RESPONSE OF NUCLEAR STRUCTURAL SYSTEMS TO TRANSIENT AND RANDOM EXCITATIONS, USING BOTH DETERMINISTIC AND PROBABILISTIC METHODS

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SUMMARY

The finite element is commonly used for the static analysis of reactor components and the same ideas can be applied to the dynamic analysis. The formation of various damping matrices for both internal and external damping is discussed. For the most part only the linearised form of the equations of motion is considered giving:

$$M\ddot{x} + C\dot{x} + (K + iG)x = F(t).$$

The type of solution adopted depends upon both the force input and the required response. For a short period transient input usually only the initial response is required which is best found from a step-by-step integration. A method involving a Taylor series and curve fitting over a series of steps is developed and applied to the solution of a cylindrical vessel subject to a pressure transient.

For a periodic force input the steady state response is needed requiring the determination of the damped eigen vectors. An efficient algorithm, using the undamped vectors, is given and the response to an arbitrary force input developed. This method is applied to the previous example and the two solutions are compared illustrating the advantages and disadvantages of each method.

For random excitation methods of presenting information are discussed standard results for the stationary problem are presented in terms of the first part of the paper. As an example the spectral density of the response due to random imposed movements is given illustrated by a series of heat exchange tubes excited by random end movements. The probability that any interference will occur with clashing at a particular velocity is determined.

The response due to a non-stationary force input such as an earthquake is discussed. The step-by-step method of analysis is used to predict the probability density function of the response of a simple system at any time under such a non-stationary input.

Possible modes of failure with reference to reactor components are discussed and indications of how the proceeding theory can be applied to predicting failures are given.

1. Introduction

In recent years the degree of detailed analysis of nuclear structures has increased enormously. This has been due in the main to the advent of large digital computers and the development of general finite element programs [1] , [2] . Most development has so far been in the field of linear static analysis. However, as plant gets larger and more complicated the efforts of dynamic loads is increasingly important. This paper is the outcome of a survey of the requirements for an analysis of the dynamic response of nuclear structures. Emphasis has been placed upon the less well developed parts of the problem and upon the particular aspects of nuclear plant.

Finite element theory has proved most useful in the static analysis and is briefly developed here for dynamic work. The formation of the damping matrix is considered in some detail. Two methods of solution for the deterministic problem are given, either numerical integration [8] or a normal mode solution [3] . Two parallel types of solution are developed for random loads [4], [5] together with a discussion upon methods of presenting data for random loads.

2. Formation of the Equations of Motion Using Finite Elements

Following the usual finite element method, an interpolation polynomial $[w]$ is assumed relating internal values of a variable to the values at the nodes. The variables can be displacements, velocities and accelerations for the stiffness, damping and mass matrices respectively. If the same interpolation is used for all three variables, the resulting matrices are said to be kinematically equivalent. In some cases only the surface values $[w_s]$ of the interpolation matrix are used.

2.1 The Element Stiffness Matrix $[k]$

If the nodal point displacements are $[r]$ and the displacements at any point within the element are $[u]$ then

$$[u] = [w][r]$$

These can be operated on by a differential matrix $[\partial]$ to give the strains $[\varepsilon]$ as

$$[\varepsilon] = [\partial][w][r] = [\alpha][r]$$

Application of the principle of virtual displacements leads to the element stiffness matrix $[k]$

$$[k] = \int_V [\alpha]^t [\kappa] [\alpha] dV$$

where the stresses $[\sigma]$ are $[\sigma] = [\kappa][\varepsilon]$

2.2 The Element Mass Matrix $[m]$

The velocity of any point within the element is

$$[\dot{u}] = [w][\dot{r}]$$

The kinetic energy of a small elemental volume dV is

$$\delta T = \tfrac{1}{2} [\dot{u}]^t \rho [\dot{u}] \, dV = \tfrac{1}{2} [\dot{r}]^t [w]^t \rho [w] [\dot{r}] \, dV$$

where ρ is the material density (in some cases there is a matrix at the centre of the quadratic product, e.g. if rotary inertias are included in a beam analysis). Integrating gives

$$T = \tfrac{1}{2} [\dot{r}]^t \int_V [w]^t \rho [w] \, dV [\dot{r}] = \tfrac{1}{2} [\dot{r}]^t [m] [r]^t$$

The element mass matrix is identified as

$$[m] = \int_V [w]^t \rho [w] \, dV$$

2.2.1 Kinematically Equivalent Matrix for a Concentrated Mass

Most structures have mass concentrations at various points. If these coincide with nodal points, the extra mass can be added directly to the corresponding leading diagonal terms of $[m]$. If the concentrated mass is not at a node a kinematically equivalent mass matrix can be formed by making the kinetic energies of the real and equivalent masses the same.

Say there is a concentrated mass $[m_c]$ (a diagonal matrix corresponding to the mass values in each co-ordinate direction) at some point i. The value of the interpolation matrix $[w]$ at point i is $[w]_i$. Then from the kinetic energy argument the kinematically equivalent mass matrix is

$$[m] = [w]_i^t [m_c] [w]_i$$

2.3 The Element Damping Matrix $[c]$

The finite element method can also be used to determine the damping matrix of a structure where definite damping mechanisms can be recognised. In nuclear reactors the main damping mechanisms are:-

1) Radiation of energy by a travelling wave (usually a sound wave)
2) Fluid flow damping
3) Internal energy dissipation of the material composing the structure
4) Coulomb friction.

Coulomb friction is highly non-linear and is difficult to include. It will not be discussed further except to say that an equivalent linearisation similar to that for material damping can be used for a sinusoidal response.

2.3.1 Energy Dissipated per Cycle by a Viscous Dashpot

A viscous dashpot with a damping co-efficient c when subject to a sinusoidal displacement $u_o \sin \omega t$ has an energy dissipation per cycle E of

$$E = \pi \omega c \, u_o^2$$

This is now used to relate the actual damping mechanisms to equivalent viscous damping.

2.3.2 Damping Matrix Arising from a Travelling Wavefront

A plane wavefront travelling at a constant velocity a through an infinite medium will impart energy to the medium through which it passes. This will appear as an energy loss, i.e. a damping, to the system that is causing the wave. It can be shown that the total energy per cycle passing through an area ds as the wavefront moves is

$$E = \pi \rho_o a \omega u_o^2 \, ds$$

where ρ_o is the density of the medium through which the wave passes. Hence the plane wave appears as a viscous damper with a co-efficient $\rho_o a$ per unit area.

In finite element terms the velocity u at some point on the surface of the structure (normal to the surface), is

$$\dot{u} = [w_s] [\dot{r}]$$

Hence the damping force per unit surface area is

$$f = \rho_o a \dot{u} = \rho_o a [w_s] [\dot{r}]$$

Applying the principal of virtual displacements leads to an element damping matrix

$$[c] = \int_s [w_s]^t \rho_o a [w_s] \, ds$$

These matrices are only formed for the surfaces of the structure that radiate a travelling wave.

2.3.3 Damping Matrix Arising from Fluid Flow

Only forces in the direction of the flow velocity will be considered; any lift forces normal to the flow direction are not considered, although they are very important since they can give rise to self-sustaining vibrations. In this case the force on a length dx of the body as the fluid flows past it is

$$f = \tfrac{1}{2} C_D \rho_o d (\dot{u} - V)^2 dx$$

where ρ_o is the density of the fluid and V its flow velocity. C_D is the drag co-efficient of the body and d the dimension associated with this. Assuming the fluid velocity is much greater than that of the body then

$$f = C_D d \rho_o V \dot{u} dx - \tfrac{1}{2} C_D d \rho_o V^2 dx$$

The first term represents a damping force. The second gives a static force and is not considered further. Again the principle of virtual displacements gives the damping matrix as

$$[C] = \int_{\ell} [w_1]^t C_D d \rho_o V [w_1] dl$$

2.3.4 Damping Matrix Arising from Internal Energy Dissipation

Internal energy dissipation within the material composing a structure is called variously internal, material or hysteric damping. It is essentially non-linear and in order that it might be included, an equivalent linearised form is used. It has been found experimentally that the energy dissipated per cycle by most materials is independent of frequency. For a viscous dashpot, this energy is proportional to frequency hence as a simple model structural damping is taken as viscous damping divided by frequency. For a harmonic response this is equivalent to a complex stiffness. The model can be used in the stress-strain relationships as

$$[\sigma] = [\kappa] [\varepsilon] + 1/\omega [\gamma] [\dot{\varepsilon}] = ([\kappa] + i [\gamma])[\varepsilon]$$

Values of $[\gamma]$ (assumed to be diagonal) for most materials are given in reference [6].

It can now be included directly into the finite element analysis usually as a complex stiffness giving

$$([k] + i [g]) = \int_V [\alpha]^t ([\kappa] + i [\gamma]) [\alpha] dV$$

It must be emphasised that this idealisation is only meaningful if a frequency ω can be defined. It is valid for the steady state periodic response or for the stationary random response in the frequency domain. It has no meaning for a transient response or a random analysis in the time domain.

2.3.5 The Importance of the Damping Matrix

Before discussing methods of solving the equations of motion, it is worth considering the importance of the damping matrix. For most structural problems the damping is small (of the order of one percent of critical) implying that damping is not usually important. It is important however, if the structure is subjected to a periodic force containing at least one frequency component coinciding with a structural resonance. Even so the distribution of damping throughout the structure is not usually important provided each mode contains damping to the correct order of magnitude. If the excitation is slightly off-resonance, the response will be almost entirely controlled by the mass and stiffness distribution. All of this implies that an exact determination of $[C]$ and $[G]$ is not normally necessary. The usual assumption that $[C] = \alpha [K] + \beta [M]$ after assembly where α and β are factors to give a correct order of magnitude for $[C]$ will be adequate in most cases.

However, there are occasions where an exact knowledge of the distribution of the damping is important, in particular where it forms a boundary condition. A case in point is the

idealisation of an infinite medium where a wavefront will continue to infinity and will not return to its starting point. Finite elements, by their very nature, cannot be used to model an infinite region directly. Without the correct boundary conditions waves will be reflected from the boundaries of the region back to where they came from. If, however, damping is distributed around the surface of the finite region such that the energy in the wavefront is all absorbed, no reflections occur and the infinite region is modelled. The correct distribution of damping is that given previously for the travelling wave. If the magnitude (either more or less) or the distribution of this damping is altered, then reflections occur and the model is not correct. This boundary condition has been used successfully in acoustic problems [7] and will probably be useful in earthquake analysis though here the effect of shear waves would have to be included.

3. Methods of Solving the Equations of Motion

Having established the element mass, damping and stiffness matrices these can be assembled in the usual manner to give the equations of motion as

$$[M]\,[\ddot{x}] + [C]\,[\dot{x}] + ([K] + i\,[G])\,[x] = [F(t)]$$

where $[F(t)]$ is the force input. Again it is worth emphasising that the complex part of the stiffness matrix only has meaning if a frequency can be defined.

There are basically two methods of solving the above equations. Either the complete set is integrated step-by-step obtaining the response at a series of sequential time intervals, or a co-ordinate transformation can be applied such that the equations become uncoupled in the new set of co-ordinates. This is the normal mode method. The choice of which type of analysis is used depends upon both the type of force input and what is required in the response. Also to a lesser extent upon the computer facilities available.

3.1 The Step-by-Step Solution

This is a solution in the time domain, that is the force is specified at certain times and the response is found usually at the same times. Frequency has little or no meaning here so that the equations to be solved are

$$[M]\,[\ddot{x}] + [C]\,[\dot{x}] + [K]\,[x] = [F(t)]$$

A polynomial is assumed to relate the accelerations at a series of sequential time intervals. This is differentiated and used in a Taylor series to give the velocity and displacement at a future time step.

3.1.1 Mathematics of the Step-by-Step Solution

The method is now developed assuming a parabolic variation of acceleration but it is easily modified for any other assumption. Let the acceleration at the three times $t - \Delta$ t and $t + \Delta$ be $[\ddot{x}]_-$, $[\ddot{x}]_o$, and $[\ddot{x}]_+$ respectively. The acceleration at any time over the interval is then

$$[\ddot{x}(\tau)] = -\tfrac{1}{2}(1-\tau)\,\tau\,[\ddot{x}]_- + (1-\tau)(1+\tau)\,[\ddot{x}]_o + \tfrac{1}{2}(1+\tau)\,\tau\,[\ddot{x}]_+ \qquad (1)$$

where $\tau = 1/\Delta\ (t - t_1)$ and $t_1 - \Delta \leq t \leq t_1 + \Delta$

This can be differentiated twice giving at the time t ($\tau = o$)

$$[\dot{x}]_o = 2\ 1/2\Delta\,([\ddot{x}]_+ - [\ddot{x}]_-); \quad [\ddot{x}]_o = 1/\Delta^2\,([\ddot{x}]_+ - 2\,[\ddot{x}]_o + [\ddot{x}]_-)$$

Substituting these into a Taylor series gives the velocity and displacement at the time $t + \Delta$ as

$$\begin{bmatrix} [x]_+ \\ [\dot{x}]_+ \end{bmatrix} = \begin{bmatrix} [I] & \Delta[I] \\ [0] & [I] \end{bmatrix} \begin{bmatrix} [x]_o \\ [\dot{x}]_o \end{bmatrix} + \begin{bmatrix} -\Delta^2/24\,[I] & 5\Delta^2/12\,[I] & \Delta^2/8\,[I] \\ -\Delta/12\,[I] & 2\Delta/3\,[I] & 5\Delta/12\,[I] \end{bmatrix} \begin{bmatrix} [\ddot{x}]_- \\ [\ddot{x}]_o \\ [\ddot{x}]_+ \end{bmatrix} \qquad (2)$$

Initially some guess is made for $[\ddot{x}]_+$. The above equation then gives $[\ddot{x}]_+$ and $[\dot{x}]_+$. A better estimate for the acceleration can then be made from the equation of motion

$$[\ddot{x}]_+ = [M^{-1}]\,([F]_+ - [K]\,[x]_+ - [C]\,[\dot{x}]_+)$$

This is repeated until $[\ddot{x}]_+$ converges. The process can then be stepped to find the acceleration at the end of next time interval.

3.2.1 Comments on the Step-by-Step Method

As presented above, the method only involves the inversion of the mass matrix. However, it does require iteration at each step and this will later be shown to be equivalent to another matrix inversion. A modification is required to start the process since $[\ddot{x}]_-$ does not exist. Probably the simplest approach is to use a linear polynomial for the first step since this does not involve $[\ddot{x}]_-$. The assumption of a continuous polynomial over a series of time steps means that an impulse cannot be applied. They can, however, be applied by restarting the process at this time.

One of the main advantages of the step-by-step method is that non-linearities can be easily included by changing $[K]$ and $[C]$ at each stage of the iteration. This gives the basic method great versatility. Here, though, only the linear problem is considered.

The choice of the time interval is somewhat arbitrary but ideally it should be less than the time period of the highest natural frequency of the structure to enable the complete transient response to be determined. This can be found from the largest eigen value of the structure, but the calculation of this value can be time consuming for many degrees of freedom. It is suggested instead that a more economical method is to invoke a theorem due to Rayleigh which says that if a constraint is applied to a system then the highest eigen value after the constraint is applied is lower than before. Hence if the highest eigen value of a 'typical' element is found before assembly it will almost certainly be higher than the largest eigen value of the final structure since the process of assembly improves a series of constraints.

3.1.3 Extension of the Step-by-Step Method

Integrating the accelaration polynomial and using the equation of motion allows the velocity and displacement to be found without iteration within each step. The response at $t + \Delta$ is related to the response at time t and the force input at times $t - \Delta$, t and $t + \Delta$.

Integrating equation (1) gives the velocity at any time in the inverval $t - \Delta$ to $t + \Delta$

$$[\dot{x}\,(\tau)] = -\Delta/2\,(\tau^2/2 - \tau^3/3)\,[\ddot{x}]_- + \Delta(\tau - \tau^3/3)\,[\ddot{x}]_o + \Delta/2\,(\tau^2/2 + \tau^3/3)\,[\ddot{x}]_+ + [\dot{x}]_o$$

and the displacement

$$[x\,(\tau)] = -\Delta^2/2\,(\tau^2/6 - \tau^4/12)\,[\ddot{x}]_- + \Delta^2(\tau^2/2 - \tau^4/12)\,[\ddot{x}]_o + \Delta^2/2\,(\tau^3/6 + \tau^4/12)\,[\ddot{x}]_+ + \Delta\tau\,[\dot{x}]_o + [x]_o$$

This allows the displacement and velocity to be found at times $t - \Delta$, t and $t + \Delta$. Substituting these into the equation of motion gives three equations which can be written in matrix form as

$$\begin{bmatrix} [M] - 5\Delta/12\,[C] - \Delta^2/8\,[K] & -4\Delta/3\,[C] + 5\Delta^2/12\,[K] & \Delta/12\,[C] - \Delta^2/24\,[K] \\ [O] & [M] & [O] \\ -\Delta/12\,[C] - \Delta^2/24\,[K] & 2\Delta/3\,[C] + 5\Delta^2/12\,[K] & [M] + 5\Delta/12\,[C] + \Delta^2/8[K] \end{bmatrix} \begin{bmatrix} [\ddot{x}]_- \\ [\ddot{x}]_o \\ [\ddot{x}]_+ \end{bmatrix}$$

$$= \left[\begin{array}{ccc|cc} [I] & [O] & [O] & -[K] & -[C] + \Delta[K] \\ [O] & [I] & [O] & -[K] & -[C] \\ [O] & [O] & [I] & -[K] & -[C] - \Delta[K] \end{array}\right] \left[\begin{array}{c} [F]_- \\ [F]_o \\ [F]_+ \\ \hline [x]_o \\ [\dot{x}]_o \end{array}\right]$$

or $[D]\,[\ddot{x}] = [F] + [A]\,[x]$

where $[\ddot{x}] = \{[\ddot{x}]_- \; [\ddot{x}]_o \; [\ddot{x}]_+\}$ $[F] = \{[F]_- \; [F]_o \; [F]_o\}$ $[x] = \{[x] \; [\dot{x}]_o\}$

Also equation (2) can be written as

$$[y] = \{[\ddot{x}]_+ \; [\dot{x}]_+\} = [B]_1\,[x] + [B]_2\,[\ddot{x}]$$

hence $$[y] = \left[[B]_1 + [B]_2[D^{-1}][A]\right][x] + [B_2]\,[D]^{-1}\,[F] \qquad (3)$$

The response at $t + \Delta$ can now be calculated without any iteration but it does require the inversion of the (3n x 3n) non symmetric matrix $[D]$.

The iterative solution is essentially the same as the Adams-Bashforth method. The well-known Nemark β - method [9] can be shown to be the same method with a linear interpolation if $\beta = 1/12$. A similar approach is also described by Argyris and Chan [8] . The direct solution has similarities with that described by Argyris and Scharpf [10].

3.2 The Normal Mode Solution

Most engineers are familiar with the idea that the eigen values of the equations of motion are the squares of the resonant frequencies. Associated with the eigen values are a set of eigen vectors. If these vectors are used to transform the variables (displacements etc.) from the original set $[x]$ to a new set $[\phi]$ (say) it can be shown that the equations in the new variables are all uncoupled. Each one can then be solved as a single degree of freedom and finally the results transformed back to $[x]$. The same ideas can be applied to the damped equations of motion except that they have to be written in a slightly unusual form and the resulting eigen values and vectors are complex.

3.2.1 The Undamped Equations of Motion

If the damping is less than critical then the eigen vectors of the system will be determined largly by the distribution of mass and stiffness throughout the structure. The solution of the undamped equations are therefore a good approximation to the damped results. The undamped homogeneous equation of motion is

$$[M]\,[\ddot{x}] + [K]\,[x] = [0]$$

This leads to the standard eigen value problem [11] with undamped vectors $[P_u]$. It can be shown that since $[M]$ and $[K]$ are real symmetric and at least positive semi-definite then the eigen values are always positive and the vectors wholly real. Transforming the variables with $[x] = [P_u]\,[\phi]$

leads eventually to the equation

$$[P_u^t]\,[M]\,[P_u][\ddot{\phi}] + [P_u^t]\,[K]\,[P_u][\phi] = [m][\ddot{\phi}] + [k][\phi] = [P_u^t]\,[F]$$

It is a property of the vectors that both $[m]$ and $[k]$ are diagonal matrices.

3.2.2 The Damped Eigen Value Problem

Here again the aim is to determine a co-ordinate transformation that uncouples the equations of motion. In general it is not possible to find a transformation which makes more than two matrices diagonal simultaneously so the damped problem is re-written as

$$\begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}\begin{bmatrix} [\ddot{x}] \\ [x] \end{bmatrix} + \begin{matrix} -[M] & [0] \\ [0] & [K] + i\,[G] \end{matrix}\begin{bmatrix} [\dot{x}] \\ [x] \end{bmatrix} = \begin{bmatrix} [0] \\ [0] \end{bmatrix} \qquad (4)$$

This has a solution of the form $[x] = [x_i]\exp\{\lambda_i t\}$ which leads to the standard eigen value problem

$$\begin{matrix} -[M] & [0] \\ [0] & [K] + i\,[G] \end{matrix}\begin{bmatrix} \lambda_i\,[x]_i \\ [x]_i \end{bmatrix} = \lambda_i \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}\begin{bmatrix} \lambda_i\,[x]_i \\ [x]_i \end{bmatrix}$$

Note however that although the matrices are symmetric, they are not positive definite

and in general the roots and vectors will be complex, even with $[G]$ zero.

3.2.2.1 Solution of the Damped Equations by Inverse Iteration

Although the damped eigen value problem can be solved using standard methods, it is much more economical to use the undamped vectors as an approximate solution in the method of inverse iteration [11]. Say q_j is some approximation to the root λ_j. Then treating the components of the vector $\{\lambda_j [x]_j \quad [x]_j\}$ as two independent quantities $\{[y]_j \; [x]_j\}$ and applying inverse iteration gives

$$\left\{\begin{bmatrix} -[M] & [O] \\ [O] & [K] + i\,[G] \end{bmatrix} + q_j \begin{bmatrix} [O] & [M] \\ [M] & [C] \end{bmatrix}\right\}\begin{bmatrix} [y]_{jr} \\ [x]_{jr} \end{bmatrix} = \begin{bmatrix} [O] & [M] \\ [M] & [C] \end{bmatrix}\begin{bmatrix} [y]_{jr-1} \\ [x]_{jr-1} \end{bmatrix}$$

The subscript r refers to the current estimate and r-1 to the one before.

This can be expanded into two equations. Since $[M]$ is always positive definite and can be inverted, the first equation can be substituted into the second to give

$$(q_j^2[M] + q_j [C] + ([K] + i\,[G]))\,[x]_{jr} = -[M]([y]_{jr-1} + q_j [x]_{jr-1}) - [C][x]_{jr-1}$$

also $$[y]_{jr} = [x]_{jr-1} - q_j [x]_{jr}$$

This gives an iterative algorithm for finding the damped vectors. The terms are normalised with respect to the largest absolute term in $[x]_{jr}$. The eigen value can then be found from the ratios of corresponding terms in $[y]_{jr}$ and $[x]_{jr}$

$$\lambda_{jk} = y_{jrk}/x_{jrk}$$

This provides a convenient check for convergence by confirming that λ_{jk} is the same for all k ($1 \leq k \leq n$).

A good estimate for the j'th eigen value q_j can be obtained using the j'th undamped vector $[P_{uj}]$ giving $q_j = 1/2m_j\,(c_j \pm i\,(4k_j m_j - c_j^2)^{\frac{1}{2}})$

where $m_j = [P_{uj}]^t [M] [P_{uj}]$ $c_j = [P_{uj}]^t [C] [P_{uj}]$ $k_j = [P_{uj}]^t ([K] + i[G])[P_{uj}]$

This gives two estimates for q_j, both equally valid and leads to the required 2n eigen values and vectors. The process can be started with

$$[x]_{uo} = [P]_{uj} \quad ; \quad [y]_{jo} = q_j [P]_{uj}$$

3.2.2.2 Comments upon the Inverse Iteration Procedure

If $[P]_{uj}$ happens to be a vector of the damped system, then the estimate q_j is the exact eigen value and the process converges immediately. If the damping is small $[P]_{uj}$ is a good approximation to the damped vector and the convergence is rapid.

Once the estimate q_j is found the matrix $(q_j [M] + q_j [C] + [K] + i\,[G])$ need only be inverted once for each iteration. It is complex but symmetric and usually quite heavily banded. If the matrix is not inverted but factorised using a Cholesky decomposition, the banding is retained and the size of problem that can be solved is increased.

In putting the damped equations into an eigen value form the size of the matrices was expanded from (n x n) to (2n x 2n). This gives 2n eigen values and associated eigen vectors. However, it is not necessary to solve the (2n x 2n) problem; instead two separate (n x n) problems are solved. The theory has been developed for the general case for both $[C]$ and $[G]$ non-zero. If $[C]$ is zero, there are only n eigen values and vectors and the procedure given here gives one set of roots and vectors minus the other. Alternatively, if $[G]$ is zero, one set of roots and vectors is the complex conjugate of the other. In the general case the two sets of roots and vectors are different.

3.2.3. Solution of the Damped Equations of Motion

The damped eigen vectors are used to transform the variables to give a set of 2n uncoupled equations of motion. Each one of these can be solved for an arbitrary force input by means of a convolution (Duhamel) integral. These solutions are transformed back to the original variables giving the response for an arbitrary force input.

The damped eigen vectors can be written in the form

$$[P_D] = \begin{bmatrix} [X_1] & [\lambda_1] & [X_2] & [\lambda_2] \\ [X_1] & & [X_2] & \end{bmatrix}$$

where

$$[\lambda]_i = [\lambda_{i1} \quad \lambda_{i2} \quad \cdots\cdots\cdots\cdots \lambda_{in}]$$

$$[x]_i = [[X]_{i1} \quad [X]_{i2} \quad \cdots\cdots\cdots\cdots [X]_{in}]$$

The co-ordinate transformation is then

$$\begin{bmatrix} [\dot{x}] \\ [x] \end{bmatrix} = \begin{bmatrix} [X]_1[\lambda]_1 & [X]_2[\lambda]_2 \\ [X]_1 & [X]_2 \end{bmatrix} \begin{bmatrix} [\dot{\phi}]_1 \\ [\phi]_2 \end{bmatrix}$$

Substituting this into equation (4) and pre-multiplying by $[P_D]^t$ leads eventually to the equation

$$\begin{bmatrix} [a]_1 & [o] \\ [o] & [a]_2 \end{bmatrix} \begin{bmatrix} [\ddot{\phi}]_1 \\ [\ddot{\phi}]_2 \end{bmatrix} + \begin{bmatrix} [b]_1 & [o] \\ [o] & [b]_2 \end{bmatrix} \begin{bmatrix} [\dot{\phi}]_1 \\ [\dot{\phi}]_2 \end{bmatrix} = \begin{bmatrix} [f_1 \cdot (t)] \\ [f_2 (t)] \end{bmatrix}$$

where

$$[a]_i = [X]_i^t [M] [X]_i \quad [\lambda]_i + [\lambda]_i^t [X]_i^t [M] [X]_i + [X]_i^t [C] [X]_i$$

$$[b]_i = [\lambda]_i^t [X]_i^t [M] [X]_i [\lambda]_i + [X]_i^t ([K] + i [G]) [X]_i$$

$$[f]_i = [X]_i^t [F]$$

It can be shown from the properties of the eigen vectors that both $[a]_i$ and $[b]_i$ are diagonal. Hence the r'th equation in the above set can be written as

$$a_{ir} \ddot{\phi}_{ir} + b_{ir} \dot{\phi}_{ir} = f_{ir}(t)$$

This is a first order differential equation with complex coefficients. For zero initial conditions it can be solved to give ϕ_{ir} at some time t as

$$\phi_{ir}(t) = \int_o^t a_{ir}^{-1} f_{ir}(\tau) \exp\{\lambda_{ir}(t-\tau)\} \, d\tau$$

where

$$\lambda_{ir} = -a_{ir}^{-1} b_{ir}$$

Every equation in the set can be solved in the same manner. Transforming the results back to the original variables leads to

$$[x(t)] = \int_o^t [W(t-\tau)] [F(\tau)] \, d\tau \qquad (5)$$

where

$$[W(t-\tau)] = [X]_1 [a]_1^{-1} [E\{\lambda_1 (t-\tau)\}] [X]_1^t + [X]_2 [a]_2^{-1} [E]\{\lambda_2(t-\tau)\} [X]_2^t$$

and

$$[E\{\lambda_i (t-\tau)\}] = [\exp\{\lambda_i (t-\tau)\} \cdots\cdots\cdots \exp\{\lambda_{in} (t-\tau)\}]$$

This gives the complete response of the damped system to any arbitrary force input. It is worth noting that if only viscous damping is present the second term in $[W]$ is the complex conjugate of the first, hence $[W]$ itself is wholly real. It follows that for viscous damping the response $[q]$ is also wholly real. $[W]$ is called the impulse response function.

The convolution integral can be written in various forms. One form to be used later is

$$[x(t)] = \int_{-\infty}^{\infty} [W(t-\tau)] [F(\tau)] \, d\tau$$

where $[F(\tau)]$ is zero for $\tau < 0$

and $[W(t-\tau)]$ is ignored for $\tau > t$.

3.3 Comparison between the Step-by-Step and the Normal Mode Methods

Both the step-by-step and the normal mode methods need a large amount of work before results are obtained. The normal mode solution requires a great deal of preliminary effort to find the eigen vectors but once these are obtained, the response to any force input at any

time can be quickly found. Conversely the step-by-step method produces the response over th initial time relatively quickly but the amount of work required increases linearly with the length of response required. Also all of the work has to be repeated for another force input. Usually no attempt is made to reduce the number of equations in the step-by-step method since these can all be stored in banded form within the computer. The eigen value problem in general loses any banding of the matrices so that only relatively small problems can be solved without some form of condensation. For a lightly damped system if the force input only contains frequencies up to some value then modes with resonances greater than this need not be included in the solution.

From these considerations it will be seen that the step-by-step method is most useful when only the initial transient response is required for a small number of loading cases. If there are many loading cases or if the steady state response is required, then the normal mode solution is to be preferred.

4. Response to Random Force Inputs

So far the theories relating to the response of structures to deterministic force inputs have been developed. There are, however, many cases where the forces are not deterministic in that they are not completely predictable. Such forces are random and although the force at any time cannot be specified, it is usually possible to say something about its average values, its likely magnitude and its frequency content. Random forces are then described in terms of these average parameters and the foregoing theories have to be modified to allow working with this data. In general, if the force input is random then the response will also be random. If the average values are constant with time the process is said to be stationary (analogous with the steady state case in the deterministic response) and if not it is non-stationary (transient for the deterministic case).

4.1 Comparison of Methods for Presenting Data for a Periodic and a Random Signal

Initially, the steady state random case is considered. This can be compared directly with a deterministic periodic force input. If the signal is periodic, then the frequency content of the signal can be found by performing a fourier transform on one complete period. The resulting curve of component magnitude against frequency is called the spectrum of the signal and will in general be in the form

$$f(\omega) = \tfrac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \sin n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

It has both in-phase and out-of-phase components, (i.e. real and imaginary parts). Similarly, if two periodic signals occur together, there will be relative phases between the same frequency component of each signal.

For any periodic signal, no matter how complicated, there is a unique fourier transform which specifies it completely. It is a common practice to try to apply the same ideas to a random signal. Here a portion of the signal, over a time T say, is taken and a fourier transform performed on it. It must be emphasised that having done this the transformed signal ceases to be random; it is now periodic and the portion of the signal that was analysed is repeated every T seconds. If a different record or a different time period is chosen, the resulting spectrum will be different from the first. Since the choice of the original record was arbitrary, the situation is not satisfactory and becomes even worse where two or more signals occur simultaneously since the relative phasing between each can be very important.

It is possible to remove some of these objections by smoothing the spectrum, and carrying this idea to its logical conclusion leads to the systematic and mathematically acceptable idea

of averaging. Here a fourier transform is performed not on just one record, but upon a whole series of them (m say). At a frequency ω the k'th record will have a Fourier component

$$f_k(\omega) = (a_k(\omega) + i\, b_k(\omega))$$

Taking the modulus of this and averaging over all m records gives the average value of the fourier amplitude

$$S_{xx}(\omega) = 1/m \sum_{k=1}^{m} f_k \bar{f}_k = 1/m \sum_{k=1}^{m} (a_k + b_k)$$

This is called the spectral density of the signal x. The subscript xx denotes the direct spectral density of x with itself. Note that $S_{xx}(\omega)$ is wholly real. It is the square of the average value of the spectrum at the frequency ω .

If there are two signals, x and y say, the same idea can be applied to obtain the cross-spectral density between the two signals as

$$S_{xy}(\omega) = 1/m \sum_{k=1}^{m} f_{xk} \bar{f}_{yk} \quad ; \quad S_{yx}(\omega) = 1/m \sum_{k=1}^{m} f_{yk} \bar{f}_{xk} = \bar{S}_{xy}(\omega)$$

Note that $S_{xy}(\omega)$ is generally complex giving a measure of the relative phase between the two signals at the frequency w.

To actually calculate spectral densities of signals from experimental results more refined techniques are used [12] . It will be appreciated that the spectral density is a measure of the frequency content of a random signal. It contains no ambiguity about phasing or record length provided enough samples have been taken. It has also been implicitly assumed that any one record is representative of them all, that is the signal is stationary.

In practice, the spectral density is the most useful form of presenting and manipulating data. However, in developing the equations, the idea of the correlation function has to be introduced.

4.2 The Correlation Function

Consider two signals x and y which occur over the same time. M samples of these can be taken and by assuming all of the samples occur simultaneously a series of averages (or means) can be obtained. In particular, the average of the produce of x at time t_1 and y at time t_2 can be found. Such a mean is called the correlation between x and y and is denoted by the symbol $R_{xy}(t_1\ t_2)$.

$$R_{xy}(t_1\ t_2) = E[x(t_1)\, y(t_2)]$$

where $E[z]$ denotes the mean of z. In general, the correlation is a function of the two times t_1 and t_2 . If however, both x and y are stationary, it becomes a function of the time difference $t_1 - t_2$ $(= \tau)$ only.

$$R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$$

The correlation of x with itself $R_{xx}(\tau)$ is called the auto-correlation.

The correlation function is not of much practical use in itself but it can be shown that its fourier transform gives the spectral density of the signal

$$S_{xy}(\omega) = 1/2\pi \int_{-\infty}^{\infty} R_{xy}(\tau)\ e^{-i\omega\tau}\ d\tau$$

Similarly, an inverse fourier transform of $S_{xy}(\omega)$ gives the correlation $R_{xy}(\tau)$.

Note that the mean square σ_x^2 of a signal is thus given by

$$\sigma_x^2 = R_{xx}(o) = \int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega \qquad (6)$$

4.3 Response of a System with Many Degrees of Freedom to a Random Force Input

Equation (5) of the normal mode solution gives the structural response at any time to any force. This can be used to construct the correlation matrix of the response. Applying a

Fourier transform to this allows the spectral density of the response to be calculated, know ing the spectral density of the input.

The correlation matrix is

$$[R_x(\tau)] = [R_{x_i x_j}(\tau)] = E\left[[x(t)]\,[x^t(t+\tau)]\right]$$

$$= E\left[\int_{-\infty}^{\infty}[W(t-\tau_1)]\,[F(\tau_1)]\,d\tau_1 \int_{-\infty}^{\infty}[F^t(\tau_2)]\right][W(t+\tau-\tau_2)]\,d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[W(t-\tau_1)]\,[R_f(\tau)]\,[W^t(t-\tau_2)]\,d\tau_1\,d\tau_2$$

Applying the fourier transform to this leads to [4], [5]

$$[S_x(\omega)] = [H(\omega)]\,[S_f(\omega)]\,[\overline{H}^t(\omega)] \qquad (7)$$

where $[S_f(\omega)]$ is the matrix of spectral densities of the force input $[S_x(\omega)]$ is the spectral density of the response. Also

$$[H(\omega)] = \int_{-\infty}^{\infty}[W(\tau)]\,e^{i\omega\tau}\,d\tau = [X]_1[R]_1[X]_1^t + [X]_2[R]_2[X]_2^t$$

where $$[R]_i = \left[\frac{1}{(b_{ir} - i\,a_{ir}\omega)}\right]$$

Equation (7) is a standard result written in matrix form. From the previous discussion on spectral densities, it can be seen that $[S_f(\omega)]$ is an Hermitian matrix (the real part is symmetric, the imaginary part is skew symmetric). The above transformation preserves this form hence $[S_x(\omega)]$ is also Hermitian.

4.3.1 The Cross-Spectral Density between a Signal and its Time Derivative

If the spectral density of a signal x is known, then implicitly the time behaviour of the signal and hence its derivative $\dot{x}$ is also known. The cross-spectral density between x and $\dot{x}$ can thus be found from $S_{xx}(\omega)$.

Consider first the auto-correlation of x $R_{xx}(t_1\ t_2)$ differentiating this with respect to t_1 gives $d/dt_1\ R_{xx}(t_1\ t_2) = d/dt_1\ E[x(t_1)\,x(t_2)]$

$$= E[\dot{x}(t_1)\,x(t_2)] = R_{\dot{x}x}(t_1\ t_2)$$

For a stationary signal $\tau = t_2 - t_1$ and the spectral density is

$$S_{\dot{x}x}(\omega) = 1/2\pi \int_{-\infty}^{\infty} R_{\dot{x}x}(\tau)\exp(-i\omega\tau)\,d\tau$$

$$= -1/2\pi \int_{-\infty}^{\infty} d/d\tau\,(R_{xx}(\tau))\,\exp(-i\omega\tau)\,d\tau$$

Integrating this by parts and noting $R_{xx}(\pm\infty)$ tends to zero gives

$$S_{\dot{x}x}(\omega) = -iw\ S_{xx}(\omega)$$

Similarly $S_{x\dot{x}}(\omega) = iw\ S_{xx}(\omega)$

4.3.2 The Response of a Structure to Imposed Movements

The displacement vector can be partitioned into free displacements $[x]_1$ and imposed movements $[x]_2$. Obviously, if the displacement is known as a function of time then so is the velocity and acceleration. The complete equation of motion can then be written as

$$\begin{bmatrix}[M]_{11} & [M]_{12}\\ [M]_{21} & [M]_{22}\end{bmatrix}\begin{bmatrix}[\ddot{x}]_1\\ [\ddot{x}]_2\end{bmatrix} + \begin{bmatrix}[C]_{11} & [C]_{12}\\ [C]_{21} & [C]_{22}\end{bmatrix}\begin{bmatrix}[\dot{x}]_1\\ [\dot{x}]_2\end{bmatrix} + \begin{bmatrix}[K]_{11} & [K]_{12}\\ [K]_{21} & [K]_{22}\end{bmatrix}\begin{bmatrix}[x]_1\\ [x]_2\end{bmatrix} = \begin{bmatrix}[0]\\ [F]_2\end{bmatrix}$$

$[F]_2$ are the reaction forces and it is assumed that no other forces are present. The first equation gives

$$[M]_{12}[\ddot{x}]_1 + [C]_{11}[\dot{x}]_1 + [K]_{11}[x]_1 = -([M]_{12}[\ddot{x}]_2 + [C]_{12}[\dot{x}]_2 + [K]_{12}[x])$$

The left-hand side forms a standard equation of motion in $[x]_1$ with an impulse response function $[W]_{11}$ Hence the response is

$$[x]_1 = -\int_{-\infty}^{\infty}[W]_{11}([M]_{12}[\ddot{x}]_2 + [C]_{12}[\dot{x}]_2 + [K]_{12}[x]_2)\,d\tau$$

The correlation matrix is then $[R_{x_1x_1}(\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[W]_{11}[B][R_{ii}(\tau)][B]^t[W]_{11}^t\,d\tau_1\,d\tau_2$

where $[\bar{B}] = [\,[M]_{12}\ \ [C]_{12}\ \ [K]_{12}\,]$

and $[\,R_{ii}(\tau)\,]$ is the matrix of correlations of the applied movements including the velocities and accelerations.

Applying the fourier transform the spectral densities of the response is

$$[S_{x1\,x1}(\omega)] = [\bar{H}]_{11}(\omega)]\ [\bar{B}]\ [S_{ii}(\omega)]\ [B^t]\ [\bar{H}]^t_{11}(\omega)$$

Using the theory of section 4.3.1

$$[S_{ii}(\omega)] = \begin{bmatrix} [S_{\dots xx}] & [S_{\cdot\cdot\cdot xx}] & [S_{\cdot\cdot xx}] \\ [S_{\cdot\cdot\cdot xx}] & [S_{\cdot\cdot xx}] & [S_{\cdot xx}] \\ [S_{\cdot\cdot xx}] & [S_{\cdot xx}] & [S_{xx}] \end{bmatrix} = \begin{bmatrix} -\omega^2 [\bar{I}] \\ i\omega\ [\bar{I}] \\ [\bar{I}] \end{bmatrix} [S_{x2\,x2}(\omega)] \begin{bmatrix} -\omega^2[\bar{I}] & -i\omega\ [\bar{I}] & [\bar{I}] \end{bmatrix}$$

where $[S_{x2\,x2}(\omega)]$ is the spectral density of the applied displacements (i.e. it does not include velocities and accelerations). Giving finally the spectral density of the response as

$$[S_{x1\,x1}] = [\bar{H}]_{11}\ [\bar{B}]_1\ [S_{x2\,x2}]\ [\bar{B}_1]^t\ [\bar{H}_{11}]$$

where $[B]_1 = [\,-\omega^2\ [M_{12}] + i\omega\ [C_{12}] + [K_{12}]\,]$

4.3.3 Spectral Density of a Moving Random Force

If an earthquake travels across the base of a structure force at one point at a time, t is repeated exactly at another point at a later time (t + τ). The same type of loading occurs as traffic moves over a bridge. The force input is defined by the spectral density of the force at any point and the matrix of spectral densities of the forces at all points can be specified in terms of this and the velocity of the earthquake. Say the force at the p-th nodal point at a time t is $[F_p(t)]$. This force moves with a velocity v in the direction x. If the q'th node is a distance x_{pq} from the p'th then the force at this node is $[F_q(t + x_{pq}/v)]$. The cross-correlation between the two points p and q is

$$[R_{pq}(\tau)] = E[\,[F_p(t)]\ \ [F_q{}^t(t+\tau)]\,]$$
$$E\,[\,[F_p(t)]\ \ [F_p{}^t(t+\tau-x_{pq}/v)]\,] = [R_{pp}(\tau - x_{pq}/v)]$$

Applying the fourier transform

$$[S_{pq}(\omega)] = \int_{-\infty}^{\infty} [R_{pp}(\tau - x_{pq}/v)] e^{-i\omega(\tau - x_{pq}/v)}\, e^{-i\omega x_{pq}/v}\, d(\tau - x_{pq}/v)$$
$$= \text{esp}\,(i\,\omega x_{pq}/v)\ \ [S_{pp}(\omega)]$$

Similarly, $[S_{qp}(\omega)] = \exp(i\omega\, x_{pq}/v)\ \ [S_{pp}(\omega)]$

For the force input at a series of points, the spectral density matrix is

$$[S_{ff}(\omega)] = [\,[S(\omega)]\ \ \exp\{-i\omega\, x_{pq}/v\}\,]$$

where $[S(\omega)]$ is the spectral density of the force at any particular point.

Note that this is also an Hermitian matrix.

5. Presentation of Results for Failure Estimates

When carrying out any analysis exactly what is trying to be done must always be borne in mind. In most cases the aim is not to find the frequency content of the response or to find most of the results given in the previous sections. What really is being done is to access the integrity of the structure and to answer such questions as 'are the displacements large enough to prevent proper operation of the plant?' or 'is the stress level sufficient to cause a fatigue failure in the plants' lifetime?' For a deterministic input it is possible to solve for displacements and stresses as such and, using some chosen criteria, to estimate the integrity of the plant.

However, for a random input, by its very nature, it is not possible to solve the problem to give definite figures for displacements and stresses. In the preceeding sections it has been shown how to calculate the spectral density of the response given that of the input. Unfortunately this in itself cannot be used to give failure estimates and the idea of the pro-

bability density function has to be introduced. It is not possible to say that any specific value of the variable x will occur but again by averaging various records of x, an estimate can be made of the chance of it being any particular value. The curve of x against the chance of x occuring is called the probability density function p(x). It is normalised such that

$$\int_{-\infty}^{\infty} p(x)\ dx = 1.0$$

For a deterministic problem there is a definite value for any variable either given or derived. For a random problem the equivalent statement gives the probability that the variable has a particular value. This is given by the probability density function. Returning to the problem of failure, it is possible to say definitely if a specific criteria is exceeded for the deterministic case. In the random response it is only possible to give the chance that the criterion has been exceeded. An acceptable level of probable failure then has to be decided. If for a given x the probability density function $p(x) \to 1.0$ then the chance of x occuring is very likely. If $p(x) \to 0.0$ then it is very unlikely.

At first sight this appears a much less definite conclusion than that of the deterministic case. However, this is not so when the criterion that a failure is judged against is considered. The laws relating to say a fatigue failure or a stress fracture are certainly not well enough defined to be as precise as the calculation of the response can be. In fact all that can be really done is to say that the structure is likely not to fail. When expressed in these terms the random calculation can be used to make a much more precise statement of the likelihood of failure, particularly if the experimental failure results are presented in a probability form.

5.1 Probability Density Function of the Response

So far the response has been determined in terms of correlations and spectral densities. This in general does not determine the probability density function of the response except in the particular case where a gaussian distribution is assumed. If such a process has zero mean then $p(x) = \exp(-x^2/2\sigma^2) / \sigma\sqrt{2\pi}$ where σ^2 is the mean square of the signal. This can be found from the spectral density using equation (6).

It is fortunate that the gaussian distribution also has the property that if the signal is subjected to a linear process, then the resulting signal is also gaussian. For naturally occuring phenomena the gaussian distribution tends to be a good approximation making the above approach realistic.

6. Probability Density Function of the Response for a Non-Stationary Input

So far all of the equations have been developed for a stationary force input and a stationary response. Unfortunately there are cases where the input and the response are not stationary. In particular, for nuclear work earthquakes form just such an input. The ground movements rise from zero in a random manner to a maximum and then die away again. In most cases the duration of the earthquake is such that it can neither be considered as a short period shock nor a long period steady state input.

The ideas of correlations and spectral densities can be extended to the non-stationary case. However, the information contained in such a formulation of the response is not easily applicable to predicting a failure. It is better now to work directly in terms of the probability density function and to make this a function of time. The problem then reduces to 'given the probable amplitudes of the force input what is the probable amplitude of the response?' To be able to do this the idea of functions of random variables has to be introduced [4]

6.1 Functions of Random Variables

Given a set of variables x_1 to x_n then let these be combined to form m functions ($m \leq n$) y_1 to y_m such that $y_i = f_i(x_1 \ldots\ldots x_n)$

These functions can be of any form but in the following discussion they are linear.

If the variables x are random with a joint probability density function (PDF) of $p(x_1 \ldots x_n)$, then it can be shown that the PDF of the dependent variables y is

$$p(y_1 \ldots y_m) = \int\cdots\int p(x_1 \ldots x_n)\,|J|\; dx_{m+1} \ldots dx_n \qquad (8)$$

i.e. the 'excess' variables are integrated out. $[J]$ is the Jacobian

$$[J] = \begin{bmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 & \ldots & \partial x_1/\partial y_m \\ \partial x_2/\partial y_1 & & & \\ \partial x_m/\partial y_1 & \ldots & & \partial x_m/\partial y_m \end{bmatrix}$$

It is found by partial differentiation of the functions f_i. The PDF is normalised such that $\int\cdots\int p(y_1 \ldots y_m)\, dy_1 \ldots dy_m = 1.0$

6.2 The Step-by-Step Method with Random Variables

Given a force input and the displacement and velocity at some time then equation (3) of the step-by-step method allows the response to be calculated at a later time. If now the force input and the displacements are random then knowing the PDF of these at time t allows the response PDF to be found at time $t + \Delta$ using equations (3) and (8). The theory is correct for any form of the PDF but is now developed in detail for a gaussian distribution.

6.2.1 The Response Assuming a Gaussian PDF

For n variables $[u]$ say the gaussian PDF with zero mean can be written as

$$p([u]) = \frac{1}{(2\pi)^{n/2}\,|V_u|^{\frac{1}{2}}} \exp\left(-\tfrac{1}{2}\,[u]^t\,[V_u]^{-1}\,[u]\right) \quad \text{where } [V_u] = [\sigma_{ij}{}^2]$$

$\sigma_{ii}{}^2$ is the variance (or mean square) of the i'th term in $[u]$.

$\sigma_{ij}{}^2$ is the co-variance and is equal to the mean of the product u_i and u_j at the time t (i.e. the cross-correlation when $t_1 = t_2 = t$).

In the present case the PDF is

$$p([x][F]) = \frac{1}{(2\pi)^{5n/2}\,|V|} \exp\left\{-\tfrac{1}{2}\,[[x]^t\;[F]^t]\begin{bmatrix}[V_{11}] & [V_{12}] \\ [V_{21}] & [V_{22}]\end{bmatrix}^{-1}\begin{bmatrix}[x] \\ [F]\end{bmatrix}\right\}$$

Hence the PDF of the response at the time $t + \Delta$ is

$$p([y]) = \int_{[F]} \frac{1}{(2\pi)^{5n/2}\,|V|} \exp\left\{-\tfrac{1}{2}\,[x^t]\;[F^t]\begin{bmatrix}[V_{11}] & [V_{12}] \\ [V_{21}] & [V_{22}]\end{bmatrix}^{-1}\begin{bmatrix}[x] \\ [F]\end{bmatrix}\right\} d[F]$$

where $\int_{[F]} \ldots\ldots d[F] = \int_{F_1}\int_{F_N} \ldots\ldots dF_1 \ldots\ldots dF_N$

However, equation (3) gives $[x] = [[B]_1 + [B]_2[D]^{-1}[A]]^{-1}[[y] - [B]_2[D]^{-1}[F]] = [E]_1[y] + [E]_2[F]$

or $$\begin{bmatrix}[x] \\ [F]\end{bmatrix} = \begin{bmatrix}[E_1] & [E_2] \\ [0] & [I]\end{bmatrix}\begin{bmatrix}[y] \\ [F]\end{bmatrix}$$

In this case the transformation is linear hence $[J]$ is $[E_1]$.

Substituting gives

$$p([y]) = \frac{|E_1|}{(2\pi)^{5n/2}\,|V|}\int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}[[y]^t\;[F^t]]\begin{bmatrix}[E_1^t] & [0] \\ [E]^t & [I]\end{bmatrix}\begin{bmatrix}[V_{11}] & [V_{12}] \\ [V_{21}] & [V_{22}]\end{bmatrix}\begin{bmatrix}[E_1] & [E_2] \\ [0] & [I]\end{bmatrix}\begin{bmatrix}[y] \\ [F]\end{bmatrix}\right\} d[F]$$

$$= \frac{|E_1|}{(2\pi)^{5n/2}\,|V|}\int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}[[y]^t\;[F^t]]\begin{bmatrix}[f_{11}] & [f_{12}] \\ [f_{21}] & [f_{22}]\end{bmatrix}\begin{bmatrix}[y] \\ [F]\end{bmatrix}\right\} d[F]$$

It can be shown that the integration eventually leads to

$$p([y]) = C \exp\left\{-\tfrac{1}{2}\;[y]^t\;[V_y]^{-1}\;[y]\right\}$$

where $[V_y]^{-1} = [f_{11}] - [f_{12}]\,[f_{22}]^{1}\,[f_{21}]$

and the constant C is found from normalisation as

$$C = \frac{1}{(2\pi)^{3n/2}\,|V_y|}$$

6.2.2 Response with Gaussian White Noise

So far the variance matrix $[V]$ is undefined. If a white noise is assumed then there are by definition no cross-correlations and hence no co-variance terms between the forces at different times. There will in general be cross-coupling between the displacement and velocity but this is known from the previous step. It is assumed that for a small Δ the forces over the time $t - \Delta$ to $t + \Delta$ have such a small effect on displacement and velocity (these representing the integrated effects of the force) that the co-variance between force and response over this interval is zero. The variance matrix then becomes

$$[V] = \left[\begin{array}{ccc|ccc} [V_{xx}] & [V_{x\dot{x}}] & & [0] & [0] & [0] \\ [V_{\dot{x}x}] & [V_{\dot{x}\dot{x}}] & & [0] & [0] & [0] \\ \hline [0] & [0] & & [V_{F-}] & [0] & [0] \\ [0] & [0] & & [0] & [V_{FO}] & [0] \\ [0] & [0] & & [0] & [0] & [V_{F+}] \end{array}\right] = \left[\begin{array}{c|c} [V_{11}] & [V_{12}] \\ \hline [V_{21}] & [V_{22}] \end{array}\right]$$

The matrices $[V_{F-}]$, $[V_{FO}]$ and $[V_{F+}]$ are the input data for the problem representing the (mean square) force level at each time.

The matrices $[V_{xx}]$ $[V_{x\dot{x}}]$ and $[V_{\dot{x}x}]$ $(= [V_{x\dot{x}}])$ are given as data for the first step (representing boundary conditions) and is given for each subsequent step by the previous response.

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