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2. Present address: Department of Mathematics, Ohio State University, Columbus, Ohio 43210.

A CLASS OF GEOMETRIC LATTICES BASED ON FINITE GROUPS

by

T. A. Dowling[1,2]

Department of Statistics
University of North Carolina at Chapel Hill

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ABSTRACT

For any finite group G and positive integer n a finite geometric lattice $Q_n(G)$ of rank n, the lattice of partial G-partitions, is constructed. Let P_{n+1} be the lattice of partitions of an $(n+1)$-set. There exists a surjection $\pi: Q_n(G) \to P_{n+1}$, and an injection $\iota: P_{n+1} \to Q_{n+1}(G)$, each of which preserves order and rank. When G is the trivial group, $\pi = \iota^{-1}$ reduces to an isomorphism. The interval structure, Möbius function, and characteristic polynomial of $Q_n(G)$ are determined, and Stirling-like identities for the Whitney numbers obtained. The existence of a Boolean sublattice of modular elements in $Q_n(G)$ is established, implying that $Q_n(G)$ is supersolvable. It is further shown that nonisomorphic groups give nonisomorphic lattices, and the representation problem is solved completely: $Q_n(G)$ is representable over a field when $n \geq 3$ if and only if G is isomorphic to a subgroup of the multiplicative group of the field. Consequently, $Q_n(G)$ is representable over no field iff G is noncyclic, and if G is cyclic of order m, then $Q_n(G)$ is representable over (a) every field iff $m = 1$, (b) a finite field of order q iff m divides $q-1$, (c) the rational or real fields iff $m = 1$ or 2, and (d) the complex field for all m.

1. INTRODUCTION

The set P_n of all partitions of an n-element set, when ordered by refinement, is a well-known geometric lattice enjoying a number of structural properties. Every upper interval of a partition lattice is a partition lattice, and in general, every interval is a direct product of partition lattices. The Whitney numbers of the partition lattices are the familiar Stirling numbers, and the characteristic polynomial is simply a descending factorial, hence all its roots are integers. The set of partitions with a single nontrivial block is a Boolean sublattice of modular elements, so the partition lattice is supersolvable in the sense of Stanley [10]. Because of these and other structural properties, the partition lattices occupy a middle ground between the highly-structured projective (connected modular) geometric lattices and arbitrary geometric lattices, thereby exhibiting some of the consequences of the departure from modularity while retaining sufficient structure to facilitate their study and test conjectures.

We describe in this paper for any finite group G a class of finite geometric lattices, here called the partial G-partition lattices, which share a number of the properties of the partition lattices. Following a review in Section 2 of preliminary results on ordered sets and geometric lattices, the lattice $Q_n(G)$ of partial G-partitions of an n-set, a geometric lattice of rank n, is defined and its structure investigated in Section 3. There the existence of a surjective map $Q_n(G) \to P_{n+1}$ and an injective map $P_{n+1} \to Q_n(G)$, both of which preserve order and rank, is demonstrated. The injection embeds P_{n+1} in $Q_n(G)$ both as a sublattice and a subgeometry, and both maps reduce to isomorphisms when G is the trivial group of one element. The nature of covers and the interval structure in $Q_n(G)$ is also examined in Section 3. In Section 4, we prove the existence of a Boolean sublattice of modular elements in $Q_n(G)$, implying its supersolvability, determine its Möbius function and characteristic polynomial, and show that the Whitney numbers of the partial G-partition lattices satisfy recursions and inverse relations analogous to those of the Stirling numbers. Section 5 is devoted primarily to the representation problem of $Q_n(G)$, following a description of the structure of the rank three (planar) geometries $Q_3(G)$ and a proof that nonisomorphic groups of the same order result in nonisomorphic lattices with the same Whitney numbers. We show that when $n \geq 3$, $Q_n(G)$ is representable over a field F iff G is isomorphic to a subgroup of the multiplicative group of F. As a result, $Q_n(G)$ is not representable over any field unless G is cyclic. Thus simply by taking G noncyclic, we obtain an infinite class of moderately-structured geometric lattices which are not subgeometries of any projective geometry.

The results of this paper generalize to arbitrary finite groups many of the results in our earlier paper [5], which in the present context dealt with the case where G is the multiplicative group of a given finite field. Theorem 6 and the specializations of Theorems 1-5, 7 and 10 to that case appear in [5]; Theorems 8, 9, and 11 have no counterpart there.

Although most of the extensions of the results in [5] to an arbitrary finite group are straightforward, we include them here not only to make the present paper self-contained, but also because of differences in notation, terminology, and definitions required for the general case.

2. PRELIMINARIES

We collect in this section a number of results and definitions required later. For further details the reader is referred to [2,7].

A *preordered set* $(P', \le)$ is a set P' together with a reflexive, transitive relation, written $x \le y$. When the order is implicit, we write simply P' for $(P', \le)$. P' is an (partially) *ordered set* if $\le$ is also antisymmetric. Every preordered set P' is canonically associated with an ordered set, namely the quotient set P of E-classes of P', where E is the equivalence relation $x \, E \, y$ iff $x \le y$, $y \le x$. An ordered set P is *finite* if P is a finite set. With the exception of the projective lattices $L_n(F)$ in Section 5, in the case where F an infinite field, all ordered sets considered here are finite. The *direct product* of two ordered sets P, Q is the set $P \times Q$ with order $(u,v) \le (x,y)$ iff $u \le x$ in P and $v \le y$ in Q. An *interval* $[x,y]$ of an ordered set P is the ordered subset $[x,y] = \{z | x \le z \le y\}$ with the order of P restricted to $[x,y]$, defined whenever $x \le y$ in P. An element y in P is a *cover* of x (or *covers* x) iff $y > x$ and $[x,y] = \{x,y\}$. A finite or countable subset $C = \{x_0, x_1, \ldots\}$ of P is a *chain* if it is totally ordered in P: $x_0 < x_1 < \ldots$. If C is finite, the *length* of C is one less than its cardinality. P has *finite height* if all chains in P are finite. Suppose P has finite height and $C = \{x_0, x_1, \ldots, x_n\}$ is a chain in P. Then C is a *maximal chain* in $[x,y]$ iff $x_0 = x$, $x_n = y$, and x_i covers x_{i-1} for all $i = 1, \ldots, n$. An ordered set P satisfies the *chain condition* if it has finite height and all maximal chains in any interval $[x,y]$ have the same length. If P has a *zero element* 0 ($0 \le x$ for all $x \in P$) and satisfies the chain condition, the *rank* $\rho(x)$ of an element $x \in P$ is the length of all maximal chains in $[0,x]$. If P has a *unit element* 1 ($x \le 1$ for all $x \in P$) and satisfies the chain condition, the *corank* of an element $x \in P$ is the length of all maximal chains in $[x,1]$.

Let P have a 0 and 1. An *atom* of P is an element covering 0. A *coatom* of P is an element covered by 1. P is a *lattice* iff any two elements x, y have a unique minimal upper bound $x \vee y$, called their *supremum*, and a unique maximal lower bound $x \wedge y$, called their *infimum*. A subset M of a lattice P is a *sublattice* of P iff M is a lattice under the order of P and suprema and infima in M agree with those of L.

If P, L are ordered sets, a function $\phi: P \to L$ is *order-preserving* when $x \leq y$ implies $\phi(x) \leq \phi(y)$. If both P, Q satisfy the chain condition, ϕ is *rank-preserving* if $\rho(\phi(x)) = \rho(x)$ for all $x \in P$. P and L are *isomarphic*, written $P \cong L$, iff there is a bijection $\phi: P \to L$ such that both ϕ and ϕ^{-1} preserve order.

A lattice L is *complete* if every subset has a supremum and infimum, and *atomic* if every element x is the supremum of the set of atoms in $[0,x]$. L is *semimodular* if $x \vee y$ covers both x and y whenever x and y cover $x \wedge y$. A *geometric lattice* is a complete, atomic, semimodular lattice of finite height. A finite lattice is geometric when y covers x iff $y = x \vee p$ for some atom $p \nleq x$. A geometric lattice L satisfies the chain condition and its rank function obeys the *semimodular inequality*:
$\rho(x \vee y) + \rho(x \wedge y) \leq \rho(x) + \rho(y)$ for all $x,y \in L$. Elements of rank 1 (atoms), 2, 3 are points, lines, planes, respectively, and elements of corank 1 (coatoms), 2, 3, are copoints, colines, coplanes, respectively. Every interval of a geometric lattice is geometric.

A *combinatorial geometry* is a set S of "points" together with a closure operator $A \mapsto \bar{A}$ on subsets of S satisfying (a) the *exchange property:* if $p,q \in S$, $A \subseteq S$, and $q \in \overline{A \cup p}$ but $q \notin \bar{A}$, then $p \in \overline{A \cup q}$, (b) the *finite basis property:* if $A \subseteq S$ there exists a finite subset A_0 of A such that $\bar{A}_0 = \bar{A}$, and (c) the empty set and all singleton subsets of S are closed. A subset A of S is *independent* if $\overline{A-p} \neq \bar{A}$ for all $p \in A$, and *dependent* otherwise. All maximal independent subsets of any set A, called *bases* of A, have the same cardinality, the *rank* of A. A *subgeometry* of a combinatorial geometry on S is a subset T of S with closure operator $A \mapsto \bar{A} \cap T$. A subset of T is independent in the subgeometry on T iff it is independent in the original geometry.

The set of closed sets of a combinatorial geometry, ordered by inclusion, is a geometric lattice. Conversely, every geometric lattice L defines a geometry on its set S of points by $\bar{A} = \{p \mid p \leq \sup A\}$. The lattice of the subgeometry on T consists of all elements $x \in L$ such that $x = \sup A$ for some subset A of T. It is not in general a sublattice of L. We shall identify a geometry with its (geometric) lattice of closed sets. A *minor* of L is a subgeometry of some interval of L.

If P, L are geometric lattices, an *injective strong map* is an injection $\sigma: P \to L$ which takes points to points and preserves suprema: $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$. In this case P is isomorphic to its σ-image in L, the latter a subgeometry of L. A projective geometry of dimension n-1 over a field F is a combinatorial geometry of rank n. We denote its lattice by $L_n(F)$. A *representation* of a rank n geometry P over F is an injective strong map $\sigma: P \to L_n(F)$. Equivalently, P is representable over F iff there exists an injection (called a *coordinatization*) $\phi: S \to F^n$, where S is the point set of P, such that a subset A of S is independent in P iff its image $\phi(A)$ is linearly independent in F^n. If P is representable over F, every minor of P is representable over F.

Let X be a finite set of n elements. A *partition* of X is a set $\pi = \{A_1, .., A_r\}$ of disjoint, nonempty subsets of X with $\bigcup_{j=1}^{r} A_j = X$. The subsets A_j are the *blocks* of π. There is an obvious correspondence between partitions of X and equivalence relations defined on X, the blocks of the partition being the equivalence classes.

The set P_n of all partitions of X is (partially) ordered by *refinement*: $\pi_1 \leq \pi_2$ iff every π_2-block is a union of π_1-blocks. So ordered, P_n is a geometric lattice of rank $n-1$, with zero element the partition of X into singleton subsets (the identity relation) and unit element the single block partition $\{X\}$ (the universal relation).

The supremum and infimum of two partitions $\pi_1 = \{A_1,\ldots,A_r\}$, $\pi_2 = \{B_1,\ldots,B_s\}$ is easily found by means of the *intersection graph* of the A_j $(j = 1,\ldots,r)$ versus the $B_k(k = 1,\ldots,s)$. This is the bipartite graph with vertices $A_1,\ldots,A_r$, $B_1,\ldots,B_s$, and edges the set of all pairs $\{A_j,B_k\}$ such that $A_j \cap B_k \neq \emptyset$. Then a block of $\pi_1 \wedge \pi_2$ is a subset $A_j \cap B_k$, for any edge $\{A_j,B_k\}$. A block of $\pi_1 \vee \pi_2$ is a union $\cup A_j$ over all A_j in a connected component of the graph.

The lattice P_n of partitions of X is a geometric lattice of rank $n-1$. The rank function is $\rho(\pi) = n-|\pi|$. A partition π_2 covers π_1 iff π_2 can be obtained from π_1 by replacing two π_1-blocks by their union.

3. THE LATTICE $Q_n(G)$ OF PARTIAL G-PARTITIONS

Let $X = \{x_1,\ldots,x_n\}$ be a finite set of n elements. By a *partial partition of* X we shall mean a set $\alpha = \{A_1,\ldots,A_r\}$ of disjoint, nonempty subsets of X, i.e. a partition of a subset $\cup_{j=1}^{r} A_j$ of X. The subsets A_j are the *blocks* of α. The set Q_n of all partial partitions of X is (partially) ordered by $\alpha \leq \beta$ iff every β-block is the union of a set of α-blocks, i.e. iff for each $B_k \in \beta$ there exists a nonempty subset α_k of α such that $B_k = \cup_{\alpha_k} A_j$. So ordered, Q_n is isomorphic to the lattice P_{n+1} of partitions of an $(n+1)$-set $X \cup \{x_0\}$, the isomorphism $P_{n+1} \to Q_n$ is given simply by deleting from each partition $\{A_0 \cup \{x_0\}, A_1,\ldots,A_r\}$ of

$X \cup \{x_0\}$ the distinguished block $A_0 \cup \{x_0\}$ containing x_0. We refer to the block of any partition of $X \cup \{x_0\}$ which contains x_0 as the *zero block* of the partition. Formally, we define the inverse map $\phi\colon Q_n \to P_{n+1}$ by

$$(3.1) \qquad \phi(\alpha) \;=\; \{A_0 \cup \{x_0\}, A_1, \ldots, A_r\},$$

where $\alpha = \{A_1,\ldots,A_r\}$, $A_0 = X - \bigcup_{j=1}^{r} A_j$. The partition $\phi(\alpha)$ is the supremum in P_{n+1} of all completions of α to a partition of $X \cup \{x_0\}$. Note that ϕ takes an r-block partial partition of X to an (r+1)-block partition of $X \cup \{x_0\}$, so α has Q_n-rank $\rho(\alpha) = (n+1)-(r+1) = n-r$, i.e.

$$(3.2) \qquad \rho(\alpha) \;=\; n - |\alpha|.$$

Thus the empty partial partition is the unit element of Q_n and the partial partition $\varepsilon = \{E_1,\ldots,E_n\}$ of X into its singleton subsets $E_i = \{x_i\}$ is the zero element of Q_n. Every subset of a partial partition α is a partial partition $\geq \alpha$ in Q_n.

Covers in Q_n are of two types. A I-cover of $\alpha = \{A_1,\ldots,A_r\}$ is obtained by deleting some block A_j from α, while a II-cover is obtained by replacing two blocks A_j, A_k of α by their union $A_j \cup A_k$. The ϕ-image of a I-cover of α is obtained simply by combining some $A_j \in \alpha$ with the zero block of $\phi(\alpha)$.

Now let G be a finite (multiplicative) group, with unit element 1. Elements of G will be denoted $\kappa, \lambda, \mu, \ldots$ with or without subscripts. We define a *partial* G-*partition of* X as a set

$$(3.3) \qquad \alpha \;=\; \{a_j\colon A_j \to G \mid j = 1,\ldots,r\}$$

of functions into G for which the domains A_j are disjoint, nonempty subsets of X. Thus

$$(3.4) \qquad \pi(\alpha) = \{A_j | j = 1,\ldots,r\}$$

is a partial partition of X. Let $Q_n'(G)$ denote the set of all partial G-partitions of X. The map $\pi: Q_n'(G) \to Q_n$, defined by (3.4), takes each partial G-partition of X to its underlying partial partition of X.

To simplify expressions encountered below, we adopt the convention that the domains of functions a_j, b_k, c_ℓ etc. are always denoted by the capitals A_j, B_k, C_ℓ, etc. of the letters denoting the functions, with appropriate subscripts. Thus for example the element $\alpha \in Q_n'(G)$ given by (3.3) may be written simply $\alpha = \{a_j | j = 1,\ldots,r\}$.

If $\alpha = \{a_j | j = 1,\ldots,r\}$ is a partial G-partition of X, and α_k is any non-empty subset of α, a (left-) *linear combination* (over G) *of* α_k will be a function $b_k: B_k \to G$, where $B_k = \bigcup_{\alpha_k} A_j$, such that the restriction of b_k to A_j is a (left-) G-multiple $\lambda_j a_j$ of a_j, i.e., $b_k(x_i) = \lambda_j a_j(x_i)$ for all $x_i \in A_j$. In this case we write $b_k = \sum_{\alpha_k} \lambda_j a_j$. The summation sign is to be interpreted as the "domain-disjoint union" of the functions following it; no addition operation in G is assumed.

Let $E_i = \{x_i\}$, $i = 1,\ldots,n$, and define the *unit functions* $e_i: E_i \to G$ by $e_i(x_i) = 1$ for each $i = 1,\ldots,n$. Let $\varepsilon = \{e_i | i = 1,\ldots,n\}$. Then any function $a_j: A_j \to G$ may be written as a linear combination $a_j = \sum_{\varepsilon_j} \kappa_i e_i$ of the unit functions, where $\varepsilon_j = \{e_i | E_i \subseteq A_j\}$ and $\kappa_i = a_j(x_i)$.

The analogue in $Q_n'(G)$ of the order relation of Q_n is then the following: $\alpha \le \beta$ iff every β-function is a linear combination of a set of α-functions, i.e., iff for each $b_k \in \beta$ there exists a subset α_k of α and elements $\lambda_j \in G$ such that $b_k = \sum_{\alpha_k} \lambda_j a_j$. The relation $\le$ is clearly reflexive and transitive, hence is a preorder on $Q_n'(G)$. Suppose $\alpha \le \beta$. Then for each $b_k \in \beta$, $b_k = \sum_{\alpha_k} \lambda_j a_j$, so $B_k = \biguplus_{\alpha_k} A_j$. It follows that $\pi(\alpha) \le \pi(\beta)$ in Q_n. Thus if $\alpha \le \alpha'$, $\alpha' \le \alpha$, then $\pi(\alpha) = \pi(\alpha')$ and there exists a bijection $a'_j \leftrightarrow a_j$ such that $a'_j = \lambda_j a_j$, $a_j = \lambda_j^{-1} a'_j$. Let E denote this equivalence relation: $\alpha E \alpha'$ iff $\alpha \le \alpha'$, $\alpha' \le \alpha$, and let (α) denote the E-class containing α. Any member α' of an E-class (α) is uniquely determined up to scalar multiples of its elements. The situation is analogous to that of a set of homogeneous coordinate vectors for some set of points in a projective geometry. The preorder $\le$ on $Q'_n(G)$ induces a partial order, also denoted $\le$, on the quotient set $Q_n(G) = Q'_n(G)/E$ of E-classes in the usual way:

$$(\alpha) \le (\beta) \quad \text{iff} \quad \alpha \le \beta.$$

We will be concerned primarily with the ordered set $Q_n(G)$ henceforth, but proofs will often be given in terms of $Q'_n(G)$ and its preorder, with E-equivalence replacing equality.

Any function f on $Q'_n(G)$ which is constant on E-classes will be taken as a function on $Q_n(G)$. To avoid double parentheses, we write $f(\alpha)$ for the f-image of $(\alpha) \in Q_n(G)$. Then from the above, $\pi\colon Q_n(G) \to Q_n$ is order-preserving: $(\alpha) \le (\beta)$ implies $\pi(\alpha) \le \pi(\beta)$.

Given any nonempty subset A_j of X, the *indicator function* of A_j we define as the function $\iota_{A_j}\colon A_j \to G$ with $\iota_{A_j}(x_i) = 1$ for all $x_i \in A_j$. The set of indicator functions $\{\iota_{A_j} | j = 1,\dots,r\}$ of a partion partition $\{A_j | j = 1,\dots,r\}$ of X will be called the *indicator set* of $\{A_j | j = 1,\dots,r\}$. The E-class of the indicator set thus consists of all partial G-partitions of constant functions with π-image $\{A_j | j = 1,\dots,r\}$. The map $\iota\colon Q_n \to Q_n(G)$ assigning to each partial partition of X the E-class of its indicator set is clearly injective and order-preserving, as disjoint unions $\cup A_j$ correspond to sums $\sum \iota_{A_j}$. The composite $\pi \circ \iota\colon Q_n \to Q_n$ is the identity on Q_n, so π is surjective. If $G = 1$ is the trivial group, the ι-image of a partial partition is its only π-preimage, so π is an isomorphism $Q_n(1) \cong Q_n$, hence $\phi \circ \pi$ is an isomorphism $Q_n(1) \cong P_n$.

As in Q_n, covers in $Q_n(G)$ are of two types. A I-cover of (α) is obtained by deleting some α_j from α, while a II-cover of (α) is obtained by replacing two functions α_j, α_k of α by a linear combination $\alpha_j + \lambda\alpha_k$ (by $\mathcal{E}$-equivalence, the coefficient of α_j may be taken as 1). In either case, the covering element (β) has $|\beta| = |\alpha|-1$. Thus $Q_n(G)$ satisfies the chain condition: all maximal chains in any interval $[(\alpha),(\beta)]$ of $Q_n(G)$ have the same length $|\alpha|-|\beta|$. The zero element of $Q_n(G)$ is (ε), where $\varepsilon = \{e_i | i = 1,\ldots,n\}$ is the set of unit functions. The rank function of $Q_n(G)$ is therefore

(3.5) $\rho(\alpha) = n - |\alpha|$.

Note that (3.5) is also the rank (3.2) of $\pi(\alpha)$ in Q_n, so π preserves rank. Clearly, ι does also. We summarize these results in

THEOREM 1: Let $X = \{x_1,\ldots,x_n\}$ be a finite set of n elements, G a finite multiplicative group, P_{n+1} the lattice of partitions of $X \cup \{x_0\}$, $Q_n \cong P_{n+1}$ the lattice of partial partitions of X, and $Q_n(G)$ the ordered set of $\mathcal{E}$-classes of partial G-partitions of X. Then

(a) An element (β') in $Q_n(G)$ covers (α) iff $(\beta') = (\beta)$ for some β of the form

(3.6) $\beta = \alpha - \{a_j\}$, (I-cover)

or

(3.7) $\beta = \alpha - \{a_j,a_k\} \cup \{a_j+\lambda a_k\}$, (II-cover)

where $a_j,a_k \in \alpha$, $\lambda \in G$.

(b) $Q_n(G)$ satisfies the chain condition, with rank function

$$\rho(\alpha) = n - |\alpha|.$$

(c) The map $\pi: Q_n(G) \to Q_n$, which assigns to each $\mathcal{E}$-class of partial G-partitions its underlying partial partition, is surjective and preserves order and rank.

(d) The map $\iota: Q_n \to Q_n(G)$, which assigns to each partial partition the $\mathcal{E}$-class of its indicator set, is injective and preserves order and rank.

(e) If $G = \mathit{1}$ is the trivial group, $\pi = \iota^{-1}$ is an isomorphism $Q_n(\mathit{1}) \cong Q_n$, so $\phi\circ\pi$ is an isomorphism $Q_n(\mathit{1}) \cong P_{n+1}$, where $\phi: Q_n \to P_{n+1}$ is the isomorphism (3.1).

COROLLARY 1.1: Each element of rank $n-r$ in $Q_n(G)$ is covered by

$$\binom{r}{1} + m\binom{r}{2}$$

elements of rank $n-r+1$, where m is the order of G.

COROLLARY 1.2: The atoms of $Q_n(G)$ are $(\alpha^{(i)})$, $(\alpha^{(ii')}(\lambda))$, where

$$\alpha^{(i)} = \varepsilon - \{e_i\},$$
$$\alpha^{(ii')}(\lambda) = \varepsilon - \{e_i, e_{i'}\} \cup \{e_i + \lambda e_{i'}\},$$

defined for all $1 \le i,\ i' \le n$, $i \ne i'$, $\lambda \in G$. Note that $(\alpha^{(ii')}(\lambda)) = (\alpha^{(i'i)}(\lambda^{-1}))$.

Our next theorem describes the structure of upper and lower intervals of $Q_n(G)$. From these the structure of an arbitrary interval can be obtained.

THEOREM 2: (a) If $(\alpha) \in Q_n(G)$ is of corank r, then

$$[(\alpha),1] \cong Q_r(G).$$

(b) Let $(\beta) \in Q_n(G)$, where

$$\beta = \{b_k\colon B_k \to G | k = 1,\ldots,s\}.$$

Let $B_0 = X - \biguplus_{k=1}^{s} B_k$ and $n_k = |B_k|$, $k = 0,1,\ldots,s$. Then

$$[0,(\beta)] \cong Q_{n_0}(G) \times P_{n_1} \times \ldots \times P_{n_s}.$$

Proof: (a) Let $\alpha = \{a_j\colon A_j \to G | j = 1,\ldots,r\}$. Then $(\beta) \in [(\alpha),1]$ iff each b_k in $\beta = \{b_k\colon B_k \to G | k = 1,\ldots,s\}$ is a linear combination $b_k = \sum_{\alpha_k} \lambda_j a_j$ of a subset α_k of α. Every such function b_k corresponds in a one-one manner to a function $b_k^{(\alpha)}\colon \{A_j | A_j \subseteq B_k\} \to G$ on a nonempty subset of the r-set $\{A_1,\ldots,A_r\}$, namely $b_k^{(\alpha)}(A_j) = \lambda_j$. In particular, the $a_j^{(\alpha)}$ are the unit functions of $\{A_1,\ldots,A_r\}$. This correspondence preserves linear combinations: $(\sum \lambda_k b_k)^{(\alpha)} = \sum \lambda_k b_k^{(\alpha)}$, so the map $(\beta) \to (\beta^{(\alpha)})$, where $\beta^{(\alpha)} = \{b_k^{(\alpha)} | b_k \in \beta\}$ is an isomorphism $[(\alpha),1] \cong Q_r(G)$.

(b) For any $(\alpha) \in [0,(\beta)]$, we have $\pi(\alpha) \leq \pi(\beta)$. Every partial partition $\xi \leq \pi(\beta)$ consists of partitions $\xi_k = \{A_{kj} | j = 1,\ldots,r_k\}$ of B_k, $k = 1,\ldots,s$, together with a partial partition $\xi_0 = \{A_{0j} | j = 1,\ldots,r_0\}$ of B_0. If $(\alpha) \in [0,(\beta)]$ and $\pi(\alpha) = \xi$, then α must be of the following form: a_{0j} ($j = 1,\ldots,r_0$) is arbitrary, while a_{kj} ($k = 1,\ldots,s$; $j = 1,\ldots,r_k$) is uniquely determined up to scalar multiples as the restriction of b_k to A_{kj}. Hence by E-equivalence there is a one-one correspondence between $[0,(\beta)]$ and $Q_{n_0}(G) \times P_{n_1} \times \ldots \times P_{n_s}$. The order in $[0,(\beta)]$ is the product of the orders in B_0 and $B_1,\ldots,B_s$. Clearly the order in B_0 is that of $Q_{n_0}(G)$, and the order in B_k $(k = 1,\ldots,s)$ is that of P_{n_k}. □

COROLLARY 2.1: Let $(\alpha) \leq (\beta)$ in $Q_n(G)$, where $\alpha = \{a_j | j = 1,\ldots,r\}$, $\beta = \{b_k | k = 1,\ldots,s\}$. If b_k is a linear combination of $\alpha_k \subseteq \alpha$, let

$r_k = |\alpha_k|$, $k = 1,\ldots,s$, and $r_0 = |\alpha_0|$, where $\alpha_0 = \alpha - \cup_{k=1}^{s}\alpha_k$. Then

$$[(\alpha),(\beta)] \cong Q_{r_0}(G) \times P_{r_1} \times \ldots \times P_{r_k}.$$

COROLLARY 2.2: Let (β) be a copoint of $Q_n(G)$, where $\beta = \{b\colon B \to G\}$. If $B = X$, then

$$[0,(\beta)] \cong Q_{n-1} \cong P_n,$$

while if $B = \{x_i\}$, then

$$[0,(\beta)] \cong Q_{n-1}(G).$$

COROLLARY 2.3: Let $(\beta) \in Q_n(G)$, where

$$\beta = \{b_k = \sum_{\varepsilon_k} \kappa_i e_i\colon B_k \to G | k = 1,\ldots,s\}.$$

Let $B_0 = X - \cup_{k=1}^{s} B_k$. Then the atoms of $[0,(\beta)]$ are

(a) $(\alpha^{(ii')}(\kappa_i^{-1}\kappa_{i'}))$, for all i, i' such that $x_i, x_{i'} \in B_k$, $k = 1,\ldots,s$;

(b) $(\alpha^{(i)})$, for all i such that $x_i \in B_o$;

(c) $(\alpha^{(ii')}(\lambda))$, for all i, i' such that $x_i, x_{i'} \in B_0$, and all $\lambda \in G$.

THEOREM 3: $Q_n(G)$ is a geometric lattice.

Proof: We prove first that $Q_n(G)$ is a lattice. Let $(\alpha),(\beta) \in Q_n(G)$. Since π preserves order, $\pi(\gamma) \geq \pi(\alpha) \vee \pi(\beta)$ for any upper bound (γ) of (α), (β). Let $\pi(\alpha) \vee \pi(\beta) = \{C_\ell | \ell = 1,\ldots,t\}$, and suppose $\{C_\ell | \ell = 1,\ldots,u\}$, where $u \leq t$, are the blocks of $\pi(\alpha) \vee \pi(\beta)$ such that there exists a function $c_\ell\colon C_\ell \to G$ which is simultaneously a linear combination of $\alpha_\ell \subseteq \alpha$ and $\beta_\ell \subseteq \beta$, where α_ℓ, β_ℓ are defined by

$$C_\ell = \bigcup_{j:a_j \in \alpha_\ell} A_j = \bigcup_{k:b_k \in \beta_\ell} B_k$$

Then if $\gamma = \{c_\ell \colon C_\ell \to G | \ell = 1,\ldots,u\}$, (γ) is clearly a minimal upper bound of (α), (β) in $Q_n(G)$. To show that $(\alpha) \vee (\beta)$ exists, we must prove that the c_ℓ, $\ell = 1,\ldots,u$, are uniquely defined up to scalar multiples. Suppose then that

$$c_\ell = \sum_{\alpha_\ell} \kappa_j a_j = \sum_{\beta_\ell} \lambda_k b_k,$$
$$c'_\ell = \sum_{\alpha_\ell} \kappa'_j a_j = \sum_{\beta_\ell} \lambda'_k b_k$$

are two such functions $C_\ell \to G$. Let $x_i \in C_\ell$, and define A_j, B_k by $x_i \in A_j \cap B_k$. Then

$$c_\ell(x_i) = \kappa_j a_j(x_i) = \lambda_k b_k(x_i),$$
$$c'_\ell(x_i) = \kappa'_j a_j(x_i) = \lambda'_k b_k(x_i),$$

hence

$$a_j(x_i)[b_k(x_i)]^{-1} = \kappa_j^{-1}\lambda_k = \kappa_j'^{-1}\lambda'_k,$$

or $\kappa'_j\kappa_j^{-1} = \lambda'_k\lambda_k^{-1}$. This latter equality must hold whenever $A_j \cap B_k$ is non-empty. But since C_ℓ is a block of $\pi(\alpha) \vee \pi(\beta)$, the intersection graph of the A_j $(j\colon a_j \in \alpha_\ell)$ versus the B_k $(k\colon b_k \in \beta_\ell)$ is connected. It follows then that the elements $\kappa'_j\kappa_j^{-1}$, $\lambda'_k\lambda_k^{-1}$ are equal, say to μ, for all j, k. Thus $\kappa'_j = \mu\kappa_j$, $\lambda'_k = \mu\lambda_k$, so $c'_\ell = \mu c_\ell$.

Consider next the infimum of (α) and (β). Let $\alpha = \{a_j | j = 1,\ldots,r\}$, $\beta = \{b_k | k = 1,\ldots,s\}$, and define $A_0 = X - \bigcup_{j=1}^r A_j$, $B_0 = X - \bigcup_{k=1}^s B_k$. As π preserves order, $\pi(\gamma) \le \pi(\alpha) \wedge \pi(\beta)$ for any lower bound (γ) of (α) and (β). We obtain the blocks of $\pi(\alpha) \wedge \pi(\beta)$ by deleting the zero block of $\phi\ (\pi(\alpha) \wedge \pi(\beta))$. The blocks of $\phi(\pi(\alpha) \wedge \pi(\beta)) = \phi \circ \pi(\alpha) \wedge \phi \circ \pi(\beta)$ are the non-empty intersections of the blocks of

$$\phi\circ\pi(\alpha) \;=\; \{A_0 \cup \{x_0\},\ A_1,\ldots,A_r\}$$

with the blocks of

$$\phi\circ\pi(\beta) \;=\; \{B_0 \cup \{x_0\},\ B_1,\ldots,B_s\}.$$

Let C be a block of $\pi(\alpha) \wedge \pi(\beta)$. If $C = C_{j0} = A_j \cap B_0$, for some $j = 1,\ldots,r$, define $c_{j0}\colon C_{j0} \to G$ as the restriction $a_j|C_{j0}$ of a_j to C_{j0}. Similarly, if $C = C_{0k} = A_0 \cap B_k$ for some $k = 1,\ldots,s$, define $c_{0k}\colon C_{0k} \to G$ by $c_{0k} = b_k|C_{0k}$. Finally if $C = C_{jk} = A_j \cap B_k$ for some $j = 1,\ldots,r$; $k = 1,\ldots,s$, define an equivalence relation R_{jk} on C_{jk} by

$$x_i R_{jk} x_{i'} \quad \text{iff} \quad \lambda_i\kappa_i^{-1} = \lambda_{i'}\kappa_{i'}^{-1},$$

where $a_j|C_{jk} = \sum_{\varepsilon_{jk}} \kappa_i e_i$, $b_k|C_{jk} = \sum_{\varepsilon_{jk}} \lambda_i e_i$. Then if $\mu_{jk\ell}$ is the value of $\lambda_i\kappa_i^{-1}$ on an R_{jk}-block (equivalence class) $C_{jk\ell}$ of C_{jk}, $\lambda_i = \mu_{jk\ell}\kappa_i$ for all $x_i \in C_{jk\ell}$. Thus $b_k|C_{jk\ell} = \mu_{jk\ell}\,(a_j|C_{jk\ell})$. Define $c_{jk\ell}\colon C_{jk\ell} \to G$ by $c_{jk\ell} = a_j|C_{jk\ell}$. It is clear that the partition of C_{jk} into its R_{jk}-blocks $C_{jk\ell}$ is the maximal partition of C_{jk} for which functions may be defined on the blocks which are simultaneously G-multiples of the restrictions of both a_j and b_k. Thus if γ is the set of functions c_{j0}, c_{0k}, $c_{jk\ell}$ defined above, then (γ) is the unique maximal lower bound of (γ) and (β), i.e., $(\gamma) = (\alpha) \wedge (\beta)$. It follows that $Q_n(G)$ is a lattice.

Recall now the form (3.6), (3.7) of covers of (α), and the definition of the atoms $(\alpha^{(i)})$, $(\alpha^{(ii')}(\lambda))$ of $Q_n(G)$ in Corollary 1.2. Then it is easily verified that (β) is a I-cover (3.6) of (α) iff

$$(\beta) \;=\; (\alpha) \vee (\alpha^{(i)})$$

for any i with $x_i \in A_j$, while (β) is a II-cover (3.7) of (α) iff

$$(\beta) \;=\; (\alpha) \vee (\alpha^{(ii')}(\lambda_{ii'}))$$

for any i, i' such that $x_i \in A_j$, $x_{i'} \in A_k$, where $\lambda_{ii'} = [a_j(x_i)]^{-1}\lambda a_k(x_{i'})$. Thus $Q_n(G)$ is geometric. □

COROLLARY 3.1: The partition lattice P_{n+1} is both a sublattice and a subgeometry of $Q_n(G)$.

Proof: It is clear from the proof of Theorem 3 that the injective map $\iota: Q_n \to Q_n(G)$ preserves suprema and infima, so the ι-image of Q_n is a sublattice of $Q_n(G)$. Since ι is also rank-preserving, it takes points to points, so $\iota(Q_n)$ is also a subgeometry of $Q_n(G)$. But then $\iota(Q_n) \cong Q_n \cong P_{n+1}$. □

We refer to $Q_n(G)$ as the lattice of partial G-partitions. The elements are, of course, E-classes of partial G-partitions.

4. THE MÖBIUS FUNCTION, CHARACTERISTIC POLYNOMIAL AND WHITNEY NUMBERS OF $Q_n(G)$

A *modular element* [9] of a geometric lattice L with rank function ρ is an element $x \in L$ such that the modular identity

$$\rho(x \vee y) + \rho(x \wedge y) = \rho(x) + \rho(y)$$

holds for $y \in L$. If x is a modular element, the map $z \mapsto x \vee z$ is an isomorphism $[x \wedge y, y] \cong [x, x \vee y]$ with inverse $w \mapsto w \wedge y$, for any $y \in L$. Every point of a geometric lattice is a modular element.

THEOREM 4: Let $\varepsilon = \{e_i: E_i \to G \mid i = 1,\dots,n\}$ be the set of unit functions of X, i.e., $E_i = \{x_i\}$, $e_i(x_i) = 1$, $i = 1,\dots,n$. Then the subset

$$M = \{(\alpha) | \alpha \subseteq \varepsilon\}$$

is a (Boolean) sublattice of $Q_n(G)$. Every element of M is modular in $Q_n(G)$.

Proof: Let $(\alpha) \in M$, say $\alpha = \{e_i | i = 1,\ldots,r\}$, and let $(\beta) \in Q_n(G)$, where

$$\beta = \{b_k\colon B_k \to G | k = 1,\ldots,s\}.$$

Then the blocks of $\pi(\alpha) \wedge \pi(\beta)$ are E_i, $i = 1,\ldots,r$, and $B_k \cap A_0$ for all $k = 1,\ldots,s$ such that $B_k \nsubseteq \{x_1,\ldots,x_r\}$, where $A_0 = \{x_{r+1},\ldots,x_n\}$. The blocks of $\pi(\alpha) \vee \pi(\beta)$ are all B_k, $k = 1,\ldots,s$, such that $B_k \subseteq \{x_1,\ldots,x_r\}$. It is clear from the proof of Theorem 3 that $\pi((\alpha)\wedge(\beta)) = \pi(\alpha) \wedge \pi(\beta)$ and that $\pi((\alpha)\vee(\beta)) = \pi(\alpha) \vee \pi(\beta)$. The total number of blocks in these two partial partitions is $r+s = |\alpha|+|\beta|$, so $|\alpha\vee\beta|+|\alpha\wedge\beta| = |\alpha|+|\beta|$. Since $\rho(\gamma) = n-|\gamma|$ is the rank function of $Q_n(G)$, (α) is modular. If also $(\beta) \in M$, then each b_k is a unit function e_k, so each B_k is a singleton subset E_k, and we have $(\alpha) \wedge (\beta) = (\alpha\cup\beta) \in M$, $(\alpha) \vee (\beta) = (\alpha\cap\beta) \in M$. Thus M is a sublattice, and the map $(\alpha) \mapsto \{e_i | e_i \notin \alpha\}$ is an *anti-isomorphism* from M to the Boolean lattice of subsets of $\varepsilon = \{e_1,\ldots,e_n\}$. □

The *Möbius function* [7] $\mu\colon L\times L \to \mathbb{Z}$ of a finite partially order set L is defined recursively by $\mu(x,x) = 1$, $\mu(x,y) = 0$ if $x \nleq y$, and

$$\mu(x,y) = - \sum_{z:x\leq z<y} \mu(x,z)$$

if $x \leq y$. If L is a geometric lattice of rank n with rank function ρ, the *characteristic polynomial* [4] of L is

$$p_L(\nu) = \sum_{x\in L} \mu(0,x)\nu^{n-\rho(x)}.$$

The characteristic polynomial extends to geometric lattices the notion of the chromatic polynomial of a graph. In particular, if L is the lattice of contractions [7] of a linear graph Γ with k components, then the chromatic polynomial of Γ is $\chi(\Gamma) = \nu^k p_L(\nu)$. The lattice of partitions P_{n+1} is (isomorphic to) the lattice of contractions of the complete graph K_{n+1} with chromatic polynomial $\nu(\nu-1) \ldots (\nu-n)$, so the characteristic polynomial of P_{n+1} is $(\nu-1)(\nu-2)\ldots(\nu-n)$, i.e., the falling factorial $(\nu-1)_{(n)}$. We may obtain the characteristic polynomial $p_n(\nu,m)$ of $Q_n(G)$ for an arbitrary finite group G of order m (recall $P_{n+1} \cong Q_n(1)$) with the aid of the following special case of a result due to Crapo ([3], Th. 6, Cor. 5):

If L is a finite geometric lattice of rank n and c is a copoint of L, then

$$(4.1) \qquad \nu p_{[0,c]}(\nu) = \sum_{x:x\wedge c=0} p_{[x,1]}(\nu),$$

where $p_{[a,b]}(\nu)$ is the characteristic polynomial of the interval $[a,b]$ of L.

THEOREM 5: If G is of order m, the characteristic polynomial of $Q_n(G)$ is

$$(4.2) \qquad p_n(\nu;m) = \prod_{i=0}^{n-1} (\nu-1-mi) = m^n\left(\frac{\nu-1}{m}\right)_{(n)},$$

where $(x)_{(n)}$ is the falling factorial $x(x-1)\ldots(x-n+1)$.

Proof: We take as our c in (4.1) the copoint (β), where $\beta = \{e_1\}$. By Corollary 2.2, $[0,(\beta)] \cong Q_{n-1}(G)$. Since $(\beta) \in M$ is modular, by Theorem 4, $(\beta) \wedge (\alpha) = 0$ iff $(\alpha) = 0$ or (α) is an atom of $Q_n(G)$ not in $[0,(\beta)]$. The number of such atoms is

$$\binom{n}{1} + m\binom{n}{2} - \binom{n-1}{1} - m\binom{n-1}{2},$$

i.e., $1+m(n-1)$. By Theorem 2, $[(\alpha),1] \cong Q_{n-1}(G)$ for every atom (α). Thus (4.1) gives

$$(4.3) \qquad p_n(\nu;m) = (\nu-1-m(n-1))p_{n-1}(\nu;m).$$

Since $p_1(\nu;m) = \nu-1$, we obtain (4.2) by iteration of (4.3). □

Stanley [10] has recently investigated the class of finite geometric lattices containing a maximal chain $0 = x_0 < x_1 < \ldots < x_n = 1$ of modular elements. Such lattices, called *supersolvable*, have the property that all zeros of the characteristic polynomial are positive integers, namely,

$$p(\nu) = (\nu-a_1)(\nu-a_2)\ldots(\nu-a_n),$$

where a_i is the number of atoms in $[0,x_i]$ but not in $[0,x_{i-1}]$. By Theorem 4, $Q_n(G)$ is supersolvable, with $a_i = 1+(i-1)|G|$

COROLLARY 5.1: Let μ be the Möbius function of $Q_n(G)$, and let $u_n^m = \mu(0,1)$, where m is the order of G. Then

$$u_n^m = (-1)^n \prod_{i=0}^{n-1} (1+mi)$$
$$= (-m)^n \left(\frac{1}{m}\right)^{(n)},$$

where $(x)^{(n)}$ is the rising factorial $(x)^{(n)} = x(x+1)\ldots(x+n-1)$.

Proof: Set $\nu = 0$ in (4.2). □

When $m = 1$, i.e., G is the trivial group 1, $u_n^1 = (-1)^n n!$ is the value of $\mu(0,1)$ for the partition lattice $P_{n+1} \cong Q_n(1)$. Since the Möbius function is multiplicative over direct products, we obtain from Corollaries 2.1 and 5.1, COROLLARY 5.2: Let $(\alpha) \leq (\beta)$ in $Q_n(G)$, where α, β are as in Corollary 2.1. Then

$$\begin{aligned}\mu((\alpha),(\beta)) &= u_{r_0}^{m}\, u_{r_1}^{1} \cdots u_{r_s}^{1} \\ &= (-1)^{r-s}\, m^{r_0} \left(\frac{1}{m}\right)^{(r_0)} \prod_{k=1}^{s} (r_k - 1)!\end{aligned}$$

The *Whitney numbers* of a finite geometric lattice L of rank n are defined by

$$(4.4) \qquad w(n,r) = \sum_{x:\rho(x)=n-r} \mu(0,x), \qquad \text{(First kind)},$$

the coefficient of ν^r in the characteristic polynomial, and

$$(4.5) \qquad W(n,r) = \sum_{x:\rho(x)=n-r} 1, \qquad \text{(Second kind)},$$

the number of elements of corank r. The most well-known examples are the following:

(1) If $L = B_n$, the lattice of subsets of an n-set,

$$w(n,r) = (-1)^{n-r}\binom{n}{r}, \qquad W(n,r) = \binom{n}{r}.$$

(2) If $L = L_n(F)$, the lattice of subspaces of an n-dimensional vector space (or (n-1)-dimensional projective space) over a finite field F of order q,

$$w(n,r) = (-1)^{n-r}\, q^{\binom{n-r}{2}} \binom{n}{r}_q, \qquad W(n,r) = \binom{n}{r}_q,$$

where $\binom{n}{r}_q$ is the Gaussian coefficient [6],

$$\binom{n}{r}_q = \frac{(q^n-1)\dots(q^{n-r+1}-1)}{(q^r-1)\dots(q-1)}.$$

(3) If $L = P_{n+1}$, the lattice of partitions of an (n+1)-set,

$$w(n,r) = s(n+1,r+1), \qquad W(n,r) = S(n+1,r+1),$$

the Stirling numbers of the first and second kind, respectively.

Each of these examples, as well as the lattices $Q_n(G)$ considered here, are classes of geometric lattices which satisfy the hypotheses of

THEOREM 6: Let $\{L_n | n = 1,2,\ldots\}$ be a class of geometric lattices such that L_n is of rank n, and for all $x \in L_n$ of corank r $(0 \le r \le n < \infty)$, the interval $[x,1]$ is isomorphic to L_r. Let $w(n,r)$, $W(n,r)$ be the Whitney numbers of L_n. Then

$$\sum_r W(n,r)\, w(r,s) = \delta(n,s),$$

$$\sum_r w(n,r)\, W(r,s) = \delta(n,s),$$

where $\delta(a,b) = 1$ if $a = b$, and 0 otherwise, and the numbers $w(n,r)$, $W(n,r)$ satisfy the inverse relations

$$(4.6) \qquad a_n = \sum_r W(n,r) b_r, \qquad b_n = \sum_r w(n,r) a_r.$$

Proof: We use the identities

$$\delta(0,y) = \sum_{x:x \le y} \mu(x,y) = \sum_{x:x \le y} \mu(0,x).$$

Then

$$\begin{aligned}
\sum_r W(n,r)\, w(r,s) &= \sum_{x \in L_n} \sum_{y \ge x} \mu(x,y)\, \delta(s,n-\rho(y)) \\
&= \sum_{y \in L_n} \delta(s,n-\rho(y)) \sum_{x \le y} \mu(x,y) \\
&= \sum_{y \in L_n} \delta(s,n-\rho(y))\, \delta(0,y) \\
&= \delta(n,s).
\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_r w(n,r)\ W(r,s) &= \sum_{x\in L_n} \mu(0,x) \sum_{y\geq x} \delta(s,n-\rho(y)) \\ &= \sum_{y\in L_n} \delta(s,n-\rho(y)) \sum_{x\leq y} \mu(0,x) \\ &= \sum_{y\in L_n} \delta(s,n-\rho(y))\ \delta(0,y) \\ &= \delta(s,n).\end{aligned}$$

Then if $a_n = \sum_r W(n,r)b_r$,

$$\begin{aligned}\sum_r w(n,r)a_r &= \sum_r w(n,r) \sum_s W(r,s)b_s \\ &= \sum_s b_s \sum_r w(n,r)\ W(r,s) \\ &= \sum_s b_s\ \delta(n,s) \\ &= b_n.\end{aligned}$$

The converse is proved analogously. □

COROLLARY 6.1: Let $t_m(n,r)$, $T_m(n,r)$ be the Whitney numbers of $Q_n(G)$, for any fixed group G of order m. Then

$$\begin{aligned}m^n\left(\frac{\nu-1}{m}\right)_{(n)} &= \sum_r t_m(n,r)\nu^r, \\ \nu^n &= \sum_r T_m(n,r)\ m^r\left(\frac{\nu-1}{m}\right)_{(r)}.\end{aligned}$$

Proof: Set $a_r = \nu^r$ in (4.6). Then b_n is the characteristic polynomial $p_n(\nu;m)$, given by (4.2). □

Observe that the inverse relations in Corollary 6.1 are the analogs of the defining relations of the Stirling numbers, which are obtained from these by setting $m = 1$ and multiplying both equations by ν. The analogs of the

Stirling recurrences are given in

THEOREM 7: The Whitney numbers $t_m(n,r)$, $T_m(n,r)$ of $Q_n(G)$, for any fixed group G of order m, satisfy the recursions

$$(4.7)\qquad T_m(n,r) = T_m(n-1,r-1) + (1+m(r-1))T_m(n-1,r),$$

$$(4.8)\qquad t_m(n,r) = t_m(n-1,r-1) - (1+m(n-1))t_m(n-1,r).$$

Proof: A partial G-partition of X of size r is obtainable from a unique partial G-partition of $X-\{x_n\}$ of size $r-1$ by adding e_n, or from a unique partial G-partition α of $X-\{x_n\}$ of size r by replacing some $a_j \epsilon \alpha$ by $a_j+\lambda e_n$ $(\lambda \epsilon G)$, or else is equal to a partial G-partition of $X-\{x_n\}$ of size r. This proves (4.7), while (4.8) follows from a comparison of the coefficients of ν^r in (4.3). □

5. REPRESENTATION OF $Q_n(G)$

In this section we solve the representation problem (Theorems 9, 10,11) of $Q_n(G)$ after first considering whether nonisomorphic groups can result in isomorphic lattices (Theorem 8). The structure of the rank three geometries $Q_3(G)$ will be required in the proofs of Theorems 8 and 9, and we begin with a description of their structure. Let G be of order m, and assume a representative α of each E-class is fixed. The element (α) will be denoted by its chosen representative α throughout. The particular representatives chosen will be those given below.

Recall the Boolean sublattice $M = \{(\alpha)|\alpha \subseteq \varepsilon\}$ of $Q_3(G)$. The lines of M are $\{e_3\}$, $\{e_2\}$ and $\{e_1\}$. These are modular lines, so every line of $Q_3(G)$ meets each $\{e_i\}$. The three M-lines $\{e_i\}$ intersect in pairs; we regard them as the sides of a triangle. The vertices of the triangle are the three M-points $\{e_1,e_2\}$, $\{e_1,e_3\}$ and $\{e_2,e_3\}$, $\{e_j,e_k\}$ being the intersection of $\{e_j\}$ and $\{e_k\}$.

In what follows, (i,j,k) will denote an arbitrary element of

$$\{(1,2,3),\ (2,3,1),\ (3,1,2)\}.$$

In addition to its two vertices $\{e_i,e_j\}$ and $\{e_i,e_k\}$, each side $\{e_i\}$ of the triangle contains m other points, called its *interior points*, of the form $\{e_i,e_j+\lambda_i e_k\}$, one for each $\lambda_i \in G$. The set of interior points of $\{e_i\}$ will be denoted S_i. Thus

$$S_1 = \Big\{\{e_1,\ e_2+\lambda_1 e_3\}\,|\,\lambda_1 \in G\Big\},$$

$$S_2 = \Big\{\{e_2,\ e_3+\lambda_2 e_1\}\,|\,\lambda_2 \in G\Big\},$$

$$S_3 = \Big\{\{e_3,\ e_1+\lambda_3 e_\lambda\}\,|\,\lambda_3 \in G\Big\}.$$

There are 3m trivial (2-point) lines in $Q_3(G)$. The m lines $\{e_j+\lambda_i e_k\}$, $\lambda_i \in G$, join the vertex $\{e_j,e_k\}$ to the interior points $\{e_i,\ e_j+\lambda_i e_k\}$ of the opposite side (i.e., to S_i).

The remaining lines are m^2 in number. These are the lines, to be called *transversal lines*, of the set

$$T = \Big\{\{\lambda_2 e_1 + \lambda_1^{-1} e_2 + e_3\}\,|\,\lambda_1,\lambda_2 \in G\Big\}.$$

Each transversal line $\{\lambda_2 e_1+\lambda_1^{-1} e_2+e_3\}$ contains three points, one interior point on each side of the triangle, namely

$$(5.1)\qquad \begin{aligned} &\{e_1,\ e_2 + \lambda_1 e_3\} \in S_1,\\ &\{e_2,\ e_3 + \lambda_2 e_1\} \in S_2,\\ &\{e_3,\ e_3 + \lambda_3 e_2\} \in S_3,\end{aligned}$$

where $\lambda_3 = (\lambda_1\lambda_2)^{-1}$. Then the subset

$$L = \{(\lambda_1,\lambda_2,\lambda_3)|\lambda_1\lambda_2\lambda_3 = 1\}$$

of $G\times G\times G$ is the image of T under the injection $\{\lambda_2 e_1+\lambda_1^{-1}e_2+e_3\} \mapsto (\lambda_1,\lambda_2,(\lambda_1\lambda_2)^{-1})$. Each interior point $\{e_i,e_j+\lambda_i e_k\}$ on the side $\{e_i\}$ is incident with m transversal lines, joining it to the interior points of the remaining two sides. Combinatorially, the m^2 transversal lines of $Q_3(G)$ represent the triples of a latin square of order m, with S_1, S_2, S_3 the sets of rows, columns and symbols (in any order). Clearly any latin square (algebraically, a quasi-group) can be used to construct a planar geometry with the incidence properties described above for $Q_3(G)$. However, the nonassociativity of quasi-groups prevents generalizing the construction of $Q_n(G)$ by letting G be a quasigroup, for dimensions $n > 3$.

It is evident from the results of Section 4 that the Whitney numbers, hence the characteristic polynomial, depend on G only through its order m. Further, if G, G' are two groups of the same order m, the number of π-preimages of any partial partition of X is the same in $Q_n(G)$ and $Q_n(G')$, and the number of elements covering and covered by any element depends only on m and its π-image. These similarities naturally suggest the question as to whether two groups of the same order give rise to isomorphic lattices. The answer is given by

THEOREM 8: If $n \geq 3$ and $Q_n(G) \cong Q_n(G')$ for two groups G, G', then $G \cong G'$

Proof: Clearly, G, G' must be of the same order m. Let σ: $Q_n(G) \to Q_n(G')$ be an isomorphism. The restriction of σ to an upper interval $[(\alpha),1]$, where (α) is of corank three, is an isomorphism $[(\alpha),1] \cong$

$[\sigma(\alpha),\sigma(1)]$. By Theorem 2(a), $[(\alpha),1] \cong Q_3(G)$ and $[\sigma(\alpha),\sigma(1)] \cong Q_3(G')$. Thus it is sufficient to prove the theorem for $n = 3$. Clearly σ must take the sublattice M of modular elements in $Q_3(G)$ onto M in $Q_3(G')$, and we may assume without loss of generality that $\sigma|M$ is the identity on subsets of $\{e_1,e_2,e_3\}$. Then for each $(i,j,k) \in \{(1,2,3),(2,3,1),(3.1.2)\}$, the σ-image of a point $\{e_i,e_j+\lambda_i e_k\}$ in S_i is a point of the form $\{e_i,e_j+\lambda_i' e_k\}$, where $\lambda_i \in G$, $\lambda_i' \in G'$. The restriction of σ to S_i we may thus regard as a bijection $\sigma_i\colon \lambda_i \mapsto \lambda_i'$ of $G \to G'$. Clearly σ must take transversal lines of $Q_3(G)$ to transversal lines of $Q_3(G')$, so that for all $(\lambda_1,\lambda_2,\lambda_3) \in G^3$,

$$\lambda_3 = (\lambda_1\lambda_2)^{-1} \quad \text{iff} \quad \sigma_3(\lambda_3) = [\sigma_1(\lambda_1)\sigma_2(\lambda_2)]^{-1},$$

i.e.

$$\sigma_3((\lambda_1\lambda_2)^{-1}) = [\sigma_1(\lambda_1)\sigma_2(\lambda_2)]^{-1}.$$

Let $\tau_3\colon G \to G'$ be the bijection $\tau_3(\lambda) = [\sigma_3(\lambda^{-1})]^{-1}$. Then for all $\lambda_1,\lambda_2 \in G$,

$$\begin{aligned}\tau_3(\lambda_2\lambda_1) &= [\sigma_3((\lambda_1\lambda_2)^{-1})]^{-1} \\ &= \sigma_2(\lambda_2)\sigma_1(\lambda_1).\end{aligned}$$

The proof is then completed by application of the following

LEMMA: If G, G' are two (multiplicative) groups, and there exist three bijections ϕ_i: $i = 1,2,3$, such that for all $\kappa,\lambda \in G$,

(5.2) $$\phi_3(\lambda\kappa) = \phi_1(\lambda)\phi_2(\kappa),$$

then $G \cong G'$.

Proof of Lemma: Taking $\kappa = 1$, $\lambda = 1$, respectively in (5.2), we have $\phi_3(\lambda) = \phi_1(\lambda)\phi_2(1)$, $\phi_3(\kappa) = \phi_1(1)\phi_2(\kappa)$. Thus

$$(5.3)\qquad \begin{aligned} \phi_1(\lambda) &= \phi_3(\lambda)[\phi_2(1)]^{-1} \\ \phi_2(\kappa) &= [\phi_1(1)]^{-1}\phi_3(\kappa). \end{aligned}$$

Let $\phi = [\phi_1(1)]^{-1}\ \phi_3[\phi_2(1)]^{-1}$. Then $\phi\colon G \to G'$ is bijective, and by (5.2) and (5.3),

$$\phi(\lambda\kappa) = \phi(\lambda)\phi(\kappa),$$

so ϕ is an isomorphism. □

We turn now to the representation problem for $Q_n(G)$. If F is any field, denote by $F^* = F-\{0\}$ the multiplicative subgroup of F. We then have

THEOREM 9: If $Q_n(G)$ is representable over a field F and $n \geq 3$, then G is isomorphic to a subgroup of F^*.

Proof: Since $Q_3(G)$ is isomorphic to the minor $[(\alpha),1]$ of $Q_n(G)$ for any (α) of corank three, it is again sufficient to prove the theorem for $n = 3$. We assume a coordinatization of $Q_3(G)$ over F is given and choose a fixed coordinate vector in F^3 from the homogeneous set representing each point of $Q_3(G)$ by taking one of the coordinates unity, according to the conventions described below. The representation may then be regarded as an injection σ into F^3 such that any subset of three or fewer points of $Q_3(G)$ is independent iff its σ-image is linearly independent over F.

Again let $(i,j,k) \in \{(1,2,3),(2.3.1),(3,1.2)\}$. We may assume that the three (independent) M-points of $Q_3(G)$ are represented by the unit vectors of F^3, with $\sigma(\{e_j,e_k\})$ having 1 in position i and 0 in positions j and k. Then since an interior point $\{e_i,e_j+\lambda_i e_k\} \in S_i$ is collinear with

$\{e_i,e_j\}$ and $\{e_i,e_k\}$, it is represented by a vector with 0 in position i and nonzero elements in positions j and k. By convention we assume the coordinate in position j is 1. Denote nonzero elements of F by $a,b,c,\ldots,$ and let

$$H_1 = \{a_1 | (0,1,a_1) \in \sigma(S_1)\},$$
$$H_2 = \{a_2 | (a_2,0,1) \in \sigma(S_2)\},$$
$$H_3 = \{a_3 | (1,a_3,0) \in \sigma(S_3)\}.$$

Each H_i is thus an m-subset of F^*. Define bijections $\phi_i\colon G \to H_i$, $i = 1,2,3$, by $\phi_i(\lambda_i) = a_i$, the coordinate in position k of $\sigma(\{e_i,e_j+\lambda_i e_k\})$. Then the σ-images of the three points (5.1) of the transversal line $\{\lambda_2 e_1+\lambda_1^{-1}e_2+e_3\}$ are $(0,1,\phi_1(\lambda_1))$, $(\phi_2(\lambda_2),0,1)$, $(1,\phi_3(\lambda_3),0)$, respectively, where $\lambda_3 = (\lambda_1\lambda_2)^{-1}$. It is easily verified that the three vectors $(0,1,a_1)$, $(a_2,0,1)$, $(1,a_3,0)$ of F^3 are linearly dependent iff $a_1a_2a_3 = -1$, so for all $\lambda_1,\lambda_2,\lambda_3 \in G$, $\phi_1(\lambda_1)\phi_2(\lambda_2)\phi_3(\lambda_3) = -1$ iff $\lambda_1\lambda_2\lambda_3 = 1$, i.e.,

$$\phi_3((\lambda_1\lambda_2)^{-1}) = -[\phi_1(\lambda_1)\phi_2(\lambda_2)]^{-1}. \tag{5.4}$$

Interchanging λ_1 and λ_2 does not affect the right side of (5.4), since F^* is abelian. Thus $\lambda_2^{-1}\lambda_1^{-1}$ and $\lambda_1^{-1}\lambda_2^{-1}$ have the same ϕ_3-image. Since $\phi_3\colon G \to H_3$ is bijective, it follows that G is abelian.

We next prove that each H_i is a coset of a subgroup of F^*. Define the m^2-subset U of $H_1 \times H_2 \times H_3$ by

$$U = \{(a_1,a_2,a_3) | a_1a_2a_3 = -1\}.$$

Note that U is the image under the bijection $\phi_1\times\phi_2\times\phi_3\colon G^3 \to H_1\times H_2\times H_3$ of the subset

$$L = \{(\lambda_1,\lambda_2,\lambda_3) | \lambda_1\lambda_2\lambda_3 = 1\},$$

corresponding to the transversal lines of $Q_3(G)$. Thus for any permutation (i,j,k) of $(1,2,3)$, $a_i \in H_i$ iff there exist $a_j \in H_j$, $a_k \in H_k$ such that $a_i = -1/a_j a_k$. Let a_i, b_i, c_i be any three (not necessarily distinct) elements of H_i. Choose any $a_j \in H_j$. Then

(i) $a_j \in H_j$, $a_i \in H_i$ imply $-1/a_j a_i \in H_k$,

(ii) $-1/a_j a_i \in H_k$, $c_i \in H_i$ imply $a_j a_i/c_i \in H_j$,

(iii) $a_j a_i/c_i \in H_j$, $b_i \in H_i$ imply $-c_i/a_j a_i b_i \in H_k$,

(iv) $-c_i/a_j a_i b_i \in H_k$, $a_j \in H_j$ imply $a_i b_i/c_i \in H_i$.

Thus

(5.5) $\quad a_i, b_i, c_i \in H_i$ imply $a_i b_i/c_i \in H_i$.

Choose fixed elements $c_i \in H_i$, $i = 1,2,3$, and let

$$G_i = \{a_i/c_i | a_i \in H_i\}.$$

We claim each G_i is a group. Clearly $1 \in G_i$, and if $a_i/c_i \in G_i$, (5.5) implies $c_i^2/a_i \in H_i$, so $(a_i/c_i)^{-1} = c_i/a_i \in G_i$. Finally, if a_i/c_i, $b_i/c_i \in G_i$, then by (5.5), $a_i b_i/c_i \in H_i$, so $(a_i/c_i)(b_i/c_i) \in G_i$. Thus G_i is a subgroup of F^* of order m and $H_i = G_i c_i$ is a coset of G_i. We next prove $G_1 = G_2 = G_3$.

Let $a_i = \kappa_i c_i \in H_i$, $a_j = \kappa_j c_j \in H_j$. Then $-1/a_i a_j = -1/\kappa_i\kappa_j c_i c_j \in H_k$, so $-1/\kappa_i\kappa_j c_1 c_2 c_3 \in G_k$. Define $\lambda_0 = -c_1 c_2 c_3$. Then for any $\kappa_i \in G_i$, $\kappa_j \in G_j$, we have $1/\kappa_i\kappa_j\lambda_0 \in G_k$. Putting $\kappa_i = \kappa_j = 1$, $1/\lambda_0 \in G_k$, so $\lambda_0 \in G_k$. But then $1/\kappa_i\kappa_j\lambda_0 \in G_k$, $\lambda_0 \in G_k$ imply $1/\kappa_i\kappa_j \in G_k$, so $\kappa_i\kappa_j \in G_k$. Thus $\kappa_i \in G_i$, $\kappa_j \in G_j$ imply $\kappa_i\kappa_j \in G_k$. Putting $\kappa_j = 1$, $\kappa_i \in G_i$ implies $\kappa_i \in G_j$, so $G_1 = G_2 = G_3 = H$, say, and $\lambda_0 = -c_1 c_2 c_3 \in H$. Thus H_1, H_2, H_3 are three cosets of the subgroup H of F^*, such that in the quotient group F^*/H, $H_1 H_2 H_3 = -H$, the coset of H containing -1.

Choose any three elements $\mu_1, \mu_2, \mu_3 \in H$ such that $\mu_1\mu_2\mu_3 = \lambda_0$, and define bijections $f_i\colon H_i \to -H$ by $f_i(a_i) = -\mu_i(a_i/c_i)$. Then

$$f_1(a_1)f_2(a_2)f_3(a_3) = a_1a_2a_3,$$

so the image under $f_1 \times f_2 \times f_3\colon H_1 \times H_2 \times H_3 \to (-H)^3$ of the subset $U = \{(a_1,a_2,a_3) \mid a_1a_2a_3 = -1\}$ is a set in $(-H)^3$ with the same property. It follows that the maps

$$(0,1,a_1) \mapsto (0,1,f_1(a_1))$$
$$(a_2,0,1) \mapsto (f_2(a_2),0,1)$$
$$(1,a_3,0) \mapsto (1,f_3(a_3),0)$$

give a representation of $Q_3(G)$ with $H_1 = H_2 = H_3 = -H$. We may assume, therefore, that the given representation is of this form.

Recall now the bijections $\phi_i\colon G \to -H$ satisfying (5.4). Let $\tau_i = -\phi_i\colon G \to H$. Then by (5.4), for all $\lambda_1, \lambda_2 \in G$,

(5.6) $$\tau_3((\lambda_1\lambda_2)^{-1}) = [\tau_1(\lambda_1)\tau_2(\lambda_2)]^{-1}.$$

Let $\tau\colon G \to H$ be defined by

$$\tau(\kappa) = [\tau_3(\kappa^{-1})]^{-1}.$$

Then from (5.6),

$$\tau(\lambda_1\lambda_2) = \tau_1(\lambda_1)\tau_2(\lambda_2),$$

so by the lemma, $G \cong H$. □

The converse of Theorem 9 is

THEOREM 10: If F is a field and G is isomorphic to a subgroup of F^*, then $Q_n(G)$ is representable over F.

Proof: Let $L_n(F)$ denote the lattice of subspaces of the projective geometry of rank n (projective dimension $n-1$) over F, and S the set of points of $L_n(F)$. The set $X = \{x_1,\ldots,x_n\}$ we take as a fixed basis of $L_n(F)$, and assume that $L_n(F)$ is coordinatized over F with respect to any system of reference containing X, with the unit vectors of F^n representing the $x_i \in X$. Then if p is a point with coordinate vector $(\kappa_1,\ldots,\kappa_n) \in F^n$, we write $p = \sum_{i=1}^n \kappa_i x_i$. Since the coordinates are homogeneous, the vector $(\kappa_1,\ldots,\kappa_n)$ is determined only up to a constant nonzero scalar multiple.

Any function $f\colon X \to F$, not identically zero, represents a copoint of $L_n(F)$, namely

$$f \quad = \quad \bigvee\{p \mid p \in S(f)\},$$

where $S(f)$ is the *point set* of f:

$$S(f) \quad = \quad \{p = \textstyle\sum \kappa_i x_i \mid \sum \kappa_i f(x_i) = 0\}.$$

As in the case of points, two functions f, f' represent the same copoint of $L_n(F)$ iff $f' = \lambda f$ for some $\lambda \in F^*$.

The set $\{e_1,\ldots,e_n\}$ of unit functions, with $e_i(x_{i'}) = 1$ if $i = i'$, 0 otherwise, is the copoint basis *dual* to X:

$$e_i \quad = \quad \bigvee\{x_{i'} \mid i' \neq i\},$$
$$x_i \quad = \quad \bigwedge\{e_{i'} \mid i' \neq i\}.$$

A copoint $f\colon X \to G$ is thus a linear combination $f = \sum \lambda_i e_i$, where $\lambda_i = f(x_i)$. Then f is *minimally dependent* on the subset $\{e_i \mid \lambda_i \neq 0\}$ of $\{e_i,\ldots,e_n\}$. That is, $f \geq \bigwedge\{e_i \mid \lambda_i \neq 0\}$ but $f \ngeq$ the infimum of any proper subset of

$\{e_i|\lambda_i \neq 0\}$, since a point $p = \sum \kappa_i x_i$ is $\leq \wedge\{e_i|\lambda_i \neq 0\}$ iff $\{i|\lambda_i \neq 0\} \subseteq \{i|\kappa_i = 0\}$, so $\sum \kappa_i \lambda_i = 0$. Conversely, $x_i \nleq f$ for any i such that $\lambda_i \neq 0$. More generally, if $\{f_1,\ldots,f_r\}$ is any independent set of copoints, then g is dependent on $\{f_1,\ldots,f_r\}$ iff g is a linear combination $g = \sum \lambda_j f_j$ iff $g \geq \wedge\{f_j|j = 1,\ldots,r\}$, and in this case, g is minimally dependent on $\{f_j|\lambda_j \neq 0\}$.

Assume without loss of generality that G is a subgroup of F^*. For any element $(\alpha) \in Q_n(G)$, $\alpha = \{a_j: A_j \to G|j = 1,\ldots,r\}$, we now regard the $a_j \in \alpha$ as functions $X \to G \cup \{0\} \subseteq F$, simply by extending the domain from A_j to X and defining $a_j(x_i) = 0$ for all $x_i \in X-A_j$. Note that any two E-equivalent partial G-partitions represent the same set of copoints in $L_n(F)$. Further since the subsets $A_j = \{x_i|a_j(x_i) \neq 0\}$, $j = 1,\ldots,r$, are disjoint, the subsets $\{e_i|a_j(x_i) \neq 0\}$, $j = 1,\ldots,r$, of $\{e_1,\ldots,e_n\}$ on which the a_j minimally depend are disjoint. Hence $\alpha = \{a_j|j = 1,\ldots,r\}$ is an independent set of copoints in $L_n(F)$. Ti follows that the map $\sigma: Q_n(G) \to L_n(F)$, where

$$\sigma(\alpha) = \wedge\{a_j|a_j \in \alpha\},$$

is well-defined and preserves rank,

(5.7) $\rho_L(\sigma(\alpha)) = n - |\alpha| = \rho_Q(\alpha).$

If $(\beta) \geq (\alpha)$ in $Q_n(G)$, then each $b_k \in \beta$ is a linear combination $b_k = \sum \lambda_j a_j$ $(\lambda_j \in G)$, so in $L_n(F)$, $b_k \geq \wedge\{a_j|j = 1,\ldots,r\}$ for all $b_k \in \beta$. Thus $\sigma(\beta) \geq \sigma(\alpha)$, i.e., σ is order-preserving.

It follows from (5.7) that σ takes points to points. Recall that the points of $Q_n(G)$ are $(\alpha^{(i)})$, $(\alpha^{(ii')}(\lambda))$, defined in Corollary 1.2, where $1 \leq i,\ i' \leq n$, $i \neq i'$, $\lambda \in G$, and $(\alpha^{(ii')}(\lambda)) = (\alpha^{(i'i)}(\lambda^{-1}))$. It is easily verified that

$$(5.8)\qquad \begin{aligned} \sigma(\alpha^{(i)}) &= x_i, \\ \sigma(\alpha^{(ii')}(\lambda)) &= x_i - \lambda^{-1}x_{i'}. \end{aligned}$$

Suppose $(\alpha),(\beta) \in Q_n(G)$, and $\sigma(\alpha) = \sigma(\beta)$. Since $x_i \le \sigma(\alpha)$ iff $x_i \in X-\bigcup_{j=1}^{r}A_j$, it follows that $X-\bigcup_{j=1}^{r}A_j = X-\bigcup_{k=1}^{s}B_k = X_0$, say. Thus $\pi(\alpha)$, $\pi(\beta)$ are each partitions of X_0. If $\pi(\alpha) \neq \pi(\beta)$, then there exists a block of one, say $A_1 \in \pi(\alpha)$, containing two elements x_i, $x_{i'}$ in different blocks of the other, say $x_i \in B_1$, $x_{i'} \in B_2$, where $B_1,B_2 \in \pi(\beta)$. Let $\lambda = a_1(x_{i'})/a_1(x_i)$. Then $(\alpha^{(ii')}(\lambda)) \le (\alpha)$, so $\sigma(\alpha^{(ii')}(\lambda)) = x_i-\lambda^{-1}x_{i'} \le \sigma(\alpha)$. But $b_1(x_i)-\lambda^{-1}b_1(x_{i'}) = b_1(x_i) \neq 0$, so $x_i-\lambda^{-1}x_{i'} \not\le b_1$. It follows that $x_i-\lambda^{-1}x_{i'} \not\le \sigma(\beta)$, a contradiction, and we conclude $\pi(\alpha) = \pi(\beta)$. If $(\alpha) \neq (\beta)$, then there exists a block of $\pi(\alpha)$, $A_1 = B_1$, say, containing two elements x_i, $x_{i'}$ such that $a_1(x_{i'})/a_1(x_i) \neq b_1(x_{i'})/b_1(x_i)$. But then if $\lambda = a_1(x_{i'})/a_1(x_i)$, we have $x_i-\lambda^{-1}x_{i'} \le \sigma(\alpha)$, $x_i-\lambda^{-1}x_{i'} \not\le \sigma(\beta)$, a contradiction. Thus $(\alpha) = (\beta)$, so σ is an injection.

We now prove that σ preserves suprema. It will then follow that $\sigma\colon Q_n(G) \to L_n(F)$ is an injective strong map, so that $Q_n(G)$ is isomorphic to its σ-image, the latter a subgeometry of $L_n(F)$. Let $(\alpha),(\beta) \in Q_n(G)$. Since σ is order-preserving,

$$(5.9)\qquad \sigma(\alpha) \vee \sigma(\beta) \le \sigma((\alpha)\vee(\beta)).$$

To prove the reverse inequality, suppose $c\colon X \to F$ is a copoint of $L_n(F)$ and $c \ge \sigma(\alpha) \vee \sigma(\beta)$, i.e., there exist $\kappa_j,\lambda_k \in F^*$ such that $c = \sum_j \kappa_j a_j = \sum_k \lambda_k b_k$. Let $C = \{x_i \mid c(x_i) \neq 0\}$. Then C is the union of the $\pi(\alpha)$-blocks $\{A_j \mid \kappa_j \neq 0\}$ and also of the $\pi(\beta)$-blocks $\{B_k \mid \lambda_k \neq 0\}$, hence is the union of a set of $(\pi(\alpha)\vee\pi(\beta))$-blocks $\{C_\ell \mid \ell = 1,\ldots,v\}$, say. Assume first that

$v = 1$, i.e., that C is a single block

$$C = A_1 \cup \ldots \cup A_t = B_1 \cup \ldots \cup B_u$$

of $\pi(\alpha) \vee \pi(\beta)$. Then

$$c = \sum_{j=1}^{t} \kappa_j a_j = \sum_{k=1}^{u} \lambda_k b_k,$$

where $\kappa_j, \lambda_k \in F^*$. Let $x_i \in C$, and suppose $x_i \in A_j \cup B_k$. Then $\kappa_j a_j(x_i) = \lambda_k b_k(x_i)$, so $\kappa_j/\lambda_k = b_k(x_i)/a_j(x_i) \in G$. Thus the cosets $G\kappa_j$, $G\lambda_k$ of G are equal whenever $A_j \cap B_k$ is nonempty. But since C is a block of $\pi(\alpha) \vee \pi(\beta)$, the intersection graph of the A_j $(j = 1,\ldots t)$ versus the B_k $(k = 1,\ldots,t)$ is connected. Hence all κ_j, λ_k are contained in a common coset $G\lambda$ of G. Thus the *function* c is an F^*-multiple $c = \lambda c'$ of a function c' into $G \cup \{0\}$, hence represents the same copoint of $L_n(F)$. It is clear from the proof of Theorem 3 that $c' \in \gamma'$ for some partial G-partition γ' such that $(\gamma') = (\alpha)\vee(\beta)$. Thus $c \geq \sigma(\gamma') = \sigma((\alpha)\vee(\beta))$. In general, if $C = C_1 \cup \ldots \cup C_v$, where $v \geq 2$, then $c = \sum_{\ell=1}^{v} c_\ell$, where $c_\ell(x_i) = c(x_i)$ if $x_i \in C_\ell$, 0 otherwise. By the preceding, each c_ℓ is a multiple $c_\ell = \lambda_\ell c'_\ell$, where $c'_\ell \in \gamma'$, $\ell = 1,\ldots,v$, for some partial G-partition γ' such that $(\gamma') = (\alpha)\vee(\beta)$. Thus $c = \sum_{\ell=1}^{v} \lambda_\ell c'_\ell$, so $c \geq \sigma((\alpha)\vee(\beta))$. It follows that $\sigma(\alpha) \vee \sigma(\beta) \geq \sigma((\alpha)\vee(\beta))$, and hence by (5.9), that σ is supremum-preserving. This completes the proof. □

COROLLARY 10.1: If G is a subgroup of F^*, then

$$(\alpha^{(i)}) \mapsto x_i, \quad (\alpha^{(ii')}(\lambda)) \mapsto x_i - \lambda^{-1} x_{i'},$$

where $1 \leq i$, $i' \leq n$, $i \neq i'$, $\lambda \in G$, and $X = \{x_1,\ldots,x_n\}$ is a basis of F^n, is a coordinatization of the points of $Q_n(G)$ over F.

THEOREM 11: Let G be of order m, and let $n \geq 3$. Then if G is noncyclic, $Q_n(G)$ is representable over no field. If G is cyclic, then $Q_n(G)$ is representable over

(a) every field iff $m = 1$ (i.e., G is trivial),

(b) a finite field of order q iff m divides $q-1$,

(c) the rational or real field iff $m = 1$ or 2,

(d) the complex field for all m.

Proof: If F is any field, and G is a finite subgroup of F*, then G is necessarily cyclic (see, e.g. [1], Thm. 17). Thus by Theorem 9, $Q_n(G)$ is representable over no field if G is noncyclic. Conversely, if G is cyclic of order m, then G is a subgroup of the multiplicative group of every field iff $m = 1$, of a finite field of order q iff m divides $q-1$, of the rational or real field iff $m = 1$ or 2, and of the complex field for every m. Thus (a)-(d) follow form Theorem 10. □

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FOOTNOTES

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2. Present address: Department of Mathematics, Ohio State University, Columbus, Ohio 43210.