

A NEW METHOD FOR ANALYZING FLUID-STRUCTURE INTERACTION USING MSC/NASTRAN

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A popular method for analyzing compressible fluids in flexible containers is to represent the fluid by a three-dimensional finite element model in which the pressure is the unknown nodal point variable, and to represent the structure by another finite element model in which displacement components are the unknown nodal point variables. This method has the computational drawback that the matrix terms coupling the fluid to the structure are unsymmetric.

This paper shows that symmetric fluid-structure coupling can be achieved if either the fluid or the structure is represented by its uncoupled vibration modes, and if additional auxiliary variables are defined. The resulting system equations can be solved efficiently for the coupled vibration modes and for the coupled dynamic response by a general purpose finite element program, such as MSC/NASTRAN.

1. Introduction

The interaction between an acoustic medium and an elastic shell structure has recently been given considerable attention, particularly in the nuclear industry. For many systems, such as nuclear containment vessels, a need exists to determine the dynamic response of a structure and fluid coupled together. For some problems, the inertia effects of the contained fluid are all that is required in the dynamic analysis. In many other problems, however, a compressible acoustic representation of the fluid is necessary.

The NASTRAN structural analysis program [1] has often been used to solve fluid-structure interaction problems. Although it was not originally developed with this application in mind, NASTRAN's flexibility with regard to abstract matrix algebra and with regard to the addition of new capability have facilitated its extension to fluid dynamics. Soon after its initial public release in 1970, capabilities were added to analyze fluid-structure coupling for compressible fluids in axisymmetric containers, and to analyze the acoustic vibrations of slotted cavities which typify the construction of solid propellant rocket motors. Both of these efforts employed finite elements within the fluid. Due to their specialized nature, neither of these capabilities has been widely used. More recently, the Helmholtz integral method [2] for the representation of both compressible and incompressible fluid masses of arbitrary shape has been added to the proprietary MSC version of NASTRAN [3]. The latter capability, which does not utilize finite elements within the fluid, is well suited for the analysis of submerged structures because it easily handles an infinite fluid medium. It has, however, the drawback that the resulting fluid mass matrix is fully populated and, in the case of compressible fluids, is complex as well. The latter feature discourages its use for transient analysis.

Several investigators have used the general matrix handling and finite element capabilities of NASTRAN to solve fluid-structure interaction problems without incorporating specific new code into the program. In these efforts, the fluid has usually been represented by "mock" structural elements in which the solid material properties are modified to simulate fluid properties. In the most straightforward application of this approach [4] the fluid is treated as a degenerate solid continuum for which the shear modulus is zero and the bulk modulus is finite. The unknown degrees of freedom within the fluid are its three components of motion at nodal points connecting finite elements. This approach has the advantage that the fluid-structure coupling terms are symmetric and transparently simple, but it has the disadvantage of an unnecessarily large number of degrees of freedom, and it exhibits difficulties associated with the suppression of free vortex modes due to zero shear modulus.

A more popular approach [5, 6] which is also the basis of the early built-in NASTRAN capability for axisymmetric fluid containers, is to express the fluid field equations in terms of the pressure, which becomes the unknown degree of freedom at nodal points in the finite element model and which is treated by the program as if it were a structural "displacement." In the resulting structural analogy, the reciprocal of the fluid mass density becomes an equivalent elastic modulus and the reciprocal of the bulk modulus becomes an equivalent structural mass density. This approach is easy to implement and it is relatively efficient provided that the container is rigid and immobile. It has, however, the drawback for the analysis of flexible containers that the coupling terms between the fluid and the structure are non-symmetric. As a consequence, relatively slow unsymmetric eigenvalue extraction methods are required to find the coupled modes of the system.

Variations of the approach defined in the previous paragraph are available which overcome the presence of non-symmetric fluid-structure coupling. The fundamental reason for the unsymmetric coupling is that the problem is formulated in terms of variables on the two sides of the interface which are different in nature, one being a pressure and the other a displacement. The approach may, accordingly, be classified as a hybrid method. Symmetry can be restored by matrix manipulations which produce completely full symmetric matrices of the same order as the original matrices, or it can be restored by condensing out interior degrees of freedom in either the fluid or the structural field, and reducing the system to purely structural or purely fluid equations. The latter method is particularly straightforward when the fluid is incompressible and it results in a symmetric fluid mass matrix which can be added to the structure at its boundary with the fluid [7]. Unfortunately, the fluid mass matrix is full, so that approximate methods of computation, such as static condensation [8], are required to provide efficient solutions of the resulting structural equations.

The main purpose of the present paper is to describe a new method whereby the formulation of the hybrid approach can be modified for the general case of a compressible fluid of arbitrary shape to produce symmetrical equations which are inexpensive to solve. The method utilizes an uncoupled modal formulation of the fluid equations, or of the structural equations, or of both, and it introduces a set of auxiliary variables which restore symmetry to the coupled equations. The standard formulation of the hybrid approach [5], will be reviewed before the new method is introduced.

2. Standard Formulation of the Hybrid Approach

The governing equations for acoustics are the equilibrium equation,

$$\nabla p + \rho \ddot{\vec{w}} = 0 \quad (1)$$

and the compressibility equation

$$p = -\chi \nabla \cdot \vec{w} \quad (2)$$

where p is the fluid pressure, \vec{w} is the particle displacement within the fluid, ρ is the density, χ is the bulk modulus, and ∇ is the gradient operator.

Differentiation of eq. (2) with respect to time and substitution for $\ddot{\vec{w}}$ from eq. (1) produces the scalar wave equation:

$$\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) = \frac{1}{\chi} \ddot{p} \quad (3)$$

or, in Cartesian coordinates,

$$\frac{\partial}{\partial x} \left(\frac{1}{\rho} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\rho} \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial p}{\partial z} \right) = \frac{1}{\chi} \frac{\partial^2 p}{\partial t^2} \quad (4)$$

An analogous equation in structural mechanics is the equation for the equilibrium of stresses in a particular fixed direction:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho_s \frac{\partial^2 u_x}{\partial t^2} \quad (5)$$

where u_x is the structural displacement in the x-direction; σ_{xx} , τ_{xy} and τ_{zx} are stress components, and ρ_s is the structural mass density.

In order to establish the acoustic-structural analogy, take

$$u_x = p \quad (6)$$

$$\rho_s = \frac{1}{\chi} \quad (7)$$

$$\sigma_{xx} = \frac{1}{\rho} \frac{\partial p}{\partial x} = -\ddot{w}_x \quad (8)$$

$$\tau_{xy} = \frac{1}{\rho} \frac{\partial p}{\partial y} = -\ddot{w}_y \quad (9)$$

$$\tau_{zx} = \frac{1}{\rho} \frac{\partial p}{\partial z} = -\ddot{w}_z \quad (10)$$

where the second forms of eqs. (8, 9 and 10) are obtained from eq. (1), and \ddot{w}_x is the x-component of fluid acceleration, etc.

In order to complete the analogy, note that, if structural displacement components u_y and u_z are set equal to zero, the general stress-strain relationship becomes

$$\begin{Bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{14} & G_{16} \\ & G_{44} & G_{46} \\ & \text{SYM} & G_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \gamma_{xy} \\ \gamma_{zx} \end{Bmatrix} \quad (11)$$

where G_{ij} is an element of a 6x6 anisotropic elastic material matrix relating $[\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{zx}]$ to $[\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]$, and the strain-displacement relationships are

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} ; \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} ; \quad \gamma_{zx} = \frac{\partial u_x}{\partial z} \quad (12)$$

Thus, if we take

$$G_{11} = G_{44} = G_{66} = \frac{1}{\rho} ; \quad \text{and} \quad G_{14} = G_{16} = G_{46} = 0 \quad (13)$$

eqs. (11, 12 and 6) will produce eqs. (8, 9 and 10) exactly. The other components of $[G_{ij}]$ may be set to any values, including zero, since ϵ_{yy} , ϵ_{zz} , and γ_{yz} are all zero.

All that is required to solve the acoustic vibration problem with the solid elements provided in NASTRAN is to employ the stress-strain relationship indicated by eqs. (11 and 13) (which can be done with the MSC/NASTRAN MAT9 card) and to suppress all grid point degrees of freedom except u_x (which can be done with a single SPCL card). Standard solution formats will provide vibration modes, transient response and frequency response for the fluid. Note also, from eqs. (8, 9 and 10), that normal NASTRAN stress data recovery will produce the acceleration components within the fluid, with a change in sign.

Boundary conditions are treated as follows:

1. At free surfaces, set $u_x = p = 0$.
2. At rigid walls, take no action. The acoustic boundary condition,

$$\frac{\partial p}{\partial n} = n_x \frac{\partial p}{\partial x} + n_y \frac{\partial p}{\partial y} + n_z \frac{\partial p}{\partial z} = 0 \quad (14)$$

where n_x , n_y , and n_z are the direction cosines of the normal, implies an analogous structural boundary condition

$$f_x = n_x \sigma_{xx} + n_y \tau_{xy} + n_z \tau_{zx} = 0 \quad (15)$$

where f_x is the x-component of traction at the boundary. Satisfaction of eq. (15) requires only that no structural forces be applied to grid points on the boundary.

3. At a surface where the pressure is a known function of time, set $u_x = p(t)$.

4. At a surface where the normal component of the displacement, velocity or acceleration is known, apply a grid point load

$$F_f = -A \ddot{w}_n(t) \quad (16)$$

where $\ddot{w}_n(t)$ is the outward component of acceleration, and A is the area associated with the grid point. Eq. (16) follows from eqs. (8, 9, 10 and 15).

5. At a surface where the pressure and the normal velocity are linearly related, i.e., where

$$p = -C \dot{w}_n \quad (17)$$

connect a viscous damping element to u_x with a damping coefficient

$$B = \frac{A}{C} \quad (18)$$

where, again, A is the area associated with the grid point.

6. Boundary conditions for other simple cases can easily be worked out. For example, if

$$p = -k w_n \quad (19)$$

connect a scalar mass, $M = \frac{A}{k}$, to the boundary grid point.

At a grid point on a fluid-structure interface, the boundary conditions can be expressed as a real force

$$F_s = Ap \quad (20)$$

acting normal to a structural surface element of area A, and an acoustic pseudo-force

$$F_f = -A \ddot{u}_n \quad (21)$$

where \ddot{u}_n is the normal acceleration of the structure. Eq. (21) follows directly from eq. (16).

The coupled matrix equations for the fluid-structure system can now be written in the form

$$\begin{bmatrix} Ms^2 + K_s & -A^T \\ -s^2 A & Cs^2 + K_f \end{bmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \quad (22)$$

where $[M]$ = structural mass matrix, $[K_s]$ = structural stiffness matrix, $s = d/dt$, $[A]$ = matrix of surface contact areas, $[C]$ = acoustic "mass" matrix (proportional to the reciprocal of the bulk modulus), $[K_f]$ = acoustic "stiffness" matrix (proportional to the reciprocal of fluid density), $\{u\}$ = vector of structural displacements, $\{p\}$ = vector of acoustic pressures, $\{F\}$ = vector of applied forces.

Note that $\{p\}$ includes pressures at points within the fluid and on the surface. Likewise, $\{u\}$ includes normal components of motion at the surface, and other components of motion that do not couple with the fluid. Thus, $[A]$ is a matrix with either one or no nonzero terms in each row. It will not be restricted to be a diagonal matrix in order to accommodate cases where $\{u\}$ and $\{p\}$ are not ordered in the same way.

Eq. (22) is difficult to solve directly because the coupling is unsymmetric. The extraction of eigenvalues is particularly difficult.

3. The New Method

Symmetry can be restored to the coupled fluid-structure equations, expressed by eq. (22), by reducing either the structural displacements or the acoustic pressures to modal coordinates. The reduction of both to modal coordinates leads to an attractive and efficient solution.

The modes are computed with the coupling terms absent, which means that the acoustic modes assume a rigid container and that the structural modes assume an empty container. The modal representation of displacements and pressures takes the form

$$\{u\} = [\phi_s]\{\xi_s\} \quad (23)$$

$$\{p\} = [\phi_f]\{\xi_f\} \quad (24)$$

where $[\phi_s]$ and $[\phi_f]$ are matrices of eigenvectors.

The resulting modal matrix equations are

$$\begin{bmatrix} m_s s^2 + k_s & | & -\bar{\phi}^T \\ \hline s^2 \bar{\phi} & | & m_f s^2 + k_f \end{bmatrix} \begin{Bmatrix} \xi_s \\ \xi_f \end{Bmatrix} = \begin{Bmatrix} \phi_s^T F \\ 0 \end{Bmatrix} \quad (25)$$

where

$$[\bar{\phi}] = [\phi_f]^T [A] [\phi_s] \quad (26)$$

and

$$[m_s] = [\phi_s]^T [M] [\phi_s] ; \quad [k_s] = [\phi_s]^T [K_s] [\phi_s] \quad (27)$$

$$[m_f] = [\phi_f]^T [C] [\phi_f] ; \quad [k_f] = [\phi_f]^T [K_f] [\phi_f]$$

are diagonal matrices.

Since $[m_f s^2 + k_f]$ is a diagonal matrix, it is inverted simply by replacing each term by its reciprocal. Thus, from the bottom row of eq. (25),

$$\{\xi_f\} = - \left[\frac{s^2}{m_f s^2 + k_f} \right] [\bar{\phi}] \{\xi_s\} \quad (28)$$

An uncoupled equation for $\{\xi_s\}$, obtained by inserting eq. (28) into the top row of eq. (25), is

$$\left[m_s s^2 + k_s + \bar{\phi}^T \left[\frac{s^2}{m_f s^2 + k_f} \right] \bar{\phi} \right] \{\xi_s\} = \{\phi_s^T F\} \quad (29)$$

If the fluid is incompressible, then $m_f = 0$ and eq. (29) is seen to reduce to a modal structural equation with an added fluid mass term.

As an alternative procedure, an uncoupled equation for $\{\xi_f\}$ can be obtained by solving the top half of eq. (25) for $\{\xi_s\}$

$$\{\xi_s\} = \left[\frac{1}{m_s s^2 + k_s} \right] [\bar{\phi}]^T \{\xi_f\} + \left[\frac{1}{m_s s^2 + k_s} \right] \{\phi_s^T F\} \quad (30)$$

and inserting it into the bottom half of eq. (25) to obtain

$$\left[m_f s^2 + k_f + \bar{\phi} \left[\frac{s^2}{m_s s^2 + k_s} \right] \bar{\phi}^T \right] \{\xi_f\} = - [\bar{\phi}] \left[\frac{s^2}{m_s s^2 + k_s} \right] \{\phi_s^T\} \quad (31)$$

Eqs. (29 and 31) are both symmetrical and either may be used in further work. If eq. (29) is selected, and if the following auxiliary variables are introduced,

$$\{\zeta\} = [\bar{\phi}] \{\xi_s\} \quad (32)$$

$$\{\eta\} = \left[\frac{\bar{k}}{m_s s^2 + \bar{k}} \right] \{\zeta\} = - \left[\frac{1}{m_s s^2} \right] \{\xi_f\} \quad (33)$$

$$\{\lambda\} = [\bar{k}]\{\zeta - \eta\} \quad (34)$$

where $\bar{k} = 1/m_f$ and $\bar{m} = 1/k_f$, then eq. (29) can be expanded into the following form

$$\begin{bmatrix} m_s s^2 + k_s & 0 & 0 & -\bar{\phi}^T \\ 0 & \bar{k} & -\bar{k} & -I \\ 0 & -\bar{k} & m_s s^2 + \bar{k} & 0 \\ \bar{\phi} & -I & 0 & 0 \end{bmatrix} \begin{Bmatrix} \xi_s \\ \zeta \\ \eta \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \phi_s^T \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (35)$$

which may be verified by eliminating the auxiliary variables from eq. (35). All of the matrix quantities appearing in eq. (35) are diagonal matrices except $[\bar{\phi}]$ and $[\bar{\phi}]^T$. Equations of this form are very easily solved by NASTRAN. $[\bar{k}]$ is recognized as a diagonal matrix of the stiffness constants of a set of uncoupled springs connecting $\{\zeta\}$ and $\{\eta\}$; $[\bar{m}]$ is recognized as the mass matrix of a set of uncoupled masses connected to $\{\eta\}$. $\{\lambda\}$ is seen to be a Lagrange multiplier associated with the set of constraints expressed by eq. (32). These constraints are normally eliminated first in NASTRAN to form a reduced set of symmetric equations involving only $\{\xi_s\}$ and $\{\eta\}$. Although the mass and stiffness matrices of this reduced set are full, their order is only equal to the sum of the number of important structural and fluid modes. Eq. (33) provides the means to evaluate the modal coordinates of the fluid once $\{\xi_s\}$ and $\{\eta\}$ have been obtained.

It should also be clear that there is no need to reduce the structure to its modal coordinates because, by using eq. (23) and the definition of $[\bar{\phi}]$ in eq. (26), eq. (35) can be replaced by

$$\begin{bmatrix} M_s s^2 + K_s & 0 & 0 & A^T \phi_f^T \\ 0 & \bar{k} & -\bar{k} & -I \\ 0 & -\bar{k} & m_s s^2 + \bar{k} & 0 \\ \phi_f^T A & -I & 0 & 0 \end{bmatrix} \begin{Bmatrix} u \\ \zeta \\ \eta \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (36)$$

which may be more convenient, for example, when $\{F\}$ includes nonlinear terms.

An analogous set of operations applied to eq. (31) results in a set of equations in standard form in which the modal coordinates of the fluid, or alternatively the pressures within the fluid, appear explicitly. In this case we define the following auxiliary quantities:

$$\{\bar{\zeta}\} = \left[\frac{1}{k_s} \right] [\bar{\phi}]^T \{\xi_f\} = \left[\frac{1}{k_s} \right] [\phi_s]^T [A]^T \{p\} \quad (37)$$

$$\{F_{\xi}\} = [\bar{\phi}] \left[\frac{k_s}{m_s} \right] [\phi_s]^T \{F\} \quad (38)$$

$$\{\bar{\lambda}\} = [\bar{k}]\{\tau - \xi_s\} \quad (39)$$

$$\bar{m} = k_s \quad (40)$$

$$\bar{k} = k_s^2/m_s \quad (41)$$

With these definitions, eq. (31) can be expanded into the form

$$\begin{bmatrix} m_f s^2 + k_f & 0 & 0 & \bar{\phi} \frac{1}{k_s} \\ 0 & \bar{k} & -\bar{k} & -I \\ 0 & -\bar{k} & m_s s^2 + \bar{k} & 0 \\ \frac{1}{k_s} \bar{\phi}^T & -I & 0 & 0 \end{bmatrix} \begin{Bmatrix} \xi_f \\ \bar{\zeta} \\ \xi_s \\ \bar{\lambda} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F_{\xi} \\ F_{\xi} \\ 0 \end{Bmatrix} \quad (42)$$

Again, it is seen that there is no need to reduce the fluid to its modal coordinates, because, by using eq. (24) and the definition of $[\bar{\phi}]$ in eq. (26), eq. (42) can be replaced by

$$\begin{bmatrix} C_s^2 + K_f & 0 & 0 & A\phi_s \frac{1}{k_s} \\ 0 & \bar{k} & -\bar{k} & -I \\ 0 & -\bar{k} & m_s s^2 + \bar{k} & 0 \\ \frac{1}{k_s} \phi_s^T A^T & -I & 0 & 0 \end{bmatrix} \begin{Bmatrix} p \\ \bar{\zeta} \\ \xi_{fs} \\ \bar{\lambda} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F_{\xi} \\ F_{\xi} \\ 0 \end{Bmatrix} \quad (43)$$

The form of eq. (43) is advantageous if interest centers in the fluid rather than the structure. Note also that the $\{\xi_s\}$ vector appears directly in eqs. (42 and 43) which is convenient for subsequent recovery of the structural displacements by means of eq. (23).

4. Practical Implementation of the New Method

The procedures described in the preceding section can be implemented with a number of finite element codes, including MSC/NASTRAN. For example, if eq. (43) is selected for implementation, the following steps should be carried out:

1. Set up a finite element model for the empty container and calculate enough vibration modes to adequately represent its structural behavior. Record the natural frequencies, ω_s , the eigenvectors, ϕ_s , and the generalized masses, m_s , (which can be normalized to unity for convenience). The generalized stiffness, $k_s = \omega_s^2 m_s$.
2. Compute the contact surface areas which are elements of the $[A]$ matrix. This calculation has been automated in MSC/NASTRAN for the Helmholtz integral method [2] discussed earlier, but has not as yet been automated for the new method.
3. Compute the matrix product $[\frac{1}{k_s}][\phi_s]^T[A]^T$ appearing in the last row of eq. (43). This may be done in NASTRAN with DMAP macro-instructions.
4. Set up a finite element model for the fluid as discussed in Section 2. Append to this model the additional scalar degrees of freedom $\{\bar{\zeta}\}$ and $\{\xi_s\}$ together with the scalar springs \bar{k} and the scalar masses \bar{m} . Also indicate the existence of the equations

of constraint given by eq. (37), whose coefficients were computed in step 3. Equations of constraint are implemented in NASTRAN with MPC (Multipoint constraint) cards.

5. Analyze the resulting system for its vibration modes, or for its response to dynamic excitation. In the latter case, dynamic structural forces are represented in the model as indicated by the vector $\{F_{\xi}\}$ in eq. (43). Dynamic excitation at fluid surfaces not in contact with the container can be represented by techniques discussed in Section 2.

6. Once $\{\xi_g\}$ has been evaluated, compute structural displacements by means of eq. (23). In NASTRAN, this operation, and the additional recovery of internal forces and stresses, can be achieved by reentering the structural analysis described in step 1.

With some modest coding effort, the above computational steps can be combined into a single MSC/NASTRAN execution, but this has not yet been done.

The new method has been used to analyze several problems for practical nuclear containment vessels, but these results cannot be published at present due to proprietary restrictions. The new method has also been validated with simple examples for which results are available by other methods. These results are of little interest because the new method employs no unusual assumptions and hence should be expected to produce results which are consistent with other methods.

The main advantage claimed for the new method is that it removes the unsymmetric coupling present in fluid-structure interaction in such a way that the resulting system equations can be easily and efficiently solved by a general purpose finite element program.

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