

A SIMPLIFIED METHOD FOR SHAKEDOWN ANALYSIS

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INTRODUCTION

We propose a simplified method for the analysis of structures under strain or displacement controlled cyclic loading. This work is based on the method introduced by Mr Zarka J. [7]. After defining the theoretical context of the study, we introduce the concept of generalized standard materials [2] which allows us to transpose a certain number of results, corresponding to the elastic shakedown theory, to elastoplastic materials with strain hardening. On the basis of this theoretical context and certain hypothesis on loading, we demonstrate a proposition concerning the convergence of transformed parameters towards a stable position in the case of plastic shakedown.

The simplified method described in this paper will be used for structural analysis which will then be compared to a complete elastoplastic analysis.

I HYPOTHESES AND NOTATIONS

We are in the usual context of solid mechanics.

V is the vectorial space of displacements; Φ is the vectorial space of loads;

Σ is the vectorial space of stresses; E is the vectorial space of strains.

We make the classic hypothesis of small displacements, small strains and quasi-static loading of the structure. The loads and displacements imposed vary slowly enough for the forces of inertia to be ignored.

In what follows, we used the notations used by Nayroles B [1] and the concept of generalized standard materials introduced by B. Halphen and Nguyen Quoc Son [2].

II GENERAL THEORITICAL CONTEXT

Statement of a mechanical problem

Any mechanical problem is stated in the following manner.

Problem 1

Given the external load $\phi_d \in \Phi$, an imposed displacement $U_d \in V$, find the fields of displacement U , strain ϵ and stress σ , solving:

$$\epsilon \in \epsilon_0 + I \quad (1)$$

$$\sigma \in \sigma^* + J \quad (2)$$

$$\sigma \in k(I) \quad (3)$$

With : J is a subspace of Σ of self equilibrated stresses and σ^* a particular solution.

I is a subspace of E of compatible strains and ϵ_0 an imposed strain.

k is an operator of E in Σ characterizing the behaviour of the material.

Each mechanical problem has a corresponding elastic problem defined by the following specific behaviour law : There is a linear one to one operator, k_e , of E in Σ , called stiffness, which is assumed to be, self-adjoint and defined as positive. Equation (3) of problem 1 is replaced by: $\sigma = k_e \cdot \epsilon$ (4)

The elastic problem defined in this manner allows only one solution [3], [4], noted ϵ^{el} and σ^{el} . For what follows, we will state: $k_e^{-1} = L$ (flexibility operator)

III VARIATION OF AN ELASTOPLASTIC SOLID WITH STRAIN HARDENING

Let us assume that all the hypothesis relative to problem 1 are verified and let us define the elastoplastic constitutive law with strain hardening. For this purpose, we separate the strain, ϵ , into an "elastic" part, e , and a "plastic" part, p . (5) $\epsilon = e + p$ with $\sigma = k_e \cdot e$

We then introduce the concept of generalized standard strain-hardenable elastoplastic materials. This concept allows the variation of a strain-hardenable structure to be written in a coherent manner with perfect plastic behaviour. In this way, most of the theorems, and particularly those related to elastic shakedown theory, can be transcribed with a minimum brainwork on generalized standard materials.

Definition 1 : Generalized standard materials [2] :

There is a strain hardening parameter $\beta = \{\beta_i; i = 1, \dots, n\}$ and a convex function of β , $\phi(\beta)$ which is such that if we state : $\alpha = \partial\phi / \partial\beta$,

then, generalized strain rate $(\dot{p}, -\dot{\beta})$ is an external normal at a convex domain C of the space $[\Sigma \times X]$, containing the origin, called the elasticity domain of the material.

$$C(x) = \{ (\sigma(x), \alpha(x)) \in [\Sigma \times X] / f(\sigma(x), \alpha(x)) \leq \sigma_y \} \text{ where } f \text{ is convex function and } \sigma_y \in R^* +$$

The plastic flow rule and the strain hardening rule are given by : $(\dot{p}(x), -\dot{\beta}(x)) \in \partial \psi_{C(x)}(\sigma(x), \alpha(x))$ (ψ_C being an indicator function)

In this context, the mechanical problem (1) is stated in the following manner :

Problem 2

Given an external load $\phi_d \in \Phi$, and an imposed displacement $U_d \in V$, find the fields of displacement U , generalized strain $(\dot{p}, -\dot{\beta})$ and generalized stress (σ, α) , solving:

$$\begin{aligned} \epsilon &= \epsilon_0 + l \\ \sigma &= \sigma^0 + J \\ \sigma &= p + e \text{ with } \sigma = k_e \cdot e \\ (\dot{p}, -\dot{\beta}) &\in \partial \psi_{C(x)}(\sigma, \alpha) \end{aligned}$$

Problem 2 allows only one solution [5]

An auxiliary problem can be defined by the difference between problem 2 and the corresponding elastic problem.

Problem 3 (auxiliary)

Given null boundary conditions in terms of load and displacement, find the fields of displacement U_{ine} , of generalized strain $(\dot{p}, -\dot{\beta})$ and generalized stress (ρ, α) , solving:

$$\begin{aligned} \epsilon - \epsilon^{el} &= \epsilon_{ine} \in l \\ \sigma - \sigma^{el} &= \rho \in J \\ \epsilon_{ine} &= p + L\rho \\ (\dot{p}, -\dot{\beta}) &\in \partial \psi_{C(x)}(\sigma(x), \alpha(x)) \end{aligned}$$

Note : $-L\rho$ is the projection of p on $L(J)$

Put $Sp = \{ \sigma \in \Sigma / \sigma = T \cdot Id \cdot T \in R \}$, Id being the identity tensor.

The deviatoric space is defined by : $D = \Sigma / Sp$

Similarly, we assume that space X is deviatoric, isomorphic to D . We note $S = dev \sigma$.

Put $H = \{ (\sigma, \alpha) \in D^2 / \sigma = \alpha \}$.

In what follows, the generalized stress space will be defined by : $D^* = D^2 / H$

Put $Y = \alpha - dev \rho$, and $S = S^{el} + dev \rho$ (with $S^{el} = dev \sigma^{el}$). With these definitions we have:

$$C(x) = \{ (S, \alpha) \in D^* / f(S, \alpha) \leq \sigma_y \} \iff C(x) = \{ (S^{el}, Y) \in D^* / f(S^{el}, Y) \leq \sigma_y \}$$

The constitutive law is then written :

$$(\dot{p}, -\dot{\beta}) \in \partial \psi_{C(x)}(S^{el}(x), Y(x))$$

Problem 3 is written as follows :

Problem 4

$$\begin{aligned} \epsilon_{ine} &\in l \\ \rho &\in J \\ \epsilon_{ine} &= p + L\rho \\ (\dot{p}, -\dot{\beta}) &\in \partial \psi_{C(x)}(S^{el}, Y) \end{aligned}$$

Note 1

$C(S^{el}) = \{ Y \in D / f(S^{el}, Y) \leq \sigma_y \}$ is a mobile convex with centre $S^{el}(x,t)$ and radius σ_y .

Note 2

Only $Y(x,t)$ need be known in order to completely solve the problem.

IV APPLICATION TO PERIODIC LOADING

Elastic shakedown theory

Definition 2

Let Ω be a volume of generalized standard material and let $(\sigma(t), \alpha(t)), (\dot{p}(t), -\dot{\beta}(t))$ be the response of volume Ω in terms of generalized stresses and strain rate when it is subjected to a given loading path from fixed initial conditions. We say that volume Ω is in an elastic shakedown state for this loading path, for these initial conditions, if the integral:

$$\int_0^{+\infty} \langle (\sigma, \alpha) ; (\dot{p}, -\dot{\beta}) \rangle dt \text{ has a finite value.}$$

$\langle (\sigma, \alpha) ; (\dot{p}, -\dot{\beta}) \rangle$ represents the generalized dissipation. This is the power dissipated otherwise than by conduction.

Melan-Koiter theorem [6]

If there is a number $m > 1$ and a generalized self-stress field (ρ^*, α^*) independent of time for the loading process of Ω such that:

$$\forall x \in \Omega \quad \forall t \in [0, T] \quad m [(\rho^*, \alpha^*) + (\sigma^{el}(t), 0)] \in C(x)$$

then, volume Ω is in an elastic shakedown state.

Proposition 1

$$\text{If } \forall x \in \Omega \quad \forall t \in [0, T] \quad f(S^{el}(t), 0) \leq 2\sigma_y$$

then, volume Ω is in an elastic shakedown state.

Demonstration

The Melan-Koiter theorem is written as follows :

$$\exists \rho^* \in J \text{ and } m > 1 \text{ such that : } mf(S^{el}(t) + \text{dev } \rho^*, \alpha^*) \leq \sigma_y$$

$$\Leftrightarrow f(S^{el}(t), Y^*) \leq \sigma_y/m$$

$$f(S^{el}(t), 0) \leq 2\sigma_y \Leftrightarrow \bigcap_{t \geq 0} C(S^{el}) = \emptyset$$

$$\text{then } Y^* \in \bigcap_{t \geq 0} C(S^{el}) \text{ and } m_x > 1 / f(S^{el}, Y^*) \leq \sigma_y/m_x$$

$$m \text{ must simply be chosen so that: } m = \min_{x \in \Omega} m_x$$

Proposition and definition

For a generalized, standard elastoplastic material subjected to periodic loading and if the stress space is finite-dimensional, any solution of the problem of variation tends towards a periodic solution in terms of stress and strain.

This proposition leaves us to define the plastic shakedown state. We say that volume Ω shakes down plastically for a periodic loading path if it is not in the elastic shakedown state.

V SIMPLIFIED METHOD

On the basis of the work by Mr. Zarka, we propose to solve problem 4 by a simplified method based on elastic calculation under the following hypotheses.

Hypothesis 1 Prager model

Function f is identified with the Von Mises function : $f(S, \alpha) = (3/2 (S - \alpha) : (S - \alpha))^{1/2}$

let : $\dot{\alpha} = C \dot{\beta}$ with $C \in R^*_{+}$, and $\beta = \dot{p}$.

With this hypothesis we have: $\dot{p} = C^{-1} (\dot{Y} + \text{dev } \dot{\rho}) = -\dot{k} \cdot \partial f / \partial Y$

Hypothesis 2

The external loading will be assumed to depend only on a single parameter.

It must be periodic and in the following form:

$$\lambda : [0, T] \rightarrow [0, 1]$$

$$t \rightarrow \lambda(t)$$

Given force : $F_d = C^t e$; Given displacement: $U_d = \lambda(t) U_2 + (1 - \lambda(t)) U_1$

Given temperature: $\theta_d = \lambda(t) \theta_2 + (1 - \lambda(t)) \theta_1$

This implies that S^{el} is written: $S^{el}(t) = \lambda(t) S^{el}_2 + (1 - \lambda(t)) S^{el}_1$

According to these hypotheses, problem 4 is written as follows :

Problem 5

$$\begin{aligned} \dot{\epsilon}_{ine} &\in I \\ \dot{\rho} &\in J \\ \dot{\epsilon}_{ine} &= [L + C^{-1} dev] \dot{\rho} + C^{-1} \dot{Y} = L' \dot{\rho} + C^{-1} \dot{Y} \\ (\dot{\rho}, -\dot{\beta}) &\in \partial \psi_{C(x)}(S^{el}, Y) \end{aligned}$$

and we can read the following proposition :

Proposition 2 ($\dot{Y} = \dot{S}^{el}$) \Rightarrow ($\exists \zeta \in R^*_+ / \dot{S}^{el} = \zeta \dot{\alpha}$)

Demonstration

We have: $E = I + (L J)$, and thus $C^{-1} \dot{\alpha} \in E$, there is only one $\dot{\epsilon}_{ine} \in I$ and

$\dot{\rho} \in J$ such that $\dot{\epsilon}_{ine} = L \dot{\rho} + C^{-1} \dot{\alpha}$. Owing to the linearity of the deviatoric operator (dev) we have:

$$dev(\dot{\epsilon}_{ine}) = dev(L \dot{\rho}) + C^{-1} \dot{\alpha}$$

Whence: $dev(L \dot{\rho}) = -proj(C^{-1} \dot{\alpha})$, projection on $dev(L J)$ in parallel to $dev(I)$.

then $\dot{Y} = \dot{S}^{el} = \dot{\alpha} - dev \dot{\rho} \iff \dot{\alpha} = \dot{S}^{el} + dev \dot{\rho}$ and thus $\dot{\alpha} \in dev(J)$ (thanks to hypothesis 2)

hence: $dev(L \dot{\rho}) = ((1 + \nu)/E) dev \dot{\rho} = -C^{-1} \dot{\alpha}$. (E is the young modulus and ν the poisson modulus)

Finally, we obtain:

$$\dot{S}^{el} = \dot{\alpha} (1 + (C^{-1}.E) / (1 + \nu))$$

Note 1: This proposition allows us to state that if "Y" no longer varies on the surface of plasticity ($\dot{S}^{el} = \dot{Y}$), then it is located on the axis defined by \dot{S}^{el} and S^{el}_1 .

Note 2: If $\dot{\alpha}$ is different from zero, then "Y" is on the boundary of the convex and $\dot{S}^{el} \cdot \dot{\alpha} \geq 0$.

Hypothesis 2 allows us to write: $\Delta S^{el} \cdot \Delta \alpha \geq 0$.

Note 3: If $C(S^{el}_1(x)) \cap C(S^{el}_2(x)) \neq \emptyset$ then $Y_{mean} \pm 1/2 \Delta Y \in C(S^{el}_1(x)) \cap C(S^{el}_2(x))$.

These remarks lead us to propose a simplified method of calculation with the purpose of forecasting the stabilized state of the structure submitted to periodic loading. If the structure is adapted (proposition 1), we apply the method given by J. Zarka [7]. Otherwise, we apply the following method:

We initially have three areas. With: $C_1 = C(S^{el}_1(x)) \cap C(S^{el}_2(x))$

Area 1: $C_1 \neq \emptyset$ and $Y_o \in C_1$ we take: $\alpha = 0$ and $\Delta \alpha = 0$

Area 2: $C_1 \neq \emptyset$ and $Y_o \in C_1$ we take: $Y_{mean} = proj(Y) / C_1$ and $\Delta Y = 0$

Area 3: $C_1 = \emptyset$ we take: $Y_{mean} = S^{el}_{mean}$ and $\Delta Y = \Delta S^{el} (1 - ((2\sigma_y) / ||\Delta S^{el}||))$

Y_o being the initial value of Y obtained by an elastoplastic calculation or taken as being equal to zero. A preliminary calculation is carried out with this option. The algorithm figure 1 is applied. (see also figure 2)

VI EXAMPLE OF APPLICATION

1) Hollowed plate

We consider a hollowed plate submitted to displacement controlled loading. (See Figure 3).

We have plotted the amplitudes for plastic strain and stress along line L_2 .

2) Bitubes structure

The structure comprises two coaxial tubes linked rigidly at the top and submitted to an axial load which is constant with respect to time, on one hand, and cyclic heating of the outer tube, on the other hand. (See Figure 4).

CONCLUSION :

We have proposed a simplified method for structural analysis under cyclic loading.

This method is based on various propositions relative to elastic shakedown theory and the existence of stable points for the transformed parameters.

The introduction of the criterion $\Delta S^{el} \cdot \Delta \alpha \geq 0$ considerably improves the prediction of amplitudes of stress and strain.

As well as giving extremely good results on the stabilized cycle (with an error of less than 5 % in the most heavily loaded areas for the structures studied), this method provides an appreciable saving in terms of calculation time (and thus in terms of cost). The saving made corresponds to a factor of 10 for the plate calculation and a factor of 15 for the bitube calculation.

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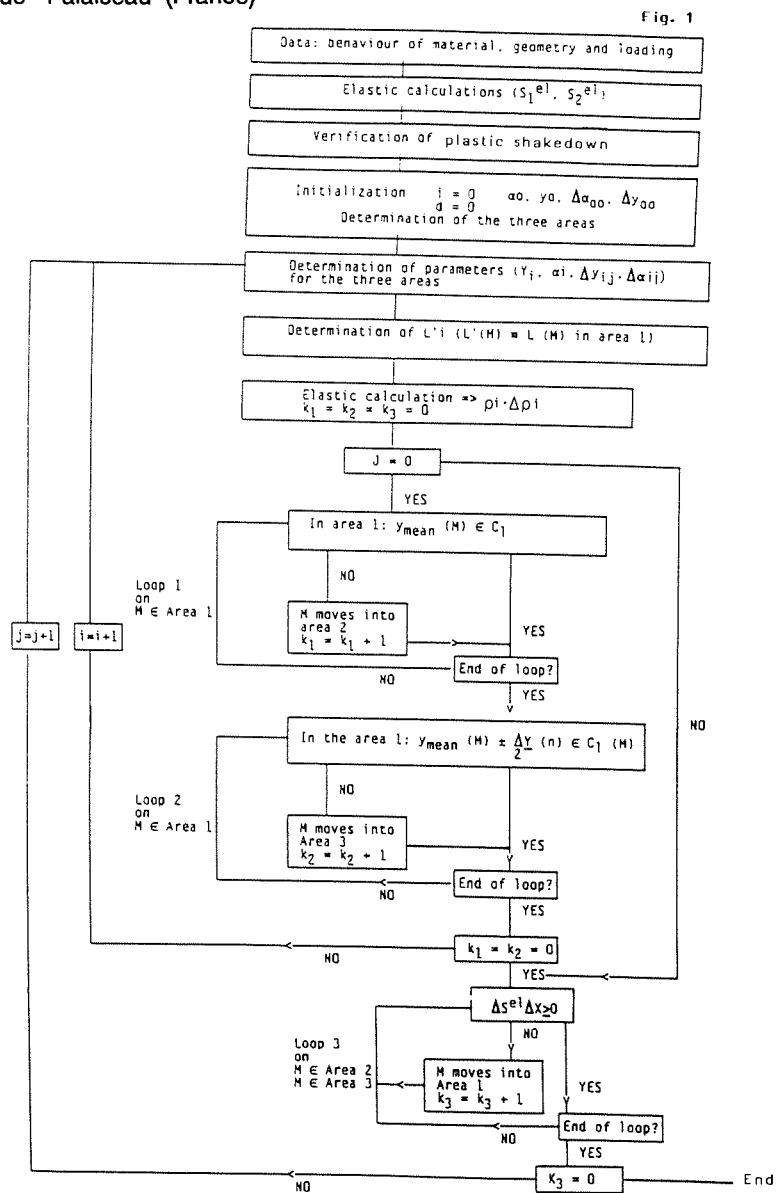


Fig. 2 : SCHEMATIZATION OF ALGORITHM

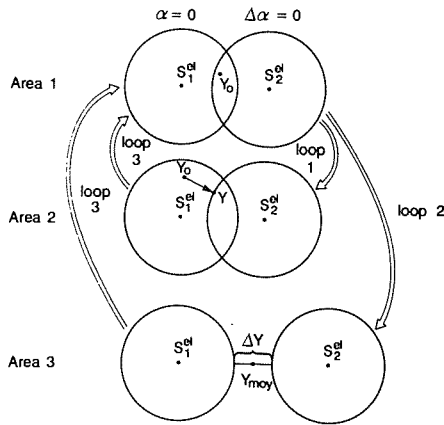


Fig. 3 : HOLLOWED PLATE

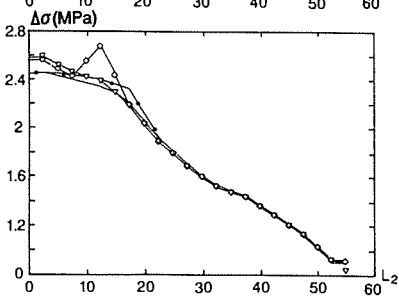
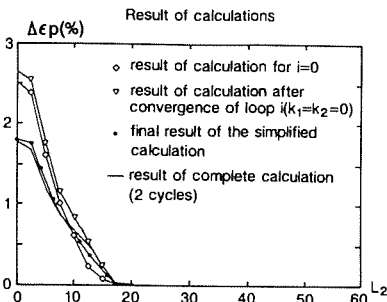
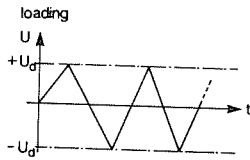
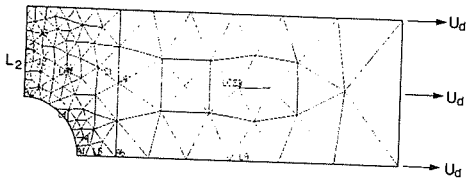
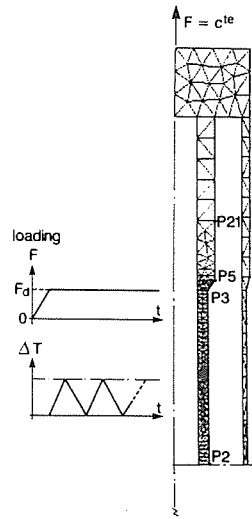


Fig. 4 : BITUBES STRUCTURE



RESULT OF CALCULATIONS

Point of structure	$\Delta\epsilon_p(\%)$	$\Delta\sigma(\text{MPa})$	$\epsilon_{moy}(\%)$	$\sigma_{moy}(\text{MPa})$
P2 Simplified method.	0	519	0.62	141
P2 Complete calculation.	0	515	0.58	142.5
P3 Simplified method.	0.069	656	0.64	157
P3 Complete calculation.	0.072	653	0.59	157.5
P21 Simplified method.	0	282	0	78
P21 Complete calculation.	0	280	0	78

The complete calculation is an elastoplastic calculation on six cycles (not yet stabilised).