ABSTRACT

NGUYEN, KENNY HUY. Investigating the Role of Equipartitioning and Creating Internal Units in the Construction of a Learning Trajectory for Length and Area. (Under the direction of Dr. Jere Confrey.)

Under current learning trajectories for length and area, researchers examine how students coordinate and systematize measurement using a given external iterating unit. This study investigates the possible cognitive behaviors missing in that approach by giving students the larger task of comparing areas to see how they construct internal units based on related ideas of equipartitioning (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). It investigates three case studies from a teaching experiment with four third-grade students and examines the strategies, mathematical reasoning practices, and emergent relations and properties that they generated. The results were synthesized into a ten-level learning trajectory: (1) Identification, (2) Conservation of Rigid Transformations, (3) Direct Comparison of Length and Area, (4) Conservation of Decomposition/Composition over Breaking, (5) Indirect Comparison of Length, (6) Non-congruent Rectangles Can Have Equal Areas, (7) Measurement of Area Is Relative to Unit Size, (8) Qualitative Compensation of Area Units, (9) Measurement with Area Units, and (10) Ratio Relationship Between Different Sized Area Units. The research suggests that this learning trajectory can be used in the elementary school curriculum to give students an internal measurement structure that allows for switching between different area units and to concomitantly introduce fractions and ratio, and set the foundation for quantitative area measurement.

Investigating the Role of Equipartitioning and Creating Internal Units in the Construction of a Learning Trajectory for Length and Area

by
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BIOGRAPHY

Kenny Huy Nguyen was born on August 10, 1979 in Hue, Vietnam. He immigrated to the United States in 1983 and grew up in Portland, Oregon graduating in 1997 from Woodrow Wilson High School. He attended the University of Chicago and became passionate about urban education and social justice while working as a technology intern at Kozminski Community Academy during his third and fourth years of college. He graduated in June 2001 with a Bachelor of Science degree in mathematics. Upon graduation, he worked as Technology Assistant for the Joyce Foundation and helped the foundation upgrade its technology infrastructure and grants database. During this time, he continued pursuing his interest in education by conducting research for the Foundation's education program and teaching summer school.

Kenny returned to school in August 2004 to pursue a Master of Arts degree in Learning Technologies at the University of Michigan, graduating in August 2005. He joined the GISMO (Generating Increased Science and Mathematics Opportunities) research team at Washington University in St. Louis and began full time doctoral studies there in the fall of 2006. He moved with GISMO to North Carolina State University in the fall of 2007. In September 2010, Kenny accepted a position as Research Associate at the Friday Institute for Educational Innovation to research the impact of mobile devices on formative and diagnostic assessment.

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CHAPTER 1

INTRODUCTION

The DELTA (Diagnostic E-Learning Trajectories Approach) research group, led by Dr. Jere Confrey and Dr. Alan Maloney, are a group of researchers[1] at North Carolina State University engaged in a program to construct learning trajectories and diagnostic assessments for six domains in rational number reasoning: equipartitioning; division and multiplication; length, area, and volume;[2] fractions, ratio, proportion, and rate; decimal and percent; and similarity and scaling. To assist with this complex work, the group created a methodology for constructing learning trajectories (Confrey, Maloney, Wilson, & Nguyen, 2010; Maloney & Confrey, 2010). The five steps in this methodology are:

1. Synthesize the research literature in a domain.
2. Conduct clinical interviews and teaching experiments to supplement the literature.
3. Describe a first version of the proficiency levels and outcome spaces[3] for the trajectory.
4. Engage in a cycle of item development, model fitting, and validation using methods both qualitative—e.g., evidence from clinical interviews and teaching experiments—

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[1] Former and current members of the DELTA research group include Dr. P. Holt Wilson, Dr. Gemma Mojica, Marrielle Myers, Cyndi Edgington, Ryan Pescosolido, Ayanna Franklin, Zuhal Yilmaz, and Nadia Monrose. The pronoun "we" and references to the research group acknowledge that this work is conducted in collaboration with other members of the team and/or is built on Dr. Confrey's work on splitting and equipartitioning.

[2] We are choosing to initially restrict this trajectory to length and area and extend to volume at a later time.

[3] A delineation of the different types of outcomes ordered from lower to higher sophistication.

and quantitative—e.g., analysis of student responses on field test items using item response theory (Confrey, Maloney, Pescosolido, & Rupp, in progress; Confrey et al., 2010), descriptive statistics (Pescosolido, 2010), and classical test theory.

5. Build learning progress profiles (Confrey & Maloney, 2010) and resources based on the validated trajectory.

This dissertation study is intended to inform the first three steps of the DELTA research program in the construction of a learning trajectory for length and area.

We documented three approaches to studying measurement in the literature.

1. Using a given external unit to coordinate measurement. In the case of length, students are usually given either a unit length (such as a one-inch straw) to iterate with no gaps or overlaps (Battista, 2007), or a longer length with unit lengths clearly marked or "notched" (Barrett, Clements, Klanderman, Pennisi, & Polaki, 2006). In the case of area, students are usually given a unit square (such as a 1x1 tile) to tile with no gaps or overlaps (Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred & Mitchelmore, 2000).
2. Creating an internal unit from the object to be measured or compared and using that unit as a basis for measurement (Lehrer, Jenkins, & Osana, 1998; Strom, Kemeny, Lehrer, & Forman, 2001).
3. Coordinating measurement and algebraic coding (Davydov & Tsvetkovich, 1991; Dougherty, 2007, 2010).

There are four potential problems with limiting instruction to solely the external unit approach. First, Kamii and Kysh (2006) argue that even by seventh grade, students do not think of a square unit as the unit of area and infer, “If seventh graders do not understand units of length as we saw earlier, they cannot be expected to understand ‘length x width’” (p. 113). Second, the tasks that accompany this approach always involve measuring a length or an area that can be covered by a whole number of units. Indeed, this may explain Kamii and Kysh’s results, because students never encounter problems where the unit does not fit exactly. If students were presented with inexact lengths or areas, they would be challenged to equipartition the unit and wrestle with how to name the fractional result. Seminal work by Davydov (Davydov, 1991/1969; Davydov & Tsvetkovich, 1991) and recent work by Dougherty (Dougherty, 2010; Dougherty & Venenciano, 2007) suggests that elementary students are capable of solving these problems and developing fractions from measurement. Third, we may be missing the chance for students to develop understanding between the unit and the whole and to learn how to switch between different-size units. Fourth, the DELTA team’s research on equipartitioning suggests that this could be a potential foundation on which to build length and area measurement.

From *equipartitioning*, the cognitive behaviors that have the goal of producing equal-size groups (from collection) or pieces (from continuous wholes) as *fair shares* for each set of individuals (Confrey et al., 2009), we saw evidence of children bringing together area and quantity. When students shared a collection among a number of people, they encountered the idea that fair shares had to be *the same*. For some students, *the same* meant an equal count in each person's share. Other students used visual patterns, stacks, or arrays to justify that their fair shares were *the same*. When students shared a continuous whole, they initially believed that shares that were *the same* had to be congruent. However, they eventually encountered instances in which non-congruent shares were equivalent in terms of area (Confrey et al., 2009; Franklin, Yilmaz, & Confrey, 2010; Pescosolido, 2010). For example, in the assessment item shown in Figure 1, students were told that the two rectangles represented equal-size brownies and were asked to compare the sizes of the two shaded pieces.

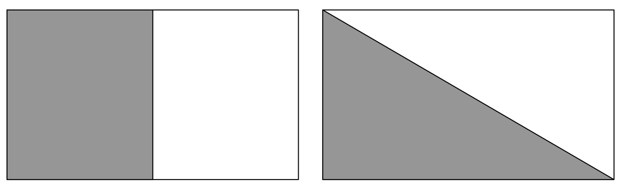

Figure 1. An example from equipartitioning of two equal areas that are not congruent.

Results from field testing this item revealed four levels of understanding (Franklin et al., 2010; Pescosolido, 2010). Students at the lowest level stated that the problem could not be solved or that one of the shaded pieces was bigger than the other. Students at level two reasoned that the two shaded pieces were the same size using a compensation argument, yet could not provide further empirical basis for this conclusion. Students at the third level used decomposition and composition (decomposing one object into pieces and reforming it to compose the other) to show that the two shaded pieces were the same size. Level four students argued for equality of the shaded pieces because both brownies, which were the same size, were fairly shared for two people. Hence, each shaded piece is half of the same whole. We call this understanding *same splits of equal wholes are equal* (SSEWE). That is, students at this level understand that, for example, half of a whole is the same no matter the method of splitting. Hence, equipartitioning also provides an approach to measurement.

Thus, this study first seeks to gain insight into the cognitive behaviors that may be overlooked when researchers use an external unit approach to area measurement. That is, instead of giving students an area unit *a priori*, what would happen if they were given an area comparison task that required them to construct an internal unit? Would they develop stronger understandings of the relationship between the unit and the whole and schemes to predict the change in measure (e.g., number of units) based on changes to the size of the unit? Second, this study investigates the role of equipartitioning in developing understanding of length and area measurement.

Research Questions

Specifically, the research questions for this dissertation study are:

1. What is the viability of a learning trajectory for length and area based on a foundation of equipartitioning that requires students to construct the unit and prepares them for the learning of fractions and ratio? That is, once proposed, through synthesis and theoretical work, how does the initial learning trajectory change as a result of empirical study through interviews of children?
2. What strategies and mathematical reasoning practices do students use to construct internal units, what types of emergent relations and generalizations do students make by comparing multiple units, and does this help students avoid the misconception that units are discrete and inviolable?

Additionally, this study lays the foundation for answering the larger question of whether the field can combine the until-now-disparate approaches of equipartitioning, the external unit approach, the internal unit approach, and the algebraic coding approach into a robust learning trajectory for measurement that combines the strengths while mitigating the weaknesses of each approach.

Plan of Thesis

Following this introduction, an abbreviated version of a longer synthesis (Confrey & Nguyen, in progress) justifying the need for this study will be presented and discussed in Chapter Two. The context of the research and the methods used to collect and analyze data will be discussed in Chapter Three. Chapter Four presents analysis of the three teaching experiments conducted for the study. Observed student strategies, mathematical reasoning practices, emergent relations, and generalizations from these teaching experiments will be used to construct a learning trajectory with delineation of proficiency levels and outcome spaces. Finally, Chapter Five discusses the findings from the study, explores the implications of the research for practice, and addresses unresolved issues and future research related to the study.

CHAPTER 2

RESEARCH SYNTHESIS OF LENGTH AND AREA

Method

Many studies have been conducted in the field of mathematics education on the topics of length and area measurement since Piaget, Inhelder, and Szeminski's (1960) seminal work. In general, five types of research articles have been published in journals, conference proceedings, book chapters, and dissertations since Piaget's et al. work: 1.) Examinations of specific measurement phenomenon (e.g., students' reliance on perception in early measurement tasks), 2.) Examinations of a specific misconception or conceptual obstacle (e.g., students' inability to understand the area = length x width formula), 3.) Broader attempts to delineate levels or learning trajectories in measurement (e.g., students' development of linear measurement from perceptual cues to identifying, iterating, and coordinating a unit), 4.) Studies that examine curriculum and assessment in measurement, and 5.) Research syntheses.

Both Cooper (1998) and Kennedy (2007) have addressed the issues of the function of literature syntheses and the considerations a researcher must take into account when reviewing the literature in a field. Cooper argues that research syntheses are different than literature reviews because, beyond identifying and reviewing key studies in an area of research, syntheses must identify common significant themes and findings, introduce new distinctions to resolve differences or integrate results, discern the most compelling results among controversial competing theories, and pave the way for researchers to implement robust results in practice. He summarizes, "the investigator must propose overarching schemes that help make sense of many related but not identical studies" (p. 12) while introducing new schemes may then lead to a, "circumstance in the social sciences … when a new concept is introduced to explain old findings" (p. 17). Kennedy notes that researchers must address five considerations when reviewing the literature in a field: 1.) bounding the literature, 2.) distinguishing studies from citations, 3.) distinguishing literature from lore, 4.) deciding the sources to include, and 5.) deciding which studies are anomalous.

In our conduct of syntheses, the DELTA group chooses to review only articles that appear in peer-reviewed journals, conference proceedings, and edited books and volumes. We emphasize articles that are more widely referenced and those that are considered *seminal* (e.g., Davydov, 1975b; Davydov & Tsvetkovich, 1991; Piaget et al., 1960). For this synthesis, we started by looking at extant literature reviews of length and area and their bibliographies in the *International Handbook of Mathematics Education* (Bishop, Clements, Keitel, Kilpatrick, & Leung, 2003), the *Handbook of Research on Mathematics Teaching and Learning* (Grouws, 1992; Lester, 2007), and the *Handbook of Research on the Psychology of Mathematics Education* (Gutiérrez & Boero, 2006). Next, we conducted a comprehensive search of the major journals in mathematics education (e.g., *Journal for Research in Mathematics Education*, *Educational Studies in Mathematics*, *Journal of Mathematical Behavior*, *Mathematical Thinking and Learning*), international journals (e.g., *International Journal of Science and Mathematics Education, ZDM*), other relevant journals (e.g., *Cognition and Instruction*, *Educational Psychology*), and historical examples (e.g., Davydov, 1975a, 1975b, 1991/1969; Davydov & Tsvetkovich, 1991; Piaget & Inhelder, 1948; Piaget et al., 1960). We also looked at conference proceedings (e.g., *Psychology of Mathematics Education*, *American Education Research Association*, *National Council of Teachers of Mathematics, International Congress of Mathematics Instruction*) but were careful not to include preliminary work that was further expounded in journal articles or edited books. The bibliographies of these works provided further references. In group meetings, members of the DELTA team discussed and summarized these articles.

The synthesis begins by examining the reasons behind U.S. students' struggles with length and area measurement in elementary and middle school. Next, because learning trajectories have been proposed to ameliorate student learning, a comprehensive review of the learning trajectories in length and area measurement is presented from Piaget's et al. (1960) seminal work to more recent work by Clements and Sarama (2009). This review will show that the majority of the work in the field relies on an external unit approach. This work will then be contrasted with the seminal work of Davydov (Davydov & Tsvetkovich, 1991) and recent work by Dougherty (2010) which takes the approach that rather than being taught as a separate content strand, measurement should be taught as a basis for all elementary mathematics, including the construction of fraction. Finally, the DELTA perspective is used to point out unresolved issues in the literature, potential avenues for future investigation, and setup the dissertation study.

Introduction to the Problem of Measurement

Numerous studies and reports show that U.S. students perform poorly on measurement items in large scale assessments. In the 2003 National Assessment of Educational Process (NAEP) national study, performance of fourth, eighth, and twelfth grade students on measurement was low in comparison to other mathematics subtopics—e.g., algebra, data and probability, and number properties (Smith et al., 2008). Performance has not increased dramatically, as the 2007 NAEP national study showed that measurement and geometry were the weakest sub-strands in fourth grade[4] and measurement was the second weakest sub-strand in twelfth grade[5] (National Assessment of Educational Progress, 2009). Additionally, Smith et al. reported that on the 1997 Trends in International Mathematics and Science Study (TIMSS), the gap between U.S. eighth grade students and their international peers was greatest in geometry and measurement. Most distressing of all, performance is weakest for low SES and minority students. The achievement gap between these students and their high SES and white counterparts is greatest on measurement than on any other mathematics content strand (Lubienski, 2002).

In one widely cited example, only 41% of eighth grade students taking the 2003 NAEP could correctly determine the length of a toothpick placed above a ruler with the left endpoint aligned with eight inches and the right endpoint aligned with 10½ inches. U.S. students showed equally poor performance on other items that assess measurement of area

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[4] Measurement–278; Algebra–285; Data and Probability–285; Geometry–278; Number and Operations–279
[5] Measurement–299; Algebra–302; Data and Probability–300; Geometry–303; Number and Operations–295

and volume (especially non-rectangular figures and objects), problems that relied on using the "length x width" formula (Kamii & Kysh, 2006), and problems that relied on measurement estimation (Joram, Gabriele, Bertheau, Gelman, & Subrahmanyam, 2005).

These assessment results are not surprising in light of the misconceptions research in mathematics education. For example, students' struggles on the toothpick problem may partially be explained by Lehrer's (2003) research, which showed most students are unaware that any point on the scale can serve as a starting point for measurement. Lehrer et al. (1998) found that first, second, and third grade students in their study did not adequately understand iteration and could not identify core units[6], two prerequisites for understanding length measurement. For example, when asked to measure a 7-inch piece of wood that was placed between the two and nine inch marks on a ruler, 21% of the students in their study focused on the left endpoint or added the two endpoints together while 41% counted from the one-inch mark. By the third year of the teaching experiment, 80% of the students counted from the 2-inch mark, but 20% persisted in counting from the one-inch mark. Another common problem was documented by Boulton-Lewis, Wilss, and Mutch (1996) where students counted the tick marks on a scale and not the space between the tick marks. A combination of these struggles were found on a rational number reasoning pre-test that Confrey, Maloney, & Nguyen (2007) administered to rising sixth grade students in St. Louis demonstrating that

[6] Lehrer et al. (1998) use the term core unit to denote the 1x1 square unit.

these problems persist through elementary and into middle school if they are not ameliorated through diagnosis and relevant instruction.

Students' struggles with area measurement are also well documented. Although most U.S. curricula and research focuses on students' understanding of area by iterating square units into an array structure (Smith et al., 2008), Battista and colleagues (Battista, 2004; Battista et al., 1998) found that students had difficulty visualizing the row-by-column array structure of tiled squares as area measure and did not understand that multiplying two length units produces a square area unit. They said that when shown an incomplete array structure, only 19% of second graders, 31% of fourth graders, and 78% of fifth graders could correctly predict the number of squares in the array. Kamii and Kysh (2006) came to the same conclusions and further showed that even by seventh grade, students do not think of a square unit as the unit of area and inferred, "If seventh graders do not understand units of length as we saw earlier, they cannot be expected to understand length x width" (p. 113). This is problematic, they said, because typical instructional activity to support students' understanding of area as "length x width" involves providing students with unit squares to cover an area. However, while squares are discrete units, length and width are continuous linear quantities (i.e. lines) and it is not straightforward for students to see that two lines can produce an area when multiplied. They remarked, "it is very easy for children to cover an area empirically with squares and to use multiplication to know how many were used … according to this view, what is impossible for children to understand is how two lines (the length and the width) can produce an area when they are multiplied" (p. 108).

Students also have difficulty distinguishing between length and area units. The eighth grade students in Chappell and Thompson's study (1999) confused length and area units and expressed perimeters as square units. Battista (2004) and Barrett, Sarama, and Clements (2009) documented that students had difficulty relating length units with area units and sometimes tried to add these two incommensurate quantities.

Many scholars attribute students' struggles with length and area to curricular issues. Thompson, Phillip, Thompson, and Boyd (1994) argued that the push toward "calculational" proficiency in the curriculum divorces the value of measurement from its spatial conceptions. This orientation, they claimed, results in a premature rush toward calculating and representing length and area with formulas and number sentences. Similarly, Kamii and Kysh (2006) and Smith et al. (2008) cited that most instruction on area measure is procedurally focused and instruction takes for granted that without an understanding of unit area, tasks that require students to iterate unit areas is problematic. Sfard and Lavie (2005) argued that measurement topics pose a special problem in the elementary school curriculum because measurement involves three types of intersecting language: number, geometry, and measurement. Children are often confused because of ambiguous references to spatial quantities and numbers. Tarr, Chavez, Reys, and Reys (2006) claimed that the problem is not just how measurement topics are presented but when they are presented. In an analysis of elementary school textbooks, they found measurement topics were usually located at the end of textbooks and, hence, taught at the end of the school year when the attention spans of students and teachers are attenuated.

Ongoing work by Smith et al. (2008) is analyzing to what extent weaknesses in the K-8 mathematics curricula contributes to students' low performance on measurement. They are currently conducting a detailed content analysis of the treatment of measurement concepts across six (three elementary and three middle school) curricula. Preliminary results have shown that while there was agreement in the coverage of measurement topics in upper elementary (third to fifth grade) and middle school, there was wide variation in the topics emphasized in the lower elementary (kindergarten to second grade) curricula. A more in-depth analysis of these lower elementary curricula revealed that all three curricula (Everyday Mathematics, Saxon, and Scott Foresman/Addison Wesley) were dominated by the presentation of procedural knowledge. (Procedural knowledge was defined as tasks that asked students to *do* measurement as opposed to *understanding* measurement.) For example, of all knowledge elements coded in the first grade curriculum, 78% were coded procedural for Everyday Mathematics and Scott Foresman/Addison Wesley while 91% were coded procedural for Saxon.

Teachers' measurement pedagogical content knowledge has also been called into question by researchers. Simon and Blume (1994) contended that most prospective elementary school teachers do not understand why the length and width relationship for area is modeled by multiplication. When asked to explain why multiplying the length and width to compute the area of a desk is equivalent to creating an array and counting the number of units in the array, teachers responded that this was simply a law that required no further justification. Baturo and Nason (1996) also found that teachers had a procedural view of area measure and regarded area formulas as an arbitrary collection of rules. Their lack of understanding of area meant that they could not derive the formula for the areas of triangles and circles. When formulas were presented to them, they accepted them without regard to their meaning and viewed them as something to be memorize and would only “come in handy” in certain mathematical situations. Similarly, Ma’s (1999) study of U.S. and Chinese teachers’ pedagogical content knowledge echoed Baturo and Nason’s (1996) claim that U.S. teachers thought of mathematics as an arbitrary set of rules. In a task that asked teachers to evaluate the relationship of the area of a rectangle to its perimeter, 22 out of 23 U.S. teachers believed that when you increase the perimeter of a rectangle, its area always increases and were satisfied with their conclusion after constructing a finite number of examples[7]. More recently, Clements and Sarama (2009) observed that in instruction, many elementary school teachers simply tell students that they are “wrong” if their measurement procedures are incorrect, and do not possess the skills in their toolkit to help these struggling students by more deeply diagnosing and treating the problem.

To summarize, the research base in mathematics education has identified a number of student misconceptions in learning length and area including not comprehending that length is the space between tick marks on a ruler, believing that a square unit is discrete and inviolable, and confusing (and hence sometimes trying to combine) length and area units.

[7] To understand, for instance, how a dynamic representation could help, imagine one has a flexible chain of a given length and forms it flexibly into a rectangle by pulling it out to surround four corners. As one stretches it longer and therefore narrows it, one can see that the area can approach zero for a given perimeter. This is the kind of informal proof that is missing for most teachers.

Researchers also contend that students' measurement weaknesses can be attributed to weaknesses in teachers' knowledge of measurement and curricular issues including a rush to get to "calculational" proficiency, overemphasizing procedural understanding of measurement, and weaknesses in the K-8 written curriculum. As Smith et al. (2008) point out, however, "A recognized problem is not a solution" (p. 18). The question for the field moving forward is how to address these known problems in length and area measurement.

The creation of detailed *learning trajectories* for content areas in mathematics has been proposed as one possible solution to address low student achievement (Clements & Sarama, 2004; Corcoran, Mosher, & Rogat, 2009; Steffe, 2004). In the next section, learning trajectories are defined and past and current trajectories in length and area measurement are reviewed.

Learning Trajectories in Length and Area Measurement

Learning trajectories have been defined and used for myriad purposes by researchers in mathematics and science education (Brown & Campione, 1996; Catley, Lehrer, & Reiser, 2004; Clements, Wilson, & Sarama, 2004; Confrey, 2006; Confrey et al., 2010; Simon, 1995). A common theme among these definitions is that knowledge progresses from least to more sophisticated ideas of understanding in predictable ways over time. Learning trajectories have been cited as useful for informing work at the level of curriculum (Clements et al., 2004), assessment (Battista, 2004; Confrey et al., 2010), teacher education (Mojica, 2010; P. H. Wilson, 2009; P. H. Wilson, Mojica, & Confrey, 2010), and standards development (Confrey & Maloney, in press; Corcoran et al., 2009). Furthermore, Confrey

(Confrey, 2008; Confrey, Maloney, Nguyen, Wilson, & Mojica, 2008) has argued that learning trajectories can help promote integration across mathematical content strands because they allow for vertical (across grades) and horizontal (within grades) visualization of standards.

Based on Confrey's (2006) work and definitions in the field (e.g., Catley et al., 2004; Clements & Sarama, 2004; Simon, 1995), the DELTA research group has defined a learning trajectory to be:

> a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (Confrey, 2008; Confrey et al., 2009)

This definition shares many commonalities with those in the field: a.) a learning trajectory is an ordered network of constructs that builds student understanding from less sophisticated to more sophisticated knowledge over time, b.) instruction and tasks are part of the trajectory and development through a trajectory does not happen in the absence of instruction, c.) learning trajectories are constructed from a synthesis (Cooper, 1998) of the literature, and d.) a learning trajectory is a hypothesis that must be strengthened and verified through empirical research.

In this section, a review of the major learning trajectories for length and area is conducted starting with seminal work by Piaget et al. (1960) to more recent work in the field (e.g., Barrett & Clements, 2003; Battista, 2007; Clements & Sarama, 2009; Outhred & Mitchelmore, 2004). Although Piaget's et al. (1960) work is not considered to be a learning trajectory, it provides a basis for understanding the current work in the field and was the first work of its kind to examine how young children developed measurement based reasoning.

Seminal Work by Piaget

Piaget believed, fundamentally, that "L'intelligence … organise le monde en s'organisant elle-même.", i.e. "Intelligence organizes the world by organizing itself" (von Glasersfeld, 1982, p. 613). That is, he rejected the notion of *progressive absolutism* (Confrey, 1980) where an individual can make finer and finer approximations to approach the truth and that, as humans, we can get better and better at understanding the nature of reality. Rather, he believed that knowledge is an interaction between our ways of knowing and seeing and interacting with each other and the world *out there* and that this is *the best* that we can do as there is no way of absolutely checking that ontological reality corresponds with an external reality *out there*. So, for Piaget, cognitive adaption is not the generation of knowledge that matches more and more with an external world. Instead, he believed that, "all knowledge consists of invariants which the experiencer creates and maintains in the face of changing experience" (p. 616). His theory of learning is known as the process of adaptation. That is, a cognizing subject's experiences help to assimilate into that subject certain structures (i.e. invariants or theories). These structures are perpetuated if they are successful in preserving the subject's internal equilibrium. However, they are accommodated (i.e. modified) if they are unsuccessful. Hence, learning occurs when individuals respond to perturbations in their structures that prompt accommodations appropriate in degree and magnitude to the regulations triggered by the failed assimilatory process. The process of adaptation and accommodation enhances and enriches structures for the cognizing agent.

Due to this theory of adaptation, Piaget was interested in studying the genesis of knowledge in individuals and how their cognitive structures are open to development which he labeled *genetic epistemology* (Piaget, 1970). To this end, he gave his subjects tasks and watched them solve the tasks to determine the *fit* of the task to an individual's current collection of schemes or cognitive structures. By extending conflicts beyond the current structures for a given child, Piaget hoped to study the genetic epistemology of a domain. Applied to measurement, Piaget claimed that children's understanding of measurement would follow from a process of equilibration within settings involving three components: "measuring, grouping of changes in position, and coordinate systems" (Piaget et al., 1960, p. 30). Hence, he gave children common tasks that gave them the opportunity to compare lengths and area by visual estimation, to relate smaller objects to larger objects, to segment an object by comparing its length or area to a smaller object, and to place objects end to end (in the case of length) or in overlap (in the case of area) in anticipation of the development of an abstract and reversible scheme for spatial measurement. By studying these three components, he hoped to see how children build up change of position and subdivision operations into effective metric operations that would provide a system that was transitive and reversible.

To study spontaneous length measurement, Piaget et al. (1960) placed two tables two meters apart with the second table being 90 centimeters shorter than the first table. They constructed a block tower on the first table and asked the children to build a tower on the shorter table (in vertical height) that was the same height as the original tower. They believed this task would necessitate spontaneous measurement for the children. Overall, Piaget et al. found three levels of sophistication. Level I, divided into two sublevels, was characterized by inaccurate visual comparisons between the two towers and the inability to conserve length because the children focused on the endpoints of the towers, not the total length of the towers. Children at level IA made inaccurate approximations based on perceptual distortions in visually transferring the height of the two towers. Children at level IB began to notice the tension caused by the uneven heights of the tables and tried using their hands and arms to physically transfer the heights as a means to compare two towers, albeit unsuccessfully.

Level II was characterized by intuitive attempts to coordinate changes of position, setting up a foundation for more sophisticated strategies relying on the iteration of a universal frame of reference (i.e. iterating length). Although children at Level 2 exhibited the ability to subdivide lengths, they could not reverse the operation. Some children did compare specific blocks to one another to find whether the one tower or the other might have taller parts. Like Level 1 children, Level II children used their hands and forearms as an intermediary object for moving between the two towers without comparing. Piaget et al. (1960) note the beginnings of using a mediating object as the basis for an operational comparison between the two towers. This move was facilitated by the progression from visual comparison, to body transfer, and then to using a third object as a reference which is a precursor to using an iterating unit.

Level III, divided into two sublevels, was characterized by formal operations, transitive reasoning, and the ability to subdivide the whole into parts and reverse this action. Children at Level IIIa were able to use an intermediate object of the same length or longer as the tower as a reference, but not a shorter object. They reasoned by using the intermediate object to transfer the length to the other object while attending to the conservation of the measurement. Children at Level IIIb, the most sophisticated level, were able to use a reference object shorter than the towers and iterate that unit along the length of the towers[8].

Conservation of length was studied by presenting children with two rows of congruent matches, placed end to end, with the same number of matches in each row. Once the child confirmed that the two rows were of equal length, one of the rows was rearranged so that the matches "zigzagged," but all of them were still touching end-to-end. Some matches were also broken so that not all individual stick lengths were congruent. The child was then asked to compare the length of the two rows of sticks. If needed for clarification, the researcher would ask whether an ant traveling along the rows of sticks would travel the same distance.

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[8] Note, this task was intentionally set up so that reference objects measured the towers an integer number of times.

At Levels I and II, children demonstrated conservation of length only when the matches were symmetrically arranged and the endpoints coincided. When children were directed to think in terms of movement along a path, they tended to judge paths with many zigzags as longer. Because of their difficulties with combining lengths, they also tended to judge lengths with the longest straight section as longer. Some older children also counted the number of congruent segments of each length in order to compare lengths. However, these children failed to consider segment length and thus declared shorter lengths with more segments as longer. Piaget et al. (1960) connected these misconceptions to children's incomplete understanding of the conception of regular intervals, which relies on the coordination between concepts of subdivision and order.

At Level II, children moved from oscillation between non-conservation and conservation towards conservation through a process of trial and error. As they moved towards conservation, they tended to justify their conclusions by returning segments to their original positions either physically or mentally or by recognizing compensation between transformations. Their growing understanding of reversibility and coordination of order and change of position enabled the conservation of length. By Level III, children had complete coordination between order and change of position and thus demonstrated conservation of length. They also showed signs of differentiating between movable objects and an external system of references.

For Piaget et al. (1960), conservation of area was the idea that the child recognized that decomposing a whole and rejoining the decomposed parts in any configuration conserves the area of the whole. In the task, children were shown an area made up of several separate subareas. The subareas were decomposed and rejoined as the children watched. The researchers wanted to know if the whole remained invariant for the children and asked how the parts compared to each another. The results of this study were analogous with those obtained in the tasks involving compositions of length. Piaget et al. (1960) reported that this task was inaccessible to children at Level I who use only perceptual and visual judgments and arguments. At this level, children make perceptual judgments and do not believe that area is conserved when the parts are rearranged into a different configuration.

Children at Level II exhibit understanding that the modified shape is the result of an altered arrangement of parts, however they have not yet progressed to the point of claiming that the areas are equal. At level IIA, children respond at an intuitive level and by inspection. They recognize that the areas are equal when the parts are rearranged in the same configuration. However, variation in the configurations leads to an immediate denial of equality. Finally conservation of area is achieved at level IIIA, where children reason that the transformation does not affect the total area of the whole.

To summarize the three levels that emerge through Piaget's et al. (1960) studies on length and area, in Level 1, children use only visual perception to compare the length or area of two objects or two objects composed of several parts (e.g., line segments with bends or perimeters). At this stage, children do not move the objects suggesting the lack of understanding of conservation of identity. Lack of conservation of identity means that children believe that a length or area may change under simple rigid transformations (e.g.

translation, rotation, and reflection). At Level IIA, children begin to understand conservation of identity. For example, asked to evaluate the relative lengths of two objects, distant objects are brought closer together, but are not necessarily aligned and are still perceptually evaluated. In the latter progressions of stage II thinking (IIB), children begin to use their body parts as a third object and referent for measurement through trial and error. In stage III thinking, a common measure is introduced and children understand transitivity[9] and sometimes bring in a middle term (e.g., a third object) as a referent. At first, the middle term must be longer than the two lengths being compared but this idea progresses so that a shorter middle term can also be used. The culmination of stage III thinking (stage IIIB) occurs when children can construct a metric unit and iterate it. Although these stages happen in succession, children that have progressed to stages II and III may still rely on visual perception and even believe that visual perception is more accurate than stage II and III actions. However, Piaget et al. (1960) state that, "inasmuch as they are willing to try auxiliary methods, they are no longer at stage I" (p. 33).

Current Learning Trajectories on Length

There are three major learning trajectories on length in the literature. The first is a research-based learning trajectory posited by Clements, Barrett, and colleagues (Barrett &

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[9] The idea of transitivity is tricky throughout the literature. Although Piaget et al. (1960) and Clements and Sarama (2009) ostensibly use transitivity to mean that given quantities A, B, and C if $A \leq B$ and $B \leq C$, then by logical necessity, $A \leq C$. However, these authors also imply that this logical transitivity is a prerequisite for using a third object to measure (e.g, using an iterating unit or mapping to a physical object). While it is true that a third object serves as the intermediary in the transitivity definition, it is not clear if students that iterate or physically map two lengths to a third object are actually aware of this logical transitivity.

Clements, 2003; Barrett et al., 2006), the second by Battista and colleagues (Battista, 2003, 2007), and the third is an attempt by Clements and Sarama to make the research based trajectory more assessable to teachers.

Clements, Barrett, and colleagues (Barrett & Clements, 2003; Barrett et al., 2006) have synthesized the results of their research on children's understanding of length into a learning trajectory with three levels (with Level 2 consisting of two sublevels). At Level 1, students make only visual comparisons between objects. If number quantities are used at all, Level 1 students only estimate or guess at numerical measurements without justification or reference to any unit. At Level 2a, students realize the necessity of movement (sometimes physical such as a step) along an object to measure length or perimeter. At this level, quantification occurs through counting, but the unit to be counted may be inconsistent even over a straight path, not to mention when the path bends as in perimeter. At Level 2b, students successfully and consistently integrate a sequence of equal sized units in a single direction (but not in a bend as in perimeter – for example a student may use a consistent unit for one side of a triangle but use a different unit for another side but keep the same count). At Level 3, students can reason about measurement without perceptual objects. At this level, students exhibit the ability to reason about multi-directional paths and perimeter, and express additivity of lengths (the idea that if a whole length AB is made up of sub-lengths AC and CB, then length(AC) + length(CB) = length (AC)).

Overall, Barrett and Clements (2003) claim that movement in this trajectory relies heavily on the correspondences between children's counting, partitive, and iterative schemes.

They conclude that instruction should allow students to explain measurement within connected representations, attend to their ways of enumerating length and explain the coordination between length units, side lengths and overall perimeter, and (a) verbally emphasize features and properties of objects they employ as units to focus students' attention to a measured dimension; (b) coordinate length measurement around a polygon by describing perimeter both as a single number and as a composite sum of side lengths (students need to grapple with counting objects that are ambiguous in their linear dimensions, such as square tiles as tools for finding perimeter); (c) identify relevant and irrelevant attributes of objects for measuring. For example, when measuring a collection of links arranged in a rectangular shape the child must isolate the length of each link in the chain, but disregard the angles at the corners of the rectangle and the width of the chain links; (d) help students move physically (walking or tracing) along the objects they are measuring, being quick to differentiate between counting incremental movements along a linear path, and counting the parts of the path; (e) ask students to make and label drawings as records of measurement activity to emphasize part–whole connections between the perimeter and the sides; (f) contextualize students' measuring activities by creating narratives or stories involving comparison (Barrett & Clements, 2003, p. 517).

Taking the work of Barrett and Clements (2003) and Clements et al. (1997) into consideration, Battista (2003, 2007) attempted to synthesize this work into a broader characterization of students' length measurement. He distinguished between two fundamentally different types of reasoning: non-measurement reasoning and measurement reasoning. Non-measurement reasoning does not take into account the assignation of a number to a quantity and relies solely on visual inferences or direct comparison[10]. Measurement reasoning directly involves unit-length iteration utilizing a fixed iterating unit that is fitted end-to-end with no gaps or overlaps. This reasoning takes into account this action coordinated with counting and requires conservation of position and knowledge that lengths are additive. Battista conjectured three levels of non-measurement reasoning and five levels of measurement reasoning.

At Level 0 of non-measurement reasoning, students reason solely based on the appearance of length objects. If a length object is composed of several smaller parts, then students at this level focus on the whole length rather than systematically on the smaller parts. At Level 1 of non-measurement, students can compose and decompose a length object composed of several smaller parts to compare the overall length of two objects. They start by physically rearranging path pieces and compare the rearranged paths as a whole and later on are able to compare two paths by matching pairs of pieces that are the same length without having to transform one path onto another. Finally, at Level 3 of non-measurement reasoning, students understand conservation of rigid transformations. They compare path lengths by translating, rotating, and reflecting parts and make inferences based on transformations. That is, if one path can be transformed into another via rigid transformations, then the two paths or equivalent in terms of length.

[10] Note, however, that one could use an object as a referent for another object without assigning a numeric quantity.

Battista's (2003, 2007) measurement reasoning trajectory parallels Barrett and Clements's (2003) work with some slight modifications. At Level 0, students count to assign a number to a length but they may vary their unit size or leave gaps. For example, some students may use their fingers and count as they move their figures on a path but their counts and movements are not coordinated. At Level 1, students use an iterating unit to measure a path but they do not properly coordinate that unit and their iteration contains gaps, overlaps, or use of an inconsistent iterating unit. At Level 2, students understand that an iterating unit must have the same length and the coordination of the unit must be such that there are no gaps, overlaps, or variations in size of the iterating unit. At Level 3, students understand that lengths are additive. In both the work of Battista and Clements, additivity refers to quantitative additivity and facilitates the solving of "missing measures" and "missing perimeter" problems. Finally, at Level 4, students can numerically operate on length measurement without iterating any unit lengths and extend their knowledge to various geometric perimeters and transformations. Battista also claims that at this level, they fully coordinate the non-measurement trajectory with the first four levels of the measurement trajectory resulting in a robust length measurement schema.

Clements and Sarama (2009) wanted to synthesize previous research into a developmental trajectory with appropriate ages and tasks to help preschool and elementary teachers develop children's early recognition of length into measurement with a ruler. They posited a learning trajectory consisting of eight levels. Children at Level 1 (age 2) cannot identify length as an attribute. Level 2 children (age 3) can identify length and distance as attributes and use absolute descriptors (e.g., tall, big) but cannot yet compare lengths. Level 3 children (age 4) can compare two length objects directly by aligning them to determine which one is longer. Level 4 children (ages 4-5) can compare the length of two objects by representing them with a third object and may assign a number to a length by moving a non-consistent unit along a length object while counting. At this level, given a ruler, children ignore the starting point and, if they measure with a ruler at all, can only measure starting from 0. Level 4 children (age 5) can order lengths that are marked with, at most, six units. Level 5 children (age 6) begin to understand unit iteration although they still do not understand the need for the same size iterating unit. Level 6 children (age 7) can relate the size and number of units and fully understand unit iteration with no gaps and overlaps. They also understand that lengths are additive and that using a different unit to measure will result in a different measurement. Level 7 children (age 8) fully understand that the length of a bent path is the sum of the parts and not the distance between the endpoints. They understand the need for identical units, iterate without gaps or overlaps, understand the relationship between different units, can partition a unit into smaller units, and can begin to estimate lengths accurately. Finally, Level 8 children can mentally move an iterating unit along an object and accurately count the segments. They can also fully operate arithmetically on lengths. That is, they can solve “missing measures” problem without physically measuring as they fully understand that lengths are additive.

In current learning trajectories for length measurement, students coordinate an external iterating unit and learn to iterate this unit without gaps and overlaps. They coordinate this action with number and the ultimate goal is to abstract length to a quantity that can be seen as additive. At the highest levels, students no longer need to iterate and can consistently operate with length as number.

Current Learning Trajectories on Area

Two trajectories on students' development of area share the idea of an iterable unit leading to student development of an array structure. The first, posited by Outhred and Mitchelmore (2000), uses an iterable unit given *a priori* to cover an area. In this learning trajectory, students progress from uneven covering, covering with no gaps and overlaps, and ultimately creating an array of equal rows and columns and generalizing this to an interiorization of the "area = length x width" formula. Clements, Battista, and their colleagues (Battista, 2007; Battista et al., 1998; Clements, Battista, Sarama, Swaminathan, & McMillen, 1997) detail a similar learning trajectory for the development of the array model.

Outhred and Mitchelmore's (2002) trajectory comes from a study that investigated young children's strategies to find the number of unit squares that cover a rectangle. Through this work, they posit a learning trajectory toward conceptual understanding of the area formula. Working with 115 elementary school children in Australia (grades 1 – 4), they asked them to cover a square area using wooden tiles, paper tiles, and by drawing. Although the wooden and paper tiles allowed children to construct arrays, those at the first level of the trajectory do not see the inherent structure of the array as equal rows and equal columns. The drawing task is even more difficult because of the unevenness of students' array drawings. In the end, children often did not relate the area formula to their work with the manipulatives.

Three tasks were devised to elicit student reasoning. In the first task, students were given a movable iterable unit. In the second task, the same sized unit was given but it was not movable. Finally, in the third task, children were asked how many two centimeter squares would be needed to cover an 8x10 centimeter rectangle without the use of manipulatives.

From the data gathered, a four-stage learning trajectory was conjectured. At Level 1 (incomplete covering), children constructed an incomplete covering with gaps and overlaps. At Level 2 (visual covering), children's visual coverings contained no gaps but the iterable units were not the same size or the columns and rows were not equally aligned. At Level 3 (concrete covering), children completely covered the rectangle with arrays that contained equal rows and columns. Finally, at Level 4, (measurement) children had interiorized the "length x width" model and used the iterable unit to measure the length and width of the rectangle and multiplied to obtain the area. Children's drawings at level 4 only showed arrays down the length and width of the rectangle or just dots to indicate where students had measured the individual linear side lengths.

Hence, the idea of constructing an array is powerful in Outhred and Mitchelmore's (2002) learning trajectory. Once interiorized, students no longer need to draw out the entire array. They are convinced that they can innumerate their iterable unit across the rectangle by knowing only the length and width of the rectangle. Outhred and Mitchelmore believe that the iterable row is the basis for the array structure and that, furthermore, knowing the relationship between the size of the iterable unit in relation to the rectangle is important. That is, the smaller the iterable unit, the more of them you will need to cover a surface . They conclude that going through this trajectory is essential to children's success in learning area. Rote memorization of "area = length x width" does not allow children to achieve complex understanding. Of particular interest is the formation of the iterable row in a rectangle whose structure is not intuitively obvious to children. Finally, although essential to learning the area formula, they claim that knowledge of multiplication is not essential for progressing through this trajectory. This is an interesting claim and one conjectures that this is because, initially, finding area is additive in this trajectory refers to their count of the number of tiles in the array. Their move to multiplication seems to be presented only as a shorthand and faster way to arrive at the number of tiled squares than counting.

The array structure also occupies a prominent place in the area trajectory of Battista, Clements, and their colleagues. Battista et al. (1998) saw three levels of sophistication in coming to an array model in their work with 12 second graders in an interview study. At Level 1, students have no row-column structure and cannot even identify squares in an array or count the number of square tiles that are in a rectangle. At Level 2, students begin to see the row or column as a composite unit but the composite is not successfully used to cover an entire rectangle. At the beginning stages of Level 3 (called Level 3A), students completely cover a rectangular area with their composite units but the array drawn may not have equal-sized composite units. At Level 3B, rows and columns are correctly iterated. At Level 3C, students no longer need to use a composite unit as a referent and, "uses the *number* of squares in an orthogonal column or row to determine the iterations" (p. 517). They contend that because of the various stages of thinking and interiorization that must occur for students to firmly grasp the array model, we cannot simply expect students to automatically understand the “area = length x width” formula. Furthermore, “if students do not see row-by-column structure in these arrays, how can using multiplication to enumerate the objects in the array, much less using area formula, make sense to them?” (p. 531).

Measurement from Constructing Internal Units

As shown by the previous learning trajectories, the majority of work in the field involves measurement using a given external unit. Curriculum (Smith et al., 2008) mirrors this approach. However, the work of Lehrer and colleagues (Lehrer et al., 1998; Strom et al., 2001) suggests that students can succeed at constructing their own common units of measure without being given them *a priori*, leading to ideas of how different common units affect the whole measure, the equivalence of common units, and transitivity of area equivalence (e.g., if A, B, and C are the areas of non-congruent rectangles and A = B and B = C, then A = C).

To develop students’ understanding of area measure, Lehrer conducted a set of teaching experiments (Lehrer et al., 1998; Strom et al., 2001) and sought to explore the role that splitting (Confrey, 1989) might have had in that conceptual development of area (R. Lehrer, personal communication, 1999). Students were shown three rectangles (1x12, 2x6, and 4x3), told they were quilts, and were asked to order them from smallest to largest in relation to which one “covers the most space.” This task stand in stark contrast to most textbook curricula where a core unit (usually a 1x1 square) is brought in externally for students to use and iterate. Instead, by allowing students to construct their own core unit, ideas of how different core units affect the whole measure, the equivalence of core units, and transitivity were brought into the classroom discourse.

Although most students initially thought that the 2x6 or 4x3 rectangles covered the most space because they “looked fatter,” additional work with composition and decomposition aided by instruction led them to a notion of “additive congruence” whereby students would, for example, fold the 1x12 rectangle in half producing a 1x6 rectangle and lay this on exactly twice to cover the 2x6 rectangle. They reasoned that since the 1x12 rectangle was folded in half and exactly covered the 2x6 rectangle twice, that the two rectangles were equivalent. Students began to believe, because these area quantities were equal, that they could turn any of the rectangles into another through folding and rotating. Eventually, students progressed to the construction of “units of measure,” or core 1x1 units that they could verify were the same number across all three rectangles. After moving to the construction of core units, students were given the additional task of seeing how many different core units they could construct and if these core units had anything in common.

In this teaching experiment, Lehrer et al. (1998) also found evidence that students understood logical transitivity of equality. First, students empirically verified that the 1x12 quilt and the 2x6 quilt covered the same amount of space, and that the 2x6 quilt and the 1x12 quilt covered the same amount of space. Then, a student remarked that this must imply that the 1x12 quilt and the 2x6 quilt covered the same space. The teacher used this as an opportunity to ask the class if they could verify this claim without measuring using their constructed units.

Strom et al. (2001) replicated Lehrer's et al. (1998) study and synthesized student strategies into five levels of development. At Level 1, consistent with Piaget's et al. (1960) stage I, students perceptually observed that each rectangle covered area and conjectured that, "they cover up the same space" (p. 737). At Level 2, they believe that congruence is the only way of determining equal area. This led some of the students to conjecture that the 1x12 and 3x4 rectangles did not cover the same space, "cause one is skinny and one is fat" (p. 737). At Level 3, students understand the idea of "additive congruence." That is, they believed that congruence can be established by decomposing and rearranging the parts, and if the rearranged parts cover the same space, then the two objects are equal in area. At Level 4, they relate the size and counts of units. At Level 5, they establish a unit area and the area of a rectangle is the count of how many unit areas are contained in the rectangle. Although preliminary, this sequence offers an alternative trajectory that does not require starting with an iterable unit and is similar to dissection theory (Eves, 1972).

The Davydov Measurement and Algebraic Coding Approach

Use of measurement as a way to introduce the rational numbers in the elementary school curriculum was central in the seminal work of Davydov (Davydov, 1975b; Davydov & Tsvetkovich, 1991) who challenged the notion that fractions should be taught via part-whole methods. Davydov argued that such a method, "substitutes searching for the relationship of various units of measurement with the purely external act of singling out and designating equal parts of a whole" (Davydov & Tsvetkovich, 1991, p. 27). He also noted that, for the Greeks, the rational number system came out of the necessity of measurement.

Hence, he claimed that divorcing rational number from its measurement roots and teaching it as theoretically derived from division stretches the psychological abilities of elementary school children. In the preface to Lebesque's *On Measuring Magnitudes*, Kolmogrov also argued against the idea that such a theoretical treatment was the most "scientific" approach of introducing the rational numbers, felt that it was an arbitrary way to broaden the domain of whole numbers, and concluded that it is the scientific construction of the rational numbers must involve measuring magnitudes (Davydov & Tsvetkovich, 1991).

Due to these beliefs about measurement, Davydov and Tsvetkovich (1991) created a curriculum that used measurement as a basis to introduce fractions. Through measurement tasks that emphasized the ratio relationship between a unit and its whole, students learned to code the relationship between the iterating unit and the whole measure as an algebraic equation. They claimed that the necessity of fraction (and hence rational numbers) comes when students encounter the situation where measuring a quantity *A* with a unit *i* leaves a remainder *r* as in the case illustrated by Figure 2.

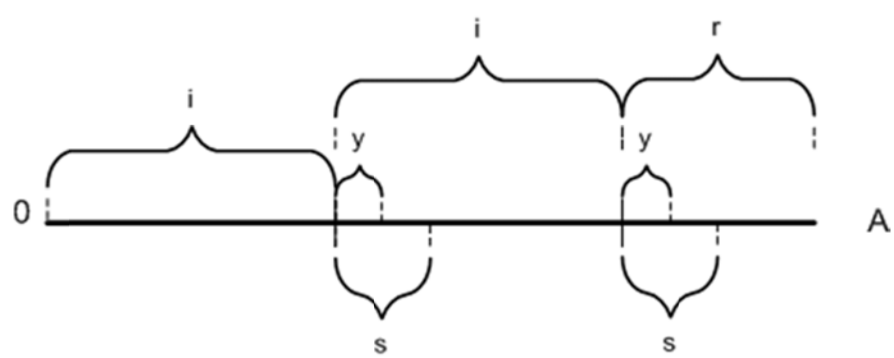

Figure 2. Example of a situation where a unit *i* leaves remainder *r* when measuring *A*.

Before instruction, students were asked how they would code *A* in terms of *i*. As a class, the students in the teaching experiment decided on the notation $A = 2i + r$ which the teacher helped them understand to mean two units of length *i* plus some remainder *r* measured *A*[11]. The teacher then led students in a task where they equipartitioned *i* to obtain a smaller unit to measure *A*. Hence, students formed *s* and *y* such that $3s = i$ and $6y = i$ (i.e. *s* measures *i* three times and *y* measures *i* six times – but again students were not introduced to this as a multiplicative relationship). Using these new iterating units to measure *r,* students found that $2s = r$ and $4y = r$ (i.e. *s* measures *r* 2 times and *y* measures *r* 4 times). Students concluded that $A = 2i + 2s$ or $A = 2i + 4y$. Working with the first equation, they were instructed that another way of writing that *s* measures *i* three times was *s* is 1/3rd the size of *i* or $s = (1/3)i$ and, hence, that $A = 2i + (2/3)\, i$. Thus, for Davydov, fractions are necessitated when an iterating unit *i* fails to measure *A* and leaves a remainder *r* where the construction of a smaller iterating unit *s* from *i* does measure *r*. In the more advanced case, students realized that another way to code A was $A = 2i + (4/6)\, i$ necessitating the equivalence of the similar fractions $2/3 = 4/6$. Similar measurement situations then necessitate addition of fractions and the fraction property $a/b = a/b * d/d = a/b * e/e$.

The design of the *Measure Up* (MU) curriculum by Dougherty (Dougherty, 2007, 2010; Dougherty & Venenciano, 2007) builds from the Davydov approach. Her main claim is that, contrary to current work in the field, elementary mathematics need not focus on

[11] Note that although the equation $A = 2i + r$ implies a *multiplicative* relationship between 2 and *i*. Davydov (Davydov & Tsvetkovich, 1991) chose to interpret it additively, however. That is, instead of teaching the children that the length *2i* is *two times as large as i*, he elected to teach it to them as two units *of size i*.

discrete counting units but can begin with the comparison of continuous quantities (Dougherty & Venenciano, 2007). Indeed, instead of quantity coming out of number, in the MU approach number comes out of the study of quantities. Before the introduction of numbers, students are taught how to informally measure and code the results of their measurement as less than, equal to, or greater than. Number is not introduced until unit and, "Number now represents a way that students can express the relationship between a unit and some larger quantity, both discrete and continuous. Conceptually, the introduction of number in this manner offers a more cohesive view of number systems in general" (p. 19). Hence, the idea of qualitative and quantitative compensation of units is strongly embedded within this MU curriculum.

Additionally, Dougherty's (2010) work differs from current work in the field due to the heavy reliance on symbolization. Because quantity and symbolization are introduced before number, ideas such as additivity of length and area are no longer dependent on number. For example, in Clements and Sarama's (2009) learning trajectory for length measure, additivity of length specifically means, for example, recognizing that if a whole length measures 10 units and one of its parts measures 8 units then the other part measures 10 units – 8 units = 2 units. Hence, the idea of additivity is strongly tied to number. However, Dougherty (2010) points out that in her work, "You might notice that the take-away model of subtraction, as well as the comparison and missing-addend models, were represented with these tasks, again, without number. It is worthwhile to note that this also represents the joining model of addition. Thus addition and subtraction are represented concurrently so that children see the links between the two computations" (p. 65).

Commentary on Current Trajectories from a DELTA Perspective

The majority of the research in length and area measurement is based on measurement and comparison of lengths and areas using a given external unit. The end goal of many of these trajectories is for students to learn how to iterate the units with no gaps and overlaps and coordinate this action with number. In the case of length, students eventually move to a quantitative and additive notion of length which no longer requires unit iteration. In the case of area, the goal is for students to generalize counting unit squares in an array to the formula area = length x width.

From a DELTA perspective of rational number reasoning[12], the main critique is that current trajectories largely ignore that the source of an iterating unit comes from equipartitioning leading to privileging addition and subtraction and neglecting ratio, multiplication, and division (Confrey, 1994; Confrey & Smith, 1995). That is, by pre-selecting the unit and focusing solely on its iterative structure, the external unit approach misses the multiplicative aspects of measurement. Indeed, the only time this multiplicative

[12] This section reflects elements of the results coming from the reported research and elements from a revised trajectory developed by Confrey and Nguyen (2010) subsequent to the dissertation research based on participation in a mini-conference held in East Lansing, Michigan on September 9, 2010. The first two elements on the role of equipartitioning and on inverse relationships were investigated in the research. The issues of distinguishing additive and multiplicative and ratio relations and connections of measurement to the broader issues of measurement and statistics were articulated as a modified trajectory was proposed by Confrey to build on this dissertation's work and develop a theory integrating the work of Davydov and Dougherty and extant measurement learning trajectories. It is included here to complete the argument in this dissertation, as it has already evolved in the DELTA research group.

structure seems to be addressed is through the generalization of repeated addition (e.g., counting rows and columns) as multiplication. Missing, then, is the idea that equipartitioning a rectangle with dimensions *a* x *b* results in *ab* equal parts of size 1/*ab*th and that the area of the whole rectangle is *ab* times as large as one of the parts, a key proficiency in the equipartitioning learning trajectory (Confrey et al., 2008; Confrey et al., 2010). This implies that if equipartitioning is used, then the construction of a unit is organically created from equipartitioning. The challenge for comparison rests in finding common units, however, and this approach may also strengthen the understanding of the unit within a single measurement. Furthermore, students are never faced with the problematic of what to do if unit iteration fails to completely cover the whole. This problematic was used by Davydov (1991/1969) to necessitate the introduction of fractions. Paired with area units and equipartitioning, these types of problems may help students to avoid the documented misconceptions in the literature such as seeing a unit as discrete and inviolable (Kamii & Kysh, 2006).

Another idea that is largely missing from the literature involves the inverse relationship between the size of the unit and the number of units needed to measure a whole. Of the trajectories reviewed, only Clements and Sarama's (2009) trajectory explicitly addresses this issue within the trajectory. However, it is presented only at a qualitative level. That is, the goal is for students to understand and state that as the unit increases/decreases, the number of units needed to completely cover decreases/increases. However, in the DELTA work on equipartitioning, we have found that the power of compensation (predicting the result on one quantity as another quantity changes) comes from its multiplicative aspect

(Confrey et al., 2010; P. H. Wilson, Edgington, Nguyen, Pescosolido, & Confrey, in press). In the case of equipartitioning, quantitative compensation requires that students compensate for factor or split-based changes in the number of objects being shared, the number of people sharing, or the size of fair shares (P. H. Wilson et al., in press). Because the relationship between the size of the unit and the number of units needed to cover is multiplicative, it seems to us that it would be valuable for students to have a robust *quantitative* notion of this relationship and be able to predict, for example, the change to the number of units needed to cover based on a factor based increase or decrease to the size of the unit.

Another issue that is largely ignored in current trajectories is the inherent ratio relationship between the iterating unit and the whole which is why length, area and volume were originally included in the DELTA selection of learning trajectories. Piaget et al. (1960) hinted at this relationship when they remarked that unit iteration gives one a larger unit that is an integer multiple of the iterating unit. This multiplicative relationship is ignored in current trajectories and comparisons of length are largely treated as additive. Again, taking into account Confrey's (1989) splitting conjecture and refinement that additive and multiplicative worlds should develop concomitantly (Confrey, 2008), she conjectured that it would be valuable to see the effects on children's reasoning if they were required to make comparisons between two lengths or two areas both additively and multiplicatively. Indeed, this multiplicative relationship is also a ratio relationship if students are given the opportunity to construct multiple units that measure a whole and code the relationship.

Lastly, although conservation of rigid transformations, conservation of subdivision and reassembly, and transitivity are cited in the literature and current trajectories (Clements & Sarama, 2009) as prerequisites to measurement, the properties seem to be hidden within the trajectories. We see this as a weakness, because we feel that teachers need to know the formal mathematical properties and language they are teaching to their students. Other prerequisites such as attributes, partitioning, and relationship between number and measurement are also discussed but hidden. Furthermore the work of Lehrer and colleagues on measurement and statistics (Lehrer, Kim, & Schauble, 2007) suggests that students are not being introduced to questions such as: 1.) What is measurement? 2.) Why is there variability in measurement? 3.) What is the importance of precision in measurement? 4.) Where do units come from and how are they embedded in the tools that we use in our world (i.e. rulers, grids, road maps, etc). In a way, an external approach to measurement where the unit always measures the whole obviates these questions which we know students struggle with from elementary school to college (Smith et al., 2008).

A final note should be mentioned about the terms conservation and transitivity and the ways they are discussed in the literature and will be developed in the subsequent DELTA work. Conservation is used in two ways. One meaning is that conservation means a quantity (i.e. length or area) is invariant under rigid transformations (e.g., rotation, reflection, and translation). Piaget et al. (1960) used the term *conservation of identity* to describe this type of conservation. However, conservation is also used to mean that length or area does not change over decomposition/composition and rejoining. These two ideas of conservation seem to be used interchangeably and we suggest that they be differentiated within the literature with the terms *conservation of rigid transformations* and *conservation of decomposition/composition*.

Transitivity is also problematically covered in the literature. For Piaget et al. (1960) and Davydov (1991/1960), transitivity was defined as logical mathematical transitivity. That is, given quantities A, B, and C such that A (≤, ≥) B and B (≤, ≥) C, then it logically follows that A (≤, ≥) C. The observation of this is that the child does not need to measure to check but can state that A(≤, ≥) C as a logical necessity. This is also the notion of transitivity that is used by Lehrer and colleagues (Lehrer et al., 1998; Strom et al., 2001). However, other scholars (Battista, 2007; Clements & Sarama, 2009) use transitivity analogously with using a third object (such as an iterating unit or longer third object) to facilitate measurement. In a way, however, this makes sense mathematically but perhaps not from a child's perspective. That is, suppose a length of 10 cm is being compared with a length of 12 cm. A child uses a one cm iterating length and concludes that the 12 cm length is longer because it takes 12 of the one cm lengths to measure. Is that conclusion necessarily a transitive argument or is transitivity hidden and lost within the understanding? This is another interesting and potential site of further investigation. For now, we propose that the field use the term transitivity to mean logical mathematical transitivity and perhaps use a term such as *mediating measure* to describe the situation where a third object is used to help measure.

Setting up the Study

In preparing for the empirical work of this study, I started with the assumption that equipartitioning forms a basis for rational number reasoning (Confrey, 2008) and wondered how the concepts of equipartitioning and splitting might give new insights into how students develop skills in length and area measurement. Based on Confrey's conjectures that there exists a contrast between recursive and iterative approaches to reasoning that would show up explicitly as one moved from splitting to the use of split of splits (Confrey et al., 2009), which she had previously identified as relevant to exponential functions (Confrey & Smith, 1995), I sought evidence if making an assumption of a primary role for iteration in measurement tasks without consideration of a recursive option might cause students to miss important concepts in such as notions of *times as many*, especially as it occurs at multiple levels (i.e. *split of splits*).

Also based on the DELTA framework, I sought to learn if and how children's actions might be strengthened by using measurement to understand the concepts of fractions and ratio earlier than they are addressed in the traditional elementary curriculum. Drawing upon the work of Davydov (1991/1960), I wondered if there was a way to ground student reasoning in physical measurement activities before abstraction. Indeed, in his studies, children were explicitly told the ratio relationship between abstract units and used algebra to simplify these relationships. I believe that children should be given physically materials in order to test out their theoretical conjectures so they have the opportunity to tie together physical actions with their algebra. This could potentially create a stronger sequence of tasks than in Davydov's sequence while still considering the development of the unit and not undermining the relational structures between units and measures.

Finally, I made careful note to examine the ways that the underlying structures of measurements such as conservation of rigid transformations, conservation of decomposition/composition, additivity of lengths and areas, logical transitivity, and the relationship between the unit and the whole emerged in students' thinking.

CHAPTER 3

METHODOLOGY

This study is situated within the overall DELTA methodology for learning trajectory creation as shown in Figure 3, below.

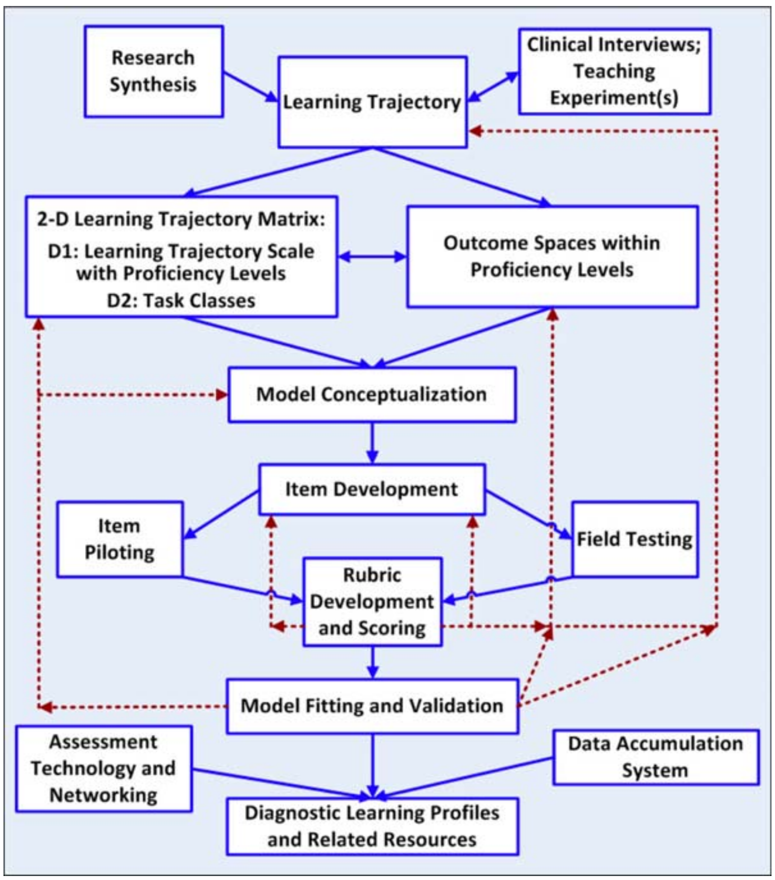

Figure 3. DELTA method for learning trajectory creation (Maloney & Confrey, 2010).

As stated in the introduction, the method includes the following steps:

1. Synthesize the research literature in a domain.
2. Conduct clinical interviews and teaching experiments to supplement the literature.

3. Describe a first version of the proficiency levels and outcome spaces[13] for the trajectory.
4. Engage in a cycle of item development, model fitting, and validation using methods both qualitative—e.g., evidence from clinical interviews and teaching experiments—and quantitative—e.g., analysis of student responses on field test items using item response theory (Confrey, Maloney, Pescosolido, et al., in progress; Confrey et al., 2010), descriptive statistics (Pescosolido, 2010), and classical test theory.
5. Build learning progress profiles (Confrey & Maloney, 2010) and resources based on the validated trajectory.

As this study seeks to inform the first three steps in the DELTA methodology, this section will address literature syntheses, clinical interviews, and teaching experiments and explain how these three pieces of evidence are used to create an initial learning trajectory.

Synthesis

An initial learning trajectory is first constructed from a synthesis of the literature (Cooper, 1998). More than ever, research syntheses are important to expanding the knowledge base in educational research. As the National Research Council (2002) stated in *Scientific Research in Education*, "Rarely does one study produce an unequivocal and durable result; multiple methods applied over time and tied to evidentiary standards are

[13] A delineation of the different types of outcomes ordered from lower to higher sophistication.

essential to establish a base of scientific knowledge. Formal syntheses of research findings across studies are often necessary to discover, test, and explain the diversity of findings that characterize many fields" (p. 2). Hence, syntheses are different than traditional literature reviews; beyond identifying and reviewing key studies in an area of research, they must identify common significant themes and findings, introduce new distinctions to resolve differences or integrate results, discern the most compelling results among controversial competing theories, and pave the way for researchers to implement robust results in practice. Cooper (1998) also argues that a synthesis must go beyond a literature review saying, "the investigator must propose overarching schemes that help make sense of many related but not identical studies" (p. 12) while introducing new schemes may then lead to a, "circumstance in the social sciences … when a new concept is introduced to explain old findings" (p. 17).

Although our synthesis of length and area (Confrey & Nguyen, in progress) helped us in delineating the levels and providing initial empirical support for the ordering of the levels, in order to delineate the learning trajectory, we found it useful to use a design method expounded by Clements and Sarama (2009) and Barrett (personal communication, August 10, 2009). To construct the top levels of a learning trajectory, they find it useful to articulate a *target concept*; that is an answer to the question: What it is that students should know after progressing through this trajectory? This should not only be articulated as a specific behavior but also as a mathematical concept. For example, in their length work, Clements's et al. (2009) target was the number line. To construct the bottom level of the learning trajectory, it is useful to look at observations of young children as they begin to develop informal ideas toward the target concept. The initial learning trajectory should then *build up* from these observations while *pulling up* toward the target concept. In the DELTA framework, the ultimate goal of measurement should be the flexible ability to make comparisons on a wide variety of commensurable quantities (for this dissertation length and area). To do so, students need to recognize qualitative differences among continuous quantities (i.e. length as path, area as space covering), understand what "less" and "more" of that quantity means, assign a number[14] to that quantity, build a robust notion of rational number, and understand the ratio relationship between the whole measure and the referent unit building towards a robust understanding of unit. The starting point is students' informal reasoning about quantities and our trajectory attempts to *build* from this *pulling up* toward setting up ratio and fraction. Although the literature informs this initial pass, validation requires knowledge gained from teaching experiments, and, later, through measurement models (Confrey, Maloney, Pescosolido, et al., in progress). Similarly to how the DELTA work on equipartitioning has evolved, (Confrey et al., 2009; Confrey, Maloney, Nguyen, et al., in progress) students' work on the designed tasks is expected to reveal other possibilities and pathways that have not yet been considered while illuminating which parts of the initial trajectory are viable.

Clinical Interviews and Teaching Experiments

Although an initial learning trajectory is constructed with best evidence from the empirical literature, it must be further modified and strengthened using a phenomenological approach (Neuman, 1999; M. Wilson, 2005). This step provides the necessary existence

[14] Initially a whole number and building to rational and real numbers.

proof that students will actually act in the ways hypothesized by the initial learning trajectory. This phase includes developing tasks, hypothesizing initial outcome spaces of student thinking and behavior on these tasks, conducting clinical interviews or teaching experiments, and analyzing these data to further modify the trajectory and outcome spaces.

Tasks are developed to target students' thinking on pertinent levels of the initial learning trajectory. They are informed by previous work in the field and the researchers' own experiences and hypotheses. After adapting or conjecturing tasks that might elicit student thinking on a level or multiple levels of the initial learning trajectory, we theorize an initial outcome space of possible student responses on a task.

Opper (1977) defines the clinical method as, "a diagnostic tool applied to reasoning in children. It takes the form of a dialogue or conversation held in an individual session between an adult, the interviewer, and a child, the subject of study" (p. 92). Piaget (1976) developed the clinical method as a means to combine the strengths while reducing the weaknesses of the extant methods of standardized testing and natural observation. The clinical method involves setting up an initial problematic, making hypotheses about how students will react to the problematic, and testing these hypotheses against students' action and thinking during an interview. During the interview the researcher may ask probing questions to further clarify student responses, rephrase questions according to the student's responses, follow up with additional questions if the student makes a particularly novel response, and ask the student to explain her use of a mathematical term or idea in her own words rather than erroneously concluding that the student is referring to a textbook definition. Student responses and reaction to the task allows the researcher to reformulate her original hypotheses and further probe the student until she is satisfied that she has explored the student's thinking as much as possible. Although the interviewer has freedom in the choice of probes and directions the interview can take, it does not mean that clinical interviews are not systematic. Indeed, in order to make results comparable from student to student, Opper (1976) argued for a standardized version of the method in which a standard task with identical manipulations and prompts are initially presented to each student. A number of common questions are also presented to each student although some of these questions may only be asked if certain conditions in the interview are met.

The constructivist teaching experiment, "is a technique that was designed to investigate children's mathematical knowledge and how it might be learned in the context of mathematics teaching. In a teaching experiment, the role of the researcher changes from an observer who intends to establish objective scientific facts to an actor who intends to construct models that are relative to his or her own actions" (Steffe, 1991, p. 177). Teaching experiments are also characterized by the study of *how*, not just if, students dynamically move from one state of knowledge to another and qualitative data consists of exchanges between teachers and students in descriptive instructional contexts (Cobb & Steffe, 1983). The teaching experiment evolved from and draws upon Piaget's clinical method in that the teacher/researcher formulates and tests hypotheses about aspects of a student's mathematical task solving in order to infer the student's mathematical knowledge. Confrey (2006) expounds further and also traces the evolution of teaching experiments from Constructivism and Socio-cultural theory. These theories called into question the applicability of research theories developed in confined out-of-school laboratory settings and the realization that research in classrooms is not "deterministic but complex and conditional." Cobb and Steffe (1983) explain that although clinical interviews are well suited to investigating the sequence of steps that children take when constructing a new concept, the methodology also has weaknesses. These weaknesses include the inability to account for when specific cognitive restructuring takes place, the fact that clinical interviews are devoid of the classroom context which is important information for teachers to know in planning interventions, and that some students take different mathematical steps depending on the learning context. They propose that teaching experiments reduce these weaknesses because of the teaching context and the model building that is inherent within the method. The researcher's role also changes in a teaching experiment and she does not act as merely a participant (Steffe, 1991). The teacher-researcher seeks to understand students' mathematics in their ways and how they come about their thinking. Learning occurs as a result of students' interactions with particular tasks that help them alter their existing schemes. Variations in the situations with respect to their context, material, and scope of the task is also a critical component of teaching experiments (Steffe & Thompson, 2000). However, tasks should not go too far beyond a student's current scheme but, rather, should cause enough of a perturbation that forces students to reorganize their existing schemes in novel ways.

Confrey and Lachance (2000) have further moved the field in the teaching experiment methodology with the idea that teaching experiments should be conjecture-driven in their description of “transformative teaching experiments.” Their use of the word “conjecture” does not mean an assertion that must be proved or disproved. Rather, conjecture guided research, “seeks to revise and elaborate the conjecture while the research is in progress” (p. 235). Hence, a conjecture constantly evolves through the research process whereas a research hypothesis remains static. Although conjectures are formed through theory, they seldom burst forth fully formed from the researcher’s mind. Indeed, they are usually formed only after a thorough review of the literature “to discern an anomaly that has been overlooked, unsolved, or addressed inadequately by one’s colleagues” (p. 236).

Initial Learning Trajectory and Task Creation

In the context of this dissertation, an initial DELTA learning trajectory for length and area was constructed from a review of the literature in consultation with Confrey. This trajectory serves as the evolving conjecture in the conjecture-driven teaching experiment. This initial learning trajectory is presented in Appendix H and a discussion of how the trajectory changed in light of the results of the teaching experiment is presented in Chapter 5. The initial learning trajectory contains a brief description of each proficiency level, one or two questions that targets the student issue[15] at the level, the cognitive aspects (including any conjectured emergent properties) at the level, and a brief description of a task that is conjectured to elicit reasoning at the level. These tasks, some of which are adapted from previous studies (e.g., Clements & Sarama, 2009; Lehrer et al., 1998; Piaget et al., 1960), were developed in consultation with Confrey and members of the DELTA team during our

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[15] That is, a question that might be directly posed to a student or asks how students would respond to a task.

weekly group meetings. For the first five levels of the initial trajectory, tasks were adapted from Piaget et al. (1960), Clements and Sarama (2009), and Barrett and Clements (2003). The exceptions were the first level (identification of length and area) and the third level (conservation of rigid transformations). For the first level, the DELTA team believed that a task asking students to sort objects into categories they thought were "the same" would allow students to more flexibly talk about length and area, including identifying length and area on the same object. For conservation of rigid transformations, we asked students what actions could be done to a straw to not change its length and a rectangle to not change its area. For initial levels 6-10, we started with Lehrer's et al. (1998) three rectangles task and added additional probes to investigate issues of conservation, co-variation, and equipartitioning. Tasks for initial levels 11-15 were developed by Confrey and Nguyen. Tasks for levels 1-10 were piloted with a small sample of students (n = 10) before the teaching experiments.

Validity

In the last steps of the DELTA Method for Learning Trajectory Creation, item response theory models are used as an addition way to validate the trajectory, as more difficult items targeting the upper levels of a trajectory should be more difficult for students to answer (Confrey et al., 2010). However, literature synthesis, clinical interviews, and teaching experiments are no less important in establishing the validity of a trajectory. As numerous researchers in mathematics education have reported (Confrey, 1998, 2006; Ginsburg, 1997; Opper, 1977; Simon, 1995; Steffe & Thompson, 2000), as one conducts numerous clinical interviews and teaching experiments with students using the same task, noticeable patterns in student strategies and misconceptions and barriers can be observed. While no one would claim that all strategies or misconceptions will eventually be captured, researchers are confident that a majority of strategies are captured using these methods.

To further contribute to the validity of the analyzed interviews, triangulation strategies and peer review of the transcript and video were utilized in this study (Merriam, 2002). In addition, as part of the DELTA team's work on the construction of learning trajectories, select videos were reviewed by the entire DELTA team.

Participants and Method for Dissertation

The participants were four third-grade students (8-year olds), three boys and one girl (Peter, Raima, Moemen, and Nabeel; all names are pseudonyms). The sample was selected as a convenience sample. Peter was recruited from the community while Moemen, Nabeel, and Raima were recruited through their classroom teacher at a local charter elementary school during the spring of 2010. During the study, interviews and teaching experiments occurred in the parent-teacher conference room during the school's mathematics class period.

For this study, I acted as the teacher-researcher and worked with the students on tasks designed to illicit their thinking on the initial DELTA length and area learning trajectory. Nine teaching episodes of approximately 40 minutes were conducted with each child (Moemen and Nabeel were taught as a pair). Data collected included videos of the teaching episodes, students' written work from the teaching episodes, my planning notes going into the teaching episodes, field notes taken during and after the teaching episodes, and revisions to the tasks based on reflecting on each day's teaching.

For this conjecture-driven teaching experiment (Confrey & Lachance, 2000) the initial DELTA learning trajectory for length and area served as the initial conjecture. Before each teaching episode, the task, hypotheses and teaching goals were reviewed. After each teaching episode, a set of field notes were written, reflecting on the day's teaching episode and outlining what changes were spontaneously made in the planned episode and why. A summary of what students appeared to learn and how this might influence the next teaching episode was also recorded. A detailed plan for the next day's teaching episode was created from these field notes. Ongoing analysis of the teaching episodes was conducted between teaching episodes. Each day's teaching episode was watched and analyzed. In addition, a working log of generalizations observed throughout all the teaching episodes for all students along with variations and working explanations was kept.

The conjecture driven-teaching experiment methodology was used to justify changes to the initial learning trajectory. Analyzing student thinking on various sequences of specific tasks targeted toward the initial learning trajectory from the teaching experiments allowed a first approximation of the possible outcome spaces for each level of the initial trajectory. Data for the dissertation was analyzed using the DELTA Framework for Understanding (Confrey et al., 2010). This framework describes how students' approaches and patterns of responses to tasks mature over time due to instruction and cognitive development. The framework posits students' development progressing through a series of: (a) strategies and representations, (b) mathematical reasoning practices, (c) emergent relations and properties, (d) generalizations, and (e) misconceptions and barriers. At first, student approaches focus primarily on generating solutions to problems. Instruction helps them master these strategies and introduces them to best practices in the field (for example naming to describe the outcome of a strategy and justification of strategies). As students develop a repertoire of strategies and solve increasingly complicated tasks, they begin to develop nascent elements of mathematical relations that emerge from their initial strategies. These mathematical relations form the basis for generalizations that often codify complex understanding of several proficiencies. Finally, as students progress, they encounter common misconceptions and barriers.

This framework is useful in the modification of a learning trajectory and construction of initial outcome spaces because it anticipates the role of a learning trajectory as encompassing a description of the means by which a student progresses, rather than simply a sequence of behaviors or observations. Because students not only demonstrate increasing success in generating strategies to solve tasks, their development of improved reasoning practices, generation of emergent relations properties, and engagement with critical barriers represents ways in which they shift to higher order thinking in mathematics as they move through the learning trajectory. Analyzing the ways that students move from strategies to emergent properties using Confrey's (1997) *Voice and Perspective* ensures that the constructed outcome space respects the inventions and perspectives of students, and not the expert understandings and mathematical structures and biases of the researchers. Such a heuristic will be vital in understanding how students naturally and spontaneously handle measurement tasks and examine whether or not they naturally bring in ideas and resources that we have missed in our standard teaching of measurement.

The analysis included analyzing transcriptions along with the artifacts and videotape records for students' strategies, mathematical reasoning practices, emergent relations, and misconceptions. Short memos were recorded with the *Transana* software to identify themes and categories in the transcripts. After the initial coding, all transcripts and artifacts were reviewed to extract major categories. I sought patterns within the categories along with variations to specify properties and dimensions of each category.

The second stage of analysis for the development of the trajectory was to identify exemplars of each of the strategies, practices, properties and misconceptions and incorporate these into the learning trajectory. Next, revisions to the levels of the initial learning trajectory and their descriptions were undertaken. Finally, each teaching experiment was written up as an individual case, the findings which are presented in Chapter 4.

CHAPTER 4

RESULTS OF CASE STUDIES

In this chapter, I will present each of my three teaching experiments as a case study. Two of the teaching experiments were one-on-one (Peter and Raima) while the third teaching experiment was conducted with a pair of students (Moemen and Nabeel) to enable the tracking of interactions. Peter attended public school and was recruited through a local church group while Raima, Moemen, and Nabeel attended a private charter school (K-8). All of the students were in third grade.

Among the four students, Peter's approaches were most likely to be focused at the level of strategies and representations as he illustrated the transition between conservative and non-conservative measurement strategies. He did developed several emergent relations and properties from his strategies, although he would often reconsider them in later teaching episodes and needed more examples and time before he fully believed in his conjectures. Working with Peter on the mathematical reasoning practices associated with equipartitioning (e.g., times as many and composition of splits) allowed him to express emergent relations and properties within an equipartitioning context. For example, he thought about the difference between area as enclosed space and measurement of area as number of units by considering two congruent cakes. He explained if one cake were shared for six people and the other for two, he could eat more if he had a slice from the cake that was shared for two because its slices were bigger even though the other cake had more total slices.

Moemen and Nabeel's approaches also functioned mostly at the level of strategies and representations and when their strategies led to an emergent relation and property, they seldom relied on further examples to be convinced of their conjectures. Their equipartitioning skills included equipartitioning single wholes and naming the results. Through instruction, they learned multiple methods of sharing a single whole, composition of splits, and *n* times as much. As a result, they developed the relationship between unit and measure and were able to compare the size of different units when the multiplicative difference was a whole number and they had manipulatives.

In contrast, Raima consistently developed many of her strategies into emergent relations and properties and then into generalizations. Her approach to the tasks reflected these abstractions and she primarily used the manipulatives to check her conjectures. She had a robust understanding of equipartitioning and could use composition of splits to share a whole multiple ways (exhausting all factor pairs) and believed that a whole cake could be shared with any number of people. By the end of the teaching experiment, she approached a quantitative notion of compensation of area units[16]. Although she did not state the full generalization (nor would I expect her to), given two rectangles equal in area (not necessarily congruent) where the first rectangle was equipartitioned into *m* parts and the second into *n* parts, she could compare the relative sizes of the parts. She did this first for whole number

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[16] If a rectangular area unit *N* measures a rectangular area *n* times and another rectangular area unit *M* measures a rectangular area *m* times, then Size (unit *N*) : Size (Unit *M*) :: *m* : *n*. For example, a 3x4 rectangle is measured 6 times by a 1x2 unit and 12 times by a 1x1 unit. It follows that Size (1x2 unit) : Size (1x1 unit) :: 12 : 6. That is, the 1x2 unit is two times as large as a 1x1 unit.

ratios and eventually for rational number ratios after I taught her the concepts of ratio boxes and daisy chains[17] (Confrey, 1989).

In this chapter, I present the cases for the four children in the following sequence: Peter, Raima, and finally Moemen and Nabeel. I begin with Peter as the student whose approaches were mostly at the level of strategies and then move to Raima, the most advanced, to provide contrast. The variations of approaches are represented by the case of Moemen and Nabeel. Then, I synthesize the strategies and representations, mathematical reasoning practices, emergent relations and properties, generalizations, and misconceptions and barriers that emerged from these teaching experiments into a hypothetical learning trajectory using exemplars from the teaching experiments to delineate initial outcome spaces for each proficiency level in the trajectory. Chapter 5 discusses the implications to the extant research in the field of measurement in light of this dissertation study and poses further questions for investigation.

Peter

Peter began the teaching experiment without conservation of length and area. He also did not exhibit the characteristic of understanding the reasoning behind logical mathematical transitivity. His case illustrates how equipartitioning can be used as a rich context to develop

[17] In her teaching experiment, Confrey (1989) challenged her third grade students to take an integer *a* and, using only the operations of multiplication and division, form a "chain" of operations transforming *a* into an integer *b*. Students were able to generalize that they could solve any daisy chain problem in two steps by dividing *a* by itself and multiplying the result by *b*. They learned to code this as multiplying by the rational number *b*/*a*.

area measurement. Indeed, the careful sequencing of equipartitioning tasks helped him to express his emergent relations in terms of equipartitioning contexts.

Teaching Episode One

In this teaching episode, I worked with Peter on Task 1[18] to study his understanding of length and area as attributes of an object. In this task, students were given eight objects (a rectangular sheet of aluminum foil, a rectangular piece of paper towel, an index card, a rectangular sheet of plastic, a triangular piece of cloth that was half the size of the rectangular sheet of plastic, a piece of dental floss, a bendy straw, and a wooden dowel) and asked: "Sort the objects into piles of objects that you think are the same. You have to sort all of the objects."

He made four piles by grouping the straw and dowel together, putting the dental floss by itself, putting the index card by itself, and grouping the plastic sheet and triangular cloth together. When I asked him why he grouped the straw and dowel together, he explained that one was a straw and the other was a stick, they both were round on the side, and you could use them to drink from a glass if the dowel was hollow. Hence, he was attending to their material and usage and did not necessarily perceive them as having length. Unable to get him to explain further about length as an attribute, I asked him why he had grouped the plastic sheet and triangular cloth together.

P: Because this [the plastic sheet] is like ... like a ... this is like a cover to cover something and this [the cloth] is also like a cover to cover something.

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[18] See Appendix A for a copy of the full task.

I: What does cover mean to you?

P: It means like you put something over something.

I: Can something cover something more than something else?

P: Yes.

I: Okay, can you explain for me how something might cover something more than something else?

P: Umm ... like something might be under something and something else covers something. But that's like a bigger and then you put this over it and the other thing covers it more because it's bigger.

I: Which one of these covers more [pointing to the plastic sheet and cloth]?

P: This [picking up the sheet of plastic].

I: That? And how do you know?

P: Cause ... this [the cloth] is like a triangle and it only covers half of the square.

I: And how do you know that it only covers half of the square?

P: Because if you did that [placing the cloth on top of the plastic] it would cover half.

In this exchange, Peter demonstrated that he understood area as a covering, that covering is an attribute of an object which can be compared, and a strategy to directly compare the cover of two objects is to place them on top of each other. For him, the "bigger" object will completely cover the smaller object. He also demonstrated, in the case of one-half, that he could use a fraction to compare the relative size of two objects. This is analogous to the example in equipartitioning where students were able to share a single whole among two people, named each person's share as one-half, and stated that the whole was two times as big as each share (Confrey et al., 2010). In the area context, Peter made an analogous argument by claiming that the cloth was half as big as the plastic sheet.

Next, I asked him why he had grouped the foil and paper towel together but had placed the index card by itself. Again, paying attention to material and usage, he explained that the foil and paper towel could both be used to cover food while the index card could be written on. When I asked him if these three objects could also be used to cover, he said they could, described them as a group sharing the quality of covering, and decided they could also be grouped together with the plastic sheet and cloth.

I asked him more about the length objects, asking him to first explain why he placed the floss by itself. He explained that the floss couldn't be used to cover anything (and hence could not be placed with the covering objects) and that you couldn't drink out of it like the straw. I asked him if the floss shared any qualities with the covering objects, to which he responded no. His response provides some evidence that he understood that one cannot directly compare the magnitudes of length and area.

Next, because he was able to compare the size of coverings, I pointed back to the straw and dowel and asked, "Is there anything that you can tell me about their size?" He aligned the two objects side-by-side, observed that the straw extended further, and concluded that the straw was longer. Moving the objects side-by-side demonstrated that Peter understood that length is conserved under translations[19]. Next, I asked him to define longer and shorter.

P: Like if this [the dowel] was also a straw you could get more out of this [the straw] because it's [the dowel] shorter.

I: What do you mean by more than? What more can you get?

P: Like something to drink.

I: So if this were a straw [pointing at the dowel], you could get more out of this straw [the straw] than that straw [the dowel]. Why do you think that?

P: This [picking up the dowel] would take longer … [changing his mind] *this* [picking up the straw] would take longer [pauses] because this [the straw] is longer. This [the straw] would take longer to get water or something to drink but this [the dowel] would take less amount of time because it's shorter than the straw.

Although he had initially compared the straw and dowel by placing them side-by-side and claimed the straw was longer because it extended further, he switched to a rate argument in his explanation of longer and shorter. He claimed that if the dowel were used as a straw, it would take more time for a drink to reach his mouth with the straw than the dowel because the straw was longer[20].

[19] This understanding was originally posited to come after direct comparison but evidence from all three teaching experiments suggests it is a prerequisite for direct comparison.

[20] Confrey (personal communication, 2010) has claimed that children reason about rate and scaling at much earlier ages than in their mathematical instruction because both are necessary to navigate in a three-dimensional world. Peter's explanation of length as rate provides evidence for this claim and warrants further investigation.

Although I had not intended to work with him on direct comparison until the second teaching episode, I asked him if he could compare the length of the floss, straw, and dowel. He used the same direct comparison strategy that he had used to compare the straw and dowel by lining the three objects side-by-side (pulling the floss taut), noticed that the dental floss extended further than the straw and dowel, and concluded the floss was the longest object. He further claimed that if you put the straw and the skewer together, they would still be shorter than the floss. I asked him to explain, and he placed the straw and dowel end-to-end, aligned the floss side-by-side them, and showed that the floss still extended further. Hence, Peter demonstrated a *non-numeric* additive notion of length believing that a length AB can be combined with a length CD end-to-end with B and C touching to create a longer length AD such that AB + CD = AD[21].

To conclude this teaching episode, I investigated if Peter could identify length and area on various objects. I gave him a ball of playdough and asked him if the playdough had "longness." He said yes, and explained that an inch was a type of length. I asked him what a length was and he responded that a length was how long something was. I was puzzled by how he defined an inch as a type of length and asked him if he could show me an inch. He showed me that two of his fingers were an inch. I asked him if the playdough had any other

[21] In the DELTA methodology, we place mathematical reasoning practices and emergent relations as separate proficiency levels even if they are initially used to solve a task at a lower proficiency level. Even though students may use an emergent relation to solve a task, they have often not interiorized and cannot articulate that emergent relation. When we observe this, we construct a new task that specifically targets the development of the observed emergent relation. So while students in my teaching experiment demonstrated additivity of length and area, for the purposes of the teaching experiment I did not construct a new task to test this understanding but rather flagged it as an area of further investigation for the DELTA group.

lengths and he said that it also had centimeters. I realized that for Peter, a *type of length* does not mean that an object could have several length dimensions (i.e. width and diameter) but that inches and centimeters were *types* of lengths. I asked him what a centimeter was and he responded that it was a type of length shorter than an inch and the width of one finger. He demonstrated that he could use his fingers as an iterating unit (albeit with gaps and overlaps) when I asked him to show me length on a rectangular piece of construction paper. Iterating two fingers down the side of the rectangle he concluded that it was "12 inches long."

To summarize, in this teaching episode Peter demonstrated that he could distinguish qualities of length and area and understood length as an extension and area as a covering. He also provided some evidence that a length could not be directly compared with an area. He exhibited strategies to directly compare two straight lengths and two completely overlapping areas and, in the case of one-half, could make a claim about the relative size of two covers. He also demonstrated non-quantitative additivity of lengths by joining two lengths together and claiming that together they created a new length.

Teaching Episode Two

In this teaching episode, I worked with Peter on Tasks 2 and 3[22] which investigated direct comparison of straight lengths and completely overlapping areas and invariance of rigid transformations (length and area stay the same under translations, rotations, and reflections).

[22] See Appendix A for a copy of Task 2 and Appendix B for a copy of Task 3.

As expected from the first teaching episode, Peter was able to order two or more straight lengths and two or more completely overlapping areas. As he employed his previous strategies (aligning lengths side-by-side and placing area objects on top of each other), I will not report on his work on Task 2.

However, after he had ordered the area objects from smallest to largest, I wondered if he understood that areas were also additive because in the first teaching episode he demonstrated that lengths were additive when he combined the dowel and the straw's length and compared it to the length of the floss. I used the objects shown in Figure 4: a pink, blue, and yellow rectangle, and a pink half-circle and quarter-circle (cut from the same circle).

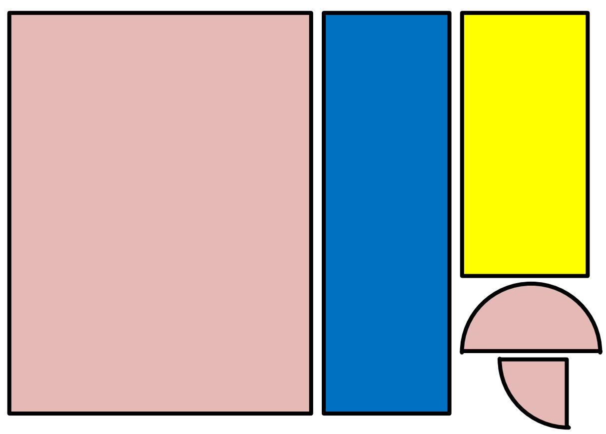

Figure 4. Objects used in additivity of area investigation.

I started the investigation by asking him if the blue and yellow rectangles, combined, were bigger than the pink rectangle. He placed the blue and yellow rectangles side-by-side (non-overlapping) on top of the pink rectangle, saw that the pink rectangle had more "space showing," and concluded that the pink rectangle was bigger. Next, I asked him if the blue and yellow rectangles along with the quarter circle were bigger than the pink rectangle. He placed the quarter circle above the yellow rectangle and concluded that the pink rectangle was still bigger. Finally, I asked him to consider those three objects plus another half-circle. The half-circle was problematic for him because he could not fit the blue and yellow rectangles, the quarter-circle, and half-circle on the pink rectangle without overlap. Although he conjectured that he thought the four objects covered more space than the pink rectangle, he wasn't sure. When I asked him if there was any way to find out for sure, he said that it couldn't be done without overlapping. After giving him some time to think, I asked him if he could figure it out if he were allowed to cut the objects. He thought about this for a few seconds and exclaimed, "Yes!"

I gave him scissors and he cut the quarter circle into strips and placed them to the right of one of the yellow rectangle as shown in Figure 5.

Figure 5. Peter's use of decomposition to compare areas.

After placing all the quarter-circle strips, he concluded that the pink rectangle was larger than all of the other objects combined because there was still space left over. His strategy implied that understood conservation of decomposition/composition of area over breaking. That is, he had to believe that no area was lost when an area is decomposed and the pieces[23] are joined back together. I wanted to see if he could formally explain this emergent relation and property.

I: So, these four pieces here [of the quarter-circle] ... do they cover the same amount of space, more space, or less space than before you started cutting?

P: The same!

I: Okay, the same. And how do you know that?

P: Because even if I cut all of it into different pieces they still cover up the same because it's not like a different size of paper and [you can] cut it into more pieces.

Peter's response confirmed his understanding of conservation of under decomposition[24] over breaking. Next, I worked with him on Task 3. In the first teaching episode, he demonstrated that length and area were conserved under rigid transformations. As expected, when asked, "What are some things you can do to a straw that won't change the length of the straw?" he showed me that moving it, throwing it, flipping it, and turning it would not change the length because, "it's not like it's [the straw] going to get shorter or the

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[23] For clarity, I will use *decomposition* from this point forward to imply that a length or area was broken up into different sized *pieces*. This is in contrast to my use of *equipartitioning,* when a length or area is equipartitioned into *n* same-sized *parts*. Hence, in the context of the narrative, *pieces* are implied to be unequal while *parts* are implied to be equal. Note, however, that within the interview transcripts I am often inconsistent with my language. I believe that to avoid confusion in instruction, teachers should be aware of this language, be consistent in their use of it, and define these terms for their students.

[24] Although I did not ask him to rejoin the decomposed parts, his response indicated a more sophisticated form of decomposing and rejoining. For him, the four decomposed pieces, when abstractly added together, have the same area as before he cut. That is, he considered all of the pieces together abstractly as the pieces were not joined when he placed them on top of the pink rectangle. Hence, he should conclude that the area is conserved under any rejoining of the pieces (not just into the original configuration).

table is magic and it's going to eat the straw! It's not like when you move it, it's going to eat some of the straw." I report now on how the question, "Anything else we could do to it so its length won't change?" launched an investigation of conservation of length decomposition/composition over breaking.

I: Anything else we could do to it so its length won't change?

P: Cut it! [I gave him a pair of scissors and he decomposed the straw into two pieces.]

I: Okay. So how did you change the length?

P: Because I got the scissors and I cut it.

I: And so now is it longer or shorter than it was before?

P: Shorter

His response surprised me because previously, for area, he had stated that area is conserved under decomposition. I wondered if he meant that the two decomposed straw pieces, taken together, were shorter than the original straw or if he was only comparing one decomposed piece of the original straw.

I: What if we took these two pieces and what if we glued them back together? Would it be longer, shorter, or the same as before you cut it?

P: The same.

I: The same? And why do you think it would be the same?

P: Because when I cut it the ... these two pieces didn't get like smaller so they still stayed the same size so when you glue them back … the straw … together it's still the same size as before I cut it.

He demonstrated that he understood decomposing a length and joining it back in the same configuration conserves length. Hence, he was previously comparing one of the decomposed pieces to the original straw. However, the action of bending the straw gave him trouble and showed that Peter was still developing conservation of length.

I: Anything else you could do to it to change the length besides cutting?

P: Bend it.

I: So when you bend it does it make it longer or shorter?

P: Shorter.

From my pilot interviews and the literature (Clements & Sarama, 2009), a common misconception for young children when they first encounter bent paths is to consider the endpoints of the path rather than the total length of the path.

I: Shorter. So right now it's this length, right [using my fingers to indicate the straw's length]? So now you bent it like this [bending the straw]. So you think it's shorter. Why do you think it's shorter?

P: Cause when I bent it, it … I lost some of the stick.

I: Could you use your fingers to show me how long it is before you bent it [unbending the straw].

He placed his thumb on one endpoint and his index finger on the other endpoint.

I: Okay so it's that long. So bend it now. And ... how long is it now?

He bent the straw and again placed his fingers on the endpoints.

Thus, he was observing the length between the endpoints and not the total path length. I tried a probe that Piaget et al. (1960) had used in their investigation of conservation of length, asking him to consider the straw as a path that an ant walks. I unbent the straw.

I: Okay, so let's say this [straw] is a path in the park. And let's say an ant is walking on this path. So the ant walks from here down to here [running my finger down the straw]. Let's say the path is slightly bent like this [bending the straw]. Okay? So now the ant is going to walk down like this [running my finger from an endpoint to the bend] and then he's going to turn on the path and then walk like that [running my finger from the bend to the other endpoint]. Do you see what I'm saying?

P: Um hmm.

I called the unbent path Path A and the bent path Path B and asked him, "Do you think the ant walks further on Path A or Path B?"

P: A.

I: Okay. And why do you think A?

P: Because the path got smaller like when you bent it. The path didn't like bend and go like up. So, when it was bent it just stayed and it didn't go up like this.

I: Can you clarify what you're saying? I'm not getting exactly what you're saying. So when I bent it, it didn't go up? What do you mean by it didn't go up?

P: It didn't do like diagonal ... it didn't go like diagonal.

I: And so you think the path becomes shorter?

P: Um hmm.

I: How much shorter do you think it becomes?

P: It gets this much [uses his fingers to show the approximate length of the second piece of the straw after the bend]!

I explored this apparent contradiction.

I: Does it get shorter by this amount right here [using my fingers to denote the length of the second piece of the straw after the bend]? Is that how much shorter it gets?

P: Yes.

I: I see. Interesting. But isn't the ant still walking down this way and then walking all the way down this way [using my finger to trace the two parts of the bent path]?

P: Yeah ...

I: So how could it get this much shorter if the ant is still walking all the way down this way?

P: [Gasp!] It couldn't …

I: What do you think?

P: He walked the same amount of … he walked the same amount on both paths!

I: So why did you change your mind?

P: Because even when he was on Path A, he was doing like that [uses his fingers to trace the length of the unbent straw] and then on Path B he did that [bending the straw]. And it's still … he's still walking as much as Path A but it's just a bend. So it doesn't …. so it means it goes like that and then it doesn't go farther. It turns.

I: I see … so does it matter how we bend it? I think I bent it like this before [unbending the straw]. If we bent it like this [bending the straw the other way] does it matter? Or do you think the paths are still the same?

P: The paths are still the same!

I: And what made you change your mind? Cause I think before you said that it was going to be shorter.

P: Because … at first I thought it was shorter because it … it went like that [bends the path]. He walked more because he did that [straightens it out and references the straight path]. He went on Path B and it only has the bend so the path wouldn't get any shorter.

Originally, I thought that he was observing the endpoints and expected him to make a subtraction argument, arguing that the length of the bent path was the distance between the endpoints and that he would compare that length with the length of the unbent straw[25]. His response indicated that this was not the case. Pointing out the contradiction and showing him that the ant must walk down both pieces of the bent path caused a re-accommodation for him. As a result, he now believed that bending a straw does not change its length. Based on this re-accommodation, I conjectured that he would demonstrate conservation on Piaget's length conservation tasks (Task 4) in the next teaching episode.

To summarize, in this teaching episode Peter used his direct comparison strategies to correctly order a set of straight lengths and completely overlapping areas. He also recognized

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[25] Another common misconception, which I encountered in my pilot interviews, is for children to reason that the bend causes the path to get slightly longer. Children that exhibit this misconception will state that the path gets longer by a very minute amount.

that an area can be decomposed and the decomposed pieces, together, have same area as the original whole. This seems to be a more powerful form of conservation of area composition/decomposition over breaking because it implies that rejoining in any configuration also conserves area. Despite this area conservation, however, he believed that a straw's length is not conserved under bending. Asking him to consider the straw as a path and that an ant must walk on both pieces of the bent path helped him rethink the problem.

Teaching Episode Three

In this teaching episode, I worked with Peter on Task 4[26] which was based on Piaget's et al. (1960) conservation of length tasks. In the task, students are given matchsticks (I used straws) which they are asked to cut. The sticks are rejoined in different configurations and the student is asked a question to investigate their understanding of conservation. I used the following scenarios: 1.) A straw is decomposed into two pieces and rejoined in its original configuration. The student is asked to compare the original straw with the rejoined straw. 2.) Same as (1), but the straw is decomposed into three pieces. 3.) The student is given two equal-length straws. One straw is decomposed into two pieces. The pieces are rejoined with a bend. The student is asked to compare the length of the uncut straw with the rejoined straw. 4.) Same as (3), but the straw is decomposed into three pieces. 5.) Same as (4), but the other straw is decomposed into two pieces. Based on the last teaching episode, I hypothesized that Peter would assert and explain conservation in all five scenarios.

[26] See Appendix C for a copy of the full task. I provide only a brief description in this narrative.

I first verified that he could assert conservation for scenarios one and two (which he did in the previous teaching episode). I told him to again consider a straw as a path that an ant walks on (as this had helped him in the previous episode). He asserted conservation explaining, "This time you cut it but it's like before when you bent it. When you bended it, the path wouldn't get any shorter and when you cut it wouldn't get any shorter." He reasoned similarly for scenario three.

Next, I posed to him scenario four. After decomposing one of the straws into three pieces and rejoining them in a bent pattern I asked him, "On which path does the ant walk further?" He replied that they walked the same because, "before you cut the paths, they're the same length." For this scenario, he went further and justified his reasoning by straightening the three straw pieces that composed the bent path, placed them next to the uncut straw, and showed that the straws were still equal in length. This further verified that he understood additivity of lengths. He approached scenario five similarly, asserting that the two rejoined paths were equal and justified by arranging them back into straight paths.

To summarize, in this teaching episode I verified that Peter understood conservation of length decomposition/composition over breaking. As expected from the previous episode, he was able to assert conservation in all five scenarios. In scenarios four and five, he further demonstrated his understanding of additivity of lengths by showing that the bent paths, when rejoined in a straight line, were equal.

Teaching Episode Four

In this teaching episode, I worked with Peter on Task 5 to investigate his strategies for indirectly comparing two, possibly bent, line segments. I gave him an unmarked straightedge, a pencil, scissors, a piece of string (cut to be shorter than any of the segments he was asked to compare), and different length straw pieces (to see if he would use them as an iterating unit[27]). He was then asked to compare three pairs of line segments (one pair at a time). He was first given two straight unequal line segments: path A (aligned diagonally on a sheet of paper) and path B (aligned vertically on another sheet of paper[28]) and asked which segment was longer (path A was longer).

First, he tried to put the two segments side-by-side to use his direct comparison strategy. However, he noticed a limitation in his ability to directly compare the paths (which were intentionally constructed to differ by less than one inch) because he used scissors to cut around the vertical path. He was then able to use his direct comparison strategy and concluded that path A was longer by one inch (he used his two fingers = one inch rule). In hindsight, I should not have permitted him to move the paper or cut the paths as these actions facilitated a direct comparison strategy. However, his strategy was viable and reconfirmed his understanding of conservation of rigid transformations (he cut and moved the vertical

[27] In the DELTA group meetings that followed my data collection, Confrey (personal communication, 2010) noted that giving students external iterating units privileges the iteration with no gaps and overlaps strategy (e.g., Outhred & Mitchelmore, 2000) over other viable strategies such as mapping to a longer object and selecting an internal unit. I acknowledge that giving students the straw pieces also causes the loss of parallel structure in this work because I did not give students an iterating unit when I later ask them to compare areas.

[28] Please see Appendix D for the full task including copies of the line segment paths used in the task.

path knowing this would not change its length) and that he could take a more complicated task (comparing two non-aligned paths) and turn it into one that he knew how to solve (direct comparison of lengths).

Next, I gave him paths C and D (path D was longer), where path C had one bend and path D had two bends. Again, I did allow him to move and cut the paths. Appealing to the overall shape of the paths, he initially told me that path C was longer, “because this [path C] doesn’t curve [pointing to the second straight part of path C] and … if it curved like this [pointing to the second bend in path D] ... it might not be as long because it would have to curve.” His explanation seemed to imply that if path C had a second curve like path D, then it might not be as long. Unsure if the misconception was perceptual or if he really believed that additional curves could shorten a path (which would contradict his understanding of additivity of length), I asked him, “Is there any way that you could check to make sure?” He proceeded to use his previous strategy and decomposed path D into its three pieces. Instead of decomposing path C and using a direct comparison strategy, however, he placed the decomposed path D pieces on top of path C to compare. Because one of the pieces of path D was the same length as one of the straight pieces of path C, this strategy was viable, and he did not have to decompose path D further to compensate for the additional bend. He concluded that his initial perceptual guess was incorrect and that D was longer than path C. I asked him how he knew and he replied, “Cause some of this piece was going off of this line.” That is, the third piece of path D extended past the second piece of path C. This is not a direct comparison strategy per se, but a combination of a decomposition and additive strategy.

Last, I gave him paths E and F (path E was longer), where path E had three bends and path F had four bends. This time, he did not use his decomposition strategy but instead relied on using two of his fingers (which he called an inch) as an iterating unit. He measured both paths using finger iteration (trying his best not to have gaps or overlaps) and concluded that path F was 13 and path E was 16 (note that he named the path lengths as a magnitude without reference to a unit). His strategy demonstrated successful application of unit iteration without gaps and overlap. However, because he named the lengths using only a magnitude, I was unsure if he understood that measurement (as a number) is always relative to a unit size or if he had the misconception that unit size did not matter. I asked him if there was any other way to measure path F.

P: It could be centimeters.

I: So ... how many centimeters is this path [F]?

P: Twenty-six.

I: Twenty-six? Okay, and how did you know that it was 26?

P: Cause 13 times two is 26.

This seemed to imply that he understood 1 inch = 2 centimeters and to convert from inches to centimeters, one needs to multiply by two. However, he immediately said, "Wait …" and thought about it some more. He was still struggling with the relationship between inches and centimeters and proceeded to check his conjecture by iterating a finger down path F. "Twenty-six centimeters," he concluded. I asked him if it would be possible to compare 16 inches with 26 centimeters by just paying attention to the magnitude (i.e. 16 vs. 26).

I: Okay, so this path [F] is 26 centimeters. But this path [E] is 16 inches. So what if I said that this path [F] is longer? Would you believe me?

P: No.

I: No? And why not?

P: Because you probably didn’t measure this one [path E] in centimeters yet.

I: So if I measured this [path E] in centimeters, do you think it would be more than 26 centimeters, 26 centimeters, or less than 26 centimeters?

P: More.

I: You think it would be more? Okay, and why do you think it would be more?

P: Because 16 times two is bigger than 13 times two.

This time, he didn’t need to check to be confident that path E, if measured in centimeters, would be more than 16 centimeters. He showed an understanding that measure is not solely dependent on the magnitude but that unit size also mattered when he implied that I could not directly compare 26 centimeters with 16 inches. Moreover, he believed that path E would still be longer because if you measured it in centimeters you would get 16 times two which is bigger than 13 times two.

Having demonstrated that he was comfortable using different iterating units to measure a path, I checked to see if he could use a consistent iterating unit on bent paths (including perimeter) and coordinate iteration correctly with number. I asked him if he could measure the length around (i.e. the perimeter) a box of crackers using a straw piece. He began by iterating the piece down one of the sides of the box and, interestingly, at the corner he bent the piece of straw, marked where it ended on the new side, and began the next iteration at his mark. He maintained his strategy, correctly handling the corners. His only misconception came after counting five straws and seeing that it would take another half straw to completely measure the perimeter. But instead of concluding that the length was 5½, straws he said it was 6½ straws. I conjectured that the half straw might be throwing him off and asked him to reiterate after five straws. He realized that 6½ straws would be longer than the box's perimeter and concluded that it was 5½ straws around the box.

To summarize, in this teaching episode Peter demonstrated two viable strategies to compare complex paths: a) decomposing the paths and using the property of additivity, and b) consistent unit iteration (his fingers and a straw). His iteration strategy correctly compensated for bends and, with scaffolding, fractional units. He also demonstrated the ability to convert between units. It would be interesting to see which of these strategies and emergent relations and properties he would use when comparing areas.

Teaching Episode Five

In this teaching episode, I worked with Peter on Task 6[29], the comparison of three rectangular "cakes": a 12x1 (white cake), a 6x2 (purple cake), and a 3x4 (yellow cake). I asked which cake he would choose to eat if he wanted to eat the most cake. The tools available to him were scissors, a straightedge, and a pencil.

He started the task by juxtaposing all of the cakes as shown below in Figure 6.

[29] See Appendix E for a copy of the full task.

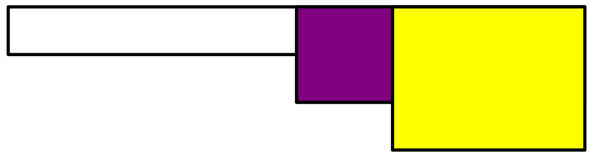

Figure 6. Peter’s initial juxtaposition to compare the three cakes.

This juxtaposition gave him trouble because in this teaching experiment (and perhaps in schooling), he had only compared completely overlapping areas. Scholars refer to these moments as *critical junctures* (Katz, 2007; Stephens, 2008), moments in instruction that require students to modify their existing schemes or invent new ones to solve a more complex task. In this case, could Peter use decomposition/composition or invent a new strategy to help deal with non-completely overlapping areas? After a few minutes of thinking, I asked him to temporarily ignore the yellow cake and concentrate on comparing the white and purple cakes. After some visual inspection and looking at the juxtaposition of the white and purple cakes, he stated that the white one was bigger. I asked him if there was any way that we could check to find out for sure. He again juxtaposed the white cake over the purple cake and reached for the scissors. I asked him to explain what he was thinking and he said, “I’m gonna cut the cake and if I cut it, I’m gonna see.” He equipartitioned the white cake into two 1x6 parts and laid them over the purple cake, completely covering it. He seemed surprised that the white cake parts completely covered the purple cake saying, “They’re both … the same …”

This may have been the first time that he encountered two non-congruent rectangles that had the same area. I asked, “That’s interesting … they’re both the same. So how can that be, Peter? I'm going to show you two fresh cakes." I placed a second purple and a second white cake on the table. "How is it possible that these two cakes that don't look alike at all, do they? But how is it that they can be the same?" He covered the first purple cake with the second purple cake and covered the second white cake with the two first white cake pieces, as if to verify that the cakes I had just given him were the same as the first cakes. This verification made him confident of his finding and he said, "Then these … [the white and purple cakes] are both the same … I cut them apart and put them together and they were the same" referring to his action of cutting the white cake into two 1x6 parts and using those parts to completely cover the purple cake.

Next, I asked him to consider the white cake and the yellow cake. He juxtaposed the two cakes as shown in Figure 7 below. The dotted line indicates where he marked the white cake.

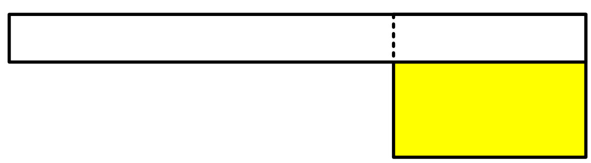

Figure 7. Peter's method of comparing the white cake and the yellow cake.

Instead of cutting the white cake on the dotted line, however, he translated the white cake, aligning his mark with the end of the yellow cake, and made another mark revealing two 1x4 parts. Examining the marks, he cut the white cake into three 1x4 parts, placed them over the yellow cake and said, "It's a tie again …" Again, he seemed surprised.

Having established the equality of the three cakes in terms of area, I decided to explore if he could reason qualitatively and quantitatively about area units in the same way he was able to reason about length units. Recall in the length example, he understood that the total length depended on the size of the unit and further was able to reason with his estimate that one inch is two centimeters to predict the outcome of using different units. That is, he could predict the effect on the number of units needed to measure a rectangular area when the unit was increased or decreased in size in a similar way.

I: So we have three pieces of this size [1x4] that go into this yellow cake. Can you predict what would happen … if I took one of these pieces and cut it in half? [equipartitioning[30] one of the 1x4 units[31] in half creating a 1x2 unit.] How many of these [1x2] pieces would fit?

Without relying on the manipulatives, he immediately responded, "Six!"

I: So how did you know that?

P: Because if you cut ... three times two is six. And you cut each piece in half then it would be six.

Peter demonstrated an elementary version of Level 7 understanding (predicting the outcome on a composition of splits[32]) in the DELTA equipartitioning learning trajectory (Confrey et al., 2010). In Peter's case, he was able to predict the number of pieces produced

[30] I will use the word equipartition to refer to any action that creates equal sized groups or shares.
[31] I will use the word unit from this point forward when referring to parts resulting from equipartition used for measurement.
[32] An n split followed by an m split yields nm parts.

after a split. This anticipates multiplication, as he was able to reason that the action of splitting each of three parts into two parts yields six total parts. Moreover, he associated this action with a multiplication fact. He also understood this from an area perspective: if three 1x4 parts cover the purple cake and each part is equipartitioned into two parts, yielding six total parts, then the six parts also cover the purple cake.

I: What if I cut each of those six pieces in half? How many of those pieces would go in here [the purple cake]?

P: Hmm ...

I: So what if I took this piece [the 1x2 unit] and I cut this piece in half [equipartitioning one of the 1x2 units in half creating two 1x1 units]. How many of these [1x1] pieces do you think would go in?

P: Umm ...

He picked up the 1x1 unit and floated it above one of the 1x4 units and said, "Twelve!" I conjectured that when he floated the 1x1 unit over the 1x4 unit he was trying to figure out (either through counting or multiplication) how many 1x1 units would fit into a 1x4 unit.

I: Twelve? And how do you know 12?

P: Because, umm ... these three pieces [1x4] are four so three times four is 12.

So he saw that equipartitioning the 1x4 unit twice (in half) yielded four 1x1 units and since there were three 1x4 units, this would yield twelve total units.

I concluded the teaching episode by investigating if Peter could apply transitivity to the case of area. That is, he knew that the white cake was the same size as the purple and yellow cakes. When asked to compare the purple and yellow cakes, would he need to empirically check this relationship or would he deduce that, logically, the purple cake and the yellow cake must be equal.

I: You told me that this white cake is the same size as this yellow cake. And you also told me that the white cake is the same size as the purple cake. Do you think the purple cake is bigger, smaller, or the same size as the yellow cake?

P: Bigger.

I: You think it's bigger? And why do you think that?

P: Because the ... wait ... the yellow cake is bigger.

To check, he juxtaposed the purple cake on top of the yellow cake as shown in Figure 9 below. Instead of cutting, he mentally compared the non-overlapping areas of the purple and yellow cakes and incorrectly concluded: "Yep, this [the yellow cake] is bigger."

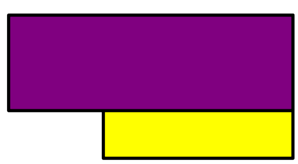

Figure 8. Peter juxtaposing the purple cake on top of the yellow cake.

Again, I encouraged him to check to make sure. He cut off the non-overlapping piece of the purple cake creating a 2x2 unit. Realizing that this piece overlaps the non-overlapping yellow part, he equipartitioned the 2x2 unit into two 1x2 units and used them to completely cover the non-overlapping yellow part. He concluded: "Really close!" and yet again seemed surprised by this result.

To summarize, in this teaching episode Peter encountered a situation in which non-congruent rectangles had the same area. He utilized two strategies to verify that this could happen: (1) Decomposing and composing the first rectangle and showing that the recomposed first rectangle was congruent to the second rectangle[33] and (2) juxtaposing the two rectangles and comparing the areas of the non-overlapping regions (subsequently by using composition and decomposition). I refer to both of these strategies as *methods of exhaustion* because students employ a continual strategy of overlap and comparison until they can conclude using the strategy of direct comparison of completely overlapping areas. He also demonstrated the ability to predict the effect on the number of units needed to cover an area when that unit was halved and write an associated multiplication fact.

Teaching Episode Six

I began this teaching episode by formalizing some mathematical vocabulary. Previously, Peter had told me that area could be used to talk about a spatial location such as in the phrase, "The area around this table." I told him that area could also be used to describe the size of a rectangle. I referred him back to the three cakes task.

I: You figured out that this white cake has the same area as this purple cake. Tell me how you did that.

P: I cut it and then I put the pieces in the cake.

[33] This strategy is referred to as *additive congruence* by Lehrer et al. (1998).

I: Okay. So you cut this in half [I equipartitioned the 1x12 cake into two 1x6 units and used them to cover the purple cake]. So do you think we could use “two” to describe the area of this [purple] cake? Two pieces of this [1x6] size?

He thought about this for a moment and agreed.

I: Okay. So it looks like this purple cake is two pieces, right? One way we could also do it is to draw it like this. [I removed the two 1x6 units and drew a line down the middle of the purple cake showing him how to mark to see that two 1x6 units composed the purple cake.] Is there any way we could make the area of this [purple] cake four?

He took the purple cake that I had already split down the middle and performed a composition of splits by drawing a line perpendicular to my line creating four parts. I asked him, “So why is it four?” He showed me by covering the purple cake with the two 1x6 white units, equipartitioned each unit in half to create four 1x3 units, cut out the units, and covered the purple cake with them. I asked him if he could do eight and he performed another composition of splits on the purple cake (which was already marked to be equipartitioned for four), splitting each fourth in half to create eight total parts.

Then he said, “And I don’t think I could make 10.” I challenged him to try. He started with the four-split, equipartitioned two of the fourths in half, and said “Six …” He split the other fourths in half and said, “Eight …” puzzled that he had created eight again. He tried to fix this by drawing another line, as shown in Figure 9, and concluded that he had made 10.

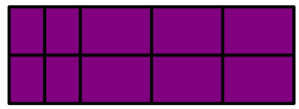

Figure 9. Peter's attempt to make an area of 10.

He looked at the cake and commented, "Some of them are uneven." I asked, "Is there some way we can make it 10, but make these [pointing to the 10 pieces] so they're even?" He began again with the four-split, not realizing that this strategy was not viable because four was not a factor of 10. He proceeded to perform two more 2-splits, creating 16 parts. He was excited that he had made 16 and we recorded his successful equipartitions: 2, 4, 8, and 16. Again, I challenged him to make 10 and again he started with the four split. This time, he split one of the halves into thirds creating six shares on one-half of the cake. After thinking for a minute he shouted, "I can make 12!" He drew another three-split on the other half of the cake and counted to show me that he had made 12. For him, this was the beginning of understanding composition of splits, as anticipated that he would create 12 parts.

I ended the teaching episode by checking if he was cognizant of composition of splits. That is, did he understand that the initial two-split combined with making n parallel cuts[34] would give him 2 (n + 1) parts? I began by showing him the cake he had shared for 12 and asked, "How many lines did you draw to make six in a row?" He replied, "Five!" I asked him how many parts that created and he said, "Six!" I pointed out that he had created six parts in a row and 12 parts altogether. Next, I showed him the cake he had shared for eight

[34] It is also not trivial for students to develop the understanding that n parallel cuts yields n + 1 parts (Confrey et al., 2010).

and, going through a similar process, he noticed that he had drawn three parallel lines creating four parts per row and eight parts altogether. Time had run out for this teaching episode, so I asked one last set of questions to help him equipartition for 10.

I: In order to get 10 total parts, how many have to go in a row?

P: Five!

I: Five. So how many lines do you think you should draw to make five?

P: Four!

I conjectured that he now had a viable scheme that he could use to create 10 during the next teaching episode. To summarize, in this teaching episode I began by introducing Peter to a way we could use numbers to talk about area, although I intentionally left out mentioning the referent unit. I asked him to use the purple cake to "make" areas of *n*, that is equipartition the cake for *n* people. He was able to equipartition for 2, 4, and 8 but could not equipartition for 10, although his attempts to make 10 allowed him to make 16 and 12. This was because his understanding of composition of splits only extended to composing a two-split with an even split. The 10-split was difficult for him because he could not yet create an odd split on a rectangle.

The importance of equipartitioning in this teaching episode caused me to revise my plan for Peter's teaching experiment. Originally, I had planned to move quickly through levels 7 – 10 of the initial learning trajectory (Appendix H). I planned to explore area = length x width by introducing the 1x1 unit as a convenient unit and, through a comparison task, conjectured that the row and column structure of the array would be an accomplishment for students as they compared different rectangular areas using the 1x1 unit. However, I decided to explore equipartitioning further with Peter to ensure that he could create all splits on a rectangle. Thus, the investigations of levels 7 – 10 became more important, and I conjectured that understanding how to split a rectangle into any number of equipartitions would allow Peter to construct and make meaning from multiple area units.

Teaching Episode Seven

Seeing the importance of composition of splits and equipartitioning in creating equal area units, I continued working with Peter to see if he could develop strategies to equipartitioning a rectangle for any number. I started by showing him the purple cakes that he had previously equipartitioned for 2, 4, 8, 12, and 16 and reviewed his strategies. We discussed the conclusion of the last teaching episode where he told me that drawing four lines to create five equal parts in a row would help him make 10. He struggled with the task of creating five equal shares, however, and I asked him if perhaps instead of five he could equipartition for three, knowing from the DELTA equipartitioning work that, in general, 3-splits on a rectangle are easier than 5-splits on a rectangle (Confrey et al., 2010). I also hoped that if he succeeded in creating a 3-split, he would be able to perform the 5-split.

His insistence on starting with the 2-split caused him to struggle with the 3-split. After several unsuccessful attempts he said, "I don't think you can do it [a 3-split]." I wondered if moving to a dynamic representation, where he could move the cuts, would help him visualize the 3-split. He was currently marking on the cake with a pencil to show where he would cut. I put out straws and told him that since marking on the paper was permanent, we would try using the straws to show where he would have drawn the lines. As an example, I showed him his last unsuccessful attempt where he had drawn two parallel lines to create three unequal pieces. I placed a straw on each line he had marked and asked, “Is there any way we could make it equal by moving the straws?” His face lit up and he said, “We *can* do three!” Using a new purple cake, he drew two parallel lines, but this time the parts were equal.

I: How did you do that?

P: When you were doing this [pointing to the straws I had placed on his last unsuccessful attempt] it gave me the idea that I can make three cause I saw three pieces and I thought, huh! I can make three pieces with it. Then I turned it over and made three pieces!

The dynamic straw representation helped Peter to create the 3-split. As an aside, he suddenly conjectured that although he was still unsure if he could equipartition a rectangle for five, he could make five on a circle. I drew a circle and he demonstrated this for me using radial cuts to equipartition the circle for five explaining, “When I made this cake [3-split on the purple cake] I thought … I’ll bet I could make five equal pieces if I made the circle cake. But if I made this type of cake [rectangle] I might not be able to.” Hence, Peter knew something inherent about the shape of the circle (perhaps radial symmetry), which helped him equipartition a circle for five. I now refocused his attention back to equipartitioning for 10. “Any way to use that strategy [for three] and make five?” He took the cake he had shared for three and split it down the middle. I observed, “Interesting. How many pieces did you make by drawing a line through the middle?” He replied, “I made five … I mean six!” I showed him the purple cake he had equipartitioned for four by drawing three parallel cuts and asked, “If I drew a line here [down the middle], how many pieces would there be?” He replied, “Eight!” I showed him a cake he had shared for two when he interjected, “weren’t we trying to do 10 before?” Using a new cake, he drew a five split by drawing four parallel lines. Then, with glee in realizing he had solved the task, he drew a line down the middle to create 10 parts.

I: How did you do that?

P: I don’t know ... I was just .. I wasn’t trying to do five. I was just trying to do the lines for 10 and I ... hah!... and I did five. That was before I did the big line. And it was five pieces and I thought, huh, I got five!

Next, I asked if he could share for any number of people. He replied that he could share for even numbers and small odd numbers, but not large odd numbers like seven or nine. Using straws, he was able to share for 7 and 9 and then made the observation that he could always share for *n* times two by “drawing a line down the middle” after sharing for *n* using the parallel cuts strategy. I asked him if he now believed he could share for any number of people and he replied, “Except 40 and up. Probably 20 and up are hard … because it would be hard to do 20 pieces or more just on this tiny piece of paper.” He did agree, however, that if I gave him a bigger piece of paper that he could share for 20[35].

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[35] Peter’s development mirrors the DELTA research on continuity (Confrey et al., 2010) where students first believe that a single whole can only be equipartitioned for certain numbers (e.g., even or odd), then a range of numbers (e.g., all numbers < 40), then any number provided the medium is large enough, and finally any number regardless of medium.

Next, I addressed the issue of naming area. Thus far, he and I had used the number of equipartitions in a cake as the name of the area of the cake regardless of the size of the equipartitions. Having previously demonstrated an understanding between different units of length, I wondered how he would respond when asked to consider the contradiction of using two different numbers to describe the same area. I showed him the purple cakes that he had equipartitioned for two and six.

I: So all of these rectangles you've been drawing on, they're the same size, right?

P: But they don't have the same pieces in them.

I: Yeah, so that's what I wanted to ask you about.

I: How is it that we can call this [the 2-split cake] two, but we can call this [the 6-split cake] six? Because ... you and I know that two doesn't equal six. But we use two to think about this [the 2-split cake] and we use six to think about this [the 6-split cake]. And these are the same. So how is it that we can use different numbers to talk about two things that are the same?

P: This [the 2-split cake] might have less numbers, but they still have the same things inside of them. Like this [the 6-split cake] has more slices cut out of it in the cake. This [the 2-split cake] has more like space on each.

I: What do you mean by space on each?

P: I mean like there's more ... there's more ... you can eat like if I was at a party and they had cake. And this was a cake and this was a cake. If I got a slice out of both, I'd be able to eat more out of a slice from this [2-split cake] cake than a slice from this [6-split cake] cake because these [2-split cake] slices are bigger than that [6-split cake slices].

One notes the extent to which Peter's understanding of the relative relationship between the unit size and the whole is facilitated by the tasks he had done with equipartitioning, suggesting a key role for this construct in the development of the length and area trajectory. Indeed, he expressed his understanding of this emergent relation using the language of equipartitioning. Additionally, equipartitioning a whole into n parts yields n parts of size $1/n$ such that the whole is n times as large as each part, another important construct in learning measurement. I concluded this teaching episode by exploring this idea with him.

I asked, "So if we know that two slices in this [2-split] cake equals six slices in this [6-split] cake, which cake has the bigger slices?" He pointed to the 2-split cake. Next, I asked, "Can you figure out for me ... how much bigger is a slice of this [2-split cake] cake than a slice of this [6-split cake] cake?" He used his hand to cover half of the 6-split cake to examine one of the rows of 3 parts. He answered, "by three pieces" noticing that three slices from the 6-split cake was congruent to one slice from the 2-split cake. His use of the word "more" suggests that although he answered three, he may not have seen the relationship between the units as multiplicative or he did not have the language to distinguish between the additive "more" and the multiplicative "times as much." I asked him to justify: "By three pieces? How do you know that?" His response demonstrated the tension between multiplicative and additive reasoning when comparing the relative size of area units.

P: Because there are three pieces on one side of the cake. But there's one slice on the side of this cake [he pauses for a few seconds]. So you'd need … two more slices ... if you had this slice [pointing to a slice from the 6-split cake] you'd need the other two more slices to equal this [a slice from the 2-split cake].

He was unsure whether to describe the relationship between the two units as "by three pieces" or as "two more slices." That is, he knew three units from the 6-split cake equaled one unit from the 3-split cake, but if he started with one unit from the 6-split cake he would need two more of them to equal one unit from the 3-split cake. This illustrates the need to teach *times as many* to students as a mathematical reasoning practice (Confrey et al., 2010) as different from unit iteration. Measurement, as argued by Piaget et al. (1960) and Davydov (1991/1969), is a *times as many* relationship between the unit and the whole. That is, five inches means the whole (five inches) is five times as large as an inch even though one could arrive at five inches by starting with the unit inch and iterating four times. I ended the teaching episode by teaching Peter this mathematical reasoning practice.

To summarize, in this teaching episode, using movable straws instead of drawing lines on the cakes helped Peter to equipartition a rectangle into three and five parts and, combined with his earlier demonstration of composing 2-splits with even splits, allowed him to create a 10-split. He then came to believe that he could equipartition a rectangle into any number of parts, provided that the rectangle was large enough. He then demonstrated an emergent relation and property by explaining the relationship between a unit and the whole using the language of equipartitioning. Finally, asking him to compare the relative size between area units demonstrated the tension in his mind between additive and multiplicative comparison.

In this teaching episode, Peter used equipartitioning as a context to talk about the difference between area as enclosed space and area as a number of units. However, I conjecture that this episode shows the beginning of the movement toward measurement based reasoning which is different from equipartitioning based reasoning. In equipartitioning, the mathematical reasoning practice of naming introduces unit fractions (Confrey, 2008; Confrey et al., 2008). For example, a student that equipartitions a rectangular cake for five might call each person's share one-fifth. The mathematical reasoning practice of naming in measurement, however, is a count relative to a unit setting up a ratio relationship between the whole measure and the unit. Hence, whereas naming an area "12" is confusing, naming it as 12 inches or 12 units of size A removes the ambiguity. As will be further explained in the discussion section of Chapter 5, I approached this idea with Peter in the next teaching episode by exploring an idea that was originally called *Transitivity of Area Units.* For reasons explained in the next section and in Chapter 5, this understanding was renamed M*easurement of Area*.

Teaching Episode Eight

In this teaching episode, I explored an emergent relation and property that was originally called *Transitivity of Area Units* because it seemed to closely align with the *Transitivity of n-Equipartitions* proficiency level in the DELTA learning trajectory for equipartitioning (Confrey et al., 2010; Franklin et al., 2010)[36]. The emergent relation and property states that if two non-congruent cakes have the same area and a unit fills one cake *x* times, it must also fill the other cake *x* times. In equipartitioning, the related concept is called *Same Splits of Equal Wholes are Equal* (SSEWE), and means that given two congruent shapes, an *n*-split performed in multiple ways yields fair shares. For area measurement, the claim is given two shapes equal in area, a unit must measure both the same number of times. Otherwise, the hypothesis would be incorrect as the two areas were not originally equal. Although this emergent relation and property seems to be logically based, its converse and contrapositive are true and point to a complete notion of quantitative area measurement. This will be further discussed in Chapter 5 and it will be explained why the property was renamed Measurement *of Area*. In this section, I will report on my work with Peter on the original statement of the property.

I began by reviewing Peter's strategies for determining the equality of the three cakes and equipartitioning the purple cake (2x6) for 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, and 18. Equipartitioning created different sized units, which was key for this episode. To examine the effect that different sized units have on the measure (as a number of units) of an object, I constructed a table with Peter. In the first column (labeled Cake), I had him trace the purple cake. In the second column (labeled Slice), I had him trace the 1x6 unit that was created from equipartitioning the white cake for two. In the third column (labeled Number of Slices in

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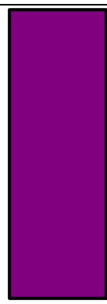
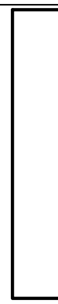
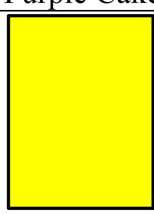
[36] As explained in the introduction, the equipartitioning version that states same splits of equal wholes are equal was renamed from transitivity to SSEWE

Cake), I asked him to predict how many times the unit in the second column fit into the cake in the first column. For the first row, he knew that two 1x6 units filled the purple cake. Proceeding in this manner, I asked him to predict how many of Slices A, B, C, and D went into the purple and yellow cakes. Table 1 shows the table we created with his estimates.

Table 1. Peter's predictions of how many times a slice goes into a cake

Cake	Slice	Number of Slices in Cake
 Purple Cake (2x6)[37]	 Slice A, EP 2, (1x6)[38]	2
Purple Cake	Slice B, EP 4, (1x3)	4
Purple Cake	Slice C, EP 8, (1x1.5)	8
Purple Cake	Slice D, EP 6, (1x2)	6
 Yellow Cake (3x4)	 Slice A (1x6)	3

[37] The dimensions were not provided for Peter.

[38] This information denotes that Slice A was created by equipartitioning the white cake for two and yielded a 1x6 unit.

Table 1 Continued

Yellow Cake	Slice B, EP 4, (1x3)	4
Yellow Cake	Slice C, EP 8, (1x1.5)	7
Yellow Cake	Slice D, EP 6, (1x2)	8

Checking the number of Slices A-D needed to fill the purple cake was trivial for him, and he verified that his predictions were correct. However, as indicated by his predictions, the yellow cake was not. I noted to him that Slice C fit into the purple cake eight times and asked him to check if his prediction (that Slice C would fit into the yellow cake seven times) was correct.

I: So eight, right? And you thought you would need seven of them to fit onto the yellow cake. So let's see if you're right.

P: How many is this for?

I: Remember, this one was size C. So eight of them would fit into the purple cake and you had guessed that seven of them would fit into the yellow cake.

He counted as he filled the yellow cake with units of Slice C: "4, 5, 6, 7, 8 … it took eight! Huh!" He was surprised by this result. I asked him, "Why do you think it took eight?" Suddenly, he had a revelation: "Oh! Then I bet … it takes two of these [Slice A] pieces to go in that [the yellow cake]." I asked him how he knew.

P: Because ... I chose eight and it was really ... I mean I chose seven and it was really eight. And that was all of the cake … the whole cake fit into the yellow cake and then I'm thinking the white cake was gonna fit into the yellow cake.

I: So you thought ... I think you had said originally that you had guessed three of A was going to fit into the yellow cake.

P: Yes.

I: So why do you think it's only two now?

P: Because it took eight of those pieces ... to go into the yellow and that was all the cake … so it's going to take all of this cake to go into the yellow.

I: Okay, should we check?

P: Huh! It does take all of this ... because I remember earlier when I was talking with you that cake [the white cake] ... it went into this cake [the yellow cake]. So I'll bet this cake [the white cake] will fit into this cake [the yellow cake]!

I: Shall we give it a try? Should we try out your idea?

I gave him scissors and he decomposed two slices of size A, showing that they completely covered the yellow cake as shown in Figure 10 below.

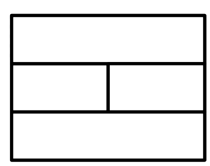


Figure 10. Peter's demonstration that two slices of size A fits into the yellow cake.

P: It does! I was right!

Again, it appears that this property is not trivial for children but is vital in their development of measurement. Even Peter, who had demonstrated an understanding of the relationship between the unit and the whole, could not logically reason that for two rectangles to be equal, a unit must measure both the same number of times[39]. And even though he demonstrated this emergent property in the vignette above, the concept would not be fully solidified for him until the next teaching episode.

I conclude by briefly describing a misconception that followed. Peter looked at the decomposed pieces (Figure 10) and said that it looked like four slices of size A went into the yellow cake. That is, he didn't realize that when one decomposes two slices of size A and joins them together, one still has two slices *of size A*, even though there are four total pieces.

I provided an opportunity for him to consider this misconception by giving him multiple units of size A and asked him to construct a cake that was made up of four slices of size A. He laid down four of the slices side-by-side to construct a 4x6 cake. Realizing that a cake that was four slices *of slice A* was this large, he realized the rejoined cake in Figure 10 could not be four slices *of slice A*. He explained, "It's [the rejoined cake] two and not four because it's four of *these* pieces [the broken up pieces]."

To summarize, in this teaching episode Peter showed that a 1x½ unit fits into the purple cake eight times and predicted that this unit would fit into the yellow cake seven times. When he checked and discovered that the unit fit into the yellow cake eight times, he revised his previous conjecture. Previously, he had predicted that a 1x2 unit that fit into the purple cake two times would fit into the yellow cake three times. He explained that because the white cake and the yellow cake were equal, and two 1x2 units made up the white cake,

[39]If one thinks of this relationship as ratio, then this means that the ratio of a unit to a unit is equal to the ratio of a whole to a whole.

then the entire white cake (or both 1x2 units) must fit into the yellow cake. He also exhibited a naming misconception, but reconsidered when I pointed out the contradiction.

Teaching Episode Nine

In my last teaching episode with Peter, I continued working with him on the idea of measurement with area units, checked his understanding of the relationship between the unit and the whole, and asked him if he could construct fractional areas given a reference unit of one. I began by reviewing his predictions from Table 1. Next, I told him to imagine a slice of cake that went into the purple cake 15 times and asked, “How many times would that slice go into the yellow cake?” He immediately said, “Fifteen.”

I: So you said if we were to make a slice that fit in here [the yellow cake] 15 times, would it be bigger or smaller than this slice [showing him a 1x3 unit that had been created from equipartitioning the purple cake for four]?

P: Smaller.

I: And why do you think smaller?

P: Because it can’t be this size [picking up the 1x3 unit] ... and 15 of it fit in it because only four of these [1x3] slices would fit in.

I: What I want to figure out is why is it that if we have a piece of Size B (1x3) and it goes into the purple cake four times, why is it that it goes into the yellow cake four times?

P: Because these two cakes [the purple and yellow cakes] are the same size ... but you just can’t tell they’re the same size because this one [the purple cake] is longer than this one. So you can’t really tell that they’re the same size cause this one [the purple cake] is longer ...

so they're the same size so that's why the slices are the same [pointing to the third column of the table that indicated four 1x3 slices filled both cakes]."

That is, he justified the emergent relation and property that if you start with two non-congruent rectangles that are the same in area, then the number of units filling them must be the same. Next, I asked him if he thought this emergent relation and property always worked.

I: Do you think this will always work?

P: Yeah ...

I: So even if I made 100 slices [pointing to the purple cake] ... is that gonna be really big or really small?

P: REALLY small! Not even the size of my finger!

I: Do you think 100 of those little small pieces would fit into the yellow cake?

P: Yeah!

Next, I probed to see if his understanding of this emergent relation and property allowed him to use internal units to compare the size of different rectangles.

I: I know that two [1x6] slices of size A go into the yellow. How many [1x6 units] would go into the purple?

P: Two.

I: What if I told you that there was another cake out there where three [1x6] slices go in. Do you think that cake is bigger, smaller, or the same size as these two cakes?

P: Bigger.

I: And why do you think it's bigger?

P: Because it needs more pieces to … like this one [the yellow cake] only needs two [1x6] pieces to cover it up but that one [the imaginary cake] needs three pieces. So that's probably bigger.

I: What if I had another cake and I told you that it takes one of these [1x6] slices to fill the entire cake. Do you think that cake …

P: Smaller!

I: And why do you think smaller?

P: Because it only needs this [he shows me a 1x6 unit] ... it only needs this to fill it up.

I: What if I had a cake that was one-and-a half slices of size A [1x6]? Do you think it would be bigger, smaller, or the same size as these two [yellow and purple] cakes?

P: Smaller.

I: Could you draw me a cake where it's one and a half [of size A]?

He traced the 1x6 unit onto a piece of paper and extended the unit.

I: So tell me what you did.

P: This is one and a half. [He places two 1x6 units on top of his drawing to demonstrate.]. I drew this [1x6] piece and then I drew more ... I drew more so it would be one and a half pieces of A.

I: What if I asked you to draw something that was one and a-fourth of A?

P: I think that would be two of these [slice A] … Actually I don't think I can draw it.

I: Can you show me a fourth of A?

He used his fingers to denote a fourth of the 1x6 unit. I asked him to show me a half. He moved his hand over the middle of the unit. However, he was unable to show me a third and a fifth of A. Given further time to work with him, I would have investigated this in more detail. However, the vignette illustrates that the task of having students use a unit to construct cakes of fractional sizes could be potentially useful for diagnosing fractional misconceptions. I would explore this task further in the other two teaching experiments.

I concluded the teaching experiment by asking, "Is there a way that we could draw two cakes that are different in size but have the same measure, that is the same number of equal slices in each cake?" and "Is there a way that we could draw two cakes that are the same in size but have different measures?" To answer the first question, he took a purple cake and a yellow cake and cut off some of the yellow cake, explaining the cakes were now unequal. He then performed a 5-split on each cake and said, "The cakes are different sizes but each has five." I asked him which cake he would rather have a slice from and he replied the purple cake, because it was bigger. To answer the second question, he took two purple cakes and equipartitioned one for two and the other for four. He further told me that he would rather have a slice from the cake that was shared for two because he knew those cake slices were bigger.

To summarize, in this last teaching episode Peter was able to justify the emergent property that two cakes of the same size must be measured by a unit the same number of times. Given a unit, he was able to construct some fractional areas, although one-third and one-fifth remained problematic. Although the task of using a unit to construct areas with fractional measures is useful in diagnosing student misconceptions about fractions, I do not claim that this necessarily leads to a full understanding of fractional area: a rectangular area with dimensions 1/*a* and 1/*b* yields an area of 1/*ab*.

Although I was unable to fully explore area = length x width with Peter, in this teaching experiment, Peter demonstrated a solid foundation that I believe will aid in the learning of more formal measurement constructs. This foundation included qualitative and quantitative (for factor based changes of two) compensation of area units, demonstration of the difference between area as enclosed space and a count of units, and the recognition that two areas that are equal must be measured by the same unit the same number of times.

Raima

Raima entered the teaching experiment with a strong understanding of equipartitioning including the ability to equipartition a rectangle for any number (with composition of splits), a multiplicative understanding of *times as many*, and conservation of composition/decomposition over breaking for length and area. This allowed me to work deeper on the area tasks with her and I tried modifying the Davydov (1991/19649) tasks to see if she could compare the relative sizes of different units.

Teaching Episode One

In this teaching episode, I worked with Raima on Task 1[40]. She started by putting the red cloth, foil, paper towel, plastic sheet, and card together. She then placed all of the length objects together and explained, “I put all the flat stuff over here. And then the different not-

[40] See Appendix A for a copy of the full task.

flat stuff over here." "Tell me about the flat stuff, what's the same about the things that are flat?" I asked. She responded that all of the area objects were flat. For the length objects she remarked, "They are the same because they are kind of round except for the floss." I asked if there was any way she could compare the length objects. She nodded. "Which one of those [length objects] is the biggest?" I asked.

She identified the floss as the longest and the other length objects as not as long. I asked if any of the "flat objects" had longness. She responded that they have "width" and "tallness" and that she could sort the area objects by "tallness." Here, she demonstrated the ability to categorize by length and to see length in two dimensional objects. This is an example of a student that understands and distinguishes between these two qualities.

Next, I asked her if she could compare the area objects. She said that the foil was the biggest one. I asked, "What are you using to help decide that the foil is bigger than everything else?" She responded, "I would put like put the bottom on the top to see if it goes over the end of it" demonstrating a strategy to compare the area of two entirely overlapping areas.

I: What does cover mean to you?

R: Cover means like to go on top of it like doesn't show ...

I: Do any of these objects have cover?

R: Yes, this (the plastic sheet) covers this much but not as much as the red cloth.

To summarize, in this teaching episode Raima demonstrated not only the ability to distinguish length and area, but could also distinguish between them on a two dimensional object. She also directly compared straight lengths and entirely overlapping areas.

Teaching Episode Two

In this teaching episode, I worked with Raima on Task 2[41]. However, based on the previous teaching episode I included two area objects that did not completely overlap to see if she could demonstrate conservation of area decomposition/composition over breaking and that areas were additive.

She began by easily sorting the straight lengths through direct comparison. When she started sorting the area objects, I asked her to explain why she thought the quarter-circle was smaller than the half-circle. She responded, “If you put him [the quarter-circle] in here (on top of the half-circle), you need more than two of them [quarter-circles] to cover this guy [the half-circle].” I probed, “How much bigger is this (the quarter-circle) than this (the-half circle)?”

R: Like about two times.

I: What do you mean by two times?

R: Because these two sides are almost like two halves of this [the half-circle] ... so it would take about two of these [quarter circles].

I: About how much bigger is this rectangle than the half-circle?

[41] See Appendix A for a copy of the full task.

She "iterated" the half-circle on the rectangle I gave her, saw that four half-circles would fit and said, "There is some left over so it would be a little bit more than four times as big." Here, she seemed to demonstrate the ability to answer *times as many* questions. However, it is difficult to claim for sure if her understanding is multiplicative versus iterative (and hence, additive). However, she did not show the tension of struggling with *n* times as many versus $n - 1$ more.

She completed sorting the area objects and I questioned her about the objects that did not completely overlap.

I: And how do you know that this rectangle is bigger than the triangle?

R: Because if you put it here [placing the triangle so its right angle aligned with the bottom left corner of the rectangle] there's just this much left [pointing to the part of the triangle that goes beyond the rectangle] from here there's a lot you can take out [indicating that she could cut it] again.

I asked her if there was a way to check for sure. She used scissors to decompose the triangle, placing the decomposed piece on the region of the rectangle not covered by the triangle as shown in Figure 11.

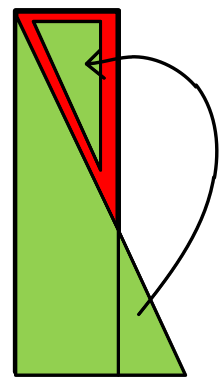


Figure 11. Raima's decomposition strategy.

She explained, "Cause this [the triangle] is big like this to here [shows where she cut off the triangle] … this is too small [showing me the part of the triangle she cut off] ... and doesn't cover the whole thing [showing me it doesn't cover the region of the rectangle not covered by triangle].

To summarize, in this teaching episode Raima demonstrated direct comparison of straight lengths and completely overlapping areas. She used a decomposition/composition and additive area strategy for areas that did not entirely overlap demonstrating conservation of area decomposition/composition over breaking. She also demonstrated the ability to compare two areas using *times as many*.

Teaching Episode Three

In this teaching episode I worked with Raima on Tasks 3 and 4[42] exploring conservation of rigid transformations and conservation of area decomposition/composition over breaking. She demonstrated complete mastery of these ideas.

She first verified that she could translate, rotate, and reflect a bendy straw without changing its length. However, she stated that while the length of the straw does not change when you bend it, stretching the bendy part of the straw would increase the length of the straw. She also said she could add another piece of straw to the end of the bendy straw to increase its length, and cut (and throw away) a piece of the bendy straw to decrease its length confirming her understanding of additivity of length.

She asserted area conservation under rigid transformations and reconfirmed her understanding of conservation of area decomposition/composition. I probed to see if this understanding extended to quantity.

I: What if I took this [rectangle] and I cut it into four equal sized pieces? And I put the four equal sized pieces back together. Does it cover more, less, or the same amount of space as before I cut it?

R: It would be the same.

I: So four of these pieces put back together would create the whole. What if I cut it again [I cut one of the fourths in half]. Is it still four pieces of this size [holding up a fourth of the original whole]?

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[42] See Appendix B for a copy of Task 3 and Appendix C for a copy of Task 4.

R: No. It's five: three of the same size [pointing to the three-fourths of the original whole] and two of these sizes [pointing to the halves of a fourth that I had created].

I: How many pieces of this size [showing her a fourth of the whole] are there in the whole sheet of paper?

R: Well, four.

I: Even if I cut that piece? It's still four?

R: Well, yeah, because this is the same [she covers one of the fourths over the two eights, making the claim that they are equivalent]. If you put them back together (the eighths) it would be the same size [as the fourth]. This demonstrated that she had both a qualitative and quantitative understanding of area conservation. A student without quantitative understanding would have said that decomposing the whole into three-fourths and two-eighths yields five-fourths.

To summarize, in this teaching episode Raima demonstrated understanding of conservation of length and area under rigid transformations and quantitative understanding of area decomposition/composition over breaking.

Teaching Episode Three

I report only briefly on Raima's work with Task 4[43]. For each of the conservation of length scenarios, she rejoined the decomposed straw or straws back into their straight configuration to justify her assertion of conservation. She explained, "Because for example if

[43] See Appendix C for a copy of the full task.

somebody made the paths straight again it would probably be the same … it is the same because these are the same types of straws and the same length."

Teaching Episode Four

In this teaching episode, I worked with Raima on Task 5. She was able to compare each set of paths through a decomposition/composition and additive strategy and an external unit iteration strategy (using one of the given straw pieces).

For the first two paths[44], she used the straightedge to physically map the length of each line to a straightedge, saw that Path A extended farther on the straightedge, and concluded that it was longer. Next, I gave her Path C and Path D (Path D was longer), where Path C had one bend and Path D had two bends. She employed a decomposition strategy for these paths and began decomposing Path D into its three pieces.

I: Can you tell me what you're thinking about as you're cutting?

R: I am thinking about cutting out [Path D] the middle part and cutting out around it so I can cut out the lines and then cut like each edge off. Make it straight and do the same thing to this [Path C] and see which one is longer.

By decomposing each path into his pieces and rejoining the pieces into straight paths, she demonstrated the ability to use decomposition/additivity to change the problem into one that could be solved using direct comparison. Doing so, she concluded that Path D was longer

I: Is there any other way that you could have figured that out?

[44] Recall that Path A (diagonally aligned) was longer than Path B (vertically aligned).

R: Yes ... you can measure around it ... if you had an inch or centimeter ruler and see which one has more inches or centimeters.

I: Anything you could do if you didn't have an inch ruler or a centimeter ruler?

R: You could use one of your fingers as a centimeter and two as an inch.

I asked her to try this strategy with Paths C and D. She used two of her fingers as an inch and iterated it along the paths with no gaps or overlaps, successfully navigating the bends. She concluded that Path C measured 6½ inches and Path D measured 7½ inches.

I: Tell me what you were thinking when you were going around the corner. It looks like when you were going around the corner, you couldn't fit two of your fingers. So how did you decide what you had to do to go around that corner?

R: Well, I decided to put one finger and then another finger.

I: So one of your fingers is how many inches?

R: Is half an inch.

I: Do you know how many centimeters this (Path C) would be then?

R: I would probably have to times this.

I: Times this. Why would you times it?

R: By two because two centimeters is one inch and that's two of them ... so you would have to times it by two or do it again with one finger.

I: If you were to times 6½ by two, do you know what you would get?

R: Twelve.

I: Twelve? Okay, do you want to try it with your fingers?

When she iterated with her fingers she was surprised that she measured 13. I briefly worked with her on doubling 6½ and showed her that it was like adding the combination of doubling 6 and doubling one half which is why 6½ doubled is $2 * (6 + ½) = (2*6) + (2 * ½) = 12 + 1 = 13$. I asked her how many centimeters were in the second path. She said, "I would have to double 7½ and that would be 15." Here, she demonstrated the ability to quantitatively compensate for length units given the assumption that 2 centimeters = 1 inch.

For paths E and F (path E was longer), where path E had three bends and path F had four bends, she once again used two of her fingers as an "inch" and measured the paths with no gaps or overlaps concluding that Path E was 11 inches and Path F was 10 inches.

I: So which one is longer?

R: This one [Path E].

I: If my fingers were twice as large as yours, how many inches would I have measured for this one (Path F)?

R: Five.

I: How do you know it's five?

R: Because 10 divided on two is five.

I: Okay, how about this one [pointing to path E].

R: Five and a-half.

In summary, in this teaching episode Raima demonstrated the ability to compare two bent paths using a decomposition/additive strategy along with unit iteration. Using division and multiplication (and the ratio relationship between her inch and centimeter), she also demonstrated quantitative compensation of length units.

Teaching Episode Five

In this teaching episode, I worked with Raima on Task 6[45], the comparison of three rectangular cakes: a 12x1 (white cake), a 6x2 (green cake), and a 3x4 (brown cake). I asked her how she would figure out which cake to eat if she wanted to eat the most cake.

R: I could ... I don't know if this [white cake] is bigger or smaller than this [green cake] so I will cut it in half and see if it covers up this part too. [She cuts the white cake into two 1x6 units and uses them to cover the entire green cake]. It does ... so now I have to see if this [green cake] or this [white cake] is as big as this [brown cake].

She placed the two 1x6 units from the white cake over the brown cake and decomposed and composed them as shown in Figure 12.

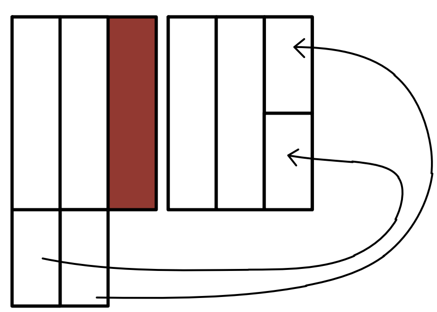

Figure 12. Raima's decomposition/composition strategy for comparing the white and brown cakes.

[45] See Appendix E for a copy of the full task.

“I figured out all the cakes are the same,” she claimed. “Why are all the cakes the same size?” I asked. She replied, “They’re all the same size because the cakes are all the same size.” As she had never directly compared the green cake with the brown cake, I conjectured that she must have used logical transitivity to conclude that all three cakes were the same size.

I: You said that all three cakes are the same but it seems to me that you only showed me that the vanilla cake is the same size as the lime cake and the vanilla cake is the same size as the chocolate cake. You never showed me that the lime cake was the same size as the chocolate cake. How do you know that those two are the same?

R: Because this is as big as this [the white and green cakes] so even if you used this [green cake] instead of that [pointing to the decomposed white cake that she had used to cover the brown cake] you would still get same size cake.

I: So you don’t have to actually check?

R: No. You could to make sure.

I: But do you have to?

R: I’m sure already!

I: Is there any other way you could have figured that out?

R: Maybe I could.

She equipartitioned the brown (3x4) for four and thought, “Could be divided into four with the same size pieces from other cakes.”

I: So you just shared that [the brown cake] for four. How is that going to help you figure out that the chocolate cake is the same size as the other cakes?

R: Cause I could cut [the white cake] into eight pieces and put two in each one [taking the white cake and juxtaposing it on the brown cake as shown in Figure 13] and see if they fit.

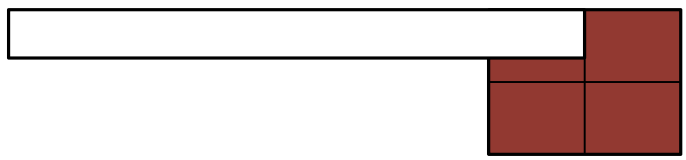

Figure 13. Raima's juxtaposition of the brown and white cakes.

She looked at her juxtaposition for a minute and concluded, "But it probably wouldn't work." I asked her why she thought it her idea wouldn't work. She replied, "Cause this is almost already more than half of this ... so you can't do it." That is, she conjectured that equipartitioning the white cake for eight, and showing that two of those one-eighth parts fit into one part of the brown cake (which she had already equipartitioned for four) would be another way of solving the task. However, she didn't realize that the unit created by juxtaposing the white cake on the brown cake was a 1x2 unit that fit into the brown cake six times, not eight. Hence, she thought something had gone wrong when she noticed this 1x2 unit (which she thought was going to be 1x3/4 unit) covered more than half of one-fourth of the brown cake. However, I decided to probe her on the emergent relation and property that two equal rectangles have to be measured by a unit the same number of times.

"So you shared that [brown cake] for four. Can you share this [white cake] for four for me?" I asked. She equipartitioned the white cake into four parts by composing two 2-splits. "How big is a slice of this [white] cake versus a slice from this [brown] cake?" I asked. "I think … they are probably going to be the same size," she replied. "Why do you think they're going to be the same size?" I asked.

R: Cause they're from the same size cake and they should have even, like these slices.

I: Can you explain a little bit more about what you mean? I'm a little confused.

R: Let's say all of them are the same size …

I: All of the pieces?

R: Yes. So, you could, to make sure, you could cut one slice out [of the white cake] and then if it is too long or something you could cut out the piece... and then cut it again.

This was her way of explaining decomposition/composition which she used to show that a 1x3 unit from the white cake was the same as a 1.5x2 unit from the brown cake. In an unexpected way, Raima had come to the conclusion of the emergent relation and property that two same sized rectangles had to be measured by a unit the same number of times. I was unsure, however, if she demonstrated that the 1x3 unit was the same as the 1.5x2 unit because I asked, or if she needed empirical evidence to confirm her conjecture.

I: So what if you shared the green cake for four? Do you think a slice of the green cake would be bigger, smaller, or the same size as the slices that you created from the white and chocolate cakes?

R: I think it would be the same size still.

I: And why do you think they would be the same size still?

R: I still think it would be the same size because they are still going to be the same size cakes ...

I: What do you mean they're still going to be the same size cakes? Do you mean the whole cakes are the same or the pieces?

R: The whole cake is the still the same. And the pieces are going to be the same.

I: So you shared all of those [cakes] for four. What if I shared ... can you share this [green cake] for five for me?

She equipartitioned the green cake for five by making four parallel cuts.

I: So if you were at a birthday party and I said you can a slice from this cake [the 4-split brown cake] or a slice from this cake [the 5-split green cake]. Which slice would you choose if you wanted the most cake?

She pointed at the brown cake.

I: Why?

R: Because it [the green cake] takes up a lot of slices .... would need to ... they would have to have smaller pieces to fit all the slices [of the green cake] to be equal so that's what I think and why these [green cake slices] are going to be smaller.

The equipartitioning tasks helped to facilitate Raima's understanding that as the measure of a rectangular area (as number of units) increases, the size of that unit must decrease if the area (as space enclosed) is to stay invariant. She justified her reasoning in the language of equipartitioning.

I ended the teaching episode by exploring composition of splits with her. First, I found that she believed she could equipartition a rectangle for any number of people. I asked her how many different ways she could equipartition for ten. At first, she only used her parallel cuts strategy. I referred her to the way she had composed two 2-splits to equipartition the brown cake for four. She thought about it for a second and was able to compose a 2-split and a 5-split to equipartition for ten. Finally, I asked her for all the ways to equipartition for 12. She was able to exhaust all factor pairs for 12 (i.e. 12x1, 3x4, and 2x6). She was also able to associate her equipartitions with a multiplication fact. I took the rectangle that she had composed a 3-split and a 4-split.

I: What would you call this piece?

R: One-twelfth.

I: How do you know to call it one-twelfth?

R: Let's say I take one piece, I took one of 12.

I: How big is one of these pieces compared to the whole cake?

R: Like 12 times more.

I: How do you know that?

R: Because they were the same size – all of them. I would take one and put it keep doing it and seeing how much there were.

To summarize, in this teaching episode Raima was able to compare the three cakes using composition/decomposition and a logical transitivity argument. She also demonstrated, rather early, that she understood the emergent relation and property of measurement with area units: two equal areas must be measured by the same area unit an equal number of times. She could also describe the relationship between the unit and the whole and had a multiplicative understanding of *times as many*.

In Peter's teaching experiment, I eschewed an exploration of area = length x width (Level 11 in the initial learning trajectory) for a more detailed examination of initial levels 7 – 10. In the altered trajectory, Peter developed an internal measurement structure, although he did not fully demonstrate that he understood measurement as the association of a number of units with a given quantity. Reflecting on Peter's teaching experiment and Raima's current progress, the goals of the teaching experiment and the learning trajectory changed from understanding area as the enumeration of units in a rectangular array to the ability to create and move among units as measures of area. That is, equipartitioning allowed the students to see that different sized units could be used to measure an area. Instead of moving immediately to generalizing to the 1x1 unit, and using that unit to enumerate rectangular arrays, I decided to explore if students could switch between different sized area units.

For Raima, my new plan was to explore the multiplicative comparison of different sized area units to see that the ratio between area units answers the question, "How much bigger or smaller is one unit compared to the other?" In a way, this is similar to the Davydov (1991/1969) approach without the use of symbolic algebra. Instead, I would rely on equipartitioning as a context for the problems and introduce ratio boxes as the notational system (Confrey & Scarano, 1995).

Teaching Episode Six

In this teaching episode, I further explored with Raima qualitative and quantitative compensation of area units. In the last teaching episode, she had expressed this emergent relation and property using the language of equipartitioning.

I began my investigation by asking her to equipartition a green cake (2x6) for four which she did by composing two 2-splits. I asked, “If that slice [pointing at one of the slices] of cake were twice as big, would it take more, less, or the same slices of those cakes to fill the entire cake?”

R: It would take less.

I: Do you know how many less?

R: Two less because there are four [1x3 units] and to make this [1x3 unit] twice as big, you can just erase the line [i.e. take away the second 2-split] and the other side would be two.

I: What if that slice of cake [the 1x3 unit] were half as big? Would it take more, less or the same?

R: More.

I: How many more?

R: Four more.

I: How do you know that?

R: There are two lines [indicating the lines drawn to create two 2-splits], so you divide each line and cut each in half, so you'd have four. And you do the same to this one and you have eight pieces.

Raima's response and explanation warrant further analysis. Note, that her response "four more" at first indicates an additive, not multiplicative, understanding of compensation. However, her explanation and work indicates multiplicative thinking. Figure 14 shows her words coordinated with her actions.

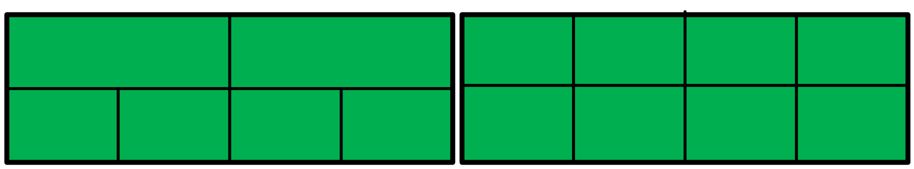

Figure 14. Raima's explanation of "four more" if the 1x3 units were halved.

She began with the idea that equipartitioning the two units in the bottom row would yield four 1x1.5 units, each unit being half the size of the original 1x3 unit. By multiplicative action, she created "four more" of these units in the top row and it is this "four more" to which she originally referred in answer to my original question. Another possible conjecture is that "four more" could refer to a percent increase, however she does not have the mathematics or language to express as the increase (i.e. four) as a fraction of the original. However, her action suggests a multiplicative compare, similar to percent increase. Finally, a last conjecture is that she was answering the question in the way that I phrased it. That is, I asked, "How many more" not "How much more" or "How many total slices", which she could have interpreted as my request for an additive, rather than a multiplicative, compare. Indeed, I switched my language and the next question I asked her was, "What if each slice was one-third as big. How many slices would fill the entire cake?"

R: Let's see ... 12.

I: How do you know?

R: Because there are six on each [she had equipartitioned each fourth into thirds]. There are six on each half so six times two equals 12.

So in response to the question, "How many slices …" she replied 12, not six. I wonder now if she would have answered "two times as much" if I had asked her, "How much more slices would you have?"

Next, I checked if she could still demonstrate understanding of the transitivity of area units. I asked, "How many of those slices [pointing to her one-twelfth units] would fill the brown cake?" "Twelve," she replied, "because if they're [the green and brown cake] all the same size, they should be able to fit all the same size pieces."

I: What if there were a cake out there and it takes 13 of these [1/12th] slices to fill. Would that cake be bigger, smaller, or the same size as the brown cake?"

She reasoned bigger, "Because this [the brown cake] can only fit 12 of these pieces. So it would have to have a bit more." The context of a piece of cake filling an area a set number of times would figure very prominently in her subsequent thinking. At this point in the teaching experiment, I asked her if she would mind if I shifted the language and started using the word unit for slice, like we had introduced in the teaching episode with length. She agreed that this would be okay. I asked her, “Can you construct for me a cake that is 14 of these [one-twelfth] units?” She thought about it for a minute and said the new cake would have to be a bit bigger than the brown cake. She reached for the brown cake and two one-twelfth units from the green cake. “Can you tell me what you’re doing?” I asked. She replied, “This [brown cake] fits twelve [of the one-twelfth units]. And 12 plus two equals 14.” She then pointed to the two one-twelfth units she took from the green cake. Further demonstrating her flexibility with units and understanding of transitivity of area units, she added that she also could have equipartitioned the white cake for 12, taken two of those one-twelfth units, and joined them onto the brown cake to create a cake that measured 14 [one-twelfth] units.

“Could you make me one [cake] that is 20 of these [one-twelfth] units?” Undeterred by my question, she demonstrated that she knew two of the one-twelfth units was the same as one of the $1/6^{th}$ units by creating and joining together 10 units of size one-sixth. “How about 21?” I asked. She took another one-sixth size unit, split it in half, and added it to the 20 cake. “Okay, how about 21½?” She took a one-twelfth size unit, splits it in half, and joins it to the 21 cake. I asked her, “Can you explain to me how you did that, what you were thinking, and how you know it’s 21 ½?” Her response indicated that she knew the one-sixth piece has a measure of two if the one-twelfth unit has a measure of one. “I took a piece like this [takes a one-sixth unit and cuts it in half] and I cut it in half [cutting one of the one-twelfth units in half].”

I concluded this teaching episode by posing her a question that had come up in Moemen and Nabeel's teaching episode. I told her that Moemen and Nabeel had both created different sized units to help them measure a blue (6x4) cake.

I: One of Nabeel's units fits this blue cake eight times. And one of Moemen's units fits this cake 20 times. Which unit was bigger, Nabeel's unit or Moemen's unit?

R: One of Nabeel's units.

I: And how do you know that?

R: Because his was eight units to fill one of these [blue cakes] ... and Moemen's was 20 so they would …. say it would take 20 of these [Moemen's units] and if it took eight of one size unit, it would probably be a bit bigger to fit all of the thing [cake].

Her explanation was similar to the reason she had given earlier for qualitative compensation of area units.

I: So here's what I want to figure out with you. Is there any way at all that we could figure out how much bigger ... so we know that Nabeel's unit is bigger than Moemen's unit ... is there a way we could figure it out exactly?

I realized immediately that this was not a good question because the ratio between Nabeel's unit and Moemen's unit was 2.5, a rational number. However, she immediately grabbed two blue cakes and began working on the problem, so I waited to see what she would do. Using composition of splits, she equipartitioned the blue cakes for eight and 20, respectively. She stared for a long at the cake equipartitioned for eight.

R: Less than half of the size of this [pointing to a unit from the 8-split cake] because if you cut all of these [points to all the units in the 8-split cake] in half you, would only have 16.

I: Interesting ... how did you figure that out?

R: Because I know that if you take eight and cut it in half [i.e. if you split each one-eighth unit in half] … like you would have more … and if you counted you would have 16.

I: So based on the fact that you know that half of one of Nabeel's units takes 16, how did you then figure out that one of Nabeel's units has to be smaller than one half [of Moemen's units]?

R: Because … if I made his just half of the size of Nabeel's [indicating that if she created a unit that was half the size of Nabeel's by halving each unit in the 8-split cake] ... and I made a lot of them to fill this [points at the 20-split cake], I would only have 16 ... not 20 and that's enough.

That is, Raima was able to bound the problem. First, she realized that if she halved each of Nabeel's units, 16 of them would fill the blue cake. However, she knew that Moemen's unit went into the blue cake 20 times. Hence, she reasoned that Moemen's unit had to be even smaller than the 1/16th unit. She concluded that this meant Moemen's unit had to be more than twice as small than one of Nabeel's units because if Moemen's unit were *exactly* twice as small as Nabeel's unit, it would have gone into the blue cake exactly 16 times.

I ended the teaching episode here. In my planning session, I carefully planned the next three teaching episodes to explore with Raima a full and robust version of quantitative compensation. My conjecture for the learning trajectory was that, at first, children first develop qualitative compensation of units. That is, as both Peter and Raima demonstrated, they can reason that increasing/decreasing the size of a unit inversely (i.e. decreases/increases) the number of units needed to cover an invariant area. They are then able to quantitatively predict the effect on the measure (as number of units) for whole number factor-based changes to the size of the unit. For example, if the unit is doubled/halved, then the number of units needed to cover halves/doubles. At the highest level of quantitative compensation, students should be able to predict the effect on the measure for all rational factor based changes to the size of the unit. Moreover, if unit 1 measures an area n times and unit 2 measures the same area m times, then

$$\text{Size (unit 1) : Size (unit 2) :: } m : n.$$

That is, they can conclude that given m and n in the above ratio equation, the size of unit 1 is m/n times as large as the size of unit 2[46].

Teaching Episode Seven

In this teaching episode I worked with Raima on Task 8, comparison of five rectangular cakes: a blue (8x3), white (2x12), orange (4x6), yellow (5x5), and red (4x7) cake[47]. Note that she was not given the dimensions of the cakes.

[46] For example, suppose unit A measures an area 8 times. Unit B measures the same area 20 times. Then unit A is 20/8 or 2.5 times larger than unit B.

She compared the cakes two-at-a-time using direct comparison of completely overlapping areas. If the areas did not completely overlap, she relied on methods of exhaustion. She first compared the white and blue cakes, concluding they were the same size. Next, she noted that the orange cake completely overlapped the red cake, so the red cake was bigger. She then compared the red and yellow cakes using exhaustion and concluded that the red cake was bigger, "Because if you put the red on top of this yellow … one and the other small pieces it will like ... the second piece will stick out and so red is bigger. Red is bigger than the yellow." (See Figure 15.)

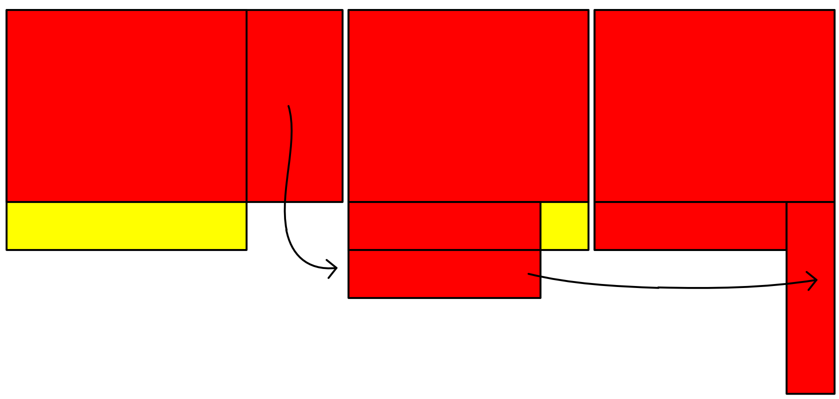

Figure 15. Raima's comparison of the red and yellow cakes.

Next, she compared the orange and blue cakes, discovering they were the same size. Finally, she figured out that the yellow cake was larger than the orange cake but smaller than the red cake.

R: Red is the biggest, then yellow, then all of these: orange, blue, and white!

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[47] See Appendix F for a copy of the full task.

I: Is there another way to figure it out? You were able to cut up shapes and put them on top of each other. Can we use the method of units like last time?

She nodded and picked up a piece of the orange cake that had been decomposed from an exhaustion strategy. It happened to be a 1x3 unit. She proceeded to tile [by tracing the unit] the blue cake with these 1x3 units, calculating that eight of them go in.

I: If eight of those units go into the blue cake, how much bigger is the blue cake compared to the unit?

R: Eight because it takes eight of these [1x3 units]. And it's one-eighth of the whole. Or this one ... [points at the white cake and giggles].

I: So you said it's one-eighth of the blue cake. But now you said it's one-eighth of what?

R: The white cake!

I: How did you know that?

R: Because a few minutes ago, I found out that the blue, the white, and the orange were all the same size. And so this should be able to fit the same as all the others.

She once again reconfirmed her understanding that if two cakes have the same area, then a unit must measure both the same number of times.

I: How many of those [1x3] units do you think will go into the red cake? Do you think it will be more than eight, less than eight, or exactly eight?

R: More than eight!

I: How do you know that?

R: Because it's [the red cake] bigger than all of these [other cakes].

I: How about the yellow cake?

She replied that the 1x3 unit would go into the yellow cake more than eight times. I asked to show me. Here, she encountered the situation where a unit does not fit exactly into an area. However, she reasoned that it takes, "about eight and a-quarter," explaining that a-quarter was half of a half, and the un-tiled space on the yellow looked to be one-fourth as big as her 1x3 unit. I asked her to estimate and then figure out how many 1x3 units fit into the red cake. She estimated, "a little more than 10," reasoning, "it has to be more than eight and a-quarter because the red cake is bigger." After tiling, she figured out that it took nine and a-third of the 1x3 units to fill the red cake.

I decided to move toward a ratio box representation to compare the relative size of units. I asked, "What if I shared the blueberry cake for two people and the orange cake for four people? A slice from which cake would be bigger?" She replied, "The blueberry because if I shared the blueberry for two, I would have two halves and if I shared the orange cake for four, I would have one-fourth of the cake and since the two cakes are equal, one-half is bigger than one-fourth." I asked, "How many times larger is the piece from the blueberry cake compared to a piece from the orange cake?" "Two times as large," she replied, "because to make one of these [blueberry pieces], it takes two of these [orange pieces] so I think it is two times bigger. I summarized this information in the first row of

Table **2**.

Table 2. Raima’s Summary Table.

Number of people sharing blueberry cake	Number of people sharing orange cake	Comparison	Times as big
2	4	Slice(B) > Slice (O)	2
6	3	Slice (O) > Slice (B)	2
8	2	Slice (O) > Slice (B)	4

For the second row, I asked her to share the blueberry cake for six and the orange cake for three. For the third row, I asked her to share the blueberry cake for eight and the orange cake for two. Additionally, I asked her how she knew a slice from the orange cake was four times as big as a slice from the blueberry cake.

R: I was thinking about division.

I: You were thinking about division? What kind of division?

R: Eight divided on two … because eight divided by two is four.

I: Can you tell me ... I know those division facts … but how do those division facts relate to your cutting and what you’ve been thinking about the cakes?

R: I think they relate to the cakes because I just divided this [points to the eight units in the 8-split cake] on four, and I see how many of these units would fit into the orange cake and I used division and I did not use multiplication.

To summarize, in this teaching episode, Raima used both strategies of exhaustion and, when prompted, a unit strategy to compare the sizes of the five rectangular cakes. I then worked with her on ratio problems of the form Size (unit 1) : Size (unit 2) :: m : n where m

and *n* were given, *m : n* was a whole number, and she was asked to find Size (unit 1) : Size (unit 2).

Teaching Episode Eight

In this teaching episode, I continued the investigation of comparing the relative size of units. This time, however, I used a ratio box (Confrey & Scarano, 1995) to express the relationship between the number of slices in one cake versus the number of slices in another cake.

I gave her two blue cakes asking her to share one for two (which we called Cake A) and the other for four (which we called Cake B). As if anticipating my questions, she told me that two slices from Cake B would fit into four slices from Cake A, and that a slice of Cake A was twice as large as a slice from Cake B. To confirm transitivity of area units, I asked her what would happen if Cake B were replaced with an orange cake and that orange cake was also shared for four. She replied that since the blue and orange cakes were the same size, nothing would change. We summarized this information in the first row of Table 3.

Table 3. Ratio box for Cakes A and B.

Slices from Cake A	Slices from Cake B
2	4
3	6
5	10

For the second row, I asked her, “How many slices of Cake A would we need to be the same as six slices from Cake B?” She replied it was three, “because four units of Cake B are two units of Cake A, and I know that if you have two more [pieces of Cake B] you will have 1 more piece of Cake A." That is, she figured out this question by adding ratio units. She figured the unit ratio was 1:2, or that an additional piece of Cake A must be accompanied by an addition two piece of Cake B to keep the ratio invariance. "What if we had 10 slices of Cake B, how many slices of cake A would we need?" I asked. She replied five, "because there are three [of A] in six [of B] and if you put another one [Cake A slice] and two more [Cake B slices] you would have eight [total Cake B slices] and that would be [equal to] four [total Cake A slices], and it would have 10." Noticing a pattern, she said she could do it another way by just using numbers. "Actually this [points to the second column] divided on 2." She explained that 10 divided by two equals five, four divided by two equals two, and six divided by two equals three.

I: So what you're saying is if you know the number of slices of Cake B, you just divide by two?

R: No, not if you do an odd number.

I: So if I had 15 slices of Cake B we couldn't divide on two to get [the number of slices of] A?

R: Actually, maybe if you do 15 divided on two and then you would have number point five or something.

I: So what is 15 divided by two? Do you know?

R: Seven and a-half.

She was thus able to extend her previous thinking about fractions to solve and visualize that 15 slices of Cake B were the same as seven and a-half slices of Cake A. She struggled for a second with this realization, but then smiled and said that it made sense because the reverse is true: “7.5 times 2 is 15.”

I asked her to consider the cake of sharing Cake C for nine and Cake D for three. She equipartitioned the cakes and from visual inspection deduced that three slices from Cake C were the same as one slice from Cake D. We made another ratio table as shown in Table 4.

Table 4. Ratio box for Cakes C and D.

Slices from Cake C	Slices from Cake D
3	1
3 * 2 = 6	2
9	3

She reasoned that to have two slices of Cake D you would need to perform, “Three times two equals six” slices of Cake C. She quickly moved to a general rule: “Here I think you have to divide on three. Cause here you have nine divided on three is three.”

I ended the teaching episode by seeing how she would fill in the ratio table for two blueberry cakes, one shared for 12 (Cake E) and the other shared for 18 (Cake F).

I: If I only have six slices of Cake E, how many slices of cake F would that be?

R: Nine … because 12 divided on two … so I just did 18 divided on two. If I split the number of slices here they’re the same.

Here, Raima used a strategy documented by Confrey (1989). In Confrey’s context, her students found that given any valid ratio of water to lemons, they could always make “less” lemonade by halving the amount of water and number of lemons. For example, if a recipe for lemonade called for 12 cups of water and eight lemons, one could make “less” lemonade (which tastes just as "lemoney") by using 12/2 = 6 cups of water and 8/2 = 4 lemons.

To summarize, in this teaching episode, Raima explored ratio boxes in the context of equipartitioning and area. She discovered that she could add ratio units in a ratio box and divide both quantities by two to maintain the ratio relationship. In the context of equipartitioning, she could always check her results using the rectangles as manipulatives. Although this exploration of ratio is not part of measurement, it sets up the last teaching episode in which I explore with Raima how to switch between different sized area units.

Teaching Episode Nine

For my last teaching episode with Raima, I decided to teach her *daisy chains*. Confrey (1989) used daisy chains as a way to get students to understand that they could always get from a whole number *a* to another whole number *b* using only the operations of multiplication and division. Eventually, students generalize that multiple methods of solution always reduce to dividing by *a* and multiplying by *b*, i.e. multiplying by *b/a*.

I structured this teaching episode to be very similar to Confrey's (1989) first teaching episode with students in her ratio study. I began the episode by introducing the word ratio as an "underlying relationship" between two numbers. I used the example of people and chopsticks, so the "underlying relationship" was that for every person that eats, two chopsticks are needed. Hence, the ratio is one person for every two chopsticks. She was able to reason that if two people eat, four chopsticks are needed, and so on.

I then asked her to consider a recipe for lemonade that called for two cups of water for every three lemons. I asked her how she could make more lemonade that was just as “lemony” as this recipe.

R: Four cups of water and six lemons.

I: How do you know that?

R: You have two more cups, and you will get four [cups of water] and three more lemons, so you’ll get six [lemons total].

That is, she added the ratio unit 2:3, which she learned from the previous teaching episode. I then asked her how many lemons she would need to use to make lemonade if she only had one cup of water. She replied, “One and a-half lemons,” explaining that she started with the 2 cups of water and 3 lemons and realized she could divide both on 2 to get a base ratio of 1:1.5 because she knew that “Three divided on two is 1.5.”

I then taught her how to use the ratio box notation to record the above ratios as shown in Table 5.

Table 5. Exploring the ratio box for a lemonade recipe.

Cups of Water	Number of Lemons
2	3
4	6
1	1.5

I taught her about *isomorphism of measures* (Vergnaud, 1983), that is the ratio relationship works horizontally and vertically in the ratio box. Finally, I taught her daisy chains and she was able to generalize that to get from *a* to *b* using only multiplication and division, she could divide by *a* and multiply by *b*.

Next, I asked, without using manipulatives, to consider two cakes of the same size. Cake A was equipartitioned for six and Cake B was equipartitioned for nine. I asked her if she could construct a ratio box for this problem. She began by telling me that if there were three slices of Cake A, then there would be 4.5 slices of Cake B. She justified by using the ratio table showing me that if she divided 6 on 2 to get 3, then she would need to divide 9 on 2 to get 4.5. The ratio box is shown below in Table 6.

Table 6. Ratio box for two equal area shapes shared for six and nine respectively.

Slices of Cake A	Slices of Cake B
6	9
3	4.5
1	?

She also figured out, using her knowledge of daisy chains, that she could go horizontally in the ratio box by multiplying by nine and dividing by six. I asked her if she could figure out how many slices of Cake B were equal to one slice of Cake A. She replied that she could take one and multiply it by nine to get nine and then divide it on six. However, she realized, “I don’t know how to divide nine on six.” I realized she could figure this out as an equipartitioning Case C problem (sharing multiple wholes among multiple people). I told her to consider the problem of 9 pizzas being shared among 6 friends. She dealt one pizza to each friend, realized she could split each of the remaining three pizzas in half, and each friend would get one and a-half pizzas. I related this to “nine divided on six” and she concluded that one slice of Cake A was the same as one and a-half slices of Cake B.

This last teaching episode was very experimental and I taught Raima three new topics: daisy chains, ratio boxes, and equipartitioning Case C. A post interview would be needed to see the extent to which these ideas stayed with her. However, it provides an example that ties together the ideas of equipartitioning, area units, ratio, and rational number (i.e. daisy chains). Although the episode does not seem to contribute to the overall learning trajectory for length and area, Raima demonstrated that she had flexibility in switching between area units and I conjecture this would not have been possible without explicit discussion of ratio and using ratio boxes.

Moemen and Nabeel

Moemen and Nabeel entered the teaching experiment with the ability to distinguish between length and area, conservation of rigid transformations, direct comparison, conservation of decomposition/composition over breaking, and a unit iteration scheme (although at first they showed the misconception of gaps and overlaps). During the teaching experiment, they strengthened their abilities to reason about units and equipartitioning helped them to express their conjectures in terms of fair sharing.

Teaching Episode One

In this teaching episode, I worked with Moemen and Nabeel on Task 1[48], the sorting objects task. After asking them to, "Sort the objects into piles of objects that you think are the same. You have to sort all of the objects", Moemen took the lead and put together the foil and plastic sheet saying, "These are things to cover something. Foil covers food and the plastic sheet can be used to cover." I asked Nabeel if he agreed with Moemen, what cover meant, and if there were any other objects on the table that could be used to cover. Nabeel nodded and replied, "Cover means to put something over something" and he proceeded to add the index card, triangular cloth, and paper towel with the foil and plastic sheet. They both agreed that there were no more objects on the table that could be used to cover.

I then asked them to sort the remaining objects. Moemen put together the bungee cord and straw saying, "They're both like lines." I asked Nabeel what a line was and he replied, "It's something that goes up or down." Moemen then proceeded to add the dental floss and dowel to the pile saying, "These are all lines." I asked them if the lines could be used to cover or if any of the objects in the other pile could be thought of as lines. Nabeel shook his head and said, "Lines are not big enough to cover" suggesting that he understood some qualitative difference between lines and covering objects. "These all have length," Moemen interjected, as he pointed to the length objects. "Okay, what's length?" I asked. "It's like how

[48] See page 76 for an informal description of the task or Appendix A for the full task description. Note that I added a bungee cord to the objects as a way to perhaps talk about scaling a line and inadvertently had another straw in the length object set which Nabeel would use to demonstrate additivity of lengths during the teaching episode.

long something is … like if you put these together like here," Moemen replied while aligning the bungee cord and the straw side-by-side, "then this [the bungee cord] is longer by like two centimeters." I asked him what a centimeter is and he told me that a centimeter is one finger while an inch is two fingers.

Nabeel had been silent during this exchange and I noticed he was playing with the two straws. "What are you thinking about with those straws?" I asked. "Putting them together," he replied. "Okay," I said, "So if you put those two together do you think they would be longer, shorter, or the same length as the cord?" He put the two straws end to end and aligned them side-by-side the bungee cord and replied that they would be longer. I asked him to justify his response. "Cause if you put these two together you can see that there's extra here," he said while using his fingers to indicate where the second straw extended past the bungee cord. "And if you put this [the dowel] with this [the bungee cord] then they will be the same length," Moemen added as he put the dowel end-to-end with the bungee cord. Thus far, they both demonstrated understanding area as a cover, length as how long an object is, direct comparison of length (and hence conservation of translation), and that lengths are additive.

Next, I moved to the area objects and asked if something can cover more than something else. "Which object is bigger? The foil or the card?" I asked. "The foil," Nabeel replied, "because you can put this in here [he placed the card on the foil] and you can see some of the aluminum foil." I asked them to compare the other area objects (two at a time) in terms of cover. They demonstrated the ability to directly compare completely overlapping areas by placing the smaller object on top of the larger object and saying that they could "see" some of the larger object over the smaller object.

I ended the teaching episode by exploring how they would compare the size of one of the length objects and one of the area objects. "Is the bungee cord bigger than the index card?" I asked. "Yeah it is," Moemen said, but then he paused and shook his head. "No, it's [the bungee cord] not bigger but it's taller. This [the card] is bigger but not taller than this [the bungee cord]," he explained. "What were you looking at to help you decide that?" I asked. Nabeel placed the bungee cord on top of the card with the two ends hanging over the card and then Moemen took the card and demonstrated (by rotating the card) that each side was shorter than the bungee cord. Convinced of their argument that the bungee cord was taller than the card I asked, "And how do you know that the card is bigger than the bungee cord?" Nabeel replied, "You can put this in this," and placed the bungee cord over the card, "and it's got more space right here," and gestured to indicate the extra space on the card that the cord was not overlapping. "What's space?" I asked. Nabeel thought about it for a minute and then Moemen interjected, "It's like a gap." "Okay," I said. "So what if I told you that I think the cord is bigger than the card because it's over here [showing one end where the bungee cord extended past the cord] and over here [showing the other end]?" Moemen replied, "Because it [the card] has more space than this. This [the bungee cord] is just a line." In this exchange, both students demonstrated that the bungee cord has length and the sides of the card has length and can be compared. They both believed that the card was bigger than the cord because it has more space while Moemen also demonstrated that, in a way, he thought lines don't have space.

To summarize, in this teaching episode Moemen and Nabeel demonstrated the ability to distinguish among qualities of length and area, to directly compare lengths and completely overlapping areas, and conservation of length over translations. They also distinguished between comparing lengths and areas by comparing the length of a bungee cord with the length of the sides of a card, but claiming that the card was bigger because it had more space. Moemen demonstrated evidence of seeing that a line does not have space.

Teaching Episode Two

In this teaching episode, I[49] worked with Moemen and Nabeel on Task 2 (direct comparison of length and completely overlapping area objects), Task 3 (the conservation of rigid transformations), and Task 4 (conservation of composition/decomposition over breaking). Based on the first teaching episode, I conjectured that they understood direct comparison and conservation of translations. To induce a *critical juncture* (Dougherty, personal communication, 2010), I deliberately included a smaller rectangle among the objects that did not completely overlap with the triangle.

I started with Task 2[50] and asked Moemen to sort the length objects from shortest to longest and Nabeel to sort the area objects from smallest to biggest. As expected, Moemen used his direct comparison strategy to correctly sort the length objects and Nabeel used his

[49] Dr. P. Holt Wilson assisted me in videotaping this teaching episode and also assisted in the interviewing.
[50] See Appendix A for a copy of the full task.

direct comparison strategy to correctly sort the completely overlapping area objects. However, he struggled with the ordering of the where the smaller rectangle (that I had included) relative to the other objects. I focus my analysis of this *critical juncture*.

Nabeel had originally placed the triangle on the side and I asked him to clarify how it compared to the other objects. He took the triangle and placed it on top of the large rectangle saying, “You need another one of this [pointing to the triangle] to make it here [using his hands to indicate the region of the larger rectangle that the triangle did not cover].” I asked Moemen if he agreed with Nabeel and he added that, “If you have two of these [the triangle] it will cover the whole thing. It’s like a half.” Next, I asked Nabeel to compare the triangle with the smaller rectangle. Using perceptual cues, they guessed that the triangle was larger than the smaller rectangle.

Dr. Wilson had noticed that Nabeel was holding up the hypotenuse of the triangle against the sides of the smaller rectangle and asked him what he was looking at. He replied that, “I’m looking at which one is taller.” After struggling for a few minutes on whether the tallness would help him I interjected, “When you guys were comparing these two [the triangle and the larger rectangle], I believed you. When you [Nabeel] put this [the triangle] on top of here [the larger rectangle] and said that this one [the larger rectangle] was bigger, I really believed you. Is there something we can do to show that the pink [triangle] one is bigger than the red one? Can we try that same strategy?” Nabeel placed the triangle over the rectangle and stared at it for a minute. “Which one’s bigger? Can you tell?” I asked. “No, but you could cut this [indicating the part of the triangle that hung over the small rectangle] and then put it right here [indicating the part of the small rectangle that was not covered by the triangle]." I gave him scissors and asked him to try this strategy. He decomposed the overhanging triangle region, placed it on top of the non-overlapping rectangle region, and noticed that the decomposed triangle piece was smaller. "The pink one is bigger," he concluded.

Having observed Nabeel decompose the triangle and use an additive area strategy to compare two non-entirely overlapping areas, I asked a question to investigate their understanding of conservation composition/decomposition over breaking for area. I told them to think about the two pieces of the triangle as cake and asked, "Do I have more cake, less cake, or the same amount of cake as before you [Nabeel] cut it?" Moemen replied that you still had the same cake because, "Here [composing the pieces back into the original triangle] … either way it's going to be the same amount [moving the decomposed piece and joining it to the top of the main piece]. If you took it [the decomposed piece] it would be less but if it's together it's going to be the same."

Next, I worked with them on Task 3[51], conservation of length and area under rigid transformations. I comment only briefly on this task, as they confirmed their understanding of this concept from the previous teaching episode. For the straw, they believed that moving, turning, flipping, throwing, and standing it up did not change the length. Moemen explained, "It's still the same length … you're just moving it." They also believed that moving, turning, flipping, and throwing the piece of paper did not change the area. They said that the only way

[51] See Appendix B for a copy of the full task.

to change the length of the sting or the area of the paper would be to cut a piece and take it away.

Next, I worked with them on Task 4, the conservation of length task[52]. They both asserted conservation of length for all the scenarios[53]. I report on their justifications for assertion of conservation. For the single straw decomposed into two pieces and rejoined in the same orientation Nabeel reasoned, “Because if you take this [one of the pieces] away it won’t be the same, but if you put it back together it will be the same length,” and Moemen added, “If you took a three inch straw and cut it and glued it back together it will still be three inches.” Moemen extended length conservation using a quantitative argument that also required an understanding that lengths are also quantitatively additive[54].

For the single straw decomposed into two pieces and joined with a bend Moemen again justified using his notion of quantitative conservation, “You’re still putting it together. For example if this [one straw piece] is one inch then it’s still one inch [gesturing to indicate that he meant it’s still one inch after you cut it] and if this [the other straw piece] is two inches then it’s still two inches and this is all three inches.”

For the comparison of two straws (one left intake and the other decomposed into two pieces and rejoined in the same orientation), Nabeel reasoned, “They’re the same length because even if you still cut it, it will be the same length,” and demonstrated by aligning the

[52] See page 92 for a brief description of this task and Appendix C for a copy of the full task.
[53] See the brief description of this task on page 92 for the scenarios. I will informally describe the scenarios as I go through their justifications for conservation.
[54] Indeed, this quantitative understanding of length additivity is implied in the Common Core State Standards (CCSSO, 2010).

straws together to touch and showing that they were still the same length. For the comparison of the whole straw and the decomposed straw that was joined with a bend, Nabeel took the pieces of the decomposed straw and joined them in a straight line to compare with the whole straw and said, “It will still be the same length.”

Instead of using scenarios four and five, I made the last comparison more difficult I started with two equal paths, each made up of three straws attached end-to-end in a straight line (I refer to this scenario as Part 3 in Appendix C). I arranged the three straws that made up the first path in a zigzag. I took two of the straws from the second path and decomposed one into two pieces and one into three pieces (leaving one straw uncut), and arranged these six pieces into a second zigzag path. They both asserted that the paths were the same length. Moemen explained that the second path was still made up of three straws: “This is one straw [pointing at the uncut straw], this is another straw [pointing at two pieces], and this is another straw [pointing at the last three pieces]. Then this one [referring to the first path] is one, one, one [pointing at the three straws that made up the first path].” I chose not to work them on the last part of the task as they had already demonstrated conservation of composition/decomposition over breaking for area.

To summarize, in this teaching episode Moemen and Nabeel demonstrated the ability to directly compare lengths and completely overlapping areas. With some scaffolding, Nabeel was able to use composition/decomposition and his understanding of the additivity of areas to compare two non-completely overlapping areas. They both demonstrated conservation of length and area under rigid transformations. Finally, they were able to assert the conservation of composition/decomposition over breaking for length under a number of scenarios. Moemen applied a quantitative justification for conservation that also demonstrated his understanding of additivity of length.

Teaching Episode Three

In this teaching episode, I worked with Moemen and Nabeel on Task 5[55]. Nabeel compared path A (longer than path B and aligned diagonally on a sheet of paper) and path B (aligned vertically on another sheet of paper[56]) by mapping the lengths of the two lines onto the straightedge and claimed path A was longer. I asked him to explain his strategy and he said, "Because I put this right next to it [placing the straightedge next to path B] and I used my pencil to mark where it ends. And then I went to the other one [path A] and marked it. And then it was here [showing me the mark for path A extended past the mark for path B] and it was about one inch away from each other." He showed me that two of his fingers fit the gap between the two marks. Moemen had a different strategy. He used one of the straw pieces (which he called one inch) as an iterating unit but overlapped the unit as he iterated. Despite overlapping his unit, he measured path A as "six inches long", path B as "five inches long", and concluded that path A was longer.

To compare paths C and D (path D was longer, path C had one bend, and path D had two bends) they both relied on a unit iteration strategy. Using the same piece of straw that he had used to compare paths A and B, Nabeel claimed that path D was "three and a-half

[55] See Appendix D for a copy of the full task.
[56] Please see Appendix D for the full task including copies of the line segment paths used in the task.

inches" and path C was "three and a-quarter inches." I asked him to show me how he arrived at these measurements. Unlike last time, this time he had attended to precision and marked where his straw piece ended so his iteration did not contain any overlaps. He was also able to attend to the bend in the path and name the path length as a fraction of his iterating unit. Figure 16, shown below, is a copy of his work. I use the label "A" to denote where he placed and marked his iterating unit. When he iterated the longer segment of path C noticed the third iteration extended past the bend. He turned his unit and noticed that it matched perfectly with the shorter segment of path C.

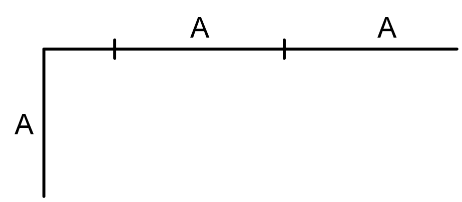

Figure 16. Moemen's justification that path C measured three and a-quarter inches.

Nabeel used his thumb as an iterating unit and said that path D measured 10 inches and path C measured eight and a-half inches. "So this is interesting guys," I pointed out. "So you both decided that this path [D] was longer, right? But you [Moemen] got three and a-half and three and a-quarter and Nabeel you got 10 and eight and a-half. So can we think about how you got different numbers?" The two vociferously argued, each claiming the other had "measured wrong." Moemen picked up his iterating unit and said, "This is an inch!" "What did you use as an inch?" I asked Nabeel. "My thumb," he replied. "So why do you think you got different measurements?" I asked. They argued some more without answering and so I told them to think about the question and moved onto the last two paths.

For paths E and F (path E was longer), where path E had three bends and path F had four bends, they both measured with different iterating units but both concluded that path E was longer. Nabeel said path E measured 16 inches and path F measured 13 inches. Moemen said path E measured six inches and path F measured seven inches. Again, I posed the question, “How did you both get different numbers but arrive at the same conclusion [as to which path was longer]?” After a few more minutes of unconstructive arguing, I told Nabeel to pretend that my pinky was half the length of his thumb and that I used my pinky as “my inch.” I asked, “How many inches do you think I would get if I measured [path F] with my pinky which is half the length of your thumb?” He replied, “I think you would get 26.” I asked him how he knew and he explained, “Because if it’s half the size you would double it since it’s two.”

To summarize, in this teaching episode Nabeel used a physical mapping strategy to compare paths A and B and used his thumb as an iterating unit to compare paths C and D and E and F. Moemen used a straw piece as an iterating unit to help him compare. Since they both called their iterating unit an inch, their measures (as number of units) were different even though they both agreed on which path was longer. I pointed this fact out to them and Nabeel determined that halving the size of his unit would result in doubling the measure.

Teaching Episode Four

In this teaching episode, I worked with Moemen and Nabeel on Task 6, the comparison of three rectangular cakes: a 12x1 (white cake), a 6x2 (green cake), and a 3x4 (brown cake). I had them work separately on the task for five minutes and then asked what they had decided. Moemen replied that he had found out that the brown and white cakes were the same size and demonstrated by showing me that he had equipartitioned the brown cake into three 1x4 units and that completely covered the white cake: “I would choose either this [the brown cake] or this [the white cake] because they’re both the same cause I put them like this and it’s one, and it’s two, and it’s three.”

I asked Nabeel if he agreed. He said, “No!” and Moemen explained, “I think this [the green cake] is the smallest.” I asked Moemen why he disagree and he replied, “They’re all the same because I put this one [the brown cake] and I cut it into pieces [he had decomposed the brown cake into a 4x2 piece and two 1x2 pieces and placed the 4x2 piece on the green cake] and I put this one here [a 1x2 piece of the brown cake on the green cake] and this one here [the other 1x2 piece of the brown cake on the green cake to completely cover]. And I put this one here [a 1x6 part equipartitioned from the white cake] and this one here [the other 1x6 white cake part] and they’re all the same.” Figure 17 shows his method.

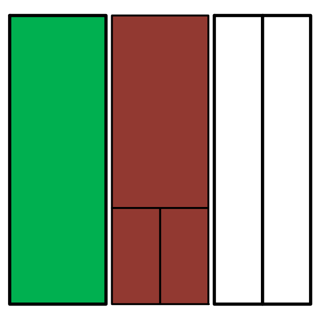

Figure 17. Nabeel’s method of showing that the three cakes were the same size.

I asked Moemen if he accepted Nabeel’s explanation. Moemen was already at work, however. “Wait … I want to see …” he said as he equipartitioned the green cake into two

1x6 pieces which he used to cover the white cake. “If I put this like that [placing the two 1x6 pieces on the white cake] … Nabeel’s right!”

Next, I worked with them on creating other sized units that could be used to completely cover the cakes. I asked Nabeel, “So, you created two pieces of this size [1x6 unit created from equipartitioning the white cake into two parts] and you found that you could take these two pieces and it would fit into the green cake. How many times do you think this piece [1x2 unit] would fit into the green cake?” Nabeel proceeded to treat this problem as a tiling task. He tiled the 1x2 unit three times completely down one column of the green cake, marking with his pencil before moving the unit to ensure that he tiled without any gaps or overlaps. Demonstrating that he might be cognizant of the array structure he answered, “Six of these,” without having to tile the other column. I asked how he knew and he replied, “Because I marked where it is and then I multiplied it by two and that equals six.”

I pointed out that, thus far, they had created two different sized pieces (i.e. units): the 1x6 unit and the 1x2 unit. I asked, “So we have different sizes of different piece. Are there any other pieces that we can make that will fit perfectly into, say, the green cake?” I had them work for five minutes and Nabeel demonstrated that he could split the 1x6 unit in half creating a 1x3 unit that fit into the green cake four times.

I ended this teaching episode by having them consider equipartitioning as a way to create different sized units. I pointed out that they had now created pieces that fit into the green cake two, four, and six times and challenged Nabeel to create a piece that would fit five times and Moemen a piece that would fit eight times. After a few minutes of working, Nabeel found that he could cut make four parallel cuts on the green cake to equipartition it for five (hence creating a piece that would fit into that cake five times) but Moemen struggled with the misconception that *n* parallel cuts creates *n* pieces. I had Nabeel explain to him how he had made five equal parts and Nabeel was then able to make an eight-split by cutting seven parallel cuts and asked, "So now we've done it for two, four, five, six, seven, and eight. I want you to think about this question. Do you think you could do this for any numbers or just some numbers?" Nabeel initially whispered, no, but then changed his mind and Moemen said he thought he could do it for any numbers. I asked them how they might create a piece that fit in 50 times and Nabeel replied that you could take the brown cake and make 49 cuts.

Having observed that Moemen created the 1x2 unit through decomposition/composition, I wondered if they understood composition of splits and so I ended the teaching episode by asking them if there was another way to make eight. They were only able to do this with the parallel cuts strategy, however.

To summarize, in this teaching episode Moemen and Nabeel used decomposition/composition strategies to determine that the three rectangular cakes were the same size by creating 1x6, 1x2, and 1x3 units. By reposing the task as an equipartitioning problem they were able to create a 1x(3/4) unit (by equipartitioning the green cake for eight using parallel cuts) and a 1x(6/7) unit (by equipartitioning the green cake for seven using parallel cuts). They also believe that they could use this parallel cuts strategy equipartition a rectangular cake for any number. However, they had not yet demonstrated that they were able to equipartition using composition of splits.

Interlude

I was gone from the school for one week to attend the 2010 National Council of Teachers of Mathematics conference. Because of possible time constraints with Moemen and Nabeel's End of Grade test, I asked DELTA team members Dr. Holt Wilson and Cyndi Edgington to work with Moemen and Nabeel on composition of splits. Dr. Wilson met with them once and Ms. Edgington met with them twice. During these sessions, Moemen and Nabeel learned how to compose splits, code the splits as multiplication facts, and used composition of splits to exhaust all factor pairs of 12 and 16.

Teaching Episode Five

In this teaching episode, I worked with Moemen and Nabeel in reviewing what they had learned about composition of splits, naming the results of equipartitioning as times as many, and began discussing the relationship between the unit created from equipartitioning and its relationship to the whole.

I began by asking them how many different ways they could share a rectangular cake (the green 6x2 cake) for 20. Having worked with Dr. Wilson and Ms. Edgington on associating multiplication facts with composition of splits they told me they could do it using 2x10 and 5x4. I asked them to tell me what they meant by this. Moemen said, "Well you could split the cake in half and then draw 10 lines," demonstrating that although he understood that composing a 2-split with a 10-split would be a way to share for 20 but that he still had the misconception that creating *n* parallel cuts yields *n* parts. I brought this to his attention by having him perform this action. To his surprise, he had created 22 parts. Realizing this, he remembered the last time we had worked together where Nabeel had told him to always make one less parallel cut than the number of parts you want to create. Moemen then showed me he could make 20 by composing a 5-split and a 4-split. Finally, they both said you could also make 19 parallel cuts.

Having used composition of splits to exhaust all factor pairs of 20 I asked them, “Are there any numbers that we can’t do it for?” because although they had told me in teaching episode four that they could do it for any number, Nabeel had told Dr. Wilson that you couldn’t share for 1000. “Nabeel, I think you told Mr. Holt that you couldn’t do it for 1000. If I gave you a large enough rectangle, say as big as this room, do you think you would be able to do it?” “Yeah, 999,” he said explaining that he could make 999 parallel cuts. “Any other way to do it?” I asked. They thought about it for a minute and Moemen replied that you could do 2x500 by cutting the cake in half and drawing 499 parallel cuts. I decided to move on with the teaching episode, but realized that this is another possible way of connecting equipartitioning with multiplication and division and factoring.

Next, I asked them to look at the cake they had shared for 20 by composing a 5-split and a 4-split.

I: How big is the whole cake compared to one of the pieces?

N: Five times four.

I: What do you mean by that?

He cuts out one of the parts and takes a new green cake.

N: "You need 20 of these [holding the part] to fit into this whole cake. So it's 20 times as much.

I: Why can't we call it 19 times as much?

N: We'd have a gap.

He demonstrated by counting the units in the green cake that had been equipartitioned for 20 and where he had already cut out a piece. Half-way through counting he realized because he had cut out one part, there were 19 parts left in the cake. He told me that a cake 19 times as big as the piece would look like that cake. Moemen agreed with this explanation.

Next, I explored with them the relationship between the unit and the whole. I began by asking them to share two green cakes: one for eight and the other for four.

I: So here's what I want to explore with you. We can call this eight [the cake shared for eight] and we can call this four [the cake shared for four]. These two cakes are the same, right? But eight is not the same as four. How is it that we can call this eight and this four but these two cakes are the same?"

M: Case you're dividing it into different … sizes. It's still the same sized cake. You divided them into different sizes … large and small … but you just cut them into different sizes. This [points to a piece from the cake shared for four] is larger than this [points to a piece from the cake shared for eight].

I asked Nabeel if he agreed and if he could explain Moemen's thinking to me.

N: They're [the cakes] the same size but this one [the cake shared for eight] has smaller … you draw it smaller. And this one [the cake shared for eight] you have two of these [slices from the cake shared for eight] equals one of these [slices from the cake shared for four] and if you cut this [cutting one slice of the cake shared for four in half] it would be the same size as this [a slice from the cake shared for eight]. And this one it's both the same size [showing me that a slice of the cake shared for four, when split in half, is the same size as a slice from the cake shared for eight] and if you cut all of them [pointing to all the slices in the cake shared for four] and put them in there [the cake shared for eight] it would be the same size.

Here, they both demonstrated in the language of equipartitioning the relationship between the unit and the whole demonstrating the distinction between area as enclosed space and measurement. Nabeel went further and demonstrated that splitting the 1x1.5 unit (created from equipartitioning the 2x6 green cake for four) in half created a unit that was the same size as equipartitioning the green cake for eight. He also recognized the multiplicative relationship between the two units: two 1x1.5 units were the same as one 1x.75 unit. I checked their understanding of this and asked them how many 1x1.5 units would be the same as 40 1x.75 units. Moemen replied 20 and Nabeel explained, "It's 20 because two of these [1x.75 units] equals one of these [1x1.5 units]. I multiplied two times 20 to get 40."

I ended the teaching episode by asking them, "Show me two cakes that are equal but have different measures, that is, different numbers of equal slices in the cakes." They both took two brown cakes each. Moemen finished the task first explaining he had made seven pieces in one cake and nine pieces in the other. I noticed that Nabeel had used a composition of two 3-splits to equipartition for nine in one of his cakes and asked them to think at home about how a piece from Moemen's cake shared for nine (which he had equipartitioned using the parallel cuts strategy) compared to apiece from Nabeel's cake shared for nine. As I was cleaning up they both claimed that a slice from Nabeel's cake would be larger. Suddenly, Moemen said, "No. I think they would be the same." "How do you know?" I asked. He proceeded to cut out a slice from each cake and, using decomposition and composition, showed that the two pieces were equivalent. I left them with the question of how this could be.

To summarize, in this teaching episode Moemen and Nabeel demonstrated they could use composition of splits to equipartition a rectangle for 20 in three different ways, exhausting all factor pairs, and that the whole was 20 times as large as a piece. They also confirmed that they understood that a rectangular cake could be equipartitioned for any number of people. They explained why measure is dependent on unit size. Nabeel was able to quantitatively explain this using the example of two green 2x6 cakes, one that was equipartitioned for four and the other for eight. Finally, Moemen demonstrated knowledge of *measurement of areas* by using decomposition/composition to show that a slice from a brown cake shared for nine using composition of splits was equal to a slice from a brown caked shared for nine using parallel cuts.

Teaching Episode Six

In this teaching episode, I further explored *measurement of areas* with Moemen and Nabeel. I began by asking them how many different ways they could share the brown [3x4] cake for four. Figure 18 shows the two ways they shared the cake.

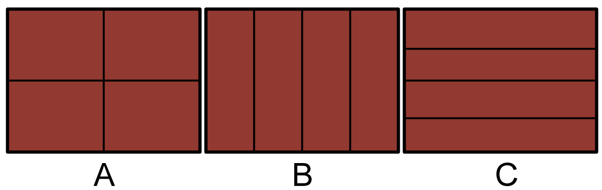

Figure 18. Three ways to share a 3x4 rectangle cake for four.

I asked Nabeel how a slice from cake A compared to a slice from cake B. He cut out the four pieces from cake A.

N: Two of these halves [pointing to a slice from cake B] equals one of these [pointing to a slice from Cake A]

I: How do you know that?

N: Because this [points to a slice from cake B] is skinny and this [points to a slice from cake A] is fat, so it would be this would be a half of this. Two halves of this [points to two slices from cake B] would equal one of these [points to a slice of cake A].

He then placed a slice from cake A slice on top of two Cake B halves as shown in Figure 19 to demonstrate what he meant.

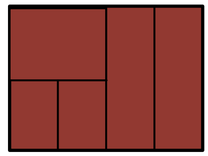

Figure 19. Nabeel's visual justification that two halves equals a whole

This justification was slightly different from the two highest levels in the DELTA proficiency level for SSEWE. While Nabeel does not need to decompose a slice of Cake A to show that it is equal to a slice of Cake B, he does not necessarily imply that he knows that slices from cakes A and B are one-fourth of the same whole. However, he seemed to be on his way to reaching this understanding because he realized (and checked using visual inspection) that two halves of a cake B slice (i.e. [(2*(1/2)) of B] equals a slice of cake of cake A [1 of A].

Next, I asked Moemen how a slice from Cake B compared to a slice from Cake C. Using decomposition/composition he showed that they were the same. I realized I had a chance to pose a logical transitivity problem and asked them to consider how a slice from cake A compared with a slice from cake C given what they had figured out. When they reached for scissors, I asked them if there was any way they could figure it out without cutting. They thought for a very long time shifting from thinking that there was no way they could figure it out without cutting to believing that it was probably equal but that they would have to check. Finally, Moemen demonstrated logical transitivity explaining, "If C and B are equal, then A and C are equal because A is equal to B and C is equal to B. So it has to be C is equal to A because they're all equal. C is the same as B. A is the same as B. So A, C and, D altogether are equal."

To summarize, in this teaching episode Nabeel demonstrated a potential level before logical mastering of SSEWE for congruent wholes and Moemen demonstrated full logical transitivity of area as applied to equipartitioned parts showing that full logical transitivity can be explored within the context of an equipartitioning task.

Teaching Episode Seven

In this teaching episode, I further explored the idea of cake slices as units and using units to create wholes with fractional measures. I began by showing them two 3x4 brown cakes and asking, "If I shared one of these cakes for two and the other for three, how would the size of the slices compare?" They both agreed that the cake shared for two would be bigger. Nabeel explained, "If I shared this one for two [performing a 2-split on one cake] and this for three [performing a 3-split on the other cake], this one [the 2-split cake] would be bigger [indicating the slice] because the smaller the number of pieces, the more the space the piece would have." Nabeel demonstrated a form of the emergent relation of qualitative compensation of area units. That is, he knew that the decreasing the number of units (formed from equipartitioning) to fill an area means the size of that unit must increase because, "the more the space the piece would have."

I then asked them if they could show me an example of two shapes with different areas but the same measurement. Nabeel did this in an unexpected way taking a yellow and a black piece of construction paper and cutting the four corners off the yellow piece and then splitting each piece of paper in half showing that the areas were different but they both had a measure of two. Next, I introduced the word unit to them, saying that we would use that word now instead of cake slices. They both had heard the word before, but in the context of length units.

I concluded the teaching episode by exploring if they could start with a given unit and construct a whole with a fractional measure. I showed them a 3x4 brown rectangle that I split in half and explained, “This is the unit I want you to think about [tracing around one of the units in the rectangle]. Let’s say that this unit is size one. So this entire cake has a measure of two. I want you to make a cake that measures two and-a-half of these units. How would you do that?”

Moemen drew a line splitting one of the units in half and said that he was done. I asked him to show me the cake that was two and-a-half units and he pointed to the unit he had split in half. I pointed out that while I agreed the piece he was showing me was half a unit, the whole rectangle still measured two units. I gave them more time to think about the task but the struggled to come up with a strategy.

M: This is hard.

I: What were you thinking about? That’s a half, right [pointing to the unit he had drawn in half]? That unit there is a half. This unit is one [pointing to the whole unit]. Could you cut out a half for me? Is there any way to add on to the cake to make two and a-half?

Moemen reached for another 3x4 brown rectangle. He split this in half creating a 3x2 piece and joined it the first brown rectangle.

M: This is two and a-half.

I: How do you know?

M: Because if you put it together this [pointing to the 3x2 piece that he had joined] is a half and this is one and a-half [pointing at one of the 3x2 units in the first brown cake] and this is two and a-half [pointing at the other 3x2 unit in the first brown cake].

I pointed out to him the contradiction in this.

I: But a unit, remember, is this [tracing around the 3x2 unit in the first brown cake]. So you just made something that's three: one, two, three [counting the 3x2 units in Moemen's joined rectangle]. How could you make two and a-half?

He took the 3x2 unit he had joined, cut this in half, and joined one of the halves to the first brown cake, this time correctly counting the units.

Next, I asked Nabeel to equipartition a 2x6 green rectangle for four. He did this by composing two 2-splits. I told him we would call each part a unit.

I: So this is four units, right? I want you to show me what five and a-fourth looks like.

N: I don't want a hard one!

I: Okay, well first, show me what five would look like.

N: Wait, wait. I want to do a hard one.

I: Well, let's start with an easier one first. So my unit is this piece [tracing around one part of the rectangle].

M: It's one fourth of the whole.

I: Good. It's one fourth of the whole. But I want you to add on to make five.

Interestingly, Nabeel took another green rectangle but instead of equipartitioning it for four like the first rectangle, he used the straightedge to physically map one the length and width of one of the parts of the first rectangle onto the second rectangle. He then cut out this piece and added it to the first rectangle. I then challenged him to make a rectangle that measured five and a-half units. He took the second rectangle (the one with a fourth of the whole already cut out), halved one of the fourths and added that piece to the first rectangle as shown in Figure 20.

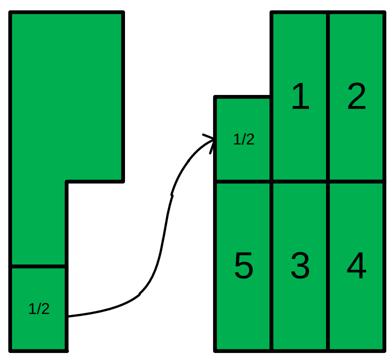

Figure 20. Nabeel's method of constructing an object with measure 5½ units.

When I asked him how he knew, he explained, "Because [counting] this is one, two, three, four, five, and a half [pointing to the numbered parts in Figure 20]." I concluded the teaching episode by asking them to each construct two objects: one with a measure of 5½ units and the other with a measure of 7½ units to see if they would use a consistent unit to construct their objects. They were both able to do this.

To summarize, in this teaching episode I confirmed their understanding of the relation of the unit to the whole. Nabeel was able to explain that if one decreases the number of units that covers a whole (keeping the whole constant), the unit size must increase: a form of qualitative compensation of area units. I then had them use their knowledge of area units and measure to construct regions with fractional measurements.

Teaching Episode Eight

In this teaching episode I worked with Moemen and Nabeel on Task 8, the comparison five rectangular cakes: a blue (8x3), white (2x12), orange (4x6), yellow (5x5), and red (4x7) cake[57]. After letting them work alone for 12 minutes, I asked Nabeel which cake he thought was the biggest. He replied that he thought the red cake was the biggest followed by the white cake.

M: No!

N: Yes! White and yellow are the same.

M: Orange and yellow are the same!

N: Blue and orange are the same!

Nabeel then demonstrated that the blue and orange cakes were the same size using decomposition/composition as shown in Figure 21.

[57] See Appendix F for a copy of the full task.

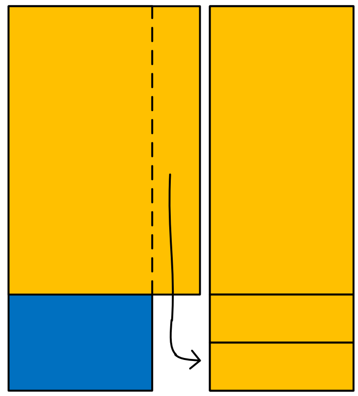

Figure 21. Nabeel's decomposition/composition justification.

I decided to leave the ordering of the other rectangles open and see if they could compare the rectangles using a unit approach. "You both think the orange and the blue are the same right? If two things are equal, and a unit and it measures this [the blue cake] ten times, then how many times would it need to measure this [the orange cake]?" They were unsure and so I asked, "What if a unit measured this [the blue cake] ten times and this [the orange cake] nine times? Would the cakes be equal?" They both immediately said, "No!" explaining that the two cakes would have different areas if the same unit measured them a different number of times. This statement is, in fact, the converse of the statement that two areas are equal if a unit measures both the same number of times. That is, they had demonstrated this emergent relation and property, but in a different way than Peter and Raima. I reflect on these distinctions in Chapter 5.

Next, I challenged them to show that the blue and orange cakes were the same area but by using a unit approach. I told them that they could use any unit they wanted. Nabeel used a 1x3 unit that he had decomposed from the orange cake (see Figure 21). Tiling this unit on another orange cake and tracing each time, he saw that it covered the orange cake eight times. “It should fit in here [the blue cake] eight times,” he whispered. Doing so, he confirmed the two cakes were the same size because his unit filled each eight times as shown in Figure 22.

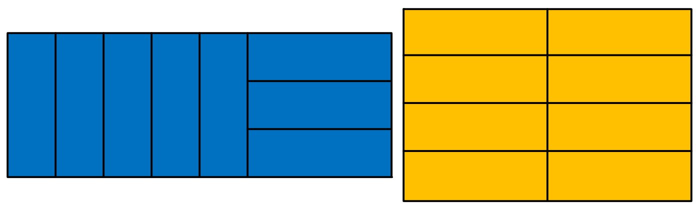

Figure 22. Nabeel’s unit justification.

I asked Nabeel to see how many times his unit would measure the yellow cake but first asked, “How many do you think will go in?” He replied, “I think about nine.” “Why do you think nine?” I asked. “Because … yellow is bigger than blue and orange so if the yellow is bigger than more units will go in,” he explained.

While Nabeel verified his conjecture, I went over to work with Moemen.

M: Why did I have to use something so small? I should have chosen something bigger.

I: So how do you know if you use …

M: It will make it longer.

I: It will make what longer?

M: It’ll take longer to do it [tile].

I: Why does it take longer to do it?

M: Cause they're [the units] smaller but if you chose something bigger you would be finished.

He demonstrated by cutting a larger unit from the orange cake and showing me that it would take less of those units to cover the blue cake than the unit he had chosen. Moemen had explained qualitative compensation of area units using a rate argument, essentially saying the more units you have to tile the longer it will take to tile them. He eventually discovered that it took 20 of the units he chose to fill the blue cake and went to work to show that it would also fill the orange cake 20 times.

I turned back to Nabeel: "Nabeel, what happened with yours?" I asked. He replied, "I was doing it [tiling his 1x3 unit] but I ran out of space because it didn't fit." Unlike tiling tasks found in most curriculum, Nabeel had encountered a situation where his unit did not fit exactly into an area he was trying to measure. I challenged him to try to find a number to describe the area of the yellow cake.

I: How many units do you have right now?

N: Seven.

I: Any way you could use what you learned about fractions last time to help you?

He struggled with this question and so I had him examine his tiling (shown on the left in Figure 23.)

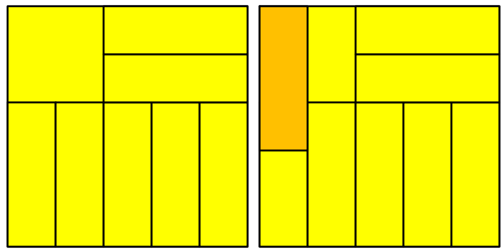

Figure 23. Nabeel’s attempt to fill a 5x5 cake with 1x3 units

I asked him again how many units he had. “Seven,” he replied. “So I think you’re struggling with this,” I said as I pointed to the region not covered by his unit. “Which is bigger? This space left over or your unit?” He thought for a while and pointed at the region on the yellow cake. “How do you know?” I asked. He took his 1x3 unit and placed it on top of the yellow region indicating with his fingers that he believed if he decomposed the unit, the decomposed parts would fit in the yellow region. “So how many of your units do you think will go into the yellow cake? Will it go in exactly eight, more than eight, or less than eight?” “More than eight,” he replied. “How do you know it’s more than eight?” I asked. “If you lay it down like this [shown on the right in Figure 23] you see that this space but if you cut it here [showing where his unit goes over the yellow region] and you had two of those, it would probably fit in here [pointing at the yellow region].”

In the same manner, I alternated working with Moemen and Nabeel and using this unit approach they concluded that the red cake was the largest followed by the yellow cake, and the white, blue, and orange cakes were the same size. I concluded this teaching episode by asking them if they could compare Moemen’s unit (which fit into the orange cake 20 times) and Nabeel’s unit (which fit into the blue orange eight times). Running out of time, and realizing I had given them a fractional ratio, I told them we would think about this more in our last teaching episode.

To summarize, in this teaching episode, I asked Moemen and Nabeel to use an internal unit approach to sort five rectangles by area. Although they each selected a unit that was the result of a decomposition (and not equipartitioning), the decomposed part happened to be a common unit for the blue and orange cakes (and indeed could have been obtained via equipartitioning). Giving them the task of using their unit to measure the other cakes led to the situation where Nabeel's unit did not fit completely into the yellow cake giving me the opportunity to work with him on finding the measure of the yellow cake in terms of his internal unit.

Teaching Episode Nine (Nabeel)

Moemen was absent during the last teaching episode and so I worked with Nabeel one-on-one. I wanted to see to what extent he needed to use manipulatives to help him compare the relative sizes of different units and to what extent he could abstract this concept. I began by asking him to equipartition two 3x8 blue cakes, one for eight (which we called Cake A) and the other for four (which we called Cake B).

I: How many slices in this cake [A] are needed to fill a slice from this cake [B]?

N: Two.

I: And how do you know it's two?

N: Because if I wanted to turn this [Cake B] one into eight, I would just cut it in half [indicating he could halve each of the four parts of Cake B].

I: How many pieces in Cake A would be the same as two pieces in Cake B?

N: Four.

I: And how do you know it's four?

N: Because if two of these [Cake A pieces] equals one of these [Cake B pieces], another two of these [Cake A pieces] would equal another one of these [Cake B pieces] so four slices of this one [Cake A] would go into two slices of this one [Cake B].

He then replied that six slices of Cake A would be the same as three slices of Cake B. I continued recorded his responses into a ratio table which is shown in Table 7. However, he struggled when I asked him how many slices of Cake A would be in 20 slices of Cake B.

Table 7. Ratio box for cakes A and B.

Cake A	Cake B
2	1
4	2
6	3
8	4
10	5
?	20
20	?

After some thinking he replied "Forty," explaining he had known to multiply by two because for the previous problem (five slices of Cake B), he had multiplied five times two to get ten. Next, I asked him how many slices of Cake B were in 20 slices of Cake A. He thought for a long time before replying, "Ten slices." I asked him how he knew and he replied, "Because if you count by 2's, five times two equals ten so if you do another one: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20. So there's ten slices." This strategy, a precursor to division, was to skip count by 2's (because he knew two slices of Cake A were equal to one slice of Cake B) until he got to 20 and then count the number of times he had skip counted.

Next, I asked him the same set of questions but asked him to think about two 4x6 orange cakes which I also asked him to equipartition for eight (Cake C) and four (Cake D). We constructed the ratio table for Cakes C and D. I then wanted to know if he could coordinate transitivity of area units with today's task.

I: So you shared this [Cake B] for four and this [Cake C] for eight. How many slices from this cake [B] did you say fit into a slice from this cake [A]?

N: Two.

I: So how many slices from this cake [B] would fit into a slice from this cake [C]?

N: Two.

I: How do you know that?

N: Because if they're both the same size, four … I mean two of these [slices from Cake B] should fit into the orange slice [from Cake C].

I thought that his response indicated the coordination of this ratio task with area measurement because he did not need to use decomposition/composition to check that two slices from Cake B were the same as a slice from Cake C. However, it turned out that he either did not have this understanding or that the sequence of tasks leading up to that question aided him. This was evident during the next problem, when I told him to equipartition an orange cake for three and a blue cake for nine, and asked "How many slices of the blue cake go into one slice from the orange cake?" He was able to solve the problem, however, when I asked him to consider two blue cakes: one equipartitioned for three and the other for nine. This time, he relied on the geometry of the cakes to figure out that three slices from the cake shared for nine was equal to one slice from the cake shared for one.

To summarize, in this teaching episode, Nabeel was able to compare the size of two units whose ratios were a whole number exhibiting some example of quantitative compensation of area units. The manipulatives helped him to visualize the scenarios, however, he was unable to transfer his knowledge of area measurement to the task.

Learning Trajectory Delineation

Based on my three teaching experiments with Peter, Raima, and Moemen and Nabeel, I modified the initial learning trajectory (Appendix H) into the following ten-level hypothetical learning trajectory for length and area. I do not claim the trajectory to be exhaustive of all topics in measurement, but only that it captures the most salient observed strategies, mathematical reasoning practices, emergent relations and properties, and generalizations that came from the task sequences in my teaching experiments. Indeed, the goal of this study was to examine the potential cognitive behaviors missing from an external unit approach to measurement and the role of equipartitioning. The proficiency levels of this trajectory reflect these purposes. However, a number of topics that are prevalent in measurement trajectories that enter from an external unit approach (e.g., unit iteration, tiling, etc.) were observed in the data. The initial 15-level trajectory was reduced to 10 because of the change of the goals of the trajectory from an enumeration of area units to a flexible understanding of area units. Chapter 5 contains a detailed discussion of the changes made to the trajectory.

First, the proficiency levels in the trajectory are listed. The full trajectory is briefly described including brief justifications for the ordering of the levels. Next, outcome spaces for each proficiency level in the trajectory are presented, ordered from low to high, with examples from the teaching experiments as page numbers to the cases.

Length and Area Learning Trajectory Levels

1. (Identification) Distinguishes among qualities of length and area.
2. (Conservation of Rigid Transformations) Conserves length and area through simple rigid transformations (translations, rotations, and reflections).
3. (Direct Comparison of Length and Area) 3a. Direct comparison among two or more extensible or straight lengths. 3b. Direct comparison among two or more entirely overlapping areas.
4. (Conservation of Length and Area Decomposition/Composition over Breaking) Recognizes different ways to break a continuous length or area into components, and joins them into a whole, and asserts conservation of length and area).
5. (Indirect Comparison of Length) Compares two or more line segments (including bent paths) using a third object.
6. Demonstrates that non-congruent rectangles can have the same area by composition/decomposition or by finding a commensurate unit.

7. Demonstrates that measurement of area (number of units) is relative to unit size and that the total area is "*n* times as many" as the unit size if an *n*-split is used to create the unit.
8. (Qualitative and Doubling/Halving Compensation of Area Units) Predicts (qualitatively or for doubling/halving) that increasing/decreasing the size of an area unit inversely (decreasing/increasing) affects the number of units needed to cover and justifies.
9. (Measurement of Area) Recognizes that if two non-congruent rectangles (A and B) are equal in area and a unit measures Rectangle A *n* times then this unit also measures rectangle B *n* times.
10. (Quantitative Compensation of Area Units) If unit 1 measures an area *n* times and unit 2 measures the same area *m* times then:

$$\text{Size (unit 1)} : \text{Size (unit 2)} :: m : n.$$

The Levels of the Length and Area Learning Trajectory

The learning trajectory begins with identification among the qualities of length and area. Students should learn to move beyond informal adjectives (such as thinness, longness, fatness, etc.) to describe length and area. They begin to see length as an extension or path and area as a cover or space enclosed by an object, start to identify lengths and areas as components on two-dimensional objects, and recognize that it is impossible to compare or combine a magnitude of length with a magnitude of area.

The next level, conservation of length and area under translations, rotations, and reflections prepares students to directly compare straight lengths and completely overlapping areas, for it would be impossible to use a direct comparison scheme if students believed that moving a length or area changes its magnitude. At the next level, students develop strategies to directly compare line segments and completely overlapping areas. Conservation of length and area decomposition/composition refers to the ability to break a length or area into components and rejoin these components back into a whole in any configuration while asserting that length and area is conserved. It sets up comparison strategies for bent line segments and non-completely overlapping areas, as well as iterating external units, and constructing internal units. There is a possible quantitative level to this emergent property for length and area involving units. That is, any length (or area) measuring X units, when decomposed into pieces of length (or areas) of magnitude X_1, X_2, …, X_n, must satisfy the property that $X_1 + X_2 + \ldots + X_n = X$. Students then develop strategies to indirectly compare lengths by either mapping to a longer external length or by iterating an external length unit with no gaps or overlaps. Although not explored in this study, an analogous task sequence should exist for indirect comparison of lengths using an internally created length unit.

Next is the problem of showing that two non-congruent rectangles can have the same area through either decomposition/composition or constructing an internal unit from equipartitioning. After creating different internal units, students come to understand that the measurement of area is relative to the unit size and not solely named based on the number of units in the magnitude. Moreover, it is important for them to realize that if an n-split is used to create an internal unit, then the whole is *n* times as large as the unit and the unit has a relative size of 1/*n*. Next, students are able to reason and justify that to keep the area enclosed invariant, there exists an inverse relationship between the number of units needed to cover the area and the size of the unit. At this level, they are able to reason that doubling/halving the size of a unit halves/doubles the number of units needed to cover an invariant area. At the penultimate level, students recognize that if two rectangles (not necessarily congruent) are equal in area and a unit measures the first rectangle *n* times, then that unit must also measure the second rectangle *n* times. This emergent property's converse and contrapositive are also true, although I did not explore these with the students in my teaching experiment. Finally, the top level of the learning trajectory is the generalization of quantitative compensation. That is, if unit 1 measures an area *n* times and unit 2 measures the same area *m* times then there exists a ratio relationship in the form of Size (unit 1) : Size (unit 2) :: *m* : *n*. This sets up the basis for ratio comparison of units. Most importantly, it implies that one can always find the *times as many* relationship between the size of unit 1 and the size of unit 2 by examining the quotient *m*/*n* and allows students to convert between any commensurate units if the ratio between the relative size of the units is known.

Outcome Spaces

Level 1: (Identification) Distinguishes Among Qualities of Length and Area

Table 8. Level 1 Outcome Space

Description	Exemplars
Fully distinguishes between qualities of length and area by, for example, stating that a line has no area or identifying length in two dimensional objects.	Peter (evidence of partial understanding), p. 64 Raima, p. 109 Moemen, p. 146
Defines length as a path or extension and area as a cover, an enclosed space, or a sweep.	Peter (area), p. 62 Peter (length), p. 64 Moemen and Nabeel (area), p. 144 Moemen and Nabeel (length), p. 145
Informally describes the qualities of length (skinniness, thinness) or area (fatness, wideness, space) using qualitative descriptors.	
Cannot identify length or area as attributes.	

Description. At the lowest level, students cannot identify a length or an area and have no conception of them as quantities. At the next level, students may describe length or area using informal qualitative terms including “skinniness, thinness, and longness” for length and “fatness, wideness, and space” for area. This indicates that while they have some notion of descriptors for length and area, they cannot formally express these ideas. This level may also be observed in children that are unable to distinguish various dimensions of lengths believing, for example, that lengths only run horizontally or there are certain conditions for an to be longer or shorter than another. At the second highest level, students are able to formally describe length as extension or path and area as a cover, space enclosed or a sweep. All of the students in my teaching experiment achieved this level. At the highest level, students should be able to distinguish length and area as different attributes on a two-dimensional object. Moemen, for example, was able to compare the length of a bungee cord to the side lengths of an index card. Raima and Peter (although with less evidence) demonstrated that magnitudes of length cannot be combined with magnitudes of area.

Level 2: (Conservation of Rigid Transformations) Conserves length and area through simple rigid transformations (translations, rotations, and reflections).

Table 9. Level 2 Outcome Space

Description	Exemplars
Believes that length or area does not change under translations, reflections, or rotations.	Peter (length under rigid transformations), p. 70 Raima, p. 113 Moemen and Nabeel (conserves length under translations), p. 145 Moemen and Nabeel (length and area under rigid transformations), p. 149
Believes that length or area does change under translations, reflections, or rotations.	

Description. Originally, this level was placed at level 3, however, all of the students in my teaching experiment as well as all of the students in the DELTA team's length and area interviews asserted this conservation. It also became clear during the teaching experiment, that without conservation of rigid transformations direct comparison of length and area are not possible.

Level 3a: Direct Comparison among two or more extensible or straight lengths and Level 3b: Direct comparison among two or more entirely overlapping areas

Table 10. Level 3a Outcome Space

Description	Exemplars
Aligns two length objects end-to-end or nests one length object inside another and notes overlaps as a way to determine which is longer or equal.	Peter, p. 64 Raima, p. 109 Moemen and Nabeel, p. 145 Moemen, p. 147
Projects length using an informal or non-consistent measure (i.e. spacing between fingers) one object onto another without close alignment to determine which is longer or equal.	
Uses only perception or visual cues to compare two or more extensible or straight lengths.	

Table 11. Level 3b Outcome Space

Description	Exemplars
Compares two or more entirely overlapping areas by placing one object on top of the other and comparing their relative cover.	Peter, p. 63 Raima, p. 109 Moemen and Nabeel, p. 145 Nabeel, p. 147
Compares two or more entirely overlapping areas (by matching side lengths or using an additive length + width strategy).	
Uses only perception or visual cues to compare two or more entirely overlapping areas.	

Description. Level 3 is broken up into two levels: 3a and 3b. For both length and area, the lowest level occurs when very young children use only perception to compare length and area without making any projections, imagined or real, possibly due to lack of the ability to conserve across rigid transformations. In the case that a length or area is much bigger or much smaller than another, this strategy is viable (and indeed, Peter, Moemen, and

Nabeel used this strategy on discrepant objects) but becomes less viable as the two objects being compared are closer in measure and their distances increase, as the child cannot compensate for perspective.

All of the students in the teaching experiment achieved the highest level whereby they directly compared lengths by aligning them side-by-side and noticing the object that extended further. For area, they all compared completely overlapping areas by placing one object on top of the other.

Level 4: (Conservation of Length and Area Decomposition/Composition over Breaking) Recognizes different ways to break a continuous length or area into components, and to join them into a whole, and asserts conservation of length and area).

Table 12. Level 4 Outcome Space

Description	Exemplars
Believes breaking a length or area and then reassembly in any orientation (i.e. by bending or non-congruent rearrangement) does not change the length or area.	Peter (length, final understanding), p. 75 Raima (area), p. 112 Raima (length), p. 114, p. 115 Nabeel and Moemen (area), p. 149
Believes that breaking a length or area and then reassembling in the *exact orientation* does not change the length or area but reassembling in a different orientation (i.e. by bending or non-congruent rearrangement) does change the length or area.	Peter (length, initial understanding), p. 71
Believes that breaking and putting a length or area back together (regardless of bending or congruence) changes the length or area.	

Description. At the lowest level, students believe that breaking a length or an area and rejoining causes a change in length or area regardless of how the objects are joined back together. At the next level, students believe that if the length or area is reassembled *exactly* in the orientation it was before breaking then length and area are conserved. However, they believe that bending a length or reassembling an area into a configuration that is not congruent with its original configuration *does* change the length or area. Peter was initially at this level until I brought to his attention that an ant must walk down both parts of a bent path. At the highest level, students believe that breaking and reassembling *regardless of orientation* conserves the length and area.

Level 5: Comparison of two line segments (including bent paths) using a third object.

Table 13. Level 5 Outcome Space

Description	**Exemplars**
Iterates a third object with no gaps or overlaps along the path coordinating iterative action with number (including fractional parts), correctly associates the quantity with the unit of measure, and attends to possible bends in the path.	Peter (no naming of units, possibly a lower level), 80 Raima, p. 116 Moemen and Nabeel, p. 153
Directly compares the two line segments by either straightening a bent path (if necessary) and applying additivity of lengths or mapping the lengths onto a longer length.	Peter (additivity), p. 79, p. 79 Nabeel (mapping), p. 152
Uses a precise iterating unit for each part of a path but changes the unit for each bend.	
Uses an imprecise iterating method which may include the misconceptions of overlapping units, unequal units, or marking inconsistent tick marks on a length.	Moemen (initially, self corrects), p. 152
Cannot compare the length of two paths or uses only perceptual cues to compare.	

Description. This level provides an interesting challenge in learning trajectory creation because most extant trajectories describe this level as the mathematical transitive reasoning level (Battista, 2007; Clements & Sarama, 2009).

At the lowest level, students use only perception to compare the length of paths. At the next level, students use an imprecisely iterating procedure by using a non-consistent iterating unit, creating gaps and overlaps in their iteration, or draw non-equal tick marks (perhaps as a failure of equipartitioning). Moemen initially iterated a straw piece using gaps and overlaps, but the next time he self-corrected and used a pencil to mark where each iteration ended. At the next level, as proposed by Barrett and Clements (2003), students use a precise iterating procedure but only on component lengths and not for an overall length. For example, when trying to determine the perimeter of a triangle, students may use a consistent iterating procedure for one side but then switch the unit but precisely use that unit for another side. This was not seen in my teaching experiments. At the highest level, students use a consistent iterating procedure for the entire path length or perimeter of an object and can relate the magnitude of their measure to a unit.

Level 6: (Strategy) Demonstrates that non-congruent rectangles can have the same area by composition/decomposition or by finding a commensurate unit.

Table 14. Level 6 Outcome Space

Description	**Exemplars**
Finds a commensurate unit through decomposition/composition or equipartitioning.	Nabeel and Moemen (decomposition and composition), p. 155, Raima, p. 131
Methods of exhaustion.	Peter, p. 83, p. 84 Raima, p. 118, p. 131
Compares the area of non-congruent rectangles through informal gridding.	
Cannot compare or uses only perception to compare.	

Description. At the lowest level, students use only perception to compare the three areas. During my pilot interviews, many young children up to third grade exhibited this behavior and would often use the fatness or skinniness of a cake as their reason for why certain cakes were larger than another. At the next level, hypothesized but not observed, I expect some students to use informal gridding strategies similar to students in Outhred and Mitchelmore's (2000) study where students created inconsistent rows and columns. At the next level, observed from all of the students in my teaching experiment, students essentially overlap the objects using a strategy that was viable at Level 3, but discover that the objects do not completely overlap. The students then know that they have common overlapped area and must compare the non-overlap area. They do this by either cutting off both non-overlapping areas and comparing them or by cutting off one of the non-overlap areas and try to use it to cover the other non-overlap area. This strategy is summarized in Figure 24.

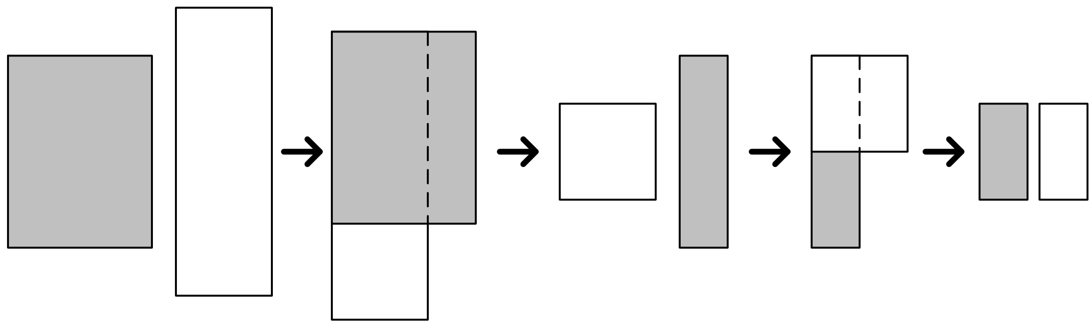

Figure 24. Level 6 Exhaustion Strategy.

At the top level of the trajectory, either through composition and decomposition or by equipartitioning, students construct an internal unit and use it to measure each of the rectangles and reason that because they are using a unit, the counts of those units can be compared across the rectangles.

Level 7: (Mathematical Reasoning Practice) Demonstrates that measurement of area (number of units) is relative to unit size and that the total area is "*n* times as many" as the unit size if an *n*-split is used to create the unit.

Table 15. Level 7 Outcome Space

Description	Exemplars
Demonstrates the distinction between area as enclosed space and measurement of area by constructing an example of different areas that have the same measure based on the unit size and an example of same areas that have different measures. Reasons that two areas that are equal in measure are also equal in terms of space enclosed if the units are the same. Can state that the total area is *n* times as many as the unit.	Peter, p. 95 (in between these levels: struggling with the tension between additive and multiplicative) Raima (times as many), p. 123 Moemen and Nabeel (unit size), p. 160 Moemen and Nabeel (times as many), p. 160
Believes that measurement of area is based on the measure (# of units) regardless of unit size hence reasoning that the bigger area is always the bigger measure (# of units) regardless of unit size. States that the total area is (*n* - 1) times as many as the unit size exhibiting an additive misconception.	
Cannot construct a unit to compare area.	

Description. This level was originally Levels 8 and 9 in the initial learning trajectory (Appendix H). However, in light of the DELTA framework for mathematical understanding, I have merged them and moved them to level 7 because *times as many* is a mathematical reasoning practice and is important in understanding the idea that measurement of area is based on the size of consistent iterating units. It also is the key mathematical reasoning practice that is needed to achieve the highest level in the previous proficiency level. That is, in order to compare two or more areas using a unit, students must believe that that unit size matters.

All of the students in my teaching experiment demonstrated understanding at the highest level. At the conjectured lowest level, students would be unable to construct or use a unit at all to compare area. At the next level, they would believe that regardless of unit size it is the measure that determines the area. These students would fail on Task 7 because they could believe thattwo2 different sized cakes both shared for 8 people would be the same. The corollary to this is that given two different sized cakes shared for *n* people, a slice from the larger cake is always preferable, in terms of equipartitioning.

At the highest level, students understand that the unit size matters. They are able to construct the scenario where two cakes with different areas can have the same measure. I place Peter at this level, even though he showed evidence of struggling with the tension of *times as many* being multiplicative versus additive.

Level 8: (Emergent Relation) (Qualitative and Doubling/Halving Compensation of Area Units) Predicts (qualitatively or for doubling/halving) that increasing/decreasing the size of an area unit inversely (decreasing/increasing) affects the number of units needed to cover.

Table 16. Level 8 Outcome Space

Description	**Exemplars**
States and justifies that doubling the size of the area unit halves the number of units needed to cover and that halving the size of the area unit doubles the number of units needed to cover.	Peter, p. 85 (can also code as multiplication) Raima (informal expressed as equipartitioning), p. 122
Can predict that increasing/decreasing the size of an area unit inversely affects the number of units needed to cover.	Nabeel, p. 165 Moemen, p. 172
Cannot predict that increasing/decreasing the size of an area unit inversely affects the number of units needed to cover.	

Description. This level was originally listed as Level 7 in the initial trajectory (Appendix H), however, in light of the DELTA framework on mathematical understanding I moved it above *times as many* because it describes an emergent relationship between the size of a unit and the number of units needed to cover. At the lowest level, students cannot predict that increasing the size of an area unit decreases the number of units needed to cover and that decreasing the size of an area unit increases the number of units needed to cover. At the next level, they can make this qualitative statement and predict that if an area unit is increased then less of it is needed to fill an area and if an area unit is decreased than more of it is needed to fill an area. At the highest level, students are able to extend this to a quantitative statement but only for the case where the iterating unit is doubled or halved. This anticipates full quantitative compensation.

Level 9: (Emergent Relation) (Measurement of Area) Recognizes that if two non-congruent rectangles (A and B) are equal in area and a unit measures Rectangle A *n* times then this unit also measures rectangle B *n* times.

Table 17. Level 9 Outcome Space

Description	Exemplars
Predicts that the unit measures rectangle *B n* times by making a transitive argument that since the unit measures *A n* times, it must also measure *B n* times.	Peter, 105 Raima, p. 121, p. 133 Moemen and Nabeel (contrapositive), p. 170
Predicts that the unit measures rectangle *B n* times through composition and decomposition.	Peter, p. 102 Moemen, p. 162
Uses perceptual cues to predict the number of times the unit measures *B* or predicts by comparing the relative sizes of the unit to another common unit of rectangle *B*.	
States that this cannot be done or would be difficult to figure out because the unit would not fit perfectly into Rectangle *B*.	

Description. While closely related to SSEWE from equipartitioning, the emergent relation and property at this level approaches an external system for measurement. Indeed, the statement at this level (if two non-congruent rectangles (A and B) are equal in area and a unit measures Rectangle A *n* times then this unit also measures rectangle B *n* times) along with its converse (If a unit measures Rectangle A *n* times and rectangle B *n* times, then the area of rectangle A = the area of rectangle B) and contrapositive (If a unit measures Rectangle A and Rectangle B a different number of times, then the two rectangles are not equal in terms of area) sets the foundation for measurement of area with units.

At the lowest level, students are unable to figure out that the problem either because of lack of composition/decomposition knowledge or believe that it is impossible to determine. At the next level students use perception to examine the relative size of the units and make informal comparisons. At the next level students predict that the unit measures *n* times based on using composition and decomposition to compare the relative sizes of the units.

At the highest level, students reason through logical necessity that it must be the case that the n number of unit fills rectangle B. Interestingly, there seems to be two ways to reach this highest level. One is through logical necessity and justification which Peter and Raima both demonstrated. However, Moemen and Nabeel came at the idea from a unit level approach. That is, if a unit measures one rectangle n times and measures another rectangle m times such that n is not equal to m, then the rectangles cannot be equal.

Level 10: Quantitative Compensation of Area Units) If unit 1 measures an area n times and unit 2 measures the same area m times then:

$$\text{Size (unit 1)} : \text{Size (unit 2)} :: m : n.$$

Table 18. Level 10 Outcome Space

Description	Exemplars
Can solve for any quantity given the other quantities in the ratio box.	
Can compare the relative size of unit 1 and unit 2 for cases m:n is a rational number.	Raima, p. 136 Nabeel, p. 175
Can compare the relative size of unit 1 and unit 2 for cases where m:n is a whole number.	Raima, p. 136
Cannot quantitatively reason or attempts to reason but uses only perception.	

Quantitative compensation in this learning trajectory is closely tied to the idea of comparison between two different sized units. At the lowest level, students cannot quantitatively reason or uses only perception to compare the two units. At the next level, students can reason about the relative size of the units if that ratio relationship is a whole number. At the second highest level, students should be able to reason if the ratio relationship is any rational number. Finally, in the hypothesized highest level, students should be able to solve the ratio box problem given any three of the quantities.

In this chapter, three cases were presented comprised of four students who participated in the teaching experiment. While different students proceeded at different rates and exhibited a variety of strategies, common strategies, mathematical reasoning practices, emergent relations and properties, and generalizations were noted across the three teaching experiments. The results were used to modify the initial learning trajectory into a new 10-level trajectory for length and area measurement.

CHAPTER 5

REVIEW, DISCUSSION AND CONCLUSIONS

Answers to the Research Questions

The central aims of this study were to examine the potential cognitive behaviors missing from approaching area measurement from a strict external unit approach and the role of equipartitioning in a learning trajectory for length and area. Specifically, the two research questions were:

1. What is the viability of a learning trajectory for length and area based on a foundation of equipartitioning that requires students to construct the unit and prepares them for the learning of fractions and ratio? That is, once proposed, through synthesis and theoretical work, how does the initial learning trajectory change as a result of empirical study through interviews of children?
2. What strategies and mathematical reasoning practices do students use to construct internal units, what types of emergent relations and generalizations do students make by comparing multiple units, and does this help students avoid the misconception that units are discrete and inviolable?

Research Question One

Viability of a Learning Trajectory for Length and Area based on EP

Based on the analysis of the three teaching experiments in this study, there is evidence to suggest a learning trajectory for area measurement whose goal is not the enumeration of the number of units in a rectangular array (Battista, 2007; Outhred & Mitchelmore, 2000).

The findings for area however suggest: 1) a central role for equipartitioning in the development of such a trajectory, 2) student construction of internal units need not obviate students from learning tiling and unit iteration, and 3) there are numerous opportunities in such a trajectory to extend toward fractions and ratio from within the context of equipartitioning and area measurement.

Changes to the Initial Learning Trajectory. Fifteen levels comprised the initial learning trajectory: (1) Identity, (2) Direct Comparison, (3) Conservation of Rigid Transformations, (4) Conservation of Reassembly, (5) Indirect Comparison, (6) Comparison of two non-congruent rectangles, (7) Construction of Multiple Common Units, (8) Qualitative Compensation, (9) Quantitative Compensation, (10) Times as Many, (11) Area = Length x Width, (12) Common Measures, (13) 1/m x 1/n Area Unit, (14) Fractional Length, and (15) Fractional Area.

Two changes were made to the first five levels of the trajectory. First, levels two and three were swapped because the task of directly comparing lengths and completely overlapping areas required that students understand that length and area are invariant under translation. Second, *conservation of reassembly* was renamed *conservation of decomposition/composition over breaking* because in the latest version of the equipartitioning learning trajectory (Confrey et al., 2010), we use the word reassembly to refer to the inverse of equipartitioning, an early form of multiplication.

As discussed in Chapter 4, the major change in the learning trajectory was made due to a change in the main goal of the trajectory. Whereas the goal of the initial trajectory was the enumeration of 1x1 area units in a rectangular array, the goal of the revised trajectory was for students to acquire the ability to create and move among different area units as measures of area. This eliminated Levels 10 – 15 in the initial trajectory, including a formal level for area = length x width and the formal introduction of fractional length and area. Level 10 (times as many) was incorporated into a new Level 7 that combined understanding that measurement is relative to unit size and, for area units created by using an *n*-split, that the whole is *n* times as large as the unit. Level 12 (Common Measures), the ability to distinguish between, for example, 12 inches and 12 centimeters was also incorporated into Level 7 in the statement that measurement is relative to unit size.

Last, Level 9 (Quantitative Compensation) was broken up into two levels. In the revised trajectory, Level 9 is an elementary version of measurement with area units without its converse and contrapositive and Level 10 is understanding the ratio relationship between different sized area units. This allows for flexibly switching between area units.

Equipartitioning. Based on the analysis of the teaching experiments, I conclude that equipartitioning played a key role in helping students in the teaching experiment recognize that an area can be measured using different sized area units. They recognized that although the number of units changed depending on the split, the area as enclosed space stayed the same. This allowed them to talk about the emergent relation and property that measurement of area is based on the size of the unit and not just the count of units. Although each student in the teaching experiment brought their own unique approaches to solving the tasks, their prerequisite knowledge of equipartitioning played a role in the extent to which I was able to explore ratio and switching between different area units.

In addition, equipartitioning a rectangular whole helped students to come to an understanding of the emergent relation and property of qualitative compensation of area units. I realized early in Peter's teaching experiment that his inability to perform certain equipartitions on a rectangle would limit his ability to create multiple internal units because equipartitioning a whole leads directly to the creation of different sized area units. I conjecture that equipartitioning the same sized rectangle for different numbers of people assists in the development of qualitative compensation of area units. That is, students note that the more people sharing, the smaller the size of the shares. This is another way of expressing the inverse relationship between the unit and the number of units needed to cover. For example, note how Peter stated the relationship between the unit size and the whole. He noticed that when sharing the same size cake for different people numbers of people, cakes that have "less numbers" (i.e. shares) have "more space" (i.e. larger units) because "they still have the same things inside of them" (i.e. the area is invariant in these rectangles). He concluded that given the choice between two cakes of the same size, he would rather have a slice from the cake with the fewer number of slices because those slices would be bigger.

Relationship to Tiling. As hypothesized in the last section of the literature review, student selection or creation of an internal unit need not obviate the development of important constructs in extant external unit learning trajectories. Once students have a unit, whether externally given or created internally through equipartitioning, they can use that unit to tile and instruction can help them to enumerate a row by column structure. In the teaching experiment, Raima, Moemen, and Nabeel used an internally created unit in the five rectangles task to compare the area of the rectangles. Granted, I did not specifically explore with them their spontaneously created arrays as is emphasized in other trajectories and their units were not 1x1 square units[58]. However, reflecting back on the data, there are two possible places where enumeration of units in an array may be explicitly treated. The first was seen during the teaching episodes where I worked with students on composition of spits. After each composition, students were asked to relate their split with a multiplication fact. For example, the multiplication fact 4x3 = 12 would be associated with performing a 12-split by composing a 4-split with a 3-split. The second was during the five rectangles task when I asked students to use a unit strategy. Nabeel, for example, used a 1x3 unit to tile the 4x6 orange cake, creating a 2x4 array of 1x3 units without any gaps or overlaps. It may be necessary, however, to first get students to understand the convenience of the 1x1 square unit as tiling non-square units is not the usual mathematical reasoning practice.

Extensions to Fractions and Ratio. Proceeding from the last point, student creation of an internal unit also facilitates the understanding of fractions. In the teaching experiments, this occurred when students encountered the situation where their internal unit did not

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[58] I would also argue, however, that equipartitioning provided a stronger multiplicative base for the students than generalizing the count of rows and columns as multiplication. Indeed, I saw evidence of them explicitly using multiplication facts during tiling to obviate the need to tile all rows and columns. For example, Nabeel only needed to tile down one column (which he did four times) and simply stated that since he had two columns, the total number of tiles would be 8 = 4 * 2.

completely cover a rectangle that they were measuring. To figure out how many of their units fit into the remaining area necessitated either an equipartitioning strategy or making an explicit comparison of their unit to the remaining area. Either way, fractions were needed to name this new area in terms of the unit. The other opportunity for the introduction of fractions occurred after students had understood the relationship between the unit and the whole. The task posed to them was to use a unit to construct an area with a fractional measure. For example, Moemen and Nabeel were able to construct a cake with measure "four and a-half" by cutting out four reference units and cutting a fifth unit in half and joining the four reference units with the half unit.

Ratio was introduced to students via comparison of different sized units. Indeed, the highest level in the trajectory, quantitative compensation, means that students understand that the ratio between two different sized units is the same as the inverse ratio of the number of times those units measure a common area. That is, if Unit A is twice as large as Unit B, this means that to fill any common area requires twice as many units of size B than size A. All students in the teaching experiment were able to quantitatively reason when the relationship between Unit A and B was two. Nabeel was able to reason for whole number values and Raima was able to reason for rational number values.

Research Question Two

Evidence of Progress on the Framework for Understanding and a Challenge to the Idea that Units are Inviolable

The learning trajectory synthesized from this study and presented in the previous chapter provides an answer to this research question. Two strategies for creating and using units to compare the area of rectangles were observed. The most common strategy was to use a "unit" that was leftover from a dissection or decomposition/composition. Another strategy was to use a unit from equipartitioning. Because of the relative sizes of the rectangles used in the tasks, it conveniently turned out that these decomposed pieces happened to be common units for the rectangles being compared in the three rectangles task. In the five rectangles task, however, this was not always the case but the students were able to successfully compare the areas of the rectangles by bounding the measure of a rectangle. For example, Nabeel reasoned that the yellow 5x5 cake was larger than the orange 4x6 cake because his unit measured the orange cake eight times but measured the yellow cake more than eight times.

Emergent relations and properties from the learning trajectory include the demonstration that measurement of area is related to unit size and the previously mentioned relationship between the unit and the whole, if the unit was created using equipartitioning. Students also developed understanding of the inverse relationship between the size of the unit and the number of units needed to cover an area. All of the students in the teaching experiment were able to express this emergent relationship qualitatively, often using the context of equipartitioning to help them explain. They were also able to express this relationship quantitatively for whole numbers. One of the students, Raima, through use of a ratio teaching sequence, was able to extend quantitative compensation to include rational number compensation factors including fractions and mixed numbers.

There was evidence from the teaching experiments to suggest that the answer to the second part of this research question is, yes, students did avoid the misconception that units are discrete and inviolable. This was demonstrated by all students when I asked them to use a unit to create a “cake” with a fractional measure. Raima, Moemen, and Nabeel demonstrated that they could solve the case in which their internal unit did not completely cover the area they were measuring. Through scaffolding, Nabeel was able to partition his iterating unit and bound the measure between two integers. Raima went further and was able to accurately name the area in relation to her unit. She also demonstrated the ability to accurately name splits of splits[59]. The teaching experiments also provide evidence that students should be given the opportunity to encounter this problem. Otherwise, tiling areas or iterating paths with units that completely cover them have the potential of perpetuating this misconception.

[59] I would argue that, given more time, I could have worked with Peter, Nabeel, and Moemen further on equipartitioning and they, too, would have been able to do this. Indeed, it would be interesting to perform a teaching experiment where the equipartitioning learning trajectory was fully developed and follow with the teaching sequence in this trajectory to develop power in measurement.

Discussion and Conclusions

Defining Measurement

The results of this dissertation study suggest a different definition of measurement than is generally taught in the school curriculum. In most school curricula (Smith et al., 2008), measurement is defined and taught as the association of a number of units with a given quantity. If one uses this definition, than the exhaustion strategy used to compare rectangular areas in the three and five rectangles tasks would not be considered formal measurement but, instead, an intermediary step toward measurement. However, if one defines measurement more broadly as the development of systematic processes to compare the amount of two or more quantities, than the previous definition (associating the number of units with a given quantity) is a strategy to compare two or more quantities. Furthermore, the mathematical practice of using 1x1 square units to measure area is a curtailment of using non-square rectangular units to measure. Note that none of the students in my teaching experiment generalized to this 1x1 unit but were able to "measure" using non-square rectangular units.

In this teaching experiment, my own notions of measurement changed from the first to the second definition when I shifted the goal of the teaching experiment away from the enumeration of 1x1 square units in a rectangular array. Hence, I believe that strategies of exhaustion and bounding the numeric area of a rectangle (as opposed to finding the precise measure) are examples of measurement. Indeed, students did not construct a "numerical measure" to solve the three and five rectangles task except when I explicitly asked them to use a unit strategy. And in the five rectangles task, it was not necessary to find an *exact* measure when the unit did not fit into a rectangle a whole number of times. Eventually, students should come to understand why the 1x1 square unit is used in mathematical practice to measure and compare areas. However, there is evidence from this dissertation study that students would miss the opportunity to develop a strong internal measurement structure if they are rushed toward this goal and were unable to explore other strategies for comparing areas.

Implications for Practice

This study provides evidence that students should be provided with a variety of instructional strategies regarding area measurement. Restricting instruction to iteration through given external units means that students miss the chance to develop understanding of units and qualitative and quantitative compensation of units.

There is also evidence from the investigation of the first five levels of the trajectory that length and area can be taught in parallel through the levels of identification, direct comparison, conservation of rigid transformations, conservation of decomposition/composition, and indirect comparison. In current curriculum (Smith et al., 2008), and standards (CCSSO, 2010), area measurement is not addressed until a complete treatment of length measurement. However, in the teaching experiment Peter was able to reason that area was a covering without a complete “mastery” of length measurement. In levels 1-4, length and area were also successfully concurrently investigated. Although there was no parallel structure in my indirect comparison tasks for length and area, the idea should be investigated if indirect comparison of length and area could be developed together too.

Finally, this research suggests the importance of equipartitioning within the school curriculum. The DELTA work on equipartitioning (Confrey et al., 2010) provided evidence of equipartitioning's key role in developing rational number and fraction. This study suggests that equipartitioning plays a key role in developing measurement concepts such as measure and qualitative/quantitative compensation of area units.

Implications for Research

As with most exploratory research, this study suggests that there is still work to be done in the study of length and area measurement. First, research needs to be more open in allowing for the wide range of student strategies to be applied to measurement contexts. Because this study was restricted to exploring student creation and application of internally created units, other student strategies such as physical mapping and gridding were not fully explored. An additional consequence was the failure to keep parallel structure between the comparison of length and comparison of area tasks. That is, the question also needs to be asked what would happen if students were asked to compare lengths without an external unit. There is evidence from the teaching experiments that analogous questions regarding the relationship of internally created length units and the whole and qualitative/quantitative compensation of length units would also apply and perhaps setup students to be better prepared for the area equivalents of these emergent relations and properties.

In Chapter 2, synthesized research showed that students had difficulty (1) calculating the length of a matchstick that was not aligned with zero on a ruler, (2) distinguishing between length and area, (3) recognizing the array structure, and (4) understanding non-whole number areas. Although (1) was not addressed in this teaching experiment, there was evidence that the students in this dissertation study made progress on difficulties 2-4. All of the students demonstrated to various degrees that it is impossible to compare a magnitude of length with a magnitude of area. Raima, Moemen, and Nabeel showed evidence of recognizing the array structure in the five rectangles task. For example, when tiling with a rectangular unit, Nabeel only needed to tile one "column" of a rectangle. He explained that he could "multiply by two" to calculate the total number of units in the rectangle because there were two columns of four units. This example, along with the ability to reason that doubling the size of an area unit halves the number of units needed to measure, provides evidence that students in the teaching experiment developed stronger notions of multiplication than in extant learning trajectories that rely on the introduction of an external unit. Finally, given a reference unit, students were able to construct fractional areas.

Limitations and Future Research

There are three major limitations to this study. As discussed earlier in this chapter, selection of the rectangles for the comparison of three rectangles task allowed any rectangle formed from equipartitioning to be used as a *common unit* among the three rectangles limiting the opportunity to explore any emergent relations and properties that might emerge from a task requiring construction of a common area unit among two or more rectangles.

Second, an external iterating length unit was given to students to explore Level 5, restricting the exploration of Levels 6 – 10 to area units. Finally, exploration of the trajectory stopped short at articulating a full system for quantitative area measurement.

Future research should address these three limitations. To address the first limitation, the DELTA framework for understanding should be applied to a task requiring students to construct a common unit to compare two or more rectangles where not all equipartitions of the rectangles results in the creation of a common unit. To address the second limitation, parallel versions of the tasks investigating levels 6 – 10 should be extended to length measurement to see if there is parallel structure between length and area for indirect comparison. Finally, the converse and contrapositive for Level 9 in this learning trajectory should be further investigated along with the 1x1 square unit. These last two steps, tied to the research in this dissertation, has the potential to articulate a full quantitative system of area measurement that is tied to equipartitioning.

Conclusions

Measurement has been a domain on which American students have historically poorly performed. However, it is a subfield that can be further examined, not only to investigate what it means to measure, but also how to use measurement to reason towards more sophisticated ideas in rational number reasoning. Learning trajectory creation and delineation can provide a scaffold to inform this examination. First, it permits one to articulate and empirically test the viability of extant research in the field. Second, it allows for the modification of tasks to investigate if possible avenues of student thinking are missing. Third, it allows one to understand how to increase the power and reach of student reasoning by examining and synthesizing different approaches toward measurement.

Construction of the learning trajectory in this dissertation began with the question of what cognitive actions were potentially missing from systematizing measurement solely through the iteration of a given external unit. The results were a trajectory whose goal was the ability to create and switch between different sized area units. Although the trajectory proved viable in the three teaching experiments in the study, it will be important to further test the validity of the trajectory through item response theory models (Confrey et al., in progress) and teacher professional development (Wilson, Sztajn, & Confrey, 2011) to see how the trajectory holds in the context of real classrooms.

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APPENDICES

Appendix A
Tasks 1 and 2

Examining Attributes of Various Objects and Ordering Various Objects by Length and Area

Task for Length/Area Levels 1 and 2

2	Can order by direct comparison among 2 or more extensible or straight lengths or entirely overlapping areas. Compares and codes these relationships with the symbols <, >, and =.	Can students compare 2 or more extensible or straight lengths (by left end justification) and overlapping areas (by noticing if one area covers another without overlap) qualitatively and code these relationships? Which are longer and which are bigger?	Development of the properties and axioms of an equivalence class: 1) <u>Trichotomy</u> (Either A = B, A < B, or A > B), 2) <u>Symmetry</u> (If A = B, then B = A), and 3) <u>Reflexivity</u> (No matter A, A = A).	
1	Can identify and distinguish among qualities of length and area.	Can students identify lengths and area on a given figure or picture?	Understanding of length as a path and area as either a sweep or enclosed space. Can distinguish between length and area on various objects. The child describes attributes that are associated with each measure (length, space, cover, location) and objects (yard, roof, house).	

<u>Goal:</u> Students distinguish various attributes (i.e. length, area, volume) on various objects. Students compare objects based on length and area using terms such as “longer, shorter, more, less, greater, etc.” Students distinguish the appropriateness of comparing objects based on length and area and understand how the two quantities are incommensurate.

Teaching Experiment Goals: In the level 1 task come to a negotiated meaning of length and area – length as "longness" and area as "space covered by" in order to set up the comparison tasks (level 2) and conservation tasks (level 4). In the level 2 task, determine how students compare objects on length and area. Goal is for students to compare length by lining up length objects and comparing endpoints and compare area by looking at which object "completely covers" the others.

Hypothesized Outcome Spaces, Difficulties, and Assorted Notes Based on Literature and Pilot Interviews: In the Level 1 task, very young students will tend to sort on material and color rather than by attributes such as length and area. Keep asking them if there's another way. However, the younger children may not nudge on this. In that case, go to the next prompt where the interviewer sorts the objects herself. Watch for students' language here and get them to talk about any common words they use that describe length, area, quantity, etc.
In the Level 2 task for length, children may also believe that under certain conditions (i.e. the dimension being measured) one object is longer than the other but not under other conditions one object is shorter than the other. For example, in a few of the interviews, in comparing a piece of string to a bungee cord, some students said that the string is shorter than the bungee cord because they are looking at thickness but the string is longer than the bungee cord when looking at longness. It's important for the remainder of the teaching experiment to tease out this early idea and it can lead to a fruitful discussion of how certain objects might have different dimensions of length or that one can locate many lengths on one object. Something else that may be brought up is the idea of perspective and that a straw is "shorter" when moved farther away. In this case, it's important to get into a discussion of conservation of identity even though this is a level 3 task by probing if the child really believes that the length changes when moved or if it depends on perspective. (May also be interesting to note this as Karen Hollebrands wondered about the ordering of compare (level 2) with invariance under translation (level 3). Jere also believes that similarity and scaling are in play for children that notice perspective – might be something to track and probe more on.) Children at the top outcome space for length comparison behave similarly to how Clements and Sarama (2009) describe: lining both objects end to end and saying the longer object is the one that "sticks out". For the purposes of later tasks I have found it useful to get the child to articulate "how much more" out it sticks and use their fingers or another object to represent. This helps get a common language and will help immensely in the conservation tasks for children that believe that bending a path makes it shorter or longer.

In the Level 2 task for area, 2 of the students in the pilot interviews came to the idea that the object with the greater area will have both a greater length and a greater width and will not stack the objects on top of each other to compare area. This is an interesting idea because, in a way, they are right in that if this condition is fulfilled it does mean that one is greater than the other. But if it fails it doesn't necessarily mean that one is greater or you can't tell. Other younger students compare one dimension only and it is necessary to give them contexts (more cake, more room to clean in a house) to help them think about space rather than focusing exclusively on the linear dimensions. Although outside of the dissertation study, I am fascinated now by how children go from thinking about linear dimensions to area dimensions and how this develops. There doesn't seem to be any literature on this except for the early Piaget (1960) work where he conjectures that children don't see area until they can envision an infinite "sweep" of infinitesimal lengths. Students with more sophisticated understandings will lay area objects on top and compare and come to the possibility that this may not be a necessary and sufficient condition and that under some type of composition/decomposition two objects that are not congruent may have equal area – but this is in the early stages. Allow the students to experiment with this idea if it comes up, however.

Finally, for one child (fourth grade) the topic of mathematical transitivity (rather than logical transitivity which is how I am starting to describe the DELTA version of transitivity) was brought up as he ordered the objects and remarked that he didn't have to make pair-wise comparisons of all of the objects to order. Might be interesting to see if this develops in younger children and ask the question (after they ordered) if they can compare two objects without empirically comparing.

Preparation

Materials Needed

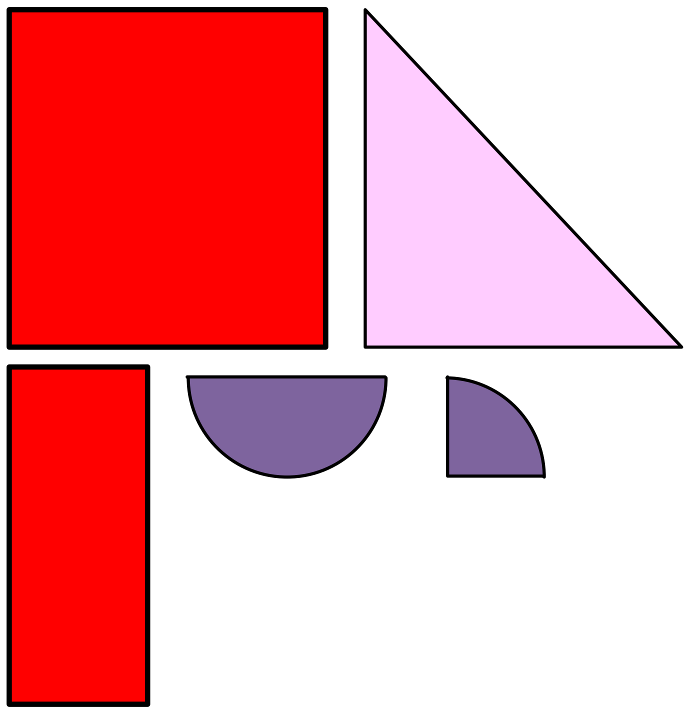
For Task 1

- Sheet of cloth
- Sheet of aluminum foil
- Sheet of paper towel
- 3x5 index card
- Sheet of plastic
- Dental floss
- Bendy straw
- Skewer

For Task 2

- Length Objects
    - 3 Skewers (cut off sharp ends): 2 of the same length (3 inches), 1 longer (7 inches)
    - 1 coffee stirrer, uncut (approx. 5.5. inches)
    - 2 pieces of string (1 the same length as the coffee stirrer, and one that's long (8 inches))
    - 2 pieces of straw (2 inches and whatever is left over)
    - 1 whole bendy straw
- Area Objects (see below for image)

  - One semi-circle and one quarter-circle (cut from the same circle)
  - One Rectangle
  - One Right Triangle (half of the rectangle)
- Task Sheet (for you)
- Video equipment, including camera, DV tape (if applicable), and tripod



Make sure that you take notes on:

- Probes that were especially helpful for eliciting student strategies, especially if they helped them get “unstuck.”
- Questions/probes that were not as helpful for eliciting student strategies.
- Specific language that students use to describe length (longness, bigness, etc.) and area (roofs, houses, location, bigness, smallness).
- Student strategies to solve tasks.
- Any emergent properties/generalizations that they made.
- Any misconceptions or critical barriers.

Task 1: The Activity

Sample Introductory Statement for Clinical Interviews and Teaching Experiments (Will be good to reiterate the important points before each session too): Thank you so much for helping me today. My job is to help kids like you learn math. Today I'm going to give you some things to do and I'm going to watch how you solve them and as you solve them I'll ask you to talk out loud. I'm really interested in *how* you think so there are not going to be any right or wrong answers. If I ask you a question it's not because I think you're right or wrong, I just want to see how you're doing it. And if there's any time that you want to stop you can just say so. It's all up to you how much we do and how long we go for. Do you have any questions before we start?

1. Before the activity, lay out all of the Task 1 objects on the table.

2. **Say:** Sort these objects any way that you want. You have to sort *all* the objects.

Probe to whatever extent possible (students may order by color and material type), but if students do not sort by length and area objects use the Step 3 Probe.

3. **Say**: **"Nice job doing that. I'm going to sort these objects into 2 groups by how I think they are the same."**
Sort objects into two piles: one for "length" objects and another for "area" objects.
Ask: How do you think I decided into which group to place each object?"

Try to establish some language here regarding length (i.e. longness), area (bigness), and volume (bigness). Use that language on the questions below or whatever you and the child have negotiated.

4. Show the student the ball of playdough.
a.) **Ask: "Can you show me length on this piece of playdough?"**
b.) **Ask: "Can you show me area on this piece of playdough?"**
c.) **Ask: "Can you use the playdough to make something that has length?"**
d.) **Ask: "Can you use the playdough to make something that has area?"**

5. Give the student 6 bendy straws [make sure multiple straws are available in case the child wants to construct something].
a.) **Ask: "Can you show me area on this straw?"**
b.) **Ask: "Can you show me volume on this straw?"**
c.) **Ask: "Can you use the straw to make something that has area?"**
d.) **Ask: "Can you use the straw to make something that has volume?"**

6. Show the student a sheet of paper [make sure multiple sheets are available].
a.) **Ask: "Can you show me length on this piece of paper?"**
b.) **Ask: "Can you show me volume on this piece of paper?"**
c.) **Ask: "Can you use the paper to make something that has length?"**
d.) **Ask: "Can you use the paper to make something that has volume?"**

Task 2: The Activity

Place the length and area objects for Task 2 on the table.

1. **Ask: "Can you order these objects [point to the length objects] from smallest to largest [or use their language for length if they have introduced it]?" Good Probe to Use: "How do you know this is smaller than that?"**

2. **Repeat with the area objects. Ask: "Order these objects [point to the area objects] from smallest to largest." Probe: "How do you know this is smaller than that?"**

After the sorting activity:

3. Show students a straw and a rectangle. **Ask: "Is a straw bigger than a rectangle?"**

If Time Allows:

4. Show students a circle and a piece of playdough. **Ask: "Is a circle bigger than this piece of playdough?"**

Appendix B
Task 3

Examining Attributes of Various Objects
Ordering Various Objects
Task for Length/Area Level 3

3	<u>Conservation</u>: can conserve length and area under simple rigid transformations and distinguish them from scaling.	What different ways can I adjust or move a line segment or polygonal figure without changing its length or area? What different ways can I adjust or move a line segment or polygonal figure which would result in changing its length or area?	Understanding that when a length or area is moved, rotated, or reflected, its length or area does not change.	Visually transform a length or area by translation, rotation, and reflection and ask students if the length or area has changed. Use an overhead projector/smart board to "scale" a length or area and asks students if the length or area has changed. Stretch a bungee cord and ask if the length has changed.

<u>Goal:</u> Students recognize that lengths and area are conserved through translation, rotation, and reflection and are not conserved through scaling and dilation.
<u>Teaching Experiment Goals:</u> Investigate students' understanding of conservation of translation, rotation, and reflection and see if these understandings come before or after direct comparison (level 2).

<u>Hypothesized Outcome Spaces, Difficulties, and Assorted Notes Based on Literature and Pilot Interviews:</u> In the pilot interviews, all students were able to state conservation through translation, rotation, and reflection. Scaling has been much more difficult to get at as young students are struggling with: 1.) the pre-image, 2.) the scaling action, and 3.) the image. Jere's suggestion is to do the shape similarity clinical interview with students in the teaching experiment before asking about non-conservation of scaling/dilating in order to establish the common language that "the same as" can mean the same as in terms of similarity.

Preparation

Materials Needed

- Pens
- Scissors

- Straight Edge
- Coffee Stirrers
- Bendy Straws
- 8.5x11 Sheets of Paper
- Task Sheet (for you)
- Video equipment, including camera, DV tape (if applicable), and tripod

Make sure that you take notes on:

- Probes that were especially helpful for eliciting student strategies, especially if they helped them get "unstuck."

- Questions/probes that were not as helpful for eliciting student strategies.

- Specific language that students use to describe length (longness, bigness, etc.) and area (roofs, houses, location, bigness, smallness).

- Student strategies to solve tasks.

- Any emergent properties/generalizations that they made.

- Any misconceptions or critical barriers.

Task 3: The Activity

1. Give students a coffee stirrer. **Ask: "What can I do to this stick without changing its tallness [or use students' language]."** You can use any of these [point to the tools] to help you if you want. Repeat with the bendy straw [pay careful attention or probe if they say bending the bendy straw results in a change of length.] Repeat with the bendy straw and the bungee cord.

2. Give students the rectangular sheet of paper. **Ask**: **"What can I do to this piece of paper without changing the space it covers? [or use students' language]."** Repeat with the rectangular playdough cake.

3. Repeat question 1 with the coffee stirrer and the prompt: **"What can I do to this stick to change its tallness?"**
Repeat with the bendy straw and the bungee cord.

4. Repeat question 2 with the rectangular sheet of paper. **Ask: "What can I do to this sheet of paper to change the space it covers?"** Repeat with the rectangular playdough cake.

Appendix C
Task 4

Conservation of Equipartitioning/Breaking and Reassembly

Task for Length/Area Level 4

4	Conservation of Equipartitioning and Reassembly: Recognizes that breaking and joining or equipartitioning and reassembling conserves the length or area.	Do students understand that splitting or cutting and reconnecting lengths and areas do not change the length or area?	If a length or area A is translated or broken/ equipartitioned into any number of parts length or area remains the same as long as nothing has been added or taken away.	Break and equipartition various lengths and areas including curved paths and irregular areas (including using common examples from DELTA equipartitioning Case B) and ask students if reassembly preserves length and area.

Goal: Students recognize that splitting or cutting and reconnecting lengths and areas do not change the length or area.
Note to the interviewer: Read Piaget et al. (1960) pp. 104-127 for explication of outcome spaces for this task for length.

Teaching Experiment Goal: The main goal is to establish conservation of length and area. A second goal that has come out of the pilot interviews is to establish that lengths are additive and two smaller lengths can be combined to form a larger length.

Hypothesized Outcome Spaces, Difficulties, and Assorted Notes Based on Literature and Pilot Interviews: From the Piagetian work and clinical interviews the outcome space is: 1.) Does not a recognize reassembly, i.e. conservation of identity. A length or area that is broken up and then reassembled has a different length/area than before breaking. 2.) Recognition of conservation of identity but not general conservation. For young students this means that when a straw is broken up it has the same length as long as it is lined up end to end, but not when it is bent. 3.) Complete recognition of conservation of identity and general conservation.
Students at Outcome Space 1 and 2 are often examining endpoints and not considering the entire path length. It is helpful here to switch to Piaget's language of "on which path does an ant have further to walk". However, it is necessary to make sure that understanding is consistent with students' understanding of length and longness. For some students it seems like these are three different understandings. It's useful to ask students to use their fingers to show you what they may about each.

In the pilot interviews it was found that moving students from Outcome Space 2 to Outcome Space 3 was difficult and will be an important part of the teaching experiment. A few strategies that seemed to help: 1.) Go back to the level 2 tasks to reestablish what it means to be longer or larger, 2.) Ask students to cut a piece of string “as long as” a straw and by using motions on the string get the student to see that the string and straw are the same length. Then ask the student to squish the string into a path and ask if the string and straw have the same length. Students seem to make a connection that “you’re not changing the string so you’re not changing the length” with the straws “you’re just bending the straw so you’re not changing the length” but some students will still insist that the bend in the straw either slightly increase or decrease the length. A good strategy at this point seems to be asking the student to show with his/her fingers by how much it increases or decreases the length.

Preparation

Materials Needed

- 6 coffee stirrers or straws
- 6 coffee stirrers or straws each cut into 2 equal pieces
- Scissors
- Task Sheet (for you)
- Playdough or Paper Circles (3) [If using playdough provide a safety knife, if using paper circles provide safety scissors]
- Video equipment, including camera, DV tape (if applicable), and tripod.

- Probes that were especially helpful for eliciting student strategies, especially if they helped them get “unstuck.”
- Questions/probes that were not as helpful for eliciting student strategies.
- Specific language that students use to describe length (longness, bigness, etc.) and area (roofs, houses, location, bigness, smallness).
- Student strategies to solve tasks.
- Any emergent properties/generalizations that they made.
- Any misconceptions or critical barriers.

Notes: If at all possible, (especially in the context of the teaching experiment): establishment some notion of one object being longer (bigger) than the other. Go back to Task 2 if necessary to remind the child how they were looking at longness or bigness. If you do decide to use the ant probe, make sure that both you and the child have distinguished between longness (length) and “further to go.”

Task 4 Part 1 (Conservation of Length Identity): The Activity

1. Give the student a coffee stirrer and scissors. Ask: "**Can you help me cut this stick into 2 pieces for me? I'm going to hold it and you cut, okay?**" Help the child cut the stirrer into 2 parts.

2. After the student has done this, put the coffee stirrer back end to end and ask: "**Is the stick longer, shorter, or the same length as before you cut it**?"

3. Repeat, asking the student to cut the stick into 3 pieces.

Task 4 Part 2 (Conservation of Length Equivalence): The Activity

1. Arrange 2 coffee stirrers approximately 2 centimeters apart so it is clear they are the same length. Have the student verify that the lengths are the same.

Note: If the child has trouble understanding length, rephrase. While running your finger along the coffee stirrers, ask: "**What if an ant walked along these two paths. Would it have further to go one way than the other?"** It may be necessary also to verify that the child understands that one stick longer than the other is the same thing as the ant goes further on one path than the other.

2. Ask: "Can you help me cut this stick into 2 pieces for me? I'm going to hold it and you cut, okay?" Help the child cut the stirrer into 2 parts.

3. After the student has done this, arrange the pieces in a zigzag (**make sure that the zigzag path has different endpoints than the first stirrer**) and ask: "Is this stick longer, shorter, or the same length as this one?"

4. Arrange the broken stick end to end and ask: "Is this stick longer, shorter, or the same length as this one?"

5. **Repeat,** but **break the first stick into 2 pieces and break the second piece into 3 pieces**. Keep the stick broken into 2 pieces the same but arrange the stick broken into 3 pieces into a zigzag (**make sure that the zigzag path has different endpoints**).

Task 4 Part 3 (Conservation of Length Equivalence with Bends): The Activity

1. Arrange 2 sets of 3 coffee stirrers end-to-end in a straight line approximately 2 centimeters apart so it is clear they are the same length. Have the student verify that

the lengths are the same. Note: If the child has trouble understanding length, rephrase. While running your finger along the coffee stirrers, ask: "**What if an ant walked along these two paths. Would it have further to go one way than the other?"** It may be necessary also to verify that the child understands that one stick longer than the other is the same thing as the ant goes further on one path than the other.

2. Take one set of coffee stirrers and arrange them in a zigzag. For the other set of coffee stirrers, leave one uncut, cut the second into two pieces, cut the third into three pieces, and rearrange all six pieces in a zigzag pattern too. **Ask: "Do these two lines still have the same length?" or "Would two ants walking along these paths still walk the same distance?"**

Task 4 Part 4: The Activity

Give students a playdough/paper cake.

1. Say: "**Pretend that you are at a birthday party. Can you share this birthday cake fairly for 2 people?"**
2. After the student has fairly shared, ask: **"If we put the pieces back to together would we have more, less, or the same amount of cake as we started with?"**

Appendix D
Task 5

Indirect Length Measure

Task for Length/Area Level 5

5	Compares the length of two or more objects by representing them with an indirect object (possibly the beginnings of use of non-standard units).	Can students compare two or more immovable lengths by inventing a means of comparison?	Use of an outside non-standard unit (i.e. fingers, string, paperclips) to compare the length of two or more objects.	Task 1: Compare 2 or more objects with small variation in relative size. Task 2: Compare 2 or more objects with large variation in relative size.

Goal: Students can compare 2 or more (possibly immovable) lengths by inventing a means of comparison. Strategies may include transitivity arguments, use of non-standard units, unit iteration, etc.

Teaching Experiment Goals: Based on Barrett's (1998) work, Task 1 has the potential for eliciting equipartitioning of lengths, unit length creation, unit length iteration, additivity of lengths, and length/path comparison.

Hypothesized Outcome Spaces, Difficulties, and Assorted Notes Based on Literature and Pilot Interviews: Task 1 was a much more difficult task than expected as all the students in the pilot interviews did not go beyond perception even after being asked if they could provide direct evidence. One child interestingly enough used a hybrid notion of rate by running his finger on a path and trying to mentally determine which path "took more time" to go down. It was determined later by the interviewer that the students did not have a complete understanding of level 4 (conservation).

According to similar tasks that Barrett (1998) used in his work the most likely outcome space progression for this task would be: 1.) use of only perception to compare lengths, 2.) use of fingers, feet, other body parts to informally iterate and count a path, 3.) use of markings or body parts to more or less accurately determine a unit for *one side only*. That is for the bent paths the iterating unit used for one side may differ drastically from that used for another side. For comparison of the length of a door versus the length of a cabinet the unit used to measure the door might drastically differ from that of the cabinet although the students will attend to the count of those for comparison, 4.) stabilization of a unit (either standard or nonstandard) and iteration of that unit coordinated with a count.

For the teaching experiment for kids that are struggling with this task, auxiliary tasks that Barrett used may be useful including drawing the paths on the floor and allowing the child to use motor-sensory skills to connect walking or movement with their verbal and written constructions of length. In particular, this seemed to help during Barrett's experiment in that it helped the children consider length to be the "space between" two points and not rely just on tick marks or an arbitrary count to determine length. It might be interesting also to note students' early perception of perimeter as path length and reconsider Piaget's conservation task in light of Barrett's work that suggests bends in paths are difficult for students to incorporate into their length schemes.

Preparation

Materials Needed

- Pen
- Scissors
- String piece (cut to be less than any of the line segments to be compared)
- Straw Pieces of Various Lengths
- Line Segments for Students to Compare (Include 3 copies of each in case you need them and print from the **Visio** file, **NOT** the PDF as only the Visio file preserves the dimensions.)

Make sure you take notes on:

- Probes that were especially helpful for eliciting student strategies, especially if they helped them get "unstuck."
- Questions/probes that were not as helpful for eliciting student strategies.
- Specific language that students use to describe length (longness, bigness, etc.) and area (roofs, houses, location, bigness, smallness).
- Student strategies to solve tasks.
- Any emergent properties/generalizations that they made.
- Any misconceptions or critical barriers.

Task 5: The Activity

Lay out the materials for task 5.

Say: For these problems, you can use any of the things on the table.

1. Place the first pair of line segments (Paths A and B) to be compared on the table, away from each other[60]. **Ask:** "Which of the following paths is longer?"

Note: You may find it useful to go back to the ant analogy from Task 4, i.e., ask: "**What if an ant walked along these two paths? Would it have further to go one way than the other?"** Repeat with the other 2 pairs of lengths (Paths C and D and Paths E and F).

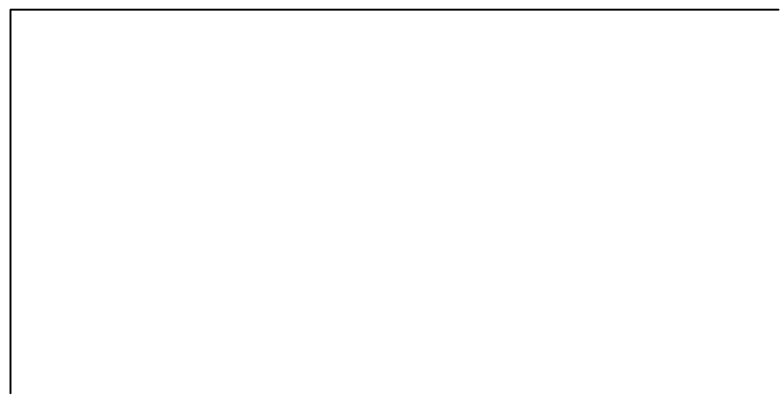
Note: these paths are displayed here only for your reference. Give the students the **VISIO** printout of the paths printed on separate pieces of paper.

[60] Revised 6/2010: Tape the paper to the table and do not allow the students to move. This induces the need for indirect comparison.

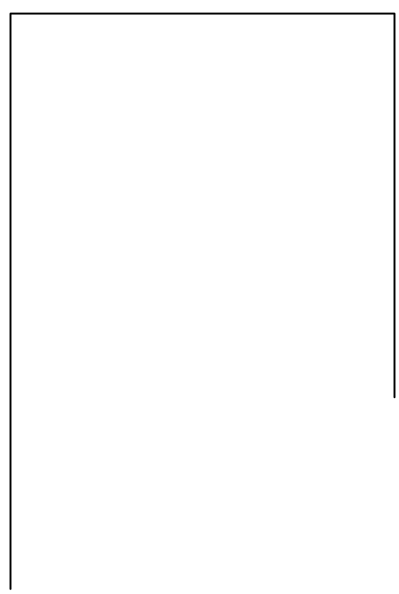
Path A

Path B

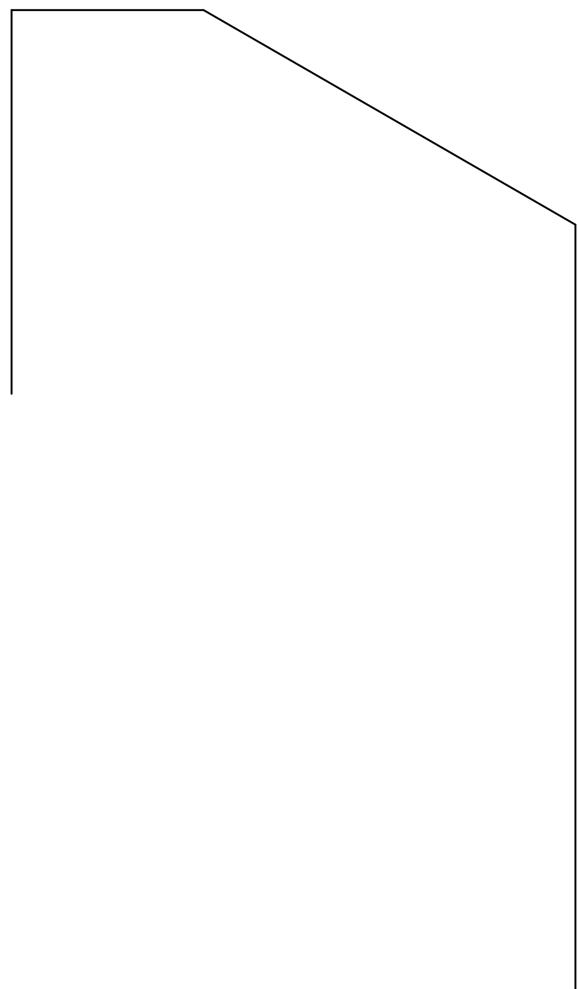
Path C



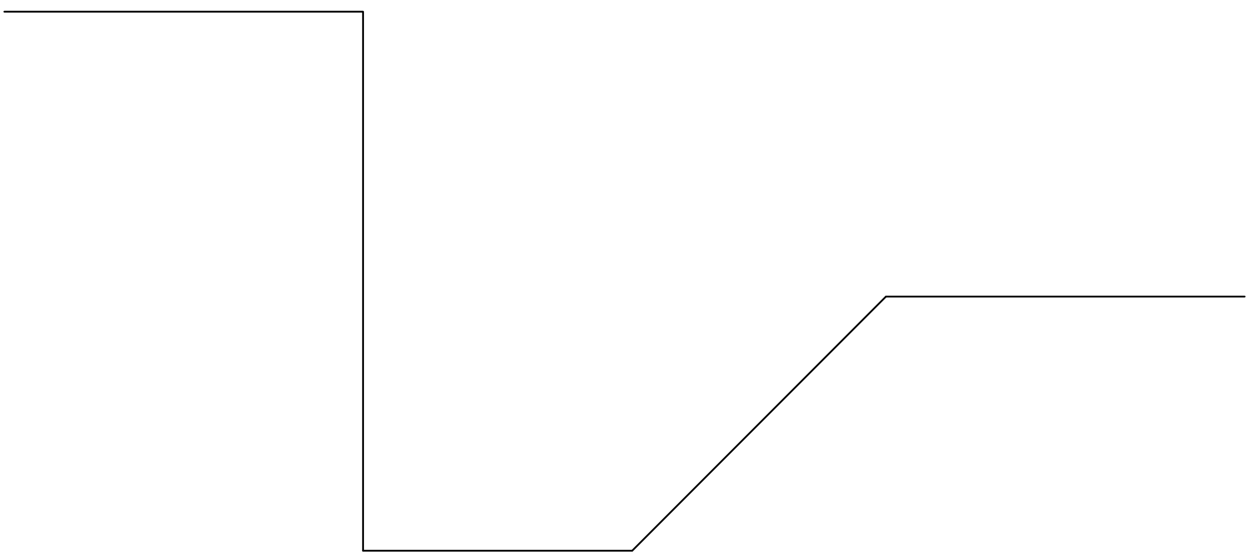
Path D



Path E



Path F



Appendix E
Tasks 6 and 7

Comparing 3 Equivalent Non-Congruent Rectangles Task

Task for Length/Area Levels 6/7

7	Constructs multiple common units to compare 2 or more non-congruent rectangles of equivalent area.	Can you find another common unit that also measures that area? How many of each common unit makes up each area?	Constructs multiple common units to measure an area. If a common unit *X* measures an area *A* *x* times and a common unit *Y* measures the same area *A* *y* times then can state this and say that *A* is *x* times as big as *X* and *y* times as large as *Y*. Transitivity of multiple common units: In the scenario above, $X * x = Y * y = A$.	Continuation of Level 6 task with the probe/question: Can you find another common unit that also measures that area?
6	Compares 2 or more non-congruent rectangles of equivalent area by finding the common unit. Note: We also conjecture that students may 1.) overlap and compare non-overlapping areas and 2.) grid but we will construct the task carefully so it is necessary for students to find a common unit.	Can students compare 2 or more non-congruent rectangular areas by finding a common unit between them?	If a common unit *X* (that measures areas *A* and *B*) measures an area *A* *x* times and measures an area *B* *y* times and $x = y$ then $A = B$. Non-congruent rectangles can have the same area.	Show students 2 equivalent non-congruent rectangular areas and ask them to order them in terms of which rectangular area covers the most space. Repeat the task with 3 equivalent non-congruent rectangular areas.

<u>Goal:</u> Students construct a common unit to compare the area of 2 and then 3 equivalent non-congruent rectangles. Students come to understand that 2 objects can be non-congruent and have the same area. The 3 rectangle task can be used to bring up notions of transitivity (Lehrer et al., 1997).

Teaching Experiment Goals: Note: As designed, expect these tasks to last for 2 or 3 teaching sessions. In particular, it may be good to do a separate session in which students compare the 5 different shapes. In the teaching experiment, it will be necessary to guide students to the construction of the 1x1 square unit or core unit and comparisons to other units that students construct. Make careful note of all the units that students construct and at the end of the task, Jere suggests having them put it in a table and get them to see a pattern between size of unit and how many units it takes to fill an area.

Hypothesized Outcome Spaces, Difficulties, and Assorted Notes Based on Literature and Pilot Interviews: Based on the literature, there are five conjectured student strategies when approaching this task.

1. Students may at first use perceptual reasoning such as one rectangle is "fatter" or "skinnier" to justify that one rectangle covers more space than the other.

   Questions: "How do you know?", "Is there any way that you can show me why you think …?" If students express a strategy encourage them by saying: "Can you use the rectangles to show me your way?"

2. *Additive Congruence* Strategies (Lehrer et al., 1998; Strom et al., 2001). Students may compare 2 rectangles by rearranging parts of one rectangle so it can be superimposed onto corresponding parts of another. For example, students may fold the 2x6 rectangle in half to create a 2x3 rectangle and discover that exactly 2 of these 2x3 rectangles can be superimposed onto the 2x6 rectangle. They then deduce that if this is so, then the two rectangles cover the same space.

3. Gridding Strategies. Students may draw or superimpose their own grid onto the two rectangles and count the units in the grid to figure out the task.

4. Comparison of non-overlapping areas. Students may overlap the two rectangles and decide to cut out the non-overlapping areas and compare those areas. They repeat until non-overlapping areas can be determined to be equal, greater than, or less than based on complete coverings.

5. Construction of common units. Students may construct a common a unit and use this to measure the two rectangles. Although the idea of multiple common units is reserved for the next level, it may be interesting to lead students to see that multiple common units are possible and that a 1x1 unit is possible without delving into the relationship between 2 common units.

From the pilot interviews, it was often necessary to use additional probes and asking the child "how would you know for sure" if they were stuck on using perceptual strategies. Sometimes, encouraging students to try a strategy helped. For example, for those using perceptual strategies and said "it looks like" – simply suggesting if they could cut it out and make certain "for sure" seemed to help them. The context of cakes was also confusing for one child so using another metaphor such as "size of a room." Finally, it is very important to have common and negotiated language around area. Younger students will still be convinced that the condition for larger area will be to compare both the length and the width of the rectangle. Jere suggests if the child is still stuck on this notion to go back to the level 2 (comparison) tasks and set up a situation when this doesn't hold.

Preparation

Materials Needed

- Each student should have at least 5 copies of the following rectangular areas:
    - 1x12, 2x6, 4x3 [Cut each cake from a different color of construction paper so you and the child can easily tell which parts come from which cake if the child decides to partition.]
- A straightedge (not a ruler), pens, scissors
- Task Sheet (for you)
- Video equipment, including camera, DV tape (if applicable), and tripod

The Activity

1. Tell the student that they are at a birthday party and there are three cakes. Show the student the three cakes (1x12, 2x6, 4x3). *Note*: It will help if you call them by their color or assign a flavor to the color. E.g. white cake, lemon cake, and grape cake if the colors are white, yellow, and purple.

2**. Ask: "Which of the following cakes would you choose if you wanted to eat the most cake?"** Tell the student to use any of the manipulatives (e.g., straightedge, pens, or scissors) to help figure it out, if the student wants.

Some Helpful Probing Questions:

- Show me why you think cake X is (bigger/smaller/the same as) cake Y.
- I don't believe you. How would you convince me that that cake is bigger than that cake?
- How many of these pieces of cake make up a whole cake?

3**.** Encourage students to move away from perceptual strategies to a unit strategy. If they discover a unit or an "additively congruent piece" (Lehrer et al., 1998) encourage them to find other units. Probe and ask them to compare the size of those units to the whole, first qualitatively: **Ask: "So you know that 4 of these pieces of cake fill up the whole cake. What if the piece of cake was bigger? Do you think it would take more, less, or the same number of those larger cake pieces to fill the whole cake? What if the piece of cake was smaller? Do you think it would take more, less, or the same number of those smaller cake pieces to fill the whole cake?**

4**.** Have students construct a table for each cake like the one below.

Whole Cake (i.e. draw the 3x4 cake)	
Unit Created	Number of units needed to fill cake
2x1 (Have students draw out the unit)	6
4x1	3
1x1	12
3x1	4

Ask students if they see any pattern or notice anything as the pieces of cake get bigger or smaller.

Repeat the task for the second set of cakes and finally the last set of cakes.

Probing for early transitivity of area:

1. If students make the claim that the three rectangles in the second set of cakes are equivalent in terms of area, ask: **"Okay, suppose you know that these two cakes (A and B) and these two cakes (A and C) are equal. Do you need to test to see of cakes A and B are equal?"**

Probing for early quantitative reasoning between a unit and total area.

2. Ask: "**Suppose that you have two cakes and you know that there are exactly 8 pieces of cake of this size (point to a square unit) in one cake. There are 10 pieces of cake of this size in the other cake. Which cake is bigger? How do you know?**

At the end of the task, ask students to reflect on the task.

If you have time, ask: "Do you think that two cakes have to look exactly the same to have the same area? Can you draw two cakes that don't look exactly the same and have the same area?"

Appendix F
Task 8

Comparing 5 Rectangles Task

Task for Length/Area Levels 6/7

<u>Goal:</u> Students use multiple methods that have learned (from Task 7) to compare five rectangles. They have the opportunity to reinvestigate their notions of qualitative comparison, quantitative comparison, and notions of unit.

<u>Teaching Experiment Goals</u>: To see how students incorporate their understanding from task 7 (comparison of rectangles, times as many, qualitative and quantitative compensation, and unit) to compare five rectangles.

Preparation

Materials Needed

- Each student should have at least 5 copies of the following rectangular areas cut from different colored construction paper:
  - An 8x3 blue cake
  - A 2x12 white cake
  - 1 4x6 orange cake
  - A 5x5 yellow cake
  - A 4x7 red cake
- A straightedge (not a ruler), pens, scissors
- Task Sheet (for you)
- Video equipment, including camera, DV tape (if applicable), and tripod

The Activity

1. Tell the student that they are at a birthday party and there are now five cakes. Show the student the five cakes.

2**. Ask: "Which of the following cakes would you choose if you wanted to eat the most cake?"** Tell the student to use any of the manipulatives (e.g., straightedge, pens, or scissors) to help figure it out, if the student wants.

[For the Teaching Experiment]: Probe, based on your conjectures from observing their strategies, mathematical reasoning practices, emergent relations, generalizations, and misconceptions and barriers.

Appendix G
Glossary of Terms

Equipartitioning – "Cognitive behaviors that have the goal of producing equal-sized groups (from collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing 'fair shares' for each of a set of individuals" (Confrey et al., 2009).

Learning Trajectory – "A researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey 2008; Confrey, Maloney, et al., 2008; Confrey, et al. 2009).

Learning Trajectory Scale – A referent (Latour, 2000) to a *learning trajectory* articulating it as qualitatively ordered *proficiency levels* from greater to less sophisticated ideas. **Note**: This was referred to in the proposal as a *Diagnostic Profile Scale* and in previous DELTA publications as a *Progress Variable*.

Proficiency Level – A level within a *learning trajectory scale*.

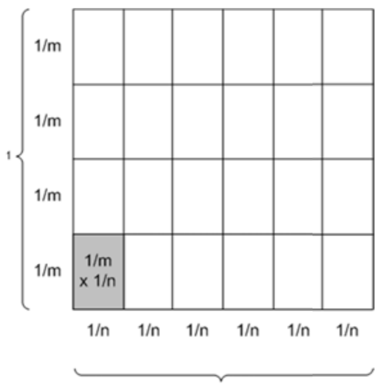
Outcome Space – Hypothesized (in the case of an initial *learning trajectory scale*) or observed (in the case of a modified *learning trajectory scale*) student strategies, mathematical reasoning practices, emergent relations, and misconceptions ordered from greater to less sophisticated ideas delineated for each *proficiency level* in a l*earning trajectory scale*.

A delineation of the different types of outcomes ordered from lower to higher sophistication.

Common Unit – A rectangular unit that can be used to measure (i.e. that completely covers) two or more rectangular areas an integer number of times without gaps or overlaps. For example, given two rectangular areas with dimensions 3x4 and 2x6, the common units are 1x1, 1x2, 1x3, and 2x3. Lehrer and colleagues (e.g., Lehrer, Jenkins, & Osana, 1998; Lehrer et al., 1998; Strom et al., 2001) refer to the 1x1 common unit as a *core square* or *core unit*.

Appendix H
Initial Learning Trajectory

L	***Description***	***Student Issue***	***Cognitive Aspect***	***Eliciting Tasks***
15	Calculates fractional area (rational numbers only, no irrationals).	How do students handle the case where an iterating area *i* fails to cover the whole area A to be measured? "Your common unit did not completely cover that rectangle. Is there anything that you can do to help you find out the *exact* area?"	Uses common unit strategies to further equipartition a common unit *i* and coordinates using ratio to determine fractional areas.	Using only a given iterating unit, calculate the area of the given rectangular regions.
14	Calculates fractional length (rational numbers only, no irrationals).	How do students handle the case where an iterating length *i* fails to cover to cover the whole length A to be measured? "Your common unit did not completely cover that rectangle. Is there anything that you can do to help you find out the *exact*	Uses common unit strategies to further equipartition a common unit *i* and coordinates using ratio to determine fractional rational number lengths.	Using only a given iterating unit (length), calculate the following lengths.

		length?"		
13	<u>Uses equipartitioning on a 1 x 1 square to produce the area unit 1/m x 1/n[61].</u> 	Can students extend their knowledge of equipartitioning, length, and area to produce a common area iterating unit of area 1/m x 1/n and recognize the properties of this special iterating unit?	If a length is *m* times as long as a "1" unit, then it is *m* x *n* as long as a 1/*n* unit. Moreover, if one side of a square is equipartitioned into *m* parts of length 1/*n* and the other side is equipartitioned into *n* equal parts of size 1/*m*, then the whole is *mn* larger than 1 or comprised of *mn* units of area 1/*mn*.	Equipartitioning Case "E" task: Suppose that one pirate equipartitioned the length of a rectangle for *n* pirates and another pirate equipartitioned the width for *m* pirates. How big is a share compared to the whole?
12	Understands common measures (i.e. feet, inches, yards, etc.) as a ratio relationship between "1" common measure and the whole measure. (e.g., 7 feet means a 7:1 ratio relationship between 7 feet and 1 foot) and uses to make comparisons between different common measures.	Is 8 centimeters bigger than 8 inches? Why?	If A is measured by *m* units of size M and B is measured by *p* units of size P and *m* > *p*, this does not necessarily imply that A > B.	Comparison problems where the units are incommensurable, e.g. Which is larger: 4 feet or 20 centimeters?
11	Area model for multiplication.	Can students calculate the area of rectangular areas using both row	Ability to see the row and column structure of a rectangular area	Give students two rectangles and ask them *exactly* how much more space is in one than the

[61] In the drawing, a 1 x 1 square is equipartitioned into *m* parts of length 1/m on the width and *n* parts of length 1/n on the length. This produces an equipartition of m x n parts each with area 1/m x 1/n.

		and column iteration and multiplication and reconcile the two methods?	and iterate both ways and coordinate actions to see squares as parts of rows and columns[62].	other.
10	Recognizes that if a *k*-split was performed to obtain the iterating unit that the whole length or area is *k* times as large as the iterating unit or that the total length or area is to the unit as k : 1. Also distinguishes that the iterating unit needs to be iterated $k - 1$ times to get to the whole length or area.	How many times as large is the area compared to this common unit that you constructed with equipartitioning? Can you write that as a ratio fact?	Constructs a commensurable iterating unit using equipartitioning and composition and decomposition for objects in the case where $A = ki$.	Allow students to perform equipartitioning to construct common units and ask them to compare the area with the common unit. As them to write as a ratio fact.
9	Quantitative Compensation of Common Units.	If I doubled the size of a common unit, how many of them would I need to measure the area A? What if I tripled it? Halved it?	If the area of a common unit is increased x times, then the number of common units needed will be decreased $1/x$ times. If the area of a common unit is decreased $1/x$ times, then the number of common units will increase x times.	Continuation of level 6 & 7 task with the probe/question: If I doubled the size of this common unit, how many of them would I need to measure the area of *A*. Without checking, can you predict how many you would need if I tripled the size of the common unit?

[62] We conjecture that in this learning trajectory, the area model of multiplication will be an accomplishment for students based on their work to construct common units from equipartitioning.

8	Conservation of Area when measuring with Multiple Common Units and Qualitative Compensation of Common Units: An area *A* is independent of the size of the common unit used to measure it and it requires fewer larger sized common units to measure an area *A*.	Does an area change based on what common unit you use to measure it with? If I doubled the size of the common unit do you think it would take more, less, or the same number of the larger common unit to measure the area?	If a common unit *X* measures an area *A x* times and a common unit *Y* measures the same area *A y* times then *A* is still equal to *A*. If a common unit *X* measures an area *A x* times, a common unit *Y* measures *A y* times and $X < Y$, then $x > y$. If a common unit *X* measures an area *A x* times, a common unit *Y* measures *A y* times and $X > Y$, then $x < y$.	Continuation of level 6 & 7 task with the probe/question: Does an area change based on what common unit you use to measure it with?
7	Constructs multiple common units to compare 2 or more non-congruent rectangles of equivalent area.	Can you find another common unit that also measures that area? How many of each common unit makes up each area?	Constructs multiple common units to measure an area. If a common unit *X* measures an area *A x* times and a common unit *Y* measures the same area *A y* times then can state this and say that *A*	Continuation of Level 6 task with the probe/question: Can you find another common unit that also measures that area?

			is *x* times as big as *X* and *y* times as large as *Y*. Transitivity of multiple common units: In the scenario above, $X * x = Y * y = A$.	
6	Compares 2 or more non-congruent rectangles of equivalent area by finding the common unit. Note: We also conjecture that students may 1.) overlap and compare non-overlapping areas and 2.) grid but we will construct the task carefully so it is necessary for students to find a common unit.	Can students compare 2 or more non-congruent rectangular areas by finding a common unit between them?	Simple area transitivity: If a common unit *X* (that measures areas *A* and *B*) measures an area *A x* times and measures an area *B y* times and $x < y$ then $A < B$. Non-congruent rectangles can have the same area.	Show students 2 or 3 non-congruent rectangular areas and ask them to order them in terms of which rectangular area covers the most space. Vary the task by: 1.) 2 equivalent rectangular areas 2.) 2 non-equivalent rectangular areas 3.) 3 equivalent rectangular areas 4.) 3 non rectangular areas (2 equivalent) 5.) 3 non-equivalent rectangular areas 6.) 5 rectangular areas (3 of which are

				equivalent)
5	Compares the length of two or more objects by representing them with an indirect object (possibly the beginnings of use of non-standard units).	Can students compare two or more immovable lengths by inventing a means of comparison?	Use of an outside non-standard unit (i.e. fingers, string, paperclips) to compare the length of two or more objects.	Task 1: Compare 2 or more objects with small variation in relative size. Task 2: Compare 2 or more objects with large variation in relative size.
4	<u>Conservation of Equipartitioning and Reassembly</u>: Recognizes that breaking and joining or equipartitioning and reassembling conserves the length or area.	Do students understand that splitting or cutting and reconnecting lengths and areas do not change the length or area?	If a length or area A is translated or broken/ equipartitioned into any number of parts length or area remains the same as long as nothing has been added or taken away. Moreover, if a length or area A is equipartitioned into m parts of size x, $x < A$ and A is m times as big a x ($A = mx$).	Break and equipartition various lengths and areas including curved paths and irregular areas (including using common examples from DELTA equipartitioning Case B) and ask students if reassembly preserves length and area.
3	<u>Conservation</u>: can conserve length and area under simple rigid transformations and distinguish them from scaling.	What different ways can I adjust or move a line segment or polygonal figure without changing its length or area?	Understanding that when a length or area is moved, rotated, or reflected, its length or area does not change.	Visually transform a length or area by translation, rotation, and reflection and ask students if the length or area has changed. Use an

		What different ways can I adjust or move a line segment or polygonal figure which would result in changing its length or area?		overhead projector to "scale" a length or area and asks students if the length or area has changed. Bungee chord
2	Can order by direct comparison among 2 or more extensible or straight lengths or entirely overlapping areas. Compares and codes these relationships with the symbols <, >, and =.	Can students compare 2 or more extensible or straight lengths (by left end justification) and overlapping areas (by noticing if one area covers another without overlap) qualitatively and code these relationships? Which are longer and which are bigger?	Development of the properties and axioms of an equivalence class: 1) Trichotomy (Either A = B, A < B, or A > B), 2) Symmetry (If A = B, then B = A), and 3) Reflexivity (No matter A, A = A).	Which is bigger, a straw or a rectangle? Show students two or more lengths and two or more areas (that can be compared by simple overlapping) and ask them to order by which one is the longer path, or which one covers the most space. Make sure you have 3 congruent objects. Note for the covering tasks: (1) a few that complete covers, (2) matches on the length or width
1	Can identify and distinguish among qualities of length and area.	Can students identify lengths and area on a given figure or picture?	Understanding of length as a path and area as either a sweep or enclosed	Show students various objects (pieces of string, skewers of various lengths, coffee

			space. Can distinguish between length and area on various objects. The child describes attributes that are associated with each measure (length, space, cover, location) and objects (yard, roof, house).	stirrers of various lengths, straws, rectangle, circle, toothpick, and triangle. String, straw, rectangle, circle, toothpick, triangle. Ask: "Sort these objects." If students have trouble sorting into piles of length versus area, a possible probe would be to sort for them and ask them how the objects in the way you sorted are similar or different. Ask the question of does an area have length? Does a length have area/cover? i.e. Ask symmetrical questions (i.e. whether they can find length on a rectangle and find area on a straw) and make sure you account for volume. Ball of playdough. Can you make

				something that has length/area/volume. See if they have certain physical behaviors (operations). Can you use length to construct area? Can you use area to construct length?