	115.	105.	105.						
				62-4	108.	105.	105.						
				52-4	114.	105.	105.						
4	4	.25	1	72-2	7.75	6.98	6.98	8.28	6.97	6.97			
				68-3	7.61	7.03	7.03	7.74	6.98	6.98			
				62-4	7.20	7.02	7.02	7.67	7.03	7.03			
				52-4	7.56	7.09	7.09	7.49	7.06	7.06			
1	1	1	1	72-2	.80	.70	.69	.83	.69	.69	.26	.29	.29
				68-3	.80	.74	.73	.79	.70	.70	.27	.30	.30
				62-4	.77	.72	.72	.79	.73	.72	.29	.32	.32
				52-4	.79	.75	.75	.78	.74	.74	.33	.36	.36
16	1	1	1	72-2	120.	106.	106.						
				68-3	118.	107.	107.						
				62-4	111.	107.	107.						
				52-4	116.	107.	107.						
4	4	1	1	72-2	8.46	7.44	7.44	8.97	7.43	7.43			
				68-3	8.30	7.55	7.55	8.41	7.47	7.47			
				62-4	7.87	7.52	7.52	8.31	7.53	7.53			
				52-4	8.18	7.61	7.61	8.10	7.58	7.58			
16	16	1	1	72-2				129.	106.	106.			
				68-3				120.	107.	107.			
				62-4				118.	107.	107.			
				52-4				115.	107.	107.			
1	1	4	1	72-2	2.00	1.44	1.43	1.97	1.43	1.42	2.44	1.97	1.97
				68-3	2.03	1.63	1.62	1.90	1.48	1.48	2.35	2.04	2.04
				62-4	1.96	1.55	1.55	1.87	1.56	1.56	2.40	2.16	2.16
				52-4	1.83	1.64	1.64	1.83	1.62	1.61	2.41	2.23	2.23

continued

Table 9.10 (continued)

Parameters				Design	Rows Procedure			Columns Procedure			Interaction Procedure		
σ_r^2	σ_c^2	σ_{rc}^2	σ_e^2		A[a]	M	$\hat{M}$	A	M	$\hat{M}$	A	M	$\hat{M}$
4	4	4	1	72-2	11.6	9.48	9.47	12.1	9.47	9.46			
				68-3	11.5	9.89	9.88	11.4	9.59	9.59			
				62-4	10.9	9.75	9.75	11.2	9.76	9.76			
				52-4	10.9	9.94	9.94	10.8	9.90	9.89			
16	4	4	1	72-2	132.	114.	114.						
				68-3	129.	115.	115.						
				62-4	122.	115.	115.						
				52-4	126.	115.	115.						

[a]Procedure A is based upon the method of fitting constants, and Procedures M and $\hat{M}$ are based upon the method of unweighted means. Procedures A and M give unbiased estimates for the variance components while Procedure $\hat{M}$ gives biased estimates.

[b]Procedures A, M and $\hat{M}$ have the properties that M.S.E.$(\hat{\sigma}_r^2)$ remains invariant over any change in σ_c^2; M.S.E.$(\hat{\sigma}_c^2)$ remains invariant over any change in σ_r^2; and M.S.E.$(\hat{\sigma}_{rc}^2)$ remains invariant over any change in σ_r^2 and σ_c^2. Thus, if a cell of the table is left blank it implies that the results for this cell have already been given.