ABSTRACT

DROUJKOVA, MARIA ALEXANDROVNA. Roles of Metaphor in the Growth of Mathematical Understanding. (Under the direction of Sarah B. Berenson.)

The purpose of this qualitative study was to investigate roles of metaphor in the growth of mathematical understanding in the area of proportionality. To this end, the task of designing software that would help other people learn about proportions was offered to six children ages 13 to 16 during individual interviews.

The process of software design helped to access metaphors students developed for thinking about proportionality. A conceptual framework for the study was based on enactivist perspective. Data analysis was based on the study of sources and targets of metaphors, as represented in learners' actions. The Pirie-Kieren Model for the Growth of Mathematical Understanding, in combination with a model for proportional reasoning development that was created for the study, were used to map learners' knowing actions.

Microworlds created by learners supported metaphoric systems, which in turn helped to coordinate the process of knowledge development in proportionality domain. The model used for mapping this process included the notions of equivalence class, relation and invariance. Findings indicate that each of these notions may be developed, during local and context-specific growth of understanding, through actions in additive, multiplicative and qualitative analogy worlds.

Metaphors serve as a tool for coordination of collecting in these worlds. Moreover, metaphor is a way new understanding grows out of old knowing. In metaphoric structures, sources fade away, and formalized targets become independent entities. This extended process of metaphorising is supported by open-ended, extended tasks.

ROLES OF METAPHOR IN THE GROWTH OF MATHEMATICAL UNDERSTANDING

by
MARIA ALEXANDROVNA DROUJKOVA

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Chair of Advisory Committee

DEDICATION

To Dmitri.

BIOGRAPHY

Maria Alexandrovna Droujkova, daughter of Elena I. Koldertsova and Alexander Z. Rozenfeld, was born on September 25, 1972 in Simferopol, Ukraine. From early on, Maria was active in children's science and mathematics organizations, and attended a mathematical magnet high school.

In 1989 Maria entered the Department of Mechanics and Mathematics at Moscow State University, the top mathematics department in the former Soviet Union. In the Fall of that year, she married Dmitri, also a student at the same department. During her studies at Moscow State, Maria specialized in differential equations and bifurcation theory. Her diploma "Heart-shaped asymptote polygons with two singularities" has won a Soros Foundation undergraduate studies award.

In 1994 Maria and her husband entered Tulane University in New Orleans, LA, as graduate students in the Mathematics Department. During her studies there, supported by a teaching assistantship, Maria realized that helping learners learn was the most meaningful part of her academic experience. At the same time, she started to work with young children and their parents, individually and in small groups, helping them make sense of mathematics. She also began to participate in on-line forums for education researchers, and read articles and books relevant to the issues of mathematics education.

When Maria finished her M.S. in Applied Mathematics programs at Tulane in 1997, she entered a practical training program at The Shodor Education Foundation in Durham, NC in the role of mathematics educator. Having had more practical experience, Maria realized that she wanted a deeper understanding of the theories of mathematics education, as well as more opportunities for research. In 1998, Maria entered the Ph.D. program in Mathematics Education at North Carolina State University. Dmitri and Maria's daughter Katherine was born two weeks into Maria's first semester at NCSU. Parenting has been a big influence on the development of Maria's understanding of education and psychology. Maria is expected to receive her Ph.D. in 2004.

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I would like to thank the members of my Ph.D. committee – Sarah B. Berenson, Glenda S. Carter, Hollylynne Stohl, Ron Tzur and Mladen A. Vouk for their advice and guidance in the dissertation process, and for their care throughout my studies. Dr. Carter helped me to clarify many theoretical points through discussions, bringing in more general approaches by providing a science education perspective. Dr. Stohl was an invaluable resource on computer microworlds and learner representations, as well as more general theories. Discussions with Dr. Tzur were extremely helpful in construction of my conceptual framework. Dr. Vouk's help allowed me to refine the software design part of my methodology. I would like to thank my advisor, Dr. Berenson, for her constant support during the years of my studies, for her ability to guide me on sometimes nontraditional paths I took, and for helping me to enter the community of mathematics education scholars.

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CHAPTER 1. INTRODUCTION

Education research studies have, by definition, a single umbrella topic – learning. "How do we begin to unravel the nature of the learning process, a relationship between an individual and the environment that results in the individual having new understandings and capacities?" asks Schoenfeld (1999) who sees hope of answering this question in integration across the social/cognitive constructivism split (p. 7). This split, reaching deeply into the field of education research, is associated with many consequential divides on all levels, from methodology to epistemology.

Initially, these separate points of view, born of analysis, create "handles" that help researchers to tackle phenomena not investigated before. An example from epistemology is the difference in views on where knowledge is situated. In radical constructivist perspectives, researchers "see knowing and knowledge as located "in the head" (or at least the body and brain) of the person" (Kieren, Calvert, Reid, & Simmt, 1995, p. 1). In social constructivism, knowledge has an already constructed existence on a social level, and "societal level constructions are seen as a necessary 'cloud' of knowledge hanging over the individuals or in another sense surrounding them" (Kieren, Calvert et al., 1995, p. 4). From the theoretical and methodological point of view, one can say that radical constructivist models focus on learners, while sociocultural models say more about pedagogy. Another take on the same split is formulated by Schoenfeld (1999), who notes that radical constructivists work on the content of learning, or "How learning works?" while sociocultural researchers concentrate on educational theories, addressing the question, "What is learning?" In their (complimentary) takes on learning, radical constructivists have "focus, detail and tunnel vision" while sociocultural constructivists work on “general ideas about processes by which learning takes place” (p. 7). Another example of the separation in methodology is the divide between structured laboratory interview methods, such as Piagetian type tasks (Goldin, 2000; Inhelder & Piaget, 1958; Noelting, 1980; Piaget & Campbell, 2001), and contextual studies of situated activities (for example, Hoyles, Noss, & Pozzi, 2001; Lave, Murtaugh, & de la Rocha, 1984; Nunes, Schliemann, & Carraher, 1993).

These tendencies to approach the complexity of education in reductionist, analytical terms can be partially attributed to mechanistic metaphors that historically dominated all branches of research (B. Davis, 1996; Goldhaber, 2000; Hartwell, 1996). In some modern conceptual frameworks, for example, (Cobb & Yackel, 1995; B. Davis, 1996; Kieren, Calvert et al., 1995; Lave & Wenger, 1991; Pirie & Kieren, 1994b; Presmeg, 1997a; Sfard, 2000; Tzur & Simon, 1999; Tzur, Simon, Heinz, & Kinzel, 2001), learning is seen as a cyclic process, where each cycle can be broadly described as having inductive and deductive, or analytic and synthetic, sides. Applying such a lens to education research itself leads to a model for reducing complexity to linearity, which serves not as a positivistic atavism, but as the analytic part of the growth of understanding cycle (Figure 1). In a visual simile for cycles of analysis and synthesis in Figure 1, prisms separate light into spectral colors, which are then recombined by another prism. Similarly, education research exhibits analytic processes, where theories and frameworks are created by "separating this from that," and synthetic ones, where theories are created by combination and integration.

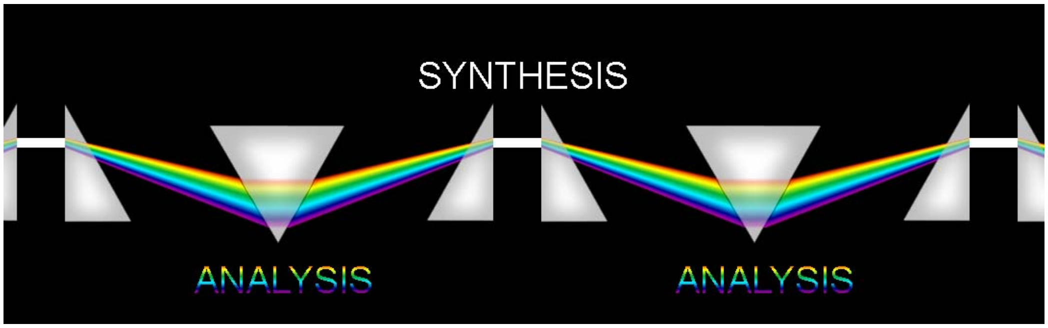

Figure 1. A simile for cycles of synthesis and analysis.

The conceptual framework for this study provides tools and mechanisms for using methods and theoretical constructs from both sides of the divides discussed above. For models of the mathematical content area of this study, proportionality, I use works done from cognitive or radical constructivist perspectives (such as Behr & Harel, 1990; Noelting, 1980; Vergnaud, 1994). On the other hand, for bridging the gap between work done by learners in particular contexts, and corresponding formal ideas and models, I used integrative research (such as Hoyles et al., 2001). The term "integrative" here refers to the fact that these researchers use laboratory-type studies and situated learning, and also context-oriented studies, in their framework. To give an example of a method used for the integration, these researchers found that they could "map" activities observed in the natural setting of the nursing practice into areas of formalized mathematical systems such as "proportionality."

To define and to chart learners' understanding, I used the Pirie-Kieren Dynamical Model For The Growth of Mathematical Understanding (Berenson, Cavey, Clark, & Staley, 2001; Cavey, 2002; Kieren, 2002; Kieren, Calvert et al., 1995; Kieren, Reid, & Pirie, 1995; Pirie & Kieren, 1994b; Pirie & Martin, 2000; Towers, 2001), developed within enactivism. The theoretical stance of enactivism aims at integrating sociocultural and radical constructivist ideas. In enactivist view, cognizing organism is inseparable from the world:

“Knowledge is not in a book or in the library, knowledge is not in our heads. Knowledge is in the inter-action!” (Kieren, Calvert et al., 1995, p. 1). Using this perspective, I study co-creation of *problems* by the learner and the context; and also co-creation of *learning* by the learner, the context and problems. In a non-linear, recursive process learners’ actions are co-defined with the environment (B. Davis, 1996; Kieren, Calvert et al., 1995; Pirie & Kieren, 1994b). Enactivist view of cognition that I use in my conceptual framework for integrating different theories can be modeled by the paradoxical Klein bottle (Figure 2): the world is contained in the person’s cognition that is contained in the world (Kieren, Calvert et al., 1995, p. 2). The paradox this view would present within either a radical constructivist or sociocultural constructivist perspective is resolved in the integrative enactivist epistemology. The Klein bottle in Figure 2, which is used to illustrate the above simile for enactivism, is a four-dimensional object that appears paradoxical in three-dimensional space, since its "inside" opens up into its “outside.” The bottle is not paradoxical in four dimensions, since the notion of the boundary between inside and outside is redefined by adding a dimension. In enactivism, the world is contained "inside" a person's cognition, which is contained in the world. Addressing the issues of integration on the levels of theory and epistemology helps to alleviate dangers, of which several researchers warn (Cobb & Yackel, 1995; Creswell, 1998; Goldhaber, 2000), of combining within one framework findings and ideas from previous research done in different methodological traditions.

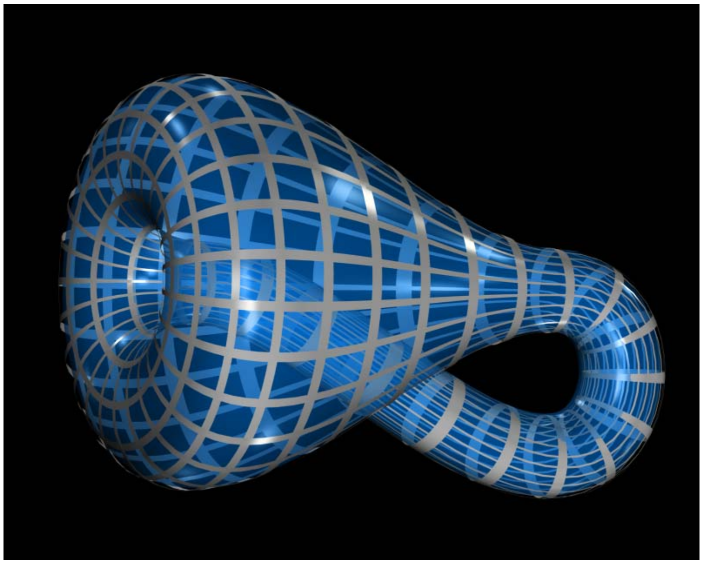

Figure 2. A simile for enactivist epistemology.

The Pirie-Kieren Dynamical Model For The Growth of Mathematical Understanding is a leveled, non-linear, recursive construct for researching learning of a particular topic by a particular person. Each level denotes a certain way of knowing, where "knowing" is understood as an action. The levels unfold from the innermost one of *primitive knowing*, in the sense of "prime" or "elemental." Then follow the levels of *image making*, *image having*, *property noticing*, *formalising*, *observing*, *structuring*, and the outermost level of *inventising*. While trajectories, or mapping onto levels, of each individual learning episode do not present a straight line from primitive knowing to inventising, the construction of each level envelops and structurally incorporates all levels that are inner to it. More formal and abstract levels, namely formalising, observing and structuring, enfold the less formal and more local levels.

The Pirie-Kieren model can be applied to all mathematical contexts and to learners of all levels of experience. For example, it has been modified for mapping mathematical and pedagogical understanding of prospective teachers (Berenson et al., 2001; Cavey, 2002). Moreover, there are complex relationships between leveled models of knowing in different content areas. For example, layers up to formalising in one content area can be considered to be "inside" of a learner's primitive knowing layer for another area (Towers, 2001). Primitive knowing denotes understandings that took place up to the beginning of the new learning, and is composed fractally of other levels from the model that correspond to other concepts.

A trajectory that maps learning into levels corresponding to several mathematical contexts may not only be recursive, that is, going back and forth between levels corresponding to one concept, but also nonlinear with regards to interacting understandings of several concepts. In this manner, the Pirie-Kieren model has been used to map "points" corresponding to complex "trajectories" among different modes of understanding. Yet little research has been done to explain *how* learners move between points on these trajectories. This study helps to promote development of tools for helping learners learn by addressing the "How?" questions that require integration of learning and pedagogical research (Cobb & Yackel, 1995; B. Davis, Sumara, & Luce-Kapler, 2000; Tzur, 1999; Tzur et al., 2001).

Research Questions

The problem I addressed in this study is: "What are roles of metaphor in the growth of mathematical understanding?" In particular, I studied roles of metaphor in the development of students' understanding of proportionality. One reason for choosing proportionality as the mathematical area of focus is that metaphor has been extensively studied as an analogical reasoning phenomenon (Ortony, 1993), and analogical and proportional reasoning are connected (Alexander, White, & Daugherty, 1997; Inhelder & Piaget, 1958; Lehrer, Strom, & Confrey, 2002; Piaget & Campbell, 2001). I investigated connections between proportional and analogical reasoning, using studies from both these areas for my framework. For the literature review, as well as for the data analysis, I used a perspective that is built on umbrella definitions of three notions: equivalency class, relation, and invariance. For short, I call the perspective ERI, after the first letters of its key notions. I call the definitions "umbrella" in that they can be used to map actions in additive, multiplicative or analogy worlds, and interactions between these worlds. *Relation* is a binary operation on numbers or other objects, such as a multiplicative or an additive operation between numbers, or a qualitative relationship. *Invariance* is the preservation of a relation between objects in a group under an operation on each object. *Equivalency class* is a set of groups of objects that have the same system of relations between objects in each group. Curly brackets {} are used to denote equivalency classes. Some examples are in order.

- Couples such as (dog; fur), (fish; scales), and (bird; feathers) belong to the qualitative equivalency class that can be described as {(type of animal; type of covering)}. This is an example of a traditional analogy.
- Couples such as (5; 7), (1.5; 3.5), and (100; 102) belong to the additive equivalency class with respect to the relation between the numbers of "two more," and can be described as {(a; a+2)}. It should be noted that I do not use the sign ":" since it usually denotes division, while statements such as 5:7=100:102, that could be considered descriptions of additive equivalence class members, are traditionally labeled "additive misconceptions" (Abrahamson, 2003; Behr, Harel, Post, & Lesh, 1993; Kaput & West, 1994)
- Couples such as (3; 6), (7; 14), and (-2; -4) belong to the multiplicative equivalency class based on the relation that can be described as "two times more": {(a; 2a)}. This is what is usually called "a proportion." Note, for the

discussion below, that the equivalency class consisting of the same members can be formalized differently, for example, as {(a:2; a)}.

Several researchers of proportional reasoning note a similarity between proportions and analogies that motivated me to include qualitative relations into the ERI perspective. For example, Piaget says: “analogies… are a sort of qualitative proportions. They are relations among relations” (Piaget & Campbell, 2001, p. 139). Piaget and neo-Piagetians such as Noelting (1980) created stage models of proportional reasoning development. In these models, the notions of equivalence, invariance and relation – all the parts required for working with proportionality – are added consequently, one by one, starting with purely qualitative analogies where equivalence is based on non-numeric relations, and ending with formal equivalence of ratios. Their theories, again, focus mostly on mathematical content rather than on mechanisms of learning (Schoenfeld, 1999). Lehrer, Strom and Confrey (2002) see analogy-based systems of scale, magnification, and classification as grounding metaphors that can serve as a basis for development of proportional reasoning.

Researchers who study analogical reasoning have noted a methodological dilemma that is based on theoretical and epistemological differences in definitions of “understanding” and “learning.” Analyzing analogical reasoning studies devoted to metaphors, Ortony (1993) divides their methodologies into “construction” and “comprehension” types. In comprehension studies, understanding of metaphors is identified via explanations of *given* metaphors. Comprehension models describe analogical thinking as appearing relatively late in life, and playing a relatively small, and mostly communicative role in the formation of cognitive structures. Studies that investigate *construction* of metaphors describe analogical reasoning as a cognitive phenomenon of much younger children, and view it as central to the development of cognition and language. This methodological distinction corresponds to the epistemological difference between *comparison* and *interaction* views on metaphor (Black, 1962; Sfard, 2000). According to the interaction view, similarity between two concepts is not simply observed, as in the comparison view, but created. Seeing the similarity is not the prerequisite, but the outcome of the metaphoric projection. Such an analysis can also be applied to studies of proportional reasoning. Laboratory proportional reasoning tasks done in the Piagetian tradition access understanding in the form of solving *given*, strictly pre-determined problems, while ethnographic research looks at understanding in the form of posing, and only then solving, problems.

I define metaphor as the recursive movement between a source and a target that are structurally similar, both changing in the dynamic process of learning (B. Davis, 1996; R. Davis, 1984; English, 1997a; Lakoff & Johnson, 1980; Lakoff & Nunez, 1997, 2000; Pimm, 1987; Presmeg, 1997b; Sfard, 1997). I adopt the notion that mathematical thinking is fundamentally metaphoric (R. Davis, 1984; Lakoff & Nunez, 2000; Sfard, 1997). As discussed above, the “construction-comprehension” divide splits studies on metaphors done outside of the field of mathematics education. However, based on a review of literature in mathematics education, I see a possibility, and a need, to situate metaphor across and above many of the dividing lines drawn by past education research dilemmas. For example, metaphors may play a role in addressing the cognitive constructivists’ learning paradox (Sfard, 1997), since they allow people to work with novel or abstract ideas by mapping them into strong, meaningful images that were originally developed in a different context or for a different purpose (R. Davis, 1984). English (1997a) sees metaphors as tools for creating formal concepts out of image schemas, and of restructuring these concepts in complex ways.

Lakoff and Nunez (2000) also argue that metaphor's primary function is "to allow us to reason about relatively abstract domains using the inferential structure of relatively concrete domains," (p. 42) with structures of image schemas preserved by this mapping. Hence metaphor can be considered as a mechanism for connecting informal and formal ways, or levels, of understanding (Pirie & Kieren, 1994b). The action of metaphoric projection (Sfard, 1997) is co-determined by learner-specific and context-specific factors. Thus sociocultural perspectives can promote the study of metaphor's grounding in culture, and its communicative, shared facets. Additionally, I follow Presmeg, who considers metaphors to be "very private, personal, and ripe with meaning for an individual" (1997b, p. 277), which means metaphors are difficult to access directly by methods such as clinical interviews and laboratory tasks, calling for more contextual, situated approaches.

Based on the above considerations, the following are my research questions:

- What are the roles metaphors might play in the growth of mathematical understanding?
- What metaphors may be involved in the growth of understanding of proportionality?

CHAPTER 2. LITERATURE REVIEW

’Tis writ, ‘In the beginning was the Word.’
I pause, to wonder what is here inferred.
The Word I cannot set supremely high:
A new translation I will try.
I read, if by the spirit I am taught,
This sense: ‘In the beginning was the Thought.’
This opening I need to weigh again,
Or sense may suffer from a hasty pen.
Does Thought create, and work, and rule the hour?
’Twere best: ‘In the beginning was the Power.’
Yet, while the pen is urged with willing fingers,
A sense of doubt and hesitancy lingers.
The spirit comes to guide me in my need,
I write, ‘In the beginning was the Deed.’
Faust (von Goethe, 2003)

After this review paragraph, I start this chapter with issues related to metaphor in mathematics education literature and in studies of analogical reasoning. Next, I review studies of proportional reasoning, and their connections to research on metaphor. I conjecture that a study of metaphor, as well as a study of proportional reasoning, may benefit from integrative approaches, drawing on research from different areas. After that, I present frameworks that support such integrative approaches, and the implications for my conceptual framework. Throughout the chapter, I also note questions, issues and concerns raised within each topic, in their relevancy for the study's methodology. These issues are discussed in detail in Chapter 3.

Definitions of Metaphorical and Analogical Reasoning

I start this part by presenting several definitions of metaphor found in literature, and reviewing categorizations and analysis of metaphors done by different authors. Then I talk about possible roles of metaphors in the growth of understanding, as it is modeled within different constructivist perspectives.

Sfard (2000, p. 68) defines metaphor as "transferring templates from discourse to discourse" and notes that the metaphorical projection with use of templates is a mechanism of creating new mathematical objects. In her view, expectations about properties of newly created mathematical objects may arise, by work of metaphors connected with a signifier, around an "empty space" held by that signifier. Sfard's work is an example of a study with the "construction" (Ortony, 1993, pp. 1-2) methodology, corresponding to the interaction theory (Black, 1962) of metaphor: here the target of the metaphoric projection is being created by the act of projection, and the source may also be modified.

An example of the "comprehension" (Ortony, 1993, pp. 1-2) methodology, based on the comparison theory (Black, 1962) of metaphor is offered by Pimm, (1987):

> One way of seeing metaphor is as a condensed analogy. From the analogy codified as A is to B as C is to D, we speak, for instance, of 'the C of B' or 'A is C'. Thus, 'he is in the winter of his life.' If we are presented only with a metaphor, then in order to

understand it we must decide on the latent analogy upon which it is based. Our interpretation of the metaphoric expression is dependent on this reconstruction of the underlying analogy (pp. 100-101).

In this case, the target of the metaphoric projection is not created by the projection; the metaphor, at most, transfers some features of the source onto the target, connecting the already existing target and the source. In another telling example, children who created their own analogies, such as geometric figures as different types of dinosaurs, were categorized as "least proficient reasoners" (Alexander et al., 1997, p. 141) by researchers focusing on comprehension of the pre-determined analogies.

As I mentioned in Chapter 1, interaction theories give metaphor a fundamental role in the development of reasoning, while comparison theories see it as playing secondary roles. As an example of this distinction, consider the Klein bottle simile I used in Chapter 1, which served as a tool for development of my understanding of enactivist ideas. In this case one can argue that the use of analogical reasoning is "figurative, poetic, colorful or fanciful (Lakoff & Johnson, 1980, p. 13) and not a matter of "ordinary" thinking or "necessary" language. This kind of analogical reasoning is studied by comparison theories. However, many of the researchers who studied metaphors argue that all of our thinking is fundamentally metaphorical: "the way we think, what we experience, and what we do every day is very much a matter of metaphor" (Lakoff & Johnson, 1980, p. 3) even though metaphor may remain largely unnoticed. As an example, consider whether the metaphoric use of the word "tool" was noticeable in the second sentence of this paragraph. In such cases, metaphoric nature of thinking works largely below the level of consciousness (Lakoff & Nunez, 2000).

English (1997a) defines analogical reasoning as "the transfer of structural information from one system, *the base*, to another system, *the target*" (p. 5). She notes that the transfer happens "through matching or mapping processes, which entail finding the relational correspondences between the two systems" (p. 5), and that mathematical thinking is based on such deep, structural correspondences. English, following Lakoff and Nunez (see next) notes that metaphorical reasoning is characterized by mapping between different domains that "can change our understanding of both the source and the target" (p. 7). In a study on word problems, English (1997b) addressed questions such as "whether children *know* to look for common structures" indicating emphasis on the comparison model of analogical reasoning. For example, she writes: "If learners are to make these mappings however, they need to clearly understand the structure of the base and must be able to recognize the correspondence between the base and target" (p. 199). The words "look for" and "recognize" are indicators of the comparison model.

Lakoff and Nunez (1997) see mathematics as "a product of inspired human imagination" (p. 29). They define metaphor as a conceptual mapping between domains that project structure from one domain to the other. Such mappings "are not isolated, but occur in complex systems and combine in complex ways" (p. 32), and these conventional systems of metaphors are used without conscious noticing. The metaphoric mapping preserves image-schemas and inferential structures of the source, adding structure to the target, unless the target structure "overrides" the mapping. Lakoff and Nunez claim that the novel metaphors created consciously, such as Pimm's (1987) "he is in the winter of his life" (p. 100) use mechanisms of the unconscious conventional metaphor system.

In their work on metaphors, researchers create categories of analysis, and the categories usually come in pairs, reflecting the dichotomizing method of analysis. Pimm (1987) talks about *extra-mathematical* and *structural* metaphors, based on their source:

> There are two main sources of metaphor which may be of interest in mathematics education. The first consists of what I term *extra-mathematical* metaphors. These attempts to explain or interpret mathematical ideas and processes in terms of real-world events, and such metaphors can involve everyday objects and processes… The second main source, which I call *structural metaphors…* involves a metaphoric extension of ideas from within mathematics itself (p. 95).

Lakoff and Nunez (1997) created similar categories for analysis, calling them *grounding* and *linking* metaphors: "Grounding metaphors ground mathematical ideas in everyday experience… Linking metaphors allow us to link one branch of mathematics to another" (p. 34). They also talk about a third kind, "concocted novel extensions of the natural grounding metaphor" that belong "to the domain of teaching" which "stand outside of mathematics proper and are part of imaginative, and sometimes forced, methods of mathematics education" (Lakoff & Nunez, 1997, p. 39). Note that in this kind of analysis, mathematics is separated from "the rest of the life" and teaching is separated from learning and from mathematics. However, there are mechanisms that may allow synthesis, since "novel metaphors that we consciously concoct use the mechanisms of our everyday unconscious conventional metaphor system." (Lakoff & Nunez, 1997, p. 32).

Unfortunately, two researchers on metaphor, Pimm and Sfard, use the same word "structural" as two totally disparate terms. Sfard (2000) makes the distinction between structural metaphors that are mapping images of objects, and operational metaphors that are mapping images of actions. In my study such distinctions correspond to objects and actions in software children design. Thus, by analyzing how learners design and develop actions and objects in their programs, I access their structural and operational metaphors. Interviews and observations during the software design process further help me to analyze these types of metaphors: "the language of object manipulation and motion can be recruited in a systematic way to talk about arithmetic" (Lakoff & Nunez, 1997, p. 38).

Categorization of analogical thinking into *conscious* and *unconscious* is where several researchers (English, 1997a; Presmeg, 1997b; Sfard, 2000) draw the line between analogies and metaphors. "Analogy enters the scene when we become aware of a similarity between two concepts that have already been created; the act of creation itself is a matter of metaphor" (Sfard, 2000, p. 345).

Another categorization into two types of metaphor is based on whether or not they "may be accepted into the "taken-as-shared" understandings" (Presmeg, 1997b, p. 268). Pimm (1987) calls these classes *idiosyncratic* and *conventional*:

> There are, therefore, personal or *idiosyncratic* metaphors which perhaps one person or only a few individuals employ, and which are often invented extemporaneously, resulting from a personal insight… The way of seeing embodied in the metaphor may become institutionalized (possibly by means of being written about in books!) in which case it would then become a *conventional* metaphor, a standard form both of conception and expression (p. 98).

Sfard (1997) writes about a distinction between visual and word metaphors. This distinction relates to several deep issues, for example, to the distinction between visual-spatial and auditory-sequential types of thinking and learning (Silverman, 2002). Another issue involves “the different manners in which hearing and sight situate us in relationships with others” (B. Davis, 1996, p. 37) and with the world, and also epistemological stances associated with visual and hearing metaphors:

> Visual metaphors, such as “the mind’s eye,” suggest a camera passively recording a static reality and promote the illusion that disengagement and objectification are central to the construction of knowledge. Unlike the eye, the ear operates by registering nearby subtle change. Unlike the eye, the ear requires closeness between subject and object. Unlike seeing, speaking and listening suggest dialogue and interaction (Belenky, McVicker Clinchy, Coldberger, & Tarule, 1986, p. 18).

A notion related to metaphor is what R. Davis (1984) calls an *assimilation paradigm*. It is a system, involving ideas for which most learners have strong representations, that is an “isomorphic image” of a mathematical domain. Such assimilation paradigms, for example, the turtle programming metaphor (Clements & Sarama, 1997; Papert, 1993), provide learners with tools for solving problems in the mathematical domain. In my view, assimilation paradigms correspond to what Lakoff and Nunez (1997; 2000) call teaching metaphors. Assimilation paradigms can also be related to what Pimm (1987) and Presmeg (1997b) call shared, conventional metaphors, since most learners must have the representations underlying assimilation paradigms. The idea of an assimilation paradigm, in my opinion, may integrate features of the comparison and interaction views on metaphor. For learners, an assimilation paradigm carries, as the metaphor from the interaction model, “a constitutive power” (Sfard, 1997, p. 344). However, people who help learners establish such an assimilation paradigm (e.g. researchers), hopefully, can consciously reason about a part of the projections happening inside the assimilation paradigm, in this part approaching the comparison model.

To summarize, I define metaphor as the recursive movement between a source and a target, both changing in the dynamic process of learning. I relate the notion of metaphor to that of an assimilation paradigm, and note that a computer microworld may serve as a basis for an assimilation paradigm. In this study, the process of designing a microworld or a computer game about proportionality serves as an assimilation paradigm.

Roles of Metaphor in Learning

The division between new and old knowing arises from definitions of learning. The key question about this division that is relevant to this study's goal is, "How does new knowing come to be?" Sociocultural and cognitive constructivists, as well as enactivists, believe that the answer is, "New knowing is born of old knowing": "The idea that new knowledge germinates in old knowledge has been promoted by all the theoreticians of intellectual development, from Piaget to Vygotski to contemporary cognitive scientists" (Sfard, 1997, p. 350). However, there are differences in hypotheses about mechanisms or ways by which new knowing is born of old. I focus on the role of metaphors in answering this question postulated by researchers.

The defining theme of the interaction perspective (Black, 1962; Ortony, 1993) is that metaphors play a constitutive role in creating new knowing. Metaphors are not just connecting old ideas in new ways, but changing both source and target ideas in a "zig-zag," cyclic process of metaphoric projection: "what is being so-called a "target concept" is often *created* by metaphor rather than being only used by it" (Sfard, 1997, p. 361). Lakoff and Nunez (1997) also write on how metaphors can "add structure to a target domain" (p. 32).

One particular example of a mechanism for making new knowing from old is the model of *bricolage,* "the process of using materials that happen to be at hand, in order to carry out some quite new process or construction that becomes possible when these old materials are combined in appropriate new ways. This suggests the image of a child's mental representation being built up in rather the way that some children build tree houses, using whatever boards, nails, and other materials are available" (R. Davis & Maher, 1997, p. 95). The metaphor of bricolage comes from a neo-Darwinian model of evolution (B. Davis, 1996; Varela, Thompson, & Rosch, 1991) that is built on survival of the *fit*, rather than fittest. What is compatible with a given context, survives. In bricolage learning process, metaphors play at least two roles: one is the role of "supply lines," or transfer tools, moving concepts into the newly emergent structure. The second role is that of "reconstruction blueprints" where metaphors change the structure of the target concept, as well as the structure of the source of the metaphoric projection. R. Davis (1984), describing a similar model, uses the terms "collage" and "assembly," as well as a lot of italics for emphasis: "*in order to "think about" abstract matters, we make use of our cognitive collages… these collages themselves must play a major role in shaping our thinking*" (p. 178). Davis and Maher (1997) underlie the role of metaphors as "tools to think with" (p. 105), not as communication devices: "We use a metaphor *in order to represent some piece of knowledge within our own mind… we use metaphors within our own minds in order to be able to think*" (R. Davis, 1984, pp. 177-178).

Another take on roles of metaphor in learning comes from frameworks involving *reification*, "in which a mathematical process becomes reified as a mathematical object, so that it can be used as a unit in a further process at a higher level of abstraction, until this process in turn becomes reified as a mathematical object, and so on" (Presmeg, 1997b, p. 275). This process corresponds to "a long chain of metaphorical projections and transformations" (Sfard, 1997, p. 351) where primary bodily experience is repeatedly transformed in the process of forming advanced mathematical concepts. Inherent in the process of reification is the serious difficulty related to the signifier of the new mathematical process or object: "In the perceptual reality, processes and objects are two different things" (Sfard, 2000, p. 50). Yet the same signifier initially plays both the operational and the structural roles: "The expression '5-8' itself could be used both operationally, as denoting and operation, and structurally, as signifying an object (the result of an operation)" (Sfard, 2000, p. 50). In the reification model of learning, metaphors play the crucial role in resolving this difficulty. During the process of reification, a new name or a signifier, which now is a noun, and a discourse template including this signifier, serves as "an act of conception rather than of baptism" (Sfard, 2000, p. 68) for the new mathematical object being reified from a process. Such templates with nouns come from the "actual reality" discourse, where perceptual mediation by material objects exists. The templates are then transferred into the mathematical discourse, the effect that is defined as metaphor. "We may say, therefore, that what we call "mathematical objects" are metaphors resulting from certain linguistic transplants" (Sfard, 2000, p. 68).

Characteristically, Sfard uses the motto of sociocultural researchers as a name for an article part: "In the beginning was the word: The role of signifier" (2000, p. 47). Similarly, in Vygotskian terms, people learn in part by re-constructing, on the intrapersonal plane, new to them knowledge that already exists on interpersonal plane (Vygotskii, 1996). Metaphors may play a part in this model in the following manner: “Like all other ‘acts of meaning,’ the metaphorical projections we perform are socially mediated… coordination of metaphors which underlies learner’s conceptual development is only made possible by his or her ongoing exchange with others” (Sfard, 1997, p. 363).

Metaphors may play a role in transitions between informal and formal understanding. English (1997a) sees metaphors as tools for creating formal concepts out of image schemas, and of restructuring these concepts in complex ways. I investigate this role of metaphor using the Pirie-Kieren model (Pirie & Kieren, 1994b), where metaphor can be considered as a mechanism for connecting informal and formal layers of knowing. Formalization may also be connected to the “death” of metaphors (Sfard, 1997), as in an example of the rational number concept: “the process of concept construction is only completed when the metaphor ‘dies’ and the learner becomes able to think about the new number as a self-sustained independent entity belonging to the abstract domain of numbers” (p. 352).

A distinction between two types of models of growth of understanding has to be made at this point. The Pirie-Kieren model presents learning as a cyclic, but not unidirectional process. For example, a learner may “fold back” (Pirie & Martin, 2000), that is, move to an inner layer from an outer layer, for example, from formalising to image making. Even though the visual representation of the layers of understanding in the Pirie-Kieren model is sequential, with more formal layers enveloping less formal layers, the trajectories of learning are not sequential. There should be a certain caution while using such a model together with developmental models. For example, Sfard’s chain of metaphoric projections in reification, or Tzur and Simon’s stages of knowing (1999), are described as developmental processes, repeated cyclically with different mathematical objects or concepts. I believe that both types of approaches can benefit, if used together for analysis of different levels of education phenomena, a study of metaphor and proportional reasoning. The Pirie-Kieren model itself, as I mentioned, shows correspondences between recursive, non-linear ways people learn on a micro-scale, or from the position of learner as observed by researchers, and a sequential model of levels of learning on a macro-scale, or from the position of pedagogy. Connecting Maturana and Varela's (Varela et al., 1991; von Glasersfeld, 1997) idea of autopoesis with metaphor, Sfard (1997) also talks about a recursive way of metaphor's development, where "creation of new concepts should be viewed as a zig-zag movement between relatively familiar source and the emergent target… this is the process of mutual adaptation which, with its every swing, does not only strengthen the target but also alters and adds new dimensions to the source" (p. 355). This way of modeling thinking parallels agile software design processes (Astels, Miller, & Novak, 2002; Beck, 1999) that I adopt for this study, as explained in more detail in Chapter 3. The zig-zag nature of metaphor can also be used for resolving "vicious circles" (Barwise & Moss, 1996), or the learning paradox (von Glasersfeld, 1998), arising within cognitive constructivists' models. I talk about it in the section on proportional reasoning, using examples from proportional reasoning literature.

To summarize, I investigate connections between metaphor's recursive movement between the source and the target, and changes in learners' understanding as modeled by the Pirie-Kieren model. I looked at the constitutive power of metaphor through interaction models, and at the reasoning applied to existing concepts through comparison models of metaphors.

Proportional Reasoning Studies

The area of proportionality attracts the interest of mathematics education researchers for many reasons, from methodological to epistemological, from historical to sociological. Theoretically, both proportionality and proportional reasoning are rich in complex phenomena ensuring fruitful studies. The subjects of proportion and ratio are included in most mathematical curricula, and in state standards and NCTM standards (2000). Proportional reasoning is one of the main arithmetic precursors to algebraic thinking. Moreover, by now the community of education scholars has conducted much research in the area, allowing new studies to start at a fairly sophisticated level. All these reasons influenced my choice of proportionality as the main mathematical area for the study.

Proportional reasoning situations have been analyzed and categorized in various ways. Frameworks that give birth to categorizations of situations can, in their turn, be categorized by, first, the degree to which they treat tasks and learners “as collections of variables” and, second, the degree to which they attempt to address tasks as “authentic, situated activity” (Kaput & West, 1994, p. 235). Not coincidently, “variable” approaches tend to involve more deductive tasks and contextual approaches tend to involve more inductive tasks. Using terminology of metaphor studies, “variable” approaches correspond to comparison theories, while contextual approaches correspond to interaction theories.

While some researchers attempt to achieve mostly “variable-based approach” by fixing many situation variables ahead of time, (such as Noelting, 1980), others strive for maximal authenticity of contexts and activities, (such as Hoyles et al., 2001; Lave et al., 1984). This choice to “take sides” or to pick out one way from either laboratory tasks or naturalistic inquiry originates in multiple theoretical, epistemological and methodological dilemmas arising from tensions between the control of variables and authenticity of the situation.

Yet excluding one of the two approaches to the study design has been only one of possible ways of resolving their contradictions. First of all, it can be argued that both contextual and "laboratory" features are necessarily present in each study, even if the degree of conscious attention the researcher pays to one or the other varies. Every study, even a most carefully controlled laboratory task, is situated in some context. Every study, even an ethnomathematical study, focuses on certain aspects of the situation – the aspects that would be construed as "variables" before the appearance of context-centered frameworks. Secondly, as pedagogy becomes science (Kuhn, 1970) and researchers become more conscious of epistemological and theoretical issues, they attempt to create frameworks based on complex integration of the two approaches, resolving theoretical contradictions on philosophical level, and methodological contradictions on the level of theories. An example of such integrative studies are laboratory tasks designed around "naturally occurring" phenomena. That is, contextual considerations are infused into the way "variables" are construed in variable-driven task design. Another synthetic example is the choice of a naturalistic setting where particular actions of interest to the researcher are sure to take place, and particular phenomena that can be mapped into pre-determined variables are sure to appear. Studies of yet another synthetic type take a contextual approach to a *created* rather than naturally occurring context, such as a computer microworld. In such studies, the laboratory-type boundaries are built into the very arena or the world where the study takes place.

In the next several parts of the chapter, studies of proportional reasoning are reviewed based on their design features. Thus descriptions of studies are grouped into categories of ethnographic methods, laboratory task designs, and “naturally occuring” variables. Next, I talk about a way definitions of proportionality from these studies can be formalized, and also the particular manner in which the learning paradox manifests itself in the area of proportionality. Finally, I describe microworld-based research as a way to integrate other methodological approaches.

Education Ethnographies

Proportional reasoning studies of this type analyze “everyday activities in context” (Lave et al., 1984, p. 67). That is, studies involve situations that are neither pre-fabricated nor negotiable by the researcher. What constitutes mathematical activity in such contexts is defined by the activity’s formal properties, and is shaped by broader activities within which it occurs, as well as the past experiences of actors as problem solvers.

An example of such an activity is price comparison observed by Lave (1984, p. 81) during the grocery shopping study. Here, mathematical tasks have a multitude of meanings, from solving “pure” mathematical problems to symbolism of power in money or in the mathematical “objectivity.” Lave found that tasks as well as solutions are established in a recursive, “gap-closing” process that involves both qualitative and quantitative considerations. Examples of activities include decision-making, such as choosing between two brands by comparing prices, and decision justifying, such as using mathematics as a declared reason for a decision based on qualitative measures of brand loyalty or package attractiveness. Tasks are co-created by setting and the person in that setting. Not only solutions, but also tasks are flexible – there is a "dialectical movement between the expected shape of the solution and the information and calculation devices at hand, all in pursuit of a solution that is germane to the activity" that gave it the initial shape (Lave et al., 1984, p. 87). Another example of this kind of mathematical ethnography is a study of money-related activities among young street sellers (Nunes et al., 1993). Researchers in these studies follow the grounded theory tradition in allowing "variables" of the study to emerge from data. For example, they used "mapping" of activities into areas of a formalized mathematical system. Researchers did not know ahead of time into *what* areas, such as proportionality or multiplication, the activities would be mapped.

Other educational ethnographies focus on actions of a somewhat pre-determined kind, allowing researchers to fix, ahead of time, some of the "variables." It should be noted that these actions are pre-determined by an authentic *context* in which they are embedded, not by researchers' creative actions. Therefore, in such research one can use frameworks that combine both situated learning and "fixed variable" features. An example of such a design is the analysis of proportional reasoning in nursing practice (Hoyles et al., 2001). More exactly, activities of nurses as they calculated drug dosages could be "mapped into" the area formalized in mathematics as proportion. The authors consider data from on-the-job dosage calculations as contextual phenomena where the results are a function of the situation. The choice of pediatric nurses for the study is an example of a context-conscious study design. Pediatric nursing is more mathematically intensive because drug dosages vary more for children than for adults. More generally, the choice of drug dosages as a high-stake task focused the research even more, downplaying the role of qualitative choice reasons (e.g. brand likes and dislikes) so pronounced in the study of shoppers (Lave et al., 1984) described above. It turned out that different tasks, as construed by different solution strategies, were associated with particular features of the situation. For example, particular drugs with known concentrations were associated with particular simpler ratios such as 1:2 and with doubling strategies, or with powers of ten. By choosing a setting where phenomena that can be construed as "proportional" occur naturally, the researchers could keep their study situated and, at the same time, could use relevant research even if it was conducted in school or laboratory settings. The second, follow-up study focused on differences between nurses' behavior "in situ" and in test situations (Noss, Hoyles, & Pozzi, 2002), and found that task authenticity dimension had a large influence on whether or not the nurses solved problems correctly.

Laboratory Task Research

Researchers represented in this section use laboratory task design that "involve minimally a subject (the problems solver) and an interviewer (the clinician), interacting in relation to one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a preplanned way" (Goldin, 2000, p. 519). Such laboratory tasks devoted to proportional reasoning can be further categorized based on formal mathematical representations of the idea of proportionality (Kaput & West, 1994, p. 236). Some of these studies were influenced by quantitative research frameworks from the mechanistic worldview (Goldhaber, 2000), where closed-ended tasks are formed around formalizations-derived variables, and "the universe of content was fixed at the beginning" (Noelting, 1980).

Tournaire and Pulos (1985, p. 183) distinguish four categories of tasks based on the situation described: physical tasks, rate problems, mixture problems, and probability tasks.

This classification may apply to other types of studies as well, as there is nothing that should restrict them to laboratory task research. For example, the type of task where ratios are construed as *probability* was used by Drier (2000) in a computer microworld study. *Physical* tasks involve a physical principle, for example, the idea of projection or gears. *Rate* problems involve ratios of "dissimilar objects," for example, assigning patients to doctors or giving food to fish. An example of a *mixture* task is mixing orange juice with water and studying concentration.

Another kind of approaches to categorizing proportional reasoning problems deals with the numerical complexity of the task, or, in other words, with task's "number structure" (Tournaire & Pulos, 1985, p. 183). For example, juice-water mixture comparison puzzles in (Noelting, 1980) study are sorted into nine structural categories, from comparing "non-mixed" pairs (0,1) and (1,0), to "equivalence classes" such as comparing (1,2) and (2,4) and to "pairs without corresponding terms multiple of one another" such as (5,7) and (3,5) (p. 228). Tournaire and Pulos (1985, p. 183) mention several "hurdles" separating problems of different numerical complexity: 1:2 ratios, 1:n ratios, non-unit ratios, and non-integer ratios. Irrational numbers, which would present the next level of numerical complexity, are not mentioned in proportional reasoning studies I have found. A distinction that relates both to the number structure and to the situation described in the task is whether the task deals with discrete (apples, pennies) or continuous (time, weight) variables (Tournaire & Pulos, 1985, p. 183), the later being less accessible to learners.

By the problem's *goal*, proportional reasoning problems have been sorted into *missing value* problems and *comparison* problems (Karplus, Pulos, & Stage, 1983, p. 220). The terms authors use to describe the difference are "extensive variable" and "intensive variable." Extensive variable is a quantitative description of an object, for example, length or time. Intensive variable is a constant ratio relationship between extensive variables. In missing value problems, the goal is to find an unknown extensive variable. In comparison problems, the goal is to compare intensive variables.

To summarize the above, task designs can be categorized based on the number structure of the task, the situation described in the task, and one of the ways the ideas of rate or ratio or proportion are formalized mathematically.

"Naturally Occurring" Variables

In this section, I review works of researchers who also study laboratory tasks with controlled variables, with a distinction from the previous section described by Kaput and West (1994) as: "our analyses are done relative to forms of reasoning that we believe are naturally occurring" (p. 236). There is a higher degree of attention to context and to learner in these studies. Researchers here use methods developed within qualitative research frameworks. Thus, studies in this section attempt to integrate contextual and "controlled variable" approaches to research by choosing their "variables" based on contextual considerations.

Lamon's (1993) framework is based on sorting tasks, by their mathematical and semantic characteristics, into four "*semantic types*." *Well-chunked measure* tasks involve "the comparison of two extensive measures, resulting in an intensive measure (or rate)" (p. 42), such as price or speed. *Part-part-whole* problems involve ratios of sub-sets, for example, boys to girls in a group of children. *Associated set* tasks deal with element pairs that are only connected in the given problem, for example, people and pizzas; they are construed similarly to the tasks that Tournaire and Pulos (1985) called “rate tasks.” The fourth type, *stretchers and shrinkers*, involves scaling up or down according to a fixed ratio, for example, the growth of trees.

A conceptualization using the terms “extensive and intensive variables” (Kaput & West, 1994, p. 239) separates *particular intensive quantities* and *rate intensive quantities*. A rate intensive quantity describes a general relationship between whole extensive quantities, for example, “four legs per every dog” refers to all dogs and all dog legs. Particular intensive quantity describes a particular instance of rate intensive quantity, such as “twelve eyes of these two dragons” – not generalizing, at least at the moment, to all dragons. The authors believe that rate intensive quantities are built from many experiences with particular intensive quantities.

Other studies create categories of proportional *reasoning*, focusing on different kinds of strategies for solving proportional problems. (Karplus et al., 1983) propose three categories of reasoning in proportional tasks: within, between and other. “*Within”* type of reasoning deals with rates of two variables within one instance of the linear relationship, for example, with the rate of distance to time. “*Between”* type of reasoning deals with ratios between corresponding quantities in different instances of the linear relationship, for example, a ratio between times or distances. The rest of strategies, for example, the cross-product algorithm, are grouped under the name “other.”

Formalizations of Proportionality and the Learning Paradox

To analyze different models of proportional reasoning found in literature, I adapted the Vergnaud’s model for formalization of proportional thinking (Behr & Harel, 1990;

Thompson & Saldanha, 2003; Vergnaud, 1983, 1988, 1994). It should be clear that this formalization is not the way learners think about proportionality; it is a way to express, in a uniform manner, different ways researchers of proportional reasoning build their theories. This formalization employs the notions of equivalency class, relation, and invariance (ERI). Vergnaud (1994), uses "a sophisticated mathematical framework to theorize about the intuitive knowledge of learners" (p. 49), that is, he *maps* learner knowledge into his model, which is formalized in mathematical terms. The framework itself involves complex relationships between four types of proportional situations, and schemes defined as "the invariant organization of action for a certain class of situations" (Vergnaud, 1994, p. 53). The four types of situations involve simple and double proportions, concatenation of simple proportions, and comparison of rates and ratios.

I adapt the formalization used by Vergnaud, applying it to different models used by other researchers. My adaptation consists of broadening the definition in such as way that it can describe not only proportions, but also equivalency classes based on non-multiplicative relations. Let ~ denote the relation, and let ≡ denote the statement of equivalence, that is, equality for quantitative relations, or "being the same" for qualitative relations. Then the classical formalizations of proportional reasoning types, now broadened to include analogical reasoning, become:

{(c; d) such that c~d≡a~b for the given (a, b, ~)}("within" reasoning), or

{(c; d) such that a~c≡b~d for the given (a, b, ~)} ("between" reasoning), or

{(c; d) such that c~d≡k for the given k and ~} ("ratio-based").

For example, an analogy equivalency class with members such as (puppy; dog) and (kitten; cat) can be formally described in the "ratio-based" style as {(c, d) such that c~d≡"child"}. In such formalizations, there are two operands, ~ and ≡, stating the *existing* proportionality. That is, for *given* numbers or objects a, b, c and d, or c, d and k, the definition can be used to determine if there is proportionality. This way of defining proportionality does not tell, explicitly, how to solve such proportional reasoning problems as missing value problems, or, in other words, how to *construct* all the members of the equivalency class described in this manner.

Vergnaud and Thompson, in their models, use a definition of proportionality that emphasizes *construction* of proportion members, explicating linearity of the constructive operation. To do so, the definition requires *three* operands: the equivalence ≡, a binary operation ~, and a linear with respect to ~ unary operation or function the authors denote *f*:

{(**k**~c, *f*(**k**~c)) such that *f*(**k**~c)≡**k**~*f*(c) for any **k** and for given (c, *f*, ~, ≡)}.

For example, if ~ denotes multiplication (as a binary operation), and *f* denotes multiplication by 3 (as a unary operation), and c=2, the proportion turns into:

{(2k, 3*2k)}

To obtain the "traditional" definition of the same proportion, take c=1 to get:

{(k, 3*k)}

Note how the "traditional" definition *implicitly* includes the linearity, or in other versions invariance under an operation, while the three-operand definition explicates it, saying how all proportion members can be found, once f is established to be linear with respect to ~. The establishment of linearity *f*(k~c)≡k~*f*(c) is defined as the statement of proportionality.

Models of proportional reasoning development can be analyzed from the ERI perspective based on the relationships between three notions present in models: equivalence, invariance and relation. The relation considered in literature is most often a multiplicative operation. For example, Behr and Harel (1990) use invariance and relation to sort proportional tasks into invariance-of-ratio and invariance-of-product categories. If the most commonly used solutions to a task are based on ratio comparison, the task is classified as invariance-of-ratio. If the most commonly used solutions are based on product comparison, the task is classified as invariance-of-product. Kaput and West (1994) see growth of proportional reasoning as development of the notion of class of equivalence or "homogeneity" from invariance. In their model, learners start from numerical invariance of particular intensive qualities, such as speed or price, and develop the idea of homogeneity of the intensive quantity. Vergnaud (1994) also construes proportion as an equivalence class of pairs of quantities, where the result of an operation within a pair is invariant under the corresponding operation on each quantity. In other words, the operation within pairs is invariant under the transformation between pairs.

Several researchers note tensions inherent in constructivist models of proportional reasoning. The notions of equivalence and invariance are construed in models as based on the operation(s). On the other hand, conceptualizing the operation within the domain of proportionality is dependent on developing these two notions. Behr and Harel (1990) provide a proportional reasoning example showing that the idea of invariance is necessary to develop the notion of decimal operations. To understand a particular operation means to be able "to determine whether the operation on initial quantities is *invariant* under the transformation," where the transformation is caused by the operation under study (Behr & Harel, 1990, p. 29). In Vergnaud's (1994) model, proportionality also may be construed as the commutative property of "within" and "between" operations, that is, invariance of their results under the switch. Understanding this property, which is a part of development of notions of operations, depends on the notion of invariance.

In the above examples, constructing ideas of operations on one hand, and invariance and equivalence on the other, form a “vicious circle” (Vergnaud, 1994). This content paradox can be considered a representation of a profound theoretical tension in radical constructivism, or the learning paradox (Tzur & Simon, 1999; von Glasersfeld, 1998), where the development of concepts seems to rely on the learner having already developed similarly advanced concepts. It should be noted that other constructivist perspectives are also driven by such deep, paradoxical tensions. In sociocultural perspectives, there is the question of the relationships between interpersonal, contextual understanding and intra-personal understanding, or the problem of internalization (Vygotskii, 1996). In enactivist theories, there is the problem of the relationships between understanding in action, co-created with context, and understandings brought to and from actions in the form of “objectified actions” (B. Davis et al., 2000; Kieren, Calvert et al., 1995). Development of the field of education research happens in large part by addressing such profound problems. I used the epigraph of the chapter as a simile for the idea of different ways in which problems of the creation of new knowledge are addressed in different perspectives.

An example, worked out in much detail, of addressing deep problems of growth of understanding analytically by developing a concept model comes from work of Piagetians. Their way of addressing tensions in the conceptualization of proportionality was to separate the conceptualization into stages within a developmental or hierarchical system. In their model, learners start from pre-proportional analogical reasoning or “eductions of correlates” (sic), such as “hair is to mammals as feathers are to birds” (Inhelder & Piaget, 1958, p. 317) thereby developing the notion of the purely qualitative equivalence based on a qualitative relation, or "correlation," first. Using this notion, learners move to the concept of invariance or "reciprocity", such as when they balance a lever by "placing a light weight at a great distance and a heavy weight at a small distance" (p. 317). At this stage, a qualitative operation takes its place in the development of the proportionality notion. Quantitative proportions, defined as the equality of two ratios, are denoted as formal operations, and appear as a development of qualitative invariance by including numerical compensations. Noelting (1980) constructed a similarly hierarchical model, where learners start from distinguishing equivalence from non-equivalence in examples such as comparing (4, 1) and (1, 4). In Noelting's later stages learners are perceived as gradually adopting "add on" strategies involving multiplication or division of terms, developing the notion of invariance under operations.

In the present study I use mathematical formalizations to map learners' work with proportionality. I hypothesize that differences between constructive and non-constructive formalizations of proportionality models can be linked with construction and comprehension models, corresponding to interaction and comparison theories discussed in the review of metaphors. A connection between proportionality, used in a non-constructive formalization model, and analogical reasoning, seen from the position of a comparison theory, is discussed by Pimm (1987). The Greek word *analogia* was originally used as a mathematical term for "proportion" or "a relation which either holds or does not hold between four quantities" (Pimm, 1987, p. 100). Note the non-constructive definition of proportion here. Only later this term entered the common use in an instance of "reverse borrowing" where mathematical term becomes common, rather than the other way around. "The relation between analogy and proportion is itself one of analogy. Thus an analogy links one relationship A:B to another C:D… While proportion is a symmetric mathematical relation, the use of analogy customarily presumes a preferred direction of application, in that it assumes more is known about one relationship than about the other" (Pimm, 1987, p. 100). Pimm defines metaphor as solution to a sort of a missing value problem in analogy: "One way of seeing metaphor is as a condensed analogy. From the analogy codified as A is to B as C is to D, we speak, for instance, of "the C of B" or "A is C". Thus, "he is in the winter of his life." If we are presented only with a metaphor, then in order to understand it we must decide on the latent analogy upon which it is based. Our interpretation of the metaphoric expression is dependent on this reconstruction of the underlying analogy" (pp. 100-101).

I investigate possible connections between learners' growth of understanding in the area of proportionality, and analogical reasoning and metaphoric thinking. In terms of metaphor studies, much work has been done on proportional *reasoning*, that is, on proportionality models that can be compared with comprehension methodologies and comparison theories on metaphors. More work is needed in the area of construction methodologies and interaction theories. Also, there is a need to investigate relationships between construction and comprehension models.

Microworld-Based Research

This section reports on studies that integrate some amount of pre-determined research goals and pre-planned variables, and some degree of problem openness and attention to context, by creating their own laboratory "worlds," that is, computer-based or manipulatives-based microworlds. These microworlds are, arguably, rich enough to allow learning phenomena similar to those observed in naturalistic settings: "the pupil would bump into imbedded mathematical ideas in the context of meaningful activity" (Hoyles & Noss, 1987, p. 142). On the other hand, by playing the role of demiurges the researchers have control over the rules of their microworlds. Moreover, most researchers observe not only "free play" of learners within the microworlds, but also, in the majority of studies, engagement of learners with pre-determined tasks within microworlds, "guided discovery" activities or "independent mathematical activities" with researcher-set goals.

One common theme in microworld studies is the possibility for learners to "try out their ideas *and simultaneously* receive feedback" (Clements, 2000a, p. 36). That is, learners have opportunities to try out their ideas and to change their approaches based on the results. This feature gives a more authentic, situated activity quality to the studies. It also complements cyclic models of learning and of metaphor described above. Mathematical thinking is implemented, and tried out, in microworld actions: "When designing the microworlds, it was our intension to create possible actions, which, when used by children with specific intension and purposes, could be implementations of their mathematical operations" (Steffe & Tzur, 1994, p. 103). Thus microworlds, by providing an open environment, allow the use of frameworks with situated learning features. However, by placing additional restrictions on goals or tools available to learners within the microworld, researchers can also use some features of laboratory task methodologies, converting spontaneous activities into guided projects, open-ended problems, or even classical laboratory tasks.

Since microworlds may allow different kinds of tasks and different kinds of learner activities, researchers were led to sorting activities into several categories. One type of activity categorization in microworlds is to divide them into those that "lead to independent mathematical activity" (Steffe & Tzur, 1994, p. 105) and those that do not. In other words, also used by these authors, the activities can be categorized as open-ended or close-ended. In open-ended activities learners can continue their mathematical work for indefinitely long periods, and can use various strategies. In the other type of tasks, children form "no goals that would sustain independent mathematical activity" (Steffe & Tzur, 1994, p. 109), and the activity is sustained by teacher guidance. An example of the first type is a project to create "a set of fraction sticks," that is, to draw sticks of equal lengths partitioned into two, three, and so on parts. The authors consider the task to be open-ended because "the children could use various strategies in making an indefinite number of fraction sticks" (Steffe & Tzur, 1994, p. 105). In the original microworld, there was a command "PARTS" that would automatically divide a given stick into equal parts. However, the researchers disabled the command for the task, which shows a particular flavor of microworld task design, where researchers use contextual approaches, but may influence the context in a directed manner. In the same study, the authors offer children to make a particular fraction within the microworld "in a different way" – that is, to create an equivalent fraction. This did not lead to independent mathematical activity. Clements (2000a) uses the phrase "learners can develop their own goals" (p. 25) to describe the next level of openness of activities in microworld, more open than "independent mathematical activity" with goals heavily negotiated by researchers. Several researchers (Hancock & Osterweil, 1996; Papert, 1993; Steffe & Weigel, 1994) call that next level of openness "play." Steffe (1994) also distinguishes between "cognitive play" and "independent mathematical activity."

Drier (2000) used a modified Steffe and Wiegel (1994) model to sort activities by the playful orientation of the children's actions, the mathematical purpose of the actions, and whether the actions were initiated and sustained by children or by the teacher. The four categories are cognitive play, mathematical activity, independent mathematical activity, and mathematical play. Cognitive play and mathematical play are initiated and sustained by a child, their difference being in levels of mathematization of the playful activity. In terms used in (Lave et al., 1984), there is a more direct mapping of a child's activity into formal mathematics when the child is engaged in mathematical play, compared to cognitive play where actions are not intentionally mathematically oriented. Mathematical activity and mathematical independent activity are goal-oriented. Both are teacher-initiated, but the first one is sustained by teacher guidance, and the second one by the child. B. Davis (1996) sees learner and teacher as parts of the context. The context, therefore, is not a setting for the learning activity and not a place for actors such as learners and teachers.

The distinction between cognitive play and mathematical play is an example of partitions being born of the divide between formal and informal. In terms of metaphors, mathematical play may be seen as the cognitive play after the metaphor-driven formalization (English, 1997b) has occurred. In relation to the Pirie-Kieren model (1994b), mathematical play corresponds to the outer layers and cognitive play to the inner layers of the model.

Another reflection of the divide between formal and informal understanding is the degree of "tangibility" in microworlds: if the microworld objects resemble anything "real," or present decontextualized abstractions. Most versions of Logo, for example, contain a turtle picture that leaves a trail as it "walks" guided by commands. The trail is an example of a metaphorical, tangible representation of the idea that can be formally construed as "drawing vectors." While Olive (1999) talks about a pizza-sharing task in his microworld Sticks, what learners actually see are more abstract objects, resembling the namesake of the world rather than pizzas.

Yet another dimension that can be construed as reflected off the formal-informal divide is related to the kinds of programming within the microworld. In some studies, learners cannot control features of the microworld via programming at all, but use a direct manipulate interface, that is, more visual, concrete, informal methods. For example, the tasks in the microworld Sticks (Olive, 1999) do not allow learners to program. In an example of another type of study, Harel and Papert (1990) studied a task where fourth-grade learners designed software to teach fractions to third graders in a microworld involving "symbol" programming, where each code command must be individually programmed in a more formal language. Hoyles and Noss (1989), in their study of learning of proportional strategies with Logo computer language, used "The N tasks" – constructing a sequence of N's with given sides and given angles, where the proportionality task, embedded in the activity, was for children to determine the size of the middle part of the letter. Children constructed N's using the Logo programming language, where the angle and the distance a turtle had to walk leaving a trace had to be determined in the program.

An advantage of the direct interface is that it takes less time to learn initially. An advantage of programming is that it allows learners more flexibility and open-ended creativity, where "the development of complexity within the application… grows from the learners' expanding ideas" (Clements, 2000a, p. 22). The division between programming and non-programming tasks is not clear-cut. It can be argued that menu and mouse manipulation in pre-programmed microworlds such as Sticks is a form of "visual programming."

Moreover, several programming languages, including all newer versions of Logo, contain programming, direct manipulation and sometimes object-oriented programming features (Blaho & Kalas, 2001). "Object-oriented" means that it is possible to create objects with their own variables, procedures, and settings; build hierarchies of objects with setting-inheritance; and allow certain other objects operations such as sending messages between objects or polymorphism, that is, allowing objects to interpret the same procedure in different ways. Such versions of Logo as Turtle math, Microworlds 2.1, Imagine, or SuperLogo (MathsNet.com, 2002) have "two-way connections between visual and symbolic representations" (Clements, 2000a, p. 24). On the level of actions within the microworld, the same principle is expressed in actions implementing dynamic links between multiple representations of objects, such as numerical, graphical and iconic (Drier, 2000). Different representations of functions were used as a learning tool in *Function Probe*, a multi-representational microworld for function exploration (Confrey, Smith, Piliero, & Rizzuti, 1991).

Borrowing role descriptions from professional software development (Astels et al., 2002), one can conclude that in iconic, non-programming microworlds learners play the role of software users, and microworld creators the roles of software designers and testers. In microworlds with accessible programming, learners may play all three types of roles. Several studies investigated the transformative function of the designer role in the student's learning (Harel & Papert, 1990; Kafai, 1995; Papert, 1993).

Qualitative research may involve integration of participatory and observational roles as "participant observation" and "teacher-researcher" (Creswell, 1998). For example, if researchers' roles include that of teachers, they can be "proactively supporting… learners' mathematical development" (Cobb & Yackel, 1995, p. 176). Also, depending on researchers' framework, one can view either social norms or individual beliefs as defining tools for roles. In an integrative stance on the issue, "social norms and beliefs are seen to be reflexively related" (Cobb & Yackel, 1995, p. 178).

Some Theoretical Considerations for Building an Integrative Conceptual Framework

Much of the proportional reasoning, analogy and metaphor research I reviewed above is focused on either content or educational theories, either cognitive or sociocultural perspectives, and either construction or comprehension methodologies. If the development of the field of mathematics education is viewed from categorical positions, then such research may be considered to be in an analytic stage, where new categories are being created by "separating 'this' from 'that'" (B. Davis, 1996, p. 2). Some work in education research is also directed toward synthesis of findings from analytic studies, where integration is achieved either by collapsing categories or by creating an extended structure encompassing the categories. Both analytic and synthetic types of research are needed for growth of understanding in education. The difficulty of synthesis is that analytic studies are rarely conducted with the purpose of future integration. Analysis can be defined as an activity directed at differentiation among theories, at underlying their differences, and even at placing them in opposition to other theories. Such analytic work is a major tool of framework creation. Findings coming from different studies of analytic kind may be too disjoint to be meaningfully integrated. Thus in synthesis there is a danger of what Cobb and Yackel (1995), in their work on integration, call "intellectually schizophrenic" frameworks (p. 184). That is, tensions between parts taken from perspectives that were originally designed as disparate have to be meaningfully addressed.

Despite the dangers and difficulties, the need for synthesis is being satisfied in a “move towards interdisciplinarity that is sweeping the social sciences and the humanities” (Ernest, 1994, p. 1). Mathematics education researchers, in particular, use tools, ideas and theories from anthropology, neurology, linguistics, biology, discourse analysis, cognitive studies and other areas of thought. In this part of the literature review, I collect tools researchers used for synthesis, creating an “epistemological toolbox” for building an integrative conceptual framework for the present study (Figure 3). Researchers whose studies gave birth to the sociocultural and cognitive branches of constructivism are used in the visual simile for integration in Figure 3: Vygotski as Yin and Piaget as Yang.

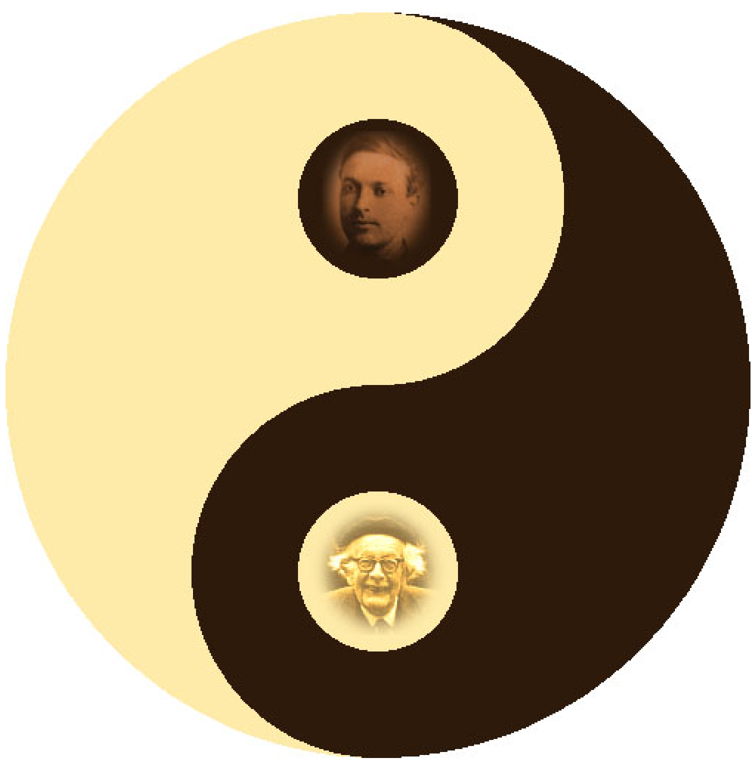

Figure 3. A playful simile for the idea of theory synthesis.

To define and to analyze the three world views of mechanism, organicism and contextualism, Pepper, cited in (Goldhaber, 2000) uses divides between universally generalizable and situation-specific theories, and between holistic and reductionist levels of analysis, claiming that incompatibility of sides in each divide makes the worldviews mutually exclusive. However, some researchers attempt "not merely healing the "gaps" but side-stepping the mode of thinking (and acting) out of which they arise" (B. Davis, 1996, p. 2). The idea of co-emergence is one of the integrating principles of enactivism (B. Davis, 1996; R. Davis, 1990). This idea is used to address a group of dilemmas. For example, knowledge is not either "out there" in the world or social context, or inside the mind. Instead, it is possible to view "the knower, the knowing and the known as emerging together" (Begg, 1999, p. 8). A dichotomous question of whether knowledge in first born in the mind or in communication, that some view as separating sociocultural and radical branches of constructivism, is thus not being posed.

Cobb and Yackel (1995) take such an integrative approach to studies, using cyclic principle to integrate theory and practice, when they write, "relationship between theory and practice is reflexive" and, "theory is seen to grow out of practice and to feed back to inform and guide practice" (p. 175). "Agent-world" and similar divisions call for, and, reflexively, come from, conducting education studies focused on either learners or teachers or social contexts as objects of analysis. From another point of view, this division causes the split betweens studies that focus on mathematical content, and studies that focus on pedagogy (Schoenfeld, 1999). For example, in studies focused on learning other contextual components, such as artifacts of the space, would be considered relevant but separate background. Some approaches claim stronger connections across these dividing lines, for example, considering "learners' mathematical development as it occurs in the social context of the classroom" (Cobb & Yackel, 1995, p. 176).

Metaphor and Analogy

Previously, I defined metaphor as the recursive movement between a source and a target that are structurally similar, both changing in the dynamic process of learning. It is both the subject of this study, and a synthesis tool that can be used for construing content models and educational theories as similar layers of a unified structure. In several models of proportional reasoning, sets of objects are united into an equivalency class defined by the invariance under an operation. Researchers agree that it takes much growth of understanding to construct operational connections between such "disparate" objects as 2, 3, 10 and 15, observing that 2/3=10/15.

As a way of viewing content models and educational theories as similar, educational research can be considered to be close to mathematical reasoning. In educational research, we observe cycles between constructing separate theoretical categories and constructing connections between such categories. The integrative part of these cycles is analogous to "proportion," or making operational connections between disparate numbers in numerical proportions. In Vosniadou and Ortony's (1989) terms, development of analogical reasoning means a move from one-place predicates that work on object *attributes*, to deep two-place predicates that involve object *relations*. Similarly, synthesis in a conceptual framework means a move from work on attributes of each theory to relations between theories. Enactivism is also founded on theories that seek "middle ways" amid different perspectives by analyzing assumptions that underlie those perspectives in "a manner of reasoning that favors analogy over logic" (B. Davis, 1996, p. xxv). Disparate theories may also be viewed within a single perspective analyzing and comparing the metaphors involved in each, as I do, for example, with the splitting conjecture (Confrey, 1994) and the counting scheme (Olive, 2001; Steffe, 1994) in a study of the role of metaphors in the development of multiplicative reasoning (M. Droujkova, 2003a).

Self-Similarity in Fractal, Leveled Structures

The fact that educational theories and proportional and analogical reasoning models can be metaphorically connected both exemplifies and helps to explain the idea of using a layered structure for integration of conceptual entities. In such a structure, different entities are construed as layers of a unity. Proportions can be viewed as "relations between relations" or second-level operations. Similarly, one can work with "theories about theories" to achieve integration. Formal set theory, applied to foundations of mathematics, as well as education research, both offer leveled structures as a method of resolving paradoxes and contradictions. In set theory, relations among relations are viewed as belonging to the level of a Tarski hierarchy that is above relations among objects (Barwise & Moss, 1996). In several education models, the next level of understanding implies relations among understandings of the previous levels, such as in the Pirie-Kieren model's nested levels (Pirie & Kieren, 1994b), the idea of semiotic chaining and reification (Presmeg, 1997b; Sfard, 1997, 2000), or conceptualization of activity-effect relationship (Tzur, 1999). Integrative approach means that movement between levels is not developmental (unidirectional), as in stage models, but rather recursive. For example, synthesizing analytic findings from different frameworks into a unified theory is not a final goal, but a step in an infinite process.

One way of constructing “infinitely developing” leveled structures is to use the idea of self-similarity, or similarity between levels, in the sense that “a portion of it resembles the whole” (B. Davis et al., 2000, p. 72). In qualitative research with conceptual framework, similar ideas should be used for theoretical support of methodologies, and for philosophical support of theories (Creswell, 1998; Eisenhart, 1991; Goldhaber, 2000; Lincoln & Guba, 1985). As related to the practice, this principle is expressed by a quotation from Rene Thom that opens a book on philosophy of mathematics education: “All mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics” (Ernest, 1994, p. 1).

Fractal geometry built on self-similarity is one of the models used to describe enactivist theory (B. Davis, 1996; B. Davis et al., 2000). Enactivism postulates “self-similarity of processes across such conceptual levels as organism, collective and species” (B. Davis, 1996, p. 15). Self-similarity here serves as a tool in the framework of “self-organizing, adaptive and spontaneously evolving” systems defined as “complex” (ibid). The world on all levels is considered “complex” in that sense, and enactivist research parallels this fractal geometry view of the world.

Ma (1999) expresses ideas related to internal similarity of a theory in a different context, talking about Profound Understanding of Fundamental Mathematics. She does not “regard improvement of teachers knowledge as necessarily preceding improvement of learner learning,” but rather claims that “work on each should support the improvement of the other,” as the two are “interdependent processes” (p. 146). She calls for a context where the same idea of profound understanding will work in both learning of the teachers and in their teaching. Berenson, Cavey, Clark, and Staley, in a similar spirit, modified the Pirie-Kieren model and used it to investigate learning of mathematics and mathematical pedagogy by beginning teachers (Berenson et al., 2001; Cavey, 2002). The Pirie-Kieren model has a nested, self-similar nature, where levels of understanding for one concept can be "contained in" the primitive knowing level of another concept, and so on. Moreover, Pirie and Kieren discuss "closeness" in functions and roles of "even-numbered" (and "odd-numbered") layers to each other (Pirie & Kieren, 1994b).

Again, there are close connections between the notion of self-similarity and the idea of leveled, hierarchical, complex multi-part structures. Note that the idea of levels may be used both for the analysis, where a unity is separated into hierarchical levels, and for synthesis, where separate entities are construed as levels of a unity. For example, mathematics has been historically separated from mathematical reasoning into an entity Lakoff and Nunez (1997) call mind-free mathematics. Now reasoning, or embodied human minds, (ibid) and mathematics are being united into mind-based mathematics, and metaphors, which are "implicit in all areas of human understanding, even in reasoning itself" (Presmeg, 1997b, p. 267) play a prominent role in this union. Analogical and proportional reasoning are structurally similar cognitive tools (Alexander et al., 1997; Inhelder & Piaget, 1958; Piaget & Campbell, 2001; Pimm, 1987). In advanced mathematical topics, including proportionality, "children need to focus on the important structural properties, these being determined primarily by how the quantities in a problem are related to each other, rather than by what the quantities are" (English, 1997b, p. 192).

<u>Cycles</u>

The idea of cyclic growth is probably present in all sizeable systems of human thought, from philosophies and religions to mathematics education theories. Circular, recursive, layered ways are also used for scholarly narrative. For example, in (B. Davis, 1996) "each chapter (and section) picks up on the ideas of the preceding chapters (and sections), thus adding to the conceptual depth. In this way, the document has taken on a sort of recursive structure… Each successive layer encompasses and expands on that which has preceded it" (p. xxx). Creswell's book on Qualitative Inquiry (1998) is also laid out in a cyclic manner, where the five traditions of research are revisited in each chapter, as the author shows them in relation to each stage of qualitative study.

The Pirie-Kieren model for the growth of mathematical understanding presents cognition and knowing as "non-linear, recursive, self-organizing processes" (Kieren, Calvert et al., 1995, p. 2). Here the word "recursive" means "series of elaborations, where the starting place of each stage is the ending place of the previous stage" (B. Davis et al., 2000, p. 70). The process of knowing, itself recursive, is mapped onto levels of the model, from primitive knowing to inventising. Kieren and Pirie consider second through fourth layers (image making, image having, and property noticing) "informal" and the next three (formalising, observing, structuring) "potential formal" (Kieren, Calvert et al., 1995, p. 3). In the process of growth of understanding, learners may "fold back" to modifying their image of the concepts (Kieren, Calvert et al., 1995; Pirie & Martin, 2000). Such recursive learning process, necessarily local and situated in a concrete context, is "the heart" out of which all cognitive activity is projected, not a mere precursor to more formal activities. Recursive movement between formal and informal levels is one manifestation of cycles in the Pirie-Kieren model. Also, the cyclic movement between the modes of understanding "possibly leads to a person developing new diverging mathematical ideas" (Kieren, Calvert et al., 1995,

p. 3), hence initiating the growth of a different concept that can be mapped into the same layers.

Clark's (1997) work in cognitive science also illustrates how the idea of cycles is related to the idea of self-similarity through the notion of complexity. Clark says that explanations emerge similarly to emergence of order in a complex dynamic system. They grow from a seed, or a starting point, producing self-similar, fractal layers of organization as more data enters the system in a cyclic process. In the same vein, Lave and her coauthors' (1984) epistemological model of situated learning assumes construction of a problem, as well as a solution, and bringing them together to make them match through a recursive process. Instead of considering the problem fixed and immobile from the beginning, this model postulates "gap-closing arithmetic" (p. 94) that is a cyclic process. During gap closing, solvers reduce the gap between the initial problem and the initial solution, cycling between modifications of both. I use this model for task construction described in Chapters 3. The task I offer to learners, with respect to the gap closing model, serves as an initial, *generative* element that starts the cyclic process where "people and settings together create problem and solution shapes, and moreover, they do so simultaneously" (Lave et al., 1984). Much like in numerical methods in mathematics, this model of cognition leaves room to be "vaguely right" (Papert, 1993, p. 172) at first. That is, the model allows the knowledge to grow continually in a recursive, approximating process. If play is seen as a "possibility of movement" and knowledge as something that can be "interpreted differently" (B. Davis, 1996, p. 147), then learning within such cyclic models is necessarily a playful process.

Types of Cycles: Recursion and Iteration

Here I explain the terms *recursion* and *iteration* in more detail, using a mathematics example of factorial, to be able to talk more precisely about different types of cycles in hierarchical systems in education theories. Computer scientists sometimes explain the difference referring to inductive, bottom-up nature of iteration and deductive, formula-driven, top-down nature of recursion. In a humorous way, it can be said, "God writes recursive formulae for the world, while we are condemned to live through their consequences as iteration" (Bird, 2001). To construe an object by *recursion* in a layered system, we start from the top layer. Then, we define "a step down," which is an action that says what to do on each layer, *in terms of the previous layer*. To give a mathematical example, the factorial n! can be recursively constructed by defining a step down the natural number sequence: $n!=n*(n-1)!$ In education research, studies are in part constructed by defining a series of "stepping down" from the level of perspectives, for example enactivism, to the level of theories and conceptual frameworks, such as the Pirie-Kieren model, and then to the level of methodology born of these theories and to the level of data collection in the field (Pirie, 1996).

Defining a step down is enough for constructing the concept of factorial, but it is not enough for finding any factorial values. To be able to compute factorials, we have to attach a particular value at the lowest level, for example, to say that $0!=1$ The recursive definition leads us from the top level down:

$n!=n*(n-1)!$

$(n-1)!=(n-1)*(n-2)!$ and so on.

When we reach the bottom level, the particular value found there is run up the levels in an *iterative* process, where every next-level value is defined by the previous-level value:

1!=1*0!=1

2!=2*1!=2

3!=3*2!=6 and so on.

Attaching a different number at the lowest level, for example, saying that 0!=42, will influence all levels, giving them different "particular values." Returning to education studies, particular activities in the field help to define particular implementations, and to construct particular interpretations, of methodological, theoretical and philosophical levels of the study. These interpretations take the form of answers to research questions. In the case of grounded theory construction, there is more stress on the bottom-up part of the cycle (Creswell, 1998).

On the level of research, Creswell (1998) closely connects "emerging" and "inductive" (p. 73) and says that "overall, the qualitative researcher works inductively" (p. 77). Yet he recommends to start the study from problem and purpose statements grounded in literature, saying: "the strongest and most scholarly rationale for a study, I believe, follows from a documented need in the literature for increased understanding and dialogue about an issue" (p. 94). That means to me that the initial deductive, analytic move is not restricted to the axiological assumption of being aware of the role of values in the study. Formulating the initial framework is the deductive, analytic part of the study. Collecting data and helping the new stage of the theories emerge from data is the inductive, synthetic part. The whole process of *co*-emerging of new theories from data and frameworks is neither purely inductive nor deductive, but a cyclic combination of both.

Possibly as an echo from the mechanistic framework, in top-down frameworks involving hypothesis testing the data becomes “condemned” to “live” through the iterative, bottom-up consequences of the recursive world-defining “formulae” of the initial theory. In studies on complex tasks such as learners designing computer games, some researchers postulated analytical models based on inductive and deductive approaches, such as “top-down” versus “bottom-up,” or “planning” versus “bricolage” (R. Davis, 1984; R. Davis & Maher, 1997; Turkle & Papert, 1990). However, others took a more integrative stance, analyzing roles of both approaches working together in learners’ thinking, for example, (Kafai, 1995). In my study, following qualitative traditions of dialogue with data, I run the top-down-bottom-up cycle multiple times, allowing data to influence particular implementations of theories, which influence further re-presentation and collection of data, and so on. The study planning described in the first three chapters of this work can be viewed as an initial, generative (Lave et al., 1984, p. 94) element that starts the cycle. The rather linear format of a thesis does not lend itself well to a depiction of such a process. Thus Chapters 4 and 5 show some of the last stages of it, while the first three chapters show the beginning stages.

CHAPTER 3. RESEARCH METHODOLOGY

…All mimsy were the borogoves…

Through the looking glass (Carroll, Gardner, & Tenniel, 2000)

In this chapter, I discuss my framework and methods of data collection. These methods served as a generative element that helped to set in motion the cyclic, gap-closing (Lave et al., 1984) process of addressing the research problem and answering the research questions. After describing the participants, I talk about establishing the roles and shared goals of the study. Then I talk about the initial, generative microworld, which was used to start establishing a shared discourse and shared meanings of the study process with learners. This discussion includes examples of proportionality tasks that can arise within the initial microworld. Such structured tasks were offered to learners at the beginning of the study to help create a shared notion of proportionality.

Conceptual Framework

In my conceptual framework, in addition to different perspectives of constructivist research in mathematics education, I use ideas from such areas as linguistics (Lakoff & Johnson, 1980; Lakoff & Nunez, 1997, 2000; Varela et al., 1991), ethnography (Belenky et al., 1986; Hoyles et al., 2001; Lave et al., 1984; Lave & Wenger, 1991; Nespor, 1994; Willis, 1977), and commercial software development (Astels et al., 2002; Beck, 1999). The design of the data collection followed and co-emerged with theoretical needs. I invited learners to participate in a software design project with the goal of creating a computer game devoted to helping other learners learn about proportionality. This task can be viewed as an intrinsic social practice, since designing real software is a goal that goes beyond purely learning or playful tasks. Thus it was possible to observe contextual phenomena, crucial for a study of metaphors, using methods similar to educational ethnography research (Belenky et al., 1986; Hoyles et al., 2001; Lave et al., 1984; Lave & Wenger, 1991; Nespor, 1994; Nunes et al., 1993; Willis, 1977), and more open-ended computer environment studies (Clements, 2000a; Clements & Sarama, 1997; Drier, 2000; Edwards, 1998; Hancock & Osterweil, 1996; Harel & Papert, 1990; Kafai & Resnick, 1996; Kerr, 1994; Papert, 1993; Turkle & Papert, 1990). In effect, computer games designed by learners represent and express "complex and observable metaphors that serve as tools for communication, formalization and development of learner's mathematical ideas" (M. Droujkova & Droujkov, 2003, p. 1). Presmeg (1997b) considers metaphors to be "very private, personal, and ripe with meaning for an individual" (p. 277), and Lakoff & Nunez (2000, p. 5) state that "most thought is unconscious… simply inaccessible to direct conscious introspection". For these reasons I concluded that metaphors should be studied in settings where individuals have a high degree of influence on the context of activities. On the other hand, I offered learners an initial example of a microworld, and a particular software development process, as "generative elements" (Lave et al., 1984, p. 94) that established and supported certain mathematical and pedagogical structures. Thus I defined the focus on the particular mathematical area of proportionality, the way foci are defined in laboratory task-based teaching experiments (Goldin, 2000).

The software development process of the study was adapted from agile processes methodologies (Astels et al., 2002; Beck, 1999; M. Droujkova & Droujkov, 2003). It is characterized by active learners' input and participation, supporting learner-centered principles of software-related research in education (Clements, 2000b; Clements & Battista, 2000; Harel & Papert, 1990; Kafai, 1995; Kerr, 1994; Papert, 1993). In turn, the fact that learners have active roles supported, and was supported by, the enactivist idea of co-construction of meaning (B. Davis, 1996; B. Davis et al., 2000; Kieren, Calvert et al., 1995), the idea needed for integration of different theories. Another feature of agile processes that I used was the idea of iterations, which parallels cyclic models of learning (Cobb & Yackel, 1995; B. Davis, 1996; Kieren, Calvert et al., 1995; Lave & Wenger, 1991; Pirie & Kieren, 1994b; Presmeg, 1997b; Tzur, 1999; Tzur et al., 2001).

To summarize, the choice of software design as a task allows using prior research with laboratory task-based, as well as contextual, designs. This choice co-emerged with the foci of the study, namely proportionality and metaphors, and with the conceptual framework. Software design work may make some of learners' metaphors accessible in discussions and in their representations as software objects and actions.

Research Design and Data Collection

The study was conducted in the tradition of teaching experiment: "teaching experiments focus on development that occurs within conceptually rich environments that are explicitly designed to optimize the changes that relevant developments will occur in forms that can be observed" (A. Kelly & Lesh, 2000, p. 192). Teaching experiment design in this study allowed "close listening" to learners, and observing some of their ideas that "experts have been trained to overlook," based on initial, generative conjectures that were revised and elaborated during the study (Confrey & Lachance, 2000, p. 234).

During the study, I met with each learner individually, from three to eight times, depending on the scope of each learner's project and the pace of progress. Each meeting was videotaped, and an additional audio recorder was set up during some of the sessions as a back-up device. Other artifacts included "user stories" created on paper as a part of an agile design process (Astels et al., 2002; Beck, 1999; Williams, Wang, & Vouk, 2002), any notes or drawings learners made, and close-up photographs of some activities. All data is kept confidential, under pseudonyms. In addition, I kept a system of files with reflections about each meeting.

The data were analyzed during the collection and after the study. One of the study foci was connection between systems of objects and actions that came up during software design, as sources of metaphors, and learners' mathematical understanding as targets. I used the Pirie-Kieren model to map learner's understanding, working directly with videotapes in a methodology similar to that used by the Pirie-Kieren group and others (Confrey & Lachance, 2000; Pirie, 1996). In addition to comments to episodes, I fully transcribed key episodes, as they were identified during preliminary and secondary analysis of the videotapes, or as needed for illustrations of key points in Chapter 4. The preliminary analysis was used to plan subsequent sessions. The data analysis was conducted as a "down-up," deductive-inductive cycle (see Chapter 2), where formulating the initial framework, and then planning subsequent sessions was the deductive, analytic part of the cycle, and collecting data and helping the new stage of the theories emerge from data was the inductive, synthetic part.

Study Planning

The emergent quality, connected to play, and the gap-closing model (Lave et al., 1984) of resolving problems (Figure 4) is reflected in the methodology of this study, echoing recursive models of learning used in the conceptual framework. Learner and researcher actions were guided by the goal of creating educational software in a process derived in part from agile processes ideas (Astels et al., 2002; Beck, 1999; Williams et al., 2002), where software development is seen as an inherently cyclic process used "to refine the product through a series of small releases" (Astels et al., 2002, p. 83). Also, while attending to mini-tasks that came up as the project progressed, learners and I were able to preserve "continuity of place, people, purpose" (Noddings, 1992, p. xii) throughout the study.

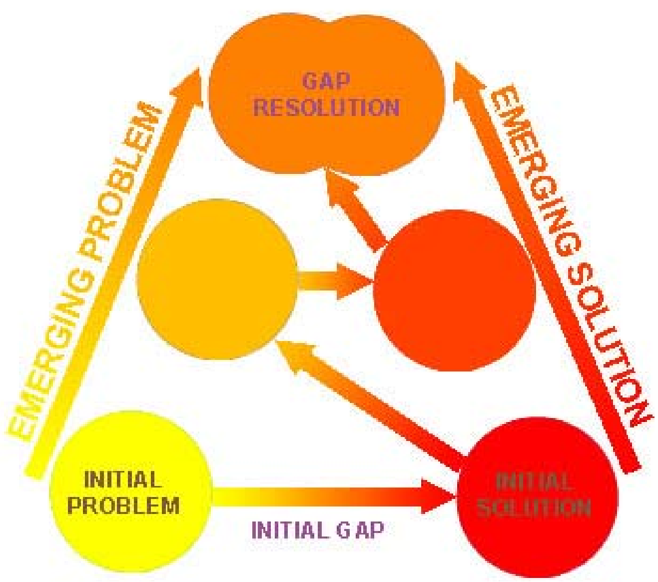

Figure 4. A diagram for the gap-closing model

In enactivist frameworks, the organism and the environment specify each other and co-emerge in this mutual specification in a fundamentally circular process from which the organism cannot be disembodied. Embodiment is one of the key ideas of my framework, and it follows that methodology had to be highly contextual. On the other hand, I wanted to address theoretical questions dealing with a certain area of mathematics, namely, proportional reasoning, and a certain area of the educational theory, namely, roles of metaphors. These goals require methodologies with relatively high levels of researchers' influence on activities during the study. This led me to consider three existing categories of the task design: ethnographies in settings with embedded phenomena of interest to the researcher; laboratory problems designed around variables observed in natural contexts; and microworld research.

The first category is problematic where children are concerned. Unlike Hoyles' (2001) nurses, or young street sellers (Nunes, 1993), children that were accessible to me as study subjects were not easily accessible in naturalistic settings rich in proportional activities. While children's home and other everyday activities such as shopping or cooking may include individual episodes that can be mapped into proportionality realm, observing children in such situations presents serious accessibility difficulties.

The second category, structured laboratory tasks with some naturalistic considerations introduced at the level of the initial task design, does not allow learners to participate in the co-creation of the study to the degree I find desirable for investigating metaphor. Metaphors, which are "very private, personal, and ripe with meaning for an individual" (Presmeg, 1997b, p. 277), should be studied in settings where individuals have high degree of influence on context of activities. Thus I came up with a method that allowed learners and myself to co-create a microworld related to proportionality.

Participants

The participants of the study were six homeschooled learners of ages 13 to 16. Their families represented a range of formal and informal parental perspectives on education, which was an enhancement to the naturalistic quality of the study. The participants were selected based on their interest in extended work on designing mathematical software. Below, I briefly describe each participant's background and relevant interests. Participants are identified by pseudonyms.

Abby, a fifteen-year-old girl, is a daughter of a building contractor and a midwife. She was homeschooled all her life, studying mathematics mostly by textbooks and workbooks that were part of structured curricula, as well as with private tutors. Abby likes to read; as her mother said, she tends to think deeply about the ideas she encounters, often asking difficult philosophical questions.

Annie, a thirteen-year-old girl, is Abby's sister. She was homeschooled in a manner similar to Abby, but she did not take to reading and using books until later. Annie likes to draw, and does it well. There are the total of nine siblings in Annie and Abby's family.

Zack, a fifteen-year-old boy, is the only son in a family of software executives. During the study, he was in transition from public school to more individualized learning. He enjoys working with his hands, and runs a successful lawn care business. One of Zack's big passions is in monster truck races.

Sofia, a thirteen-year-old girl, is a daughter of a software engineer and a homemaker. Her mother is active in homeschooling movement, and Sofia has always been homeschooled in an eclectic manner called "unschooling" (Albert, 1999; Holt, 1995). During preliminary discussions, she said that she has never done much formal mathematics. She likes animals and keeps quite a few different creatures at home.

Evans, a thirteen-year-old boy, is a son of a software engineer and an education consultant. He was always unschooled, using unit studies centered on themes he found interesting. He is interested in music, and plays percussion in a local group. He is also interested in fantasy role-play, belonging to groups formed around fantasy and science fiction activities. Evans likes to draw, and draws on a professional level.

Niky, a sixteen-year-old girl, is Evans' sister. She was always unschooled. Niky teaches piano and guitar, mostly to children. At the beginning of the interview process, she planned to enter a college majoring in music, but then she switched her goal to computers and web design. Niky is very concerned with environmental issues.

Procedures

During the study, I met with each learner individually. At the beginning of the first meeting, we established initial goals and roles. As I mentioned before, study participants were selected based on their interest in creating mathematical software and assisting in research about learning. Thus the goal of participating in a software design/education research project was established at the point of recruiting the study subjects. The slash between "software design" and "education research" in the description of the project indicates that, simultaneously, I planned to establish a strongly authentic, contextual goal of designing a computer game, and also the goal of constructing new understanding of education and of mathematics. Learners became co-creators of both goals, participating in the goals' further development during the study. Learners were supported in significantly influencing all facets of the study. Their ways of knowing, thinking and learning was accepted in the atmosphere of "epistemological pluralism" (Kafai, 1995; Turkle & Papert, 1990).

At the first meeting with each learner, I briefly reviewed the goal of developing an interesting computer game about proportionality, with a possibility that parts of it would be programmed. The learner was offered the role of a "customer" while I took on the role of a "programmer." We briefly talked about the roles in discussing our design process (Aspinwall, Shaw, & Presmeg, 1997; Astels et al., 2002; Beck, 1999; Williams et al., 2002). Agile methodologies were chosen as a basis of design, because their principles parallel many of the theoretical ideas from my conceptual framework. For example, the idea that a real customer is working directly with the project at all times parallels the idea of co-emergence of knowing from contributions all participants. Since the customer may not be familiar with programming as such, a way was created for planning what the software would do using step-by-step cards that form "user stories." In this study, learners drew and described computer objects and actions, creating a story of what should happen in their game.

My role of "a programmer" allowed me to ask clarifying questions without breaking out of the context of the authentic task. Thus my roles of an interviewer and of a helper in the process of learners' learning was integrated with my role of a programmer. "As soon as the customers see the first release, they learn what they want in the second release… or what they really wanted in the first… It is learning that can only come from experience. But customers can't get there alone. They need people who can program, not as guides, but as companions" (Beck, 1999, p. 19). Thus the cyclic nature of an agile process, and the roles established within it, supported the recursive process of learning. In an iterative process, learners and I designed software on paper, "test-run" it using the user stories thus created on paper, and redesigned it during each meeting.

In this scenario, learners participate in designing and using software, but they do not write the code. There are several reasons for this choice of methodology. Learning to program, which may take months or years, is very suitable for longitudinal studies, where researchers can investigate parallels between development of learners' cognitive structures looked upon via a microscopic lens, and development of programming, as I described in the microworld studies review. Metaphor, the topic of this study, as a constitutive entity of learning is closer to the constitutive process of software planning and design than to implementation process of writing code.

In the case when learners program themselves, the researcher takes a less active, more observational role, as the activity is mostly created by the learner within the computer environment. Thus, any interview questions are extrinsic to such an activity, which does not naturally include the researcher. By offering learners the roles of customers and users, and taking on the role of a programmer, I create an opportunity to make interviewing a part of the intrinsic task, without breaking out of the role dictated by the task.

The Initial Microworld

On the first meeting I present each learner with the initial microworld programmed before the start of the project. The initial microworld is one of generative elements for the study. The design of the initial microworld is dictated by the goal to represent some proportionality ideas, as an example. I take these ideas from research on proportional reasoning, reviewed in Chapter 2. Main goals of the initial microworld are to aid in creation of the shared mathematical discourse, as well as discourses of learning and software development, and to foster emergence of learners design goals. Wheatley (1997, p. 285) writes: “Diagrams do not communicate, they evoke,” and I plan for the initial microworld to evoke, rather than to communicate, images and ideas related to proportionality. Harel and Papert (1990) also discuss evocativeness of some materials, such as open-ended environments for microworld designing.

For the purpose of opening the dialogue on proportionality, I made it possible to realize, within the initial microworld, a few examples of proportional reasoning problems found in research literature reviewed in Chapter 2. These examples, and the microworld itself, were not designed to assist learners in construction of an extensive, multisided and interconnected concept of proportionality. As a generative element, the microworld was planned to invite development and growth in areas corresponding to all goals involved in the study: learning of mathematics, pedagogy, software development and educational research. As such, the microworld represented the idea of “requiring future development” by being visibly an unfinished project.

In the initial state, the microworld contained three moveable, changeable objects representing examples of different proportional problems, with relationships of width and length, one quantity of objects and another quantity, and speeds (Figure 5). The objects are “generic” in the sense that they do not represent specific contexts or objects, such as racing cars or candy bars. There is an eraser, which is removing objects dragged to it, and a storage area where objects get minimized.

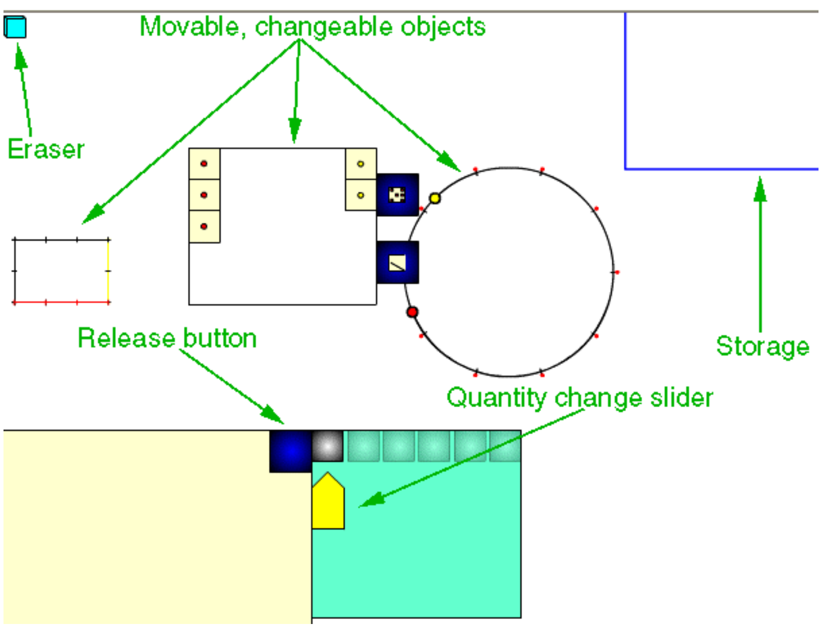

Figure 5. A screenshot of the initial microworld.

The initial microworld is rather abstract, and objects in it do not resemble concrete real-life things. This makes the microworld open to creative transformations by learners. Thus, the microworld can be compared to the nonsensical phrase in the chapter epigraph: “All mimsy were the borogoves.” The phrase follows grammatical and phonetic rules of the English language (Carroll, 2000), yet its context is open to interpretation by the reader. In fact, this phrase has been by now interpreted in many ways (Figure 6), from referring to “flimsy birds” (Carroll, 2000) to referring to “finitely many singularities” (Lewis, 1972), a nonsensical phrase in its own right until its chaos theory references enter the common discourse. Figure 6 depicts two different visual interpretation of the same “abstract” line, “…all mimsy were the borogoves…” The first one, conceived by the author of the poem, interprets "mimsy borogoves" as flimsy, miserable, thin, shabby birds (Carroll et al., 2000). The second one is based on a parody where the poem is jokingly interpreted as a proof of a theorem about "a structurally stable dynamical system" (Lewis, 1972). The graph of such a system is on the right of Figure 6. Each reader of the epigraph phrase supplies his or her own meaning to the words. Similarly, each learner may supply his or her own meaning to the abstract objects of the initial microworld, and, more broadly, to the task of creating a computer game about proportionality. To make the poem more meaningful, Lewis Carroll used nonsensical nouns and adjectives, but mostly real verbs composing "Jabberwocky" (R. Kelly, 1990), similarly to my use of abstract *objects* with "real" proportionality *actions* in the initial microworld.

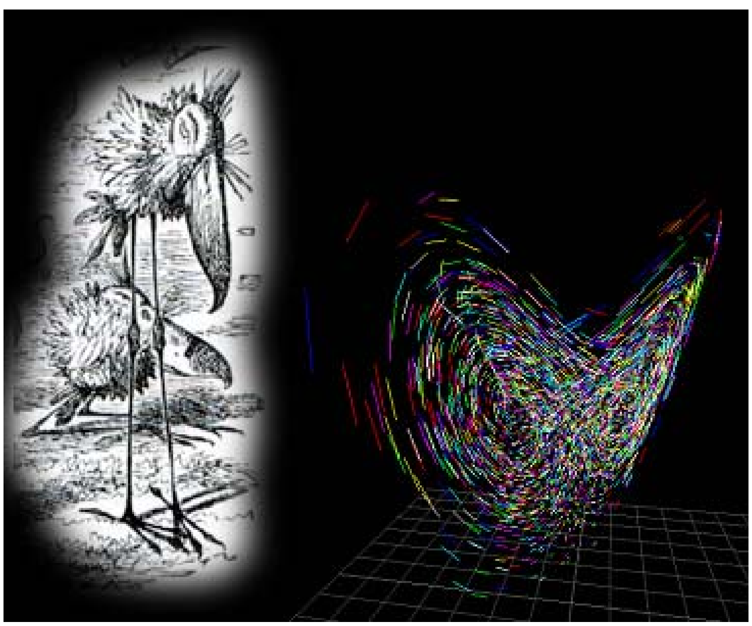

Figure 6. A simile for the idea of open-ended generative elements.

Work on making the software related to proportionality, and co-creating the meaning of the words "related to proportionality" with learners, helped to address the mathematical understanding side of research questions. Thinking about and discussing meanings of proportionality helped learners to develop their understanding of that concept (Berenson et al., 2001). On the other hand, my participation in these discussions, in the intrinsic role of a programmer, gave me an authentic access to learners' thinking *about* proportionality, as well as another window at their proportional reasoning.

Proportionality Tasks within the Initial Microworld

Initial tasks I offered learners on the first day of the study served several goals. Their first goal was to share with learners, by using several expanded examples, a part of my initial understanding of what it means for an activity in the initial microworld to be related to proportionality. The tasks also helped learners to share with me their initial discourse and ideas about proportionality, computers and learning. That is, the tasks helped learners and me to begin co-construction of context for the study, and to establish my initial access to learners' understanding construed as actions in context (B. Davis, 1996; Kieren, Calvert et al., 1995; Lave & Wenger, 1991). Understanding was communicated in actions, and to some degree in words as they accompanied and mediated actions (Figure 7).

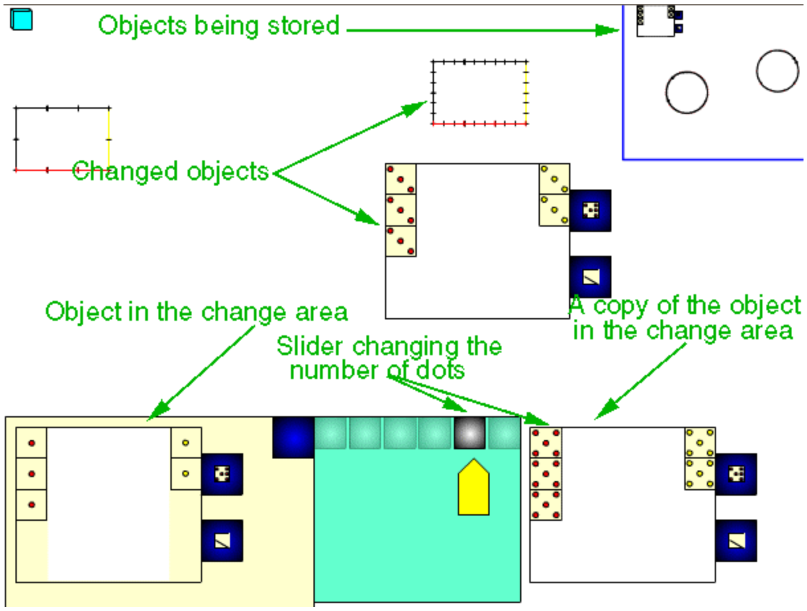

Figure 7. Microworld during the play, with some objects already modified.

Another goal of the initial tasks was to give learners some examples of possible objects and actions for their future computer games. Any object that is in common use can serve as a focus for collective actions that may lead to co-creation of shared understanding. By now, computer operating systems and typical software features became such a common object. Thus, computer features entered the pool of common, shared metaphors, such as "button" meaning "action" or "fold-out menu" meaning "making a choice." (Clements & Sarama, 1997, p. 320) write: "Computer programming is a general-purpose tool and process, and thus one capable of producing rich, flexible, interrelated metaphors." Campbell (2001), as many other modern writers, uses computer-related metaphors in his writing: "an introduction… will improve the user-friendliness of what you are about to read" (p. 1) and later, "reflecting abstraction… is expected to be "backward compatible" with Piaget's previously adopted conceptions" (p. 21). As with the idea of proportionality, I offered a starting point, or a generative element, for learners' efforts in the game design. Learners brought their own computer game experiences to the study. However, much of mathematical software is based on closed-ended tasks involving giving a numeric answer to a closed-ended question, or making a choice between several possible answers. This kind of test-oriented, or drill, software is so prevalent that many children and adults equate it with the very idea of mathematical computer games. The notion of mathematical computer game was co-constructed by learners during the course of the study – and the initial tasks served as one of generative elements for this construction.

Software Design Process

I follow Clements and Sarama (1997) in seeing development of the computer microworld as a "basic metaphor" (p. 313) for doing mathematics. They "consider Logo programming to be a *conceptual framework*, in which children construct and elaborate on schemes and thus form a structure upon which future learning and problem solving can be based" (pp. 314-315). These authors also underlie the role of researcher in "mediation based on well-developed theoretical foundation" (p. 316), which parallels the teaching experiments features of my study. The extended work in software design helped learners and me to co-create, and me to research, "a coherent system of metaphorical concepts and a corresponding coherent system of metaphorical expressions" (Lakoff & Johnson, 1980, p. 9) for proportional reasoning.

In some cases, proportionality tasks designed by learners within their microworld may stem from what Tournaire (1985, p. 183) calls "physical tasks," that is, the microworld may realize models of physical processes. Drier (2000) writes on importance of connecting the microworld to physical world simulations. Such connections may serve as an additional source of data about learners' metaphors. Thus, I planned for a possibility of physical experiments to be incorporated into the study; for example, bouncing a real ball if the microworld plans included a bouncing ball model. Such possibility to draw objects and actions on paper, to talk about them, to play with them within the microworld, and also to enact these ideas physically resonates with importance of multiple representations (Drier, 2000), and with the necessity to connect several metaphors for building a concept (Sfard, 1997, p. 362). By working on paper, within the microworld environment, in discussions and with physical objects, learners can develop rich mental images that are a basis for metaphors. Learning, learners "develop flexible and dynamic images which they can transform" (Wheatley, 1997, p. 282), a process that may be assisted by both designing the microworld and working with physical objects, which may develop more of a kinesthetic modality in addition to the auditory and visual modalities of imagery (Presmeg, 1997a, p. 302).

During detailed planning of microworld objects and actions, learners came to *create* several examples of tasks that may be related to proportional reasoning. This added to the project's "appropriability" (Harel & Papert, 1990) by learners, that is, it helped to make the microworld and any proportionality tasks arising out of its design learners' own. Moreover, "coming up with examples requires different cognitive skills from carrying out algorithms - one needs to look at mathematical objects in terms of their properties," write Selden and Selden (1998) in their MAA column on examples. Discussions of how a particular task created by learners related to proportionality, firstly, helped me access learners' understanding of and about mathematics, and, secondly, let me join learners in co-creation of their understanding. This model is similar to the emergent perspective definition of interviews: "Interviews are social events in which the researcher and learner negotiate their roles, their interpretations of tasks, and their understanding of what counts as a legitimate solution and an adequate explanation" (Cobb & Yackel, 1995, p. 185).

Learners and I wrote down "big" questions and ideas, which served as a generative element for developing overarching goals and keeping them in mind during all activities. The idea of such a list was inspired by Polya's list of key questions that apply to all problems, such as "What are you asked to find?" or "Can I think of an alternative strategy?" (Polya, 1988). I used these big questions as intrinsic probes that allowed me to ask about learners' thinking without breaking out of the context of software design. Since the questions and ideas were created in relation to software development, learners and I were able to discuss them without switching our roles to pure "subjects" and "researcher." This made the probes, and the whole study, more interesting, meaningful, and relevant for learners.

Periods between the meetings, which were a week or more apart, served several purposes. During this time, I reviewed the data collected before, which helped me to support the pedagogical goals embedded in the teaching experiment methodology. I also made some revisions of the methodology during this time, allowing data from each week to enter interpretation cycle and to influence the data collection for the next week. The idea of "thinking and learning in the context of a long-term enterprise" (Kafai, 1995, p. xvi) was thus applied both to learner projects, and to my studies of learner projects.

Data analysis

In organizing the data, I used the Pirie-Kieren model to map knowing of the learners. Metaphor, a major learning mechanism, is defined here as a structure dynamically connecting a source and a mathematical target, both changing in the process. The analysis of sources and targets served as a tool for investigating mechanisms of the growth of mathematical understanding.

The unit of analysis was each learner's microworld. The environment each participant was designing, considered together with the participants' actions related to the environment, is what I call each participant's *microworld*. It should be emphasized that this definition has two sides, which can be called the side of the *user*, and the side of the *developer* (see Chapter 3). On the side of the user, the definition of the microworld includes the snapshots of how subjects saw their software environments at different stages of development, the user actions described by them in those stages when they took on the role of the user, and the dynamic of the environment changes. On the side of the developer, the definition includes what subjects did as developers, how they changed environments, as well as their knowing actions related to changes in the environment. Some microworld researchers emphasize the *software user* role of the research subjects, since they want to control what actions and entities are available to their subjects, for example (Olive, 1999; Steffe & Tzur, 1994), while others put more stress on the designer role (Harel & Papert, 1990; Hoyles & Noss, 1989; Kafai, 1995). In this study, the designer role for the subjects was driving the main data collection on the role of metaphors in the growth of understanding, while the user role was more suitable for particular tasks to probe reasoning. Thus, as I described in more detail in Chapter 2, my microworld design methodology allows studying learning phenomena similar to those observed in naturalistic settings, as well as incorporate laboratory type tasks. Now I will briefly introduce some features of each subjects' microworlds.

I traced the role of metaphors in the growth of understanding of each learner by analyzing connections between sources and targets within metaphoric systems, and knowing as it was mapped into the Pirie-Kieren model. To explore metaphor targets related to proportionality, I used a perspective based on the notions of relation, invariance and equivalence class. In the next chapter, which provides the reader with the results, data are used to illustrate and to explicate data analysis methods.

CHAPTER 4. RESULTS

Growth of Understanding, Metaphors, and Proportions

This chapter starts from examples from the data illustrating the more detailed description of data analysis methods, organized around four main themes of the conceptual framework. The four themes are: microworlds, the Pirie-Kieren Model for the Growth of Mathematical Understanding, the ERI perspective for proportionality, and metaphor. Then the chapter moves on to roles of metaphors in knowing actions of the subjects.

In presenting my data, I used two approaches, combined as needed: sequential with respect to time, and grouped under themes that emerged during analysis. For example, in defining the recursive process of metaphorising, I used parts of Annie's interviews in sequential order. Parts of data from all cases are used to elucidate other definitions. Stories of Zack and Annie are told sequentially in the discussion of metaphoric structures for proportionality, while other learners' work is used in the form of episode descriptions.

Microworlds

To access learners' metaphors, I invited them to design software with the goal of helping other people learn proportionality. Zach's microworld was devoted to racing monster trucks. He described a racecourse (Figure 8) consisting of various obstacles such as cars to be crushed or bumpy stretches. Mathematization of his world included varying times it took the trucks to go through the obstacles. For example, the same obstacle took more time for smaller trucks.

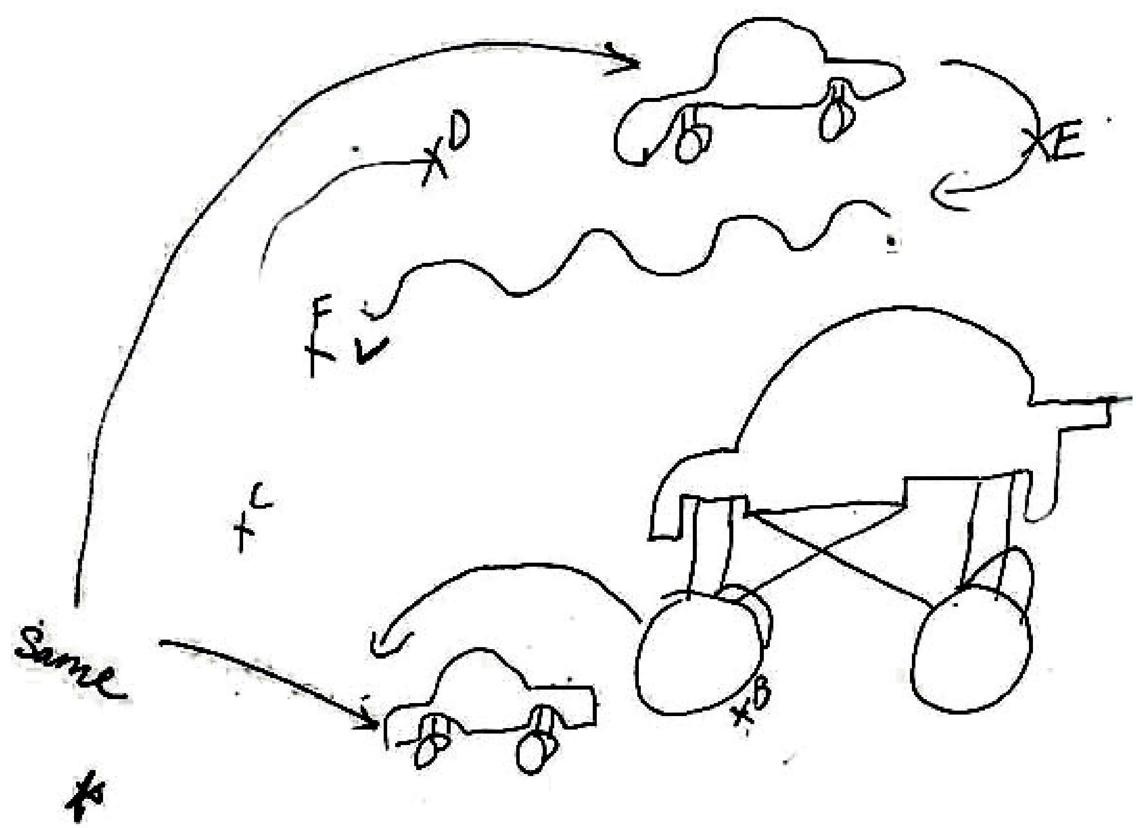

Figure 8. A truck ready to enter the racecourse

Abby's microworld was built around a circus-themed computer game. In different levels of the game, the users were supposed to solve different tasks such as finding how to make "three times the tigers for every penguin" or figuring out how many balls would four jugglers need if three used nine balls (Figure 9). Abby used a more top-down approach, starting from mathematical ideas and making models for them, rather than mathematising a given context.

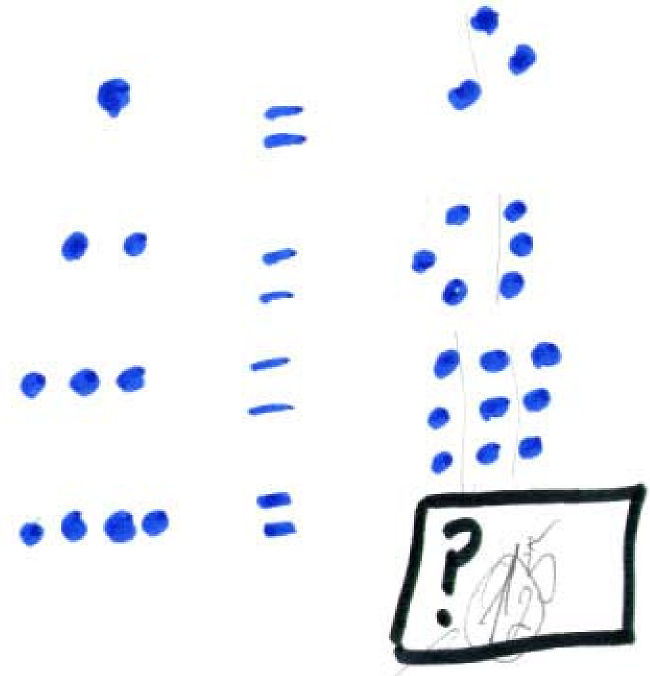

Figure 9. Abby's table with correspondences between amounts of juggling balls for different jugglers.

Evans' microworld had to do with a fantasy quest for saving a world that has become "out of proportion." Evans was not very interested in mathematization, devoting his time to creating detailed descriptions of the world's details, and commenting that proportionality can be "anywhere; it does not matter, really." He mathematized a part of his world by providing a quantitative description of an example of the "out of proportion" bush that lost some of the leaves.

Annie's world had to do with sharing objects among people. She started from describing fair sharing, and then moved on to unfair sharing based on additive and other relationships. An example of additive unfair sharing in Annie's world was when game characters described as "friends" received one more piece of candy each, compared to characters described as “others.” In a multiplicative example, friends would get twice as much candy as other characters (Figure 10).

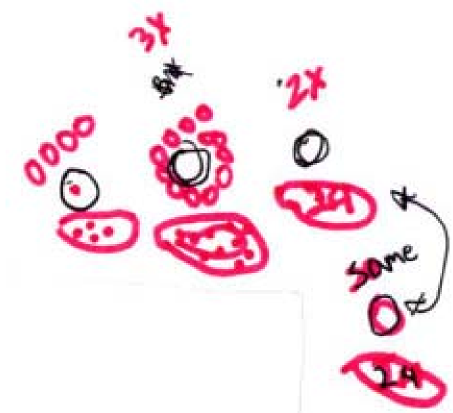

Figure 10. Annie's sharing pattern based on multiplicative relations.

Niky’s microworld was developing around designing software simulating a house with inhabitants. The goal of the user would be to observe how people live in the house, to look at some statistics such as room use or electricity consumption, and then to change the house to accommodate this particular family’s needs better. Niky’s description of software took the form of mostly word-based and number-based pop-up windows (Figure 11). She wrote in her own shorthand, explaining the features in words during interviews. Figure 11, while almost illegible, demonstrate the complexity with four different pop-up windows containing multi-item menus and symbols for time, energy consumption, and other measurements.

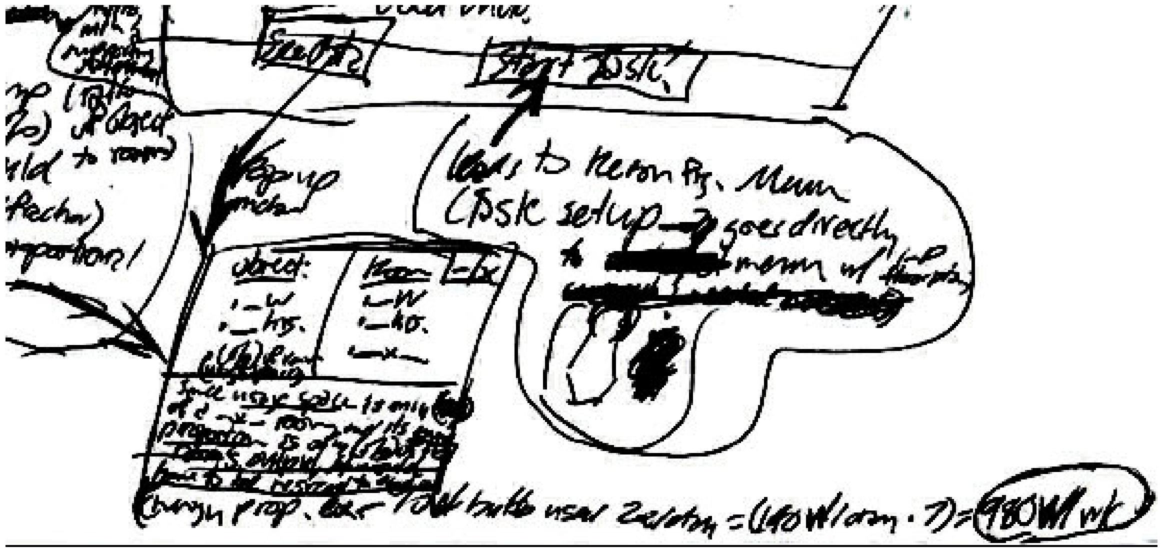

Figure 11. A fragment of Niky's screen drawing demonstrating complexity of the pop-up window structure.

Sofia's microworld was devoted to marching ants (Figure 12). The user's goal in the software she designed was to make a given number of ants using buttons and "multiplying stones." Sofia was quite concerned about the "freedom" of the user in the game, refusing to use any tasks that had a single pre-determined answer. This raised interesting issues about the nature of proportionality.

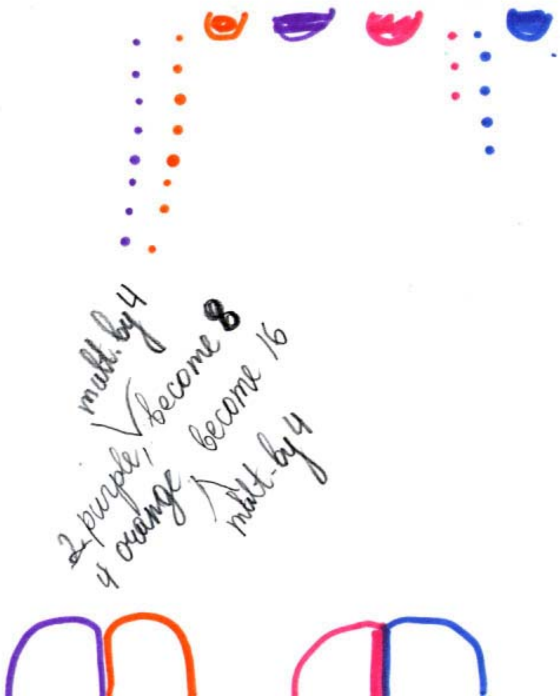

Figure 12. Sofia's marching ants and multiplying stones.

Pirie-Kieren Model for the Growth of Mathematical Understanding

The eight nested levels of the Pirie-Kieren model are co-defined by the learner and the observer. The innermost level, *Primitive Knowing*, encompasses the knowing that happened up to the point in time where the actor and the observer meet. In the case of this study, primitive knowing encompasses learner actions that have happened prior to the beginning of the data collection. Pirie and Kieren (1994) write: “Primitive here does not imply low level mathematics, but is rather the starting place for the growth of any particular mathematical understanding. It is what the observer, the teacher or researcher assumes the person doing the understanding can do initially” (p. 66).

Image Making, the next level, encompasses activities for creating an internal image, or in other terms a representation of a concept. For example, when Annie was moving from fair to “unfair with some rules” sharing, using cracker manipulatives, she was creating an image that a mathematician could call “additive equivalence.”

Image Having is the third layer, where the learner is accessing an existing image and making comparisons to it. Many examples of image having come from learners creating additional “same setting, different numbers” levels for their software, such as Abby’s examples of balance tasks. Initially, Abby came up with a task where three boxes could balance a sixty-pound lion. The goal for the user was to drag boxes to the balance scale until the balance was achieved, and then to figure out how many pounds are in one box. Once Abby set up the problem type, she came up with several more problems with different numbers, such as eleven boxes balancing a one hundred sixty-five pound elephant (Figure 13).

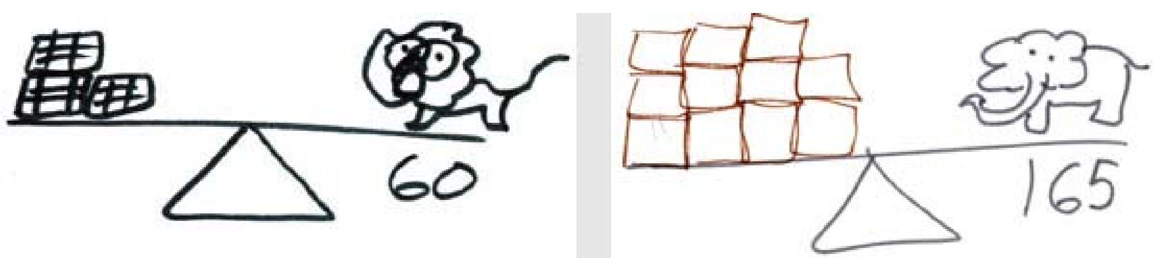

Figure 13. Two balancing levels of Abby's game.

Property Noticing, level four, is where learners begin to notice properties of images that they have. Learners who can operate at this level may be ready for formalized instruction. Pirie and Kieren also note that activities at the property noticing level involve similes, when “‘is’ becomes ‘is like’” (Pirie & Kieren, 1994a) and the notions being formed admit some use in the domain of the metaphor targets now separable from sources. In an example of property noticing, Zack assigned a value of time that was three times bigger to a racetrack stretch that was three times longer, thus noticing the multiplicative relationship. His table of time values for different stretches of track, while still related to the metaphor source of racing, became an entity in its own right, which could be manipulated by itself.

Formalising is the level where the learner can formally articulate the properties noticed at the previous level. To continue the previous example, Zack was formalising in his descriptions of combinations of monster trucks and obstacles, when he explained that moving to the next smaller size of the truck makes all times twice as long, and that a user could use this property to fill in empty spaces in racing time table. When Zack was using tables of numbers without referring to trucks and obstacles, but rather going by the numeric rules he invented, he was "generating patterns of symbolic results without necessary reference to their context-based meaning" (Pirie & Kieren, 1994a, p. 40). Formalising was in particular observed at times when subjects set up problems within already created parts of their software, while creation of new level *types* corresponded to image making, and creation of new levels *of the same type* often corresponded to image having.

Observing involves analysis of formalized understandings, and expressing it in generalized statements, or theorems. For example, Sofia was searching for combinations of numbers (a; b) such that 2a+3b=K for given numbers K. Having worked with several Ks, she attempted to formulate a general statement "the larger my K, the more combinations there are," which she did not manage to prove or disprove.

Structuring is where synthesis of observations into a theory occurs. *Inventising* is the action of creating new ideas or connecting old ideas in new ways. I have not observed structuring and inventising in this study. It should be noted that the outer layer is rarely described in literature. Formalising is already "beyond metaphors" that are based on images, since “in formalising, mathematics is no longer a metaphor for events in a physical or image world” (Pirie & Kieren, 1994a, p. 40). In Sfard’s (1997) framework, the outer layers correspond to the death of metaphor, as in her example of the rational number concept: “the process of concept construction is only completed when the metaphor ‘dies’ and the learner becomes able to think about the new number as a self-sustained independent entity belonging to the abstract domain of numbers” (p. 352). In this work, I am focusing on inner, informal levels of the model, also looking at the role of metaphor in formalising.

Folding back, another feature of the Pirie-Kieren theory (Pirie & Kieren, 1994b), is an invocative cognitive shift to an inner level by a learner functioning at an outer level of understanding. This enables the learner to use outer level knowing to inform inner level understanding. The significance of folding back to learning is that it supports further development at the outer layers by reorganizing knowledge constructed earlier, or by constructing new images as necessary for the outer layer tasks. Folding back is “an intrinsic and necessary part of the process” of growth of understanding (Pirie & Martin, 2000). *Collecting* is a special form of folding back where the learner retrieves previous knowledge for a specific purpose, reading it in the new light of that purpose. This purposefulness, or connections to outer level actions, is what distinguishes collecting from a simple recall. In the next two parts, I provide examples of folding back to collect in explaining definitions of proportionality and metaphors.

ERI Perspective for Proportionality

I define *equivalence class* as a set of groups of objects, formed with respect to a certain structure of relations within or between groups. If groups of objects are pairs of numbers, and if relations are multiplicative, the equivalence class gives us a definition of proportion found in most of the literature. Below, I describe how metaphoric structures supported collecting, in the Pirie-Kieren sense, of knowledge in qualitative, additive and multiplicative worlds for the purpose of reading it anew in light of proportionality.

In the analysis of proportionality from the point of view of the ERI perspective, in addition to equivalence class, I use the notion of relation and invariance. *Relation* is a binary operation on numbers or other objects, such as a multiplicative or an additive operation between numbers, or a qualitative relationship. For example, in the description "four is one more than three" from Annie's microworld, "one more" is an additive relation between four and three (Figure 14)

Figure 14. A representation of the relation "one more."

In "feathers cover birds," an example from (Inhelder & Piaget, 1958, p. 317), "cover" is a qualitative relation between feathers and birds. In "sixty pounds balances three twenty-pound boxes," an example from Abby's microworld, "twenty-pound box" is a multiplicative relation between sixty and three (Figure 13).

Invariance is preservation of a relation between objects in a group under an operation on each object. The notion of invariance in proportional reasoning can be explained in terms of "within" and "between" types of reasoning (Karplus et al., 1983). "Within" reasoning deals with relations of objects within a group. In above example, Abby reasoned “within,” saying that three boxes weigh sixty pounds, hence one box weighs sixty divided by three pounds. Her objects were boxes and total pounds, in pairs of (3; 60) and (1; 20), and she reasoned within each pair, considering the rate of total weight to the number of boxes. “Between” reasoning deals with relations between the corresponding objects in different groups. In an example of “between” reasoning, Sofia described two columns of ants marching by a “doubling stone” where each column doubled in quantity. Sofia focused on doubling of each ant column, that is, on a relation between the first and the first, or the second and the second quantity in each pair (Figure 15).

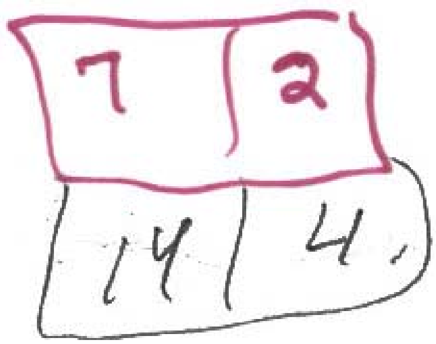

Figure 15. Sofia's "between" relations.

Invariance is a statement that a “within” relation is preserved under a “between” operation. For example, in Zack’s microworld different trucks take different times on different stretches of the same racetrack, represented in the table by capital letters AB, BC, CD etc. The small truck takes four times as much time (Figure 16) on each stretch as the large truck, which is a “between”-type statement. For example, AB takes four seconds for the large truck and sixteen seconds for the small truck. Zack’s observation that a relation within one group of distances, DE taking twice as long as CD, was preserved as the quadrupling between groups occurred, is an example of a statement of invariance. In some theories, a statement of the multiplicative invariance is taken as a definition of proportion (Behr & Harel, 1990; Thompson & Saldanha, 2003; Vergnaud, 1994), while the notion of equivalence class is seen as based on it. In the present work, invariance is seen as a property of proportions that are based on certain relations, while proportion is *defined* through the notion of equivalence class. This duality of definition vs. conjecture between equivalency class and invariance is similar to relationships between some geometric definitions and theorems. If one defines a parallelogram as a quadrilateral figure with equal opposite sides, the statement of equality of opposite angles becomes a theorem. If one defines a parallelogram as a quadrilateral figure with equal opposite angles, the statement of equality of opposite sides becomes a theorem.

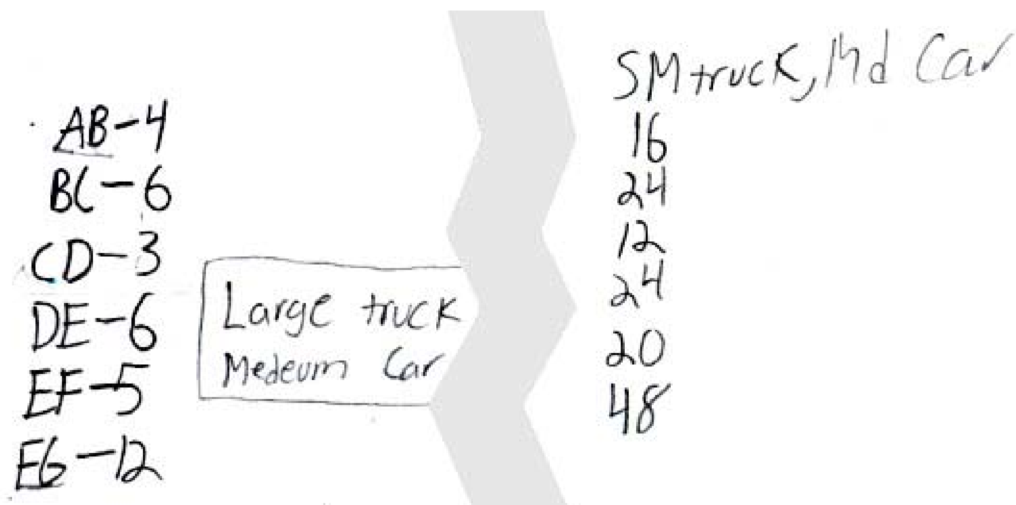

Figure 16. Two of Zack's timetables

Again, e*quivalence class*, for the purposes of this discussion, is a set of groups of objects such that the structure of relations among corresponding objects in each group is the same for every group. For example, the three groups of four numbers in Annie’s microworld are equivalent with respect to differences between numbers within each set. Annie described the structure of relations in words “you and your friends get one more candy than other people” and in gestures pointing to symbols of friends. Note that Annie has described a general rule that applies to *all* members of that equivalence class. It can be formalized as {(a, a+1, a, a+1) for any natural number a}. Thus her groups of (1, 2, 1, 2) and (3, 4, 3, 4) and (2, 3, 2, 3) all belonged to the same equivalence class, or, to use Annie’s metaphor, “had the same pattern” (Figure 17). Later, equivalence classes based on multiplicative relations also appeared in Annie’s microworld. For example, she described one multiplicative pattern as: “other people get half as much candy as you and your friends.”

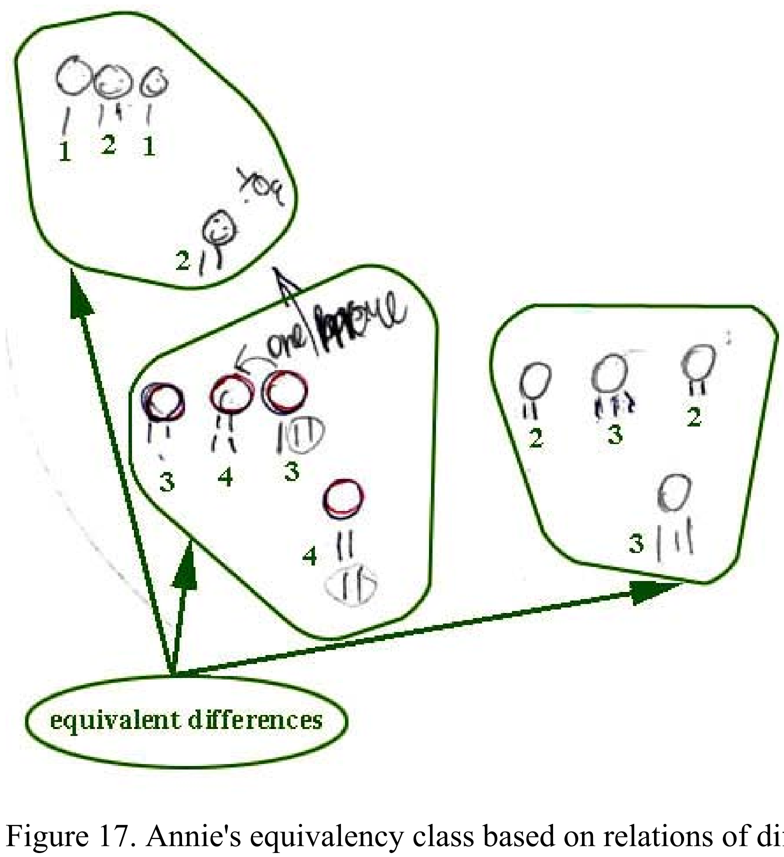

Figure 17. Annie's equivalency class based on relations of difference.

In Zack’s microworld, initial equivalence classes consisted of pairs of monster trucks and obstacles (Figure 18). They were based on the relation that I formulated as “the obstacle is one size smaller than the truck.” For example, the large truck with the medium-car obstacle belonged to the same equivalency class as the medium truck and the small-car obstacle. In another equivalence class Zack had small obstacle and small truck, medium obstacle and medium truck, and large obstacle and large truck. Large truck with small obstacle was in a class by itself. This is an example of a qualitative equivalence, or what Inhelder and Piaget (Inhelder & Piaget) called equivalence without an operation.

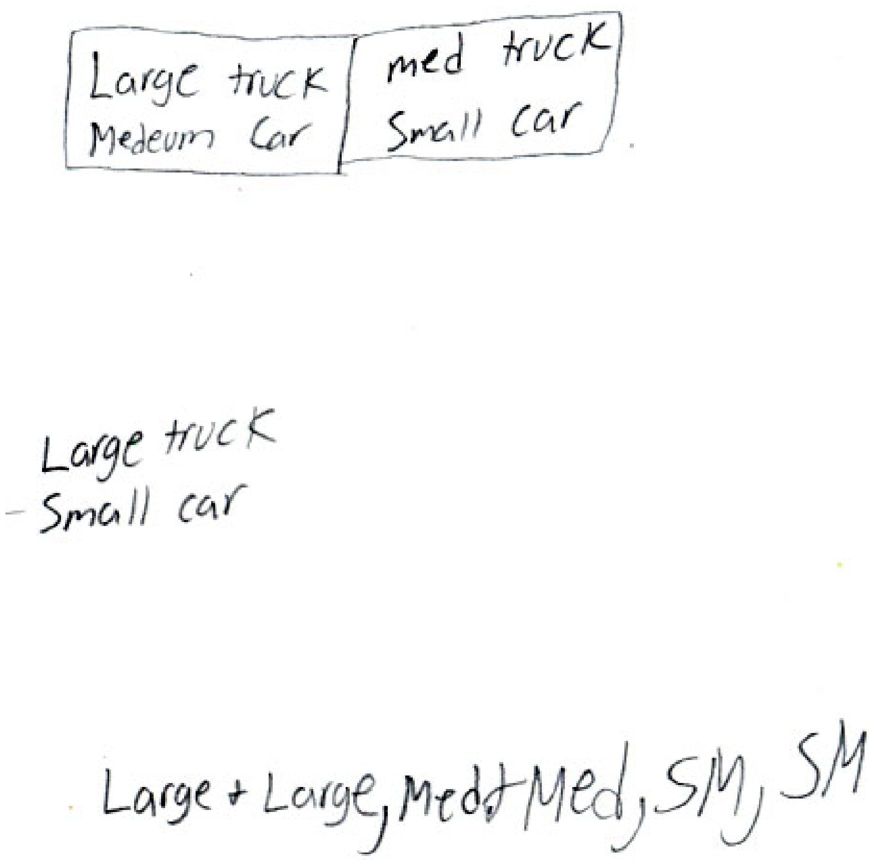

Figure 18. Zack's qualitative equivalence classes.

As is explained in more detail in Chapter 1, constructing the proportionality structure out of ideas of relation, invariance and equivalence leads to a "vicious circle" (Vergnaud, 1994) where the development of concepts seems to rely on a learner having already developed similarly advanced concepts (von Glasersfeld, 1998). Indeed, all three ideas are intricately intertwined in proportional reasoning, and each idea's development depends on development of the other two. For example, establishing invariance of a relation under an operation means establishing "sameness," or belonging to the same equivalence class. Equivalence class based on a multiplicative or additive relation is a set of, potentially all, instances that have the property of being invariant under that or an associated relation. Understanding of a relation, required for establishing the notions of invariance and equivalence, is in turn based on those notions. In this chapter, I trace the role of metaphors in building up the system of equivalence classes, relations, and invariance.

Metaphor

I define *metaphor* as the recursive movement between a source and a target that are structurally similar, both changing in the dynamic process of learning (B. Davis, 1996; R. Davis, 1984; English, 1997a; Lakoff & Johnson, 1980; Lakoff & Nunez, 1997, 2000; Pimm, 1987; Presmeg, 1997b; Sfard, 1997). I adopt the notion that mathematical thinking is fundamentally metaphoric (R. Davis, 1984; Lakoff & Nunez, 2000; Sfard, 1997).

For example, Annie's microworld can be seen as the metaphor of *sharing* with the initial source of fair sharing of objects between people, and the initial target, from the point of view of proportionality, of an equivalence class of groups of equal numbers. She drew five persons, and talked about the user helping them sharing some crackers. She picked the number thirteen, realized it would not be possible to share fairly in such a way that crackers would stay whole, counted up by sixes, and changed the number to thirty. She used Goldfish™ crackers for counters, and drew a fish shape above her total number of crackers. When I asked what would happen with twenty crackers, Annie said each of the five people would get four crackers (Figure 19).

Figure 19. Sharing 30 and 20 crackers among five characters.

Such equal sharing, or "splitting" (Confrey & Smith, 1995) is considered to be a basic operation for development of reasoning in the areas ranging from number construction to multiplicative conceptual field. Working with (1, 1) or (n, n) compositions, or in Annie's case (n; n; n; n; n), belongs to beginning stages of the development of proportional reasoning in Noelting's model (1980). Under the ERI perspective for proportionality, the description of fair sharing where "each person got the same number of crackers," is the simplest possible *system of relations*, where relations are all equalities. In the later development, when Annie also worked with other relations, the relation of "being the same" still played a special role for her, as evidenced by changes in the tone of voice, facial expressions, and speech speed when she was talking about characters who got the same number of objects. This special role of the relation of being the same, also observed in other learners, is discussed in more detail in the final chapter part on the development of proportionality.

After this initial establishment of the microworld, I intervened, using the microworld's extended metaphor of sharing as a language for "shepherding" Annie toward proportionality (Towers, 1998). *Shepherding* is a type of intervention that guides a learner towards understanding indirectly, through subtle coaxing and prompting. *Inviting* is a shepherding intervention defined as suggesting of a new and potentially useful avenue of exploration. Thus, I invited Annie to include non-fair sharing in her microworld. Non-fair sharing meant changing the relation, and Annie folded back to the additive world in order to collect other possible relations. Here is a transcript, where she initially talks about sharing among seven people.

Maria (M): Instead of sharing evenly, you can share *non-evenly.*

Annie (A): Oddly, like… Ok, ok, ok. Odd numbers.

M: It may not be fair.

A (excited): Yeah, you can have to give odd numbers to four of them, and even numbers to three of them, and that would be… Or not, you can give more to some guys, or you could do… You can make another one where you have to pick somebody that you don't want to have more.

M: Let's start, let's think about it, ok, it can be interesting. Let's think of smaller numbers.

A (readily): Yeah.

M: You were saying: "You can give smaller numbers to some…"

A: Yeah.

M: Can you make an example?

A: Yeah, yeah, I'll just use… They are giving… They are having things… (drawing, speaks slowly, pauses). They are sharing things. You have ten markers and you have five friends and you… No, three friends and you. So you have ten markers, three friends and you. They are all going to take your ten markers, and they, yeah… But they are all different colors! So you need something that's, like, all the same. All the same. (thinks, looks around, speaks softly) Something that you can give. (there are some papers with marker drawings on the wall, she looks at them for a few moments, then speaks fast, excitedly). You have three, or four, or really five pieces of paper. And everyone wants to use… Four, five, six (counts, pointing at her drawing). I'll use seven pieces of paper. It's not much this person has (laughs). You give them each one, and you give you one, that's four, that's easy. So you have three more pieces of paper, three more pieces of paper. *You* want another piece of paper, and it's *your* paper, so you get one. And all your favorite friends get one. So you just give it to your two best friends.

M: So you and your best friends get two pieces of paper each.

A: And *he* (points) does not get it (laughs, both laugh). But he only gets one.

Considering this more complicated situation, Annie folded back to image making in the additive world to collect support for unfair sharing in the form of sharing structure. She attempted to use the word “non-even” as a guide in making an image, taking it to mean “odd.” Note how she was focusing on the total amount she had to share. Initially, Annie mathematized her situation based on working with total numbers, which she counted up from each game character’s share when she was in the role of the designer, but which she set up for game users to divide. That is, as the designer she arrived at thirty crackers by counting up by sixes. However, the goal of the activity, as described by Annie, was to *share*, in other words, to divide, the total number of crackers among characters. That’s what Annie did when I asked her, placing her in the role of the user, to share twenty crackers among characters.

Since the initial relation structure within the group of character shares was a simple “they are the same,” Annie did not think of it as problematic for herself or for the user. Division, on the other hand, was still problematic for her, as evidenced by her counting up. Thus she focused on that area, division of the total number, where she could set up a *problem*. However, when my intervention called for a change in the relation structure to make some of the relations different from equality, the relation structure itself became problematic for Annie. Initially she stuck to the problem setup of sharing of the total number, and figuring out that total number was something of a challenge. But later, when Annie focused on the relation structures even more, she switched to offering the user infinite, rather than given, supply of objects, and setting up missing-value problems where the relation structure was problematic.

Also note how Annie switched from sharing markers to sharing paper, commenting that the objects to be shared had to be "the same" and markers would not work since they had different colors. Such attention to realistic detail corresponded to image making level in all learners, as I discuss in the chapter part on image making.

Thus the change of the source of metaphor to non-fair sharing led to the change of the target to "equivalence classes based on the relation of difference." Figure 20 is Annie's depiction of five groups of characters. The first and the second groups had the "sharing pattern," or the structure of relations, described by Annie as "you and two of your friends get one more cracker than the other person." The third, fourth and fifth groups had the sharing pattern of "you and a friend get one more than the other two people." I have added comments and outlined the groups in Figure 20. Annie used several other examples of such additive sharing patterns, moving to image having and property noticing (Figure 20).

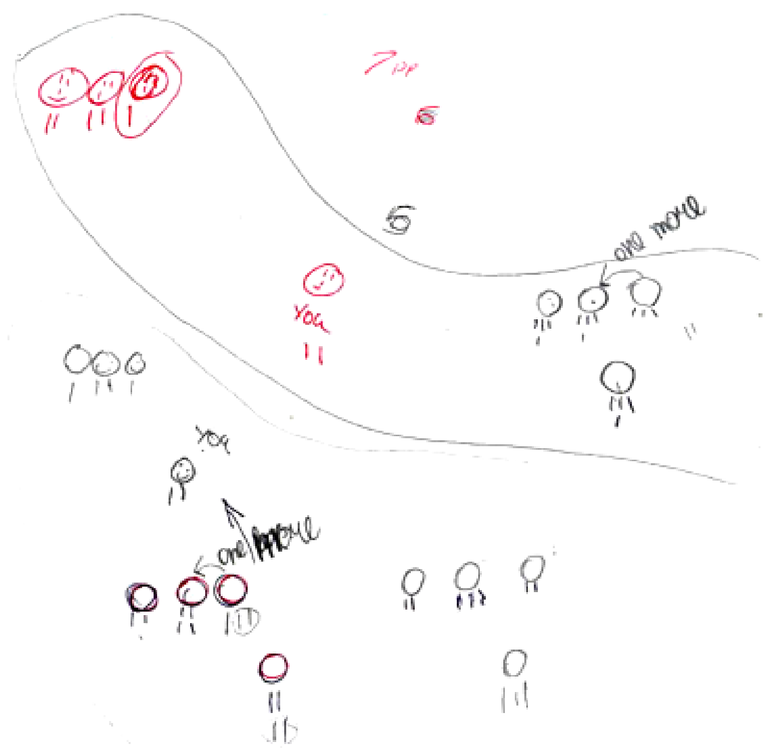

Figure 20. Two additive sharing patterns.

Later, using Annie's image having and noticed properties about the equivalence and now a more complex relation structure established through these and several other examples, I intervened again, inviting Annie to try a multiplicative, instead of an additive, relation. Thus the new version of the metaphor's source became "giving your friends twice as much as other people" and the new target was equivalency class based on ratios. It is at this point that "the basket of never-ending apples," as Annie called it, appeared in the game (Figure 21), signifying the change of focus to the relation structure.

Figure 21. The basket of never-ending apples.

On the same page, Annie drew two groups with the same differences, and two groups with the same ratios (Figure 22).

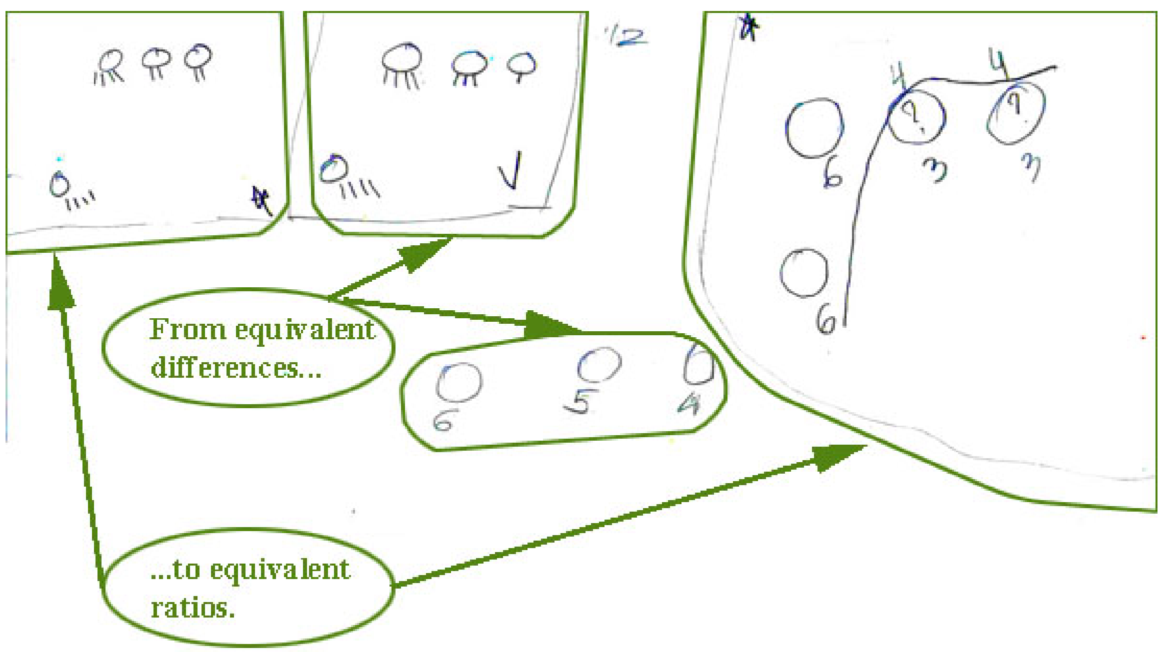

Figure 22. Sharing additively (with an error, 4 instead of 3, that was corrected in dialog) and sharing multiplicatively.

Note the top right corner of Figure 22, the part with question marks. This part of the picture, drawn by me, was used in a discussion about communicating the pattern to users of the microworld. Annie said that the task would be given to the user in words and pictures. A picture, such as the one at the top left corner of Figure 22, would be shown on the screen. Some text would also appear, saying: "Now give yourself 6 apples and keep the pattern you see for other people." I interpreted this task as corresponding to traditional missing value problems (Karplus et al., 1983). For example, the problem depicted in Figure 22 could be written as:

(4; 4; 2; 2)

(6; ?; ?; ?)

We talked about comparing the additive and multiplicative approaches, using the top right diagram. With the additive approach, the problem can be solved as (6; 6; 4; 4), and with the multiplicative approach as (6; 6; 3; 3). At this point, the conversation turned to other ways "a pattern" (Annie's word for an equivalence class) could be defined. Annie thought at first that one member of an equivalence class, or in her words "an example" is enough to define the class. The top right corner of Figure 22, drawn by me, is another shepherding intervention that Towers (1998) calls "rug-pulling" and defines as a deliberate shift of learner's attention to something that confuses the learner, and forces him or her to reassess actions. In this case, following Annie's idea in using a member of an equivalency class to define the class, I "pulled the rug" by demonstrating two possible solutions to the missing-value problem Annie was trying to set up. This time, having already folded back before and thus having made her understanding thick enough, Annie went into formalising. After our conversation, she extended her world to include three types of patterns, which she color-coded “red, green and orange” and which corresponded to additive, multiplicative and exponential relationships (Figure 23). In the picture, one can see the group in the top left corner defining a pattern, and the way three different relations can be used to solve the missing-value problem with that pattern.

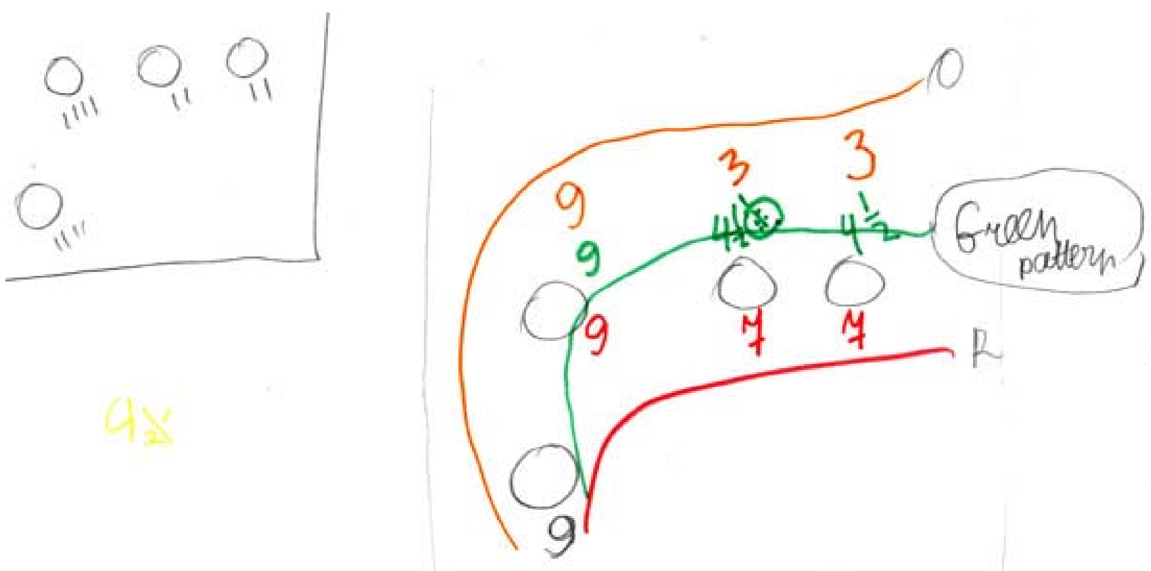

Figure 23. Additive, multiplicative and exponential patterns.

Table 1. Additive, multiplicative and exponential "patterns."

The missing value problem	**The solution using additive relation (“red pattern”)**	**The solution using multiplicative relation (“green pattern”)**	**The solution using quadratic relation (“orange pattern”)**
(4; 4; 2; 2) (9; ?; ?; ?)	(4; 4; 2; 2) (9; 9; 7; 7)	(4; 4; 2; 2) (9; 9 ; 4 ½ ; 4 ½)	(4; 4; 2; 2) (9; 9; 3; 3)
	“friends get two more crackers that others”	“friends get two times as many crackers as others”	“friends get the square of crackers that others get”

This case demonstrates how we can observe the recursive movement that changing both the source and the target in a metaphor.

Table 2. Annie's metaphor of "sharing."

Source		**Target**
Fair sharing of crackers between people		Division of whole numbers
Unfair sharing		
		Additive equivalence classes
"Patterns" of unfair sharing		
		Comparing different additive equivalence classes
"Patterns" of sharing based on different operations		
		Equivalence classes based on different operations, including multiplicative (proportional reasoning)

Metaphor and Image Making

Image making, the first of the levels in the Pirie-Kieren model that has to do with current actions rather than the past history as Primitive Knowing, is considered to be the basis and the beginning of all learning. In terms of metaphors, image making corresponds to construction, or in the case of folding back to image making, to invoking (Pirie & Martin, 2000) and re-constructing, of the source for the metaphor and its connection to the target. It should be noted that this separation of the source and the target belongs purely to the area of the data analysis and is not experienced by the learner engaged in image making. For learners in this state and the following state of image having, "mathematics *is* the image that they have and their working with that image" (Pirie & Kieren, 1994a, p. 40). In this part of the chapter I describe image making by the subjects of the study and its connection to metaphors, as expressed in construction of the subjects' microworlds.

In particular, one mechanism of metaphor source building that played a prominent role was *bricolage,* or "the process of using materials that happen to be at hand, in order to carry out some quite new process or construction that becomes possible when these old materials are combined in appropriate new ways. This suggests the image of a child's mental representation being built up in rather the way that some children build tree houses, using whatever boards, nails, and other materials are available" (R. Davis & Maher, 1997, p. 95). Bricolage took the form of using materials or images from the immediate surrounding, or bringing in materials that were important to the child at the moment. Examples of the first case include sharing crackers in Annie's microworld when there was a bowl of crackers on the table, or a microworld about ants for Sofia when she saw ants going for the cracker crumbs. Examples of the second case include Zack and his monster truck rally microworld, Niky and her environmentally friendly house microworld, and Evans and his fantasy microworld. One special case of bricolage was learners using common entities from software to modify and to expand their microworlds, such as buttons or menus.

In interaction perspective of metaphor research (Black, 1962; Ortony, 1993), metaphors play a *constitutive* role in creating new knowing. Metaphors are changing both source and target ideas in a cyclic process of metaphoric projection: "what is being so-called a "target concept" is often *created* by metaphor rather than being only used by it" (Sfard, 1997, p. 361). Design of a microworld can be considered a visible, accessible embodiment of this action of *metaphorising*. Representations used by subjects in their microworlds played two connected, or rather inseparable, roles in image making. First, these representations expressed images learners were making, and their change expressed the process of image making. Second, these representations assisted in image making, serving as external objects assisting in making an image.

Representations and Enhancing

Changes in representations between more tangible, detailed pictorial or three-dimensional objects, or movements and sounds, and more schematic iconic or abstract depictions formed a major theme in data. In this part of the chapter I discuss relationships between representations, image making and metaphorising.

To begin their microworld creation, children started from detailed, realistic, tangible images in their designs, as well as word descriptions and actions during interviews. For example, Zack spent about fifteen minutes during the first interview drawing a detailed monster truck, and even coloring it in the checkerboard scheme (Figure 24).

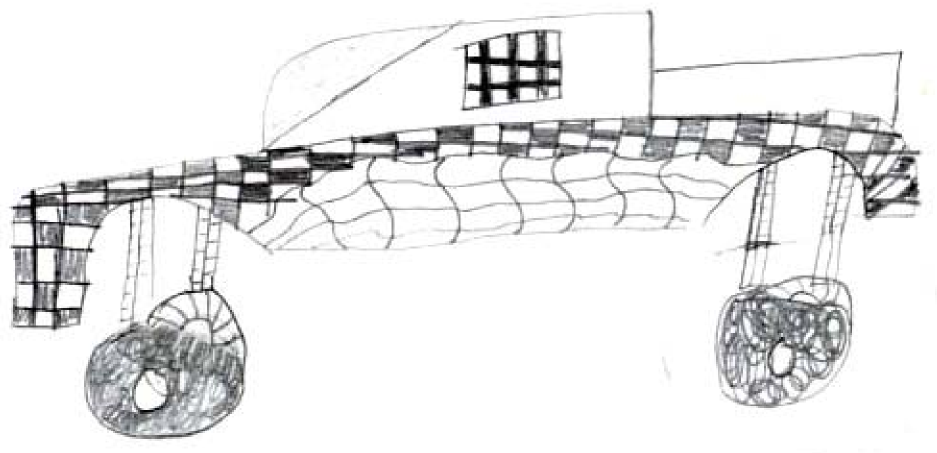

Figure 24. A highly enhanced truck.

Zack knew that he wanted his microworld to be about trucks, since he is very much interested in the topic of car mechanics and racing. To start working, he has created a very detailed representation of a truck. He also spoke about color schemes, chassis, window protection and other things that are important for a good monster truck. Later, as his microworld developed, Zack used progressively simpler, less tangible iconic drawings, and then words, word abbreviations, and numbers (Figure 25).

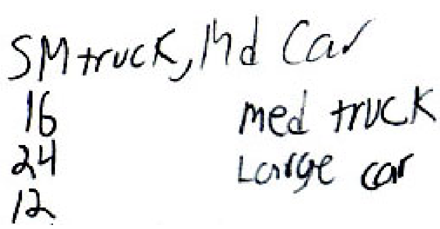

Figure 25. Words, abbreviations, and numbers: abstract representations.

At the stage of the “chessboard scheme truck” Zack’s microworld was first being created, and the source of any future metaphor was being rooted in truck pictures. To make the microworld real, and to start from anything at all, the pictures had to be very tangible. Other subjects also engaged in similar activities that I grouped under the label of *enhancing* as in “enhancing the microworld with details to make it more tangible.”

For example, Evans brought very detailed drawings of his fantasy microworld’s characters and scene layouts to the second interview. He used ideas from typical “fantasy quest” games such as Myth, filling in details of screen layouts, giving names to characters (Figure 26), and drawing game objects.

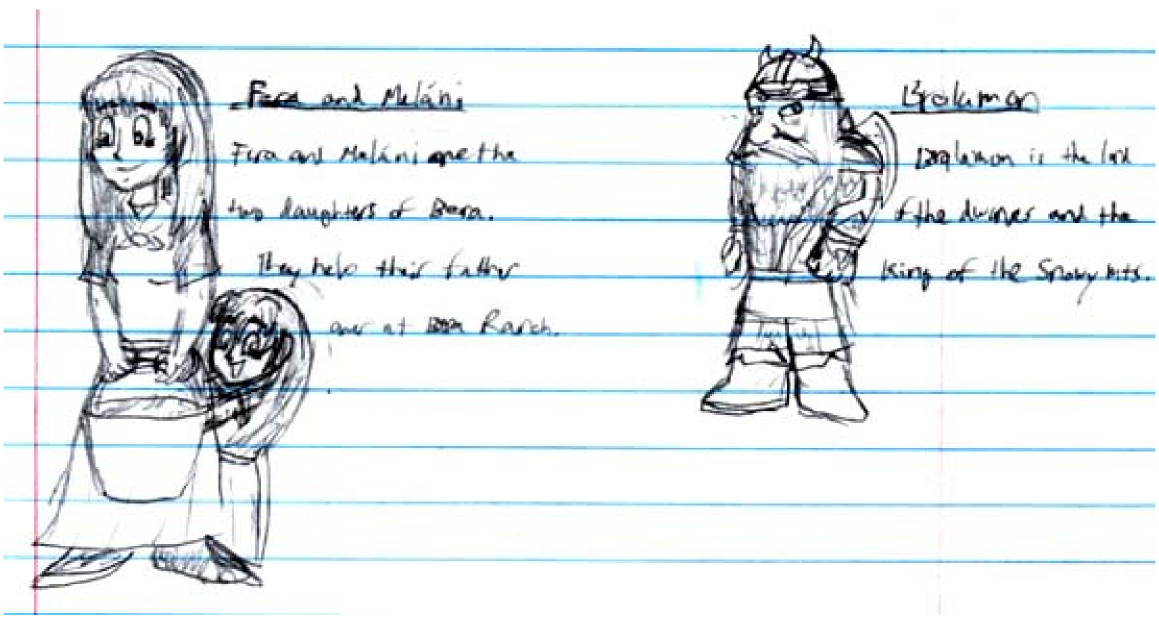

Figure 26. Fantasy characters.

None of this work, to the best of my or Evans’ understanding, was connected in any way with mathematics. It was directed at enhancing the microworld and making it more tangible, more real. Later, when mathematization started, Evans and other learners used the pictures, the words, the actions, and in general the context of their microworlds as the language to talk about mathematics. However, Evans was reluctant to do any mathematization in his microworld. In his case, enhancing “ran away” from mathematization and became too persistent and uncontrollable (Aspinwall et al., 1997). It is probable that rich images would support mathematization in a larger project. For example, it is easy to imagine many mathematical activities with elaborate collection of weaponry Evans created (Figure 27). But during this study, initial work with images did not lead to significant mathematising in Evans’ case.

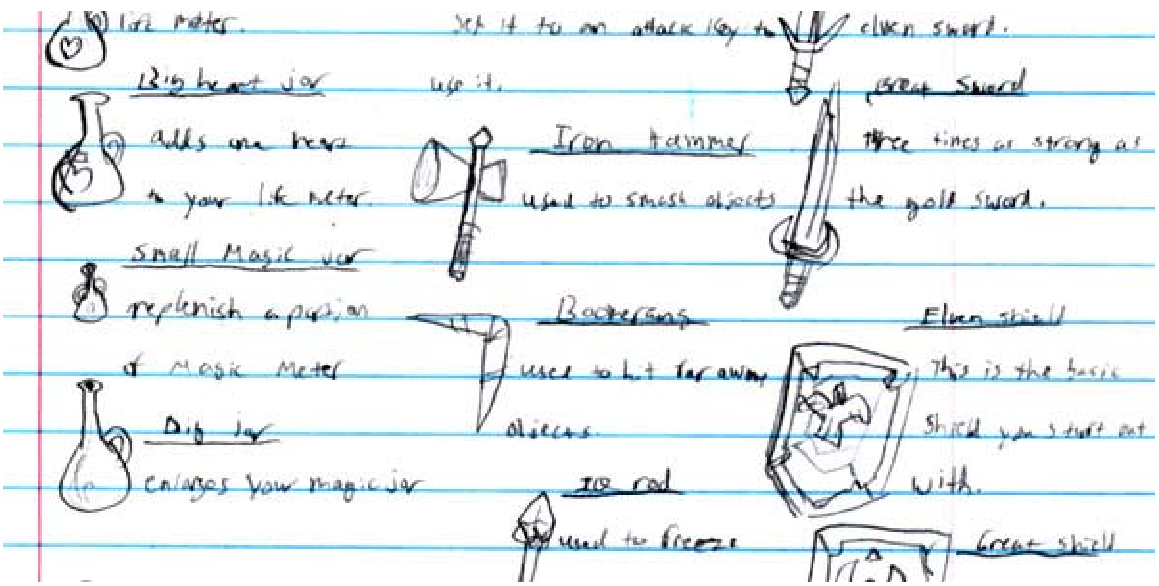

Figure 27. Weapons from Evan's game.

Other learners, even Abby with her initially strong proportional reasoning, engaged in enhancing to some degree. Abby decorated buttons in her circus microworld to resemble penguins, tigers and elephants (Figure 28) that were supposed to appear at the click of the button.

Figure 28. Penguin, tiger and elephant buttons.

Niky was enhancing her microworld with mostly verbal descriptions of philosophy of ecologically sound houses, and details of habits of people who would live there. Her drawings are an example of tangibility that does not feel too out-of place, yet does not play a clearly defined role: the garage door is striped and the house is given a 3-d appearance (Figure 29).

Figure 29. A house surrounded by menus.

Annie spent some time making her fruit baskets looking tangible with grid designs (Figure 30), and using different colors and adding hair to people (Figure 31). Again, she could have signified these objects in a simpler manner. In fact, a very simple basket was used earlier when she was property noticing while working on another level of her game.

Figure 30. Detailed drawings of fruit baskets.

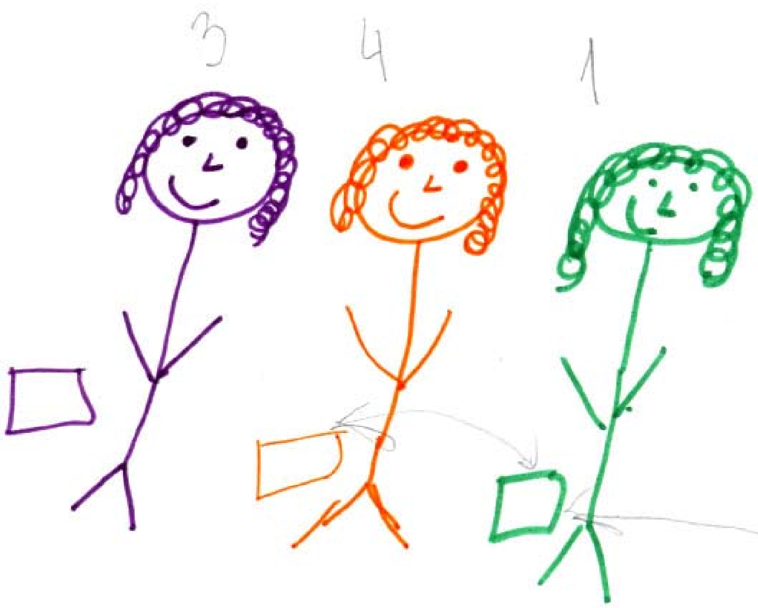

Figure 31. Enhanced stick figures.

Sofia was the only one whose use of drawings, colors and descriptions stayed very functional, with no visible desire to enhance the microworld, at the stage of planning, beyond what was needed for understanding what was supposed to happen in her software.

From the point of view of the proclaimed activity goal of designing software, drawings of "future screenshots" serve purely symbolic roles. They are placeholders for future "real" pictures that will be substituted when software is programmed. From this point of view, it makes sense to use the simplest possible diagrams that still convey the layout. Enhancing, such as drawing with more detail, or switching colors of markers, means spending much more time compared to using relatively simple abstract symbols. For example, Abby could use simple rectangles with letters instead of her decorative buttons. I think the role of detailed drawings and other enhancing actions was more prominent not only in planning the software, as was the proclaimed purpose of the activity, but also in assisting learners in constructing a virtual space for image making. Before they could start making an image, they had to invoke or to construct some entities, some context, in other words, some *world* where image making can take place. Later learners also felt the need to strengthen the world, which sometimes coincided with folding back to image making, as was the case with Annie and her relatively detailed and colorful drawings of people.

It should be noted that as the subjects' designs were developing, their depictions of game objects were changing. The pictures of objects evolved from detailed and tangible iconic symbols to increasingly schematic iconic symbols, and then to abstract symbols (Drier, 2000). By "abstract" here I mean symbols that would not seem to resemble the signified to a person who has not observed their evolution. An example of such evolution can be seen in the sequence of three of Annie's depictions of game characters (Figure 32). She starts from “stick people” and goes through a period of “smiley faces” and then uses plain circles. Note also the move from using tallies for counting to using numerals. Thus Annie went from detailed pictures, to more sketchy iconic symbols, to abstract symbols. However, starting to design a new level of the game, she goes back to more detailed stick figures in different colors.

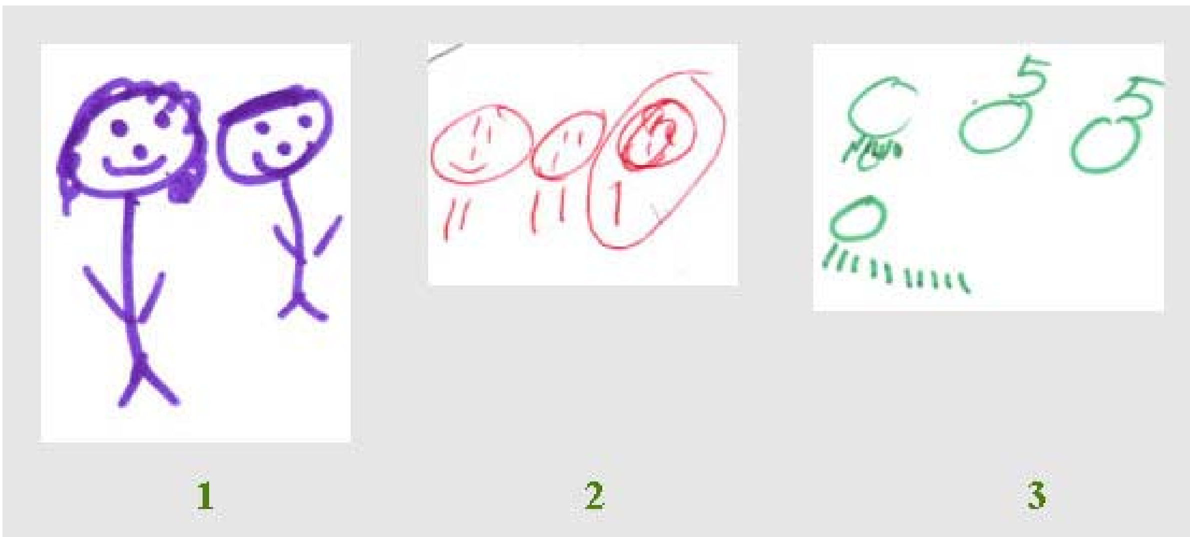

Figure 32. Evolution of Annie's representations.

A similar change in representations took place in Zack’s microworld (Figure 33). Zack moved from a very detailed, richly decorated drawing to a more schematic depiction of a truck and a car obstacle. Later, describing different combinations of truck and car obstacle sizes, he used words, and then his representation changed to abbreviations of words such as “SM truck” for “small truck.” Both Annie and Zack were property noticing and formalising while using numerals and abbreviations, and image making while using detailed drawings and iconic symbols.

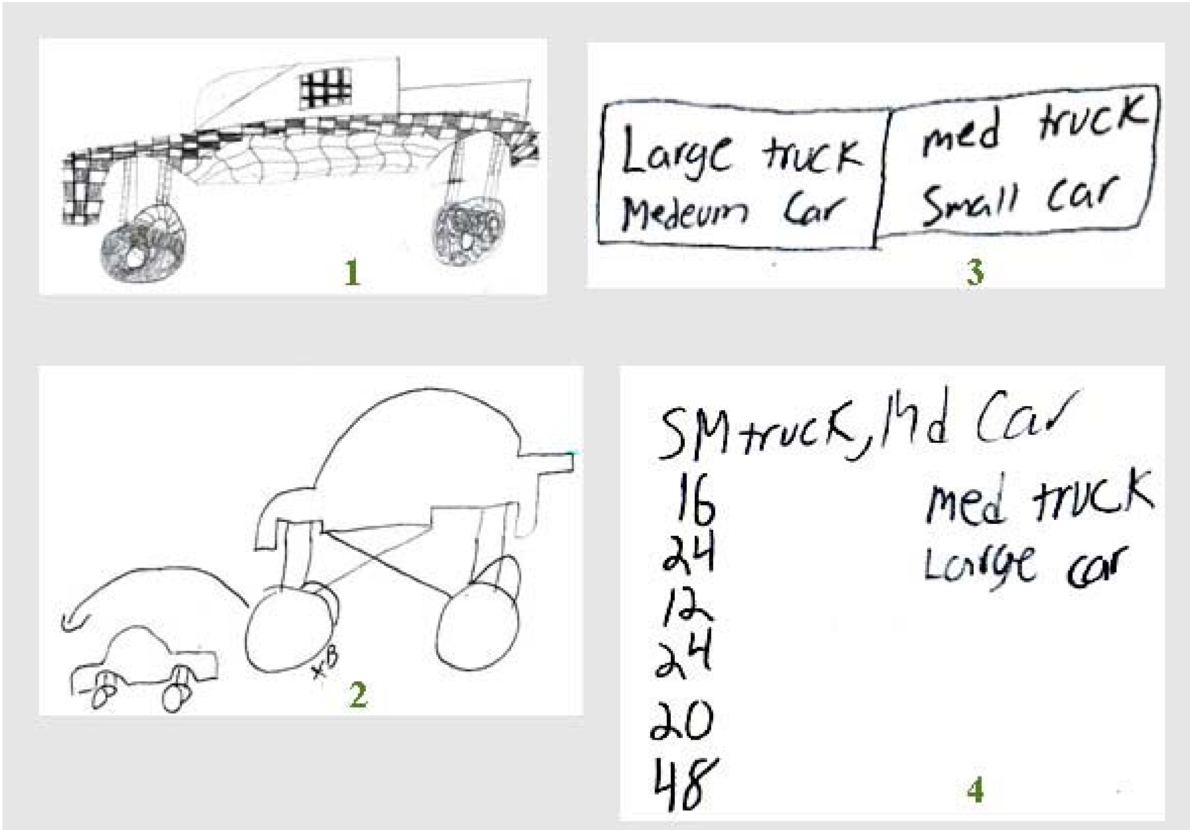

Figure 33. Evolution of Zack's representations.

Zack used an input form to set up a missing value problem about times it took for different trucks to go through different obstacles in the racetrack. When Zack was first setting up the tables with times, he was counting seconds to himself, moving his finger along a drawn stretch of a racetrack, nodding his head in unison, and saying a word “Mississippi” under his breath, which takes a second to pronounce and which he spelled “close enough that we can understand,” (Figure 34), as he explained. For example, to figure out what time to assign to the large truck going over the medium car obstacle, Zack nodded and rocked in unison with lip synching: “One Mississippi, two Mississippi, three Mississippi, four Mississippi,” and then wrote the number four into his table. In Zack’s case, making an image was not just visual, but kinesthetic, and his initial representations were auditory and kinesthetic (Presmeg, 1997a, p. 302). His sense of time measuring was tied to body movements and sounds, and later re-presented in numbers. To figure out parts of the table, Zack measured his drawing of the track with his outstretched fingers, commenting with a noted relief that DE can be made the same as BC, and FG is “three times of that” AB. He did not count to “twelve Mississippi”, but multiplied in his head. The relief had to do with his concern that he would have to figure out too many times for all combinations of trucks sizes and obstacle racecourses (Figure 34).

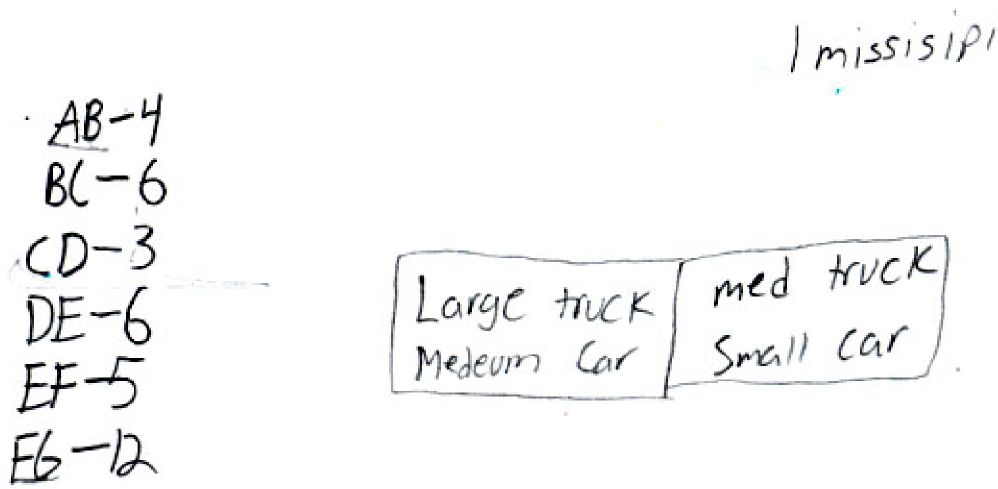

Figure 34. Zack's first timetable.

I placed this description of Zack’s kinesthetic representations here as an example of a necessity of enhancing activities. Zack could pick a random small number of seconds for his racetrack tables. In fact, I took some time to explain, with detail and examples, that it is a possibility. Zack listened to me with an expression that I later interpreted in the video as “patient,” but as soon as I finished, returned to his “Mississippi” procedure. Recently, I have read the following problem copied from a third-grade mathematics textbook: “The trip to school took Molly two hours, and the trip back took twice as long. How many hours did it take for Molly to get home?" The child who brought me the problem commented on the numbers in it not making any sense. Such "frivolous and ridiculously unrealistic problems" (Cooke, 2002, p. iii) from textbooks have been criticized (Cognition and Technology Group at Vanderbilt, 1993) for their premature, too abrupt de-contextualizing of learning. When Zack was formalising later, he sometimes did not attempt to check how realistic his racetrack times were, for example, making a track go unrealistically slow. At formalising, numeric relations fascinated him more than realism. However, when he was making an image, extremely realistic racetrack times were crucial to his growth of understanding, and he spent time and energy enhancing his microworld in this regard.

Another example of such enhancing super-realism in microworld design came from Annie's interview about sharing: "You have ten markers, three friends and you. They are all going to take your ten markers, and they, yeah… But they are all different colors! So you need something that's, like, all the same… I'll use seven pieces of paper." During image making, Annie refused to set up a problem about sharing markers, because markers usually come in different colors. It did make mathematical sense to share markers by quantity, but it did not make real-world sense for Annie to share markers of different colors among people so that none would have a full set for drawing. She took about forty seconds of thinking time to search for objects that were "the same," such as pieces of paper. Later, while formalising, Annie did not take notice of what objects, specifically, the game characters were supposed to share. However, during image making the situation had to make real-world sense.

To summarize, enhancing activities may come even before learners start to develop metaphors. Enhancing assists mathematising in the microworld, defined as making one's actions more mathematical (Freudenthal, 1968), indirectly, by creating a world where image making can take place. Enhancing also assists image making, which has practical implications I discuss in Chapter 5. Enhancing has its dangers in that a learner can engage in enhancing activities for their own sake, losing sight of any mathematics that was supposed to be involved.

Groping and Other Beginnings of Metaphor

Besides enhancing, there was another theme in somewhat random, directionless image making activities. It happens when learners used their microworld entities that were to become sources to *grope* for a mathematical target, any target at all that could be connected with the initial world. In this turn of events one can see a way metaphors are born. Learners begin to metaphorise by changing their systems, not yet developed into a source-target structure of a metaphor, by the action of trying to collect a suitable target from primitive knowing. Children "play" (Hancock & Osterweil, 1996; Papert, 1993; Steffe & Weigel, 1994), starting to give more tangibility to their microworlds, and at the same time beginning to make images and beginning to turn them into metaphor sources. When learners start from some pre-determined entities as sources, such as those microworlds initially created and enhanced without mathematising, and grope for mathematical targets, they are very likely to miss the proposed target of proportional reasoning, arriving at any concept or a composition of concepts.

For example, Sofia, when she first started to design her software, chose the theme of marching ants. In her game the user could add columns of ants of different colors to the scene by clicking buttons. Then the ants started to march down to "multiplying stones," where a set number multiplied each column, making more ants. Possible targets for metaphors from her microworld at this point included addition, multiplication and comparisons of numbers. During the first interview Sofia, as did everybody else, expressed the fact that she did not know what proportions were. Thus she started *some* mathematization, hoping it would be related to proportions. A similar effect of picking a theme and then groping for mathematization was observed in Annie's world. The initial target for her sharing source turned out to be division.

Another way learners started metaphorising was to pick a target concept and to search for sources and themes that would go with it. This way of metaphor development can be traced in Abby's microworld. She designed the first activity in her software to be structurally similar to an activity from the sample microworld I demonstrated at the beginning of the first interview. Namely, the user of her software could change the numbers of animals in each of the three rings of a circus by using buttons. The goal for the user was to make "twice as many elephants as there are tigers, and three times the tigers for every penguin." Compare this task to the sample proportional reasoning task I gave introducing in my microworld and Abby solved successfully: "If I make ten yellow dots, how many red dots will there be?" The representations were also visually similar (Figure 35). I took the similarities as a sign of attempting to understand my model of proportionality, and to reproduce it in a different setting. Thus Abby's primitive knowing included some proportionality ideas, and her initial image making in the context of the microworld included these ideas. Her metaphors had a more clearly defined target of proportionality from the beginning.

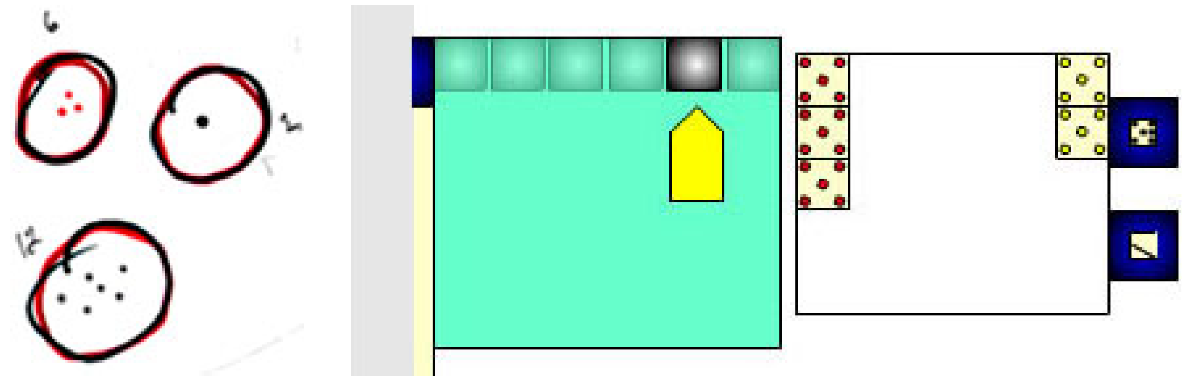

Figure 35. Similarities between Abby's first level and my microworld.

<u>Bricolage: Important Concepts, Nearby Objects, and Common Computer Entities.</u>

Bricolage, as described in the Introduction to this chapter, is using the materials and ideas at hand for the knowing. Bricolage is often a more associative (L. Vygotskii & A. Kozulin, 1986), ecological rather than pre-planned way of knowledge development. I found three major themes in the ways participants' use of bricolage affected their growth of understanding, and in the ways bricolage was related to metaphor. The themes were using concepts and contexts that are important for the learner; using objects that are present in the physical space during the interview; and using common computer entities such as drop-down menus.

The first theme, most prominent in the initial microworld creation, was *the use of contexts important in one's life*. Some of the participants chose their microworld themes based on their passions. Niky has decided to make a game simulating how a family lives in a house, analyzing patterns of their activities, and changing the house to be the most efficient for that particular family. Later, she talked about rooms being too large or too small for the percentage of the time people spend in them, measuring electricity consumption in different parts of the house, and other more detailed, and more mathematized uses of the house idea.

However, initially Niky did not know in any detail what these mathematizations would be. The house idea was used in the spirit of bricolage, not because it suited the subject of proportionality especially well, but because it was already at hand, already in Niky's thoughts.

Similar development took place in Zack's interviews. He wanted his software to be related to monster trucks, because the subject is one of his very strong interests. In the first meeting, he repeatedly said that he did not know what proportions are, or how to relate his software to proportions. Later, Zack mathematized his microworld by introducing relations between sizes and nature of obstacles on the course and sizes of trucks. Initially, choosing trucks was a bricolage action.

Finally, there is the case of Evans, who started to design software from an idea, which he expressed with the total lack of enthusiasm, that proportionality software probably could be about ovals. We pursued the subject of ovals for a few minutes, while it was quite clear that there is no interest in Evans. Finally, I suggested thinking about software that may actually be interesting to him. That's when Evans decided to make a fantasy quest game. As I described in Chapter 3, Evans likes fantasy role-playing, and spends much time designing worlds for such activities in clubs or with friends. Thus, Evans chose what he knew and liked for his software: a fantasy world where magical characters went on a daring quest. Evans' initial idea for the mathematization was that the world was overcome by a magical malady that made everything turn "out of proportion." The expression "out of proportion" was in itself used in bricolage manner, as something vaguely related to the topic, something that possibly could come in handy. Probing revealed that Evans was not initially aware of any particular mathematical notions when he used the expression "out of proportion."

For the three *bricoleurs* who created their initially non-mathematized microworlds based on their interests, this process ultimately served purposes similar to enhancing described above. That is, enhancing makes already invented microworlds more tangible and more real, so that they can better support growth of understanding by providing a context for making images that will become metaphor sources. Bricolage is one of the ways microworlds can be *invented* in the first place. Bringing in important contexts from one's life may lead to rich, meaningful images that have a hope of powerfully supporting growth of understanding. This was the case for Zack and Niky, while Evans' case demonstrates a possibility of very little mathematization.

The second bricolage-related theme was *the use of nearby objects* in learners' microworld. Sofia has selected her ant theme seeing a column of ants marching from the nearby open window in the interview setting. Annie used crackers I offered her as a snack as counters, and later she looked around the room for other ideas for things to share, and found markers and pieces of paper. Annie and Sofia used different colors of markers for color-coding when they needed to signify a distinction between objects.

The third bricolage theme was *using common computer entities* to design software. Such common objects are readily available to those who use software as handy metaphor sources, much like metaphors used in ordinary thinking such as "time is money" or "argument is war" (Lakoff & Johnson, 1980). The learners as easily used "level of the game" for "posing a different problem or set of problems" as people use the directional metaphor "times ahead" for "the future." While the subjects' microworlds themselves can be considered as extended metaphoric systems for dealing with proportionality, smaller, more specialized metaphors based on common computer entities played their role in creating the microworlds. Here I grouped several such metaphors from interviews based on their sources' common origins.

Table 3. Metaphors with sources from software and gaming.

Sources of metaphors in software features	**Targets of metaphors in mathematics and learning**
Levels	Different kinds of problems
Ending of the level	Evaluation of the problem solution
Screens 1. Change of screen when a new level starts 2. Pop-up screen	Types of problems or actions 1. Beginning to work on a new problem 2. Switching to a different action or set of actions while staying with the same problem
Buttons 1. Done! 2. Add objects 3. Go to (switch screens buttons)	Actions 1. A call for evaluation of the solution 2. Building up quantities by adding one more 3. Switch of context or activity
Dynamic counters	Calculations performed automatically by the computer
Input forms	A way to initiate the whole quantity, rather than building it up by button actions

Level of a computer game is a common way to organize the content, present in many games. The name "level" as children use it may refer to a particular theme within a game, such as the matrix level or the data trees level in Logical Journey of Zoombinis (Hancock & Osterweil, 1996). Also, a level may refer to a particular setting for the complexity or difficulty of the same-themed game, such as Zoombinis' four levels from "not so easy" to “very, very hard.” I have observed both meanings in learners’ use of the word “level.” In Abby’s software, for example, there was a level, meaning “theme,” devoted to giving balls to jugglers, and a theme of balancing boxes of feed with given weights. In each theme level, there were several “difficulty” levels corresponding to larger and smaller numbers (Figure 36).

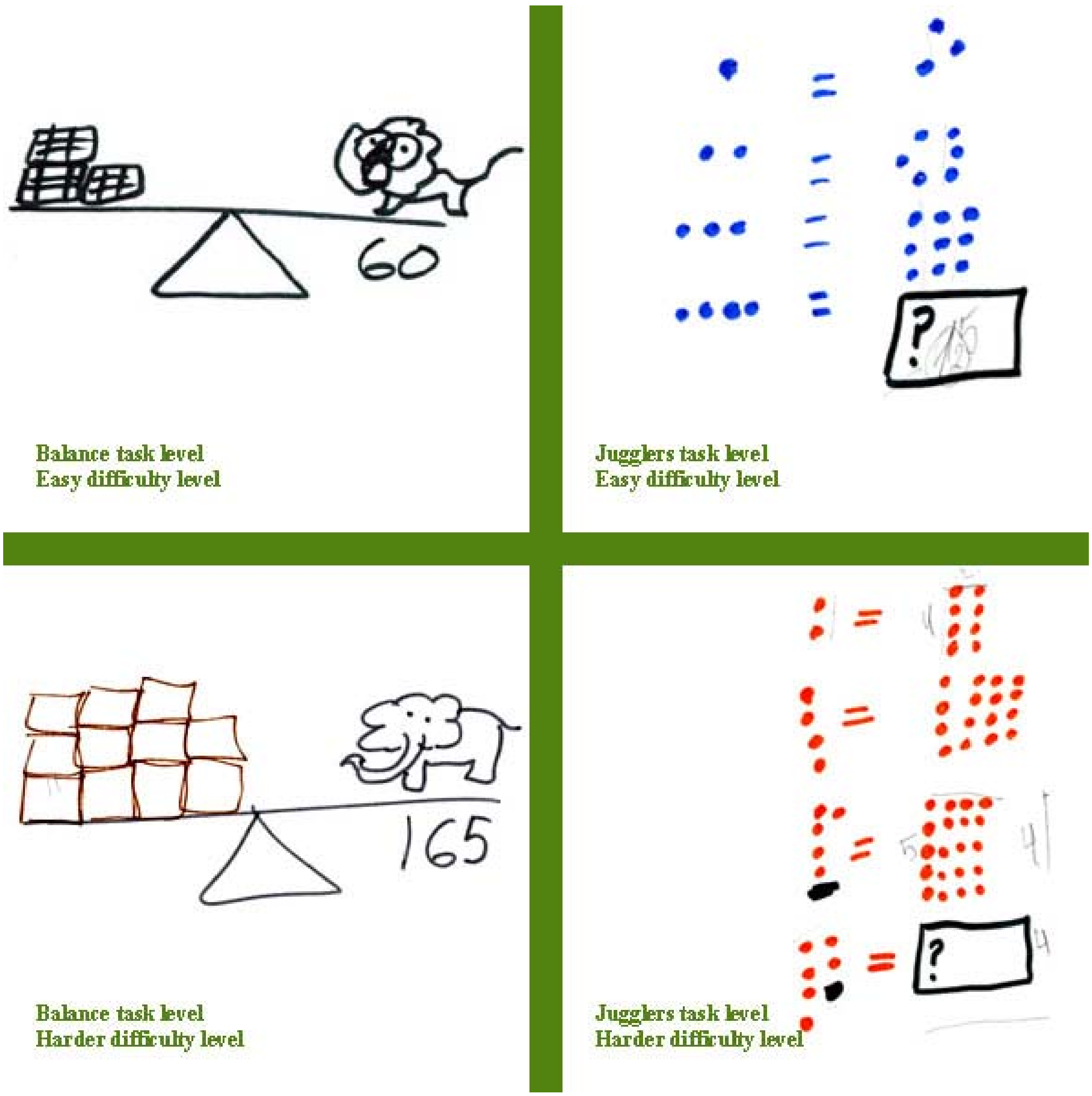

Figure 36. Abby's difficulty levels.

The notion of level helped learners to answer the question "What next?" when they finished designing a particular task for their game. Also, learners used the language of levels to talk about what kinds of problems there may be, and about features of each kind such as difficulty. The answer to the "What's next?" question typically was, "I will make another level." Thus, the metaphor of "levels as different problems" was a major organizing force in learners' designs.

There were different approaches to creation of new levels, and these approaches corresponded to different ways learners viewed proportionality and developed proportional reasoning. One way of creating new levels was to substitute different numbers into the structure of an already created level. Abby, Annie, Sofia and Zack used this approach to design new levels of their games. Learners expressed the opinion that a *more difficult* new problem means "bigger numbers." Another method for creation of a new level was to choose a different setting and context.

Learners used the idea of a *button* as the source in the metaphor with the target of "action." In all games learners instituted a "Done!" button (Figure 37). Clicking this button signaled a call to the software for a test, or an evaluation, for the user's solution of the problem. For example, in Sofia's game the "Done!" button started the cartoon of the ants marching toward the doubling and tripling stones and then being admitted, or not, into the anthill, depending on their total number. In Niky's game the "Done!" button started the simulation of the life of people in the house. In Abby's and Zack's games, the "Done!" button called up a pop-up screen that informed the user whether or not he or she filled in the input form correctly. In Annie's game, similarly to Sofia's game, once the user finished dragging apples from the "basket of never-ending apples" to game characters, he or she could click the button by the basket and be informed if the apples were distributed correctly.

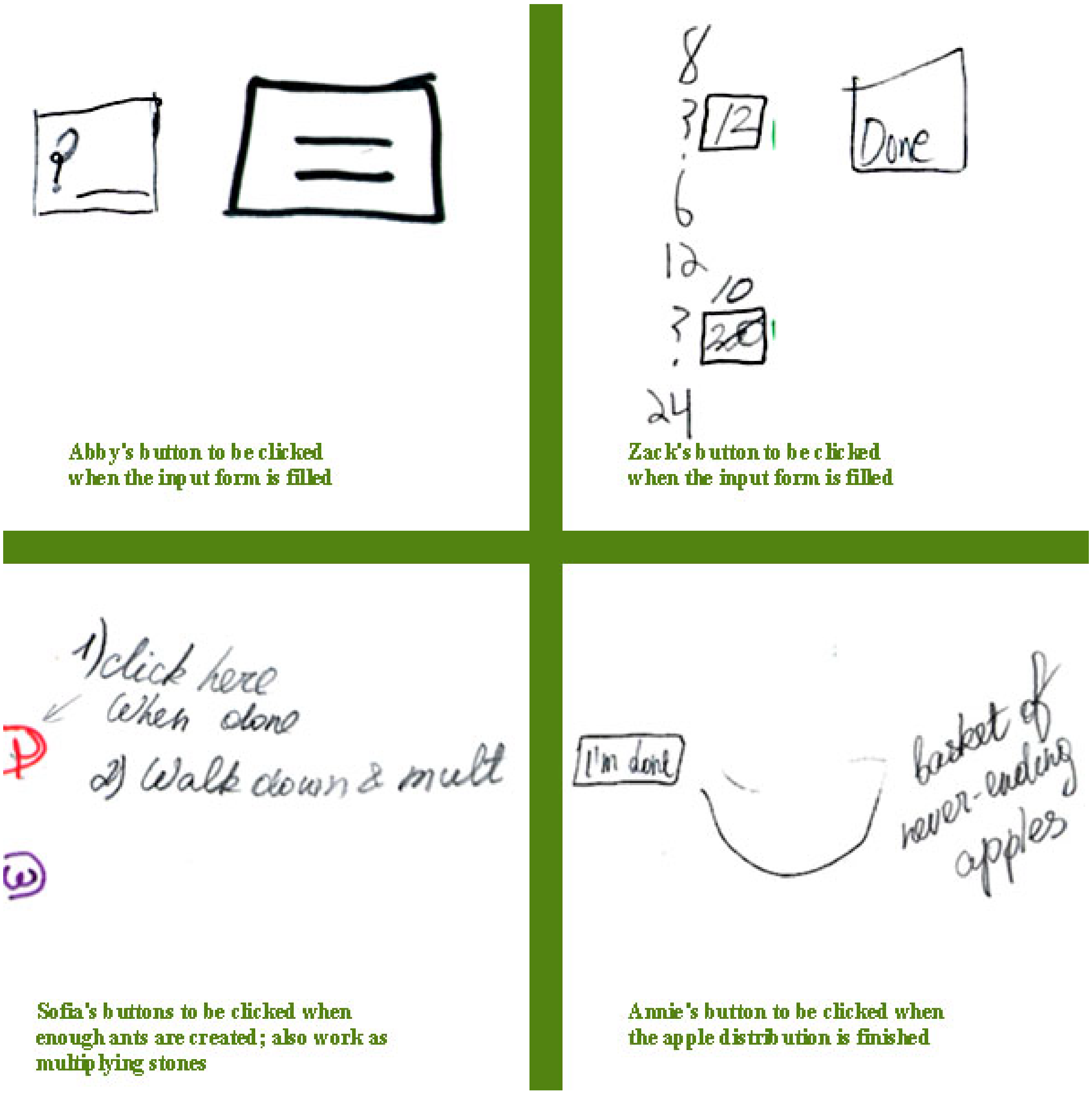

Figure 37."Done!" buttons.

Another common type of a button action instituted by learners was to add an object to the setting, or to remove all or some of the objects. For example, in Abby's three-ring level a click of the "penguin button" added a penguin, and a click on the "tiger button" added a tiger. In Sofia's game, buttons of different colors added ants of that color to the ant column under the button. There was also a "clear" button removing all ants. Annie had "less" and "more" buttons adding or removing candy to a basket selected by clicking (Figure 38).

Figure 38. Adding and clearing buttons.

Several microworld researchers discussed the importance of such buttons for learner learning, and the special role the adding of objects played in concept construction (Steffe & Tzur, 1994, p. 103). Researchers who offered their subjects pre-determined microworlds noted how each microworld object, including buttons, occasioned particular learners' actions and thinking. In the case of learners designing their own microworlds, their introduction of particular computer objects meant invoking particular actions associated with such objects in the learner's mind. Such associations came from previous experience with computers, not from current work in an existing computer environment. This button operation of "giving more" (Annie) or "making more" (Abby, Sofia), supported the idea of separately creating *different* quantities of objects. This, in turn, supported or expressed comparisons of quantities, or *relations*. Thus there was a duality in the way buttons appeared in learners' microworlds. On the one hand, learners were thinking of the roles buttons would play for users of their software. On the other hand, button metaphors helped learners in their own thinking. The two roles, which I separated for analysis, were intertwined during the interviews.

Software entities such as buttons have a special place in learning because of their dual nature: they represent *actions* and yet they are, in themselves, *objects*. Transition from actions toward objects is a key feature of several frameworks, for example, (Sfard, 1997, 2000; Tzur & Simon, 1999). I hypothesize that "objectifications" of actions in computer entities assist learners in making these transitions in their thinking.

As developments of button actions that added objects, learners instituted two other entities that can be analyzed as sources for metaphors: *dynamic counters* and *input forms*. Subjects instituted a dynamic counter when they realized that it is difficult to either remember, or to count each time, the number of objects created with the help of an object-creating button. A counter looked like a window with the dynamic text depicting the total number of created objects, either appended to the creating button, or to the group of objects. For example, in Abby's three-ring circus, the counters were attached to each ring. In Sofia's example, dynamic counters were supposed to move with the ant columns to which they were assigned. Dynamic counters, as learners explained from the position of users, were to assist the user in determining the number of created objects. Dynamic counters also served as representations in the learners' own process of growth of understanding, corresponding to the learners' roles as designers. Numbers in dynamic counters, as more abstract symbols, appeared after some work with more iconic depictions of objects to be counted, as discussed above in the part about representations. In some cases counters would seem to appear together with *partial* iconic representations, when learners wanted to communicate that in the future software there will be pictures, yet did not want to fully draw the pictures. For example, in Sofia's drawing we see the number "12" next to three dots, and the number "8" next to six dots signifying ants (Figure 39). Sofia did not want to draw all the twenty ants, using iconic placeholders together with numbers instead. Here we see another difference between my on-paper design methodology, and working with actual computer environments. While some computer environments allow learners to choose between representations (Clements, 2000b; Drier, 2000), it is hard to imagine how such partial, dual symbolization could be implemented.

Figure 39. Partial change in representation types.

In some cases, the role of numbers as dynamic counters in software, and the role of numbers as shorthand representations of quantities of objects to be drawn in software, merged so closely that learners would forget which role they are considering at the moment.

Annie's top view of plates with candy, signified by dots, changed into a plate with number "24" written on it when she realized she did not want to draw twenty-four dots (Figure 10). The writings "3x" and "2x" were used by Annie to explain relations between the numbers of candy, and were not supposed to appear on the screen in software. After using "24" for herself, she decided that it should also appear on the screen instead of twenty-four candy pieces. This example illustrates the reason why my definition of the microworld includes learners' design actions, in addition to software descriptions: design actions constituted inseparable job in student metaphor development. It also shows how it is not possible to determine where the designed software ended, and the learning actions of the learner designer began.

In Abby's three-ring circus problem, users were given multiplicative relationships ("twice as many elephants as there are tigers") and were asked to set up particular numbers of animals by clicking the "adding objects" buttons. The buttons followed Abby's own actions of drawing dots representing animals in rings. Abby also added dynamic counters to the rings *after* setting up an example. That is, the dynamic counter metaphor, corresponding to the more abstract and objectified *numeric* representations of quantities, appeared *after* the button metaphor, corresponding to the *iconic* representations of making of quantities.

Describing the next level, Abby drew an animal on a see-saw. To balance the see-saw, users could click on a "crate button" in the corner, which added crates to the other side of the see-saw until it balanced. The dynamic counter under the see-saw represented the *total weight* of crates in pounds. The user was supposed to determine the amount of pounds per crate, and to type it into an *input form*. The "Done!" button in this level had the "=" symbol on it (Figure 37). While the role of "adding objects" buttons is to *build up* a quantity, the role of an input form is to instantiate a quantity in its entirety. Learners' move to input forms from "adding objects" buttons can be considered corresponding to the movement from image making to image having and to outer levels of the Pirie-Kieren model. Zack's work on his detailed drawings of different-sized trucks and obstacles had features similar to such building up, and also corresponded to image making. However, he did not use a common *computer* object, but rather a table that he filled, building up, with trucks and cars of different sizes. In Zack's description of the game, an input form also appeared during property noticing and formalising, when Zack used an input form to set up a missing value problem about times it took for different trucks to go through different obstacles in the racetrack. Earlier, during image making, when he was first setting up the tables with times, however, Zack was counting seconds, moving his finger along a stretch of a racetrack, nodding his head, and saying a word "Mississippi," as I described above in the part on representations.

"Go to" buttons served to "navigate" between different screens of the game. Niky designed whole menus of such buttons, since her game had many different modes for different actions such as the warp mode of a pop-up dimensions calculator. Zack used "quit," "restart" and "go to menu" buttons. Annie, Abby and Sofia used "next level" buttons. These "navigation" buttons can be considered a source of the metaphor with the target of learner's control of the learning process. For example, the "next level" button corresponds to the learner choosing to move to a different problem. "Go to menu" buttons also correspond to the learner changing some settings of the problem.

Metaphors and Structures for Proportionality

In the last decades, researchers studying extended tasks situated in particular contexts have been reexamining roles of various strategies in the development of proportional reasoning. A particularly puzzling series of findings had to do with learners apparently inventing “mathematical constructs that are considerably more sophisticated than those that they seemed unable to comprehend during past histories” (Lesh & Harel, 2003, p. 169), in the matter of hours instead of years going through sequences of models similar to the stages of development described by developmental psychologists. Where laboratory-type closed-ended tasks occasioned low levels of reasoning, recursive sequences of trying and revising of different strategies within context-rich environments that can be called “microworlds” led to development of complex and appropriate models. In the present study, learners moved between different kinds of strategies, including additive and qualitative reasoning, while developing their microworlds and their notions of proportionality. In this part of the chapter, I use the Pirie-Kieren model to map “processes and paths” (Lesh & Harel, 2003, p. 169) of learners in qualitative, additive and multiplicative worlds as their notions of proportionality developed, and the role metaphors played in the process.

The Case of Annie

At the beginning, Annie, like the rest of the learners, did not know what proportionality was. She was groping for an idea to start her game. At first she tried to find proportions in objects that looked like the object from my software, which she interpreted as dice. Later, Annie claimed that proportions are “kind of fractions” and related to dividing.

During the first interview, she has developed a microworld where goldfish crackers were divided equally among the five drawn characters, which Annie called "the same proportion."

Annie (A): I like this adding of dice. I can do that. The dots here (pointing to the laptop). I can do some design, I can use that idea of what proportions are. (Pauses)

Maria (M): So you can think of what has proportions.

A: Yeah (both nod). Dominoes do, they have dots…

M: Dots.

A: Yeah, they have dots. (quietly) Proportions… It seems that there are lots of different proportions, but I can't think of anything.

…

M: You can start from the proportion idea, like you did before.

A: Yeah.

M: Then try to think where you can…

A: Yeah, I can try to think of some good things that you can do proportions on. (Pause). Do you have… Do you think you can give me some idea of what these proportions are?

M: (looks up, thinking) Hmm. I can… What are they?

A: Proportions?

M: What do you see, how do you see them?

A: How do I see proportions? Well, it's hard to explain, for me. Like, they are things that show proportions of stuff. Like, how many things you have. I, I can't really explain it.

M: Can you draw a proportion?

A: (animatedly) I think I can! But I am not sure. A proportion is something like… How much you… If you have… They are something like fractions. Do you think so?

M: Kind of?

A: Kind of fractions. If you have… Like, if you are *dividing* them, there are proportions.

M: You are dividing… What?

A: You are giving proportions to *people*.

M: Giving proportions to people. So you have people.

A: Yeah, yeah. And you are going to give them an amount, like a proportion.

M: Ok, ok, what will you be giving them?

A: Goldfish!

M: Goldfish, ok. So you have some goldfish.

A: Yeah (draws a fish shape). And there's goldfish. I have lots of them, like thirteen. And I have, let's see how many people. I have five people (shows five outstretched fingers with her left hand, while drawing five smiley faces). And so I can give them proportions or something, right?

M: Ok, so you… If you have thirteen goldfish, how would you do it with five people?

A: (laughs) How do I divide it up? I don't know. Maybe I will have more.

M: You can have thirteen, you can have more.

A: (counts, lowering and lifting her marker) You can have thirty. And you can give them each the same amount and the same proportion, right?

M: Ok, so if you have thirty, and have five people, and give them the same amount…

A: You can give each of them six.

Groping for an idea to start image making, Annie started from a picture she saw in my software, and talked about dice and dominoes since they have dots as well. She decides proportions have to do with quantities, with "how many things you have." Having established that she would be dealing with quantities of objects, Annie searches for relations she would have to introduce, naming "division" and "fractions" as candidates. Then she decides to work with "the same amount and the same proportion." To establish that amount, she counts up by sixes to get to thirty, and sets up the problem of giving thirty crackers to five people.

How can one interpret such an approach to the task of finding an example of proportionality? If this was a closed-ended task, and if I were to declare the task finished at this point, Annie would have failed at it. She did not set up a proportion, and she did not figure out what proportions are. However, Annie's work on the task was just beginning. She has established a microworld where she used a metaphor of sharing objects among people. The first source for the metaphor was the idea of giving people the same amount of something. It is a possibility that Annie used the similar-sounding word "portion" as the guidance to what is a proportion. The evidence supporting this conjecture comes from Annie's utterances such as "each gets a fair proportion" and "each gets the same proportion." The first target consisted of identical relations between the amounts each character got, and the relation between the total amount of objects and the amount each character got. In the view of the *future* development of Annie's metaphor, the first target can also be described as establishing an equivalence class consisting of groups of five equal quantities. This equivalence class that Annie called "the same proportion," can be formally expressed as {(n;

n; n; n; n)}. Thus at the first iteration, Annie established what can be formalized as the relation of identity (the same shares) and an equivalence class based on that relation.

My intervention of suggesting non-fair sharing can be interpreted as a move toward changing the relation. The idea turned out to be non-trivial, and it took some effort for Annie to built a new target for the changed metaphor source. With counting, Annie established a simple additive relation of "friends get one more than others." To incorporate this new concept of non-identity relations among quantities, Annie moved into purely additive work, making an image of non-equal relations there. The metaphor of sharing helped to scaffold that move, keeping Annie within the topic and suggesting a way of changing the relation. In my intervention, I used *the language of metaphor* to talk with Annie within the boundaries of her microworld. As a result, now Annie changed the target to the equivalency class based on the relation of "one more." Her new equivalency class can be formally expressed as {(n+1; n+1; n; n+1)}. She has also made an image of the idea of an equivalency class, which she called "pattern of sharing." Acting in the level of image having, Annie was quickly creating different equivalency classes, which in the process of interview corresponded to different levels of her software. Her examples included {(n; n+1; n; n+1)} and {(n; n+2; n+3; n+3)}. Annie introduced each "pattern of sharing" through a description of relations, such as "this friend gets three more than that person," and an example of such a pattern with particular numbers.

By this time, Annie has formed what can be called a "situated abstraction" (Hoyles et al., 2001) of a proportion, with the help of the sharing metaphor. Namely, she formed ideas of a pattern, of keeping the same pattern, and of changing the pattern. Using my terms for defining proportionality, Annie was formalising the ideas of *equivalency class* and of additive *relation* structure defining it. Formalization is evidenced by how she used the idea of unfair sharing to quickly create, using the established numeric relationships rather than drawings that corresponded to image making, more examples, or levels of the game, dealing with such equivalency class structures. While Annie was in image making, numbers were coming from drawings; at the present, more formalized state, Annie's drawings were following her work with numbers. However, she was not using multiplicative relations at the time. My next intervention was to suggest a multiplicative relation where one character got several times more than another character. While we were still using the language of "sharing patterns," structurally it was rather a relic, since now Annie worked numerical structures directly, even forgetting to specify what it is her game characters were sharing, the piece that was quite important to her during image making. My pedagogical intervention was rather formal, and Annie was ready for it since she was property noticing and formalising in her system (Pirie & Kieren, 1994a). She moved to equivalency classes with multiplicative relations. The question of other type of operations also came up at this time, which corresponds to the level of observing, where Annie was observing and comparing operation types, or "sharing patterns" of different kinds.

The Case of Zack

Initially, Zack did not know how to start his software design process. He asked if an example from my microworld, when four yellow dots and six red dots turned into six yellow and nine red dots, was a proportion. He drew a checkerboard picture (Figure 40) with different-sized squares that looked promising to me, but could not explain what to do with it. Then he said that he used to draw monster trucks a lot as a child, and went on to spend close to fifteen minutes drawing one in detail (Figure 24). Zack was mostly quiet as he drew, with occasional comments, every three or so minutes, about drawing's details or the way he depicted perspective in his drawing of "the behind wheel." One comment was "You see, the proportion comes in the size between the other things and the truck." I also asked two questions during this time, which both had to do with details of drawings. Then Zack decided to try and figure out what proportions are. He said:

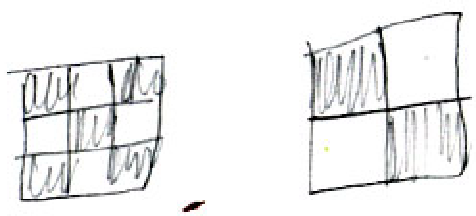

Figure 40. Zack's checkerboard idea.

> Zack (Z): I am just drawing a truck. You see, I am not good at math, or with proportions. I just like to draw.
>
> Maria (M): We can see proportions in the drawings of monster trucks, because they are everywhere.
>
> Z: What is a proportion?
>
> M: That's a good question. Let us see what your drawings had to do with proportions.
>
> Z: I would tell you what it did if I knew what a proportion is, really. Besides the algebra proportions
>
> M: Can you just say what that is, the algebra proportion?
>
> Z: It's like ten over five multiplied by ten over X (Figure 41), that's a proportion.

$$\frac{10}{5} \cdot \frac{10}{x}$$

Figure 41. "A proportion is like..."

Then he asked me, pointedly, to explain what proportions are. Other learners also asked, but they never pressed the matter in a way Zack did. I decided to pick up on his comment about proportions "coming in the size" and drew several pictures that I thought depicted his idea (Figure 42). My pictures were based on the grounding metaphor of classification, which is a way children approach proportionality (Lehrer et al., 2002). We discussed the pictures for about five minutes, and then I asked if Zack wanted to do a game about relative sizes of wheels and bodies of trucks, a subject that came up. He responded with a very confident "No" and proceeded to the idea that later became the theme of his software: a truck driving through an obstacle course.

Figure 42. My drawings of proportionality in response to Zack's request for a definition.

The first mathematising of the theme came from Zack's utterance "The truck has to be bigger" (than the obstacle). Thus at the beginning came a qualitative *relation*, represented by the word "bigger," drawings, and also gestures. Based on the established relation, Zack has created three different sizes of obstacles-cars, and three different trucks (Figure 43). Note how the gradual size increase in the drawing, signifying attention to detail at this stage, changes at the large car obstacle: it was drawn on a separate, attached piece of paper. Zack commented on the discrepancy, but decided not to spend time re-drawing, which is a beginning of movement from more tangible to more symbolic representations. The user of his software was supposed to "drive" a truck over an obstacle by dragging it with the computer mouse. It turned out the truck was supposed to go slower over obstacles than over smooth stretches of racetrack. Again, Zack was using qualitative relations to describe the situation. I thought it was a good opportunity to intervene. Trying to shepherd (Towers, 1998) toward quantitative relations, I asked how to program the speed of that movement. Zack methodically labeled key points in his track with letters from A at the start line to G at the finish line, commenting that it "will be easier." He said that he would count seconds for each stretch using the word "Mississippi."

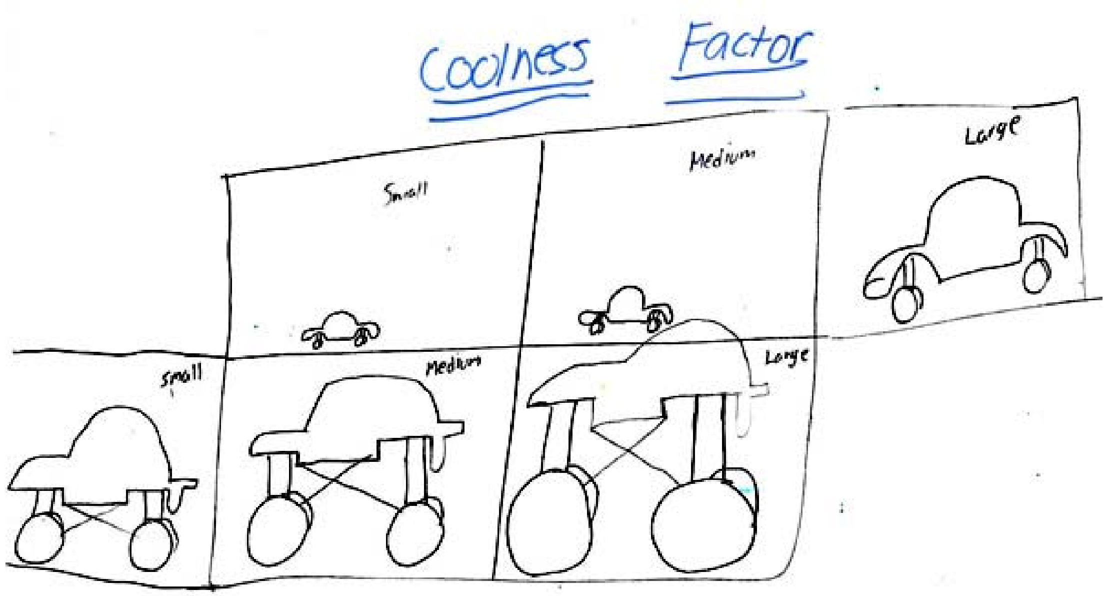

Figure 43. Monster truck sizes and car obstacle sizes.

At this point, I attempted to intervene, hoping to speed the process up. I gave a two-minute lecture on using frames in animation, saying that a frame can have different time value, and that you can just pick "some" number of frames and change the value later. This episode has significance for making sense of Zack's learning. My lecture *used* proportional reasoning, in a context not familiar to Zack, and also in a formalized manner. If Zack made *any* sense out of it I would be surprised. He did not react to my lecture in any way, listening patiently and then proceeding with his way of counting the seconds, as I explained earlier in the discussion of enhancing. Here was Zack in image making, barely starting to introduce quantification into his qualitative-relation structure; and there was my formal lecture using advanced proportional reasoning in a different context. The two did not connect.

Zack proceeded to count the seconds, making his timetable realistic. He saw that there were a lot of combinations of trucks and cars, and exclaimed: "Do we have to do all this, trucks and cars and all?' which I classify as a formalising event even though the property he noticed and communicated was qualitative. His worry was addressed by establishing equivalency classes based on the relation "the obstacle is one size smaller than the truck." As I explained before, the large truck with the medium-car obstacle belonged to the same equivalency class as the medium truck and the small-car obstacle, and so on. Now he was using qualitative relations and a qualitative notion of equivalency, or being the same, within his system.

Zack started figuring out his timetables for the class of "large truck, small car." BC and DE, representing same-size obstacles, took the same time. Zack estimated EG to be three times longer than AB, both being smooth stretches of the track, confirming his estimation with measuring by hand. Then Zack proceeded to make the time for EG three times longer.

Thus numeric relation, a multiplicative one, appeared in his system. When Zack moved on to large truck with small-size obstacle, he said that it should go “a couple Mississippies faster,” introducing an additive relationship. This decision moved Zack to property noticing and formalising actions, where he worked with timetables without necessarily referring to trucks, and discussed his numerical ideas. He made the timetable, subtracting two seconds from each value of his previous table.

I intervened, saying that I noticed something strange: now AB takes two seconds and FG ten seconds, even though FG is supposed to be three times longer. Zack immediately understood the problem, even though it was expressed formally, and started to joke about it and talk about it. This is a striking contrast with my previous attempt to intervene about time values. By now, Zack had an image and moved to formalising within the system. Thus he was ready for this kind of pedagogical intervention, as Pirie and Kieren describe in their article on formalising (Pirie & Kieren, 1994a). The problem had to do with non-commutativity between additive and multiplicative relations in Zack’s system. In Vergnaud’s (1994) terms, this lack of commutativity of “within” and “between” operations meant non-proportionality. In the terms of the ERI perspective, there was no invariance of relation, but there was a motivation for it in the context. Annie’s sharing metaphor that did not call for “between” operations, thus invariance, as a property, did not come into consideration. Zack’s setup, however, occasioned both “within” and “between” relations, as well as scaffolded the idea of invariance of “within” relations under “between” operations.

Thus Zack decided to move from the relation of “two seconds faster” to the relation of “twice faster” (Figure 44). It is interesting to note that halving or doubling played a special role in Zack’s world, as the first multiplicative “between” relation, as well as in Annie’s world as the first multiplicative relation. The two-split plays a special role in Confrey's conjecture (Confrey, 1994), and I also observed its special role in a study of multiplicative reasoning development (M. Droujkova, 2003b). Zack proceeded to make another timetable, with numbers twice as small as in the original one. Then he went on to make timetables for other combinations of truck and obstacle sizes, making up a system of between-relations based on the powers of two.

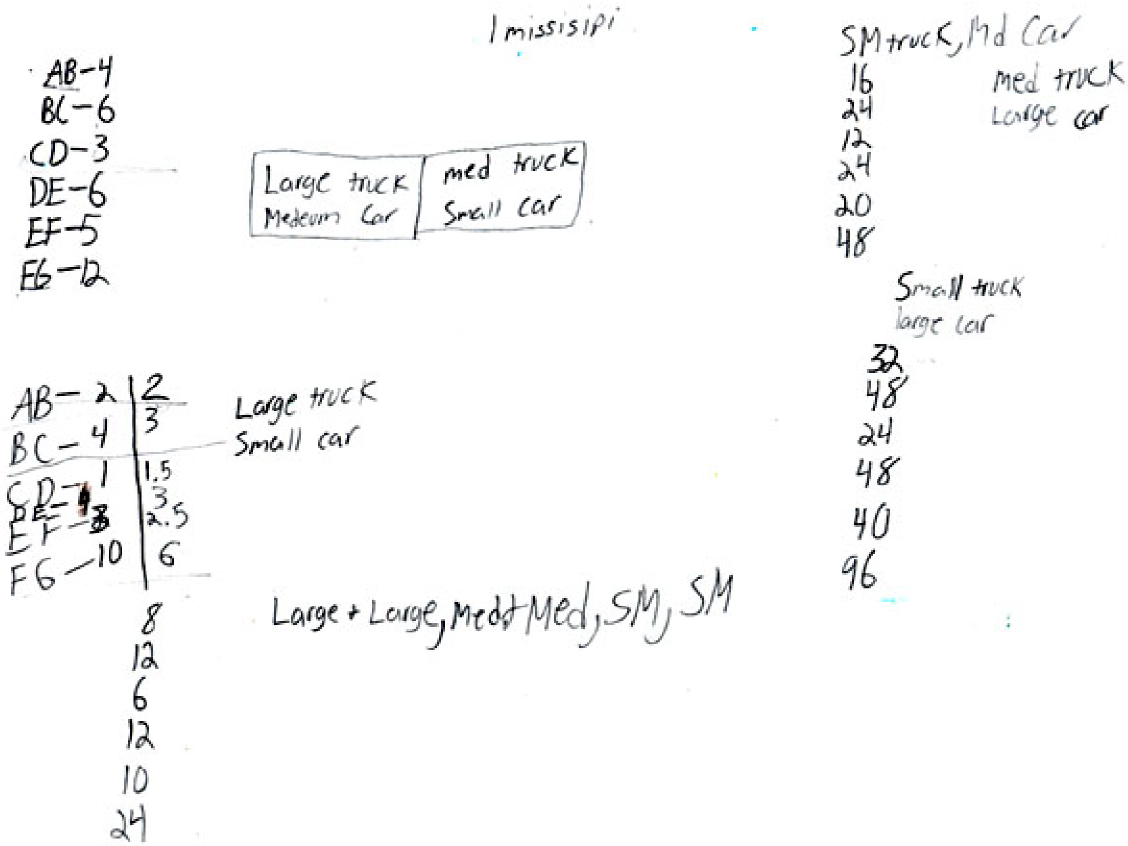

Figure 44. Zack's timetables.

Zack was formalising at the time, and his metaphor died, turning into an analogy, and a formal system was born. The death of the metaphor is further evidenced by the fact that he applied the multiplicative relation to *all* pieces of racetrack, even to non-obstacle pieces where same-sized trucks should, realistically, take the same time (Figure 44). For example, the table for the combination "small truck, medium car" indicates that AB takes 16 seconds, while the table for "small truck, large car" indicates that AB takes 32 seconds. Also, 32 seconds is a *very* long time, in terms of a computer game. If Zack continued to count seconds using his "Mississippi" method, as he did in image making, he would realize it instantly. Yet Zack was now approaching his timetables systems formally. He wasn't counting seconds, and he wasn't concerned, at the moment, that the relation structure he established went contrary to the physics of trucks.

Having organized his timetables, Zack talked about issues of designing software about them. I asked him how the user would be learning about proportions, and he started to think about the question. The task for the user, he explained, would be to select different combinations of trucks and obstacles, try them out, and then decide which one is the fastest. Zack quickly noticed that the largest truck with the smallest car would be obviously fastest. He did not know what to do. At first, he started to set up another level, with a whole new racetrack. If the task of doing timetables for the first time was daunting to Zack, going through it yet again seemed very boring. Instead, I suggested that Zack could set up a different problem about the timetables we produced. Up to now, timetables served Zack, as the designer of the program, to make sense of it. Now the idea was to make them available to the user, and to "figure out the blanks on the proportion table," in other words, to solve a missing-value problem (Figure 45). Zack was now reluctant to move to slow, "messy" image making, staying with numbers in his timetables and using his constructed system of relations to manipulate them. He said that he enjoyed the way he understood what proportions are in terms of numbers. He did not have to fold back to image making to support his formalized system, and thus did not want to return to actions that corresponded to image making.

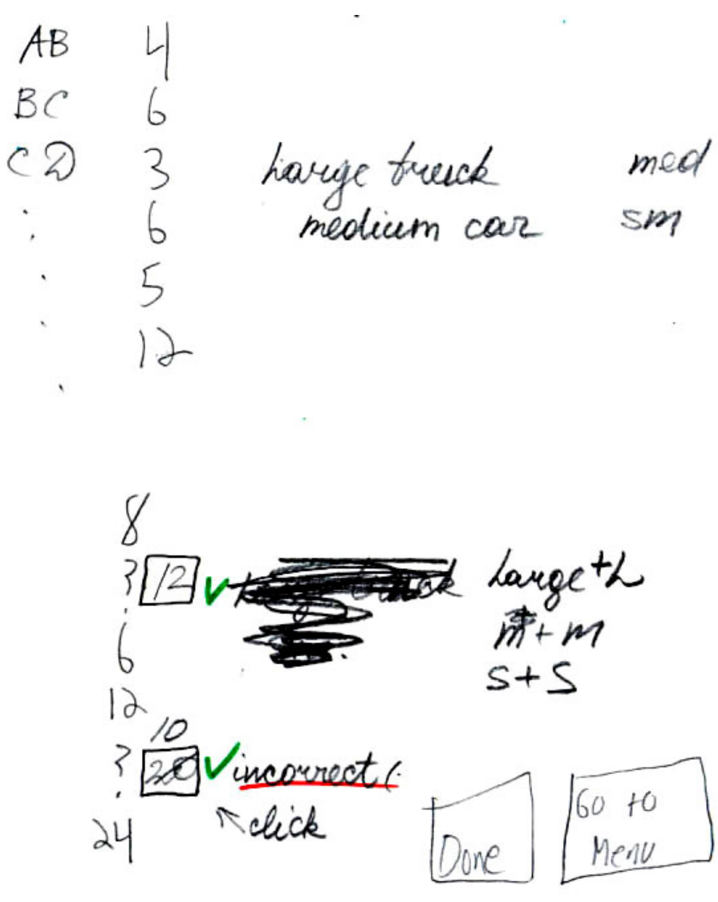

Figure 45. Zack's missing values in timetables.

<u>Source Fading: Property Noticing and Beyond</u>

When learners go into property noticing, the action of noticing brings properties to the foreground and images to the background. Property noticing pulls out some parts of the metaphor that have certain structuring qualities, and separates them from other parts that are being structured. The structuring parts *become*, for the learner, the metaphor's target, just as metaphor turns into simile (Pirie & Kieren, 1994a). During image making, learners do not separate sources and targets. It took either *retrospective analysis* or *co-creating interventions* for me, as an observer, to separate sources and targets that were created during image making. However, in property noticing sources begin to *fade*, just as targets are pulled to the foreground, and thus the two become more available to the analysis as separate entities.

When Zack was property noticing and formalising, as described above, the parts of the metaphor that could be considered sources faded, to the degree that Zack stopped caring about the situation's realism, to which he devoted much time and effort during image making. Now consistency of the newly created formal structure became more important for him than racetracks considerations. However, Zack switched to a *new* metaphor and used the metaphoric idea of a "new level" to signify a new problem he wanted to create. Then Zack realized that it would be a problem of the same kind, and he did not want to go through the same set-up process again. Thus he used the idea of a different screen, accessible via a button, which he called "figuring table," within the same level to think about a missing-value problem. Zack was using software features as *a different kind of metaphor* in his formal levels of knowing.

When Annie was making an image of sharing, the details such as reasoning behind non-fair sharing or the nature of shared objects were quite important to her. The source of her metaphor in the sharing situation, and the target in equivalency classes were not separated. However, when Annie moved to property noticing, she left these details behind, as her metaphor's sharing source faded into the background. Annie was not talking about the nature of objects to be shared, not even mentioning "sharing" at all. For her, "sharing patterns" became "number patterns" as the source and the target of her metaphors became separated.

Again, she used the metaphoric language of game levels in her formal actions on equivalency classes based on different relations.

Abby's case is telling in this regard. She started designing her microworlds with a more developed understanding of proportionality. Within the first few minutes, she created a task that incorporated an equivalency class based on multiplicative relations in her three-ring circus problem. The process of her microworld development had a feel of contexts being built around mathematics, rather than mathematics growing out of contexts. Toward the end of the first interview, having created several different types of tasks related to proportionality as different levels of the game, Abby raised the question of how many different levels can and should there be. In the brief discussion that followed, Abby acted in the level of observing, thinking of organizing her work on particular types of proportions into categories. It is interesting to note how Abby was metaphorising at the time. She was not using the language of the images that she was offering, as sources for metaphors, to users of her software. For her, proportions were only *similar* to each situation in her game. In fact, all her levels had proportionality as a somewhat extrinsic quality of the situation, where the setting resembled a decoration of a problem already set up. Such approach was also evident in the way she was describing the design process. However, *types of proportionality problems* were still an issue for her, and she, as well, talked about them in terms of a metaphor of computer game levels, not making a distinction for herself. It is my analysis that presented "game levels" as metaphor source and "proportionality problem types" as metaphor target. I introduced the idea of types of problems, that I considered to be metaphor target, into the discussion, and Abby expressed interest in it.

Sofia went into property noticing and formalising levels when she and I worked on a table of all possible pairs of ant column lengths for her game (Figure 46). The goal of the game, at this time, was to create two ant columns, send them marching to two stones that multiplied the number in the first column by two and the number in the second column by three. The total number of ants was supposed to be twenty. In working on the table of combinations, Sofia noticed and communicated the property that the column that was supposed to be multiplied by three could not have odd numbers in it. Working with the results other than twenty, she applied this method of analyzing for odd and even numbers, and tried to formulate more general conjectures about table properties. She also raised questions such as "Which total numbers produce more possible combinations?" or "Which numbers are harder?" That is, Sofia was acting in observing level, not even mentioning ants anymore, as the source of the metaphor faded.

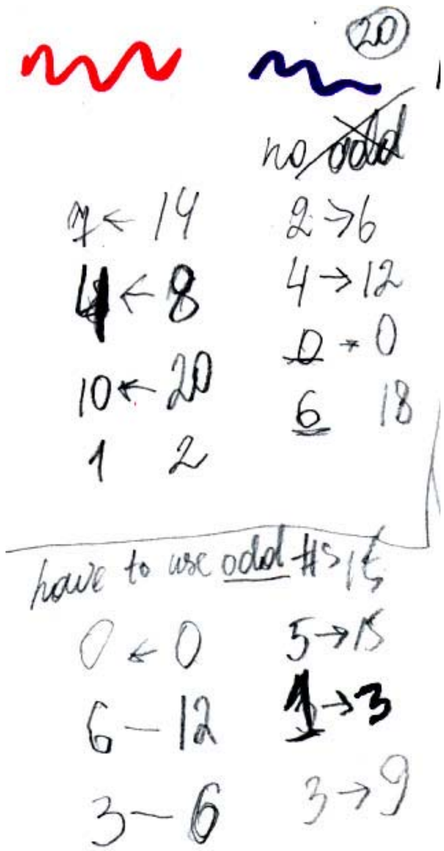

Figure 46. Possible lengths of ant columns.

While Sofia was working on setting up problems in her ant world, and I was helping her, another observing issue came up. Sofia absolutely did not want to give users of her software any tasks that had one possible answer. In my literature review of proportional reasoning, I found that most laboratory-type tasks, and also tasks coming up in microworlds, such as missing value problems, did have one possible answer. In those frameworks where proportions were defined in terms of relationships of four numbers, only such tasks are possible. But even in frameworks based on equivalency class or invariance definitions, methodologies mostly dealt with tasks that had a particular set answer. Only design-type tasks were different, for example, this study's task of designing software about proportionality. However, while the task of *designing* software was open enough for Sofia, tasks that she could invent, or those I offered in my interventions, which could be incorporated *into* her software, were not open enough at all. Sofia used the metaphor of *freedom* to talk about task openness. During these dialogues, the initial images of ants and multiplying stones did not come up, but another metaphor was used, in a process similar to Abby with her consideration of "game levels" as "problem types."

Multi-Part Structures

In much of proportional reasoning and analogical reasoning literature, basic definitions deal with *pairs* of numbers or objects, and relationships between *two* pairs. A thread that emerged from the development of several learners' microworlds was the use of groups of more than two related numbers and objects. In data analysis, I called this thread "the rule of many," where "many" denotes three or more parts.

For example, Niky set up the task of resizing rooms of complex shapes, where the room had several dimensions that had to be changed proportionally. An example of a representative, proportion-defining group of her equivalency class, that is, the description of the original room dimensions, is (25; 8; 25; 23). Annie used sharing patterns with several members, with an example of (4; 12; 24; 24) that she also described in terms of relations as "three times more, two times more, the same" or {(a; b; c; d) such that b=3a, c=2b, d=c}. Zack had systems of timetables with six stretches of racetracks in each and with different tables related multiplicatively, and also with several combinations of truck and obstacle sizes corresponding to the same table. Zack used within and between strategies to work with the whole system. Abby used two relations between quantities to describe her three-ring circus system that can be formalized as {(a; b; c) such that b=3a and c=2b} (Figure 47).

Figure 47. Multi-part object structures.

Thus learners were creating systems that had more than two groups of objects, with more than two objects in a group, and thus with more than one relation. Such complexity seemed to be supported by the metaphoric systems learners were using, and in turn was supporting, rather than confusing, the development of proportional reasoning. It seems that learners, via metaphors, were pulling in more *examples* related to the ideas they were trying to develop, while the metaphor helped to coordinate the examples. Thus the existence of several different relation examples helped to establish and to develop the very notion of "relation." This observation has direct pedagogical implications discussed in the next chapter.

Summary

In this chapter, I used the Pirie-Kieren model, analysis of metaphor sources and targets, and the equivalence class, relation and invariance (ERI) perspective for proportionality to map the results. The extended task of designing software allowed learners to develop metaphors that supported growth of understanding in proportionality. Initial image making corresponded to the beginning of metaphors. In property noticing actions, target structures were separating from sources and becoming new concepts, and metaphors were "dying," or turning into similes. In formalizing, these new concepts were manifested as stand-alone entities, now separated from sources of metaphors that gave birth to them. Metaphors also coordinated development and use of ERI structures involving qualitative, additive and multiplicative ideas, sometimes via bricolage. Within these structures additive and qualitative relations, invariance, and equivalence classes played roles not as misconceptions, but as steps towards proportionality. Chapter 5 discusses these findings and their implications in more detail.

CHAPTER 5. CONCLUSIONS AND RECOMMENDATIONS

Metaphor and Learning Cycles

In constructivist and enactivist frameworks, the growth of understanding is a non-linear, recursive process where learners' actions are co-defined with the environment (B. Davis, 1996; Kieren, Calvert et al., 1995; Pirie & Kieren, 1994b). During local, context-specific work in their microworld, learners in my study were developing and re-defining both sources and targets of their metaphors. In that recursive process of learning, conceptual structures of participants were developed and redefined. As learners worked on modifying their microworld, they modified their proportionality models in the recursive manner. Metaphors served as a scaffold supporting the organization of this recursion.

In their study of learner models of proportionality, Lesh and Harel (2003) found a similarity between stages of development described by cognitive psychologists of Piaget school (Inhelder & Piaget, 1958), and "*modeling cycles* that learners typically go through during sixty-to-ninety minute solutions to a class of problems that we refer to *as model-eliciting activities*" (Lesh & Harel, 2003, p. 157). In their framework, these researchers and their colleagues (Lester & Kehle, 2003; Zavojewski & Lesh, 2003) use a fractal approach. That is, they see a similarity between mechanisms of general conceptual development, and mechanisms of problem solving in individual sessions.

Modeling cycles researchers underlie the importance of task types. In their studies, children are working on what the authors describe as "case studies for kids" (Lesh & Harel, 2003, p. 157), an activity that is contrasted with laboratory-type, well-formulated tasks that require a brief answer. The authors structurally compare their tasks to graduate school level research. The similarity between their tasks, and those I offered to learners, is that both elicited cycles of conceptual growth that could be mapped in a system fractally similar to stages of development described by cognitive scientists. I saw learners develop proportionality systems, sometimes going from qualitative equivalences without an operation to multiplicative proportionality, as Annie and Zack did, or observing different types of proportional problems, or sorting between additive and multiplicative reasoning. I also interpret these processes as local expressions of global developmental mechanisms. The difference between local growth of understanding and the global picture of development is, in particular, in non-linearity of paths of growth of local understanding.

There are significant differences between my methodology, and Lesh and Harel's methodology (2003). The first difference is in direct vs. mediated social goals, and the second difference is in providing the purpose and the context vs. focusing on how learners develop their own. While in both methodologies the growth of understanding is seen as a social activity, in Lesh and Harel's work learners were working in groups, thus allowing researchers to observe how knowledge became sharable. In my study, the *goal* that learners saw was social, in that they were creating software for other people. However, that goal was mediated by working with the interviewer, rather than working with these other people. The reason for that change was in the second methodological difference, namely, in my desire to focus on contexts and themes, and thus metaphors, which each individual learner was developing.

Integrating the two methodological approaches, either by offering groups of learners to develop their own *common* microworlds, or by offering individuals to develop their microworlds individually and then continue working on them with others, is a promising venue for future research. Such an integrative approach is used in the Lesson Plan Study (Berenson, 2002; Berenson et al., 2001; Cavey, 2002; M. Clark, 2001, 2003; M. Clark, Berenson, & Cavey, 2003), where prospective teachers went through several lesson planning cycles, first individually and then in small groups. The cycles led to teachers' growth of understanding in the area of proportionality that can also be compared to developmental processes.

My findings suggest that if the learners are given time, and a supporting structure, for growth of understanding, they can go far beyond the stage of development that may be measured by laboratory-type task. This growth of understanding is local and situated, in that it originated in a particular context. The transfer of such understandings to different contexts may be problematic. Thus another potentially fruitful research idea related to cycles of situated learning is to investigate generalizations to multiple contexts and roles of metaphor in it.

Meanwhile, practitioners can find the analysis of roles of metaphor in learning cycles useful for setting up metaphor-soliciting tasks and projects such as software design, writing stories about mathematics, investigating mathematized situations, or building mathematical museum exhibits. When people create such design-oriented settings, they may plan for the underlying metaphors to organize the recursive learning process, in a way similar to sharing metaphor organizing Annie's growth of understanding, for example. The comparison of learning in closed-ended tasks allowing for one solution attempt, and open-ended tasks supporting multiple cycles of learning can help to inform pedagogical decisions.

Metaphor and New Knowing

Image making, the basis of all understanding, is where internal representations of concepts are formed or reexamined. It should be noted that all "new" knowing grows out of "old" knowing, thus image making envelops primitive knowing that occurred before, according to the fractal Pirie-Kieren model. During image making, metaphor sources get constructed, collected from primitive knowing, and reorganized, and metaphor targets co-develop with the sources.

During image making and image having, targets and contextual sources of metaphors are inseparable. A researcher analyzing data can *retrospectively* analyze particular entities within metaphor as corresponding, from the point of view of their future development, to particular sources and targets, using the knowledge of the *future* separation that occurs between them in property noticing level and beyond. Alternatively, a participant observer can *influence* the ways targets and sources are co-created, based on pedagogical considerations. However, during image making and image having, the source *is* the target in a very direct sense. For Zack, there was no separating of source and target when he was making an image of the system of obstacle and truck sizes. For Annie, proportions were *like* fractions when she attempted to approach them formally, yet for the times she was making an image, sources of sharing and targets of equivalence classes only appeared in my analysis retrospectively. In Annie's world at these times, there was the whole metaphor, the entity of sharing.

Thus, in terms of metaphor image making can be interpreted as the action of creating a dynamic system that is metaphor. Metaphor is a way to invoke concepts and to organize them into a system. Image having is the stage where such metaphoric systems of concepts, somewhat established, can be internally manipulated and clearly communicated to others or oneself. During these stages, the notions of metaphor source and metaphor targets are *observer phenomena*, appearing in researchers' analysis or being co-defined in pedagogical interventions, and corresponding to his or her knowledge of mathematical systems. Property noticing corresponds to the appearance, *within* the metaphoric system, of structural change that separates the source and the target. That is, in property noticing the source and the target can be analyzed as phenomena of learning (Figure 48).

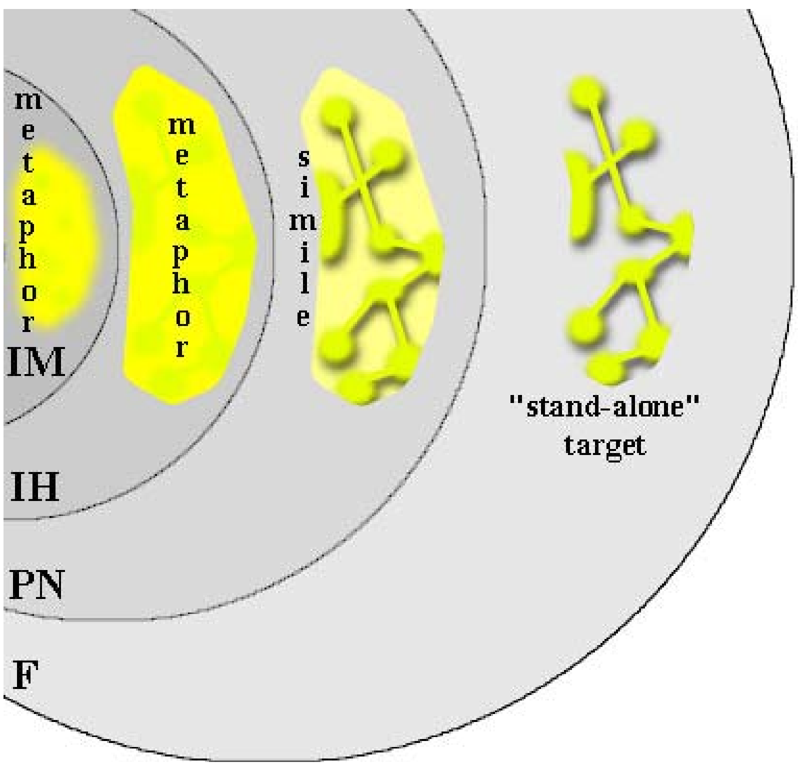

Figure 48. Source and target separation.

A theme coming from the data underlies the constitutive nature of metaphor. Formal target structures created by metaphorising could not be considered "transferred" from metaphor sources. Parts of these structures, for example, the idea of a qualitative relation, could be collected from primitive knowing. The nature of targets, when they become separated as formal systems, is different from the nature of sources. The act of *noticing* means pulling properties out into a separate domain that becomes the target of the metaphor. This is the point where metaphor, strictly speaking, turns into a simile (Pirie & Kieren, 1994a), and the learner's reasoning with the image systems becomes *analogical* rather than metaphoric. In formalising, learners can focus on the mathematical target of the former metaphor without connecting it to the source. This is, in Sfard's (1997) work, when the original metaphor "dies" and the learner begins to think of a new concept "as a self-sustained independent entity belonging to the abstract domain." For the level of property noticing, I speak about *fading* of metaphor sources, rather than something as final as death, since learners use the same sources, now as parts of similes, when they fold back from property noticing.

The nature of the metaphor analysis is paradoxical, since the analysis is only possible when the metaphor "dies" and the source and the target appear and thus become available as the subject of analysis. However, from enactive perspective, even initially, when the metaphor is not dead, there are phenomena of source and target, co-created by the environment and the metaphorising learner. In particular, the role of the mentor, in the present study the interviewer, is to interpret the initial, non-differentiated metaphor in terms of *possible* sources and targets. For example, when Annie was making an image of fair sharing, I interpreted her system as corresponding to the target concept of an equivalence class based on the relation of "the same." However, by the time Annie started talking about different number patterns for sharing, there was a concept system for her, separate from the initial source, and co-developed through interactions with my interpretations of the target and the source. The power of the widely cited book "Metaphors We Live By" (Lakoff & Johnson, 1980) comes from a similar mechanism: in describing common metaphors such as "argument is war" and a way to analyze them, the author assisted the readers in their transition "beyond metaphor" (Pirie & Kieren, 1994a), that is, beyond informal levels of understanding.

Image making and folding back to image making from levels beyond are different, from the point of view of metaphor. Having separated the source and the target, learners coordinate their image making activities within this two-part system through analogical reasoning. Thus their work on entities that formerly were categorized as metaphor sources is occasioned by the demands of the new concept systems that grew out of metaphor targets. Now the learner takes on more of the task of mathematising, whereas before, in the first journey through informal levels, the context and the environment, including the mentor, were heavily involved in this task. This process would be interpreted as internalization in sociocultural perspectives.

In the future, it may be useful to investigate the role of metaphor history, so to speak, in actions of knowing beyond the formalising level. That is, if metaphor is a primary mechanism of informal learning, and if it "dies" during the transition to formal levels of knowing, how does knowing beyond metaphor depend on its metaphoric roots? In this study, folding back meant invoking the same metaphors, or the corresponding similes, when the learner was designing the same software level. In the two cases where there was a switch between different-themed levels, learners considered metaphors born of all levels designed before. The same two learners also used, at the beginning, my microworld as a source of metaphors. Several learners, when discussing properties of tasks during observing, used the language of software development, such as levels of the computer game, in new metaphors that corresponded to formal levels of knowing. Thus they moved from metaphor coming from within the context of their games, to more decontextualized metaphors of software organization. This process should be studied in more detail. What tasks would help learners to organize the knowledge born of local metaphors into more general structures? What kind of metaphors and metaphoric structures, possibly linking metaphors (Lakoff & Nunez, 1997, 2000) or metaphoric chains of reification (Sfard, 1997, 2000), assist learners in these processes? These are questions arising for future research.

Pedagogical implications of the analysis of roles of metaphor in creation of new knowledge include the stress that should be placed on image making activities. That is where metaphor starts, to become a simile during property noticing, and to give birth to a new concept in formalising. Image making is also the level to which learners frequently fold back from property noticing, refining the target of their metaphor through working on the source. In her dissertation on teacher interventions, Towers (1998) mentions that teachers sometimes attempt to block image making activities, especially folding back to image making. Blocking such activities may be detrimental to the student's learning. My findings underlie, yet again, the importance of image making activities, in particular folding back to image making, in formation of new concepts and conceptual structures. Image making is of especially great consequence in the area of proportionality that requires complex multi-concept structures involving coordination of additive, multiplicative and analogical reasoning.

Another pedagogical possibility comes from the data on using "the language of metaphor" to communicate with learners. For example, when I was suggesting a multiplicative relation to Annie, I used the terms “a new kind of sharing pattern” which came from her microworld. Such use of the language of metaphor as a tool (L. Vygotskii & A. Kozulin, 1986), or working within what R. Davis (1984) calls an *assimilation paradigm*, is a powerful pedagogical intervention.

Metaphor and Representations

Two related themes emerged in the analysis of learners’ representations. The first theme was enhancing, that is, decorating symbols or descriptions beyond functional purposes, or pulling in themes without clear mathematical or contextual purpose. Such activities supported learning indirectly, by creating a more tangible world for image making.

The second theme is correspondence between levels of knowing, as mapped by the Pirie-Kieren model, and the level of abstraction in representations. Generally speaking, initial image making, as well as folding back to image making, corresponded to use of more tangible representations, such as detailed drawings, descriptions, or physical counters. Property noticing and formalising corresponded to more abstract and compact symbols, such as abbreviations, iconic drawings, or numerals.

There are new research questions arising from the theme of enhancing, especially from the case in my study when very little mathematising and a lot of enhancing happened during the whole data collection process. What kinds of enhancing support, and what kinds of enhancing distract from the growth of mathematical understanding? What are the criteria that would let practitioners tell the difference and thus support image making appropriately?

Related questions come from the analysis of representations in general. In my study, learners could choose their representations, even though some of them were influenced by common computer entities such as buttons, and all of them were influenced by mathematical culture at large. Thus I was able to observe links between levels of knowing and types of representations learners used. These links can be further investigated, possibly using more general semiotic frameworks, as well as microworld and other mathematics education studies that deal with representations. Also, there is a research question that can be answered in further teaching experiments with methodologies where learners can both use their own and be offered other people's representations: "What is the interplay between the kinds of representations offered by other people, the kinds of representations created by the learner, and metaphor as a mechanism of learning?"

Metaphor and the ERI Perspective for Proportionality

In the process of data analysis, I used and refined a perspective for analyzing learners' growth of understanding related to proportionality. This perspective is built on the notions of equivalence class, relation, and invariance (Figure 49) that initially came up as common threads in my review of literature on proportional and analogical reasoning (Abrahamson, 2003; Behr & Harel, 1990; Behr et al., 1993; Berenson, 2002; Campbell, 2001; Cavey, 2002; M. Clark et al., 2003; English, 1997a; Hoyles & Noss, 1989; Hoyles et al., 2001; Inhelder & Piaget, 1958; Kaput & West, 1994; Lakoff & Nunez, 2000; Lamon, 1993; Lesh, Post, & Behr, 1988; Noelting, 1980; Noss et al., 2002; Ortony, 1993; Piaget & Campbell, 2001; Pimm, 1987; Sfard, 1997; Vergnaud, 1994; Vosniadou & Ortony, 1989). These notions took their current form through multiple cycles of revisions based on data.

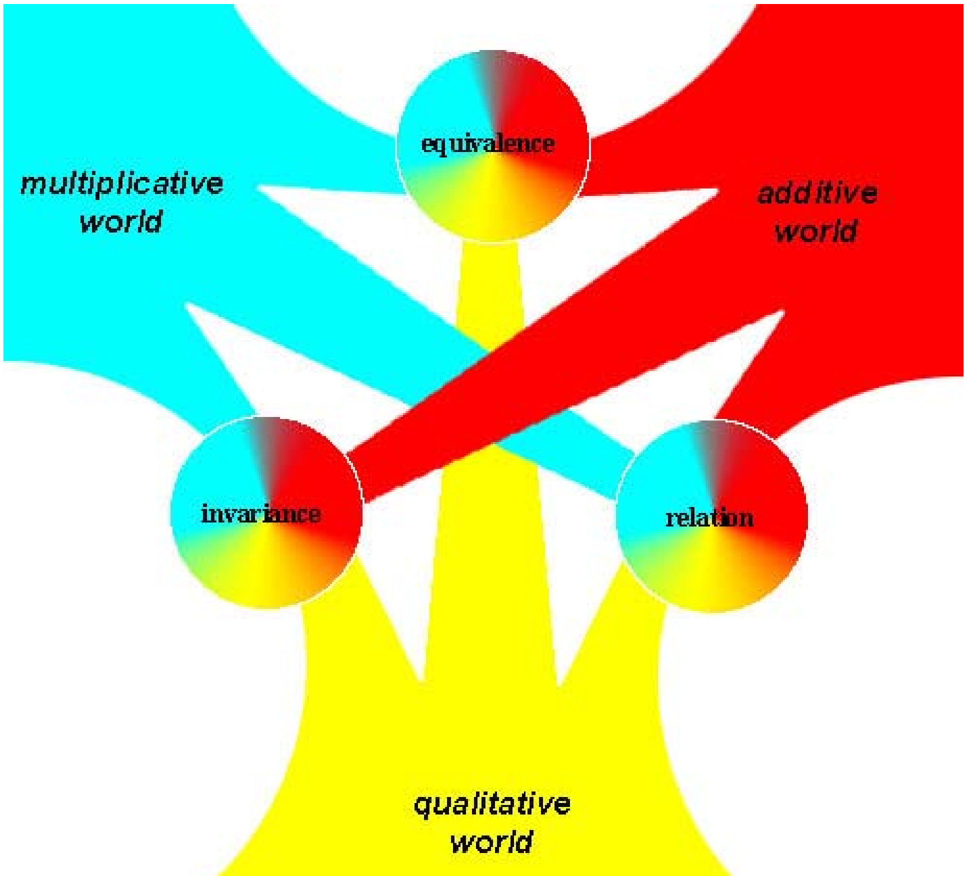

Figure 49. ERI perspective.

I used the ERI perspective for mapping my inferences of targets of learners' metaphors, in conjunction with the Pirie-Kieren model that I used for mapping my inferences of learners' actions. This combination of models allowed me to re-interpret learners' actions in additive, multiplicative and analogical worlds, mapped by the Pirie-Kieren model, from the unifying point of view of their relation to proportionality. In turn, such a unifying approach corresponded to the holistic way metaphors, embodied in microworlds, organized the growth of understanding of the study participants. The lens of the ERI perspective allowed interpreting qualitative, additive, and multiplicative worlds as a part of a structure supporting the growth of proportional reasoning in each context. Data indicates that learners collected (Pirie & Martin, 2000) threads from the three worlds, weaving them into a growing, thickening common fabric of understanding. Metaphors helped to pull new threads into the fabric, to organize the weaving process, and to hold the fabric together.

For example, I describe the multi-part, sometimes additive structures in microworlds of several learners as structures made out of several *relations* and forming an *equivalency class*. In another example, movements between additive and multiplicative worlds, which in large-scale developmental models take years and signify different stages, happened sometimes in a matter of minutes within the context of the same metaphor. The perspective helped to map and to explain such movements as a substitute of a new *relation* into an already established *equivalency class*. Similarly, I looked at *invariance* developing, at first, on the basis of additive *relations*.

For this study, the ERI perspective became a useful tool for data analysis, as well as for working with proportional reasoning models by different authors. I believe other researchers of proportionality may find it useful, as well. It can also serve practitioners, who can use it to make sense of learners' actions in qualitative, additive and multiplicative worlds. For example, the perspective may help practitioners construe folding back to additive and qualitative worlds from working with multiplicative proportionality not as a misconception, but as a way to thicken understanding. Thus by having a perspective for seeing learners' actions in all three worlds through proportionality lens, practitioners may be better able to help learners frame these actions as paths toward proportionality.

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