Dynamic Properties of Modified Systems

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Abstract

A set of methods is presented for convenient calculation of the changes in the frequency and shape of any mode of undamped vibration of a structure due to changes in the parameters of the system. Previously reported methods of similar type did not include the possibility of changes in the system that increase the number of degrees of freedom. The extension to cases in which degrees of freedom are added is effected by perturbation methods. In particular, the methods presented can treat close or multiple frequencies. Some sample numerical results are given for separated frequencies and multiple ones.

1. Introduction

In structural engineering it is often desirable to be able to predict the consequences of modifying the structural system. Such modification may be the result of the inevitable difference between the analytical model and the system as actually constructed. Non-structural components may also contribute to such deviation from the analytical predictions of behavior. When the modifications are in the form of additional masses which give rise to additional degrees of freedom that happen to have frequencies near the original ones, large, destructive motions of the "extra" masses may occur along with some noticeable effects on the structure proper.

In addition to changes in a structure that are not under the designer's direct control, there are often variations in the system that result from redesign. It is certainly an advantage to be able to use the dynamic properties of a design and to modify them simply and inexpensively to account for changes introduced in redesign. The extensive calculation made in the original design need not then be discarded, but can be modified so that it refers to the altered structure.

2. Separated Frequencies

One approach to the improvement of approximate eigenvalues and eigenvectors that can be used for any eigenpair independently of the others is essentially an application of the Newton-Raphson procedure to the equation

$$KX = \lambda MX \qquad (1)$$

where K is the stiffness matrix, M the mass matrix and λ the eigenvalue, equal to the square of the natural circular frequency. If the values are known for one value of a

parameter μ then these may serve as initial approximations for the corresponding quantities when μ_0 is changed to $\mu_0 + \Delta\mu$. What results [1] is a set of equations

$$\left[\begin{array}{c|c} K - \lambda_0 M & -MX_0 \\ \hline -X_0^T M & 0 \end{array}\right] \left[\begin{array}{c} \Delta X \\ \hline \Delta\lambda \end{array}\right] = \left[\begin{array}{c} R \\ \hline 0 \end{array}\right] \tag{2}$$

where R is the residual $K(\mu)X_0 - \lambda_0 M(\mu)X_0$.

The matrix of the system (2) consists of that of (1) bordered by one row and one column. The last equation in the system guarantees that there can be no unlimited drift in the change of the eigenvector by specifying that the change in the eigenvector is normal to its original direction. It is not difficult to show [1] that the matrix of coefficients (2) is nonsingular even if the eigenvalue is exact, provided that the eigenvalue under consideration is not multiple. If the procedure is continued by replacing λ_0 and X_0 by quantities including the increments just found, this iterative method is convergent with a rate higher than second order. The iterative scheme has the drawback of requiring a new triangularization every time that the eigenvalue is changed on the left. A method using the modified Newton-Raphson method has been used [2], but it can slow convergence considerably.

The method developed herein uses a perturbation method to extrapolate to a new value of the eigenvalue accurately before applying the iteration of (2). This application of a perturbation technique is fairly straightforward. With a perturbation technique it is also possible to treat cases where the change in the system involves the addition of new degrees of freedom as well as changes in the existing mass and stiffness matrices.

The system of Fig. 1, which may represent the simplest model of a shear frame, Fig. 2, can be treated very effectively and straightforwardly by this method. It should, however, be emphasized that the procedure is applicable for any connections in the system whatever. The perturbation process sets

$$\lambda = \lambda^{(0)} + \mu\lambda^{(1)} + \mu^2\lambda^{(2)} + \ldots \tag{3a}$$

$$X = X^{(0)} + \mu X^{(1)} + \mu^2 X^{(2)} + \ldots \tag{3b}$$

for the eigenvalue and the eigenvector, where μ is now a parameter that measures the size of the added masses and springs, or more generally, of the changes in the mass and stiffness matrices due to the additional elements. When μ approaches zero, the components are very small compared to masses and stiffnesses of the original system; when μ is of the order 1, they are comparable in size.

The equations of the sytem (1) including the new coefficients in K and M that are proportional to μ are written down along with the equation that specifies that the total change in X is orthogonal to the $X^{(0)}$ found for the original system. When the equations so formed are separated according to powers of μ, sets of equations of the form (2) are found, which always have the same coefficient matrix on the left.

These equations can be used for any of the "old" modes and any of the "new" modes that arise from the addition of degrees of freedom. For an old mode, it is easily shown that the value of $\lambda^{(0)}$ is the eigenvalue for that old mode before the augmentation of the system. The $X^{(0)}$ consists of the previous eigenvector for the masses of the old system and entries corresponding to the new masses that are obtained from the forced vibration response of the added system with the original masses moving in the mode and with the frequency of the old mode in question. Obviously this characterization breaks down when

such a forced vibration would result in infinite motions, i.e., when tuning occurs. For a new mode, a frequency is what would be found if the old masses were fixed. The mode $X^{(0)}$ involves no motion of the original masses. There are, of course, as many new modes as there are added masses, but they may be studied one at a time. As long as one of the new frequencies is not close to any of the old ones, that mode may be handled as above, no matter whether others of the new frequencies are tuned or not.

These results are neither difficult to show nor especially surprising. It might be noted, however, that the general theorems on perturbation of eigenvalues as presented in the standard treatise [3] do not include cases where the number of degrees of freedom is changed by a perturbation.

3. Close or Multiple Frequencies

The situation is much more complicated when one or more frequencies of the added system are very close to, or coincide with, one or more frequencies of the original system. The general problem of multiple frequencies was broached in [1], but in a manner that was very difficult to realize numerically. A much more useful method for dealing with multiple frequencies was presented in [2] and it is this idea that is now generalized to handle additional degrees of freedom by a perturbation process that turns out to be somewhat different from that used for the case of separated frequencies.

To explain the procedure for multiple or close roots, it is useful to recall the variational definition of an eigenvalue and the corresponding eigenvector [4]. In the case of the system of Eq. (1), we can find an X and the corresponding λ satisfying

$$KX = \lambda MX \tag{1}$$

by finding an extremum of X^TKX subject to $X^TMX = 1$. Now for multiple or close roots of the frequency equation, we must consider the entire subspace of s eigenvectors having equal or close eigenvalues. This subspace is spanned by the s orthonormalized vectors Y_1, $Y_2, \ldots, Y_s$, forming the n x s matrix Y. We now find the extremum of

$$\sum_{i=1}^{s} Y_i^T K Y_i$$

subject to the conditions of orthonormality $Y_i^T M Y_j = \delta_{ij}$, i,j = 1, 2, ..., s.

The constrained extremum problem is solved by forming the Lagrangian

$$L = \sum_{i=1}^{s} Y_i^T K Y_i - \sum_{i=1}^{s} \sum_{j=1}^{s} d_{ij} (Y_i^T M Y_j - \delta_{ij}) \tag{4}$$

where the D matrix, $[d_{ij}]$, is an s x s symmetric matrix made up of the Lagrange multipliers. If the roots are equal, the equation

$$\delta L = 0 \tag{5}$$

leads to a D matrix having only diagonal elements all equal to the eigenvalue in question. If the roots are quite close to each other, the diagonal elements are close and the off-diagonal elements of D are very small compared to the diagonal elements. Moreover, as is shown in [2], if the roots are close, the different sets of equations for the improvement in the Y vectors and the changes in one column of D do not couple with each other. Sets of equations very similar to (2) may then be written in the form

$$\left[\begin{array}{c|c} K - \lambda_0 M & -MY \\ \hline -Y^T M & 0 \end{array}\right] \left[\begin{array}{c} \Delta Y_i \\ \hline d_{1i} \\ \cdots \\ d_{si} \end{array}\right] = \left[\begin{array}{c} -R \\ \hline 0 \end{array}\right] \tag{6}$$

For the procedure to begin, it is necessary to have the trial Y_i's available as a set of orthonormal vectors. It should be emphasized that for the case of close roots, the Y_i's obtained by successive approximations are not eigenvectors and the diagonal elements of the D matrix are not eigenvalues. A second step of the process is needed in order to find the s eigenpairs. For this we set

$$X = YZ \tag{7}$$

where Z is the s-dimensional rotation solving the eigenvalue problem of order s:

$$DZ = Z\Lambda \tag{8}$$

The X's found from (7) and the diagonal Λ from (8) are the s eigenvectors and corresponding eigenvalues of the original problem.

The iterative procedure similar to the one using Eq. (2) serves to remove any components of the Y's outside the s-dimensional subspace of the s close eigenvectors, without any tendency to rotate vectors inside that subspace. The final rotation is carried out exactly by solving the eigenvalue problem (8).

The method is now adapted to a perturbation technique suitable for the study of problems in which the small parameter measures the mass and stiffness of the system containing the additional degrees of freedom. The really interesting and more complicated case is one in which the new system would have a frequency equal to one of the old ones if the large masses were held fixed. As the parameter approaches zero, we should expect equal roots, which may separate as the parameter is increased from zero. The considerations that seemed so reasonable when the frequencies were well separated now no longer hold. The new and old modes will here involve very large motions of the small added masses. The previous way of finding the new and old mode shapes fails because the forced vibrations involved would give rise to resonance.

To begin the perturbation analysis, it is first necessary to choose an orthonormal set of Y's for use in a system analogous to Eq. (6). If there are two added masses with two new frequencies tuned to an old one with mode shape ϕ_1, the Y may be written as

$$Y = \left[\begin{array}{c|cc} \phi_1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{array}\right] \tag{9}$$

for an M of the form

$$M = \left[\begin{array}{c|cc} M_0 & 0 & \\ \hline 0 & \alpha\mu & 0 \\ & 0 & \beta\mu \end{array}\right] \tag{10}$$

it is clear that the orthogonality conditions are satisfied. For normality, we must have $\alpha\mu a^2 = 1$, $\beta\mu b^2 = 1$ or, the entries in some of the Y vectors must go as $\mu^{-1/2}$. It is convenient to introduce the quantity ε defined as $\mu^{1/2}$. Then the perturbation series for each Y and for the entries of D are of the form

$$Y_i = \frac{1}{\varepsilon}Y^{(-1)} + Y^{(0)} + \varepsilon Y^{(1)} + \varepsilon^2 Y^{(2)} + \ldots \tag{11a}$$

$$d_{ij} = d_{ij}^{(0)} + \varepsilon d_{ij}^{(1)} + \varepsilon^2 d_{ij}^{(2)} + \ldots \tag{11b}$$

The variational equations and the orthonormality conditions are then written for each Y_i. These do not couple and each $Y_i^{(k)}$ can be found from equations analogous to (6). To obtain the right-hand sides for one perturbation, it is necessary to have the lower-order perturbations for all s of the Y_i's. The final rotation can be carried out either by a perturbation calculation or numerically after choosing an ε.

The fact that the frequency change goes as $\mu^{1/2}$ was shown in [5] for a single mass attached to a system and is assumed in [6] for the more general case. In both references the calculations are carried out by a very different type of perturbation calculation.

4. Numerical Results

The computational schemes outlined above may be illustrated for the very simple system of Fig. 1, which might be thought of as a model for the frame of Fig. 2. The parameters are taken as $K_1 = 2$, $K_2 = 1$, $M_1 = 2$, $M_2 = 1$, $k_1 = k_2 = (1/2)\mu$, $m_1 = 2\mu$. For $\mu = 0$, the two frequencies are 0.5 and 2. The added system also has frequency 0.5, so that there is a double root of 0.5 at $\mu = 0$ and a single one of 2. Table 1 presents the frequency approximations for the separated root for values of μ equal to 0.001, 0.01, and 0.1. The exact results are given along with the second perturbation. The error in the second perturbation involves terms in μ to the third and higher powers.

Table 2 deals with the pair of roots at or near 0.5. Results are given for the first two perturbations along with the exact results. The second perturbation has an error going as $\mu^{3/2}$ and higher-order terms. It should be noted that the proper measure of the accuracy of an approximation is the relative error in the difference between the approximation and the root in question at $\mu = 0$ (0.5 in this case). It is seen that for λ_1, the error is about 16% for $\mu = 0.1$ in the first perturbation, but only 1.4% in the second perturbation.

References

[1] Robinson, A. R. and Harris, F. F., "Improving Approximate Eigenvalues and Eigenvectors," Jnl. of the Eng. Mechanics. Div., Proc. ASCE, Vol. 97, No. EM2, Apr. 1971,pp. 457-475.

[2] Lee, I.-W. and Robinson, A. R., "Solution Techniques for Large Eigenvalue Problems in Structural Mechanics," Civil Engineering Series, SRS No. 462, University of Illinois at Urbana-Champaign, Urbana, Illinois, June 1979.

[3] Wilkinson, J. H., The Algebraic Eigenvalue Problem, Oxford University Press, London, 1965, Chapter 2.

[4] Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. I, Interscience, New York, 1953, Chapter I.

[5] Sackman, J. L., Der Kiureghian, A., and Nour-Omid, B., "Dynamic Analysis of Light Equipment in Structures: Modal Models of the Combined System," Jnl. of Engineering Mechanics, Vol. 109, No. 1, Feb. 1983, pp. 73-89.

[6] Igusa, T. and Der Kiureghian, A. "Dynamic Analysis of Multiply Tuned and Arbitrarily Supported Secondary Systems," Report No. UCB/EERC-83/07, Earthquake Engienering Research Center, University of California, Berkeley, CA, July 1983.

TABLE 1

FREQUENCY APPROXIMATIONS FOR THE SEPARATE ROOT

	$\lambda_3^{(0)} + \mu\lambda_3^{(1)} + \mu^2\lambda_3^{(2)}$	Exact
μ	λ_3	λ_3
0	2.0̇	2.0̇
.001	2.000333342	2.000333343
.01	2.003334259	2.003334261
.1	2.033425926	2.033427207

TABLE 2

FREQUENCY APPROXIMATIONS FOR CLOSE ROOTS

	$\lambda^{(0)} + \varepsilon\lambda^{(1)}$		$\lambda^{(0)} + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)}$		Exact	
μ	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
0	0.5̇0	0.5̇0	0.5̇0	0.5̇0	0.5̇0	0.5̇0
.001	.486307	.513693	.486515	.513901	.486514	.513903
.01	.458921	.543301	.458782	.545385	.458732	.545434
.1	.363069	.636931	.383902	.657764	.382291	.659282

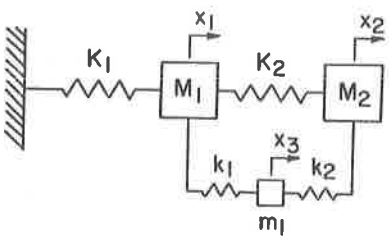

Figure 1. The Case Treated in the Numerical Example.

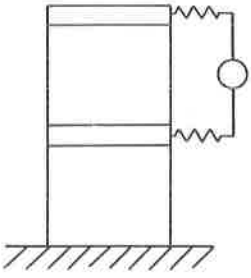

Figure 2. A Structural Frame Modeled in Fig. 2.