SOME METHODS FOR ASYMPTOTIC DISTRIBUTIONS IN THE MULTIVARIATE ANALYSIS[1]

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Summary and introduction.

Various statistics in the multivariate analysis come out as functions of Wishart matrices. Two methods for deriving asymptotic sampling distributions of these statistics are discussed.

In Part I, a method for studentized statistics and its application will be discussed. Let $Z = \{z_1, \ldots, z_m\}$ be a $p \times m$ observation matrix each column of which is independently distributed according to a p-variate non-singular normal distribution with covariance matrix Σ (>0, i.e., positive definite) and let nS_n be a $p \times p$ Wishart matrix distributed independently of Z according to the Wishart distribution $W_p(\Sigma, n)$ with n degrees of freedom and parameter matrix Σ. Then the statistics considered here are functions $T(Z, S_n)$ of the form like $tr(S_n^{-1}ZZ')$, such that the distributions of $T(Z, \Sigma)$ are known or easy to obtain. According to the basic ideas due to B.L. Welch [17, 18] and G. S. James [7] and their modifications, the distributions will be given in the expanded form

$$Pr\{T(Z, S_n) \le t(n)\} = A_0 + \frac{A_1}{n} + \frac{A_2}{n^2} + O(\frac{1}{n^3}),$$

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where A_0, A_1 and A_2 are independent of n.

After describing the general procedure of the method (§1.1), the Hotelling generalized T_0^2 statistic is treated in detail to obtain the term of order n^{-2} in the expanded form of the distribution and also to provide a concrete example of the method (§1.2.1). Some tables which may be useful when applying the method are given in Appendices 1 and 2 at the end of Part I. The evaluation of percentage points of T_0^2 in the null case is discussed briefly in §1.2.2.

In §1.3, other examples which can be treated by our method are illustrated. They are the ratio of two Hotelling's T_0^2-statistics (§1.3.1), Wald-Anderson's classification statistic (§1.3.2) and the maximum of generalized distances (§1.3.3). The last one shows that an asymptotic joint distribution of two studentized statistics can be obtained by our method.

In part II, simple but useful formulas for computing the asymptotic variances and covariances of statistics, which are functions of Wishart matrices, are given (§2.1). The need for them often arises when we want to evaluate the non-null asymptotic distributions of test statistics in the multivariate analysis by the so called delta method. In §2.2 the results obtained so far by using the formulas are listed.

To show how the formulas are used, the test statistics for symmetry in the structure of the population covariance matrix (§2.3) and for multiple independence (§2.4) are treated.

Part I. Method for a sort of studentized statistics.

1.1. The general description of the method.

Let $Z = \{ z_1, \dots, z_m \}$ be a $p \times m$ random matrix where z_i are independently distributed according to p-variate normal distributions with common covariance matrix Σ (>0) and let S_n be a $p \times p$ Wishart matrix which is independent of Z and is subject to a central Wishart distribution $W_p(\frac{1}{n}\Sigma, n)$ with n degrees of freedom and parameter matrix $\frac{1}{n}\Sigma$, i.e.,

$$\frac{|S_n|^{(n-p-1)/2} \exp\{-\frac{1}{2}tr[(\Sigma/n)^{-1}S_n]\}}{2^{np/2}\,\pi^{p(p-1)/4}\,|\Sigma/n|^{n/2}\prod_{i=1}^{p}\Gamma[(n+1-i)/2]} \tag{1.1.1}$$

Let us consider the asymptotic distribution of the statistic $T(Z, S_n)$ which is a function of Z and S_n such that

(i) the distribution of $T(Z, \Sigma)$ is know, and

(ii) $h(S_n) = Pr\{T(Z, S_n) \le c \mid S_n\}$ for a fixed S_n has the Taylor expansion about Σ. We use the notations; $\Sigma = (\sigma_{ij})$ and

$$\underset{p\times p}{\partial} = \left(\frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial\sigma_{ij}}\right) = \begin{bmatrix} \frac{\partial}{\partial\sigma_{11}} & \frac{1}{2}\frac{\partial}{\partial\sigma_{12}} & \cdots & \frac{1}{2}\frac{\partial}{\partial\sigma_{1p}} \\ \frac{1}{2}\frac{\partial}{\partial\sigma_{21}} & \frac{\partial}{\partial\sigma_{22}} & \cdots & \frac{1}{2}\frac{\partial}{\partial\sigma_{2p}} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{2}\frac{\partial}{\partial\sigma_{p1}} & \frac{1}{2}\frac{\partial}{\partial\sigma_{p2}} & \cdots & \frac{\partial}{\partial\sigma_{pp}} \end{bmatrix}$$

where δ_{ij} is Kronecker's delta. Then we have

$$\Pr\{T(Z,S_n)\le c\} = \int_{S_n>0} \Pr\{T(Z,S_n)\le c \mid S_n\}\, dW_p(\tfrac{1}{n}\Sigma, n)$$

$$= \int_{S_n>0} \exp\{tr(S_n-\Sigma)\partial\} \Pr\{T(Z,\Sigma)\le c\}\, dW_p(\tfrac{1}{n}\Sigma, n)$$

$$= \int_{S_n>0} \exp\{tr\, S_n\partial\}\, dW_p(\tfrac{1}{n}\Sigma, n)\cdot \exp\{-tr\,\Sigma\partial\} \Pr\{T(Z,\Sigma)\le c\}$$

$$= |I - \tfrac{2}{n}\Sigma\partial|^{-n/2} \exp\{-tr\,\Sigma\partial\} \Pr\{T(Z,\Sigma)\le c\}$$

$$= \Theta\cdot \Pr\{T(Z,\Sigma)\le c\} \qquad (1.1.2)$$

where

$$\Theta = \exp\{-tr\,\Sigma\partial - \tfrac{n}{2}\log|I - \tfrac{2}{n}\Sigma\partial|\}\cdot \qquad (1.1.3)$$

It should be noted that in the above and also in the sequel, $\partial_{rs} \Pr\{T(Z,\Sigma)\le c\}$ *stands for* $[\tfrac{1}{2}(1+\delta_{rs})\partial \Pr\{T(Z,S_n)\le c \mid S_n\}/\partial s_{rs}]_{S_n=\Sigma}$.

Using the formula

$$-\log|I-A| = tr\,A + \tfrac{1}{2}tr\,A^2 + \tfrac{1}{3}tr\,A^3 + \cdots \qquad (1.1.4)$$

we have, for sufficiently large n, *the following expansion* :

$$\Theta = \exp\{\tfrac{1}{n}tr(\Sigma\partial)^2 + \tfrac{4}{3n^2}tr(\Sigma\partial)^3 + \tfrac{2}{n^3}tr(\Sigma\partial)^4 + \cdots\}$$

$$= 1 + \tfrac{1}{n}tr(\Sigma\partial)^2 + \tfrac{1}{n^2}[\tfrac{4}{3}tr(\Sigma\partial)^3 + \tfrac{1}{2}\{tr(\Sigma\partial)^2\}^2]$$

$$+ \tfrac{1}{n^3}[2tr(\Sigma\partial)^4 + \tfrac{4}{3}tr(\Sigma\partial)^2 tr(\Sigma\partial)^3 + \tfrac{1}{6}\{tr(\Sigma\partial)^2\}^3]$$

$$+ O(\tfrac{1}{n^4})$$

$$
\begin{aligned}
&= 1 + \frac{1}{n}\sum_{rstu} \sigma_{ur}\sigma_{st}\partial_{rs}\partial_{tu} \\
&\quad + \frac{1}{n^2}\Big\{ \frac{4}{3}\sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}\partial_{rs}\partial_{tu}\partial_{vw} + \frac{1}{2}\sum_{rstuvwxy} \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \Big\} \\
&\quad + \frac{1}{n^3}\Big\{ 2\sum_{rstuvwxy} \sigma_{yr}\sigma_{st}\sigma_{uv}\sigma_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \\
&\qquad\quad + \frac{4}{3}\sum_{pqrstuvwxy} \sigma_{sp}\sigma_{qr}\sigma_{yt}\sigma_{uv}\sigma_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}\partial_{pq} \\
&\qquad\quad + \frac{1}{6}\sum_{lmpqrstuvwxy} \sigma_{ql}\sigma_{mp}\sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}\partial_{lm}\partial_{pq}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \Big\} \\
&\quad + O\left(\frac{1}{n^4}\right).
\end{aligned}
\tag{1.1.5}
$$

When c is constant, the desired asymptotic expansion can be obtained by (1.1.2) and (1.1.5). There are sometimes the cases where c is not constant but of the form

$$
\begin{aligned}
c \equiv c(n) &= c_0 + c^*(n) \\
&= c_0 + c_1(n) + c_2(n) + c_3(n) + \cdots,
\end{aligned}
\tag{1.1.6}
$$

where $c_i(n) = O(n^{-i})$, $i = 0, 1, 2, \ldots$. In those cases we have need to expand (1.1.2) further.

$$
\begin{aligned}
\Pr\{T(Z, S_n) \le c(n)\} &= \Theta \cdot \Pr\{T(Z, \Sigma) \le c(n)\} \\
&= \Theta \cdot \exp\{c^*(n)D\} \Pr\{T(Z, \Sigma) \le c_0\}
\end{aligned}
\tag{1.1.7}
$$

where $D = \partial/\partial c_0$. Since

$$
\begin{aligned}
\exp\{c^*(n)D\} &= \exp[\{c_1(n) + c_2(n) + \cdots\}D] \\
&= 1 + c_1(n)D + \{c_2(n)D + \tfrac{1}{2}c_1^2(n)D^2\}
\end{aligned}
$$

$$+ \{ C_3(n)D + C_2(n)C_1(n)D^2 + \frac{1}{6} C_1^3(n)D^3 \}$$
$$+ O(n^{-4}),$$

we have the formula,

$$\Pr\{T(Z,S_n) \le C(n)\} = \Theta^* \cdot \Pr\{T(Z,\Sigma) \le C_0\} \tag{1.1.8}$$

where Θ^* can be written in the following expanded form:

$$\begin{aligned}
\Theta^* = 1 &+ [\, C_1(n)D + \frac{1}{n} \sum_{rstu} \sigma_{ur}\sigma_{st}\,\partial_{rs}\partial_{tu} \,] \\
&+ [\, C_2(n)D + \frac{1}{2} C_1^2(n)D^2 + \frac{1}{n} \sum_{rstu} \sigma_{ur}\sigma_{st}\,\partial_{rs}\partial_{tu}\, C_1(n)D \\
&\quad + \frac{1}{n^2}\{ \frac{4}{3} \sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}\,\partial_{rs}\partial_{tu}\partial_{vw} + \frac{1}{2} \sum_{rstuvwxy} \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}\,\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \} \\
&+ [\, C_3(n)D + C_2(n)C_1(n)D^2 + \frac{1}{6} C_1^3(n)D^3 \\
&\quad + \frac{1}{n} \sum_{rstu} \sigma_{ur}\sigma_{st}\,\partial_{rs}\partial_{tu} \{ C_2(n)D + \frac{1}{2} C_1^2(n)D^2 \} \\
&\quad + \frac{1}{n^2}\{ \frac{4}{3} \sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}\,\partial_{rs}\partial_{tu}\partial_{vw} + \frac{1}{2} \sum_{rstuvwxy} \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}\,\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \} C_1(n)D \\
&\quad + \frac{1}{n^3}\{ 2 \sum_{rstuvwxy} \sigma_{yr}\sigma_{st}\sigma_{uv}\sigma_{wx}\,\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \\
&\qquad + \frac{4}{3} \sum_{pqrstuvwxy} \sigma_{ps}\sigma_{qr}\sigma_{yt}\sigma_{uv}\sigma_{wx}\,\partial_{pq}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \\
&\qquad + \frac{1}{6} \sum_{lmpqrstuvwxy} \sigma_{ql}\sigma_{mp}\sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}\,\partial_{lm}\partial_{pq}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \} \,] \\
&+ O(\frac{1}{n^4}).
\end{aligned} \tag{1.1.9}$$

In order to obtain the asymptotic expansion of the distribution of $T(Z,S_n)$, we need to evaluate the various derivatives of $\Pr\{T(Z,\Sigma) \le C_0\}$ with respect to elements of Σ in (1.1.2) or (1.1.8). To do this we shall use the idea of perturbation in physics and consider

$$J = \Pr\{T(Z,(\Sigma+\varepsilon)) \le t\} \tag{1.1.10}$$

where t is c in (1.1.2) or c_0 in (1.1.8) and ε is a $p\times p$ real symmetric matrix composed of small increments ε_{ij} to σ_{ij} such that $\Sigma+\varepsilon$ is still positive definite. Then by Taylor's theorem

$$J = [1 + \sum_{rs} \varepsilon_{rs}\partial_{rs} + \frac{1}{2!}\sum_{rstu} \varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu} + \frac{1}{3!}\sum_{rstuvw} \varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\partial_{rs}\partial_{tu}\partial_{vw} + \frac{1}{4!}\sum_{rstuvwxy} \varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\varepsilon_{xy}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} + \cdots]\Pr\{T(Z,\Sigma)\le t\} \tag{1.1.11}$$

On the other hand, since z_i $(i=1,\cdots,m)$ is distributed independently according to $N(\mu_i,\Sigma)$, we can express J in the form

$$J = (2\pi)^{-mp/2}|\Sigma|^{-m/2}\int_{D^*}\cdots\int \exp\{-\tfrac{1}{2}\mathrm{tr}(Z-M)'\Sigma^{-1}(Z-M)\}\prod_i dz_i, \tag{1.1.12}$$

where $M=\{\mu_1,\cdots,\mu_m\}$ and

$$D^*:\quad T(Z,(\Sigma+\varepsilon)) \le t\,. \tag{1.1.13}$$

Now we make the non-singular linear transformation : $Z = LY$ with L such that

$$L'(\Sigma+\varepsilon)^{-1}L = I, \qquad L'\Sigma^{-1}L = I - \xi \tag{1.1.14}$$

where ξ is a $p\times p$ diagonal matrix, $\mathrm{diag}\{\xi_1,\cdots,\xi_p\}$ with $|\xi_i|<1$ for all i. This is possible, because we can take ε with sufficiently small elements. Since the Jacobian of the transformation is

$$|L|^m = |\Sigma|^{m/2}|I-\xi|^{m/2},$$

J becomes

$$J = (2\pi)^{-mp/2} |I - \xi|^{m/2} \int_R \cdots \int \exp\{-\tfrac{1}{2} tr(Y-H)'(I-\xi)(Y-H)\} \prod_\alpha dy_\alpha , \quad (1.1.15)$$

where $Y = \{y_1, \ldots, y_m\}$, $H = \{\eta_1, \ldots, \eta_m\} = L^{-1}M$ and

$$R : T(LY, LL') \leq t. \quad (1.1.16)$$

In order that the problem may be feasible, $T(LY, LL')$ should be independent of L (as, for example, is $tr(\Sigma+\varepsilon)^{-1}ZZ' = tr\, YY'$), the integral in (1.1.15) should be evaluated explicitly, and J should be expanded with respect to powers of ε_{ij}'s. If this is the case we can obtain the desired derivatives of $Pr\{T(Z,\Sigma) \leq t\}$ by comparing the resultant with (1.1.11), and hence the asymptotic expansion of the distribution of $T(Z, S_n)$.

Next let us consider the case where we can make use of the characteristic function of $T(Z,\Sigma)$. Let $\phi(t)$ be the characteristic function of $T(Z,S_n)$ and $\phi(t; S_n)$ be the conditional characteristic function of $T(Z,S_n)$ when S_n is fixed, i.e.,

$$\phi(t) = E[\exp\{it\, T(Z,S_n)\}]$$

and

$$\phi(t; S_n) = E_Z[\exp\{it\, T(Z,S_n)\} \mid S_n],$$

where $i = \sqrt{-1}$. Then we have

$$\phi(t) = E_{S_n}\{\phi(t; S_n)\} = E_{S_n}[\exp\{tr(S_n - \Sigma)\partial\}] \cdot \phi(t; \Sigma)$$

$$= \Theta\, \phi(t; \Sigma) \quad (1.1.17)$$

where Θ is the same operator as (1.1.3) or (1.1.5).

In order to evaluate various derivatives of $\Phi(t;\Sigma)$ with respect to elements of Σ, we consider $\Phi(t;\Sigma+\mathcal{E})$, where $\mathcal{E}$ is the similar matrix of increments of Σ as in (1.1.10) and carry out the similar expansion as we made for J. After we have the asymptotic expansion of $\Phi(t)$ in this way, the desired asymptotic distribution of $T(Z,S_n)$ may be obtained by the inversion formula.

1.2. Hotelling's generalized T_0^2 statistics.

1.2.1. Non-null asymptotic distribution of T_0^2 up to order n^{-2}. Let us consider

$$T(Z,S_n) \equiv T_0^2 = \operatorname{tr} S_n^{-1} ZZ', \qquad (1.2.1)$$

$$= \sum_{j=1}^{m} \mathbf{z}_j' S_n^{-1} \mathbf{z}_j .$$

This is the statistic proposed by H. Hotelling [3] for testing the hypothesis $H_0: M=\{\mu_1,\cdots,\mu_m\}=0$ against $H_1: M\neq 0$. It is well known that $\operatorname{tr}\Sigma^{-1}ZZ'$ is distributed as a non-central chi-square with mp degrees of freedom and noncentrality parameter

$$\omega^2 = \operatorname{tr}\Sigma^{-1}MM' = \sum_{j=1}^{m} \mu_j' \Sigma^{-1} \mu_j \qquad (1.2.2)$$

i.e., with $\rho = mp/2$,

$$\Pr\{\operatorname{tr}\Sigma^{-1}ZZ \le t\} = e^{-\omega^2/2} \sum_{j=0}^{\infty} \frac{(\omega^2/2)^j}{j!\, 2^{\rho+j}\Gamma(\rho+j)} \int_0^t x^{\rho+j-1} e^{-x/2} dx \equiv \chi^2_{mp}(t;\omega^2). \qquad (1.2.3)$$

Since $tr(\Sigma+\mathcal{E})^{-1}ZZ' = tr\,YY' = \Sigma_{i=1}^{m} y_i' y_i$, we can evaluate J by using integral formulas of Dirichlet's type. After some computation we have the following form:

$$J = (2\pi)^{-p}|I-\xi|^{m/2}\int_{\xi\Sigma y_i'y_i\le t}\cdots\int \exp\{-\tfrac{1}{2}tr(Y-H)'(I-\xi)(Y-H)\}\prod_i dy_i$$

$$=\left\{\frac{|I-\xi E|}{|I-\xi|}\right\}^{-m/2} e^{-\omega^2/2}\exp\{\tfrac{1}{2}tr(I-\xi)(I-\xi E)^{-1}(I-\xi)HH'E\}\,\chi^2_{mp}(t;0), \qquad (1.2.4)$$

where E is an operator such that $E^r\chi^2_\nu(t;\omega^2) = \chi^2_{\nu+2r}(t;\omega^2)$ for $r=0,1,2,\cdots$. From the relations in (1.1.4), J can be expressed in terms of M, $\mathcal{E}$, Σ and E as

$$J = |I-X\Delta|^{-m/2} e^{-\omega^2/2}\exp\{\tfrac{1}{2}E\,tr(I-X\Delta)^{-1}\Omega\}\,\chi^2_{mp}(t;0), \qquad (1.2.5)$$

where $\Delta = E-1$, $X = (\Sigma+\mathcal{E})^{-1}\Sigma - I$ and

$$\Omega = \Sigma^{-1}MM\cdot \qquad (1.2.6)$$

Using the expansions

$$|I-X\Delta|^{-m/2} = 1 + \tfrac{m}{2}tr(X\Delta) + \tfrac{m}{8}\{2\,tr(X\Delta)^2 + m(tr\,X\Delta)^2\}$$
$$+\tfrac{m}{48}\{8\,tr(X\Delta)^3 + 6m\,tr(X\Delta)tr(X\Delta)^2 + m^2(tr\,X\Delta)^3\}$$
$$+\tfrac{m}{384}\{48\,tr(X\Delta)^4 + 32m\,tr(X\Delta)tr(X\Delta)^3 + 12m(tr(X\Delta)^2)^2$$
$$+12m^2(tr\,X\Delta)^2(tr(X\Delta)^2) + m^3(tr\,X\Delta)^4\} + \cdots \qquad (1.2.7)$$

$$(I-X\Delta)^{-1} = I + (X\Delta) + (X\Delta)^2 + (X\Delta)^3 + (X\Delta)^4 + \cdots \qquad (1.2.8)$$

and

$$X = -\sum_{rs} \varepsilon_{rs}\Sigma^{-1}\Sigma_{rs} + \sum_{rstu} \varepsilon_{rs}\varepsilon_{tu}\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu} - \sum_{rstuvw} \varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu}\Sigma^{-1}\Sigma_{wv}$$
$$+ \sum_{rstuvwxy} \varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\varepsilon_{xy}\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu}\Sigma^{-1}\Sigma_{vw}\Sigma^{-1}\Sigma_{xy} - \cdots \qquad (1.2.9)$$

where $\Sigma_{rs} = \partial_{rs}\Sigma$, J can be written as a series in powers of ε_{rs}'s and thus we can evaluate the desired derivatives of $Pr\{tr\,\Sigma^{-1}ZZ' \le t\}$.

When $M = 0$, i.e., in the null case

$$J = |I - X\Delta|^{-m/2}\chi^2_{mp}(t;0). \qquad (1.2.10)$$

The distribution of T_0^2 in the null case was given by K. Ito [5] in the expanded form up to order n^{-2}. The non-null distribution was treated by M. Siotani [13] in the same way discussed above and later by K. Ito[6], who used the integral representation of the characteristic function of T_0^2 due to P. L. Hsu [4]. But neither could obtain the term of order n^{-2} and even the term of order n^{-1} contained somewhat inconvenient terms for numerical work.

In what follows we will evaluate the non-null distribution of T_0^2 up to order n^{-2} based on (1.1.17), which shows, at the same time, the details of the procedure described in §1.1. To do so, we need to evaluate

$$\phi(t;\Sigma+\varepsilon) = E[\exp\{it\,tr(\Sigma+\varepsilon)^{-1}ZZ'\}]$$
$$= (2\pi)^{-mp/2}|\Sigma|^{-m/2}\int\cdots\int_{R_{mp}} \exp\{it\,tr(\Sigma+\varepsilon)^{-1}ZZ' - \tfrac{1}{2}tr\,\Sigma^{-1}(Z-M)(Z-M)'\}\prod_{\alpha} dz_{\alpha} \qquad (1.2.11)$$

where R_{mp} stands for the whole mp-space. We make the non-singular linear transformation $Z = PY$ with P such that

$$P'(\Sigma+\varepsilon)^{-1}P = I, \qquad P'\Sigma^{-1}P = I - \gamma \tag{1.2.12}$$

where γ *is a diagonal matrix,* $\operatorname{diag}\{\gamma_1, \cdots, \gamma_p\}$ *and we may take* γ_j *such that* $|\gamma_j| < 1$ *for all* $j = 1, \cdots, p$. *Then*

$$\phi(t:\Sigma+\varepsilon) = (2\pi)^{-mp/2}|I-\gamma|^{m/2}\int_{R_{mp}}\!\!\cdots\!\int \exp\{it\,\mathrm{tr}\,YY' - \tfrac{1}{2}\mathrm{tr}(I-\gamma)(Y-H)(Y-H)'\}\prod_j d\mathbf{y}_j, \tag{1.2.13}$$

where we put $H = P^{-1}M$. *Since*

$$\begin{aligned} &it\,\mathrm{tr}\,YY' - \tfrac{1}{2}\mathrm{tr}(I-\gamma)(Y-H)(Y-H)' \\ &= -\tfrac{1}{2}\mathrm{tr}(I-\gamma)HH' - \tfrac{1}{2}[\mathrm{tr}\{(1-2it)I-\gamma\}YY' - 2\,\mathrm{tr}(I-\gamma)YH'] \\ &= -\tfrac{1}{2}\mathrm{tr}(I-\gamma)HH' + \tfrac{1}{2}\mathrm{tr}(I-\gamma)\{(1-2it)I-\gamma\}^{-1}(I-\gamma)HH' \\ &\qquad\qquad - \tfrac{1}{2}\mathrm{tr}\{(1-2it)I-\gamma\}(Y-\Psi)(Y-\Psi)' \end{aligned}$$

where $\Psi = \{(1-2it)I-\gamma\}^{-1}(I-\gamma)H$, *we obtain after integration*

$$\begin{aligned} \phi(t:\Sigma+\varepsilon) = (1-2it)^{-mp/2}\left\{\frac{|I-(1-2it)^{-1}\gamma|}{|I-\gamma|}\right\}^{-m/2} \exp\{-\tfrac{1}{2}\mathrm{tr}(I-\gamma)HH'\} \times \\ \times \exp[\{2(1-2it)\}^{-1}\mathrm{tr}(I-\gamma)\{I-(1-2it)^{-1}\gamma\}^{-1}(I-\gamma)HH']. \end{aligned} \tag{1.2.14}$$

On account of the relations in (2.12), *this can be expressed as*

$$\begin{aligned} \phi(t;\Sigma+\varepsilon) = (1-2it)^{-mp/2}e^{-\omega^2/2}|I-2it(1-2it)^{-1}X|^{-m/2} \times \\ \times \exp[\{2(1-2it)\}^{-1}\mathrm{tr}\{I-2it(1-2it)^{-1}X\}^{-1}\Omega], \end{aligned} \tag{1.2.15}$$

where ω^2, Ω *and* X *are the same symbols as before. Now let us use the notations*

$$\ell(t) = (1-2it)^{-1},$$

$$\Delta(t) = \ell(t) - 1 = 2it(1-2it)^{-1}, \qquad (1.2.16)$$

$$\varphi_{mp}(t;0) = (1-2it)^{-mp/2}.$$

The $\varphi_{mp}(t;0)$ is the characteristic function of central chi-square distribution with mp degrees of freedom. Then

$$\phi(t;\Sigma+\varepsilon) = |I-\Delta(t)X|^{-m/2} e^{-\omega^2/2} \exp[\tfrac{1}{2}\ell(t)\mathrm{tr}\{I-\Delta(t)X\}^{-1}\Omega]\,\varphi_{mp}(t;0). \qquad (1.2.17)$$

It is seen that J in (1.2.5) and $\phi(t;\Sigma+\varepsilon)$ here are of the same form with the following correspondences :

$$E \leftrightarrow \ell(t), \quad \Delta \leftrightarrow \Delta(t), \quad \chi^2_{mp}(t;0) \longleftrightarrow \varphi_{mp}(t;0). \qquad (1.2.18)$$

Hence we have the similar form of expansions in the both cases.

In the null case, i.e., when $M = 0$,

$$\phi(t;\Sigma+\varepsilon) = |I-\Delta(t)X|^{-m/2}\varphi_{mp}(t;0) \qquad (1.2.19)$$

which corresponds to (1.2.10).

With the aid of the expansion formulas in (1.2.7) and (1.2.8), $\phi(t;\Sigma+\varepsilon)$ in (1.2.17) can be expanded in the following form :

$$\phi(t;\Sigma+\varepsilon) = \Big[\,1 + \{\tfrac{m}{2}\Delta(t)\mathrm{tr}X + \tfrac{1}{2}\ell(t)\Delta(t)\mathrm{tr}X\Omega\}$$

$$+\left\{\frac{m}{4}\Delta^2(t)\operatorname{tr}X^2+\frac{m^2}{8}\Delta^2(t)(\operatorname{tr}X)^2+\frac{1}{2}\ell(t)\Delta^2(t)\operatorname{tr}X^2\Omega+\frac{1}{8}\ell^2(t)\Delta^2(t)(\operatorname{tr}X\Omega)^2\right.$$
$$\left.+\frac{m}{4}\ell(t)\Delta^2(t)(\operatorname{tr}X)(\operatorname{tr}X\Omega)\right\}$$

$$+\left\{\frac{m}{6}\Delta^3(t)\operatorname{tr}X^3+\frac{m^2}{8}\Delta^3(t)(\operatorname{tr}X^2)(\operatorname{tr}X)+\frac{m^3}{48}\Delta^3(t)(\operatorname{tr}X)^3\right.$$
$$+\frac{1}{2}\ell(t)\Delta^3(t)\operatorname{tr}X^3\Omega+\frac{1}{4}\ell^2(t)\Delta^3(t)(\operatorname{tr}X^2\Omega)(\operatorname{tr}X\Omega)+\frac{1}{48}\ell^3(t)\Delta^3(t)(\operatorname{tr}X\Omega)^3$$
$$+\frac{m}{8}\ell(t)\Delta^3(t)(\operatorname{tr}X^2)(\operatorname{tr}X\Omega)+\frac{m^2}{16}\ell(t)\Delta^3(t)(\operatorname{tr}X)^2(\operatorname{tr}X\Omega)$$
$$\left.+\frac{m}{4}\ell(t)\Delta^3(t)(\operatorname{tr}X)(\operatorname{tr}X^2\Omega)+\frac{m}{16}\ell^2(t)\Delta^3(t)(\operatorname{tr}X)(\operatorname{tr}X\Omega)^2\right\}$$

$$+\left\{\frac{m}{8}\Delta^4(t)\operatorname{tr}X^4+\frac{m^2}{12}\Delta^4(t)(\operatorname{tr}X^3)(\operatorname{tr}X)+\frac{m^2}{32}\Delta^4(t)(\operatorname{tr}X^2)^2+\frac{m^3}{32}\Delta^4(t)(\operatorname{tr}X^2)(\operatorname{tr}X)^2\right.$$
$$+\frac{m^4}{384}\Delta^4(t)(\operatorname{tr}X)^4$$
$$+\frac{1}{2}\ell(t)\Delta^4(t)\operatorname{tr}X^4\Omega+\frac{1}{4}\ell^2(t)\Delta^4(t)(\operatorname{tr}X^3\Omega)(\operatorname{tr}X\Omega)+\frac{1}{8}\ell^2(t)\Delta^4(t)(\operatorname{tr}X^2\Omega)^2$$
$$+\frac{1}{16}\ell^3(t)\Delta^4(t)(\operatorname{tr}X^2\Omega)(\operatorname{tr}X\Omega)^2+\frac{1}{384}\ell^4(t)\Delta^4(t)(\operatorname{tr}X\Omega)^4$$
$$+\frac{m}{12}\ell(t)\Delta^4(t)(\operatorname{tr}X^3)(\operatorname{tr}X\Omega)+\frac{m^2}{16}\ell(t)\Delta^4(t)(\operatorname{tr}X^2)(\operatorname{tr}X)(\operatorname{tr}X\Omega)$$
$$+\frac{m^3}{96}\ell(t)\Delta^4(t)(\operatorname{tr}X)^3(\operatorname{tr}X\Omega)$$
$$+\frac{m}{4}\ell(t)\Delta^4(t)(\operatorname{tr}X)(\operatorname{tr}X^3\Omega)+\frac{m}{8}\ell^2(t)\Delta^4(t)(\operatorname{tr}X)(\operatorname{tr}X^2\Omega)(\operatorname{tr}X\Omega)$$
$$+\frac{m}{96}\ell^3(t)\Delta^4(t)(\operatorname{tr}X)(\operatorname{tr}X\Omega)^3$$
$$+\frac{m}{8}\ell(t)\Delta^4(t)(\operatorname{tr}X^2)(\operatorname{tr}X^2\Omega)+\frac{m}{32}\ell^2(t)\Delta^4(t)(\operatorname{tr}X^2)(\operatorname{tr}X\Omega)^2$$
$$\left.+\frac{m^2}{16}\ell(t)\Delta^4(t)(\operatorname{tr}X)^2(\operatorname{tr}X^2\Omega)+\frac{m^2}{64}\ell^2(t)\Delta^4(t)(\operatorname{tr}X)^2(\operatorname{tr}X\Omega)^2\right\}$$

$$+\cdots\Big]\ \varphi_{mp}(t:\omega^2) \qquad (1.2.20)$$

where

$$\varphi_{mp}(t;\omega^2) = (1-2it)^{-mp/2}\exp\{\tfrac{1}{2}\omega^2\Delta(t)\} \tag{1.2.21}$$

which is the characteristic function of the non-central chi-square distribution with mp degrees of freedom and noncentrality parameter ω^2. Using (1.2.9), $\phi(t;\Sigma+\varepsilon)$ can be written in the expanded form with respect to powers of ε_{ij}'s. Let us introduce the notations,

$$[rs] = tr\,\Sigma^{-1}\Sigma_{rs},\quad [rs|tu] = tr\,\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu},\quad [rs|tu|vw] = tr\,\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu}\Sigma^{-1}\Sigma_{vw},\cdots$$

$$(rs) = tr\,\Sigma^{-1}\Sigma_{rs}\Omega,\quad (rs|tu) = tr\,\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu}\Omega,\quad (rs|tu|vw) = tr\,\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu}\Sigma^{-1}\Sigma_{vw}\Omega,\cdots$$

Then we have

$$\phi(t;\Sigma+\varepsilon) = \Big[1 - \sum_{rs}\varepsilon_{rs}A^{(1)}_{rs}(t) + \sum_{rstu}\varepsilon_{rs}\varepsilon_{tu}A^{(2)}_{rs,tu}(t) - \sum_{rstuvw}\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}A^{(3)}_{rs,tu,vw}(t)$$
$$+ \sum_{rstuvwxy}\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\varepsilon_{xy}A^{(4)}_{rs,tu,vw,xy}(t) + \cdots\Big]\,\varphi_{mp}(t;\omega^2), \tag{1.2.22}$$

where

$$A^{(1)}_{rs}(t) = \frac{m}{2}\Delta(t)[rs] + \frac{1}{2}\ell(t)\Delta(t)(rs), \tag{1.2.23}$$

$$A^{(2)}_{rs,tu}(t) = \frac{m}{4}\{2\Delta(t)+\Delta^2(t)\}[rs|tu] + \frac{m^2}{8}\Delta^2(t)[rs][tu] + \frac{m}{4}\ell(t)\Delta^2(t)[rs](tu)$$
$$+\frac{1}{2}\ell(t)\{\Delta(t)+\Delta^2(t)\}(rs|tu) + \frac{1}{8}\ell^2(t)\Delta^2(t)(rs)(tu), \tag{1.2.24}$$

$$A^{(3)}_{rs,tu,vw}(t) = \frac{m}{6}\{3\Delta(t)+3\Delta^2(t)+\Delta^3(t)\}[rs|tu|vw] + \frac{m^2}{8}\{2\Delta^2(t)+\Delta^3(t)\}[rs][tu|vw]$$
$$+ \frac{m^3}{48}\Delta^3(t)[rs][tu][vw]$$

$$+\frac{1}{2}\ell(t)\{\Delta(t)+2\Delta^2(t)+\Delta^3(t)\}(rs|tu|vw) + \frac{1}{4}\ell^2(t)\{\Delta^2(t)+\Delta^3(t)\}(rs)(tu|vw)$$

$$+\frac{1}{48}\ell^3(t)\Delta^3(t)(rs)(tu)(vw)$$

$$+\frac{m}{4}\ell(t)\{\Delta^2(t)+\Delta^3(t)\}[rs](tu|vw)+\frac{m}{8}\ell(t)\{2\Delta^2(t)+\Delta^3(t)\}[rs|tu](vw)$$

$$+\frac{m^2}{16}\ell(t)\Delta^3(t)[rs][tu](vw)+\frac{m}{16}\ell^2(t)\Delta^3(t)[rs](tu)(vw), \qquad (1.2.25)$$

$$A^{(4)}_{rs,tu,vw,xy}(t)=\frac{m}{8}\{4\Delta(t)+6\Delta^2(t)+4\Delta^3(t)+\Delta^4(t)\}[rs|tu|vw|xy]$$

$$+\frac{m^2}{12}\{3\Delta^2(t)+3\Delta^3(t)+\Delta^4(t)\}[rs][tu|vw|xy]+\frac{m^2}{32}\{4\Delta^2(t)+4\Delta^3(t)+\Delta^4(t)\}[rs|tu][vw|xy]$$

$$+\frac{m^3}{32}\{2\Delta^3(t)+\Delta^4(t)\}[rs][tu][vw|xy]+\frac{m^4}{384}\Delta^4(t)[rs][tu][vw][xy]$$

$$+\frac{1}{2}\ell(t)\{\Delta(t)+3\Delta^2(t)+3\Delta^3(t)+\Delta^4(t)\}(rs|tu|vw|xy)$$

$$+\frac{1}{4}\ell^2(t)\{\Delta^2(t)+2\Delta^3(t)+\Delta^4(t)\}(rs)(tu|vw|xy)+\frac{1}{8}\ell^2(t)\{\Delta^2(t)+2\Delta^3(t)+\Delta^4(t)\}(rs|tu)(vw|xy)$$

$$+\frac{1}{16}\ell^3(t)\{\Delta^3(t)+\Delta^4(t)\}(rs)(tu)(vw|xy)+\frac{1}{384}\ell^4(t)\Delta^4(t)(rs)(tu)(vw)(xy)$$

$$+\frac{m}{12}\ell(t)\{3\Delta^2(t)+3\Delta^3(t)+\Delta^4(t)\}[rs|tu|vw](xy)+\frac{m}{8}\ell(t)\{2\Delta^2(t)+3\Delta^3(t)+\Delta^4(t)\}[rs|tu](vw|xy)$$

$$+\frac{m}{4}\ell(t)\{\Delta^2(t)+2\Delta^3(t)+\Delta^4(t)\}[rs](tu|vw|xy)+\frac{m^2}{16}\ell(t)\{2\Delta^3(t)+\Delta^4(t)\}[rs][tu|vw](xy)$$

$$+\frac{m}{32}\ell^2(t)\{2\Delta^3(t)+\Delta^4(t)\}[rs|tu](vw)(xy)+\frac{m^2}{16}\ell(t)\{\Delta^3(t)+\Delta^4(t)\}[rs][tu](vw|xy)$$

$$+\frac{m}{8}\ell^2(t)\{\Delta^3(t)+\Delta^4(t)\}[rs](tu)(vw|xy)+\frac{m^3}{96}\ell(t)\Delta^4(t)[rs][tu][vw](xy)$$

$$+\frac{m^2}{64}\ell^2(t)\Delta^4(t)[rs][tu](vw)(xy)+\frac{m}{96}\ell^3(t)\Delta^4(t)[rs](tu)(vw)(xy) \qquad (1.2.26)$$

Since we have, on the other hand, the Taylor expansion

$$\phi(t;\Sigma+\varepsilon)=\Big\{1+\sum_{rs}\varepsilon_{rs}\partial_{rs}+\frac{1}{2!}\sum_{rstu}\varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu}+\frac{1}{3!}\sum_{rstuvw}\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\partial_{rs}\partial_{tu}\partial_{vw}$$

$$+\frac{1}{4!}\sum_{rstuvwxy}\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\varepsilon_{xy}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}+\cdots\Big\}\phi(t;\Sigma), \qquad (1.2.27)$$

the comparison of this expansion with (1.2.20) gives us the desired derivatives of $\Phi(t;\Sigma)$. In doing so, however, we must take account of the symmetry in suffixes rs, tu, etc. Let us define

$$B^{(1)}_{rs}(t) = A^{(1)}_{rs}(t), \qquad B^{(2)}_{rs,tu}(t) = \frac{1}{2!}\sum_{(P_2)} A^{(2)}_{rs,tu}(t), \tag{1.2.28}$$

$$B^{(3)}_{rs,tu,vw}(t) = \frac{1}{3!}\sum_{(P_3)} A^{(3)}_{rs,tu,vw}(t), \qquad B^{(4)}_{rs,tu,vw,xy}(t) = \frac{1}{3!}\sum_{(P_4)} A^{(4)}_{rs,tu,vw,xy}(t)$$

where $\sum_{(P_i)}$ stands for the summation over all the permutations of suffixes rs, tu,$\cdots$ of $A^{(i)}_{\cdots}(t)$'s. Then we have

$$\partial_{rs}\,\Phi(t:\Sigma) = -B^{(1)}_{rs}(t)\,\varphi_{mp}(t;\omega^2), \tag{1.2.29}$$

$$\partial_{rs}\,\partial_{tu}\,\Phi(t;\Sigma) = 2\,B^{(2)}_{rs,tu}(t)\,\varphi_{mp}(t;\omega^2), \tag{1.2.30}$$

$$\partial_{rs}\,\partial_{tu}\,\partial_{vw}\Phi(t;\Sigma) = -6\,B^{(3)}_{rs,tu,vw}(t)\,\varphi_{mp}(t;\omega^2), \tag{1.2.31}$$

$$\partial_{rs}\,\partial_{tu}\,\partial_{uw}\,\partial_{xy}\,\Phi(t:\Sigma) = 24\,B^{(4)}_{rs,tu,vw,xy}(t)\,\varphi_{mp}(t;\omega^2), \tag{1.2.32}$$

and hence the characteristic function $\Phi(t)$ of $T_0^2 = \operatorname{tr} S_n^{-1}ZZ'$ can be written in the following expanded form by inserting these results into (1.1.17):

$$\begin{aligned}\Phi(t) = \Big[\,1 &+ \frac{2}{n}\sum_{rstu}\sigma_{ur}\sigma_{st}B^{(2)}_{rs,tu}(t) \\ &+ \frac{1}{n^2}\Big\{-8\sum_{rstuvw}\sigma_{wr}\sigma_{st}\sigma_{uv}B^{(3)}_{rs,tu,vw}(t) + 12\sum_{rstuvwxy}\sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}B^{(4)}_{rs,tu,vw,xy}(t)\Big\} \\ &+ O\Big(\frac{1}{n^3}\Big)\Big]\,\varphi_{mp}(t;\omega^2)\end{aligned} \tag{1.2.33}$$

To carry out the summation in (1.2.33), let us, first of all, simplify the terms which appear in the $B^{(j)}_{\cdots}(t)$'s, using the properties of the trace. For example, the following are easily checked :

$$\frac{1}{3!}\sum_{(P_3)}[rs|tu|vw] = [rs|tu|vw]$$

$$\frac{1}{3!}\sum_{(P_3)}[rs|tu](vw) = \frac{1}{3}\{[rs|tu](vw) + [rs|vw](tu) + [tu|vw](rs)\},$$

$$\frac{1}{4!}\sum_{(P_4)}(rs|tu)(vw|xy) = \frac{1}{3}\{(rs|tu)(vw|xy) + (rs|vw)(tu|xy) + (rs|xy)(tu|vw)\}.$$

All the reductions necessary in our computation are listed in Appendix 1 at the end of Part I.

Next we evaluate the values of various summations such as $\sum \sigma_{ur}\sigma_{st}[rs|tu]$, $\sum \sigma_{wr}\sigma_{st}\sigma_{uv}(rs|tu)(vw)$, *etc. Since*

$$[rs] = \operatorname{tr}\Sigma^{-1}\Sigma_{rs} = \sigma^{rs}, \tag{1.2.34}$$

$$[rs|tu] = \operatorname{tr}\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu} = \frac{1}{2}(\sigma^{ur}\sigma^{st} + \sigma^{us}\sigma^{rt}), \tag{1.2.35}$$

$$\begin{aligned}[rs|tu|vw] = \operatorname{tr}\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu}\Sigma^{-1}\Sigma_{vw} = \frac{1}{8}\{&\sigma^{wr}\sigma^{st}\sigma^{uv} + \sigma^{wr}\sigma^{su}\sigma^{tv} + \sigma^{ws}\sigma^{rt}\sigma^{uv} \\ &+ \sigma^{ws}\sigma^{ru}\sigma^{tv} + \sigma^{vr}\sigma^{st}\sigma^{uw} + \sigma^{vr}\sigma^{su}\sigma^{tw} + \sigma^{vs}\sigma^{rt}\sigma^{uw} + \sigma^{vs}\sigma^{ru}\sigma^{tw}\},\end{aligned} \tag{1.2.36}$$

$$\begin{aligned}[rs|tu|vw|xy] = \operatorname{tr}\Sigma^{-1}\Sigma_{rs}\Sigma^{-1}\Sigma_{tu}\Sigma^{-1}\Sigma_{vw}\Sigma^{-1}\Sigma_{xy} = \frac{1}{16}\{&\sigma^{yr}\sigma^{st}\sigma^{uv}\sigma^{wx} + \sigma^{yr}\sigma^{st}\sigma^{uw}\sigma^{vx} \\ &+ \sigma^{yr}\sigma^{su}\sigma^{tv}\sigma^{wx} + \sigma^{yr}\sigma^{su}\sigma^{tw}\sigma^{vx} + \sigma^{ys}\sigma^{rt}\sigma^{uv}\sigma^{wx} + \sigma^{ys}\sigma^{rt}\sigma^{uw}\sigma^{vx} \\ &+ \sigma^{ys}\sigma^{ru}\sigma^{tv}\sigma^{wx} + \sigma^{ys}\sigma^{ru}\sigma^{tw}\sigma^{vx} + \sigma^{xr}\sigma^{st}\sigma^{uv}\sigma^{wy} + \sigma^{xr}\sigma^{st}\sigma^{uw}\sigma^{vy} \\ &+ \sigma^{xr}\sigma^{su}\sigma^{tv}\sigma^{wy} + \sigma^{xr}\sigma^{su}\sigma^{tw}\sigma^{vy} + \sigma^{xs}\sigma^{rt}\sigma^{uv}\sigma^{wy} + \sigma^{xs}\sigma^{rt}\sigma^{uw}\sigma^{vy} \\ &+ \sigma^{xs}\sigma^{ru}\sigma^{tv}\sigma^{wy} + \sigma^{xy}\sigma^{ru}\sigma^{tw}\sigma^{vy}\},\end{aligned} \tag{1.2.37}$$

the summation containing only the terms of these types can be easily calculated, as listed in (A) *of Appendix 2.*

A great deal of labour of computation is needed for the summations containing terms of (rs), $(rs|tu)$, etc. For example let us consider

$$K = \sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}(rs)(tu|vw).$$

Let $\Sigma^{-1} = (\underline{\sigma}^{(1)}, \ldots, \underline{\sigma}^{(p)})$, i.e., $\underline{\sigma}^{(r)}$ be the r-th column of Σ^{-1} and $\Omega' = (\underline{\omega}_1, \ldots, \underline{\omega}_p)$, i.e., $\underline{\omega}'_r$ be the r-th row of Ω. Then we have

$$\begin{aligned}
(rs)(tu|vw) &= (tr\,\Sigma^{-1}\Sigma_{rs}\Omega)(tr\,\Sigma^{-1}\Sigma_{tu}\Sigma^{-1}\Sigma_{vw}\Omega) \\
&= \tfrac{1}{8}\{\underline{\omega}'_r\underline{\sigma}^{(s)} + \underline{\omega}'_s\underline{\sigma}^{(r)}\}\{\underline{\omega}'_v(\underline{\sigma}^{(u)}\sigma^{tw} + \underline{\sigma}^{(t)}\sigma^{uw}) + \underline{\omega}'_w(\underline{\sigma}^{(u)}\sigma^{tv} + \underline{\sigma}^{(t)}\sigma^{uv})\} \\
&= \tfrac{1}{8}\{\underline{\omega}'_r\underline{\sigma}^{(s)}(\underline{\sigma}^{(u)}\sigma^{tw} + \underline{\sigma}^{(t)}\sigma^{uw})'\underline{\omega}_v + \underline{\omega}'_r\underline{\sigma}^{(s)}(\underline{\sigma}^{(u)}\sigma^{tv} + \underline{\sigma}^{(t)}\sigma^{uv})'\underline{\omega}_w \\
&\quad + \underline{\omega}'_s\underline{\sigma}^{(r)}(\underline{\sigma}^{(u)}\sigma^{tw} + \underline{\sigma}^{(t)}\sigma^{uw})'\underline{\omega}_v + \underline{\omega}'_s\underline{\sigma}^{(r)}(\underline{\sigma}^{(u)}\sigma^{tv} + \underline{\sigma}^{(t)}\sigma^{uv})'\underline{\omega}_w\}
\end{aligned}$$

and

$$\begin{aligned}
K_1 &= \sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}\,\underline{\omega}'_r\underline{\sigma}^{(s)}(\underline{\sigma}^{(u)}\sigma^{tw} + \underline{\sigma}^{(t)}\sigma^{uw})'\underline{\omega}_v \\
&= \sum_{rv}\underline{\omega}'_r\{\sum_{stuw}\sigma_{wr}\sigma_{st}\sigma_{uv}\underline{\sigma}^{(s)}(\underline{\sigma}^{(u)}\sigma^{tw} + \underline{\sigma}^{(t)}\sigma^{uw})'\}\underline{\omega}_v \\
&= \sum_{rv}\underline{\omega}'_r\underline{\delta}_r\underline{\delta}'_v\underline{\omega}_v + \sum_{rv}\sigma_{vr}\underline{\omega}'_r\Sigma^{-1}\underline{\omega}_v,
\end{aligned}$$

where $\underline{\delta}'_r = (\delta_{1r}, \ldots, \delta_{pr})$ and δ_{jr}'s are Kronecker's symbols. Since

$$\sum_{rv}\underline{\omega}'_r\underline{\delta}_r\underline{\delta}'_v\underline{\omega}_v = (\sum_r\omega_{rr})(\sum_v\omega_{vv}) = (tr\,\Omega)^2 = \omega^4$$

and

$$\begin{aligned}
\sum_{rv}\sigma_{vr}\underline{\omega}'_r\Sigma^{-1}\underline{\omega}_v &= tr\,\Sigma^{-1}(\sum_{rv}\sigma_{vr}\underline{\omega}_v\underline{\omega}'_r) \\
&= tr\,\Sigma^{-1}\Omega'\Sigma\Omega \\
&= tr\,\Sigma^{-1}MM'\Sigma^{-1}\Sigma\Sigma^{-1}MM' \\
&= tr\,\Omega^2,
\end{aligned}$$

we have

$$K_1 = \omega^4 + \mathrm{tr}\,\Omega^2 \cdot$$

The similar computation gives us

$$\sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}\, \underline{\omega}_r' \underline{\Omega}^{(s)}(\underline{\Omega}^{(u)}\sigma^{tv} + \underline{\Omega}^{(t)}\sigma^{uv})'\underline{\omega}_w = (p+1)\mathrm{tr}\,\Omega^2,$$

$$\sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}\, \underline{\omega}_s' \underline{\Omega}^{(r)}(\underline{\Omega}^{(u)}\sigma^{tw} + \underline{\Omega}^{(t)}\sigma^{uw})'\underline{\omega}_v = \omega^4 + \mathrm{tr}\,\Omega^2,$$

$$\sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}\, \underline{\omega}_s' \underline{\Omega}^{(r)}(\underline{\Omega}^{(u)}\sigma^{tv} + \underline{\Omega}^{(t)}\sigma^{uv})'\underline{\omega}_w = (p+1)\mathrm{tr}\,\Omega^2,$$

and hence

$$K = \frac{1}{4}\omega^4 + \frac{1}{4}(p+2)\mathrm{tr}\,\Omega^2 \cdot$$

If we put $\Omega = I$, K should be equal to $p(p+1)/2$, the value of $\sum \sigma_{wr}\sigma_{st}\sigma_{uv}$: $[rs][tu|vw]$, which gives us a check-method for our computation. The values of the summations of this type are summarized in Appendix 2, (B).

Now the summations of the mixed type are easily calculated by first summing out the portion within brackets and then using the (B) of Appendix 2. For example,

$$\begin{aligned}\sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}[rs|tu](vw) &= \frac{1}{2}\sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv}(\sigma^{ur}\sigma^{st} + \sigma^{us}\sigma^{rt})(vw) \\ &= \frac{1}{2}(p+1)\sum_{vw}\sigma_{vw}(vw) \\ &= \frac{1}{2}(p+1)\omega^2,\end{aligned}$$

$$\begin{aligned}\sum_{rstuvwxy} \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu][vw](xy) &= \sum_{rstuvwxy} \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}\sigma^{rs}\sigma^{tu}\sigma^{vw}(xy) \\ &= p\sum_{xy}\sigma_{xy}(xy) \\ &= p\,\omega^2 \cdot\end{aligned}$$

The (C) of Appendix 2 contains all values of summations of this mixed type, which appear in our evaluation.

Now we can evaluate the characteristic function $\Phi(t)$ of T_0^2 in (1.2.33) with the aid of (1.2.24) – (1.2.26) and Appendices 1 and 2.

(i) For the term of order n^{-1},

$$\begin{aligned}
\sum_{rstu}\sigma_{ur}\sigma_{st}B^{(2)}_{rs,tu}(t) &= \sum_{rstu}\sigma_{ur}\sigma_{st}\Big\{\frac{1}{2!}\sum_{(P_2)}A^{(2)}_{rs,tu}(t)\Big\}\\
&= \frac{m}{4}\{2\Delta(t)+\Delta^2(t)\}\sum_{rstu}\sigma_{ur}\sigma_{st}\Big\{\frac{1}{2!}\sum_{(P_2)}[rs|tu]\Big\}\\
&\quad+\frac{m^2}{8}\Delta^2(t)\sum_{rstu}\sigma_{ur}\sigma_{st}\Big\{\frac{1}{2!}\sum_{(P_2)}[rs][tu]\Big\}\\
&\quad+\frac{1}{2}\ell(t)\{\Delta(t)+\Delta^2(t)\}\sum_{rstu}\sigma_{ur}\sigma_{st}\Big\{\frac{1}{2!}\sum_{(P_2)}(rs|tu)\Big\}\\
&\quad+\frac{1}{8}\ell^2(t)\Delta^2(t)\sum_{rstu}\sigma_{ur}\sigma_{st}\Big\{\frac{1}{2!}\sum_{(P_2)}(rs)(tu)\Big\}\\
&\quad+\frac{m}{4}\ell(t)\Delta^2(t)\sum_{rstu}\sigma_{ur}\sigma_{st}\Big\{\frac{1}{2!}\sum_{(P_2)}[rs](tu)\Big\}\\
&= \frac{m}{8}p(p+1)\{2\Delta(t)+\Delta^2(t)\}+\frac{m^2}{8}p\Delta^2(t)\\
&\quad+\frac{1}{4}(p+1)\omega^2\ell(t)\{\Delta(t)+\Delta^2(t)\}+\frac{1}{8}\operatorname{tr}\Omega^2\ell^2(t)\Delta^2(t)\\
&\quad+\frac{m}{4}\omega^2\ell(t)\Delta^2(t).
\end{aligned}$$

Since $\ell(t)=\Delta(t)+1=(1-2it)^{-1}$ and $\ell^j(t)$ is the characteristic function of the chi-square distribution with $2j$ degrees of freedom, it is convenient to arrange the above expression with respect to powers of $\ell(t)$. Then the above is equal to

$$\begin{aligned}
&\frac{m}{8}p(p+1)\{\ell^2(t)-1\}+\frac{m^2}{8}p\{\ell^2(t)-2\ell(t)+1\}+\frac{1}{4}(p+1)\omega^2\{\ell^3(t)-\ell^2(t)\}\\
&+\frac{1}{8}\operatorname{tr}\Omega^2\{\ell^4(t)-2\ell^3(t)+\ell^2(t)\}+\frac{m}{4}\omega^2\{\ell^3(t)-2\ell^2(t)+\ell(t)\},
\end{aligned}$$

and hence

$$\sum_{rstu} \sigma_{ur}\sigma_{st} B^{(2)}_{rs,tu}(t) = \frac{m}{8} p(m-p-1) - \frac{m}{4}(mp-\omega^2)\ell(t)$$

$$+ \frac{1}{8}\{mp(m+p+1) - 2(2m+p+1)\omega^2 + \mathrm{tr}\,\Omega^2\}\ell^2(t)$$

$$+ \frac{1}{4}\{(m+p+1)\omega^2 - \mathrm{tr}\,\Omega^2\}\ell^3(t) + \frac{1}{8}\mathrm{tr}\,\Omega^2 \ell^4(t) \qquad (1.2.38)$$

which gives us the term of order n^{-1} in the expansion of $\phi(t)$.

(ii) Some patience and care are needed to get the term of order n^{-2}. The results are

$$\sum_{rstuvw} \sigma_{wr}\sigma_{st}\sigma_{uv} B^{(3)}_{rs,tu,vw}(t) = \frac{m}{48} p(p^2+3p+4)\{3\Delta(t)+3\Delta^2(t)+\Delta^3(t)\}$$

$$+ \frac{m^2}{16} p(p+1)\{2\Delta^2(t)+\Delta^3(t)\} + \frac{m^3}{48}\Delta^3(t)p + \frac{1}{16}(p^2+3p+4)\omega^2 \ell(t)\{\Delta(t)+2\Delta^2(t)+\Delta^3(t)\}$$

$$+ \frac{1}{16}\{\omega^4+(p+2)\mathrm{tr}\,\Omega^2\}\ell^2(t)\{\Delta^2(t)+\Delta^3(t)\} + \frac{1}{48}\mathrm{tr}\,\Omega^3 \ell^3(t)\Delta^3(t)$$

$$+ \frac{m}{8}(p+1)\omega^2\ell(t)\{\Delta^2(t)+\Delta^3(t)\} + \frac{m}{16}(p+1)\omega^2\ell(t)\{2\Delta^2(t)+\Delta^3(t)\}$$

$$+ \frac{m^2}{16}\omega^2\ell(t)\Delta^3(t) + \frac{m}{16}\mathrm{tr}\,\Omega^2\ell^2(t)\Delta^3(t)$$

$$= -\frac{mp}{48}\{m^2-3m(p+1)+(p^2+3p+4)\} + \frac{m}{16}(m-p-1)(mp-\omega^2)\ell(t)$$

$$- \frac{m}{16}\{mp(m+p+1)-(3m+p+1)\omega^2+\mathrm{tr}\,\Omega^2\}\ell^2(t)$$

$$+ \frac{1}{48}[mp\{m^2+3m(p+1)+(p^2+3p+4)\} - 3\{3m^2+5m(p+1)+(p^2+3p+4)\}\omega^2$$

$$+3\omega^4 + 3(3m+p+2)\mathrm{tr}\,\Omega^2 - \mathrm{tr}\,\Omega^3]\ell^3(t)$$

$$+ \frac{1}{16}[\{m^2+3m(p+1)+(p^2+3p+4)\}\omega^2 - 2\omega^4 - (3m+2p+4)\mathrm{tr}\,\Omega^2 + \mathrm{tr}\,\Omega^3]\ell^4(t)$$

$$+ \frac{1}{16}\{\omega^4+(m+p+2)\mathrm{tr}\,\Omega^2 - \mathrm{tr}\,\Omega^3\}\ell^5(t) + \frac{1}{48}\mathrm{tr}\,\Omega^3 \ell^6(t) \qquad (1.2.39)$$

and

$$\sum_{rstuvwxy} \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx} B^{(4)}_{rs.tu.vw.xy}(t) = \frac{mp}{384}\{m^3p-2m^2(p^2+p+4)+m(p+1)(p^2+p+20)-4(2p^2+5p+5)\}$$

$$-\frac{m}{96}(mp-4)(m-p-1)(mp-\omega^2)\,\ell(t)$$

$$+\frac{m}{192}[mp\{3m^2p-(p+1)(p^2+p+4)\}-2\{4m^2p-m(p^2+p+4)-p(p+1)^2\}\omega^2$$

$$+2m\omega^4+(mp-p^2-p-4)\operatorname{tr}\Omega^2]\,\ell^2(t)$$

$$-\frac{m}{96}[mp\{m^2p+m(p^2+p+4)+4(p+1)\}-\{6m^2p+3m(p^2+p+4)-(p+1)(p^2+p-8)\}\omega^2$$

$$+2(2m+p+1)\omega^4+(2mp-p^2-p+4)\operatorname{tr}\Omega^2-\omega^2\operatorname{tr}\Omega^2]\,\ell^3(t)$$

$$+\frac{1}{384}[mp\{m^3p+2m^2(p^2+p+4)+m(p+1)(p^2+p+20)+4(2p^2+5p+5)\}$$

$$-4\{4m^3p+5m^2(p^2+p+4)+m(p+1)(p^2+p+32)+4(2p^2+5p+5)\}\omega^2$$

$$+4\{6m^2+6m(p+1)+(p^2+2p+7)\}\omega^4+4\{3m^2p+24m+10p+16\}\operatorname{tr}\Omega^2$$

$$-4(4m+p+1)\omega^2\operatorname{tr}\Omega^2-16\operatorname{tr}\Omega^3+(\operatorname{tr}\Omega^2)^2]\,\ell^4(t)$$

$$+\frac{1}{96}[\{m^3p+2m^2(p^2+p+4)+m(p+1)(p^2+p+20)+4(2p^2+5p+5)\}\omega^2$$

$$-2\{2m^2+3m(p+1)+(p^2+2p+7)\}\omega^4+3(2m+p+1)\omega^2\operatorname{tr}\Omega^2$$

$$-\{2m^2p+m(p^2+p+28)+4(5p+8)\}\operatorname{tr}\Omega^2+12\operatorname{tr}\Omega^3-(\operatorname{tr}\Omega^2)^2]\,\ell^5(t)$$

$$+\frac{1}{192}[2\{m^2+2m(p+1)+(p^2+2p+7)\}\omega^4+\{m^2p+m(p^2+p+20)+4(5p+8)\}\operatorname{tr}\Omega^2$$

$$-2(4m+3p+3)\omega^2\operatorname{tr}\Omega^2-24\operatorname{tr}\Omega^3+3(\operatorname{tr}\Omega^2)^2]\,\ell^6(t)$$

$$+\frac{1}{96}\{(m+p+1)\omega^2\operatorname{tr}\Omega^2+4\operatorname{tr}\Omega^3-(\operatorname{tr}\Omega^2)^2\}\,\ell^7(t)$$

$$+\frac{1}{384}(\operatorname{tr}\Omega^2)^2\ell^8(t) \qquad (1.2.40)$$

The desired expanded form of $\Phi(t)$ is given by inserting (1.2.38), (1.2.39) and (1.2.40) into (1.2.33). Since

$$\ell^{j}(t)\varphi_{mp}(t;\omega^2) = (1-2it)^{-(mp+2j)/2}\exp\{-\tfrac{1}{2}\omega^2\Delta(t)\} = \varphi_{mp+2j}(t;\omega^2), \qquad (1.2.41)$$

we can write $\Phi(t)$, after some arrangement, in the following form:

$$\begin{aligned}\Phi(t) = \varphi_{mp}(t;\omega^2) &+ \frac{1}{4n}\sum_{j=0}^{4} a_j(m,p)\varphi_{mp+2j}(t;\omega^2)\\ &+\frac{1}{96n^2}\sum_{j=0}^{8} b_j(m,p)\varphi_{mp+2j}(t;\omega^2)\\ &+O\left(\frac{1}{n^3}\right), \qquad (1.2.42)\end{aligned}$$

where $a_j(m,p)$'s and $b_j(m,p)$'s will be given in a while.

Let $f(x)$ be the density function of T_0^2 and let $g_\nu(x;\omega^2)$ be the density function of the non-central chi-square distribution with ν degrees of freedom and non-centrality parameter ω^2. We know that

$$\begin{aligned}g_\nu(x;\omega^2) &= e^{-\omega^2/2}\sum_{\alpha=0}^{\infty}\frac{(\omega^2/2)^\alpha}{\alpha!\,2^{(\nu/2)+\alpha}\,\Gamma[(\nu/2)+\alpha]}x^{(\nu/2)+\alpha-1}e^{-\frac{x}{2}}, \qquad (1.2.43)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_\nu(t;\omega^2)\,dt.\end{aligned}$$

Hence from (1.2.42) we have immediately the following final result:

A non-null asymptotic distribution of the Hotelling generalized T_0^2 defined by (1.2.1) is given by

$$\begin{aligned}f(x) = g_{mp}(x;\omega^2) &+ \frac{1}{4n}\sum_{j=0}^{4} a_j(m,p)g_{mp+2j}(x;\omega^2)\\ &+\frac{1}{96n^2}\sum_{j=0}^{8} b_j(m,p)g_{mp+2j}(x;\omega^2)\end{aligned}$$

$$+\ O\left(\frac{1}{n^3}\right), \tag{1.2.44}$$

where

$$a_0(m,p) = mp(m-p-1),$$

$$a_1(m,p) = -2m(mp-\omega^2),$$

$$a_2(m,p) = mp(m+p+1) - 2(2m+p+1)\omega^2 + \mathrm{tr}\,\Omega^2,$$

$$a_3(m,p) = 2\{(m+p+1)\omega^2 - \mathrm{tr}\,\Omega^2\},$$

$$a_4(m,p) = \mathrm{tr}\,\Omega^2,$$

$$b_0(m,p) = mp\{m(3mp-8)(m-2p-2) + m(p+1)(3p^2+3p-4) - 4(2p^2+3p-1)\},$$

$$b_1(m,p) = -12m^2p(m-p-1)(mp-\omega^2),$$

$$\begin{aligned} b_2(m,p) = 6m[\,&mp\{3m^2p+8m-(p+1)(p^2+p-4)\} - 2\{(mp+2)(4m-p-1)-(p-2)(p+1)(p+3)\}\omega^2 \\ &+ 2m\omega^4 + \{mp-(p^2+p-4)\}\mathrm{tr}\,\Omega^2\,], \end{aligned}$$

$$\begin{aligned} b_3(m,p) = -4[\,&mp\{m(3mp+16)(m+p+1) + 8m(p+1) + 4(p^2+3p+4)\} \\ &-3\{3m(mp+4)(2m+p+1) - m(p+1)(p^2+p-16) + 4(p^2+3p+4)\}\omega^2 \\ &+6\{2m^2+m(p+1)+2\}\omega^4 + 3\{(mp+8)(2m-p-1)+4(3p+4)\}\mathrm{tr}\,\Omega^2 \\ &-3m\omega^2\mathrm{tr}\,\Omega^2 - 4\mathrm{tr}\,\Omega^3\,], \end{aligned}$$

$$\begin{aligned} b_4(m,p) = 3[\,&mp\{m(mp+8)(m+2p+2)+m(p+1)(p^2+p+4)+4(2p^2+5p+5)\} \\ &-4\{m(mp+6)(4m+5p+5)+m(p+1)(p^2+p+14)+4(3p^2+8p+9)\}\omega^2 \\ &+4\{6m(m+p+1)+(p^2+2p+15)\}\omega^4 + 4\{3m^2p+36m+18p+32\}\mathrm{tr}\,\Omega^2 \\ &-4(4m+p+1)\omega^2\mathrm{tr}\,\Omega^2 - 32\mathrm{tr}\,\Omega^3 + (\mathrm{tr}\,\Omega^2)^2\,], \end{aligned}$$

$$\begin{aligned} b_5(m,p) = 12[\,&\{m(mp+8)(m+2p+2) + m(p+1)(p^2+p+4) + 4(2p^2+5p+5)\}\omega^2 \\ &-2\{(2m+p+1)(m+p+1)+8\}\omega^4 - \{(mp+16)(2m+p+1)+8(p+3)\}\mathrm{tr}\,\Omega^2 \\ &+3(2m+p+1)\omega^2\mathrm{tr}\,\Omega^2 + 16\mathrm{tr}\,\Omega^3 - (\mathrm{tr}\,\Omega^2)^2\,], \end{aligned}$$

$$b_6(m,p) = 2[\,6\{(m+p+1)^2+6\}\omega^4 + 3\{(mp+20)(m+p+1)+12\}\mathrm{tr}\,\Omega^2$$

$$-6(4m+3p+3)\omega^2 \mathrm{tr}\,\Omega^2 - 80\,\mathrm{tr}\,\Omega^3 + 9(\mathrm{tr}\,\Omega^2)^2\,],$$

$$b_7(m,p) = 12\{(m+p+1)\omega^2\mathrm{tr}\,\Omega^2 + 4\,\mathrm{tr}\,\Omega^3 - (\mathrm{tr}\,\Omega^2)^2\},$$

$$b_8(m,p) = 3(\mathrm{tr}\,\Omega^2)^2,$$

and $\Omega = \Sigma^{-1}MM'$, $\omega^2 = \mathrm{tr}\,\Omega$.

Let $\lambda_1, \lambda_2, \cdots, \lambda_q$ be the characteristic roots of $\Omega = \Sigma^{-1}MM'$, where q is the rank of MM' and $q \le \min(p, m)$. Since

$$\mathrm{tr}\,\Omega^j = \sum_{\alpha=1}^{q} \lambda_\alpha^j, \qquad j = 1, 2, \cdots \tag{1.2.45}$$

our result (1.2.44) tells us that, at least up to the term of order n^{-2} and for fixed p and m, the power function of the T_0^2-test depends on only the homogeneous symmetric power-sum functions of λ's, which makes the numerical work easy. The numerical comparison between powers of the T_0^2-test and the Λ-test (based on the likelihood ratio criterion) for testing the hypothesis H_0: $M = 0$ against H: $M \ne 0$ is now under investigation.

The following are special cases of (1.2.44) expressed in terms of the cumulative distribution functions,

$$F(x) = \int_0^x f(x)dx, \qquad \chi_\nu^2(x;\omega^2) = \int_0^x g_\nu(u;\omega^2)du.$$

(i) $M = 0$, i.e., when the null hypothesis H_0 is true.

$$F(x) = \chi^2_{mp}(x;0)$$
$$+ \frac{mp}{4n}[(m-p-1)\chi^2_{mp}(x;0) - 2m\chi^2_{mp+2}(x;0) + (m+p+1)\chi^2_{mp+4}(x;0)]$$

$$+\frac{mp}{96n^2}\Big[\{m(3mp-8)(m-2p-2)+m(p+1)(3p^2+3p-4)-4(2p^2+3p-1)\}\chi^2_{mp}(x;0)$$

$$-12m^2p(m-p-1)\chi^2_{mp+2}(x;0)+6m\{3m^2p+8m-(p+1)(p^2+p-4)\}\chi^2_{mp+4}(x;0)$$

$$-4\{m(3mp+16)(m+p+1)+8m(p+1)+4(p^2+3p+4)\}\chi^2_{mp+6}(x;0)$$

$$+3\{m(mp+8)(m+2p+2)+m(p+1)(p^2+p+4)+4(2p^2+5p+5)\}\chi^2_{mp+8}(x;0)\Big]$$

$$+O(\tfrac{1}{n^3}), \qquad (1.2.46)$$

which may be compared with the formula (4.3) *in* [5] *due to* K. Ito.

(ii) $m=1$: *In this case,* T_0^2 *becomes an ordinary* T^2*-statistic whose exact distribution is known (see* e.g. [1]), *and*

$$tr\,\Omega^j = \omega^{2j} = (\mu'\Sigma^{-1}\mu)^j, \quad j=1,2,\cdots$$

and hence $F(x)$ *depends on only* ω^2.

$$F(x)=\chi^2_p(x;\omega^2)$$

$$-\frac{1}{4n}\Big[p^2\chi^2_p(x;\omega^2)+2(p-\omega^2)\chi^2_{p+2}(x;\omega^2)-\{(p-\omega^2)(p+2-\omega^2)-4\omega^2\}\chi^2_{p+4}(x;\omega^2)$$

$$-2\omega^2(p+2-\omega^2)\chi^2_{p+6}(x;\omega^2)-\omega^4\chi^2_{p+8}(x;\omega^2)\Big]$$

$$+\frac{1}{96n^2}\Big[p(3p^3-8p^2+8)\chi^2_p(x;\omega^2)+12p^2(p-\omega^2)\chi^2_{p+2}(x;\omega^2)$$

$$-6\{p(p+2)(p^2-6)-2(p^3+3p^2-6p-12)\omega^2+(p^2-6)\omega^4\}\chi^2_{p+4}(x;\omega^2)$$

$$-4\{7p(p+2)(p+4)+3(p^2-7p-34)\omega^2(p+2-\omega^2)-7\omega^6\}\chi^2_{p+6}(x;\omega^2)$$

$$+3\{p(p+2)(p+4)(p+6)-4(p+2)(p+4)(p+13)\omega^2$$

$$+4(p^2+29p+95)\omega^4-4(p+13)\omega^6+\omega^8\}\chi^2_{p+8}(x;\omega^2)$$

$$+12\omega^2\{(p+2)(p+4)(p+6)-(3p+25)\omega^2(p+4-\omega^2)-\omega^6\}\chi^2_{p+10}(x;\omega^2)$$

$$+2\omega^4\{9(p+4)(p+6)-2(9p+61)\omega^2+9\omega^4\}\chi^2_{p+12}(x;\omega^2)$$

$$+12\omega^6(p+6-\omega^2)\chi^2_{p+14}(x;\omega^2)+3\omega^8\chi^2_{p+16}(x;\omega^2)\Big]$$

$$+O\left(\frac{1}{n^3}\right). \qquad (1.2.47)$$

This can be checked by comparing with the expanded form derived from the exact distribution of T^2.

(iii) $p=1$: *In this case* T_0^2 *becomes a non-central* F *statistic with* m *and* n *degrees of freedom and noncentrality parameter* $\omega^2=\Sigma_{\alpha=1}^{m}\mu_\alpha^2/\sigma^2$ *and we have* $tr\Omega^j=\omega^{2j}$, $j=1,2,\cdots$.

$$F(x)=\chi^2_m(x;\omega^2)$$

$$-\frac{1}{4n}\Big[m(m-2)\chi^2_m(x;\omega^2)-2m(m-\omega^2)\chi^2_{m+2}(x;\omega^2)$$

$$+\{(m-\omega^2)(m+2-\omega^2)-2(m+1)\omega^2\}\chi^2_{m+4}(x;\omega^2)$$

$$+2\omega^2(m+2-\omega^2)\chi^2_{m+6}(x;\omega^2)+\omega^4\chi^2_{m+8}(x;\omega^2)\Big]$$

$$+\frac{1}{96n^2}\Big[m(m-2)(m-4)(3m-2)\chi^2_m(x;\omega^2)-12m^2(m-2)(m-\omega^2)\chi^2_{m+2}(x;\omega^2)$$

$$+6m\{m(m+2)(3m+2)-4(2m^2+3m+2)\omega^2+(3m+2)\omega^4\}\chi^2_{m+4}(x;\omega^2)$$

$$-4\{m(m+2)(m+4)(3m+4)-6(3m^2+9m+8)\omega^2(m+2-\omega^2)-(3m+4)\omega^6\}\chi^2_{m+6}(x;\omega^2)$$

$$+3\{m(m+2)(m+4)(m+6)-8(m+2)(m+4)(2m+5)\omega^2$$

$$+4(9m^2+48m+68)\omega^4-8(2m+5)\omega^6+\omega^8\}\chi^2_{m+8}(x;\omega^2)$$

$$+12\omega^2\{(m+2)(m+4)(m+6)-2(3m+11)\omega^2(m+4-\omega^2)-\omega^6\}\chi^2_{m+10}(x;\omega^2)$$

$$+2\omega^4\{9(m+4)(m+6)-4(6m+29)\omega^2+9\omega^4\}\chi^2_{m+12}(x;\omega^2)$$

$$+12\omega^6(m+6-\omega^2)\chi^2_{m+14}(x;\omega^2)+3\omega^8\chi^2_{m+16}(x;\omega^2)\Big]$$

$$+O\left(\frac{1}{n^3}\right). \tag{1.2.48}$$

<u>Remark</u>. According to the correspondence between (1.2.5) and (1.2.17), if we proceed to expand (1.2.5) in exactly the same way as we did for (1.2.17), then we would have the expansion of $\Pr\{T_0^2 \le t\} = F(t)$ in the same form as (1.2.44) with $\chi^2_\nu(t;\omega^2)$ instead of $g_\nu(x;\omega^2)$.

1.2.2. <u>Percentage points of T_0^2 in the null case.</u> K. Ito [5] and M. Siotani [11, 12] gave independently an asymptotic formula for percentage points of T_0^2 in the null case. Let c_α be the upper 100 α % ($0<\alpha<1$) point of the central chi-square with mp degrees of freedom. Then

$$\Pr\{\operatorname{tr}\Sigma^{-1}ZZ' \le c_\alpha\} = \chi^2_{mp}(c_\alpha;0) = 1-\alpha. \tag{1.2.49}$$

The problem is to find $t_\alpha(n)$ such that for a fixed α,

$$\Pr\{T_0^2 = \operatorname{tr} S_n^{-1}ZZ' \le t_\alpha(n)\} = \chi^2_{mp}(c_\alpha;0) \tag{1.2.50}$$

by assuming that $t_\alpha(n)$ is of the same form as $c(n)$ in (1.1.6), i.e.,

$$t_\alpha(n) = c_\alpha + t_{1\alpha}(n) + t_{2\alpha}(n) + \cdots \tag{1.2.51}$$

where $t_{i\alpha}(n)$, $i=1,2,\cdots$ are of order n^{-i} and may depend on S_n. By expanding the left-hand side of (1.2.50) according to (1.1.8) and then comparing both sides of (1.2.50), we have the following equations to be solved with respect to $t_{i\alpha}(n)$:

$$\{ t_{1\alpha}(n)D + \frac{1}{n}\sum_{rstu}\sigma_{ur}\sigma_{st}\partial_{rs}\partial_{tu} \} \Pr\{ \operatorname{tr}\Sigma^{-1}ZZ' \le c_\alpha \} = 0, \tag{1.2.52}$$

$$\begin{aligned} \{ t_{2\alpha}(n)D &+ \frac{1}{2}t_1^2(n)D^2 - \frac{1}{n}\sum_{rstu}\sigma_{ur}\sigma_{st}\partial_{rs}\partial_{tu}t_{1\alpha}(n)D \\ &+ \frac{1}{n^2}\frac{4}{3}\sum_{rstuvw}\sigma_{wr}\sigma_{st}\sigma_{uv}\partial_{rs}\partial_{tu}\partial_{vw} \\ &+ \frac{1}{n^2}\frac{1}{2}\sum_{rstuvwxy}\sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \} \Pr\{ \operatorname{tr}\Sigma^{-1}ZZ' \le c_\alpha \} = 0, \end{aligned} \tag{1.2.53}$$

and so on.

In the present case $t_{1\alpha}(n)$ can be solved independently of Σ from (1.2.52), which enables us exchange the order of $\partial_{rs}\partial_{tu}$ and $t_{1\alpha}(n)$ in (1.2.53). The final result may be reproduced from [5] or [12]. Let $\chi^2 \equiv \chi^2_{mp}(\alpha) = c_\alpha$, i.e., the upper 100% point of the chi-square distribution with mp degrees of freedom, and

$$\chi_{2j} = \chi^{2j}_{mp}(\alpha)/mp(mp+2)\cdots(mp+2j-2), \quad j=1,2,\cdots \tag{1.2.54}$$

Then

$$\begin{aligned} t_\alpha(n) = \chi^2 &+ \frac{mp}{2n}[(p+1)(\chi_4+\chi_2) + m(\chi_4-\chi_2)] \\ &+ \frac{mp}{48n^2}\Big[3mp\Big(1-\frac{mp-2}{\chi^2}\Big)\{(p+1)(\chi_4+\chi_2)+m(\chi_4-\chi_2)\}^2 \\ &\quad - 6mp\{(p+1)(\chi_4+\chi_2)+m(\chi_4-\chi_2)\}\{(p+1)(\chi_4-1)+m(\chi_4-2\chi_2+1)\} \end{aligned}$$

$$-16\{(p^2+3p+4)(\chi_6+\chi_4+\chi_2)+3m(p+1)(\chi_6-\chi_2)+m^2(\chi_6-2\chi_4+\chi_2)\}$$

$$+3\{4(2p^2+5p+5)(\chi_8+\chi_6+\chi_4+\chi_2)+16m(p+1)(\chi_8-\chi_2)$$

$$+m(p^3+2p^2+5p+4)(\chi_8+\chi_6-\chi_4-\chi_2)$$

$$+2m^2(p^2+p+4)(\chi_8-\chi_6-\chi_4+\chi_2)+m^3p(\chi_8-3\chi_6+3\chi_4-\chi_2)\}]$$

$$+O(\tfrac{1}{n^3}). \qquad (1.2.55)$$

Tables of $t_\alpha(n)$ are given for $\alpha = 0.05,\ 0.01$; $p = 3, 4$; $m = 1\ (1)\ 10\ (2)\ 20$; $n = 10\ (2)\ 30\ (5)\ 50,\ 60,\ 80,\ 100$ in [13].

Remarks on the numerical work for the power of the T_0^2-test. Let $\tau(\alpha)$ be the upper 100α % point of T_0^2 when $H_0: M = 0$ is true, i.e.,

$$Pr\{\,T_0^2 > \tau(\alpha)\,|\,H_0: M = 0\,\} \qquad (1.2.56)$$

Then the power of the T_0^2-test is defined by

$$Pr\{\,T_0^2 > \tau(\alpha)\,|\,H: M \neq 0\,\}. \qquad (1.2.57)$$

When $p = 1, 2$ and when $m = 1$, we have the exact values of $\tau(\alpha)$ sin the exact distributions of T_0^2 under H_0 (for $p = 2$, see Hotelling [3]). Hence we can calculate the approximate values of the power by

$$Pr\{T_0^2 > \tau(\alpha)|H: M\neq 0\} = 1 - Pr\{T_0^2 \leq \tau(\alpha)|H: M\neq 0\}$$

$$= 1 - \{\chi^2_{mp}(\tau(\alpha);\omega^2) + \frac{1}{4n}\sum_{j=0}^{4} a_j(m,p)\,\chi^2_{mp+2j}(\tau(\alpha);\omega^2)$$

$$+ \frac{1}{96n^2} \sum_{j=0}^{8} b_j(m,p)\, \chi^2_{mp+2j}(\tau(\alpha) : \omega^2) \}$$

$$+ O(\tfrac{1}{n^3}), \qquad (1.2.58)$$

where $a_j(m,p)$'s and $b_j(m,p)$'s are same as before.

When $p \geq 3$ and $m > 1$, we have to use $t_\alpha(n)$ of (1.2.55) for $\tau(\alpha)$, which requires us to expand $\Pr\{T_0^2 \leq t_\alpha(n) \mid H : M \neq 0\}$ according to (1.1.8). Since $t_\alpha(n)$ in (1.2.55) does not depend on Σ, the desired expansion can be obtained from (1.2.58) with $t_\alpha(n)$ instead of $\tau(\alpha)$ by using the expansion of $\chi^2_\nu(t_\alpha(n) : \omega^2)$,

$$\chi^2_\nu(t_\alpha(n) ; \omega^2) = [\, 1 + \{t_{1\alpha}(n) + t_{2\alpha}(n) + \cdots\} D + \tfrac{1}{2}\{t_{1\alpha}(n) + \cdots\}^2 D^2 + \cdots \,]\, \chi^2_\nu(\chi^2 ; \omega^2)$$

$$= [\, 1 + t_{1\alpha}(n) D + t_{2\alpha}(n) D + \tfrac{1}{2} t^2_{1\alpha}(n) D^2 + \cdots \,]\, \chi^2_\nu(\chi^2 ; \omega^2), \qquad (1.2.59)$$

where $D = d/d\chi^2$ and $t_{i\alpha}(n)$ stand for the terms of order n^{-i} in (1.2.55).
Since

$$D\, \chi^2_\nu(\chi^2 ; \omega^2) = g_\nu(\chi^2 ; \omega^2),$$

$$D^2 \chi^2_\nu(\chi^2 ; \omega^2) = D\, g_\nu(\chi^2 ; \omega^2) = \tfrac{1}{2}\{ g_{\nu-2}(\chi^2 ; \omega^2) - g_\nu(\chi^2 ; \omega^2)\}.$$

we have

$$\Pr\{T_0^2 > \tau(\alpha) \mid H : M \neq 0\} \approx \Pr\{T_0^2 > t_\alpha(n) \mid H : M \neq 0\}$$

$$= 1 - \Pr\{T_0^2 \leq t_\alpha(n) \mid H : M \neq 0\}$$

$$= 1 - [\, \chi^2_{mp}(\chi^2 ; \omega^2) + \{ t_{1\alpha}(n)\, g_{mp}(\chi^2 ; \omega^2) + \frac{1}{4n} \sum_{j=0} a_j(m,p)\, \chi^2_{mp+2j}(\chi^2 ; \omega^2)\}$$

$$+ \{ t_{2\alpha}(n)\, g_{mp}(\chi^2 ; \omega^2) + \tfrac{1}{4} t^2_{1\alpha}(n) (g_{mp-2}(\chi^2 ; \omega^2) - g_{mp}(\chi^2 ; \omega^2))$$

$$+ t_{1\alpha}(n) \frac{1}{4n} \sum_{j=0}^{4} a_j(m,p)\, g_{mp+2j}(\chi^2 ; \omega^2)$$

$$+ \frac{1}{96n^2} \sum_{j=0}^{8} b_j(m,p) \chi^2_{mp+2j}(\chi^2; \omega^2) \}]$$

$$+ O(\tfrac{1}{n^3}) \cdot \tag{1.2.60}$$

1.3. Other examples.

1.3.1. The ratio of two Hotelling's generalized T_0^2 statistics.

The asymptotic expansion of percentage points of

$$\begin{aligned} T(Z, S_n) &= T((Z_1, Z_2), S_n) \\ &= \frac{tr\, S_n^{-1} Z_1 Z_1'}{tr\, S_n^{-1} Z_2 Z_2'} \equiv F \end{aligned} \tag{1.3.1}$$

can be treated in the similar manner as in §1.2.2, where Z_1 is $p \times m_1$ and Z_2 is $p \times m_2$. In this case $T(Z, \Sigma) = tr\, \Sigma^{-1} Z_1 Z_1' / tr\, \Sigma^{-1} Z_2 Z_2'$ has the distribution function

$$B_{P_1, P_2}(x) = \frac{1}{B(P_1, P_2)} \int_0^x \frac{t^{P_1 - 1}}{(1+t)^{P_1 + P_2}} dt, \tag{1.3.2}$$

where $P_1 = m_1 p/2$, $P_2 = m_2 p/2$ and $B(P_1, P_2)$ is the beta function. Let c_α be the value satisfying

$$B_{P_1, P_2}(c_\alpha) = 1 - \alpha \qquad (0 < \alpha < 1) \cdot \tag{1.3.3}$$

Then $t_\alpha(n) = c_\alpha + t_{1\alpha}(n) + t_{2\alpha}(n) + \cdots$ such that

$$Pr\{ F \leq t_\alpha(n) \} = B_{P_1, P_2}(c_\alpha) \tag{1.3.4}$$

may be given by solving the equations similar to (1.2.52), (1.2.53), etc. The J to be used to evaluate the derivatives of

$$\Pr\left\{ \frac{\operatorname{tr}\Sigma^{-1}Z_1Z_1'}{\operatorname{tr}\Sigma^{-1}Z_2Z_2'} \le c_\alpha \right\}$$

with respect to elements of Σ is

$$J = \Pr\left\{ \frac{\operatorname{tr}(\Sigma+\varepsilon)^{-1}Z_1Z_1'}{\operatorname{tr}(\Sigma+\varepsilon)^{-1}Z_2Z_2'} \le c_\alpha \right\}$$

$$= \left\{ \frac{|I-\xi E_1|}{|I-\xi|} \right\}^{-m_1/2} \left\{ \frac{|I-\xi E_2|}{|I-\xi|} \right\}^{-m_2/2} B_{p_1,p_2}(c_\alpha) \qquad (1.3.5)$$

$$= \{ |I - X\Delta_1| \}^{-m_1/2} \{ |I - X\Delta_2| \}^{-m_2/2} B_{p_1,p_2}(c_\alpha) \qquad (1.3.6)$$

where E_1 and E_2 are operators such that

$$\begin{aligned} E_1^{j} B_{p_1,p_2}(c_\alpha) &= B_{p_1+j,p_2}(c_\alpha) \\ E_2^{j} B_{p_1,p_2}(c_\alpha) &= B_{p_1,p_2+j}(c_\alpha) \end{aligned} \qquad j=0,1,2,\cdots \qquad (1.3.7)$$

and $X = (\Sigma+\varepsilon)^{-1}\Sigma - I$, $\Delta_i = E_i - 1$ $(i=1,2)$. Hence we can carry out the expansion of J in the analogous way as in §1.2.1 (cf. [12]).

1.3.2. Wald-Anderson's classification statistic. Let $x_\alpha^{(i)}$ $(\alpha=1,\cdots,n_i;\ i=1,2)$ be random samples drawn from p-variate normal populations π_i: $N(\mu^{(i)},\Sigma)$, $i=1,2$ and $\bar{x}^{(i)}, (i=1,2)$ be sample means. Let S_n be a Wishart matrix distributed independently of $\bar{x}^{(i)}$ according to the Wishart distribution $W_p(\frac{1}{n}\Sigma, n)$. Suppose we have another random vector x which is to be assigned to either π_1 or π_2. Then Wald-Anderson's classification statistic is defined by

$$W = \{ x - \tfrac{1}{2}(\bar{x}^{(1)} + \bar{x}^{(2)}) \}' S_n^{-1} (\bar{x}^{(1)} - \bar{x}^{(2)}) . \tag{1.3.8}$$

It is known [1, 16] that the limiting distribution of W as n_1, n_2, and n tend to infinity is $N(\frac{1}{2}D^2, D^2)$ or $N(-\frac{1}{2}D^2, D^2)$ according as x belongs to π_1 or π_2, where D^2 is the Mahalanobis distance

$$D^2 = (\mu^{(1)} - \mu^{(2)})' \Sigma^{-1} (\mu^{(1)} - \mu^{(2)}) . \tag{1.3.9}$$

These are the same distributions as those of

$$U = \{ x - \tfrac{1}{2}(\mu^{(1)} + \mu^{(2)}) \}' \Sigma^{-1} (\mu^{(1)} - \mu^{(2)}) . \tag{1.3.10}$$

From this and the form of W, it is to be expected that our method can be applied effectively.

M. Okamoto [8] gave the asymptotic expansion of $Pr\{W < \frac{1}{2}D^2 + uD \mid \pi_1\}$ with respect to n_1^{-1}, n_2^{-1}, and n^{-1} by evaluating the asymptotic expansion of the characteristic function of

$$U = \frac{W - D^2/2}{D}$$

and by inverting the resultant to the corresponding expression of the cumulative distribution function.

Let us denote the characteristic function of $(W - \frac{1}{2}D^2)/D$ given $\bar{x}^{(1)}$, $\bar{x}^{(2)}$, and S_n by $\psi(t; \bar{x}^{(1)}, \bar{x}^{(2)}, S_n)$. Then we have

$$\psi(t) = E\{ \exp[it(W - \tfrac{1}{2}D^2)/D] \mid \pi_1 \}$$

$$= E_{\bar{x}^{(1)}, \bar{x}^{(2)}, S_n}\{ \psi(t : \bar{x}^{(1)}, \bar{x}^{(2)}, S_n)\}$$

$$= E_{\bar{x}^{(1)}, \bar{x}^{(2)}, S_n}\{ \exp[(\bar{x}^{(1)} - \mu^{(1)})' \partial^{(1)} + (\bar{x}^{(2)} - \mu^{(2)})' \partial^{(2)}$$

$$+ \mathrm{tr}(S_n - \Sigma)\partial]\} \psi(t : \mu^{(1)}, \mu^{(2)}, \Sigma)$$

$$= \exp\{ \frac{1}{2n_1} \partial^{(1)\prime} \Sigma \partial^{(1)} + \frac{1}{2n_2} \partial^{(2)\prime} \Sigma \partial^{(2)}$$

$$- \mathrm{tr}\,\Sigma\partial - \frac{n}{2} \log | I - \frac{2}{n}\Sigma\partial | \} \psi(t : \mu^{(1)}, \mu^{(2)}, \Sigma), \quad (1.3.11)$$

where $\partial^{(\alpha)\prime} = (\partial/\partial\mu_1^{(\alpha)}, \partial/\partial\mu_2^{(\alpha)}, \ldots, \partial/\partial\mu_p^{(\alpha)})$, $\alpha = 1, 2$ and $\partial = (\frac{1}{2}(1+\delta_{rs})\partial/\partial\sigma_{rs})$ as before. Now it is clear that we may proceed with the evaluation in the same way as in §1.1, but the calculation is now not so difficult as in the case of the non-central T_0^2.

1.3.3. The maximum of generalized distances. Let z_i, $i = 1, \cdots, m$ be m p-component normal vectors whose joint distribution is the mp-variate normal with mean 0 and covariance matrix

$$\begin{bmatrix} \Sigma & \kappa_{12}\Sigma & \cdots & \kappa_{1m}\Sigma \\ \kappa_{21}\Sigma & \Sigma & \cdots & \kappa_{2m}\Sigma \\ \cdot & \cdot & \cdots & \cdot \\ \kappa_{m1}\Sigma & \kappa_{m2}\Sigma & \cdots & \Sigma \end{bmatrix} = \begin{bmatrix} 1 & \kappa_{12} & \cdots & \kappa_{1m} \\ \kappa_{21} & 1 & \cdots & \kappa_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \kappa_{m1} & \kappa_{m2} & \cdots & 1 \end{bmatrix} \otimes \Sigma \qquad (1.3.12)$$

where $\Sigma\,(: p\times p)$ is positive definite, $\kappa_{ij} = \kappa_{ji}$, $|\kappa_{ij}| < 1$ and symbol $\otimes$ means the direct product. Suppose we have a $p\times p$ matrix S_n distributed independently of z_i's according to the Wishart distribution $W_p(\frac{1}{n}\Sigma, n)$. Then the maximum of m generalized distances from the origin is defined by

$$T^2_{max} = \max_i \{ T^2_i \} = \max_i \{ z'_i S_n^{-1} z_i \} \cdot \tag{1.3.13}$$

As far as the present author knows, there is no elegant method by which we can obtain the sampling distribution of T^2_{max}. But we have the well known inclusion-exclusion formula

$$Pr\{ T^2_{max} > t^2 \} = \sum_{i=1}^{m} Pr\{ T^2_i > t^2 \} - \sum_{i<j} Pr\{ T_i^2 > t^2, T_j^2 > t^2 \} + \cdots \tag{1.3.14}$$

If we have a sufficientlylarge value of t^2, the first two terms of this formula may give us a good approximation to $Pr\{ T^2_{max} > t^2 \}$. Even in this case, the evaluation of the second term is quite difficult, but an approximation may be given by the asymptotic expansion of it obtained by our method.

In this connection let us consider an asymptotic expansion of

$$Pr\{ T^2_i > c_1^2 , T^2_j > c_2^2 \}$$

for fixed c_1^2 and c_2^2. First of all it is seen that we can write

$$\begin{aligned} Pr\{ T_i^2 > c_1^2, T_j^2 > c_2^2 \} &= Pr\{ z'_i S_n^{-1} z_i > c_1^2 , z'_j S_n^{-1} z_j > c_2^2 \} \\ &= \Theta \cdot Pr\{ z'_i \Sigma^{-1} z_i > c_1^2, z'_j \Sigma^{-1} z_j > c_2^2 \}, \end{aligned} \tag{1.3.15}$$

where Θ is the same differential operator as (1.1.3) or (1.1.5).

Next we have to calculate

$$\begin{aligned} J_{ij} &= Pr\{ z'_i (\Sigma + \varepsilon)^{-1} z_i > c_1^2 , z'_j (\Sigma + \varepsilon)^{-1} z_j > c_2^2 \} \\ &= (2\pi)^{-p} |\Sigma|^{-1} (1 - k_{ij}^2)^{-p/2} \times \end{aligned}$$

$$\times \int_{D^*}\!\!\cdots\!\int \exp\left[-\frac{1}{2(1-\kappa_{ij}^2)}\{z_i'\Sigma^{-1}z_i - 2\kappa_{ij} z_i'\Sigma^{-1}z_j + z_j'\Sigma^{-1}z_j\}\right] dz_i\,dz_j \tag{1.3.16}$$

where

$$D^*: \quad z_i'(\Sigma+\mathcal{E})^{-1}z_i > c_1^2\,, \qquad z_j'(\Sigma+\mathcal{E})^{-1}z_j > c_2^2\,. \tag{1.3.17}$$

Let us make the linear transformations,

$$z_i = (1-\kappa_{ij}^2)^{1/2} L\,u\,, \qquad z_j = (1-\kappa_{ij}^2)^{1/2} L\,v \tag{1.3.18}$$

with L *such that*

$$L'(\Sigma+\mathcal{E})^{-1}L = I\,, \qquad L'\Sigma^{-1}L = I - \xi \tag{1.3.19}$$

where $\xi = \mathrm{diag}\{\xi_1,\cdots,\xi_p\}$ *and* $|\xi_\alpha| < 1$ *for all* $\alpha = 1,\dots,p$. *Then* J_{ij} *becomes*

$$J_{ij} = (2\pi)^{-p}(1-\kappa_{ij}^2)^{p/2}|I-\xi| \times$$

$$\times \int_{D}\!\!\cdots\!\int \exp\left[-\tfrac{1}{2}\{u'(I-\xi)u - 2\kappa_{ij}u'(I-\xi)v + v'(I-\xi)\}\right] du\,dv \tag{1.3.20}$$

where

$$D: \quad u'u > u_0\,, \qquad v'v > v_0 \tag{1.3.21}$$

and $u_0 = c_1^2/(1-\kappa_{ij}^2)$, $v_0 = c_2^2/(1-\kappa_{ij}^2)$. *Using integral formulas of Dirichlet's type in the same way as in* (1.2.4) (*also see* [14]), *we have*

$$J_{ij} = (1-\kappa_{ij}^2)^{p/2}\left|(I-\xi)^{-1}(I-\xi E_1)(I-\xi)^{-1}(I-\xi E_2) - \kappa_{ij}^2 E_1E_2 I\right|^{-1/2} \bar{\chi}_p^2(u_0;0)\,\bar{\chi}_p^2(v_0;0) \tag{1.3.22}$$

where

$$\bar{\chi}_p^2(x;0) = \frac{1}{2^{p/2}\Gamma(p/2)}\int_x^\infty t^{\frac{p}{2}-1}e^{-\frac{t}{2}}dt \qquad (1.3.23)$$

and E_1, E_2 are operators such that

$$E_1^r\,\bar{\chi}_p^2(u_0;0) = \bar{\chi}_{p+2r}^2(u_0;0)\,,\quad E_2^r\,\bar{\chi}_p^2(v_0;0) = \bar{\chi}_{p+2r}^2(v_0;0) \qquad (1.3.24)$$

for $r = 0, 1, 2, \cdots$. If we put $\Delta_\alpha = E_\alpha - 1$ $(\alpha = 1, 2)$, $H = (1 - \kappa_{ij}^2 E_1 E_2)^{-1}$ and $X = (\Sigma + \varepsilon)^{-1}\Sigma - I$, then we can write (1.3.22) as

$$J_{ij} = (1-\kappa_{ij}^2)^{p/2}\left|I - \{X(\Delta_1+\Delta_2) - X^2\Delta_1\Delta_2\}H\right|^{-1/2} H^{p/2}\,\bar{\chi}_p^2(u_0;0)\,\bar{\chi}_p^2(v_0;0). \qquad (1.3.25)$$

So if we expand J_{ij} in power series of ε_{rs}'s in the similar manner as in § 1.2.1, we could obtain the desired derivatives of $\Pr\{\mathbf{z}_i'\Sigma^{-1}\mathbf{z}_i > c_1^2,\ \mathbf{z}_j'\Sigma^{-1}\mathbf{z}_j > c_2^2\}$ and hence an asymptotic expansion of $\Pr\{T_i^2 > c_1^2, T_j^2 > c_2^2\}$, which gives us an asymptotic expansion of the joint distribution of two Hotelling's T^2-statistics. If in particular, $\kappa_{ij} = 0$, we have the simple formula

$$J_{ij} = \{|I - X\Delta_1|^{-1/2}\,\bar{\chi}_p^2(u_0;0)\}\{|I - X\Delta_2|^{-1/2}\,\bar{\chi}_p^2(v_0;0)\}. \qquad (1.3.26)$$

Appendix 1

Tables for $B^{(2)}_{rs,tu}(t)$, $B^{(3)}_{rs,tu,vw}(t)$ and $B^{(4)}_{rs,tu,vw,xy}(t)$ in § 1.2.1.

(I) in $B^{(2)}_{rs,tu}(t)$:

$$\frac{1}{2!}\sum_{(P_2)}[rs|tu] = [rs|tu], \qquad \frac{1}{2!}\sum_{(P_2)}[rs][tu] = [rs][tu],$$

$$\frac{1}{2!}\sum_{(P_2)}(rs|tu) = (rs|tu), \qquad \frac{1}{2!}\sum_{(P_2)}(rs)(tu) = (rs)(tu),$$

$$\frac{1}{2!}\sum_{(P_2)}[rs](tu) = \frac{1}{2}\{[rs](tu) + [tu](rs)\}.$$

(II) in $B^{(3)}_{rs,tu,vw}(t)$:

$$\frac{1}{3!}\sum_{(P_3)}[rs|tu|vw] = [rs|tu|vw], \qquad \frac{1}{3!}\sum_{(P_3)}[rs][tu][vw] = [rs][tu][vw],$$

$$\frac{1}{3!}\sum_{(P_3)}[rs][tu|vw] = \frac{1}{3}\{[rs][tu|vw] + [tu][rs|vw] + [vw][rs|tu]\},$$

$$\frac{1}{3!}\sum_{(P_3)}(rs|tu|vw) = \frac{1}{3}\{(rs|tu|vw) + (rs|vw|tu) + (tu|rs|vw)\},$$

$$\frac{1}{3!}\sum_{(P_3)}(rs)(tu|vw) = \frac{1}{3}\{(rs)(tu|vw) + (tu)(rs|vw) + (vw)(rs|tu)\},$$

$$\frac{1}{3!}\sum_{(P_3)}(rs)(tu)(vw) = (rs)(tu)(vw),$$

$$\frac{1}{3!}\sum_{(P_3)}[rs|tu](vw) = \frac{1}{3}\{[rs|tu](vw) + [rs|vw](tu) + [tu|vw](rs)\},$$

$$\frac{1}{3!}\sum_{(P_3)}[rs](tu|vw) = \frac{1}{3}\{[rs](tu|vw) + [tu](rs|vw) + [vw](rs|tu)\},$$

$$\frac{1}{3!}\sum_{(P_3)}[rs][tu](vw) = \frac{1}{3}\{[rs][tu](vw) + [rs][vw](tu) + [tu][vw](rs)\},$$

$$\frac{1}{3!}\sum_{(P_3)}[rs](tu)(vw) = \frac{1}{3}\{[rs](tu)(vw) + [tu](rs)(vw) + [vw](rs)(tu)\}.$$

(III) in $B^{(4)}_{rs,tu,vw,xy}(t)$:

$$\frac{1}{4!}\sum_{(P_4)}[rs|tu|vw|xy] = \frac{1}{3}\{[rs|tu|vw|xy] + [rs|xy|tu|vw] + [rs|vw|xy|tu]\},$$

$$\frac{1}{4!}\sum_{(P_4)}[rs][tu|vw|xy] = \frac{1}{4}\{[rs][tu|vw|xy] + [tu][rs|vw|xy] + [vw][rs|tu|xy] + [xy][rs|tu|vw]\},$$

$$\frac{1}{4!}\sum_{(P_4)}[rs|tu][vw|xy] = \frac{1}{3}\{[rs|tu][vw|xy] + [rs|vw][tu|xy] + [rs|xy][tu|vw]\},$$

$$\frac{1}{4!}\sum_{(P_4)}[rs][tu][vw|xy] = \frac{1}{6}\{[rs][tu][vw|xy] + [rs][vw][tu|xy] + [rs][xy][tu|vw] + [tu][vw][rs|xy] + [tu][xy][rs|vw] + [vw][xy][rs|tu]\},$$

$$\frac{1}{4!}\sum_{(P_4)}[rs][tu][vw][xy] = [rs][tu][vw][xy],$$

$$\frac{1}{4!}\sum_{(P_4)}(rs|tu|vw|xy) = \frac{1}{12}\{(rs|tu|vw|xy) + (rs|tu|xy|vw) + (rs|vw|tu|xy) + (rs|vw|xy|tu) + (rs|xy|tu|vw) + (rs|xy|vw|tu) + (tu|rs|vw|xy) + (tu|rs|xy|vw) + (tu|vw|rs|xy) + (tu|xy|rs|vw) + (vw|rs|tu|xy) + (vw|tu|rs|xy)\},$$

$$\frac{1}{4!}\sum_{(P_4)}(rs)(tu|vw|xy) = \frac{1}{12}\{(rs)(tu|vw|xy) + (rs)(vw|tu|xy) + (rs)(tu|xy|vw) + (tu)(rs|vw|xy) + (tu)(rs|xy|vw) + (tu)(vw|rs|xy) + (vw)(rs|tu|xy) + (vw)(tu|rs|xy) + (vw)(rs|xy|tu) + (xy)(rs|tu|vw) + (xy)(rs|vw|tu) + (xy)(tu|rs|vw)\},$$

$$\frac{1}{4!}\sum_{(P_4)}(rs|tu)(vw|xy) = \frac{1}{3}\{(rs|tu)(vw|xy) + (rs|vw)(tu|xy) + (rs|xy)(tu|vw)\},$$

$$\frac{1}{4!}\sum_{(P_4)}(rs)(tu)(vw|xy) = \frac{1}{6}\{(rs)(tu)(vw|xy) + (rs)(vw)(tu|xy) + (rs)(xy)(tu|vw) + (tu)(xy)(rs|vw) + (tu)(vw)(rs|xy) + (vw)(xy)(rs|tu)\},$$

$$\frac{1}{4!}\sum_{(P_4)}(rs)(tu)(vw)(xy) = (rs)(tu)(vw)(xy),$$

$$\frac{1}{4!}\sum_{(P_4)}[rs|tu|vw](xy) = \frac{1}{4}\{[rs|tu|vw](xy)+[rs|tu|xy](vw)+[rs|vw|xy](tu)$$
$$+[tu|vw|xy](rs)\},$$

$$\frac{1}{4!}\sum_{(P_4)}[rs|tu](vw|xy) = \frac{1}{6}\{[rs|tu](vw|xy)+[rs|vw](tu|xy)+[rs|xy](tu|vw)$$
$$+[tu|vw](rs|xy)+[tu|xy](rs|vw)+[vw|xy](rs|tu)\}$$

$$\frac{1}{4!}\sum_{(P_4)}[rs](tu|vw|xy) = \frac{1}{12}\{[rs](tu|vw|xy)+[rs](tu|xy|vw)+[rs](vw|tu|xy)$$
$$+[tu](rs|vw|xy)+[tu](rs|xy|vw)+[tu](vw|rs|xy)$$
$$+[vw](rs|tu|xy)+[vw](rs|xy|tu)+[vw](tu|rs|xy)$$
$$+[xy](rs|tu|vw)+[xy](rs|vw|tu)+[xy](tu|rs|vw)\},$$

$$\frac{1}{4!}\sum_{(P_4)}[rs][tu|vw](xy) = \frac{1}{12}\{[rs][tu|vw](xy)+[tu][rs|vw](xy)+[vw][rs|tu](xy)$$
$$+[rs][tu|xy](vw)+[tu][rs|xy](vw)+[xy][rs|tu](vw)$$
$$+[rs][vw|xy](tu)+[vw][rs|xy](tu)+[xy][rs|vw](tu)$$
$$+[tu][vw|xy](rs)+[vw][tu|xy](rs)+[xy][tu|vw](rs)\},$$

$$\frac{1}{4!}\sum_{(P_4)}[rs|tu](vw)(xy) = \frac{1}{6}\{[rs|tu](vw)(xy)+[rs|vw](tu)(xy)+[rs|xy](tu)(vw)$$
$$+[tu|vw](rs)(xy)+[tu|xy](rs)(vw)+[vw|xy](rs)(tu)\},$$

$$\frac{1}{4!}\sum_{(P_4)}[rs][tu](vw|xy) = \frac{1}{6}\{[rs][tu](vw|xy)+[rs][vw](tu|xy)+[rs][xy](tu|vw)$$
$$+[tu][vw](rs|xy)+[tu][xy](rs|vw)+[vw][xy](rs|tu)\},$$

$$\frac{1}{4!}\sum_{(P_4)}[rs](tu)(vw|xy) = \frac{1}{12}\{[rs](tu)(vw|xy)+[rs](vw)(tu|xy)+[rs](xy)(tu|vw)$$
$$+[tu](rs)(vw|xy)+[tu](vw)(rs|xy)+[tu](xy)(rs|vw)$$
$$+[vw](rs)(tu|xy)+[vw](tu)(rs|xy)+[vw](xy)(rs|tu)$$

$$+ [xy](rs)(tu|vw) + [xy](tu)(rs|vw) + [xy](vw)(rs|tu)\},$$

$$\frac{1}{4!}\sum_{(P_4)} [rs][tu][vw](xy) = \frac{1}{4}\{[rs][tu][vw](xy) + [rs][tu][xy](vw)$$

$$+ [rs][vw][xy](tu) + [tu][vw][xy](rs)\},$$

$$\frac{1}{4!}\sum_{(P_4)} [rs][tu](vw)(xy) = \frac{1}{6}\{[rs][vw](tu)(xy) + [rs][tu](vw)(xy) + [rs][xy](tu)(vw)$$

$$+ [tu][vw](rs)(xy) + [tu][xy](rs)(vw) + [vw][xy](rs)(tu)\},$$

$$\frac{1}{4!}\sum_{(P_4)} [rs](tu)(vw)(xy) = \frac{1}{4}\{[rs](tu)(vw)(xy) + [tu](rs)(vw)(xy) + [vw](rs)(tu)(xy)$$

$$+ [xy](rs)(tu)(vw)\}$$

Appendix 2

Tables of values of the summations of trace functions in §1.2.1.

(The symbol Σ below stands for the summation with respect to suffixes appeared in the summand, each of which runs over $(1, 2, \cdots, p)$ independently.)

(A):

$$\sum \sigma_{ur}\sigma_{st}[rs][tu] = p, \qquad \sum \sigma_{ur}\sigma_{st}[rs|tu] = \tfrac{1}{2}p(p+1),$$

$$\sum \sigma_{wr}\sigma_{st}\sigma_{uv}[rs|tu|vw] = \tfrac{1}{8}p(p^2+3p+4), \quad \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[rs][tu][vw] = p.$$

$$\sum \sigma_{wr}\sigma_{st}\sigma_{uv}[rs][tu|vw] = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[tu][rs|vw] = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[vw][rs|tu]$$
$$= \tfrac{1}{2}p(p+1),$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|tu|vw|xy] = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|vw|xy|tu]$$
$$= \tfrac{1}{4}p(p^2+2p+1).$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|xy|tu|vw] = \tfrac{1}{4}p(p+3),$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu|vw|xy] = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][rs|vw|xy]$$
$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw][rs|tu|xy] = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[xy][rs|tu|vw]$$
$$= \tfrac{1}{2}p(p+1),$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|tu][vw|xy] = \tfrac{1}{4}p^2(p+1)^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|vw][tu|xy] = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|xy][tu|vw]$$
$$= \tfrac{1}{2}p(p+1),$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu][vw|xy] = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw][xy][rs|tu]$$
$$= \tfrac{1}{2}p^2(p+1).$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][vw][tu|xy] = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][xy][tu|vw]$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][vw][rs|xy] = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][xy][rs|vw]$$

$$= p \quad ,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu][vw][xy] = p^2 .$$

(B):

$$\sum \sigma_{rs}(rs) = tr\,\Omega = \omega^2 , \qquad \sum \sigma_{ur}\sigma_{st}(rs)(tu) = tr\,\Omega^2$$

$$\sum \sigma_{wr}\sigma_{st}\sigma_{uv}(rs)(tu)(vw) = tr\,\Omega^3 , \quad \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs)(tu)(vw)(xy) = (tr\,\Omega^2)^2$$

$$\sum \sigma_{ur}\sigma_{st}(rs|tu) = \frac{1}{2}(p+1)\,\omega^2 ,$$

$$\sum \sigma_{wr}\sigma_{st}\sigma_{uv}(rs|tu|vw) = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}(rs|vw|tu) = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}(tu|rs|vw)$$

$$= \frac{1}{8}(p^2+3p+4)\,\omega^2 \quad ,$$

$$\sum \sigma_{wr}\sigma_{st}\sigma_{uv}(rs)(tu|vw) = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}(tu)(rs|vw) = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}(vw)(rs|tu)$$

$$= \frac{1}{4}\omega^4 + \frac{1}{4}(p+2)\,tr\,\Omega^2 \quad ,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs|tu|vw|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs|tu|xy|vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs|vw|xy|tu) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs|xy|vw|tu)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(tu|rs|vw|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(tu|rs|xy|vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(vw|rs|tu|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(vw|tu|rs|xy)$$

$$= \frac{1}{4}(p+1)^2\omega^2 ,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs|vw|tu|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs|xy|tu|vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(tu|vw|rs|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(tu|xy|rs|vw)$$

$$= \tfrac{1}{4}(p+3)\,\omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs)(tu|vw|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs)(tu|xy|vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(tu)(rs|vw|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(tu)(rs|xy|vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(vw)(rs|tu|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(vw)(tu|rs|xy)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(xy)(rs|tu|vw) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(xy)(tu|rs|vw)$$

$$= \tfrac{1}{2}(p+1)\,\mathrm{tr}\,\Omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs)(vw|tu|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(tu)(vw|rs|xy)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(vw)(rs|xy|tu) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(xy)(rs|vw|tu)$$

$$= \tfrac{1}{2}\omega^4 + \tfrac{1}{2}\mathrm{tr}\,\Omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs|tu)(vw|xy) = \tfrac{1}{4}(p+1)^2\omega^4,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs|vw)(tu|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs|xy)(tu|vw)$$

$$= \tfrac{1}{4}\omega^4 + \tfrac{1}{4}(p+2)\,\mathrm{tr}\,\Omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs)(tu)(vw|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(vw)(xy)(rs|tu)$$

$$= \tfrac{1}{2}(p+1)\,\omega^2\,\mathrm{tr}\,\Omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs)(vw)(tu|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(rs)(xy)(tu|vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(tu)(vw)(rs|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}(tu)(xy)(rs|vw)$$

$$= \mathrm{tr}\,\Omega^3.$$

(C):

$$\sum \sigma_{ur}\sigma_{st}[rs](tu) = \sum \sigma_{ur}\sigma_{st}[tu](rs) = \omega^2 ,$$

$$\sum \sigma_{wr}\sigma_{st}\sigma_{uv}[rs|tu](vw) = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[rs|vw](tu)$$

$$= \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[tu|vw](rs) = \tfrac{1}{2}(p+1)\omega^2 ,$$

$$\sum \sigma_{wr}\sigma_{st}\sigma_{uv}[rs](tu|vw) = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[tu](rs|vw)$$

$$= \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[vw](rs|tu) = \tfrac{1}{2}(p+1)\omega^2 ,$$

$$\sum \sigma_{wr}\sigma_{st}\sigma_{uv}[rs][tu](vw) = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[rs][vw](tu)$$

$$= \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[tu][vw](rs) = \omega^2 ,$$

$$\sum \sigma_{wr}\sigma_{st}\sigma_{uv}[rs](tu)(vw) = \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[tu](rs)(vw)$$

$$= \sum \sigma_{wr}\sigma_{st}\sigma_{uv}[vw](rs)(tu) = tr\,\Omega^2 ,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|tu|vw](xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|tu|xy](vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|vw|xy](tu) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu|vw|xy](rs) = \tfrac{1}{2}(p+1)\omega^2$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|tu](vw|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw|xy](rs|tu)$$

$$= \tfrac{1}{4}p(p+1)^2\omega^2 ,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|vw](tu|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|xy](tu|vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu|vw](rs|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu|xy](rs|vw)$$

$$= \tfrac{1}{2}(p+1)\omega^2 ,$$

The 12 summations of the type $\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs](tu|vw|xy)$ are all equal to $\frac{1}{2}(p+1)\omega^2$.

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][vw|xy](tu) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][vw|xy](rs)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw][rs|tu](xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[xy][rs|tu](vw)$$

$$= \tfrac{1}{2}p(p+1)\omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu|vw](xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][rs|vw](xy)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu|xy](vw) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][rs|xy](vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw][rs|xy](tu) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[xy][rs|vw](tu)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw][tu|xy](rs) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[xy][tu|vw](rs)$$

$$= \omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|tu](vw)(xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw|xy](rs)(tu)$$

$$= \tfrac{1}{2}p(p+1)\,\mathrm{tr}\,\Omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|vw](tu)(xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs|xy](tu)(vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu|vw](rs)(xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu|xy](rs)(vw)$$

$$= \mathrm{tr}\,\Omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu](vw|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw][xy](rs|tu)$$

$$= \tfrac{1}{2}p(p+1)\omega^2,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][vw](tu|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][xy](tu|vw)$$

$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][vw](rs|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][xy](rs|vw)$$

$$= \mathrm{tr}\,\Omega = \omega^2$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu](vw)(xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw][xy](rs)(tu)$$
$$= p\,\mathrm{tr}\,\Omega^2 \,,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][vw](tu)(xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][xy](tu)(vw)$$
$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][vw](rs)(xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][xy](rs)(vw)$$
$$= \omega^4$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs](tu)(vw|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu](rs)(vw|xy)$$
$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw](xy)(rs|tu) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[xy](vw)(rs|tu)$$
$$= \tfrac{1}{2}(p+1)\,\omega^4 \,,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs](vw)(tu|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs](xy)(tu|vw)$$
$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu](vw)(rs|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu](xy)(rs|vw)$$
$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw](rs)(tu|vw) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[xy](rs)(tu|vw)$$
$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw](tu)(rs|xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[xy](tu)(rs|vw)$$
$$= \mathrm{tr}\,\Omega^2 \,,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu][vw](xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][tu][xy](vw)$$
$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs][vw][xy](tu) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu][vw][xy](rs)$$
$$= p\,\omega^2 \,,$$

$$\sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[rs](tu)(vw)(xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[tu](rs)(vw)(xy)$$
$$= \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[vw](rs)(tu)(xy) = \sum \sigma_{ur}\sigma_{st}\sigma_{yv}\sigma_{wx}[xy](rs)(tu)(vw)$$
$$= \omega^2\,\mathrm{tr}\,\Omega^2 \,,$$

Part II

The evaluation of asymptotic variances and covariances of functions of Wishart matrices

In multivariate analysis, many test statistics come out as functions of Wishart matrices and the delta method can be applied to obtain the normal distribution as their non-null asymptotic distribution.

Suppose we have a p-variate normal population with covariance matrix $\Sigma=(\sigma_{ij})$. Let $S=(s_{ij})$ be a sample covariance matrix with n degrees of freedom and hence nS is a Wishart matrix subject to $W_p(\Sigma, n)$. Then it is well known [1, p.75] that S converges in probability to Σ as n tends to infinity, and the distinct elements of S are asymptotically distributed according to $p(p+1)/2$-dimensional normal distribution with mean Σ and covariances

$$\mathrm{Cov}(s_{ij}, s_{k\ell}) = \frac{1}{n}(\sigma_{ik}\sigma_{j\ell}+\sigma_{i\ell}\sigma_{jk}). \tag{2.0.1}$$

Let us now consider m scalar valued functions $f_1(S), \cdots, f_m(S)$ of S, which are not algebraically dependent and have first and second derivatives in a neighborhood of Σ. Then by a slight generalization of Theorem 4.2.5 in Anderson's book [1], the limiting distribution of

$$\sqrt{n}\{(f_1(S), \cdots, f_m(S)) - (f_1(\Sigma), \cdots, f_m(\Sigma))\} \tag{2.0.2}$$

is normal with mean $(0, \cdots, 0)$. The evaluation of variances and covariances in this limiting distribution is not necessarily easy. In § 2.1 we give formulas which reduces the labor of computation considerably and in the later sections two concrete examples are treated.

2.1. Formulas.

Let us use the notation $\partial_{rs} = \frac{1}{2}(1+\delta_{rs})\frac{\partial}{\partial\sigma_{rs}}$, where δ_{rs} is the Kronecker symbol.

Formula I. Let $f(S)$ be a function of S having the first and second derivatives in a neighborhood of Σ. Then the (asymptotic) variance of $f(S)$ in its asymptotic distribution is given by

$$Var(f(S)) = \frac{2}{n}\{f(\Sigma)\}^2 tr(\Phi\Sigma)^2 \tag{2.1.1}$$

where $\Phi = (\phi_{ij})$, $\phi_{ij} = \partial_{ij}\log f(\Sigma)$.

Proof. By the usual method we have

$$\begin{aligned}
Var(f(S)) &= E\{\sum_{i\le j}^{p} \frac{\partial f(S)}{\partial s_{ij}}\Big|_{S=\Sigma}(s_{ij}-\sigma_{ij})\}^2 \\
&= E\{\sum_{i,j=1}^{p}\frac{1}{2}(1+\delta_{ij})\frac{\partial f(S)}{\partial s_{ij}}\Big|_{S=\Sigma}(s_{ij}-\sigma_{ij})\}^2 \\
&= \sum_{i,j,k,l=1}^{p}\{\partial_{ij}f(\Sigma)\}\{\partial_{kl}f(\Sigma)\}E(s_{ij}-\sigma_{ij})(s_{kl}-\sigma_{kl}) \\
&= \{f(\Sigma)\}^2\sum_{i,j,k,l=1}^{p}\phi_{ij}\phi_{kl}\frac{1}{n}(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk}) \\
&= \frac{2}{n}\{f(\Sigma)\}^2\sum_{i,j,k,l=1}^{p}\phi_{ij}\phi_{kl}\sigma_{ik}\sigma_{jl} \\
&= \frac{2}{n}\{f(\Sigma)\}^2\sum_i\sum_l(\sum_j\phi_{ij}\sigma_{jl})(\sum_k\phi_{lk}\sigma_{ik}) \\
&= \frac{2}{n}\{f(\Sigma)\}^2 tr(\Phi\Sigma)^2.
\end{aligned}$$

Formula II. Let $f(S)$ and $g(S)$ be two functions of S as in Formula I, but not algebraically dependent. Then their asymptotic covariance is given by

$$Cov(f(S), g(S)) = \frac{2}{n}f(\Sigma)g(\Sigma)tr(\Phi\Sigma\Psi\Sigma) \tag{2.1.2}$$

where Φ is as before and $\Psi = (\psi_{ij})$, $\psi_{ij} = \partial_{ij} \log g(\Sigma)$.

As an illustrative example, let us consider the determinant of correlation matrix $R = (r_{ij})$, i.e., $|R|$, which can be expressed as $|S|/|D_s|$, where D_s is the diagonal matrix with diagonal elements, $s_{11}, s_{22}, \cdots, s_{pp}$. Hence the asymptotic variance of $|R|$ can be calculated by (2.1.1) as follows:

$$Var(|R|) = \frac{2}{n}|P|^2 tr(\Phi\Sigma)^2,$$

where $|P|$ is the determinant of the population correlation matrix $P = D_{\frac{1}{\sqrt{\sigma}}} \Sigma D_{\frac{1}{\sqrt{\sigma}}}$.
Since

$$\phi_{ii} = \partial_{ii} \log|\Sigma| - \partial_{ii} \log|D_\sigma| = (\Sigma^{-1})_{ii} - \frac{1}{\sigma_{ii}},$$

$$\phi_{ij} = (\Sigma^{-1})_{ij}, \quad (i \neq j),$$

where $(\Sigma^{-1})_{ij}$ stands for the (i,j) element of Σ^{-1}, we have

$$\Phi = \Sigma^{-1} - D_{\frac{1}{\sigma}}$$

and hence

$$\begin{aligned} tr(\Phi\Sigma)^2 = tr(I - D_{\frac{1}{\sigma}}\Sigma)^2 &= tr(I - 2D_{\frac{1}{\sigma}}\Sigma + D_{\frac{1}{\sigma}}\Sigma D_{\frac{1}{\sigma}}\Sigma) \\ &= p - 2p + tr(D_{\frac{1}{\sqrt{\sigma}}}\Sigma D_{\frac{1}{\sqrt{\sigma}}})(D_{\frac{1}{\sqrt{\sigma}}}\Sigma D_{\frac{1}{\sqrt{\sigma}}}) \\ &= tr P^2 - p. \end{aligned}$$

Therefore

$$Var(|R|) = \frac{2}{n}|P|^2(tr P^2 - p). \tag{2.1.3}$$

2.2. Previous results.

I. Olkin and M. Siotani [9] gave several asymptotic distributions of functions of correlation matrix. The evaluation was based on the asymptotic joint distribution of distinct elements of sample correlation matrix R. They recognized soon after that some of those functions can be expressed as functions of S, and their asymptotic variances and covariances can be evaluated more easily by using the formulas (2.1.1) and (2.1.2) repeatedly.

In this section, we list these results as well as other results obtained by this method so far. Most of them give us asymptotic power functions of various tests in the multivariate analysis.

We use the following notations :

$$\Sigma = \begin{array}{c} \left[\begin{matrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1q} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2q} \\ \cdot & \cdot & \cdots & \cdot \\ \Sigma_{q1} & \Sigma_{q2} & \cdots & \Sigma_{qq} \end{matrix}\right] \begin{matrix} p_1 \\ p_2 \\ \cdot \\ p_q \end{matrix} \\ \begin{matrix} p_1 & p_2 & \cdots & p_q \end{matrix} \end{array} \qquad D_\Sigma = \begin{bmatrix} \Sigma_{11} & 0 & \cdots & 0 \\ 0 & \Sigma_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \Sigma_{qq} \end{bmatrix}$$

$$(p = p_1 + p_2 + \cdots + p_q)$$

$$\Sigma_{[m]} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1m} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \Sigma_{m1} & \Sigma_{m2} & \cdots & \Sigma_{mm} \end{bmatrix} \qquad \Sigma_{(m)} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix}$$

$$(m = 1, 2, \cdots, q) \qquad\qquad (m = 1, 2, \cdots, p)$$

and the corresponding notations for P, S, and R.

(I) $\sqrt{n}(|R|-|P|)$ is asymptotically normally distributed with mean 0 and variance $2|P|^2(\operatorname{tr}P^2-p)$.

(II) The asymptotic joint distribution of $\sqrt{n}\{(|R_{(1)}|,|R_{(2)}|,\cdots,|R_{(p)}|)-(|P_{(1)}|,\cdots,|P_{(p)}|)\}$ is normal with mean $(0,0,\cdots,0)$ and covariances

$$\phi_{ij}=2|P_{(i)}||P_{(j)}|\{\operatorname{tr}P_{(i)}^2-i+\operatorname{tr}P_{ij}^{*\prime}(I-P_{(i)}^{-1})P_{ij}^*\}=\phi_{ji},\quad (i\le j),$$

where $P_{ij}^*=(\rho_{\alpha\beta})$, $\alpha=1,\cdots,i$; $\beta=i+1,\cdots,j$. When $i=j$,

$$\phi_{ii}=2|P_{(i)}|^2(\operatorname{tr}P_{(i)}^2-i).$$

(III) The likelihood ratio criterion for testing the hypothesis of independence. H_0: $\Sigma_{ij}=0$ for all $i\neq j$, is

$$v=\frac{|S|}{|S_{11}||S_{22}|\cdots|S_{qq}|}=\frac{|R|}{|R_{11}||R_{22}|\cdots|R_{qq}|}. \tag{2.2.1}$$

If at least one of Σ_{ij}'s $(i\neq j)$ are not zero, then the asymptotic distribution of v is normal with mean $\lambda=|\Sigma|/(|\Sigma_{11}|\cdots|\Sigma_{qq}|)$ and variance

$$\frac{2}{n}\lambda^2\{\operatorname{tr}(D_\Sigma^{-1}\Sigma)^2-p\}. \tag{2.2.2}$$

(IV) The test statistic for sphericity

$$w=\frac{|S|}{(\operatorname{tr}S/p)^p} \tag{2.2.3}$$

is asymptotically distributed, when $\Sigma\neq\sigma^2 I$, according to $N(\omega,\Delta^2)$, where $\omega=|\Sigma|/(\operatorname{tr}\Sigma/p)^p$, and

$$\Delta^2=\frac{2}{n}\omega^2\{\frac{p^2}{(\operatorname{tr}\Sigma/p)^2}\operatorname{tr}\Sigma^2-p\}. \tag{2.2.4}$$

(V) Roy-Bargmann's step-down multiple correlations [10] are the multiple correlations between the i th component variate x_i and $(i-1)$ component variates, $\{x_1, \cdots, x_{i-1}\}$, $i = 2, 3, \cdots, p$. If we denote their coefficients by $\bar{r}_i \equiv r_{i(1,\cdots,i-1)}$ for sample and $\bar{\rho}_i \equiv \rho_{i(1,\cdots,i-1)}$ for population, then we have the relations

$$1 - \bar{r}_i^2 = \frac{|S_{(i)}|}{s_{ii}\,|S_{(i-1)}|}, \qquad 1 - \bar{\rho}_i^2 = \frac{|\Sigma_{(i)}|}{\sigma_{ii}\,|\Sigma_{(i-1)}|}, \qquad i = 2, \cdots, p \qquad (2.2.5)$$

The non-null asymptotic distribution of $\sqrt{n}\,\{(\bar{r}_2^2, \ldots, \bar{r}_p^2) - (\bar{\rho}_2^2, \cdots, \bar{\rho}_p^2)\}$ is normal with mean $(0, \ldots, 0)$ and variances and covariances

$$\phi_{ii} = 4\,\bar{\rho}_i^2 (1 - \bar{\rho}_i^2)^2, \qquad i = 2, \cdots, p \qquad (2.2.6)$$

$$\phi_{ij} = \phi_{ji} = 2(1 - \bar{\rho}_i^2)(1 - \bar{\rho}_j^2)(\rho_{ij}^2 + \rho_{j(1,\cdots,i-1)}^2 - \rho_{j(1,\cdots,i)}^2), \quad (i < j) \qquad (2.2.7)$$

(VI) Let $\dot{r}_i \equiv r_{1(2,\cdots,i)}$ and $\dot{\rho}_i \equiv \rho_{1(2,\cdots,i)}$, $(i = 2, \cdots, p)$, be coefficients of multiple correlation between x_1 and $(x_2, \cdots, x_i)$ in the sample and in the population respectively. Then $(\dot{r}_2^2, \cdots, \dot{r}_p^2)$ are asymptotically normally distributed with mean $(\dot{\rho}_2^2, \cdots, \dot{\rho}_p^2)$ and covariance matrix with elements,

$$\phi_{ii} = \frac{4}{n}\,\dot{\rho}_i^2 (1 - \dot{\rho}_i^2)^2, \qquad i = 2, \cdots, p \qquad (2.2.8)$$

$$\phi_{ij} = \phi_{ji} = \frac{2}{n}\,\dot{\rho}_i^2 (1 - \dot{\rho}_j^2)(2 - \dot{\rho}_i^2 - \dot{\rho}_j^2), \quad (i \leq j). \qquad (2.2.9)$$

(VII) Let us denote the coefficients of partial correlation between x_1 and x_i after removing the effect of $(x_2, \cdots, x_{i-1})$ by $\hat{r}_{1i} \equiv r_{1i\cdot 2,\cdots,i-1}$ for sample and $\hat{\rho}_{1i} \equiv \rho_{1i\cdot 2,\cdots,i-1}$ for population, $i = 2, \cdots, p$. Then the asymptotic joint distribution of $(\hat{r}_{12}^2, \cdots, \hat{r}_{1p}^2)$ is normal with mean $(\hat{\rho}_{12}^2, \cdots, \hat{\rho}_{1p}^2)$ and variances and covariances

$$\phi_{ii} = \frac{4}{n}\hat{\rho}_{1i}^2(1-\hat{\rho}_{1i}^2)^2, \qquad i=2,\cdots,p \tag{2.2.10}$$

$$\phi_{ij} = \phi_{ji} = -\frac{2}{n}\hat{\rho}_{1i}^2\hat{\rho}_{1j}^2\prod_{\alpha=i}^{j}(1-\hat{\rho}_{1\alpha}^2), \qquad (i<j). \tag{2.2.11}$$

2.3. Non-null asymptotic distribution of the test statistic for symmetry in the structure of covariance matrix.

Let S be the sample covariance matrix with n degrees of freedom and nS be distributed according to $W_p(\Sigma, n)$. S.S. Wilks [19] considered the hypothesis that the covariance matrix $\Sigma = (\sigma_{ij})$ in a p-variate normal population has the structure

$$H_0: \ \Sigma = \Sigma_0 \equiv \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \cdots & \rho\sigma^2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{bmatrix} = \sigma^2\{(1-\rho)I + \rho G\}, \tag{2.3.1}$$

where G is the $p\times p$ matrix whose elements are all one and $-[1/(p-1)] \le \rho < 1$, and derived the likelihood ratio criterion for H_0, which is written as a function of S,

$$t = \frac{p^p(p-1)^{p-1}|S|}{\{\mathrm{tr}\,S(pI-G)\}^{p-1}\,\mathrm{tr}\,SG}. \tag{2.3.2}$$

As an application of the formulas in §2.1, we consider the non-null asymptotic distribution of t. As before, it is normal with mean

$$\tau = \frac{p^p(p-1)^{p-1}|\Sigma|}{\{\mathrm{tr}\,\Sigma(pI-G)\}^{p-1}\,\mathrm{tr}\,\Sigma G} \tag{2.3.3}$$

and variance which we are going to compute. From (2.1.1) we have

$$Var(t) = \frac{2}{n}\tau^2 tr(\Phi\Sigma)^2 \tag{2.3.4}$$

and

$$\log t = p\log p + (p-1)\log(p-1) + \log|S|$$
$$- (p-1)\log\{tr S(pI-G)\} - \log(tr SG)\cdot$$

Hence

$$\phi_{ij} = \partial_{ij}\log|\Sigma| - (p-1)\partial_{ij}\log\{tr\Sigma(pI-G)\} - \partial_{ij}\log(tr\Sigma G)$$

$$= (\Sigma^{-1})_{ij} - (p-1)\frac{(pI-G)_{ij}}{tr\Sigma(pI-G)} - \frac{(G)_{ij}}{tr\Sigma G}.$$

Thus we have

$$\Phi = \Sigma^{-1} - (p-1)\frac{pI-G}{tr\Sigma(pI-G)} - \frac{G}{tr\Sigma G},$$

$$(\Phi\Sigma)^2 = I - 2\left\{(p-1)\frac{(pI-G)\Sigma}{tr\Sigma(pI-G)} + \frac{G\Sigma}{tr\Sigma G}\right\}$$

$$+\left\{(p-1)\frac{(pI-G)\Sigma}{tr\Sigma(pI-G)} + \frac{G\Sigma}{tr\Sigma G}\right\}^2,$$

$$tr(\Phi\Sigma)^2 = -p + tr\left\{(p-1)\frac{(pI-G)\Sigma}{tr\Sigma(pI-G)} + \frac{G\Sigma}{tr\Sigma G}\right\}^2. \tag{2.3.5}$$

Inserting (2.3.5) into (2.3.4), the asymptotic variance of t is given by

$$Var(t) = \frac{2}{n}\tau^2\left[tr\left\{(p-1)\frac{(pI-G)\Sigma}{tr\Sigma(pI-G)} + \frac{G\Sigma}{tr\Sigma G}\right\}^2 - p\right]. \tag{2.3.6}$$

It is easy to see that when $\Sigma = \Sigma_0 = \sigma^2\{(1-\rho)I + \rho G\}$, $var(t) = 0$.

When $\Sigma \neq \Sigma_0$, the asymptotic distribution of the statistic t defined by (2.3.2) is normal with mean (2.3.3) and variance (2.3.6).

2.4. Non-null asymptotic distribution of test statistics for multiple independence in the step-down case.

In this section we shall give a more complicated example in which the formulas given §2.1 are used repeatedly. Starting with the same situation as (III) in §2.2, let us consider the step-down procedure [10] for testing the hypothesis of multiple independence. The over-all hypothesis of independence

$$H_0 : \Sigma_{ij} = 0 \quad \text{for all } i, j \ (i \neq j)$$

is partitioned into $(q-1)$ subhypotheses,

$$H_0^{(i-1)} : \ \Sigma_{1i} = \Sigma_{2i} = \cdots\cdots = \Sigma_{i-1,i} = 0, \quad i = 2, \cdots, q \qquad (2.4.1)$$

Corresponding to this partition, the likelihood ratio criterion for H_0, which is given by (2.2.1), can be factorized into

$$\begin{aligned} v &= \frac{|S_{[2]}|}{|S_{11}||S_{22}|} \cdot \frac{|S_{[3]}|}{|S_{[2]}||S_{33}|} \cdots\cdots \frac{|S_{[q]}|}{|S_{[q-1]}||S_{qq}|} \\ &= v_1 v_2 \cdots v_{q-1} \end{aligned} \qquad (2.4.2)$$

where $S_{[q]} = S$. It is known that under the null hypothesis H_0, $v_1, \cdots, v_{q-1}$ are statistically independent and by G. E. P. Box [2]

$$-\{n - \tfrac{1}{2} - \tfrac{1}{2}(p_1 + \cdots + p_{i+1})\} \log v_i \qquad (2.4.3)$$

is approximately distributed as the chi-square with $p_{i+1}(p_1 + \cdots + p_i)$ degrees of freedom. Then we have the following table for analysis:

Hypothesis	Criterion	χ^2	d.f.
$H_0^{(1)}: \Sigma_{12} = 0$	v_1	$-\{n - \frac{1}{2} - \frac{1}{2}(p_1+p_2)\} \log v_1$	$p_2 p_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H_0^{(i)}: \Sigma_{1,i+1} = \cdots = \Sigma_{i,i+1} = 0$	v_i	$-\{n - \frac{1}{2} - \frac{1}{2}(p_1+\cdots+p_{i+1})\} \log v_i$	$p_{i+1}(p_1+\cdots+p_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H_0^{(q-1)}: \Sigma_{1q} = \cdots = \Sigma_{q-1,q} = 0$	v_{q-1}	$-\{n - \frac{1}{2} - \frac{1}{2}(p_1+\cdots+p_q)\} \log v_{q-1}$	$p_q(p_1+\cdots+p_{q-1})$
$H_0: \Sigma_{ij} = 0, \ i \neq j$	v	$-\{n - \frac{1}{2} - \frac{p^3 - \sum_i p_i^3}{3(p^2 - \sum_i p_i^2)}\} \log v$	$\frac{1}{2}(p^2 - \sum_i p_i^2)$

H_0 is accepted if and only if all $H_0^{(i)}$'s are accepted based on the chi-square tests indicated in the above table, and otherwise H_0 is rejected.

Now let us consider the asymptotic joint distribution of $(v_1, \ldots, v_{q-1})$ when none of $H_0^{(i)}$, $i = 1, \ldots, q-1$ is true. It is easy to see that the desired distribution is normal with mean vector $(\lambda_1, \ldots, \lambda_{q-1})$, where

$$\lambda_i = \frac{|\Sigma_{[i+1]}|}{|\Sigma_{[i]}||\Sigma_{i+1,i+1}|}, \qquad i = 1, \ldots, q-1 \tag{2.4.4}$$

Our main work is to compute the asymptotic covariance matrix of this normal distribution.

First we consider the variance of v_i, $(i = 1, \ldots, q-1)$. By the delta method we have

$$\begin{aligned}
Var(v_i) &= Var\{ \frac{|S_{[i+1]}|}{|S_{[i]}||S_{i+1,i+1}|} \} \\
&= \lambda_i^2 E[\frac{\partial \log \lambda_i}{\partial |\Sigma_{[i+1]}|}(|S_{[i+1]}| - |\Sigma_{[i+1]}|) + \frac{\partial \log \lambda_i}{\partial |\Sigma_{[i]}|}(|S_{[i]}| - |\Sigma_{[i]}|) \\
&\qquad + \frac{\partial \log \lambda_i}{\partial |\Sigma_{i+1,i+1}|}(|S_{i+1,i+1}| - |\Sigma_{i+1,i+1}|)]^2 \\
&= \lambda_i^2 E[\frac{1}{|\Sigma_{[i+1]}|}(|S_{[i+1]}| - |\Sigma_{[i+1]}|) - \frac{1}{|\Sigma_{[i]}|}(|S_{[i]}| - |\Sigma_{[i]}|) \\
&\qquad - \frac{1}{|\Sigma_{i+1,i+1}|}(|S_{i+1,i+1}| - |\Sigma_{i+1,i+1}|)]^2
\end{aligned}$$

$$= \lambda_i^2 \Big[\frac{1}{|\Sigma_{[i+1]}|^2} Var(|S_{[i+1]}|) + \frac{1}{|\Sigma_{[i]}|^2} Var(|S_{[i]}|) + \frac{1}{|\Sigma_{i+1,i+1}|^2} Var(|S_{i+1,i+1}|)$$

$$- \frac{2}{|\Sigma_{[i]}||\Sigma_{[i+1]}|} Cov(|S_{[i]}| \cdot |S_{[i+1]}|) - \frac{2}{|\Sigma_{[i+1]}||\Sigma_{i+1,i+1}|} Cov(|S_{[i+1]}| \cdot |S_{i+1,i+1}|)$$

$$- \frac{2}{|\Sigma_{[i]}||\Sigma_{i+1,i+1}|} Cov(|S_{[i]}| \cdot |S_{i+1,i+1}|) \Big] \qquad (2.4.5)$$

To evaluate each term in (2.4.5), we need the following covariances:

(i) $Cov(|S_{[i]}|, |S_{[j]}|)$, $(i \leq j)$.

From (2.1.2), we have

$$Cov(|S_{[i]}| \cdot |S_{[j]}|) = \frac{2}{n} |\Sigma_{[i]}||\Sigma_{[j]}| \, tr(\Phi \Sigma \Psi \Sigma).$$

Since

$$\phi_{\alpha\beta} = \partial_{\alpha\beta} \log \begin{vmatrix} \Sigma_{[i]} & 0 \\ 0 & I \end{vmatrix} = \begin{bmatrix} \Sigma_{[i]}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{\alpha\beta},$$

then

$$\Phi = \begin{bmatrix} \Sigma_{[i]}^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly

$$\Psi = \begin{bmatrix} \Sigma_{[j]}^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$\Phi \Sigma \Psi \Sigma = \begin{bmatrix} \Sigma_{[i]}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_{[i]} & * \\ * & * \end{bmatrix} \begin{bmatrix} \Sigma_{[j]}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_{[j]} & * \\ * & * \end{bmatrix}$$

$$= \begin{bmatrix} I_{p_1+\cdots+p_i} & * \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{p_1+\cdots+p_j} & * \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{p_1+\cdots+p_i} & * \\ 0 & 0 \end{bmatrix}$$

and

$$tr(\Phi \Sigma \Psi \Sigma) = p_1 + p_2 + \cdots + p_i .$$

Consequently, we have

$$Cov(|S_{[i]}|\cdot|S_{[j]}|) = \frac{2}{n}|\Sigma_{[i]}||\Sigma_{[j]}|(p_1+\cdots+p_i),\ (i \le j) \tag{2.4.6}$$

and in particular

$$Var(|S_{[i]}|) = \frac{2}{n}|\Sigma_{[i]}|^2(p_1+\cdots+p_i). \tag{2.4.7}$$

(ii) $$Cov(|S_{ii}|\cdot|S_{[j]}|) = \frac{2}{n}|\Sigma_{ii}||\Sigma_{[j]}|\,p_i\,, \quad (i \le j). \tag{2.4.8}$$

(iii) $Cov(|S_{[i]}|, |S_{jj}|)$, $(i < j)$.

First of all, we have

$$Cov(|S_{[i]}|\cdot|S_{jj}|) = \frac{2}{n}|\Sigma_{[i]}||\Sigma_{jj}|\,tr(\Phi\Sigma\Psi\Sigma).$$

Since

$$\Phi\Sigma = \begin{bmatrix} \Sigma_{[i]}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \left[\begin{array}{c|c} \Sigma_{[i]} & \begin{matrix} \Sigma_{1,i+1} \cdots \Sigma_{1q} \\ \cdots\cdots \\ \Sigma_{i,i+1} \cdots \Sigma_{iq} \end{matrix} \\ \hline * & * \end{array}\right] = \left[\begin{array}{c|c} I_{p_1+\cdots+p_i} & \Sigma_{[i]}^{-1}\begin{pmatrix} \Sigma_{1,i+1} \cdots \Sigma_{1q} \\ \cdots\cdots \\ \Sigma_{i,i+1} \cdots \Sigma_{iq} \end{pmatrix} \\ \hline 0 & 0 \end{array}\right],$$

and

$$\Psi\Sigma = \begin{bmatrix} 0 & 0 & 0 \\ \Sigma_{jj}^{-1}(\Sigma_{j1},\ldots,\Sigma_{j,j-1}) & I_{p_j} & \Sigma_{jj}^{-1}(\Sigma_{j,j+1},\ldots,\Sigma_{jq}) \\ 0 & 0 & 0 \end{bmatrix},$$

we have

$$\Phi\Sigma\Psi\Sigma = \begin{array}{r} \\ p_1+\cdots+p_i \\ p_{i+1}+\cdots+p_{j-1} \\ p_j \\ p_{j+1}+\cdots+p_q \end{array} \begin{array}{c} \begin{array}{cccc} p_1+\cdots+p_i & p_{i+1}+\cdots+p_{j-1} & p_j & p_{j-1}+\cdots+p_q \end{array} \\ \left[\begin{array}{c|c|c|c} I_{p_1+\cdots+p_i} & \Sigma_{[i]}^{-1}\begin{pmatrix} \Sigma_{1,i+1}\cdots\Sigma_{1,j-1} \\ \cdots\cdots \\ \Sigma_{i,i+1}\cdots\Sigma_{i,j-1} \end{pmatrix} & \Sigma_{[i]}^{-1}\begin{pmatrix} \Sigma_{1j} \\ \vdots \\ \Sigma_{ij} \end{pmatrix} & \Sigma_{[i]}^{-1}\begin{pmatrix} \Sigma_{1,j+1}\cdots\Sigma_{1q} \\ \cdots\cdots \\ \Sigma_{i,j+1}\cdots\Sigma_{iq} \end{pmatrix} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

$$\cdot \begin{array}{cccc|l} \multicolumn{4}{c|}{} & \\ \end{array}
\begin{array}{l}
\left[\begin{array}{c:c:c:c}
0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 \\ \hdashline
\Sigma_{jj}^{-1}(\Sigma_{j1},\cdots\Sigma_{ji}) & \Sigma_{jj}^{-1}(\Sigma_{j,i+1},\cdots\Sigma_{j,j-1}) & I_{p_j} & \Sigma_{jj}^{-1}(\Sigma_{j,j+1},\cdots,\Sigma_{jq}) \\ \hdashline
0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{l} p_1+\cdots+p_i \\ p_{i+1}+\cdots+p_{j-1} \\ p_j \\ p_{j+1}+\cdots+p_q \end{array} \\
\quad p_1+\cdots+p_i \qquad p_{i+1}+\cdots+p_{j-1} \qquad p_j \qquad p_{j+1}+\cdots+p_q
\end{array}$$

$$= \begin{bmatrix}
\Sigma_{[i]}^{-1}\begin{pmatrix}\Sigma_{1j}\\ \vdots \\ \Sigma_{ij}\end{pmatrix}\Sigma_{jj}^{-1}(\Sigma_{j1},\cdots,\Sigma_{ji}) & * & * & * \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

and hence

$$tr(\Phi\Sigma\Psi\Sigma) = tr\left\{\Sigma_{jj}^{-1}(\Sigma_{j1},\cdots,\Sigma_{ji})\Sigma_{[i]}^{-1}\begin{pmatrix}\Sigma_{1j}\\ \vdots \\ \Sigma_{ij}\end{pmatrix}\right\}.$$

Thus we have

$$Cov(|S_{[i]}|,|S_{jj}|) = \frac{2}{n}|\Sigma_{[i]}||\Sigma_{jj}|\, tr\left\{\Sigma_{jj}^{-1}(\Sigma_{j1},\cdots,\Sigma_{ji})\Sigma_{[i]}^{-1}\begin{pmatrix}\Sigma_{1j}\\ \vdots \\ \Sigma_{ij}\end{pmatrix}\right\},\ (i<j). \tag{2.4.9}$$

(iv) $$Cov(|S_{ii}|,|S_{jj}|) = \frac{2}{n}|\Sigma_{ii}||\Sigma_{jj}|\, tr(\Sigma_{ii}^{-1}\Sigma_{ij}\Sigma_{jj}^{-1}\Sigma_{ji}) \tag{2.4.10}$$

$$Var(|S_{ii}|) = \frac{2}{n}|\Sigma_{ii}|^2 p_i. \tag{2.4.11}$$

Now returning to the evaluation of $Var(v_i)$, we have from (2.4.6)-(2.4.11)

$$Var(v_i) = \frac{2}{n}\lambda_i^2\Big[(p_1+\cdots+p_{i+1}) + (p_1+\cdots+p_i) + p_{i+1} - 2(p_1+\cdots+p_i)$$

$$- 2p_{i+1} + 2tr\left\{\Sigma_{i+1,i+1}^{-1}(\Sigma_{i+1,1},\cdots,\Sigma_{i+1,i})\Sigma_{[i]}^{-1}\begin{pmatrix}\Sigma_{1,i+1}\\ \vdots \\ \Sigma_{i,i+1}\end{pmatrix}\right\}$$

$$= \frac{4}{n}\lambda_i^2 \operatorname{tr}\{\Sigma_{i+1,i+1}^{-1}(\Sigma_{i+1,1},\cdots,\Sigma_{i+1,i})\Sigma_{[i]}^{-1}\begin{pmatrix}\Sigma_{1,i+1}\\ \vdots \\ \Sigma_{i,i+1}\end{pmatrix}\}. \tag{2.4.12}$$

This can be written in the form

$$Var(v_i) = \frac{2}{n}\lambda_i^2\{\operatorname{tr}\left(\begin{bmatrix}\Sigma_{[i]} & 0\\ 0 & \Sigma_{i+1,i+1}\end{bmatrix}^{-1}\Sigma_{[i+1]}\right)^2 - (p_1+\cdots+p_{i+1})\}, \tag{2.4.13}$$

which may be referred to (2.2.2). Thus we have the conclusion:

<u>If $H_0^{(i)}$ is not true, the marginal distribution of v_i is asymptotically</u> normal with mean λ_i and variance (2.4.13).

Next we compute the covariance of v_i and v_j $(i<j)$. By the delta method we have

$$Cov(v_i,v_j) = Cov\{\frac{|S_{[i+1]}|}{|S_{[i]}||S_{i+1,i+1}|}, \frac{|S_{[j+1]}|}{|S_{[j]}||S_{j+1,j+1}|}\}$$

$$= \lambda_i\lambda_j\Big[\frac{1}{|\Sigma_{[i+1]}||\Sigma_{[j+1]}|}Cov(|S_{[i+1]}|,|S_{[j+1]}|) + \frac{1}{|\Sigma_{[i]}||\Sigma_{[j]}|}Cov(|S_{[i]}|,|S_{[j]}|)$$

$$+ \frac{1}{|\Sigma_{i+1,i+1}||\Sigma_{j+1,j+1}|}Cov(|S_{i+1,i+1}|,|S_{j+1,j+1}|) + \frac{1}{|\Sigma_{[i]}||\Sigma_{j+1,j+1}|}Cov(|S_{[i]}|,|S_{j+1,j+1}|)$$

$$+ \frac{1}{|\Sigma_{[j]}||\Sigma_{i+1,i+1}|}Cov(|S_{i+1,i+1}|,|S_{[j]}|) - \frac{1}{|\Sigma_{[i+1]}||\Sigma_{[j]}|}Cov(|S_{[i+1]}|,|S_{[j]}|)$$

$$- \frac{1}{|\Sigma_{[i+1]}||\Sigma_{j+1,j+1}|}Cov(|S_{[i+1]}|,|S_{j+1,j+1}|) - \frac{1}{|\Sigma_{[i]}||\Sigma_{[j+1]}|}Cov(|S_{[i]}|,|S_{[j+1]}|)$$

$$- \frac{1}{|\Sigma_{i+1,i+1}||\Sigma_{[j+1]}|}Cov(|S_{i+1,i+1}|,|S_{[j+1]}|)\Big].$$

From (2.4.6)−(2.4.11),

$$\mathrm{Cov}(v_i, v_j) = \frac{2}{n}\lambda_i\lambda_j\Big[(p_1+\cdots+p_{i+1}) + (p_1+\cdots+p_i) + p_{i+1}$$

$$+ \mathrm{tr}(\Sigma_{i+1,i+1}^{-1}\Sigma_{i+1,j+1}\Sigma_{j+1,j+1}^{-1}\Sigma_{j+1,i+1}) + \mathrm{tr}\{\Sigma_{j+1,j+1}^{-1}(\Sigma_{j+1,1},\cdots,\Sigma_{j+1,i})\Sigma_{[i]}^{-1}\begin{pmatrix}\Sigma_{1,j+1}\\ \vdots\\ \Sigma_{i,j+1}\end{pmatrix}\}$$

$$-(p_1+\cdots+p_{i+1}) - (p_1+\cdots+p_i) - p_{i+1}$$

$$-\mathrm{tr}\{\Sigma_{j+1,j+1}^{-1}(\Sigma_{j+1,1},\cdots,\Sigma_{j+1,i+1})\Sigma_{[i+1]}^{-1}\begin{pmatrix}\Sigma_{1,j+1}\\ \vdots\\ \Sigma_{i+1,j+1}\end{pmatrix}\}\Big]$$

$$= \frac{2}{n}\lambda_i\lambda_j\Big[\mathrm{tr}(\Sigma_{i+1,i+1}^{-1}\Sigma_{i+1,j+1}\Sigma_{j+1,j+1}^{-1}\Sigma_{j+1,i+1})$$

$$+ \mathrm{tr}\{\Sigma_{j+1,j+1}^{-1}(\Sigma_{j+1,1},\cdots,\Sigma_{j+1,i})\Sigma_{[i]}^{-1}\begin{pmatrix}\Sigma_{1,j+1}\\ \vdots\\ \Sigma_{i,j+1}\end{pmatrix}\}$$

$$-\mathrm{tr}\{\Sigma_{j+1,j+1}^{-1}(\Sigma_{j+1,1},\cdots,\Sigma_{j+1,i+1})\Sigma_{[i+1]}^{-1}\begin{pmatrix}\Sigma_{1,j+1}\\ \vdots\\ \Sigma_{i+1,j+1}\end{pmatrix}\}. \qquad (2.4.14)$$

Let us summarize the above argument.

When none of the $H_0^{(i)}$, $i=1,\ldots,q-1$, are true, $(v_1,\ldots,v_{q-1})$ are asymptotically distributed according to the $(q-1)$-variate normal distribution with mean vector $(\lambda_1,\ldots,\lambda_{q-1})$ where λ_i are given by (2.4.4) and covariance matrix composed of elements (2.4.13) and (2.4.14).

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