SOME RESULTS IN ROTATABLE DESIGNS OF ORDER TWO

by

Philip V. Piserchia
University of North Carolina

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DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
Chapel Hill, N. C.

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This thesis is an attempt both to explore some general properties of rotatable designs of order two and to construct some new designs by a generalization of a method given by M. N. Das and V. L. Narasimham in [9].

Chapter II is concerned with observing properties that transformed rotatable designs of order two possess. By a transformed rotatable design we simply mean that every point in the design is obtained by a transformation of a point in a rotatable design of order two.

More specifically, section 2.2 gives a method of obtaining constancy for the variance of $\hat{y}(\underline{x})$, where $\hat{y}(\underline{x})$ is the estimated response of a second degree polynomial in k factors at a point $\underline{x}$, on k-dimensional ellipses. Section 2.3 deals with the use of rotatable designs when there are linear restrictions among the factors and section 2.1 gives a theorem showing that every rotatable design of order two in k factors has, in a sense, embedded within it $\binom{k}{r}$ rotatable designs of order two in k-r factors.

In Chapter III we consider the rows of association matrices of a partially balanced association scheme as points in a v-dimensional factor space and, by appropriate algebraic manipulation, are able to construct rotatable designs of order two based on these matrices.

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Chapter I

Introduction

1.1 Definition of Rotatable Designs of Order Two

Suppose we are interested in fitting a quadratic response surface in k factors to N experimental points.

Let the w^{th} experimental point be denoted by $\underline{x}'_{(w)} = (x_{1w}, x_{2w}, \ldots, x_{kw})$ and define the matrix of experimental points D_k by:

$$D_k = \begin{bmatrix} x_{11} & x_{21} \cdots x_{k1} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ x_{1N} & x_{2N} \cdots x_{kN} \end{bmatrix} = \begin{bmatrix} \underline{x}'(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}'(N) \end{bmatrix}$$

Also let y_w be the observed response corresponding to the point $\underline{x}'_{(w)}$.

Hence, we take as our model:

$$(1.1.1) \quad \mathcal{E}(y(\underline{x})) = \beta_o + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{1 \le i<j \le k} \sum \beta_{ij} x_i x_j ,$$

with observations

$$(1.1.2) \quad y_w = \beta_o + \sum_{i=1}^{k} \beta_i x_{iw} + \sum_{i=1}^{k} \beta_i \, {}^{2}_{iw} + \sum_{1 \le i<j \le k} \sum \beta_{ij} x_{iw} x_{jw} + \mathcal{E}_w ,$$

where it is assumed that $\mathcal{E}(\mathcal{E}_w) = 0$, var $(\mathcal{E}_w) = \sigma^2$ for $w = 1,2,\ldots,N$ and that the $\mathcal{E}_w$ are independently distributed.

We are interested in estimating the response surface 1.1.1 and the variance of this estimate at all points in our k dimensional factor space.

A matrix of experimental points D_k which has the property that the variance of the estimated response at any point, $\underline{x}$, in our factor space is a function only of the distance, $\underline{x}'\underline{x}$, of that point from the origin is called a rotatable design. More specifically, if our assumed response surface is quadratic then D_k is called a rotatable design of order two. That is, if $\mathcal{E}(y(\underline{x}))$ is a quadratic response surface and if $\underline{x}_1$ and $\underline{x}_2$ are both points in our factor space (not necessarily experimental points) and if $\underline{x}_1' \underline{x}_1 = \underline{x}_2' \underline{x}_2$ implies that var $\hat{y}$ $(\underline{x}_1)$ = var $\hat{y}$ $(\underline{x}_2)$, where $\hat{y}(\underline{x})$ represents the estimated response at the point $\underline{x}$, then D_k is said to be a rotatable design of order two.

In what follows we shall refer to rotatable designs of order two and rotatable arrangements of order two (defined in section 1.4) as rotatable designs and rotatable arrangements respectively.

It is the intention of this thesis to present new methods of using rotatable designs and to construct some new designs by a generalization of a method due to M. N. Das and V. L. Narasimham in [9].

The remaining portion of this chapter is purely expository and is given in order to facilitate the reading of the following chapters.

1.2 Schlaflian Vectors and Matrices [5,8]

Although the notion of Schlaflian vectors and matrices are not original in references 5 and 8, excellent discussions and further references are given therein and are hence reccommended.

1.2.1 Definition of Schlaflian Vectors and Matrices

Let $\underline{x}' = (x_0, x_1, \ldots, x_k)$. Then the Schlaflian vector $\underline{x}^{[2]}$ of $\underline{x}$ is defined such that:

(1.2.1.1) $$\underline{x}^{[2]'}\ \underline{x}^{[2]} = (\underline{x}'\ \underline{x})^2 = \sum_{i=o}^{k} x_i^4 + 2 \sum_{0 \leq i < j \leq k} \sum x_i^2\, x_j^2.$$

For example, if k=1 then $\underline{x}^{[2]'}\ \underline{x}^{[2]} = (\underline{x}'\underline{x})^2 = (x_o^2 + x_1^2)^2$ $= x_o^4 + x_1^4 + 2\, x_o^2\, x_1^2$ and we take $\underline{x}^{[2]'} = (x_0^2, x_1^2, \sqrt{2}\, x_o\, x_1)$.

In general, if $\underline{x}' = (x_o, x_1, \ldots, x_k)$ then $\underline{x}^{[2]}$ is the $\frac{1}{2}(k+1)(k+2)$ dimensional vector consisting of all powers $x_o^2, x_1^2 \ldots, x_k^2$ and all cross-products of the form $\sqrt{2}\ x_i\, x_j$, $i < j = o, 1, \ldots, k$, and is written as:

(1.2.1.2) $$\underline{x}^{[2]'} = (x_o^2, x_1^2, \ldots, x_k^2, \sqrt{2}\ x_o x_1, \ldots, \sqrt{2}\, x_o x_k, \sqrt{2}\, x_1 x_2, \ldots, \sqrt{2}\, x_{k-1} x_k).$$

Suppose H is a matrix defining the transformation $H\underline{x} = \underline{y}$. Then the Schlaflian matrix $H^{[2]}$ of H is defined such that

(1.2.1.3) $$H^{[2]}\ \underline{x}^{[2]} = \underline{y}^{[2]}.$$

For example, if k = 1 and $H = \begin{bmatrix} h_{oo} & h_{o1} \\ h_{10} & h_{11} \end{bmatrix}$

then $H^{[2]}$ is given by:

$$H^{[2]} = \left[\begin{array}{c:c:c} h_{oo}^2 & h_{01}^2 & \sqrt{2}\ h_{00}\, h_{01} \\ h_{10}^2 & h_{11}^2 & \sqrt{2}\ h_{10}\, h_{11} \\ \sqrt{2}\, h_{oo}\, h_{10} & \sqrt{2}\ h_{01} h_{11} & (h_{01}\, h_{10} + h_{00}\, h_{11}) \end{array}\right]$$

1.2.2 Some Theorems in Schlaflian Vectors and Matrices [5]

The following theorems will be convenient for our purposes and are stated without proof.

Theorem 1.2.2.1 If H and K are matrices then
$(HK)^{[2]} = H^{[2]} K^{[2]}$.

Theorem 1.2.2.2 If H is a matrix then
$(H^{[2]})' = (H')^{[2]}$.

Theorem 1.2.2.3 If H is a matrix then
$(H^{-1})^{[2]} = (H^{[2]})^{-1}$.

Theorem 1.2.2.4 If H is a matrix and $HH' = I$ then $H^{[2]} H^{[2]'} = I^{[2]}$ where $I^{[2]}$ is the identity matrix of proper order.

1.3 The Model Expressed in Terms of Schlaflian Vectors

In 1.1.1 we saw our model expressed as:

$$\mathcal{E}(Y(\underline{x})) = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{1 \le i <} \sum_{j \le k} \beta_{ij} x_i x_j .$$

Rewriting 1.1.1 we have:

$$\mathcal{E}(Y(\underline{x})) = \sum_{i=o}^{k} \beta_{ii} x_i^2 + \sum_{o \le i <} \sum_{j \le k} (\beta_{ij}/\sqrt{2})\sqrt{2}\, x_i x_j$$

where $x_o \equiv 1$ and $\beta_{oi} = \beta_o$ if $i = o$

$= \beta_i$ if $i \neq o$

Hence we may write our model as:

(1.3.1)
$$\mathcal{E}(Y(\underline{x})) = \underline{x}^{[2]'} \underline{\beta}$$

where $\underline{\beta}' = (\beta_{oo}, \ldots, \beta_{kk}, \beta_{o1}/\sqrt{2}, \ldots, \beta_{ok}/\sqrt{2}\ \beta_{12}/\sqrt{2}, \ldots, \beta_{k-1,k}/\sqrt{2})$
and $\underline{x}^{[2]}$ is the Schlaflian vector of $\begin{pmatrix} x_o \\ \cdots \\ \underline{x} \end{pmatrix} = \begin{pmatrix} 1 \\ \cdots \\ \underline{x} \end{pmatrix}$.

Similarly we write our observations as:

$$y_1 = \underline{x}_{(1)}^{[2]'} \underline{\beta} + \varepsilon_1$$

$$y_2 = \underline{x}_{(2)}^{[2]'} \underline{\beta} + \varepsilon_2$$

$$\vdots$$

$$y_n = \underline{x}_{(n)}^{[2]'} \beta + \varepsilon_n$$

where $\underline{x}_{(i)}^{[2]} = \left(\frac{1}{\underline{x}_{(i)}}\right)^{[2]}$

or equivalently

(1.3.2) $\quad \mathcal{E}(\underline{y}) = X'\,\underline{\beta}$,

where $\underline{y}' = (y_1, \ldots, y_n)$ and $X' = \begin{bmatrix} \underline{x}_{(1)}^{[2]'} \\ \vdots \\ \underline{x}_{(n)}^{[2]'} \end{bmatrix}$.

Now, assuming the matrix XX' to be non-singular the least squares estimates of the β's are given by:

(1.3.3) $$\hat{\underline{\beta}} = (XX')^{-1} X\underline{y} \quad ,$$

and the estimated response $\hat{y}(\underline{x})$ at any point, $\underline{x}' = (x_1, \ldots, x_k)$ in our factor space is given by:

(1.3.4) $$y(\underline{x}) = \underline{x}^{[2]'} \hat{\underline{\beta}} = \underline{x}^{[2]} \; (XX')^{-1} X\underline{y} \quad ,$$

where $\underline{x}^{[2]} = \left[\frac{1}{\underline{x}}\right]^{[2]}$.

The variance of $\hat{y}(\underline{x})$ is seen to be:

(1.3.5) $$\operatorname{var} \hat{y}(\underline{x}) = \underline{x}^{[2]'} \, (XX')^{-1} \, \underline{x}^{[2]} \, \sigma^2.$$

Hence it is seen that the problem of constructing rotatable designs reduces to the problem of finding points in our k dimensional factor space such that XX' is non-singular and that the quadratic form $\underline{x}^{[2]'} \, (XX')^{-1} \, \underline{x}^{[2]}$ is constant on hyperspheres defined by $\underline{x}'\underline{x}$ = const.

1.4 Necessary and Sufficient Conditions for Second Order Rotatability.

Let us define the "moments" of our experimental points as:

$$(1.4.1)\quad M(1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k}) = \frac{1}{N}\sum_{w=1}^{N} X_{1w}^{\alpha_1}\ X_{2w}^{\alpha_2} \ldots X_{kw}^{\alpha_k}$$

with the convention that if $\alpha_i = 0$ we shall drop the symbol i^{α_i} from $M(1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k})$ (for example if $\alpha_1 = 4$ and $\alpha_2 = \alpha_3 = \ldots = \alpha_k = 0$ then

$$M(1^4) = \frac{1}{N}\sum_{w=1}^{N} X_{1w}^4 \ ).$$

With the "moments" of our experimental points so defined, necessary conditions for second order rotatability are given in [8] as:

$$(1.4.2)\quad \text{(i)}\quad M(1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k}) = 0 \text{ if } \alpha_1 + \ldots + \alpha_k \leq 4$$

and at least one of the α_i is odd.

$$\text{(ii)}\quad M(i^2, j^2) = \lambda_4 > 0 \text{ (say) for } i \neq j \text{ and } i,j = 1,2, \ldots, k$$

$$M(i^4) = 3\lambda_4 \text{ for } i = 1,2, \ldots, k$$

$$M(i^2) = \lambda_2 \text{ (say) for } i = 1,2, \ldots, k.$$

If in addition to (1.4.2) being satisfied we have:

$$(1.4.3)\qquad \gamma = {}^{\lambda_4}/\lambda_2^2 \neq {}^{k}/(k+2)$$

then the conditions (1.4.2) and (1.4.3) are both necessary and sufficient [8] for a set of experimental points to be a rotatable design.

A set of points which satisfies (1.4.2) will be called a <u>rotatable arrangement of order two</u>.

1.5 Some Well-Known Results in Rotatability

The notion of rotatable arrangements is introduced in section 1.4 because of the following simple but important theorem.

Theorem 1.5.1 Given a rotatable arrangement with N points, it can be made into a rotatable design by the addition of n_o center points.

In the above, a center point is merely the point $\underline{X}' = (0,0, \ldots, 0) = \underline{0}'$ and n_o is completely arbitrary.

Another theorem which we will find useful is:

Theorem 1.5.2 If D_k is a rotatable arrangement then $\gamma = \lambda_4/\lambda_2^2 \geq k/k+2$

λ_4 and λ_2 are as defined in (1.4.2).

For completeness the following two results are inculded [5]:

$$(1.5.3) \quad (XX')^{-1} = \frac{1}{N} \left[\begin{array}{c|c|c} U^{-1} & 0 & 0 \\ \hline 0 & \frac{1}{2}\lambda_2 I_k & 0 \\ \hline 0 & 0 & \frac{1}{2}\lambda_4 I_{\frac{k(k-1)}{2}} \end{array} \right]$$

where

$$U^{-1} = a \left[\begin{array}{c|c|c|c|c} 2(k+2)\gamma^2 & -2\gamma/\lambda_2 & -2\gamma/\lambda_2 & \cdots & -2\gamma/\lambda_2 \\ -2\gamma/\lambda_2 & \{(k+1)\gamma-(k-1)\}/\lambda_2^2 & (1-\gamma)/\lambda_2^2 & \cdots & (1-\gamma)/\lambda_2^2 \\ \vdots & \vdots & \vdots & & \vdots \\ -2\gamma/\lambda_2 & (1-\gamma)/\lambda_2^2 & (1-\gamma)/\lambda_2^2 & \cdots & \{(k+1)\gamma-(k-1)\}/\lambda_2^2 \end{array} \right]$$

and

$$a = \frac{\lambda_2^4}{2\lambda_4\{(k+2)\lambda_4 - k\lambda_2^2\}}$$

The matrices I_k and $I_{k(k-1)/2}$ are the identity matrices of order k and $k(k-1)/2$ respectively.

(1.5.4)

$$N/\sigma^2 \ \text{var}\ \hat{y}\ (\underline{x}) = \frac{2(k+2)\gamma^2 + 2(k+2)\gamma(\gamma-1)(\rho^2/\lambda_2) + [(k+1)\gamma-(k-1)](\rho^4/\lambda_2^2)}{2\gamma[(k+2)\gamma-k]}$$

where

$$\rho^2 = \underline{x}'\underline{x} = x_1^2 + x_2^2 + \ldots + x_k^2 .$$

Chapter II

Some Theorems Concerning the Structure of Rotatable Designs

2.1 An Embedding Theorem

The following theorem gives some insight into the structure of rotatable arrangements by showing that every rotatable arrangement in k factors has, in a sense, embedded within it rotatable arrangements in k-r factors.

Theorem 2.1

$$\text{If } D_k = \begin{bmatrix} x_{11} & x_{21} \cdots & x_{k1} \\ x_{12} & x_{22} \cdots & x_{k2} \\ \vdots & & \\ x_{1n} & x_{2n} \cdots & x_{kn} \end{bmatrix}$$

is a rotatable arrangement in k factors then the set of points

$$D_{k-r} = \begin{bmatrix} x_{11} & x_{21} \cdots & x_{k-r,1} \\ x_{12} & x_{22} \cdots & x_{k-r,2} \\ \vdots & & \\ x_{1n} & x_{2n} \cdots & x_{k-r,n} \end{bmatrix}$$

is a rotatable arrangement (in fact a rotatable design) in k-r factors.

Proof: Suppose $M(1^{\alpha_1}, 2^{\alpha_2}, \ldots, (k-r)^{\alpha_{k-r}})$ is such that $\alpha_1 + \alpha_2 + \ldots + \alpha_{k-r} \leq 4$ and at least one α is odd. Then define $\alpha_{k-r+1} = \ldots = \alpha_k = 0$ and we have $M(1^{\alpha_1}, 2^{\alpha_2}, \ldots, (k-r)^{\alpha_{k-r}}) = M(1^{\alpha_1}, \ldots, k^{\alpha_k})$

$$= 0$$

since D_k is a rotatable arrangement and $\alpha_1 + \alpha_2 + \ldots + \alpha_k \leq 4$ with at least on α_i odd.

Now, if D_k is a rotatable arrangement then by 1.4.2 we have:

$$M(i^2) = \lambda_2 > 0 \text{ for } i = 1, 2, \ldots, k$$

$$M(i^2, j^2) = \lambda_4 \text{ for } i \neq j \text{ and } i, j = 1, 2, \ldots, k$$

$$M(i^4) = 3\lambda_4 \text{ for } i = 1, 2, \ldots, k$$

But if this is true then certainly

$$M(i^2) = \lambda_2 > 0 \quad \text{for } i = 1, 2, \ldots, k\text{-}r$$

$$M(i^2, j^2) = \lambda_4 \quad \text{for } i \neq j \text{ and } i, j = 1, 2, \ldots, k\text{-}r$$

$$M(i^4) = 3\,\lambda_4 \quad \text{for } i = 1, 2, \ldots, k\text{-}r$$

Hence, 1.4.2 is satisfied and D_{k-r} is a rotatable arrangement.

The fact that D_{k-r} is a rotatable design is shown by observing that

$$\gamma = {}^{\lambda_4}/_{\lambda_2^2} \geq {}^{k}/_{(k+2)} > {}^{k-r}/_{(k-r+2)}$$

That is, by using theorem 1.5.2 condition 1.4.3 is also satisfied.

It will be noticed that D_{k-r} was defined by excluding the last r columns of the matrix D_k. However, this was done merely for notational convenience and, in fact, any r columns of D_k may be omitted and the theorem will still stand. Hence, we conclude that any rotatable arrangement in k factors has embedded within it $\binom{k}{r}$ rotatable arrangements in k-r factors.

A natural question, it seems, now to ask is: Is every rotatable arrangement D_k consisting of N points embedded in some arrangement $D_{k'}$, $k' > k$, also consisting of N points?

The answer to this question is, "not in general." For consider the following. We know by theorem 1.5.1 that if D_k is any rotatable arrangement then by adding a center point the arrangement is a rotatable design; that is, the matrix XX' is non-singular. But, if D_k with a center point is a rotatable design then $N + 1 \geq 1/2\ (k+2)(k+1)$ or $N \geq \frac{k}{2}\ (k+3)$

for any rotatable arrangement in k factors. Now, for k=2 we have $N \geq 5$ and for $k' > 2$ we must have $N \geq 9$. However, there does exist a rotatable arrangement in two factors with exactly five points (the vertices of a regular pentagon centered at the origin). Hence, this arrangement is not embedded in any arrangement $D_{k'}$, $k' > 2$, and the question is answered.

2.2 Constancy of var $\hat{y}(\underline{x})$ on k-dimensional Ellipses

Part of the appeal of rotatable designs stems from the fact that quite often the statistician does not know the orientation of the assumed response surface. Because of this lack of knowledge having the var $\hat{y}(\underline{x})$ be only a function of the distance of $\underline{x}$ to the origin has a great deal of heuristic appeal.

If, however, the statistician knows that the assumed response surface is constant on contours other than hyperspheres then rotatability no longer has quite the same appeal.

The following theorem gives a method of utilizing rotatable design when, in fact, the statistician knows the response surface is constant on a particular family of concentric and coaxial k dimensional ellipses. That is, the statistician knows that if $\underline{x}_1' A \underline{x}_1 = \underline{x}_2' A \underline{x}_2$ then $\mathcal{E}(y(\underline{x}_1)) = \mathcal{E}(y(\underline{x}_2))$ where A is a known positive definite matrix and $\underline{x}' A \underline{x}$ = const. is the particular family of ellipses.

Theorem 2.2 Suppose A is a known positive definite matrix defining the family of ellipses $\underline{x}' A \underline{x}$ = const. Further suppose that $D_k = \begin{bmatrix} \underline{z}'_{(1)} \\ \vdots \\ \underline{z}'_{(N)} \end{bmatrix}$ is any rotatable design in k - factors. Then there exists a matrix B such that the set of experimental points $E_k = \begin{bmatrix} \underline{x}'_{(1)} \\ \vdots \\ \underline{x}'_{(N)} \end{bmatrix}$ where $\underline{x}_{(i)} = B\underline{z}_{(i)}$

has the property that if $\underline{x}_1' A \underline{x}_1 = \underline{x}_2' A \underline{x}_2$ then var $\hat{y}(\underline{x}_1)$ = var $\hat{y}(\underline{x}_2)$.

Proof: If A is a positive definite matrix then by [1] there exists a non-singular matrix B such that $B'AB = I$. Now consider the transformation $B^{-1}\underline{x} = \underline{z}$. We immediately see that if $\underline{x}$ satisfies $\underline{x}' A \underline{x} = c$ then $\underline{x}' A \underline{x} = \underline{z}' B' AB\underline{z} = \underline{z}'\underline{z} = c$; that is, all points on the ellipse $\underline{x}' A \underline{x} = c$ are mapped into the sphere $\underline{z}'\underline{z} = c$.

Now suppose $D_k = \begin{bmatrix} \underline{z}'_{(1)} \\ \vdots \\ \underline{z}'_{(n)} \end{bmatrix}$ is any rotatable design in k factors.

Then define:

(i) $E_k = \begin{bmatrix} \underline{x}'_{(1)} \\ \vdots \\ \underline{x}'_{(n)} \end{bmatrix}$ where $\underline{x}_{(i)} = B\underline{z}_{(i)}$

(ii) $\underline{x}^{[2]} = \left(\frac{1}{\underline{x}}\right)^{[2]}$, $\underline{z}^{[2]} = \left(\frac{1}{\underline{z}}\right)^{[2]}$

(iii) $\begin{bmatrix} 1 & \vdots & \underline{0}' \\ \cdots & \vdots & \cdots \\ \underline{0} & \vdots & B \end{bmatrix} = R$

(iv) $$X' = \begin{bmatrix} \underline{x}^{[2]'}_{(1)} \\ \vdots \\ \underline{x}^{[2]'}_{(n)} \end{bmatrix}; \quad \underline{x}^{[2]}_{(i)} = \left(\frac{1}{\underline{x}_{(i)}}\right)^{[2]}$$

$$Z' = \begin{bmatrix} \underline{z}^{[2]'}_{(1)} \\ \vdots \\ \underline{z}^{[2]'}_{(n)} \end{bmatrix}; \quad \underline{z}^{[2]}_{(i)} = \left(\frac{1}{\underline{z}}\right)^{[2]}$$

Now assuming $\underline{x}_1$ and $\underline{x}_2$ both satisfy $\underline{x}' A \underline{x} = c$ we write $\underline{z}_1 = B^{-1}\underline{x}_1$ and $\underline{z}_2 = B^{-1}\underline{x}_2$ where $\underline{z}_1$ and $\underline{z}_2$ both lie on the sphere $\underline{z}'\underline{z} = c$.

Also, $\underline{x}_1^{[2]} = R^{[2]}\underline{z}_1^{[2]}$ and $\underline{x}_2^{[2]} = R^{[2]}\,\underline{z}_2^{[2]}$.

Hence we write:

$$\text{var}\ \hat{y}\ (\underline{x}_1) = \underline{x}_1^{[2]'}(X\,X')^{-1}\,\underline{x}_1^{[2]}\,\sigma^2$$

$$= \underline{z}_1^{[2]'}\,R^{[2]'}\,[R^{[2]}ZZ'\,R^{[2]'}]R^{[2]}\underline{z}_1^{[2]}\sigma^2$$

$$= \underline{z}_1^{[2]'}(\,Z\,Z')^{-1}\,\underline{z}_1^{[2]}\,\sigma^2$$

$$= \underline{z}_2^{[2]'}(Z\,Z')^{-1}\,\underline{z}_2^{[2]}\quad\sigma^2\ \text{since}$$

$\underline{z}_1'\,\underline{z}_1 = \underline{z}_2'\,\underline{z}_2$ and D_k is a rotatable design

$$= \underline{x}_2^{[2]'}\,(R^{[2]})^{-1'}\,(Z\,Z')^{-1}(R^{[2]})^{-1}\,\underline{x}_2^{[2]}\sigma^2$$

$$= \underline{x}_2^{[2]'}(R^{[2]}\,Z\,Z'\,R^{[2]'})^{-1}\,\underline{x}_2^{[2]}\,\sigma^2$$

$$= \underline{x}_2^{[2]'}(X\,X')^{-1}\,\underline{x}_2^{[2]'}\,\sigma^2$$

$= \text{var}\ \hat{y}\ (\underline{x}_2)$ and the theorem is proved.

2.3 Rotatable Arrangements and Quadratic Response Surfaces with Linear Restrictions among the Factors

2.3.1 The Model and the Problem

In this section we deal with the model:

$$(2.3.1.1)\qquad \mathcal{E}(\,y(\underline{x})\,) = \beta_o + \sum_{i=1}^{k}\beta_i\,x_i + \sum_{i=1}^{k}\beta_{ii}x_i^2 + \sum_{1\le i<j\le k}\beta_{ij}\,x_i x_j$$

$$= \underline{x}^{[2]'}\underline{\beta}$$

with the r linear restrictions among the factors $A\ \underline{x} = 0$.

Here it is assumed that A is an r x k matrix $(r < k)$ of full rank with orthogonal rows and $\underline{x}^{[2]'}\ \underline{\beta}$ is defined as in 1.3.1.

It is our intended purpose to find designs

$$D_k = \begin{bmatrix} x_{11} & x_{21} \cdots & x_{k1} \\ \vdots & & \\ x_{1n} & x_{2n} \cdots & x_{kn} \end{bmatrix} = \begin{bmatrix} \underline{x}'_{(1)} \\ \vdots \\ \underline{x}'_{(n)} \end{bmatrix} \quad ; \quad A\ \underline{x}_{(i)} = \underline{0}$$

which satisfy the following two criteria:

(2.3.1.2) If $\underline{x}$ is any point in our factor space then the response surface must be estimable at the point $\underline{x}$. That is, if $A\ \underline{x} = \underline{0}$ then D_k should have the property that $\mathcal{E}(\hat{y}(\underline{x})) = \underline{x}^{[2]'}\ \underline{\beta}$. Again $\underline{x}^{[2]'}\underline{\beta}$ is as defined in 1.3.1 and $\hat{y}(\underline{x})$ represents the estimated response surface at the point $\underline{x}$.

(2.3.1.3) If $\underline{x}_1$ and $\underline{x}_2$ are both in our factor space and if $\underline{x}_1'\ \underline{x}_1 = \underline{x}_2'\ \underline{x}_2$ then the design D_k should be such that var $\hat{y}(\underline{x}_1)$ = var $\hat{y}(\underline{x}_2)$.

To explain the above perhaps a geometric argument will suffice.

Our factor space can be looked at as a k-r dimensional plane passing through the origin. Since this is so, we are restricted in our choice of experimental points; that is, each experimental point must lie on the plane $A\ \underline{x} = \underline{0}$. Now, condition 2.3.1.2 simply says that the response must be estimable for every point in the plane while condition 2.3.1.3 requires that the var $\hat{y}(\underline{x})$ be constant on the family of k-r dimensional spheres defined by $A\ \underline{x} = \underline{0}$, $\underline{x}'\ \underline{x}$ = const.

Having explained the problem we now give the solution.

2.3.2 Solution of the Problem

Theorem 2.3.2 Let B be the orthogonal completion of A; that is let B be such that AB' = 0 and BB' = I. Also let $S_{k-r} = \begin{bmatrix} z_{11} & z_{21} & \cdots & z_{k-r,1} \\ z_{12} & z_{22} & & z_{k-r,2} \\ \vdots & & & \\ z_{1n} & z_{2n} & & z_{k-r,n} \end{bmatrix} =$

$$= \begin{bmatrix} \underline{z}'_{(1)} \\ \underline{z}'_{(2)} \\ \vdots \\ \underline{z}'_{(n)} \end{bmatrix}$$

be any rotatable design in k-r factors. Then the matrix of experimental points

$$\begin{bmatrix} x_{11} & x_{21} & \cdots & x_{k1} \\ \vdots & & & \\ x_{1n} & x_{2n} & \cdots & x_{kn} \end{bmatrix} = \begin{bmatrix} \underline{x}'_{(1)} \\ \vdots \\ \underline{x}'_{(n)} \end{bmatrix}$$

defined by $\underline{x}_{(i)} = B' \underline{z}_{(i)}$ satisfies conditions 2.3.1.2 and 2.3.1.3.

Proof: First we must show that every experimental point $\underline{x}_{(i)}$ does, indeed, lie in our factor space.

However, this is obvious since B is the orthogonal completion of A and we have $A\underline{x}_{(i)} = AB'\underline{z}_{(i)} = 0\underline{z}_{(i)} = \underline{0}$.

Now, before proceeding with the proof we make the following definitions:

(i) $$\mathbf{X}' = \begin{bmatrix} \underline{x}^{[2]'}_{(1)} \\ \vdots \\ \underline{x}^{[2]'}_{(n)} \end{bmatrix} \; ; \qquad \underline{x}^{[2]}_{(i)} = \left(\frac{1}{\underline{x}_{(i)}}\right)^{[2]}$$

$$Z' = \begin{bmatrix} \underline{z}^{[2]'}_{(1)} \\ \vdots \\ \underline{z}^{[2]'}_{(n)} \end{bmatrix} \; ; \qquad \underline{z}^{[2]}_{(i)} = \left(\frac{1}{\underline{z}_{(i)}}\right)^{[2]}$$

(ii) $$C' = \left[\begin{array}{c|c} 1 & \underline{0}' \\ \hline \underline{0} & B' \end{array}\right]$$

(iii) $\underline{x}^{[2]} = \begin{pmatrix}1\\ \underline{x}\end{pmatrix}^{[2]}$; $\underline{x}' = (x_1, \ldots, x_k)$

$\underline{z}^{[2]} = \begin{pmatrix}1\\ \underline{z}\end{pmatrix}$; $\underline{z}' = (z_1, \ldots, z_{k-r})$

Continuing with the proof we know from [3] that condition 2.3.1.2 is satisfied if and only if the set of equations $X\,\underline{c} = \underline{x}^{[2]}, A\,\underline{x} = 0$, has a solution.

However, $X = (\underline{x}^{[2]}_{(1)}, \ldots, \underline{x}^{[2]}_{(n)})$

$$= (C^{[2]'}\underline{z}^{[2]}_{(1)}, \ldots, C^{[2]'}\underline{z}^{[2]}_{(n)})$$

$$= C^{[2]'}Z.$$

Also, for any $\underline{x}$ such that $A\,\underline{x} = \underline{0}$ we can write $\underline{x} = B'\underline{z}$ for some $\underline{z}' = (z_1, \ldots, z_{k-r})$.

Therefore, 2.3.1.2 is satisfied if and only if the set of equation $C^{[2]'}\,Z\,\underline{c} = C^{[2]'}\,\underline{z}^{[2]}$ has a solution. But, $BB' = I$ implies $CC' = I$ and by theorem 1.2.2.4 we have $C^{[2]}C^{[2]'} = I^{[2]}$. Hence, condition 2.3.1.2 is satisfied if and only if the set of equations $Z\,\underline{c} = \underline{z}^{[2]}$ has a solution. However, S_{k-r} is a rotatable design. Therefore, Z is of full rank which implies that $Z\,\underline{c} = \underline{z}^{[2]}$ does have a solution. Hence, condition 2.3.1.2 is satisfied.

Now let us consider condition 2.3.1.3.

First let us notice that if $A\underline{x}_1 = A\underline{x}_2 = \underline{0}$ and $\underline{x}_1'\,\underline{x}_1 = \underline{x}_2'\,\underline{x}_2$ then $\underline{z}_1 = B\underline{x}_1$ and $\underline{z}_2 = B\underline{x}_2$ have the property that $\underline{z}_1'\,\underline{z}_1 = \underline{z}_2'\,\underline{z}_2$.

This is easily shown by the following:

$$\underline{z}_1'\,\underline{z}_1 = (\underline{z}'_1, \underline{0}')\begin{pmatrix}\underline{z}_1\\ \underline{0}\end{pmatrix} = \underline{x}'_1\,(B' \vdots A')\begin{pmatrix}B\\ A\end{pmatrix}\underline{x}_1$$

$= \underline{x}'_1\,\underline{x}_1$ since $\begin{pmatrix}B\\ A\end{pmatrix}$ is an orthogonal matrix.

Similarly, $z_2' \, z_2 = \underline{x}_2' \, \underline{x}_2$ and we have $\underline{z}_1' \, \underline{z}_1 = z_2' \, \underline{z}_2$.

Now, let us consider the variance of $\hat{y}(\underline{x}_1)$:

var $\hat{y}(\underline{x}_1) = \underline{x}_1^{[2]'} (XX')^* \, \underline{x}_1^{[2]} \sigma^2$ where $(XX')^*$ is the conditional inverse [3] of XX'. Hence, var $\hat{y}(\underline{x}_1) = \underline{x}_1^{[2]'} \, (XX')^* \, \underline{x}_1^{[2]} \, \sigma^2$

$$= \underline{z}_1^{[2]} C^{[2]} \, (C^{[2]'} \, ZZ' C^{[2]}) * C^{[2]'} \, \underline{z}_1^{[2]} \sigma^2$$

$$= \underline{z}_1^{[2]'} \, C^{[2]} C^{[2]'} \, (ZZ')^{-1} \, C^{[2]} C^{[2]'} \, \underline{z}_1^{[2]} \sigma^2$$

$$= \underline{z}_1^{[2]} \, (ZZ')^{-1} \, \underline{z}_1^{[2]} \sigma^2 \text{ from theorem 1.2.2.4}$$

$$= \underline{z}_2^{[2]'} \, (ZZ')^{-1} \, \underline{z}_2^{[2]} \sigma^2 \text{ since } \underline{z}_1' \, \underline{z}_1 = \underline{z}_2' \, \underline{z}_2$$

and S_{k-r} is a rotatable design.

Therefore,

$$\text{var } \hat{y}(\underline{x}_1) = z_2^{[2]'} \, (ZZ')^{-1} \, z_2^{[2]} \, \sigma^2$$

$$= \underline{x}_2^{[2]'} \, C^{[2]'} \, (ZZ')^{-1} \, C^{[2]'} \, \underline{x}_2^{[2]} \sigma^2$$

$$= \underline{x}_2^{[2]'} \, (XX')^* \, \underline{x}_2^{[2]} \, \sigma^2$$

$$= \text{var } \hat{y}(\underline{x}_2) \text{ and the theorem is proved.}$$

Chapter III

Construction of Rotatable Arrangements from the Association Matrices of Partially Balanced Association Schemes

3.1 Partially Balanced Association Schemes and Association Matrices [4]

Given v objects, 1,2, ..., v, a relation satisfying the following conditions is said to be an m-class partially balanced association scheme:

(i) Any two objects are either 1st, 2nd, ..., or m-th associates, the relation of association being symmetric; that is, if the object α is the i-th associate of the object β, then β is the i-th associate of the object α.

(ii) Each object has n_i i-th associates, the number n_i being independent of α.

(iii) If any two objects are i-th associates then the number of objects which are j-th associates of α and k-th associates of β is p^i_{jk} and is independent of the pair of i-th associates α and β.

The parameters of the association scheme are given by:

(3.1.1) (v, n_i, P_i) ; $i = 1,2,\ldots, m$

where $P_i = ((p^i_{jk}))$; $j, k = 1,2,\ldots, m$.

The following indentity is easily established

(3.1.2) $p^i_{jk} = p^i_{kj}$ for any i, j, k.

We now define the association matrices of a partially balanced association scheme as:

(3.1.3) $B_i = ((b^i_{\alpha\beta}))$; $i = 1,2,\ldots, m$

and $\alpha,\beta = 1,2,\ldots, v$ where $b^i_{\alpha\beta} = 1$ if α,β are i-th associates and $b^i_{\alpha\beta} = 0$ otherwise.

The following two results are easily verified:

(3.1.4) If $B_i = (\underline{b}_{1i}, \ldots, \underline{b}_{vi})$ is an association matrix then

$\underline{b}'_{\alpha i}\ \underline{b}_{\beta i} = p^{\ell}_{ii}$ if α and β are ℓ-th associates. It is assumed $\alpha \neq \beta$.

(3.1.5) If $B = (\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_v) = c_1 B_1 + c_2 B_2 + \ldots + c_m B_m$

where the B_i are association matrices then

$$\underline{b}'_\alpha\ \underline{b}_\beta = \sum_{j=1}^{m} c_j^2\, p^{\ell}_{jj} + 2 \sum_{1 \leq k <} \sum_{j \leq m} c_k\, c_j\, p^{\ell}_{kj},$$

if $\alpha \neq \beta$ are ℓ-th associates.

In this chapter we consider the rows of the matrices B_i as points in a v dimensional factor space; that is, as v-tuplets. By generalizing a method given in [9] (where the authors consider a more restrictive kind of matrix) we use these v-tuplets to construct rotatable arrangements in v factors.

3.2 Notation and Definitions

If $\underline{a}'_w = (a_{1w}, a_{2w}, \ldots a_{vw})$ is a v-tuplet consisting of $k \leq v$ non zero co-ordinates then the notation $\underline{a}'_w \times 2^k$ refers to the 2^k distinct points of the form $(\pm a_{1w}, \pm a_{2w}, \ldots, \pm a_{vw})$.

If $A = \begin{bmatrix} \underline{a}'_1 \\ \vdots \\ \underline{a}'_n \end{bmatrix}$ is a n x v matrix

each of whose rows contain exactly k non-zero elements, then $A \times 2^k$ is the set of $N = n \times 2^k$ points $\{\underline{a}'_1 \times 2^k, \underline{a}'_2 \times 2^k, \ldots, \underline{a}'_n \times 2^k\}$ where $\underline{a}'_w \times 2^k$, $w = 1, 2, \ldots, n$, is defined above.

If $(x_{1w}, x_{2w}, \ldots, x_{vw})$ is a typical member of this set then

$\frac{1}{N} \sum_{w=1}^{N} X_{1w}^{\alpha_1} X_{2w}^{\alpha_2} \ldots X_{vw}^{\alpha_v}$ is the $(\alpha_1, \alpha_2, \ldots, \alpha_v)$ -th "moment" of that set of points where "moment" is defined in 1.4.1 and the summation extends over all members of the set.

3.3 The Union Method of Construction

The following theorem gives a method of constructing rotatable arrangements which we call the union method.

Theorem 3.3 Suppose we have an m-class association scheme with parameters (v, n_i, P_i), $i = 1,2,\ldots, m$, and association matrices $B_1, \ldots B_m$. Then if there exist real number $a_1, a_2, \ldots, a_m$, not all zero, such that

$$a_1^4 p_{11}^k + a_2^4 p_{22}^k + \ldots + a_m^4 p_{mm}^k = p \text{ (say) for } k = 1,2,\ldots, m \text{ then}$$

the following sets of points form rotatable arrangements:

(i) $\{[a_1 2^{-n_1/4} B_1] X 2^{n_1}, [a_2 2^{-n_2/4} B_2] X 2^{n_2}, \ldots, [a_m 2^{-n_m/4} B_m] X 2^{n_m}\}$

if $3p = a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m$

(ii) $\{[a_1 2^{-n_1/4} B_1] X 2^{n_1}, \ldots, [a_m 2^{-n_m/4} B_m] X 2^{n_m}, bI_{vxv} X 2\}$

if $3p > a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m$

(iii) $\{[a_1 2^{-n_1/4} B_1] X 2^{n_1}, \ldots, [a_m 2^{-n_m/4} B_m] X 2^{n_m},$

$(b, b, \ldots, b) X 2^v\}$

if $3p < a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m$ where b in (ii) and (iii) is a real number determined by the moment conditions for rotatable arrangements.

Proof: In all three cases $\sum_w X_{1w}^{\alpha_1} X_{2w}^{\alpha_2} \ldots X_{vw}^{\alpha_v} = 0$ if any α_i is odd since if the v-tuplet $(x_1, x_2, \ldots, -x_i, \ldots, x_v)$ appears in the set then so does the v-tuplet $(x_1, x_2, \ldots, x_i, \ldots, x_v)$.

Case (i) $3p = a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m$

We have:

$$\sum_w X_{iw}^2 = a_1^2\, 2^{n_1/2}\, n_1 + \dots + a_m^2\, 2^{n_m/2}\, n_m$$

for all i.

$$\sum_w X_{iw}^4 = a_1^4 n_1 + a_2^4 n_2 + \dots + a_m^4 n_m \text{ for all } i$$

$$\sum_w X_{iw}^2 X_{jw}^2 = a_1^4 p_{11}^k + a_2^4 p_{22}^k + \dots + a_m^4 p_{mm}^k \text{ if } i, j \text{ are } k\text{-}\underline{th} \text{ associates}$$

from 3.1.4.

Hence, by assumption we conclude that $\sum_w X_{iw}^2 X_{jw}^2 = p$ for all i, j, $i \neq j$.

However, if $3p = a_1^4 n_1 + \dots + a_m^4 n_m$ then

$$3 \sum_w X_{iw}^2 X_{jw}^2 = \sum_w X_{kw}^4 \text{ for all } i, j, k, i \neq j.$$

Hence, 1.4.2 is satisfied and the set of points is a rotatable arrangement.

Case (ii) $\underline{3p > a_1^4 n_1 + a_2^4 n_2 + \dots + a_m^4 n_m}$

We have:

$$\sum_w X_{iw}^2 = n_1 a_1^2\, 2^{n_1/2} + \dots + n_m a_m^2\, 2^{n_m/2} + 2\, b^2$$

for all i and any b.

$$\sum_w X_{iw}^4 = a_1^4 n_1 + a_2^4 n_2 + \dots + a_m^4 n_m + 2\, b^4 \text{ for all } i \text{ and any } b.$$

$$\sum_w X_{iw}^2 X_{jw}^2 = a_1^4 p_{11}^k + a_2^4 p_{22}^k + \dots + a_m^4 p_{mm}^k \text{ if } i, j \text{ are } k\text{-}\underline{th} \text{ associates}$$

by 3.1.4. But, by assumption we conclude $\sum_w X_{iw}^2 X_{jw}^2 = p$ for all i, j.

Now, if $3p > a_1^4 n_1 + \dots + a_m^4 n_m$ then there exists a real number b such that $3 \sum_w X_{iw}^2 X_{jw}^2 = \sum_w X_{kw}^4$ for all i, j, k, $i \neq j$. That is, $3p = a_1^4 m_1 + \dots + a_m^4 n_m + 2\, b^4$ has a real solution in b and we conclude that the set of points in case (ii) with b so determined form a rotatable arrangement.

Case (iii) $\underline{3p < a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m}$

We have:

$$\sum_w X_{iw}^2 = a_1^2 \, 2^{n_1/2} n_1 + \ldots + a_m^2 \, 2^{n_m/2} n_m + b^2 \, 2^v$$

for all i and any b.

$$\sum_w X_{iw}^4 = n_1 a_1^4 + n_2 a_2^4 + \ldots + n_m a_m^4 + b^4 2^v \text{ for all i and any b.}$$

$$\sum_w X_{iw}^2 X_{jw}^2 = a_1^4 p_{11}^k + a_2^4 p_{-2}^k + \ldots + a_m^4 p_{mm}^k + b^4 2^v \text{ if i, j are k-\underline{th}}$$

associates by 3.1.4

Hence, $\sum_w X_{iw}^2 X_{jw}^2 = p + b^4 2^v$ by assumption.

However, if $3p < n_1 a_1^4 + n_2 a_2^4 + \ldots + n_m a_m^4$ then there exists a real number b such that $3 \sum_w X_{iw}^2 X_{jw}^2 = \sum_w X_{kw}^2$ for all $i,j,k,i \neq j$. That is, $3(p + 2^v b^4) = n_1 a_1^4 + n_2 a_2^4 + \ldots + n_m a_m^4 + 2^v b^4$ has a real solution in b. Hence, we conclude that the set of points in case (iii) with b so determined form a rotatable arrangement.

It is to be noticed that if $a_i = 0$ then the set of points $[a_i 2^{-n_i/4} B_i] X 2^{n_i}$ need not be included for the entire set to be a rotatable arrangement.

With the theorem now proved, it will be our concern in sections 3.4 and 3.5 to find real numbers $a_1, a_2, \ldots, a_m$ such that:

(3.3.1) $a_1^4 p_{11}^k + a_2^4 p_{22}^k + \ldots + a_m^4 p_{mm}^k = p$ (say) for all k, for particular association schemes.

3.4 The Union Method and Some Two-Class Association Schemes.

3.4.1 Two-Class Group Divisible Association Schemes [6]

In this scheme $v = mn$, and the objects are divided into m groups of n each such that any two objects of the same group are first associates while any two objects from different groups are second associates.

Here we have an association scheme with parameters given by:

(3.4.1.1) $n_1 = n-1$, $n_2 = n(m-1)$, $v = mn$,

$$P_1 = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}.$$

Hence, we wish to find real numbers a_1, a_2 such that:

(3.4.1.2) $a_1^4\,(n-2) + a_2^4\,[n(m-1)] = a_1^4\,(0) + a_2^4\,[n\,(m-2)]$.

Equivalently, we wish to find real numbers a_1, a_2 such that:

(3.4.1.3) $a_1^4\,(n-2) + a_2^4\,(n) = 0$

For $n = 2$, choose $a_2 = 0$ and a_1 arbitrary

Then, $v = 2m$ (m is ofcourse arbitrary) and $n_1 = 1$, $n_2 = 2(m-1)$.

Immediately we see that $p = 0$ and we determine the number b as in case (iii) of the theorem.

Also, no other rotatable arrangement is possible by the union method from this scheme.

3.4.2 The Triangular Association Scheme [6]

An association scheme is said to be triangular if it is an array of n rows and n columns with the following properties:

(i) The positions in the principal diagonal (running from upper left hand to lower right hand corner) are left blank.

(ii) The $n(n-1)/2$ positions above the principal diagonal are filled by the numbers 1, 2, ..., $n(n-1)/2$ which are the objects.

(iii) The $n(n-1)/2$ positions below the principal diagonal are filled so that the array is symmetric about the principal diagonal.

(iv) For any object i the first associates are exactly those objects which lie in the same row (or in the same column) as i. The second associates of i are merely those objects which are not its first associates.

We then obtain an association scheme with parameters:

(3.4.2.1) $n_1 = 2n - 4$, $n_2 = (n-2)(n-3)/2$, $v = n(n-1)/2$ and,

$$P_1 = \begin{bmatrix} n-2 & n-3 \\ n-3 & (n-3)(n-4)/2 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 4 & 2n-8 \\ 2n-8 & (n-4)(n-5)/2 \end{bmatrix}.$$

For example, with n = 4 we have

*	1	2	3
1	*	4	5
2	4	*	6
3	5	6	*

and the first associates of 2, for instance, are 1,3,4,6.

Now we find real numbers a_1, a_2 such that:

(3.4.2.2) $a_1^4 (n-2) + 1/2\ a_2^4 (n-3)(n-4) = 4a_1^4 + 1/2\ a_2^4 (n-4)(n-5)$,

or equivalently:

(3.4.3.3) $a_1^4 (n-6) + a_2^4 (n-4) = 0$.

Consequently we see for $n \geq 7$ or $n \leq 3$ no rotatable arrangements are possible.

However, for n = 4, 5, 6 we do get the rotatable arrangements given in the following:

v	n	n_1	n_2	a_1	a_2	p	b
6	4	4	1	0	arbitrary	0	case (iii)
10	5	6	3	arbitrary	a_1	$4a_1^4$	case(ii)
15	6	8	6	arbitrary	0	$4a_1^4$	case (ii)

3.4.3 The L_r Association Scheme [6]

Consider $v = k^2$ objects set forth in a kxk square. Now take r-2 mutually orthogonal squares (if such a set exists) and define the L_r association scheme by the following:

Two objects are first associates if they occur in the same row or column of the square scheme or correspond to the same symbol of one of the latin squares. Otherwise two objects are defined to be second associates.

The parameters of the scheme are:

(3.4.3.1) $v = k^2$, $n_1 = r(k-1)$, $n_2 = (k-1)(k-r+1)$,

$$P_1 = \begin{bmatrix} (k-2)+(r-1)(r-2) & (r-1)(k-r+1) \\ (r-1)(k-r+1) & (k-r)(k-r+1) \end{bmatrix}$$

$$P_2 = \begin{bmatrix} r(r-1) & r(k-r) \\ r(k-r) & (k-r)^2 + (k-2) \end{bmatrix}$$

For example, with $k = 4$, $r = 4$ we have:

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

with the two mutually orthogonal latin squares

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

and

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

Here, for instance, the first associates of 6 are,

4,5,7,2,10,14,3,9,12,0,11,13.

After some simplification, the above scheme gives a rotatable arrangement if the equation

(3.4.3.2) $a_1^4\ (k-2r) + a_2^4\ (2-r) = 0$

has a real solution in a_1, a_2.

Some values of a_1, a_2 along with the associated value of p and the method of determination of b are given below for particular values of k and r.

v	k	r	a_1^4	a_2^4	n_1	n_2	p	b
9	3	2	0	arbitrary	4	4	$2a_2^4$	case (ii)
16	4	2	arbitrary	arbitrary	6	9	$2a_1^4 + 6a_2^4$	case (ii)
25	5	2	0	arbitrary	8	16	$12a_2^4$	case(ii)
49	7	2	0	arbitrary	12	36	$30a_2^4$	case (ii)

v	k	r	a_1^4	a_2^4	n_1	n_2	p	b
49	7	3	arbitrary	a_1^4	18	30	$26a_1^4$	case (ii)
64	8	2	0	arbitrary	14	49	$42a_2^4$	case (ii)
64	8	3	arbitrary	$2a_1^4$	21	42	$68a_1^4$	case (ii)
64	8	4	arbitrary	0	28	35	$12a_1^4$	case (ii)
81	9	2	0	arbitrary	16	64	$56a_2^4$	case (ii)
81	9	3	arbitrary	$3a_1^4$	21	59	$135a_1^4$	case (ii)
81	9	4	$2a_2^4$	arbitrary	28	52	$56a_2^4$	case (ii)

3.4.4 Pseudo-Cyclic Association Schemes [6]

Let the objects be denoted by the v integers, 1, 2, ..., v. Suppose we can find a set of n_1 integers $d_1, d_2, \ldots, dn_1$ satisfying the following conditions:

(i) The d's are all different, $0 < d_j < v$.

(ii) Among the $n_1(n_1-1)$ differences $d_j - d_{j'}$, ($j, j' = 1, 2, \ldots, n_1$ and $j \neq j'$) reduced mod v, each of the numbers $d_1, d_2, \ldots, d_{n_1}$ occurs g times; whereas, each of the numbers $e_1, e_2, \ldots e_{n_2}$ occurs h times, where $d_1, d_2, \ldots, d_{n_1}, e_1, e_2, \ldots, e_{n_2}$ are all the different integers 1, 2, ..., v-1.

Then, the first associates of the object i are defined to be $i + d_1$, $i + d_2, \ldots, i + d_{n_1}$. The remaining objects are said to be second associates.

The parameters of such a scheme are:

(3.4.4.1) v, n_1, n_2 (all defined above)

$$P_1 = \begin{bmatrix} g & n_1-g-1 \\ n_1-g-1 & n_2-n_1+g+1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} h & n_1-h \\ n_1-h & n_2-n_1+h-1 \end{bmatrix}$$

Now, all the known cyclic association schemes are such that $v = 4w+1$, $n_1 = n_2 = 2w$ and $g = w-1$ for some positive integer w. Hence, we call any association scheme with the following parameters a pseudo-cyclic association scheme:

(3.4.4.2) $v = 4w+1$, $n_1 = n_2 = 2w$,

$$P_1 = \begin{bmatrix} w-1 & w \\ w & w \end{bmatrix}$$

$$P_2 = \begin{bmatrix} w & w \\ w & w-1 \end{bmatrix}$$

Hence, for any pseudo-cyclic association scheme we wish to find real numbers a_1, a_2 such that:

(3.4.4.3) $a_1^4(w-1) + a_2^4(w) = a_1^4\ (w) + a_2^4\ (w-1)$,

or equivalently:

(3.4.4.4) $-a_1^4 + a_2^4 = 0.$

That is, we must merely choose $a_1^4 = a_2^4$ and a pseudo-cyclic association scheme gives rise to a rotatable arrangement.

However, in [2] it is shown that the matrix $N = \begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix}$ is the incidence matrix of a balanced incomplete block design. Now, the case of obtaining rotatable arrangements from the incidence matrices of balanced incomplete block designs is covered in [9]; hence, this scheme is no longer of any interest to us.

3.5 The Union Method and Some Three-Class Association Schemes

3.5.1 Three-Class Group Divisible Association Schemes [10]

Consider ordered 3-tuplets (X_1, X_2, X_3), $X_i = 1, 2, \ldots, N_i$, and define a relation of association as follows:

(X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are i-th associates if they have the first 3-i co-ordinates alike.

We then obtain a three-class association scheme with the following parameters:

(3.5.1.1) $v = N_1N_2N_3$, $n_1 = N_3-1$, $n_2 = (N_2-1)N_3$, $n_3 = (N_1-1)N_2N_3$,

$$P_1 = \begin{bmatrix} N_3-2 & 0 & 0 \\ 0 & N_3(N_2-1) & 0 \\ 0 & 0 & N_3N_2(N_1-1) \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & N_3-1 & 0 \\ 0 & N_3(N_2-2) & 0 \\ N_3-1 & 0 & N_3N_2(N_1-1) \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 0 & N_3-1 \\ 0 & 0 & N_3(N_2-1) \\ N_3-1 & N_3(N_2-1) & N_3N_2(N_1-2) \end{bmatrix}$$

After some algebraic manipulation it is seen that 3.5.1.1 gives a rotatable arrangement if the set of equations

(3.5.1.2) $a_1^4(N_3-2) + a_2^4\ (N_3) = 0$

$$a_2^4[N_3(N_2-2)] + a_3^4\ (N_2N_3) = 0$$

has a real solution in a_1, a_2.

Therefore, we see that if $N_3 > 2$ no arrangements are possible while if $N_3 = 2$ we may take a_1 arbitrary and $a_2 = a_3 = 0$.

Notice that if $N_3 = 2$ then $p = 0$ and b is determined as in case (iii) of theorem 3.3.

3.5.2 The Rectangular Association Scheme [12]

Consider a rectangle with n columns and ℓ rows. Let an object correspond to one of the ℓn cells.

Now define:

(i) Two objects are first associates if they are in the same row.

(ii) Two objects are second associates if they are in the same column.

(iii) Two objects are third associates if they are not first or second associates.

We then obtain a three-class association scheme with the following parameters:

(3.5.2.1) $v = \ell n$, $n_1 = n-1$, $n_2 = \ell-1$, $n_3 = v-n_1-n_2-1$,

$$P_1 = \begin{bmatrix} n-2 & 0 & 0 \\ 0 & 0 & \ell-1 \\ 0 & \ell-1 & (\ell-1)(n-2) \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & 0 & n-1 \\ 0 & \ell-2 & 0 \\ n-1 & 0 & (\ell-2)(n-1) \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 1 & n-2 \\ 1 & 0 & \ell-2 \\ n-2 & \ell-2 & (\ell-2)(n-2) \end{bmatrix}$$

It is seen that the rectangular association scheme gives rise to a rotatable arrangement by the union method if the equations:

(3.5.2.2) $$a_1^4\,(n-2)-a_2^4(\ell-2) + a_3^4(n-\ell) = 0$$

$$a_2^4(\ell-2) + a_3^4(\ell-2) = 0$$

have a real solution in a_1, a_2, a_3.

We now notice that if $\ell > 2$ we must take $a_2 = a_3 = 0$. Hence, we get a non-trivial solution for $n = 2, \ell > 2$ where $a_2 = a_3 = 0$ and a_1 is chosen in an arbitrary fashion. Also, for $n = 2, \ell > 2$ we see that $p = 0$ and b must be determined as in case (iii) of theorem 3.3.

Now, if $\ell = 2$ we take $a_1 = a_3 = 0$ and a_2 arbitrary. Again we see $p = 0$ and b is determined as in case (iii).

Concluding we state that the rectangular association scheme gives rotatable arrangements for any v dimensional factor space where v is even, $v = 2\ell$ (say).

3.5.3 The Cubic Association Scheme [11]

Let each object correspond to the ordered triplet (X_1, X_2, X_3), $X_i = 0, 1, \ldots, S-1$.

Define (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) to be i-th associates if they have i co-ordinates different.

We then get an association scheme with parameters:

(3.5.3.1) $v = S^3,\ n_1 = 3(S-1),\ n_2 = 3(S-1)^2,\ n_3 = (S-1)^3,$

$$P_1 = \begin{bmatrix} S-2 & 2(S-1) & 0 \\ 2(S-1) & 2(S-1)(S-2) & (S-1)^2 \\ 0 & (S-1)^2 & (S-1)^2(S-2) \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 2 & 2(S-2) & (S-1) \\ 2(S-2) & 2(S-1)+(S-2)^2 & 2(S-1)(S-2) \\ (S-1) & 2(S-1) & (S-1)(S-2)^2 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 3 & 3(S-2) \\ 3 & 6(S-2) & 3(S-2)^2 \\ 3(S-2) & 3(S-2)^2 & (S-2)^3 \end{bmatrix}$$

After some calculation we see the above scheme gives us a rotatable arrangement if the equations

(3.5.3.2) $a_1^4(S-4) + a_2^4\,(S^2-4S+2) + a_3^4(S-1)(S-2) = 0$

$$a_1^4(S-2) + 2a_2^4\,(S-2)(S-4) + a_3^4(S-2)(2S-3) = 0$$

have a real solution in a_1, a_2, a_3.

Now, for $S \geq 4$ it is easy to verify that no real non-trivial solutions are possible.

However, for S = 3 the equations become:

$$(3.5.3.3)\quad -a_1^4 - a_2^4 + 2a_3^4 = 0$$

$$a_1^4 - 2a_2^4 + 3a_3^4 = 0.$$

By substitution we see that a real solution exists if we choose a_3 arbitrary and $a_2^4 = 1/3\ a_3^4$ with $a_1^4 = 2a_3^4 - a_2^4 = 1/3\ a_3^4$.

Here, we get a rotatable arrangement with $v = 27$, $n_1 = 6$, $n_2 = 12$, and $n_3 = 8$. Also, $p = 11\ a_3^4$ while $n_1 a_1^4 + n_2\ a_2^4 + n_3\ a_3^4 = 30\ a_3^4$. Hence, b is determined as in case (ii).

For S = 2 the equations reduce to:

$$(3.5.3.4)\quad -2a_1^4 - 2a_2^4 = 0$$

Hence, for S = 2 we choose $a_1 = a_2 = 0$ and a_3 arbitrary and we obtain a rotatable arrangement with $v = 8$, $n_3 = 1$, $p = 0$ and b is determined as in case (iii).

3.5.4 <u>The Tetrahedral Scheme [7]</u>

Consider the integers 1, 2, ..., m and the $v = \binom{m}{3}$ sets of size 3 that can be formed from them.

Let an object correspond to each one of these sets and define $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ as i-<u>th</u> associates if their intersection has (3-i) integers in common.

We then get an association scheme with parameters.

$$(3.5.4.1)\quad v = \binom{m}{3},\ n_1 = 3(m-3),$$

$$n_2 = 3/2\ (m-3)(m-4),\ n_3 = 1/6\ (m-3)(m-4)(m-5),$$

$$P_1 = \begin{bmatrix} m-2 & 2(m-4) & 0 \\ 2(m-4) & (m-4)^2 & 1/2\ (m-4)(m-5) \\ 0 & 1/2\ (m-4)(m-5) & 1/6\ (m-4)(m-5)(m-6) \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 4 & 2(m-4) & (m-5) \\ 2(m-4) & 1/2(m-5)(m+2) & (m-5)(m-6) \\ (m-5) & (m-5)(m-6) & 1/6(m-5)(m-6)(m-7) \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 9 & 3(m-6) \\ 9 & 9(m-6) & 3/2\,(m-6)(m-7) \\ 3(m-6) & 3/2(m-6)(m-7) & 1/6(m-6)(m-7)(m-8) \end{bmatrix}$$

The above association scheme gives us a rotatable arrangement if the equations:

(3.5.4.2) $a_1^4(m-6) + 1/2\ a_2^4(m-6)(m-7) + 1/2\ a_3^4(m-5)(m-6) = 0$

$4\ a_1^4 + 1/2\ a_2^4(m-7)(m-14) + 1/2\ a_3^4(m-6)(m-7) = 0$

have a real non-trivial solution in a_1, a_2, a_3.

It is obvious that for $m \geq 7$ or $m \leq 5$ that no real solutions are possible.

However, for $m = 6$ the equations reduce to:

(3.5.4.3) $4a_1^4 + 4a_2^4 = 0$

Hence, for $m = 6$ if we choose $a_1 = a_2 = 0$ and a_3 arbitrary we get a rotatable arrangement in $v = \binom{6}{3} = 20$ factors. Again we notice that $m = 6$ implies $p = 0$ and b is determined as in case (iii)

3.6 Construction by Linear Combinations

In the theorem which follows, we derive a method of constructing rotatable arrangement from association matrices which we call the method of linear combinations.

<u>Theorem 3.6</u> Suppose we have an m-class association scheme with parameters (v, n_i, P_i), $i = 1,2,\dots, m$, and association matrices $B_1, B_2, \dots, B_m$. Then, if there exist real numbers $a_1, a_2, \dots, a_m$, t of which are non-zero ($a_{i_1}, a_{i_2}, \dots, a_{i_t}$ (say)), such that:

$$\sum_{i=1}^{m} a_i^4 p_{ii}^k + 2 \sum_{1 \le i < j \le m} a_i^2 a_j^2 p_{ij}^k = p \text{ (say) for } k = 1, 2, \dots, m,$$

then the following sets of points form rotatable arrangements:

(i) $\{2^{-k/4} (a_1B_1 + a_2B_2 + \dots + a_mB_m) \times 2^k\}$

$\quad$ if $3p = a_1^4 n_1 + a_2^4 n_2 + \dots + a_m^4 n_m$

(ii) $\{2^{-k/4}(a_1B_1 + a_2B_2 + \dots + a_mB_m) \times 2^k,\ bI_v \times 2\}$

$\quad$ if $3p > a_1^4 n_1 + a_2^4 n_2 + \dots + a_m^4 n_m$

(iii) $\{2^{-k/4}(a_1B_1 + a_2B_2 + \dots + a_mB_m) \times 2^k,\ (b, b, \dots, b) \times 2^v\}$

$\quad$ if $3p < a_1^4 n_1 + a_2^4 n_2 + \dots + a_m^4 n_m$

where $k = n_{i_1} + n_{i_2} + \dots + n_{i_t}$ and b in cases (ii) and (iii) is determined by the moment conditions for rotatable arrangements.

Proof: As is explained in theorem 3.3, $\sum_w X_{1w}^{\alpha_1} X_{2w}^{\alpha_2} \dots X_{vw}^{\alpha_v} = 0$ if any α_i is odd.

Now, we show that the other necessary moment conditions are satisfied.

Case (i) <u>$3p = a_1^4 n_1 + a_2^4 n_2 + \dots + a_m^4 n_m$</u>

Here:

$$\sum_w X_{iw}^2 = 2^{k/2}(a_1^2 n_1 + \dots + a_m^4 n_m) \text{ for all } i = 1, 2, \dots, v.$$

$$\sum_w X_{iw}^4 = a_1^4 n_1 + a_2^4 n_2 + \dots + a_m^4 n_m \text{ for all } i = 1, 2, \dots, v.$$

$$\sum_w X_{iw}^2 X_{jw}^2 = \sum_{j=1}^m a_j^4 p_{jj}^k + 2 \sum_{1 \leq} \sum_{\ell \leq m} a_\ell^2 a_j^2 p_{\ell j}^k \text{ if i, j are K-\underline{th} associates by 3.1.5.}$$

$= p$ by assumption.

But, if $3p = a_1^4 n_1 + \ldots + a_m^4 n_m$ then

$3 \sum_w X_{iw}^2 X_{jw}^2 = \sum_w X_{kw}^4$ for all i, j, k, $i \neq j$ and we conclude that the set of points form a rotatable arrangement.

Case (ii) $\underline{3p > a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m}$

We have:

$\sum_w X_{iw}^2 = 2^{k/2} (a_1^2 n_1 + \ldots + a_m^2 n_m) + 2b^2$ for all i and any b.

$\sum_w X_{iw}^4 = a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m + 2b^4$ for all i and any b.

$\sum_w X_{iw}^2 X_{jw}^2 = \sum_{j=1}^m a_j^4 p_{ij}^k + 2 \sum_{1 \leq \ell < j \leq m} \sum a_\ell^2 a_j^2 p_{\ell j}^k$ if i, j are k-th associates by 3.1.5

$= p$ by assumption

But, if $3p > a_1^4 n_1 + \ldots + a_m^4 n_m$ then $3p = a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m + 2b^4$ has a real solution in b and the arrangement is rotatable.

Case (iii) $\underline{3p < a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m}$

Here:

$\sum_w X_{iw}^2 = 2^{k/2} (a_1^2 n_1 + a_2^2 n_2 + \ldots + a_m^2 n_m) + 2^v b^4$ for all i and any b.

$\sum_w X_{iw}^4 = a_1^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m + 2^v b^4$ for all i and any b.

$\sum_w X_{iw}^2 X_{jw}^2 = p + b^4 2^v$ by 3.1.5 and assumption.

But, if $3p < a_1^4 n_1 + a_2^4 n_1 + a_2^4 n_2 + \ldots + a_m^4 n_m$ then

$3(p + b^4 2^v) = a_1^4 n_1 + a_2^4 n_2 + \ldots + a_n^4 n_n + 2^v b^4$ has a real solution in b and we conclude that the set of points in (iii) with b so determined form a rotatable arrangement.

With the theorem now proved, we will try to obtain in sections 3.7 and 3.8 real numbers such that:

$$\sum_{i=1}^{m} a_i^4 p_{ii}^k + 2 \cdot \sum_{1 \leq i < j \leq m} a_i^2 a_j^2 p_{ij}^k = p \text{ (say)} \qquad (3.6.1)$$

for all k = 1, 2, ..., m, for particular association schemes.

However, the equations for determining $a_1, a_2, \ldots, a_m$ are quite often intractable and, as a result, not many examples are given.

Also, it is worth noting that if we choose $a_1 = a_2 = \ldots = a_m = a$ (say) then we always get a rotatable arrangement from the linear combination method. This is apparent since the matrix $N = B_1 + B_2 + \ldots + B_m =$

$= J_v - I_v$, where J_v is the v x v matrix whose elements are all 1 and I_v is the v x v identity matrix, is the incidence matrix of a balanced incomplete block design [2]. Since this case is covered in [9], such solutions will not be of interest to us.

3.7 The Method of Linear Combinations and Some Two-Class Association Schemes

3.7.1 Two-Class Group Divisible Association Schemes

If we consider the association scheme explained in section 3.4.1 of this thesis, we see a rotatable arrangement can be obtained from it by the method of linear combinations if the equation:

$$a_1^4 (n-2) - 2a_1^2 a_2^2 (n-1) + a_2^4 (n) = 0 \qquad (3.7.1.1)$$

has a real solution in a_1, a_2, a_3.

Now, if we choose $a_1 = 0$ then we immediately conclude no arrangements are possible.

If we choose $a_2 = 0$ with a_1 arbitrary then a solution exists if $n = 2$. However, the arrangement obtained is identical with that given in section 3.4.1 and is of no interest here.

Now, assuming $a_1, a_2 \neq 0$ let $y = a_1^2/a_2^2$.

Then the equations 3.7.1 become:

(3.7.1.2) $\quad y^2(n-2) -2y(n-1) + n = 0$

Solving 3.7.2 for y we get:

(3.7.1.3) $\quad y = 1, \; y = n/(n-2).$

The solution $y = 1$ corresponds to $a_1^2 = a_2^2$ and as was explained before is not of much interest here.

However, if we choose $y = a_1^2/a_2^2 = n/n-2$ with $n > 2$ then we obtain rotatable arrangements in any number of factors. Here it is noticed that $k = n_1 + n_2 = nm-1$.

3.7.2 The Triangular Association Scheme

The relation of association and the parameters for this scheme are given in section 3.4.2.

The problem of finding numbers a_1, a_2 so that 3.6.1 will hold reduces to solving the equation:

(3.7.2.1) $\quad a_1^4(n-6) + a_2^4(n-4) -2a_1^2 a_2^2 (n-5) = 0$

Notice that for $n = 6, 4$ we obtain the same arrangements as in section 3.6.1 and for $n = 5$ we must have $a_1^2 = a_2^2$ which is of no interst.

Now, assuming $a_1, a_2 \neq 0$ and letting $y = a_1^2/a_2^2$ the equation 3.7.2.1 becomes:

(3.7.2.2) $\quad y^2(n-6) - 2y(n-5) + (n-4) = 0$

Upon solving for y, we get $y = 1$, as expected, and $y = n-4/n-6$.

Hence, for $n > 6$ and $v = n(n-1)/2$ we can always obtain a rotatable arrangement by the method of linear combinations from this scheme.

3.7.3 Pseudo-Cyclic Two-Class Association Scheme

The relation of association and the parameters of this scheme are given in section 3.4.4.

Now, we wish to find a_1, a_2 such that

$$(3.7.3.1) \quad a_1^4 (w-1) + a_2^4(w) + 2a_1^2 a_2^2 (w) = a_1^4 (w) + a_2^4 (w-1) + 2 a_1^2 a_2^2 (w).$$

This is equivalent to:

(3.7.3.2) $-a_1^4 + a_2^4 = 0$ and is of no interest to us as was explained previously.

3.8 Linear Combinations and the Three-Class Group Divisible Association Scheme

In section 3.5.1 the relation of association and the parameters for this scheme are given.

After some algebraic manipulation it is seen that 3.6.1 is satisfied if we can find a real solution to the equations:

$$(3.8.1) \quad a_1^4 (N_3-2) + a_2^4 (N_3) - 2a_1^2 a_2^2 (N_3-1) = 0$$

$$a_2^4[N_3(N_2-2)] + a_3^4 (N_2N_3) + 2 a_1^2 a_2^2 (N_3-1) - 2 a_1^2 a_3^2 (N_3-1) -2 a_2^2 a_3^2 [N_3(N_2-1)]= 0$$

We see that if $a_3 = 0$ then the second equation reduces to:

(3.8.2) $1 + 2 a_1^2 (N_3-1) = 0$ which has no real solutions.

Hence, assuming $a_3 \neq 0$ and letting $X = a_1^2/a_3^2$ any $Y = a_2^2/a_3^2$ the equations become:

$$(3.8.3) \qquad X^2(N_3-2) + Y^2(N_3) - 2XY(N_3-1) = 0$$

$$Y^2[N_3(N_2-2)] + 2\,XY(N_3-1) - 2X(N_3-1)$$

$$-2Y[N_3(N_2-1)] + N_2\,N_3 = 0$$

Solving the first equation of 3.8.3 for X in terms of Y we get as solutions $X = Y$, $\quad X = \dfrac{N_3\,Y}{N_3-2}$, $N_3 > 2$.

Substituting $X = Y$ in the second equation of (3.8.3) we obtain the two solutions $X = Y = 1$ and $X = Y = {N_2\,N_3}/{(N_2\,N_3-2)}$.

The solution $X = Y = 1$ implies $a_1^4 = a_2^4 = a_3^4$ and as explained previously, is not of much interest.

However, if we choose $X = Y = {N_2\,N_3}/{(N_2\,N_3-2)}$, $N_3 > 2$, then we obtain a rotatable arrangement in any number of $v = N_1\,N_2\,N_3$ factors.

If we substitute $X = \dfrac{N_3\,Y}{N_3-2}$, $N_3 > 2$, in the second equation of 3.8.3, the expression becomes unmanageable and is omitted.

3.9 Concluding Remarks Concerning the Two Methods of Construction.

The methods of construction given in theorems 3.3 and 3.6 although interesting mathematically do not always lead to arrangements with a small number of experimental points.

If we call N the number of points obtained by using the methods of theorems 3.3 and 3.6 and if $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$ are the non-zero numbers of the set $\{a_1, a_2, \ldots, a_m\}$ where $a_1, a_2, \ldots, a_m$ satisfy 3.3.1 or 3.6.1 then we see that by the union method we have:

Case (i) $\qquad N = v \times 2^{n_{i_1}} + v \times 2^{n_{i_2}} + \ldots + v \times 2^{n_{i_t}}$

Case (ii) $\qquad N = v \times 2^{n_{i_1}} + v \times 2^{n_{i_2}} + \ldots + v \times 2^{n_{i_t}} + 2v$

Case (iii) $N = v \times 2^{n_{i_1}} + v \times 2^{n_{i_2}} + \dots + v \times 2^{n_{i_t}} + 2^v$

and by the method of linear combinations we have:

Case (i) $N = v \times 2^k$

Case (ii) $N = v \times 2^k + 2v$

Case (iii) $N = v \times 2^k + 2^v$

where $k = n_{i_1} + n_{i_2} + \dots + n_{i_t}$.

Hence, we see that if t is large and if n_{i_j}, $j = 1, \dots, t$, are large then N is extremely large and not of much practical value.

Therefore, it seems that any future work employing these two methods should, of course, deal with finding more arrangements from association schemes other than those already discussed and, perhaps more importantly, finding methods of reducing the number of points in the arrangements already constructed. Some of the notions used in fractional replication seem to be of some use in doing this; however, we will not discuss these methods in this thesis.

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