STRUCTURAL DYNAMICS BY FINITE ELEMENTS: MODAL AND FOURIER ANALYSIS

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SUMMARY

The increasing use of finite element discretization in the prediction of the dynamic behaviour of complex structrures is bounded by obvious restraints in the computational hardware and budget. Several approaches have been developed: the comparison is conditioned by the nature of the loads and by the structural type.

In the present work two approaches are briefly described in their implementation:

(i) modal superposition;
(ii) Fourier transform.

In the modal superposition, eigenvalues and eigenvectors are computed by direct or iterative methods: matrix deflation or variable metric minimization of the Rayleigh quotient. An appraisal is given of the computation time/eigenvector in the two cases. When the Fourier transform is applied to the equations of motion one obtains an equation in the complex domain with the frequency as a parameter.

The solution of the equation for various frequencies gives the coefficients of the Fourier series expansion of the unknowns. Applying the inverse Fourier transform one obtains the unknown solution as a discretized function of time. The Cooley-Tukey fast finite Fourier transform is used.

The numerical evaluation of the two approaches is extended to several structural types (frames, plates, shells) under different mesh partitions and loads, and is performed on the basis of the accuracy of the results, total computation time, and memory requirements.

Particular emphasis is placed upon the prediction of the dynamic behaviour of large scale structures under seismic type loads.

1. INTRODUCTION

Structural analysis in dynamic regime is inherently bound to a multiplicity of approaches. That is due partly to the diverse nature of the phisycal problems, partly to the complexity of the mathematical models, ultimately to the severe restraints of computational budgets.

Forcing forces (or accelerations) range from smooth varying loads to pulsating acceleration spectra; structural stiffnesses range from slender towers to massive nuclear power plants. The associated structural response calls for such a different "grain" in the time domain, to induce different patterns of interpretation of the dynamic response itself. At one end, the smooth forcing of slender bodies leads to understanding the dynamic behaviour in terms of modal responses; at the opposite end, seismic type loads on massive bodies lead to mapping the dynamic behaviour from the time to the frequency domain.

The leading differential equations are in the form of a mixed parabolic and hyperbolic initial value problem. Their numerical solution postulates the separability of the solution between time and the space variables. At each time (or frequency) step an elliptic boundary value problem is solved in the space distribution by the finite element method. The numerical solution of the (ordinary) differential equation in time is by far the most delicate part of the procedure, and is ofter plagued by overdamping or by instability.

The paper concisely reviews in pars. 2 and 3 the modal superposition and the finite Fourier transform methods of dynamic analysis. Time step integration was not taken into account due to the disheartening cumulations of round off errors. In par. 4 the implementation of the two methods within general purpose computer programs is discussed. Appendices A and B summarize the computational aspects of the conjugate gradient method and of the Fourier transforms. Numerical results obtained for a class of structures in the area of nuclear reactor technology are presented and critically evaluated in par. 5. The reliability of the mathematical model with respect to the physical problem is accounted for in the examples by means of experimental results, whenever available. Otherwise, the reliability of the mathematical model is postulated.

2. MODAL SUPERPOSITION METHOD

The topic is sufficiently established (see for instance Clough [1] , Zienkiewicz [11]) to deserve only a review of the leading equations.

The equations of motion of an elastic structure discretised by finite elements are in the form of the initial value problem:

$$M\ddot{\underline{u}} + C\dot{\underline{u}} + K\underline{u} = \underline{f}(t) \tag{1.1}$$

$$\underline{u}(o) = \underline{u}_o \tag{1.2}$$

$$\dot{\underline{u}}(o) = \underline{u}_o^1 \tag{1.3}$$

Where f(t) is the forcing load vector, $\underline{u}$, $\dot{\underline{u}}$, $\ddot{\underline{u}}$ are the nodal displacements, velocities and acceleratios, and M, C, K are the mass, damping and stiffness matrices respectively. In the case of seismic loads (accelerations), $\underline{f}(t)$ is a vector of initial forces due to ground accelerations, while $\underline{u}$ represent the nodal displacements relative to the ground. The modal superposition method assumes that the structural response is well approximated by a linear combination of the first eigenmodes $\underline{\Lambda}j$ of the structure (normalized with respect to the mass matrix), i.e.

$$\underline{u} = \underline{\Lambda}x \tag{2}$$

where $\underline{\Lambda} = [\underline{\Lambda}_1, \underline{\Lambda}_2, \ldots, \underline{\Lambda}_p]^T$ (3)

$$\underline{\Lambda}_j M \underline{\Lambda}_j = 1 \qquad j = 1, \ldots, p \tag{4}$$

The component x_i of $\underline{x}$ is called the partecipation factor of the i-th eigenmode. The computation of the first eigenmodes is performed here by conjugate gradient minimization of the Rayleigh quotient. The procedure is described in Appendix A.

Assuming that the damping matrix is of the form

$$C = \alpha M + \beta K \tag{5}$$

substituting eqs. (2) and (5) into eq. (1.1), multiplying the resulting equation by $\underline{\Lambda}_j$, $1 \leq j \leq p$, and accounting for the orthogonality of the eigenmodes, one obtains p decoupled ordinary differential equations in the partecipation factors x_i of the type

$$m_i \ddot{x}_i + c_i \dot{x}_i + k_i x_i = f_i(t) \qquad i = 1, \ldots, p \tag{6}$$

Eq. (6) can be solved in x_i either numerically or by Duhamel integrals. From eq. (2) one finally obtains the dynamic structural response.

3. FOURIER TRANSFORM METHOD

The essentials on Fourier transfoms are given in Appendix B.

Multiplying both sides of eq. (1) by $e^{-i\omega t}$, and integrating on time between $-\infty$ and $+\infty$, one obtains from eqs. (B.3) (B.4)

$$(-\omega^2 M + i\omega C + K)\, \underline{U} = \underline{F} \tag{7}$$

where $\underline{U}$ and $\underline{F}$ are the Fourier transforms of $\underline{u}$ and $\underline{f}$ respectively. In deriving (7) it was assumed that $\underline{f}$, $\underline{u}$ and its time derivatives tend to zero when $t \to \pm\infty$. In a more compact notation, eq. (7) becomes

$$K_c \, \underline{U} = \underline{F} \tag{8}$$

It is worth noticing that $\underline{U}$ and $\underline{F}$ are vector functions in the complex domain, and that the complex "stiffness" matrix K_c is singular when there is no damping, i.e. when C = 0. That places a restriction on the applicability of the method, even though of scarce practical significance. On the other hand the damping matrix C does not need to be in the form $C = \alpha M + \beta K$ as for modal superposition. Eq. (8) can be solved in $\underline{U}$ for a finite array of values of the frequency (*), $o \leq \omega_j < +\infty$, $j = 1 \ldots, n$. From the (digitised) espression of $\underline{U}$ (ω), one may derive the solution $\underline{U}$ (t) through the inverse Fourier transform (see eq. (B.5)).

Dynamic analysis is, in cases of practical interest, confined to a finite time interval, $[0, t_m]$. Thus, repeating the above procedure now for the finite Fourier transform, one obtains in analogy to eq. (8)

$$K_c \, \underline{U}_f = \underline{F}_f \tag{9}$$

where K_c is the same as in eq. (8), $\underline{U}_f$ is the finite Fourier transform of the solution $\underline{u}$ (t), while

$$\underline{F}_f = \underline{F}_f(\underline{f}) + M(\underline{u}_o' - \underline{u}_m' \, e^{-i\omega t_m}) + (i\omega M + C)\,(\underline{u}_o - \underline{u}_m \, e^{-i\omega t_m}) \tag{10}$$

The subscripts o and m refer to the values of the transform for $t=0$ and $t=t_m$.

Digitizing $\underline{u}(t)$ and $\underline{f}(t)$ at N evenly spaced time points, and substituting in eq. (9) finite transforms by discrete finite transforms, the approximate solution of eq. (9) is obtained by solving N/2 + 1 times a system of (complex) linear equations in $\underline{U}_d$ (p+1), p=0, 1, ..., N/2. Here $\underline{U}_d$ (p+1) is the finite discrete transform of the solution $\underline{u}(t)$, i.e

$$\underline{U}_d(p+1) = \sum_{q=0}^{N-1} \underline{u}(q+1) \; e^{-2\pi i \frac{pq}{N}}, \quad p = 0,1\ldots, N-1 \tag{11}$$

A similar expression applies to $\underline{F}_d$ (p+1).

Finally, applying to $\underline{U}_d$ (p+1) the inverse discrete transform (see eq.(B.12)) one obtains

$$\underline{u}\,(q+1) = \sum_{p=o}^{N-1} \underline{U}_d\,(p+1) e^{2\pi i \frac{pq}{N}}, \quad q = 0, 1, \ldots, N-1 \tag{12}$$

(*) The scanning in ω is confined to positive values, because $\underline{u}$ (t) is a vector of real functions (see eq. (B.2).

that represents $\frac{2\pi}{\Delta t \, \Delta\omega}$ $\underline{u}(t)$ $=$ N $\underline{u}(t)$, i.e the (displacements) dynamic response, digitised at discrete time intervals, $t = q\ \Delta t = q\ t_m/N$. Notice that, from the physical interpretation of the individual terms of the discrete finite Fourier transform, the module and the argument of $\underline{U}_d$ (p+1) represent the amplitude and the phase shift of the response of the structure to a periodic force $\underline{f}(t)$ with period $2\pi p/t_m$. The boundary conditions here are given, the initial and final conditions are included in the right-end side term $\underline{F}_d$ (p+1) (see eq. (10)).

Let a discretised structure be subjected to an impulsive load vector $\underline{f}_i(t)$, where all the active components have unitary values; such load is referred to here as the i-th unit impulse. The associated discrete finite transform is such that $\underline{F}_{di}$ (p+1) = 1, p = o, ... , $\frac{N}{2}$ (see eq. (B.13)). The discrete finite transform of the structural response confined to the same active components, and with zero initial and final conditions, is the "frequency response" of the structure associated with the i-th unit impulse, and is indicated by $\underline{T}_{di}$. If $\underline{F}_d$ is the discrete finite transform of a vector similar to $\underline{f}_i$ and with synchronous components under the associated initial and final conditions

$$\underline{U}_d\ (p+1) = T_d\ (p+1)\ \underline{F}_d\ (p+1) \tag{13}$$

where T_d is a diagonal matrix with $\underline{T}_{di}$ as non zero diagonal coefficients. In analogy with the theory of servomechanisms (see for instance Gille et al. [8]), $\underline{T}_{di}$ is called the "transfer function" associated with the active components. The transfer function completely defines the frequency response of the structure: the module and the argument of the complex coefficients in $\underline{T}_{di}$ (ω) represent the amplitude and the phase shift of the structural response to a unit sinusoidal load of frequency ω. From the diagrams in ω of magnitudes and phase shift (plotted with decrasing damping) one spots easily the natural frequencies of the structure.

4. COMPUTATIONAL REMARKS

Both the modal superposition and the Fourier transform method were implemented as general purpose computer programs, and were tested on a set of preliminary examples.

The two procedures, as against numerical time-step integration of the differential equations, need to assemble, for each structure a substantial amount of information on "general performance", such as the eigenmodes or the transfer functions. Most of the computational effort falls into this phase; at little extra cost, then, one computes the response to a number of dynamic loads. A direct, one-to-one comparison of computer times, as performed in static analysis, would make little sense. Evaluations of relative merits of the two approaches are confined by the structural type, the nature of the loads and by the computer configuration at hand.

Here, the most relevant aspects of the implementation are sketched, using third generation computers (such as the IBM 370/155, 1000 K memory, at CNUCE in Pisa) as reference.

Modal superposition

The computation of eigenmodes does not require the assembly of the global stiffness and mass matrices, but rather the serial processing of strings of element matrices (see Geradin [7]). The emphasis is not so on core capacity as in the availability of fast access I/O devices. Both the eigenmodes and the time history of the associated partecipation factors are stored in secondary memory. The modal superposition is performed one mode at a time for all time steps, thus calling for serial rather than random access processing.

The computation time is tied primarily to the rate of convergence of the conjugate gradient algorithm, that in turn depends on the degree of conditioning of the eigenvalue problem. Well conditioned cases lead to very fast convergence; on the other hand, none of the corrective techiques used so far seem to cope in a satisfactory manner with ill-conditioned structures.

Fourier transforms

The procedure requires the scanning of the frequency spectrum of the forcing loads; few steps are sufficient for most of the elementary loads, more is required to analyze a seismic accelerogram. For every frequency, the discrete finite transform of the load is evaluated, the complex "stiffness" matrix K_c is assembled, and the discrete finite transform of the response is computed by solving a (complex) set of linear equations, having the size of the degrees of freedom of the structure. Finally an inverse discrete transform is performed to obtain the response in time.

While discrete transforms are quite fast, the major demand of core memory and time is provided by the solution of complex sets of linear equations. A through use of the transfer functions may lead to substantial savings for each cycle; so does an "educated" reduction of the frequency spectrum to confine the number of steps. As far as the interaction with secondary memory notice that the combination of large bulks of memory, software for "capacity" shifts of data, and multiprocessing mode of operation lowers the computation time for large problems to the all-in-core limits.

5. EXAMPLES

The programs were tested on a set of simple examples. For some of the test problems experimental results were available.

Figure 1 shows the response of a simply supported fuel bar to a finite step load. The bar is composed of an inner cylinder of pellets of uranium oxide of diameter ϕ_i = 1,318 cm. encased in a pipe of zircalloy with eternal diameter ϕ_e = 1,506 cm, and Young's modulus E = 9.637 10^5 Kg/cm^2. The average density of the fuel bar is assumed to be ρ_m = 9.06 10^{-6} Kg sec^2/cm with the damping matrix proportional to the mass matrix: C = 1.288 M.

The graph shows a fair correspondence between the Fourier and the modal (6 eigenvectors) responses.

Figure 2 shows the set up of a batch of 16 fuel bars for experimental test at the Istituto di Impianti Nucleari of the University of Pisa. The lowest frequencies, as computed by the conjugate gradient, are compared with experimental data in Tab. I.

Tab. I

Frequencies of fuel bars assembly

n°	Experimental	Gradient
1	3.8	3.5
2	7.7	7.7

Figure 3 shows the frequency by Fourier analysis to sinusoidal loads, applied to node 4 having maximum modal deflection in the first mode.

Closing in to the first eigenvalue, and decreasing damping, the response blows to resonance. Such scanning technique can be used to compute eigenvalues by the Fourier transforms. Figure 4 shows the response of a shaking box (see [9] [10]) with concentrated mass in the middle of the upper layer under a finite step ground acceleration. Here a phase shift appears between the two responses. The limited number of modes (7 eigenvectors) and the ill-conditioning induced by the concentrated mass in the computation of the eigenvectors are responsible for the less satisfactory modal response.

Figure 5 shows a pipeline in a nuclear power plant, its discretization into bar elements, and the resulting lowest three modes. The diameter of the pipe is 60 cm., the wall thickness 2.3 cm and Young's modulus E = 2.1 10^{-6} Kg/cm^2. Figure 6 shows the response of the pipeline to a ground acceleration spectrum reduced from El Centro earthquake to reproduce the highest frequencies.

The accelerations are shown by the set of black dots. In order to enhance the graphical representation, a somewhat high damping coefficient was used: $C = 50\ M$.

6. CONCLUSIONS

Both modal and Fourier analysis show specific computational advantages depending on the type of the structure and on the loads applied.

The computation of eigenvectors, being independent from the loads, provides the basis to compute the dynamic response of the structure under any forcing load or acceleration. Its major limitation is in requiring that the damping matrix be of the from $C = \alpha M + \beta K$. The efficency of the numerical procedure depends on the number of modes requested and on the accuracy associated with their computation. The minimization of Rayleigh quotient by the conjugate gradient method is a simple, convenient procedure for the first modes. When the lowest modes are not sufficient to represent the dynamic response one may resort to condensation and to matrix deflation techniques.

The analysis by finite Fourier transform is best suited to represent the frequency response of a structure under a given load or acceleration. No restriction is placed on the form of the damping matrix.

The procedure can be used, through frequency scanning, to determine frequencies of resonance; here there is direct, very useful relationship with the experimental techniques that computes eigenvectors. In the case of a structure subjected to multiple dynamic loads of different types, a large number of computations would be required.

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APPENDIX A - Eigenmode computation by the conjugate gradient method.

The j-th eigenvalue associated with a structure discretized into finite elements (displacement method) can be computed by minimizing the Rayleigh gradient

$$\lambda_j = \min_{\underline{u}} \frac{(\underline{u},\, K\, \underline{u})}{(\underline{u},\, M\, \underline{u})} \tag{A.1}$$

under the restriction that the vector of (generalized) displacements $\underline{u}$ be M ortogonal to the lowest j-1 eigenmodes $\underline{\Lambda}_i$, i.e.

$$(\underline{u},\, M\, \underline{\Lambda}_i) = 0 \qquad i = 1,\ldots, j-1 \tag{A.2}$$

Here K and M are respectively the stiffness and the mass matrix associated with the finite element partition of the structure. The minimizing vector $\underline{u}$ in (A.1) is the j-th eigenmode $\underline{\Lambda}_j$.

A well tested computational procedure (see Fox [4] , Fried, [5] , Geradin [7]) solves the nonlinear programming problem (A.1) by the conjugate gradient method integrated by gradient projection techniques to account for the linear restraints (A.2).

The determination of the first eigenvalue follows an iterative procedure computing the local value at $\underline{u}_i$ of the Rayleigh quotient and of the gradient of the objective function

$$r_i = \frac{(\underline{u}_i,\, K\, \underline{u}_i)}{(\underline{u}_i,\, M\, \underline{u}_i)} \tag{A.3}$$

$$g_i = \frac{2(K\, \underline{u}_i - r_i\, M\, \underline{u}_i)}{(\underline{u}_i,\, M\, \underline{u}_i)} \tag{A.4}$$

and also generating a set of K-conjugate directions $\underline{s}_j$, i.e.

$$(\underline{s}_j,\, K\, \underline{s}_k) = 0 \qquad j \neq k \tag{A.5}$$

The steps of the procedure are the following:

1) Starting from a (random) $\underline{u}_o$ compute r_o and g_o from (A.3), (A.4), and set the first conjugate direction as $\underline{s}_o = -\underline{g}_o$.
2) Move in the direction of the conjugate direction $\underline{s}_i$

$$\underline{u}_{i+1} = \underline{u}_i + \alpha_i\, \underline{s}_i \tag{A.6}$$

The step length α_i that minimizes the Rayleigh quotient along $\underline{s}_i$ is given from the solution of the second order equation

$$a\alpha_i^2 + b\alpha_i + c = 0 \tag{A.7}$$

where

$$a = (\underline{s}_i, K\underline{s}_i)(\underline{x}_i, M\underline{s}_i) - (\underline{x}_i, M\underline{s}_i)(\underline{s}_i, M\underline{s}_i)$$

$$b = (\underline{s}_i, K\underline{s}_i)(\underline{x}_i, M\underline{x}_i) - (\underline{x}_i, K\underline{x}_i)(s_i, Ms_i)$$

$$c = (\underline{x}_i, K\underline{s}_i)(\underline{x}_i, M\underline{x}_i) - (\underline{x}_i, K\underline{x}_i)(\underline{x}_i, M\underline{s}_i)$$

3) Determine the next conjugate direction as

$$\underline{s}_{i+1} = -\underline{g}_{i+1} + \beta_i \underline{s}_i \tag{A.8}$$

where

$$\beta_i = \frac{||\underline{g}_{i+1}||^2}{||\underline{g}_i||^2} \tag{A.9}$$

Geradin [7] proposes a second order expression for β_i.

4) Check the convergence criteria

$$\frac{|r_{i+1} - r_i|}{r_i} < \varepsilon \tag{A.10}$$

$$||g_{i+1}||^2 < \eta \tag{A.11}$$

When both are fulfilled stop. Otherwise return to 2).

In the computation of the j-th eigenvalue the gradient $\underline{g}$ must be orthogonal to the lowest j-l eigenmodes. That is accomplished by premultiplying $\underline{g}$ by a projection matrix, i.e.

$$\underline{g}_p = [\, I - Q(Q^T Q)^{-1} Q^T \,]\underline{g} \tag{A.12}$$

where

$$Q \equiv [\, M\underline{\Lambda}_1, M\underline{\Lambda}_2, \ldots, M\underline{\Lambda}_{j-1} \,] \tag{A.13}$$

APPENDIX B - Fourier transforms

Given a function g(t) such that $\lim_{t\to\pm\infty} g(t)=0$, its Fourier transform F(g) is defined as

$$F(g) = G(\omega) = \int_{-\infty}^{+\infty} g(t)e^{-i\omega t}\, dt \tag{B. 1}$$

where $i = \sqrt{-1}$. For g(t) real

$$G(\omega) = G^*(-\omega) \tag{B. 2}$$

where * stays for complex conjugate.

The Fourier transforms of derivatives of g(t) are

$$F(g') = i\omega G \tag{B. 3}$$

$$F(g'') = -\omega^2 G \tag{B.4}$$

Given $G(\omega)$, the function $g(t)$ is obtained through the inverse Fourier transform

$$g(t) = F^{-1}(G) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(\omega)\, e^{i\omega t}\, d\omega \tag{B.5}$$

On a finite interval $0, t_m$ one defines the finite Fourier transform for $g(t)$, $o\ t\ t_m$, as

$$F_f(g) = G_f(\omega) = \int_o^{t_m} g(t) e^{-i\omega t}\, dt \tag{B.6}$$

The corresponding expressions for the derivatives of $g(t)$ are

$$F_f(g') = -g_o + g_m\, e^{-i\omega t_m} + i\omega G_f \tag{B.7}$$

$$F_f(g'') = -g_o + g'_m\, e^{-i\omega t_m} + i\omega g_o + i\omega g_m\, e^{-i\omega t_m} - \omega^2 G_f \tag{B.8}$$

where the subscripts and m indicate the value of the finite Fourier transform at $t=o$ and $t=t_m$.

Notice that for $\omega = \frac{2\pi}{t_m} p$, $p = 0, 1, 2 \ldots$, one has $e^{-i\omega t_m} = 1$; in addition the real and the immaginary parts of $G_f(\omega)$, multiplied by $\frac{2}{t_m}$, represent respectively the cosine and the sine coefficients of the p-th term in the Fourier series expansion of $g(t)$, considering $g(t)$ as a periodic function of period t_m. When the function $g(t)$ is given in digitized form at N evenly spaced time points, $\Delta t = t_m/N$, one defines the finite discrete Fourier transform as

$$F_d(g) = G_d(p+1) = \sum_{q=o}^{N-1} g(q+1)\, e^{-2\pi i \frac{pq}{N}}, \quad p = 0, 1, \ldots, N-1 \tag{B.9}$$

where $g(p+1)$ represents $g(t)$ at the time $t=q\Delta t=q\,\frac{t_m}{N}$ and its inverse as

$$g(q+1) = \sum_{p=o}^{N-1} G_d(p+1)\, e^{2\pi i \frac{pq}{N}}, \quad q = 0, 1, \ldots, N-1 \tag{B.10}$$

when N is a power of two, i.e. $N = 2^m$, the discrete transform $F_d(g)$ can be easily computed by a number of algorithms (see for instance, Cooley [1] , Gentleman [6] called fast Fourier transforms. Here $G_d(p+1)$ approximates the finite transform, i.e.

$$G_d(p+1) \approx \frac{G_f(\omega)}{\Delta t} = \frac{G_f(\omega)}{t_m} N \tag{B.11}$$

when $\omega = p\Delta\omega = \frac{2\pi p}{t_m}$. Thus, except for the multiplicative factor $2/N$, $G_d(p+1)$ approximates the sine and cosine coefficient of the p-th term in the Fourier series expansion of $g(t)$, considering the function $g(t)$ periodic of period t_m. When $g(t)$ is a real valued function

$$G_d(p+1) = G_d^*(N-p-1) \qquad (B.12)$$

For real functions, then, the effective computation of discrete transforms is confined to values of $p = 0, 1, \ldots, N/2$. Notice that the case when $g(t)$ is unitary initial pulse, i.e. $g(1) = 1$, $g(k) = 0$, $K = 2, \ldots, N$ then

$$G_d(p+1) = 1 \qquad P = 0, 1, \ldots, N/2 \qquad (B.13)$$

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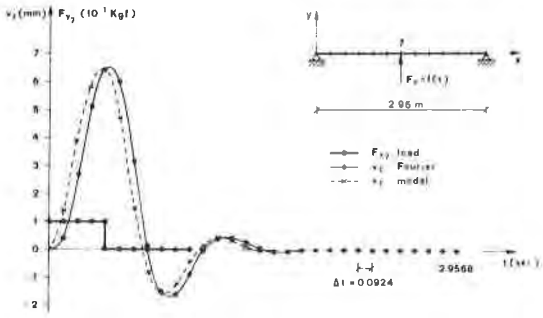

Fig. 1 - Fuel bar under a finite step load.

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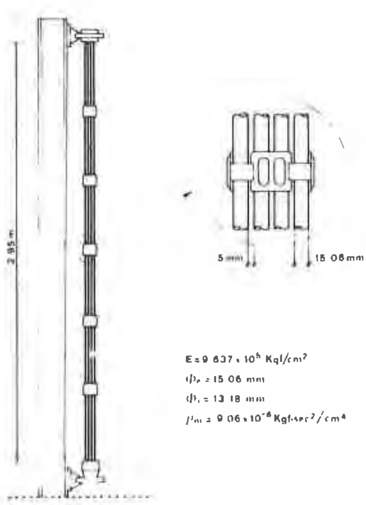

Fig. 2 - Assembly of 16 fuel bars

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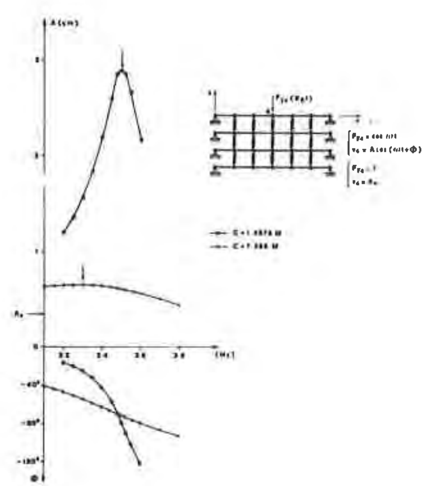

Fig. 3 - Frequency response to cosinusoidal loads

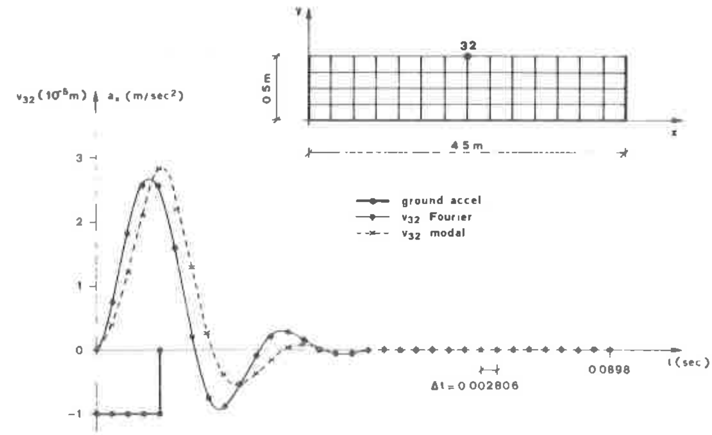

Fig. 4 - Shaking box under a finite step ground acceleration

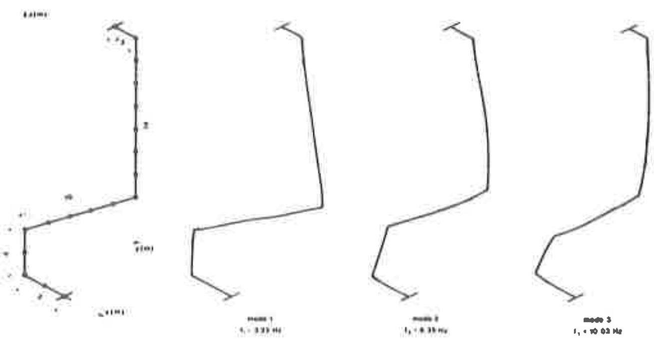

Fig. 5 - Modes of vibration of a pipeline

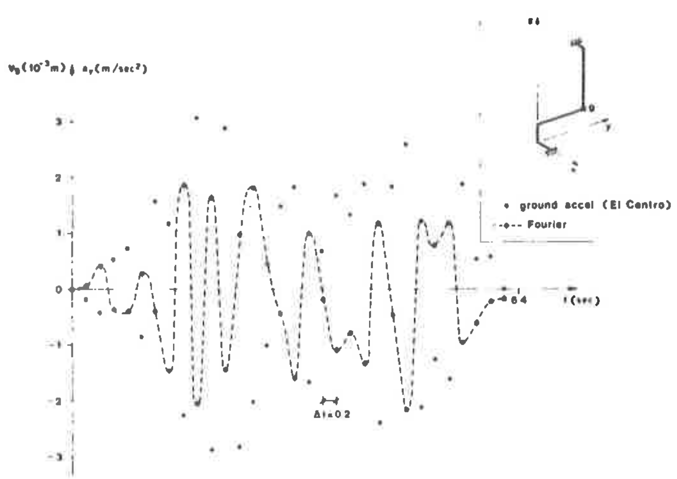

Fig. 6 - Pipeline under seismic acceleration