FINITE ELEMENT SOLUTION TO TRANSIENT CONVECTIVE-CONDUCTIVE HEAT TRANSFER PROBLEMS

J. DONEA, S. GIULIANI

Commission of the European Communities,
J.R.C., Ispra Establishment, Department A, I-21020 Ispra (Varese), Italy

L. QUARTAPELLE

Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20133 Milano, Italy

The present paper discusses finite element approximations to problems in transient convective-conductive heat transfer in a fluid region. The governing equations are considered in terms of the primitive variables; the flow is assumed to be laminar and the fluid Newtonian and incompressible within the Boussinesq approximation.

The properties of the discrete convective-diffusion equation are analysed with regard to the possible choices for mass representation (consistent or diagonal) and time integration procedure (explicit or implicit). In particular, the diagonal mass matrix and the explicit time integration method are shown to be a poor combination in terms of accuracy for meshes consisting of linear or multilinear elements. A simple remedy is suggested to improve the frequency response of such lumped-explicit schemes.

The second part of the paper is concerned with finite element formulations for the incompressible Navier-Stokes equations. The various techniques for incorporating the incompressibility constraint are briefly reviewed and those associated with explicit time integration are discussed in detail. In particular, a method is presented for solving the pressure field, which does not require inter-element continuity of the pressure, nor pressure boundary conditions. A numerical example is presented which illustrates the use of this method for the explicit solution of time-dependent natural convection problems.

1. Introduction

A great deal of effort has recently been directed toward developing finite element approximations to problems in transient convective-conductive heat transfer; see for example ref. [1-5]. The present paper discusses numerical problems encountered in this area and details some of the progress that has been made. Particular emphasis is placed on methods that are based on diagonal mass representation and explicit time integration.

The governing equations (convective-diffusion and Navier-Stokes equations) are expressed in terms of the primitive variables, velocity, pressure and temperature. The flow is assumed to be laminar and the fluid Newtonian and incompressible within the Boussinesq approximation.

Guided by the work of Krieg and Key [6] and Gresho et al. [7], we analyse the properties of the discrete convective-diffusion equation with regard to the possible choices for mass representation (consistent or diagonal) and time integration procedure (explicit or implicit).

The second part of the paper is concerned with finite element formulations for the Navier-Stokes equations. The various procedures for incorporating the incompressibility constraint are briefly reviewed. In particular, a segregated method is presented in which the solution of the pressure field does not require inter-element continuity of the pressure, nor pressure boundary conditions.

A numerical example is presented which illustrates the use of this method for the explicit solution of time-dependent natural convection problems.

2. Analysis of the Discrete Convective-Diffusion Equation

In order to make it possible to compare discrete solutions to the exact solution, we consider the one-dimensional homogeneous convective-diffusion equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \lambda \frac{\partial^2 T}{\partial x^2} = 0 \qquad (1)$$

with constant u and λ and initial conditions of the form

$$T(x, 0) = \hat{T} e^{ikx} \qquad (2).$$

For a uniform mesh of piecewise linear finite elements, the Galerkin formulation gives the following equation at any node j:

$$(1+rL)\dot{T}_j + \frac{u}{2h}(T_{j+1} - T_{j-1}) - \frac{\lambda}{h^2} L T_j = 0 \qquad (3)$$

where $h = x_{j+1} - x_j$ and $L T_j = T_{j-1} - 2T_j + T_{j+1}$. The consistent mass results are obtained with $r = \frac{1}{6}$ and the diagonal mass results with r = 0. Assuming a product solution for equation (3) in the form

$$T_j = S_j \cdot \phi(t) \qquad (4)$$

separation of variables leads to the following pair of expressions

$$\beta(1+rL)S_j + \frac{u}{2h}(S_{j+1} - S_{j-1}) - \frac{\lambda}{h^2} L S_j = 0$$
$$\dot{\phi}(t) - \beta\,\phi(t) = 0 \qquad (5).$$

With initial conditions (2), the first of equations (5) indicates that β is given by

$$\beta = -\frac{i\omega \sin kh/kh + 2D(1-\cos kh)/(kh)^2}{1 + 2r(\cos kh - 1)} \qquad (6)$$

where $\omega = ku$ is the exact frequency and $D = \lambda k^2$ is the exact damping parameter. It follows that the ratio of semi-discrete frequency $\bar{\omega}$ to exact frequency ω is given by

$$\frac{\bar{\omega}}{\omega} = \frac{\sin kh/kh}{1 + 2r(\cos kh - 1)} \qquad (7)$$

while the ratio of semi-discrete damping parameter $\bar{D}$ to exact damping parameter D, is given by

$$\frac{\bar{D}}{D} = \frac{2(1-\cos kh)/(kh)^2}{1 + 2r(\cos kh - 1)} \qquad (8)$$

The dependency of the frequency ratio on the dimensionless wave number kh is shown in Fig. 1 for $r = \frac{1}{6}$ (consistent mass) and for $r = 0$ (diagonal mass). The important point to be made here is that both mass representations depress the frequencies. The diagonal mass representation exhibits a particularly poor frequency response and severe distorsion of the exact solution is to be expected. As far as the damping parameter is concerned, the consistent mass representation shifts it upward, while the diagonal mass representation shifts it downward.

The application of a time integration operator leads to a further discretisation of the semi-discrete equations (3) and one should thus consider the effect of time discretisation on the frequency and damping responses before any final judgement is made.

When applied to the second of equations (5) with $\beta = -(i\bar{\omega} + \bar{D})$, the time integration scheme

$$\phi^{n+1} = \phi^n + \Delta t\left[(1-\theta)\dot{\phi}^n + \theta\dot{\phi}^{n+1}\right] \qquad (0 \leq \theta \leq 1) \qquad (9)$$

yields a discrete frequency $\tilde{\omega}$ given by:

$$\mathrm{tg}\,\tilde{\omega}\Delta t = \beta/\alpha \qquad (10)$$

and a discrete damping parameter $\tilde{D}$ which satisfies the relation

$$e^{-\tilde{D}\Delta t} = \sqrt{\alpha^2+\beta^2}\,/\gamma \qquad (11)$$

where

$$\alpha = \left[1-(1-\theta)\bar{D}\Delta t\right]\left[1+\theta\bar{D}\Delta t\right] - \theta(1-\theta)(\bar{\omega}\Delta t)^2$$
$$\beta = \bar{\omega}\Delta t \quad ; \quad \gamma = (1+\theta\bar{D}\Delta t)^2 + \theta^2(\bar{\omega}\Delta t)^2 \qquad (12).$$

The ratio $\tilde{\omega}/\bar{\omega}$ for pure convection ($\bar{D} = 0$) is plotted in Fig. 2 as a function of $\bar{\omega}\Delta t$ for various values of θ. It may be noted that the implicit time integration scheme (eq. (9) with

$\theta \geqslant 0.5$) always reduces the semi-discrete frequency $\bar{\omega}$. Since the effect of space discretisation is also to depress the frequencies (Fig. 1), the recommended combination is the consistent mass representation together with implicit time integration. No damping is introduced in the case of pure convection when using $\theta = 0.5$; for higher values of θ a numerical damping $\tilde{D}$ is present and the ratio $\tilde{D}/\bar{\omega}$ is plotted in Fig. 3 as a function of $\bar{\omega}\Delta t$. The insertion of damping via the time integration scheme may be an alternative to its introduction through an upwind space discretisation of the convection term.

To examine the effect of explicit time integration of pure convection problems, we consider the leapfrog scheme

$$\phi^{n+1} = \phi^{n-1} + 2\Delta t \dot{\phi}^{n} \tag{13}$$

The discrete frequency $\tilde{\omega}$ is now given by

$$\sin \hat{\omega}\Delta t = \bar{\omega}\Delta t \tag{14}$$

and, as shown in Fig. 2, the explicit scheme (13) has the effect of raising all of the semi-discrete frequencies. Comparing Figs. 1 and 2, one notes that compensatory effects may be expected when combining a diagonal mass representation and explicit time integration. However, numerical experiments indicate that in convection-dominated problems the very strong frequency depression introduced by the mass lumping process can hardly be compensated by the opposite effect introduced by explicit time integration. To improve the frequency response of lumped-explicit schemes based on linear or multilinear elements, we recently suggested [8] applying the following two-step procedure in connection with explicit time integration of equations (3):

(a) the diagonal mass representation is used to derive a first approximation to the time derivatives; in view of equation (3) with $r = 0$ this yields

$$\dot{T}_j = -\frac{u}{2h}(T_{j+1} - T_{j-1}) + \frac{\lambda}{h^2} L T_j \tag{15}$$

(b) a second approximation of the time derivatives is then sought by making explicit use of the consistent mass representation. Using equation (3) with $r = \frac{1}{6}$ we write

$$\frac{2}{3}\dot{T}_j + \frac{1}{6}(\dot{T}_{j-1} + \dot{T}_{j+1}) = -\frac{u}{2h}(T_{j+1} - T_{j-1}) + \frac{\lambda}{h^2} L T_j \tag{16}$$

and substituting $\dot{T}_{j-1}$ and $\dot{T}_{j+1}$ by the first step values (15), we obtain

$$\dot{T}_j = -u\left[\frac{3}{2}\left(\frac{T_{j+1} - T_{j-1}}{2h}\right) - \frac{1}{2}\left(\frac{T_{j+2} - T_{j-2}}{4h}\right)\right] + \\ + \lambda\left[2\left(\frac{T_{j-1} - 2T_j + T_{j+1}}{h^2}\right) - \left(\frac{T_{j-2} - 2T_j + T_{j+2}}{4h^2}\right)\right] \tag{17}$$

As equation (17) shows, the above two-step procedure yields fourth-order approximations, while the usual lumped scheme (15) has only second-order accuracy.

3. Analysis of the Discrete Navier-Stokes Equations

The mathematical description of the fluid motion is given by the Navier-Stokes equations

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} - \rho g_i + \rho g_i \beta (T - T_{ref}) - \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad (18)$$

where

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (19).$$

The condition of fluid incompressibility is enforced through the equation

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad (20).$$

T_{ref} is a reference temperature for which buoyancy forces are zero. Either the velocity components or the total surface stress are specified on the boundary.

In deriving the discrete analogue of the incompressible Navier-Stokes equations it is useful to distinguish with Tuann and Olson [9] between integrated methods in which velocity and pressure unknowns are incorporated into a single matrix system and segregated methods in which an auxiliary equation for the pressure field is derived and solved alternatively with the momentum equations.

3.1 Integrated Methods

In most integrated methods, both dynamic equilibrium and incompressibility are satisfied in the mean. That is, the Galerkin formulation of the momentum equations (18) incorporates the incompressibility (20) as a constraint upon the velocities with pressure p serving as a Lagrangian multiplier. In order to have incompressibility satisfied in the mean, the pressure must be interpolated by a polynomial of at least one order less than for velocities. That is, a mixed interpolation finite element is required. The penalty-function formulation [10-11] pertains to the same class of integrated methods. The constitutive equation (19) is replaced by

$$\sigma_{ij}^{(\lambda)} = - p^{(\lambda)}\delta_{ij} + 2\mu\, u_{(i,j)}^{(\lambda)} \qquad (21)$$

in which $p^{(\lambda)} = -\lambda u_{(i,i)}^{(\lambda)}$ where $\lambda > 0$ is a parameter.

The advantages of the penalty-function formulation are that the additional unknown p is eliminated as is the necessity of satisfying the incompressibility condition. A process similar to pressure under-interpolation must be taken, namely the λ-term should be under-integrated by one order less than that for all other terms.

3.2 Segregated Methods.

Segregated methods result in two distinct but coupled systems, of which one approximates incompressibility and the other dynamic equilibrium.

3.2.1 Poisson Equation for Pressure

The most popular segregated method in the (u, v, p) formulation has a

Poisson equation for pressure as an auxiliary equation to replace the continuity equation (20). This approach is plagued by the difficulty of supplementing the Poisson equation with proper boundary specifications. In fact, the exact boundary values of pressure are rather complex in form [13] and if they are not employed a non-zero dilatation is generated during the time integration process, leading to a violation of incompressibility. To circumvent this difficulty, the retention of the dilatation term as a source term in the numerical solution of Poisson equation has been suggested. Under these conditions the continuity equation is satisfied neither exactly nor in the mean, but only asymptotically, in the sense that the incompressibility error becomes insignificant only in the limit of converging solutions [9].

3.2.2 Galerkin Equation for Pressure

When applied to equations (18) and (20) the Galerkin formulation yields the following assembled system [12]:

$$\underset{\approx}{M}\,\dot{\underset{\sim}{v}} + (\underset{\approx}{A} + \underset{\approx}{K})\,\underset{\sim}{v} - \underset{\approx}{C}\,\underset{\sim}{p} - \underset{\sim}{F} = 0 \tag{22}$$

$$\underset{\approx}{C}^{T}\,\underset{\sim}{v} = 0 \tag{23}$$

where $\underset{\sim}{v}^{T} = (\underset{\sim 1}{v}^{T}, \underset{\sim 2}{v}^{T}, \underset{\sim 3}{v}^{T})$. $\underset{\approx}{M}$ is the mass matrix, $\underset{\approx}{A}$ the advection matrix, $\underset{\approx}{K}$ the diffusion matrix, $\underset{\approx}{C}$ the pressure matrix and $\underset{\sim}{F}$ provides the forcing function in terms of body and surface forces.

After applying the velocity boundary conditions one generally has a right-hand side for the discrete continuity equation, i.e. (23) becomes

$$\underset{\approx}{C}'^{T}\,\underset{\sim}{v} = \underset{\sim}{F_{v}} \tag{24}$$

where $\underset{\approx}{C}'$ is now the modified matrix. Once the boundary conditions have been applied, we can differentiate (24) with respect to time and substitute the accelerations given by (22) into it. This gives

$$(\underset{\approx}{C}'^{T}\,\underset{\approx}{M}^{-1}\,\underset{\approx}{C}')\,\underset{\sim}{p} = \dot{\underset{\sim}{F_{v}}} + \underset{\approx}{C}'^{T}\,\underset{\approx}{M}^{-1}(\underset{\approx}{A} + \underset{\approx}{K})\,\underset{\sim}{v} - \underset{\approx}{C}'^{T}\,\underset{\approx}{M}^{-1}\,\underset{\sim}{F} \tag{25}$$

Equation (25) is the Galerkin equation for the pressure field. Note that no pressure boundary conditions are required; they are automatically incorporated into equation (25). The pressure field must not necessarily be forced to be continuous from one element to the next. In particular, if complete incompressibility is desired, a polynomial for $p(x_i)$ must be chosen that is the same degree as that used for $u_{i,i}$. This degree will determine the number of nodal point values of p associated with each element. Once equation (25) is solved, the velocities may be obtained by integrating the momentum equations (22). In the test example to be described, the Euler first-order method has been used together with a diagonal mass matrix.

4. Test Example

We consider the standard problem of free thermal convection in a square cavity (Fig. 4). A uniform (8 x 8) mesh of four-node elements is employed with local bilinear velocity field and uniform pressure. In this case the matrix governing the pressure solution is found to split into two uncoupled sub-matrices, each being singular. To remove this "chequerboard" splitting and obtain a regular pressure solution, the condition of prescribed zero tangential velocity at the boundary nodes is suppressed when assembling the pressure matrix and simulated at each time step through the right-hand side of equation (25). The remaining singularity is eliminated by specifying a pressure datum. The transient calculations were started from an initial situation of zero velocities and temperatures and repeated until a steady-state regime was reached. Figs. 5 and 6 show plots of final profiles of velocity and temperature; they are found to be in excellent agreement with those obtained by Huyakorn et al. [14] on using a purely steady-state velocity-pressure formulation. Moreover, the present solution for pressure is quite regular, while the solution in ref.[14] exhibits large oscillations for the same type of element.

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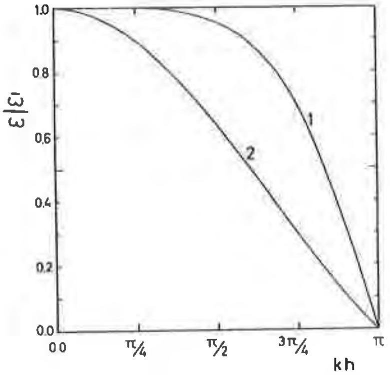

Fig. 1 - Ratio of approximate frequency $\bar{\omega}$ to true frequency ω versus nondimensional wave number for a piecewise linear spatial approximation

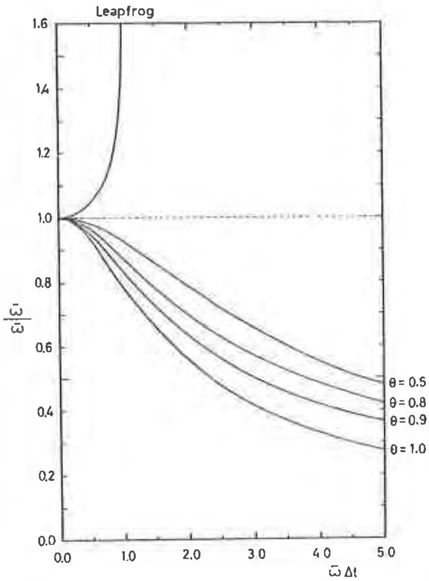

Fig. 2 - Frequency response for the implicit θ and the explicit leapfrog time integrators (pure convection)

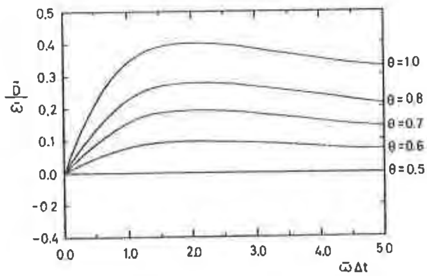

Fig. 3 - Damping introduced by the implicit θ integrator (pure convection)

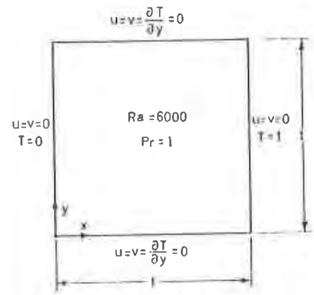

Fig. 4 - Free thermal convection in a square cavity

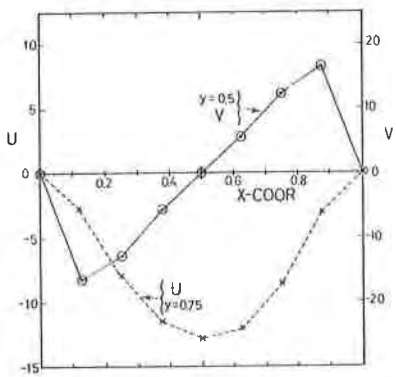

Fig. 5 - Steady state velocity profiles

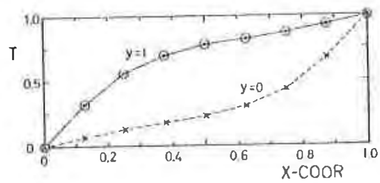

Fig. 6 - Steady state temperature profiles

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