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NONLINEAR STATIC ANALYSIS OF ELASTIC STRUCTURES UNDER GENERAL THERMAL BOUNDARY CONDITIONS

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ABSTRACT :

Nonlinear static analysis of orthotropic elastic structures determining large deflections under general thermal boundary conditions has been investigated.

1. INTRODUCTION :

In aeronautics and in nuclear power reactors many problems arise in which thermo-elastic deformation and stresses of elastic structures play an important role. Such problems are mainly restricted to isotropic cases [1 - 6]. Owing to the increased usage of macroscopically anisotropic construction materials in severe thermal environment, there is an obvious need for extensive research. Based on linearised theory thermal stresses in anisotropic elastic slab have been determined by Tauchert and Akoz [7] and a brief survey of relevant studies has been given in [8].

In this paper large thermal deflections of orthotropic elastic structure have been studied using the von Karman's coupled nonlinear partial differential equations derived in terms of normal displacement component and stress function for the orthotropic case. Considering general thermal boundary conditions the stress function for a rectangular plate with mixed boundary conditions has been determined. The thermal boundary conditions permit specification of surface temperature, heat input, thermal inclusion at each point on the boundary. Finally the Galerkin procedure has been applied leading to a cubic equation from which a load-deflection analysis can be made. Membrane stresses can conveniently be determined from the present analysis.

2. Basic Governing Equations:

Considering the equations of equilibrium for thermally-stressed orthotropic plates the following equations (in the von Karman sense) can be derived:

$$F_{,xxxx} + p^2 F_{,xxyy} + q^2 F_{,yyyy} + \lambda_1 N_{T,xx} + \lambda_2 N_{T,yy} =$$

$$= E_2 h \left(w_{,xy}^{\,2} - w_{,xx} w_{,yy} \right) \qquad (1)$$

$$D_x\, w,_{xxxx} + 2H\, w,_{xxyy} + D_y\, w,_{yyyy} + \beta_1 M_T,_{xx} + \beta_2 M_T,_{yy} =$$

$$= F,_{xx}\, w,_{yy} - 2F,_{xy}\, w,_{xy} + F,_{yy}\, w,_{xx} \quad \ldots(2)$$

where $F(x,y)$ is the stress function, $w(x,y)$ is the normal displacement, D_x , D_y , H are elastic constants and other notations are given in the Appendix-I at the end of the paper.

3. Solution for a Rectangular Plate:

We consider a rectangular plate simply-supported and clamped along the edges and occuping the space :

$$0 \leq x \leq a \;; \quad 0 \leq y \leq b \;; \quad -h/2 \leq z \leq h/2$$

The plate is subjected to a steady temperature distribution $T = T(x,y,z)$ satisfying the Fourier heat-conduction equation :

$$K_1\, T,_{xx} + K_2\, T,_{yy} + K_3\, T,_{zz} = 0 \quad \ldots(3)$$

We assume $T(x,y,z)$ in the following form such that $N_T = 0$:

$$T(x,y,z) = \sum_{m=1,3,..}^{\infty} \sin(2m\pi z/h)\, T'(x,y) \quad \ldots(4)$$

M_T is given by

$$M_T = \sum_{m=1,3,..}^{\infty} (h^2/2m)\, T'(x,y) \quad \ldots(5)$$

$T'(x,y)$ is obtained as a general form of solution from (3) and (4) by the method of separation of variables as :

$$T'(x,y) = \sum_{s=\lambda_i}^{\infty} \Big[A_1(s) \sin sx \sinh(L_m y/K) + A_2(s) \sin sx \,.$$

$$\sinh(L_m y/K) + A_3(s) \cos sx \sinh(L_m y/K) + A_4(s) \cos sx \cosh(L_m y/K) \Big]$$

where

$$K = K_2/K_1 \quad \text{and} \quad \ldots\ldots(6)$$

$$L_m = (s^2 + 4(K_3/K_1) m^2 \pi^2 / h^2)^{1/2}$$

$A_j(s)$ are to be determined from the thermal boundary conditions along the four edges of the plate which are expressed as :

$$-l_1\, T',_x + m_1\, T' = 0 \text{ on } x = 0 \;;\; l_2 T',_x + m_2\, T' = 0 \text{ on } x = a;$$

$$-l_3\, T',_y + m_3\, T' = 0 \text{ on } y = 0 \;;\; l_4\, T',_y + m_4\, T' = f(x) \text{ on } y = b \quad \ldots\ldots(7)$$

Through appropriate selection of the constants l_j and m_j ($j = 1,2,3,4$) in these equations (7) various types of ordinary boundary conditions(prescribed temperature , heat input, thermal inclusion etc.) can be obtained.

The solution of these equations for $A_j(s)$ is simplified at this stage if attention is restricted to a specific set of boundary conditions for the three edges $x = 0, a \;;\; y = 0$.

As an example we shall consider the problem of zero temperature rise on three edges, in which case

$$l_1 = l_2 = l_3 = 0 \;; \quad m_1 = m_2 = m_3 = 1 \,.$$

Substituting this choice of l_j and m_j in the first three equns.

of (7) one gets $A_1(s) \sin sa = A_2(s) = A_3(s) = A_4(s) = 0$.
For a non-trival solution $\sin sa = 0$ from which one gets

$$s_n a = n\pi , \quad (n = 1,2,3, \ldots) . \qquad \ldots \quad \ldots \quad (8)$$

From the fourth equation of (7) one gets :

$$\sum_{s=s_1}^{\infty} \left[l_4 L_m/K \cosh L_m b/K + m_4 \sinh L_m b/K \right] A_1(s) \sin sy = f(x) \qquad \ldots\ldots(9)$$

Assuming the distribution of f(x) in Fourier sine series one gets

$$T'(x,y) = \sum_{n=1,2,\ldots}^{\infty} \frac{\left[\int_0^a f(x) \sin(n\pi x/a)\, dx \right] \sin(n\pi x/a) \sinh(L_m y/K)}{l_4\, n\pi /aK \cdot \cosh(L_m b/K) + m_4 \sinh(L_m b/K)} \qquad \ldots(10)$$

Accordingly, M_T is obtained from equation (5).
The deflection w(x,y) satisfying clamped and simply-supported boundary conditions is assumed in the form:

$$w(x,y) = w_0 \sin(\pi x/a) \, (1- \cos (2\pi y / b)) \qquad \ldots (11)$$

Clearly this form of w(x,y) satisfies the S.S. boundary conditions :

$$w = 0 = - (D_x\, w,_{xx} + D_1\, w,_{yy}) + \beta_1 M_T \quad \text{at} \quad x = 0,\ a$$

and the clamped-edge conditions $w = 0 = w,_y$ at $y = 0,b$ ..(12)
The general solution of equation(1) after substituting (11) is obtained in the form :

$$F = \frac{1}{2} Ax^2 + \frac{1}{2} By^2 + \frac{1}{8} E_2 h\, w_0^2 (a/b)^2 \Big[\cos(2\pi x/a) + \frac{(b/a)^4}{q^2} \cos\frac{2\pi y}{b}$$

$$- (b/a)^4/ q^2 \cos(4\pi y/b) - \cos(2\pi x/a)\cos(2\pi y/b) / C \Big] \qquad \ldots(13)$$

where A and B are arbitrary constants to be determined from in-plane boundary conditions and $C = 1 + (a/b)^2 p^2 + (a/b)^4 q^2$.

In accordance with the conditions occuring in airplane structures the plate is considered rigidly framed, all edges thus remaining straight after deformation. Then the elongations of the plate in the directions of x and y are independent of y and x respectively [9] and one can get

$$u = \int_0^a \left[S_{11} F,_{yy}/h + S_{12}\, F,_{xx}/h - \frac{1}{2} w,_x^{\,2} \right]_{y=0} dx \qquad \ldots(14)$$

$$v = \int_0^a \left[S_{22} F,_{xx}/h + S_{12}\, F,_{yy}/h - \frac{1}{2} w,_y^{\,2} \right]_{x=0} dy \qquad \ldots(15)$$

For immovable edges of the plate $u = 0 = v$ [9] ...(16)

From equations (14) and (15) and considering (16) one gets the Constants A and B .

Putting the expressions for w(x,y), F(x,y) and M_T into equation (2) and applying Galerkin Procedure one gets, after a lengthy but simple calculation, the cubic equation involving the deflection function and temperature parameter, in the form :

$$(w_0/h)^3 \Big[4/(1-\nu_1\nu_2) + 3(b/a)^2\,\nu_2/q^2(1-\nu_1\nu_2) + 9(b/a)^4/4q^2(1-\nu_1\nu_2) + 3(b/a)^2\nu_1/(1-\nu_1\nu_2) + 2 + 17(b/a)^4/8q^2 + (b/a)^4/\left[(b/a)^4 + p^2(b/a)^2 + q^2\right] \Big] +$$

$$(w_0/h)\Big[(E_1/E_2)(b/a)/2(1-\nu_1\nu_2) + \frac{4}{3}\left(\frac{b}{a}\right)^2\left(\frac{\nu_2 E_1/E_2}{1-\nu_1\nu_2} + \frac{2G}{E_2}\right) + \frac{1}{6(1-\nu_1\nu_2)}\Big] - \frac{4\left(\frac{b}{a}\right)^2\left(\frac{a}{h}\right)^2}{\pi^5}\sum_{m=1,3..}^{\infty}\sum_{n=1,2,..}^{\infty}\frac{\frac{1}{a}\int_0^a f(x)\sin(n\pi x/a)\,dx}{m(l_4 L_m/K\cosh L_m b/K + m_4\sinh\frac{L_m b}{K})}\ \times$$

$$\times \left\{ n^2\pi^2(b/a)^2\,\beta_1/E_2 - L_m b^2/K^2\cdot\beta_2/E_2 \right\} \times$$

$$\times\left\{ \frac{K}{L_m b}(\cosh L_m b/K - 1) - \frac{L_m b(\cosh L_m b/K - 1)}{4\pi^2K^2 + b^2L_m^2} \right\} = 0 \quad \ldots(17)$$

4. Some Numerical Results:

For different aspect ratio (b/a) the variations of non-dimensional deflection (w_0/h) for different values of temperature parameter $T_0\,\beta_1/E_2$ have been presented in the following table :

Table:

w_0/h	0	0.1	0.2	0.3	0.4	0.5	0.6
$T_0\beta_1/E_2$ $(b/a)=1$	0	0.250	0.608	1.398	2.106	3.466	5.382
$T_0\beta_1/E_2$ $(b/a)=.5$	0	0.025	0.046	0.068	0.098	0.156	0.204
$T_0\beta_1/E_2$ $(b/a)=1/3$	0	0.012	0.018	0.028	0.042	0.061	0.097

For the above numerical results the following data have been used $E_1 = 1\times10^5$, $E_2 = 0.05\times10^5$, $\nu_1 = 0.2$, $\nu_2 = 0.01$, $a/h = 10$, $G = 0.05\times10^5$, $K = 0.5$, $\beta_2/\beta_1 = 0.05$, $f(x) = T_0$,

5. Membrane Stresses:

Since $F_{,yy} = N_x$ and $F_{,xx} = N_y$, therefore using equation (13), one gets the membrane stresses $(N_x)_{max}$ and $(N_y)_{max}$ which occurs at the point $x = a/2$, $y = b/2$ and can be computed quite conveniently.

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APPENDIX- I

The following notations and expressions have been used in this paper :

$$\beta_1 = (\alpha_1 S_{22} - \alpha_2 S_{11})/\nabla \quad , \quad \beta_2 = (\alpha_2 S_{11} - \alpha_1 S_{12})/\nabla$$

$$\nabla = S_{11} S_{22} - S_{12}^2 \quad ,$$ S_{ij} are elastic constants,

$$N_T = \int_{-h/2}^{h/2} T(x,y,z)dz \quad , \quad M_T = \int_{-h/2}^{h/2} zT(x,y,z)dz,$$ h = thickness ,

$$p^2 = (S_{66} + 2S_{12})/S_{22} = E_2/G - 2\nu_2 \, , \quad q^2 = S_{11}/S_{22} = E_2/E_1 ,$$

$$\lambda_1 = (S_{11}\beta_1 + S_{12}\beta_2)/S_{22} = E_2/E_1\,\beta_1 - \nu_2\beta_2 \, ,$$

$$\lambda_2 = (S_{12}\beta_1 + S_{22}\beta_2)/S_{22} = \beta_2 - \nu_2\beta_1 \, , \quad E_1 = 1/S_{11} ,$$

$$E_2 = 1/S_{22}, \quad S_{12}/S_{11} = -\nu_1 \, , \quad S_{12}/S_{22} = -\nu_2 \, , \quad G = 1/S_{66} ,$$

K_1, K_2, K_3 = thermal conductivities, $K = K_2/K_1$, $K_3/K_1 = 1$
ν_1, ν_2 = normal strains, ν_{12} = shear strains, m,n-integers.