# DESIGN RELATIONSHIPS AND FAILURE THEORIES IN PROBABILISTIC FORM

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## SUMMARY

The engineering profession has become increasingly concerned with the adequacy of design calculations. This concern indicates a need for critical evaluation of designs based on arbitrary multipliers, such as safety factors or worst-case treatment. Design has customarily been based on applied loading, geometry, and handbook property values to give a deterministic solution.

This paper presents (1) the Soderberg, Goodman, Gerber, and elliptical design relationships, and (2) the maximum shear and distortion energy failure theories in probabilistic form. Inherent in these equations are the facts that (1) design variables are generally characterized by spectra of values, rather than unique values, and (2) a small, but finite, probability of failure must be recognized in any design. By coupling the mean static and mean alternating stresses (and their standard deviations) with the strengths available in a material (and their standard deviations) in one of the four design relationships using an appropriate theory o failure, the reliability of a given design can be calculated. Conversely for a given reliability, the appropriate size can be determined.

Illustrations are provided. The influence of range (different standard deviations) of variables on reliability is demonstrated. A comparison is made between safety factor and reliability.

# 1. INTRODUCTION

The engineering profession has become increasingly concerned with the adequacy of design calculations. This concern indicates a need for critical evaluation of designs based on arbitrary multipliers, including safety factors or worst-case treatment. Freudenthal[1] has said, "Careful and rigorous analyses may be largely deprived of their merits if the accuracy of results be diluted by the employment of empirical multipliers - selected rather arbitrarily on the basis of considerations not always rational or even relevant."

Design has customarily been based on applied loading, geometry, and handbook property values. None of these account for the spectra of values associated with design variables. It is well known that conventional design practice generally gives a conservative design, yet with an occasional failure in service. It is equally well known that the degree of conservatism is not easily determined. The resulting design analyses give components and systems in which safety or adequacy is neither balanced nor clearly specified.

The concept of a linear world having single-value variables is obvious in traditional methods which have served man's needs reasonably well in the past. We recognize that most natural events are actually probabilistic. We generally recognize that any measurable parameter varies in a random manner. Design parameters, therefore, are characterized by spectra of values rather than by unique values. The safety factor concept overlooks this variability which may give different reliabilities for the same safety factor.

When a typical population of random values is assembled and plotted with frequency of occurrence as a function of magnitude, the plot tends toward a stable, predictable distribution as sample size increases. This distribution usually approximates some well known type such as normal, log-normal, beta, gamma, or exponential distribution. Many random variables encountered in the physical sciences appear normally distributed. In addition, the normal distribution gives an adequate approximation to the distribution of many other measurable random variables (life tests are one exception, being essentially log-normal). Thus a theory of statistical inference based on normal distribution is a system which can be employed quite generally.

The statistical nature of design parameters is usually ignored in conventional practice, demonstrated by the efforts made to find unique values representative of design parameters. Minimum guaranteed values, limit loads, and ultimate loads are examples of unique value representation. The large extremes of loading and the minima of strength are treated not only as representative of design situations, but also of concurrent occurrence. Actually, magnitude and frequency relationships, both load and strength, should be considered to avoid unrealistic results. If an extremely large load (of rare occurrence) must act on an extremely low permissible strength (of rare incidence) to induce a failure, the probability of this combination occurring is important.

It is often believed that use of a safety factor greater than some preconceived magnitude (for example, 2.5) will result in no failure. Actually with such high safety factors, the failure probability may vary from a satisfactory low to an intolerable high. A safety factor of one implies, to many, that failure will occur 100% of the time because there is no safety margin. Actually, if strength and stress are normally distributed, failure will occur only 50% of the time. It is well known that distributions exist in both the strength available and the load (stress) requirement. It is these distributions (as defined by mean values, standard deviations, and other parameters - depending on the specific distributions involved) with which the designer should be concerned.

## 2. PROBABILISTIC RELATIONSHIP

Assuming the variables in a given problem are normally distributed, then the Algebra of Normal Functions (2) will apply. Reliability is defined as the probability that available strength S, exceeds the load L, i.e. (S - L)   0. If both S and L are normally distributed, then the difference between them is normally distributed. This difference can be related to the standard-normalized variable z by

$$Z = \frac{S - L}{(S_S^2 + S_L^2)^{1/2}} \tag{1}$$

where

- $S$ = average strength
- $L$ = average load
- $S_S$ = standard deviation for strength
- $S_L$ = standard deviation for load

Equation 1 is called a "coupling equation", since it probabilistically relates or couples the strength and load functions. In this application, z is called the coupling coefficient. Reliability, i.e., the probability of survival is expressed by

$$R = \int_{-z}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$= \int_{-\infty}^{z} e^{-\frac{z^2}{2}} dz \tag{2}$$

Once a value for z has been determined, reliability can be found directly from standard tables of the normal function. Failure probability is

$$Q = 1 - R \qquad (3)$$

The probabilistic relationship has been applied to a number of practical situations (2, 3, 4, 5, 6, 7).

## 3. DESIGN RELATIONSHIPS

In a great number of design situations, there is fluctuating stress which is shown schematically in Fig. 1. Differences in stress wave forms are not important since the material under stress must adjust itself alternately to $\sigma_{max}$ and $\sigma_{min}$. Fluctuating stress can also be regarded as a case of alternating stress superposed on mean stress. As seen in Fig. 1, the mean stress is one-half the algebraic sum while the alternating stress is one-half the algebraic difference of the maximum and minimum stresses. The mean stress can be either positive or negative.

At least four different relationships among mean stress, alternating stress, and appropriate strength properties of a given material have been developed and used over the years. Using the nomenclature given in TableI, the deterministic equations for these relationships are:

Soderberg

$$\frac{\sigma_a}{\sigma_e} + \frac{\sigma_m}{\sigma_y} = 1 \qquad (4)$$

Goodman

$$\frac{\sigma_a}{\sigma_e} + \frac{\sigma_m}{\sigma_u} = 1 \qquad (5)$$

Gerber

$$\frac{\sigma_a}{\sigma_e} + \frac{\sigma_m}{\sigma_u} = 1 \qquad (6)$$

Elliptical

$$\frac{\sigma_a}{\sigma_e} + \frac{\sigma_m}{\sigma_u} = 1 \qquad (7)$$

A graphical representation of these equations is shown in Fig. 2. A safety factor is readily incorporated by dividing the right hand side of the equation by the chosen safety factor.

The implication of these equations is that any combination of stresses and strengths which is located in the area bounded by the axes and the curve is safe while any combination falling outside is unsafe and presumably will fail. These equations are quite acceptable if one can be satisfied with a single-value solution which ignores the random variations of the parameters involved.

Recognizing the variations in these parameters and their natural tolrances (taken as ± 3 standard deviations, which excludes 0.3 percent of all ossible values) and assuming normal distributions in all parameters, these quations can be written in probabilistic terms in the form of a coupling quation similar to eq. (1):

Soderberg

$$Z = \frac{1 - \left(\frac{\sigma_a}{\sigma_e} + \frac{\sigma_m}{\sigma_y}\right)}{\left[\left(\frac{1}{\sigma_e}\right)^2\left(\frac{\sigma_a^2 S_{\sigma_e}^2 + \sigma_e^2 S_{\sigma_a}^2}{\sigma_e^2 + S_{\sigma_e}^2}\right) + \left(\frac{1}{\sigma_y}\right)^2\left(\frac{\sigma_m^2 S_{\sigma_y}^2 + \sigma_y^2 S_{\sigma_m}^2}{\sigma_y^2 + S_{\sigma_y}^2}\right)\right]^{\frac{1}{2}}} \tag{8}$$

Goodman

$$Z = \frac{1 - \left(\frac{\sigma_a}{\sigma_e} + \frac{\sigma_m}{\sigma_u}\right)}{\left[\left(\frac{1}{\sigma_e}\right)^2\left(\frac{\sigma_a^2 S_{\sigma_e}^2 + \sigma_e^2 S_{\sigma_a}^2}{\sigma_e^2 + S_{\sigma_e}^2}\right) + \left(\frac{1}{\sigma_u}\right)^2\left(\frac{\sigma_m^2 S_{\sigma_u}^2 + \sigma_u^2 S_{\sigma_m}^2}{\sigma_u^2 + S_{\sigma_u}^2}\right)\right]^{\frac{1}{2}}} \tag{9}$$

Gerber

$$Z = \frac{1 - \left( \frac{\sigma_a}{\sigma_e} + \frac{\sigma_m^2 + S_{\sigma_m}^2}{\sigma_u^2 + S_{\sigma_u}^2} \right)}{\left[ \left( \frac{1}{\sigma_e} \right)^2 \left( \frac{\sigma_a^2 S_{\sigma_e}^2 + \sigma_e^2 S_{\sigma_a}^2}{\sigma_e^2 + S_{\sigma_e}^2} \right) + \left( \frac{1}{\sigma_u^2 + S_{\sigma_u}^2} \right)^2 \left\{ \frac{\left(\sigma_m^2 + S_{\sigma_m}^2\right)^2 \left(4\sigma_u^2 S_{\sigma_u}^2 + 2 S_{\sigma_u}^4\right) + \left(\sigma_u^2 + S_{\sigma_u}^2\right)^2 \left(4\sigma_m^2 S_{\sigma_m}^2 + 2 S_{\sigma_m}^4\right)}{\left(\sigma_u^2 + S_{\sigma_u}^2\right)^2 + \left(4\sigma_u^2 S_{\sigma_u}^2 + 2 S_{\sigma_u}^4\right)} \right\} \right]^{\frac{1}{2}}} \tag{10}$$

Ellipse

$$Z = \frac{1 - \left( \frac{\sigma_a^2 + S_{\sigma_a}^2}{\sigma_e^2 + S_{\sigma_e}^2} + \frac{\sigma_m^2 + S_{\sigma_m}^2}{\sigma_u^2 + S_{\sigma_u}^2} \right)}{\left[ \left( \frac{1}{\sigma_e^2 + S_{\sigma_e}^2} \right)^2 \left\{ \frac{\left(\sigma_a^2 + S_{\sigma_a}^2\right)^2 \left(4\sigma_e^2 S_{\sigma_e}^2 + 2 S_{\sigma_e}^4\right) + \left(\sigma_e^2 + S_{\sigma_e}^2\right)^2 \left(4\sigma_a^2 S_{\sigma_a}^2 + 2 S_{\sigma_a}^4\right)}{\left(\sigma_e^2 + S_{\sigma_e}^2\right)^2 + \left(4\sigma_e^2 S_{\sigma_e}^2 + 2 S_{\sigma_e}^4\right)} \right\} + \left( \frac{1}{\sigma_u^2 + S_{\sigma_u}^2} \right)^2 \left\{ \frac{\left(\sigma_m^2 + S_{\sigma_m}^2\right)^2 \left(4\sigma_u^2 S_{\sigma_u}^2 + 2 S_{\sigma_u}^4\right) + \left(\sigma_u^2 + S_{\sigma_u}^2\right)^2 \left(4\sigma_m^2 S_{\sigma_m}^2 + 2 S_{\sigma_m}^4\right)}{\left(\sigma_u^2 + S_{\sigma_u}^2\right)^2 + \left(4\sigma_u^2 S_{\sigma_u}^2 + 2 S_{\sigma_u}^4\right)} \right\} \right]^{\frac{1}{2}}} \tag{11}$$

4. FAILURE THEORIES

In eq. (4) through (11), there is an implication that stress application is uniaxial. In practice, however, a designer is commonly concerned with problems involving biaxial or triaxial stresses with an infinite range of ratios of the principal stresses. At the same time, the available strength data have usually been determined for unixial stress. The following question constantly recurs: If a material can withstand a known stress in unixial loading, how heavily can it be stressed in biaxial or triaxial loading? The answer to this question lies in applying a failure theory. In this context the "failure" stress is effectively an equivalent uniaxial stress.

While there are at least six such failure theories, two are in common use, namely, the Maximum Shear Stress Theory and the Maximum Distortion Energy Theory. In terms of the nomenclature in Table I, these become (in deterministic form):

Maximum Distortion Energy

$$\sigma_o = \sigma_1 - \sigma_3 \tag{12}$$

Maximum Distortion Energy

$$\sigma_o = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2} \tag{13}$$

These equations are quite acceptable if one can be satisfied with single-value solutions.

In probabilistic form, these equations become:

Maximum Shear Stress

$$\sigma_o = \sigma_1 - \sigma_3 \tag{14}$$

$$S_{\sigma_o} = \left(S_{\sigma_1}^2 + S_{\sigma_3}^2\right)^{\frac{1}{2}} \tag{15}$$

Maximum Distortion Energy

$$\sigma_o = \left[ \left\{ \frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+\left(S_{\sigma_1}^2+S_{\sigma_2}^2\right)+\left(S_{\sigma_2}^2+S_{\sigma_3}^2\right)+\left(S_{\sigma_3}^2+S_{\sigma_1}^2\right)}{2} \right\}^2 - \right. \tag{16}$$

$$\left. \left\{ \frac{4(\sigma_1-\sigma_2)^2\left(S_{\sigma_1}^2+S_{\sigma_2}^2\right)+4(\sigma_2-\sigma_3)^2\left(S_{\sigma_2}^2+S_{\sigma_3}^2\right)+4(\sigma_3-\sigma_1)^2\left(S_{\sigma_3}^2+S_{\sigma_1}^2\right)+2\left(S_{\sigma_1}^2+S_{\sigma_2}^2\right)^2+2\left(S_{\sigma_2}^2+S_{\sigma_3}^2\right)^2+2\left(S_{\sigma_3}^2+S_{\sigma_1}^2\right)^2}{4} \right\} \right]^{\frac{1}{4}}$$

$$S_{\sigma_o} = \left[ \frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+\left(S_{\sigma_1}^2+S_{\sigma_2}^2\right)+\left(S_{\sigma_2}^2+S_{\sigma_3}^2\right)+\left(S_{\sigma_3}^2+S_{\sigma_1}^2\right)}{2} - \right. \tag{17}$$

$$\left( \left\{ \frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+\left(S_{\sigma_1}^2+S_{\sigma_2}^2\right)+\left(S_{\sigma_2}^2+S_{\sigma_3}^2\right)+\left(S_{\sigma_3}^2+S_{\sigma_1}^2\right)}{2} \right\}^2 - \right.$$

$$\left. \left. \left\{ \frac{4(\sigma_1-\sigma_2)^2\left(S_{\sigma_1}^2+S_{\sigma_2}^2\right)+4(\sigma_2-\sigma_3)^2\left(S_{\sigma_2}^2+S_{\sigma_3}^2\right)+4(\sigma_3-\sigma_1)^2\left(S_{\sigma_3}^2+S_{\sigma_1}^2\right)+2\left(S_{\sigma_1}^2+S_{\sigma_2}^2\right)^2+2\left(S_{\sigma_2}^2+S_{\sigma_3}^2\right)^2+2\left(S_{\sigma_3}^2+S_{\sigma_1}^2\right)^2}{4} \right\} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}$$

## 5. DESIGN INTERACTION

No generalization can include all possible design situations. At the same time, however, it is possible to combine the above design relationships and failure theories to permit appropriate design for a large number of cases, whether on a deterministic or probabilistic basis. One determines the uniaxial equivalent mean and alternating stresses from the biaxial or triaxial stress loading using a failure theory. These two uniaxial equivalent stresses are then combined with the material strength properties using an appropriate design relationship.

6. APPLICATION

As an application of the probabilistic equations, consider a cylindrical shaft made of AISI 4340 having the strength properties given in Table II. The loading is combined bending and torsion with each having a standard deviation of 10 percent. Using the maximum shear stress theory, eq. (14) and (15), it is found that the equivalent mean stress is $80,000/(\frac{\pi}{2} r^3)$ and the equivalent alternating stress is $50,000/(\frac{\pi}{2} r^3)$. Since the radius has a standard deviation of 0.015r, the standard deviations of the stresses will be somewhat larger than the 10 percent from the loading.

It is obvious that determining the appropriate radius of the shaft from eq. (4), (5), (6), or (7) is much less tedious than from eq. (8), (9), (10), or (11) respectively. The probabilistic equations, however, can be programmed to readily give a solution. The solution for reliability as a function of the mean radius is shown in Fig. 3 for all four design relationships. Table III shows the mean radii and accompanying range (based on $\pm$ 3 standard deviations) for selected values of reliability.

There is little difference between the radii calculated from the deterministic equations (using "reasonable" safety factors) and the mean radii calculated from the probabilistic equations. Calculation of means and standard deviations, however, does permit finding the variations expected in the results by accounting for the random variations (and their ranges)in all parameters in a given problem. It is also possible to determine bands or ranges within which any given percentage of stresses (or sizes) would be expected to occur.

As further application, consider the case (Table III) of the Goodman relationship in which radius of 1.020 $\pm$ 0.046 inches is required to give a reliability of 0.9999. Using eq. (5), an equivalent safety factor can be calculated. It must be noted, however, that this radius is based on the strength properties in Table II and a standard deviation of 10% in both the mean and alternating loading components. While the ranges of strength properties are not likely to change, it would be reasonable to expect that the range of loading might well be more variable. Under the deterministic form, the safety factor will remain constant. If the ranges of the mean and alternating loadings were to change, the reliability would change. This is shown in Table IV in which the radius is kept constant. One notes that under the conditions originally stated, one failure in 10,000 would be expected. If the range in loading is tripled, then one failure in ten would be expected. The safety factor would remain constant and would not predict that the failure rate would increase by a factor of 1000.

It is obvious that analogous changes would be expected from changes in the standard deviations of the material properties and from the tolerance permitted for the radius.

7. ADDITIONAL COMMENTS

It is obvious that a number of factors have been neglected in the above calculations. These are such things as stress concentration factors, load factors (dynamic, impact, etc.), temperature factors, forming or manufacturing stress factors (residual stresses, surface treatment, heat treatment, assembly, etc.), corrosion stress factors, and notch sensitivity factors. All of these will affect the final reliability. These were not included so that the central idea of the probabilistic approach might be clearer. In no sense does the probabilistic approach eliminate consideration of such factors. They must be included. Appropriate stress concentration factors must be used to determine the effective magnitude of the applied loading. Temperature factors should be applied to the strength of the material used. In like manner, all necessary modifications can be integrated into the design solution.

8. SUMMARY

Assuming that strength properties of materials, dimensions of stressed bodies, and applied loadings are normally distributed, equations have been developed which give the means and standard deviations for the "failure" stresses in the Maximum Shear Stress and Maximum Distortion Energy failure theories. In addition, the derived probabilistic form of the Soderberg, Goodman, Gerber, and elliptical design relationships allows the "failure" stresses to be combined with strength properties of materials to permit designing for any desired degree of reliability.

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Table I Nomenclature

| Name of Variable | Symbols Mean | Standard Deviation |
|---|---|---|
| Mean stress, psi | $\sigma_m$ | $s_{\sigma_m}$ |
| Alternating stress, psi | $\sigma_a$ | $s_{\sigma_a}$ |
| Yield strength,psi | $\sigma_y$ | $s_{\sigma_y}$ |
| Ultimate strength, psi | $\sigma_u$ | $s_{\sigma_u}$ |
| Endurance strength, psi | $\sigma_e$ | $s_{\sigma_e}$ |
| Maximum principal stress, psi | $\sigma_1$ | $s_{\sigma_1}$ |
| Intermediate principal stress, psi | $\sigma_2$ | $s_{\sigma_2}$ |
| Minimum principal stress, psi | $\sigma_3$ | $s_{\sigma_3}$ |

Table II

Parameters for Illustrative Application

| | Mean | Range | Standard Deviation |
|---|---|---|---|
| Radius, in | r | ±0.045r | 0.015r |
| Yield strength, psi | 129,000 | ±9,600 | 3,200 |
| Ultimate strength, psi | 140,000 | ±9,600 | 3,200 |
| Endurance strength, psi | 71,000 | ±10,500 | 3,500 |

Table III

Mean Radius and Range

for

Selected Levels of Reliability

for

Four Design Relationships

Radius, in

| Reliability | Soderberg | Goodman | Gerber | Elliptical |
|---|---|---|---|---|
| 0.50 | 0.945±0.042 | 0.932±0.042 | 0.867±0.039 | 0.835±0.038 |
| 0.90 | 0.977±0.044 | 0.965±0.043 | 0.896±0.040 | 0.852±0.038 |
| 0.95 | 0.996±0.045 | 0.974±0.044 | 0.904±0.041 | 0.857±0.039 |
| 0.99 | 1.002±0.045 | 0.990±0.045 | 0.918±0.041 | 0.865±0.039 |
| 0.999 | 1.019±0.046 | 1.007±0.045 | 0.933±0.042 | 0.875±0.039 |
| 0.9999 | 1.033±0.046 | 1.020±0.046 | 0.945±0.043 | 0.882±0.040 |
| 0.99999 | 1.045±0.047 | 1.032±0.046 | 0.955±0.043 | 0.888±0.040 |

Table IV

Effect of Loading Deviation

on

Reliability of a Shaft

| Standard Deviation of Loading, percent: Mean | Alternating | Reliability |
|---|---|---|
| 10 | 10 | 0.9999 |
| 10 | 20 | 0.9892 |
| 10 | 30 | 0.9468 |
| 20 | 10 | 0.9964 |
| 20 | 20 | 0.9738 |
| 20 | 30 | 0.9298 |
| 30 | 10 | 0.9738 |
| 30 | 20 | 0.9443 |
| 30 | 30 | 0.9039 |

(Goodman Relationship, Max. Shear Stress Theory, r = 1.020 ±0.046 in)

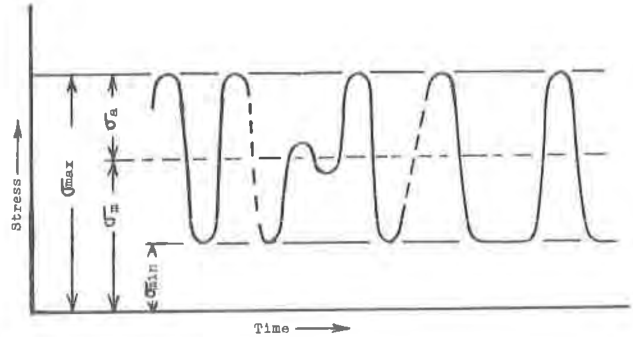


Fig. 1 Examples of Stress Reversal (Schematic)

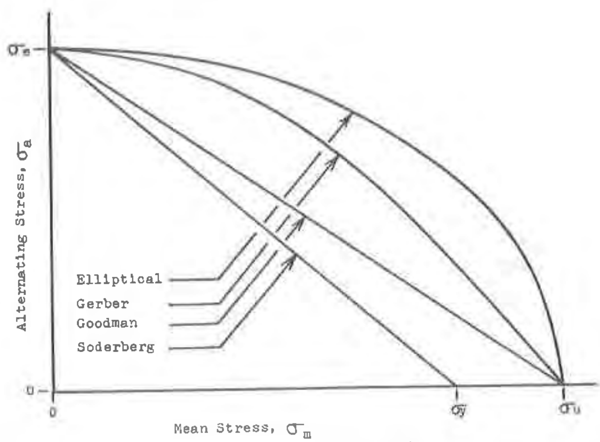


Fig. 2 Relationship between mean and alternating stresses to cause failure as determined by four design relationships

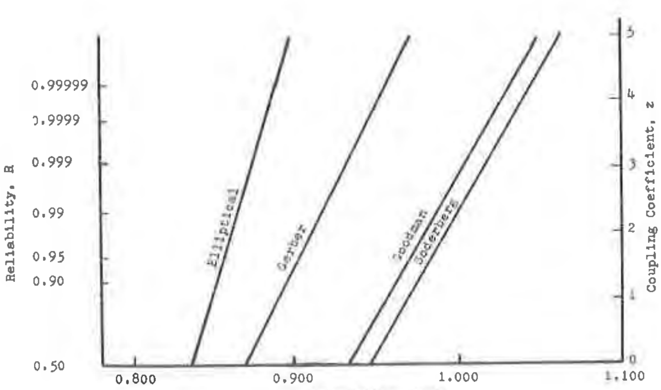


Fig. 3 Relationship between mean radius and reliability for a circular shaft under variable bending and torsion for four design relationships