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Large Amplitude Vibrations of Elasto-plastic Plates and Shells

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Abstract:: In this paper the influence of large deflections of elasto-plastic plates and shell of arbitrary shape under both static and dynamic loads have been studied by using the concept of constant deflection contour lines in conjunction with Illyushin's theory for small plastic bending. The results obtained for elliptic and circular plates and shallow dome on elliptic base are compared with the available results.

INTRODUCTION

Modern structures are often subjected to severe vibrations and high temperatures. Sometimes the magnitudes of the applied forces or loads become very large, exceeding the elastic limit of the material used . Consequently, the application of the linear theory becomes inadequate in solving problems. For large amplitude vibrations of elasto-plastic plates and shells under moderate and extreme loads, nonlinear analysis becomes enevitable to get approximate results.A few authors [1 – 5] attemped to solvesuch problems using Range-Kutta Method, Finite Element Method, Finite Difference Method,Iterative Method & Constant Deflection Contour Lines Method. But all the works are confined in problems of plates and shells with uniform thickness and mostly in static cases.The present paper aims at solving the problems relating to large amplitude vibrations of uniform elasto plastic plates and shells under both static and dynamic loads using the method of constant deflection contour lines and the results obtained are compared with the available results for structures of uniform thickness. Structures having nonuniform thickness and variable flextural rigidity are dealt in [6] by the same author.

Key words: Large amplitude, vibration, elasto-plastic, plates, shells

Derivation Of The Governing Differential Equations

A shallow shell of uniform thickness `h ` is considered. Let the equation of the middle surface of the shell, refered to an orthogonal coordinate system (x,y,z), be given by

$$Z = (x^2 / 2 R_x) + (xy / R_{xy}) + (y^2 / 2 R_y) \quad ....(1)$$

For a shallow shell $r = \sqrt{(x^2 + y^2)}$ considered small in comparison to the least of the radii of curvature, R_x , R_{xy} and R_y which are taken to constants.When the shell experiences axisymmetric free vibration the intersections of the deflected surface and the parallels z = constant yield contour lines of constant deflection. Application of D` Alembert`s principle to an element of the shell bounded by such a contour at any time τ and subsequent summation of the forces in the direction normal to the surface yields the following dynamical equations [1] :

$$\int V_n \; ds + \int\int\left[\rho h \; (\partial^2 w) / \partial \tau^2 + (N_x) / R_x + (N_y) / R_y + 2 \; (N_{xy})/R_{xy} \right] dx\, dy = 0 \quad ....(2)$$

where the transverse reaction forces $V_n = Q_n - \partial/\partial s(M_{nt})$...(2)
represents the effect of the shearing force Q_n and the edge-rate of change of twisting moment M_{nt} along the contour C_u. According to Ilyushin`s theory of the elastic plastic deformation (1948), the bending moments M_x,M_y,M_{xy} and their shear forces Q_x, Q_y are given by the following relations:

$$M_x = - D (1 - \nu) \{ (\partial^2 w/\partial x^2) + \nu (\partial^2 w/\partial y^2)\}$$
$$M_y = - D (1 - \nu) \{ (\partial^2 w/\partial y^2) + \nu (\partial^2 w/\partial x^2)\}$$
$$M_{xy} = D (1 - \nu) (1 - \Omega) (\partial^2 w / \partial x \partial y) \quad ....(3)$$
$$Q_x = (\partial/\partial x) \{ M_y\} - (\partial/\partial y) \{M_{xy}\}$$
$$Q_y = (\partial/\partial y) \{ M_x\} - (\partial/\partial x) \{ M_{xy}) \}$$
$$Q_n = Q_x \; Cos\alpha + Q_y \; Sin\alpha$$

$M_{nt} = M_{xy} (\cos^2 \alpha - \sin^2 \alpha) + (M_x - M_y) \sin \alpha \cos\alpha$ ....(6)

Where, $\cos\alpha = (dy / ds)$ and $\sin\alpha = - (dx / ds)$.

Here, ρ, h and w are, respectivly, the mass density, the shell thickness and the deflection. Using the well known expressions for the moments and shearing forces and assuming that the membrane forces N_x, N_y and N_{xy} are given by

$$N_x = (\partial^2 \Phi / \partial y^2), \quad N_y = (\partial^2 \Phi / \partial x^2), \quad N_{xy} = - (\partial^2 \Phi / \partial x \partial y) \quad ....\ ....(3)$$

Equation (2) finally reduces to:

$$(\partial^3 w/\partial u^3) \int(1 - \Omega) R ds + (\partial^2 w/\partial u^2) \int(1 - \Omega) F ds + (\partial w/\partial u) \int(1 - \Omega \quad G ds + (\partial^2 w/\partial u^2) \int D [(\partial \Omega /\partial x) (\partial u /\partial x) + (\partial \Omega /\partial y) (\partial u /\partial y)] \sqrt{t} ds + (\partial w/\partial u) \int (D / \sqrt{t}) [K(\partial \Omega /\partial x) + L (\partial \Omega /\partial y)] ds + \iint [\rho h (\partial^2 w/\partial \tau^2 (1/ R_x)(\partial^2 \Phi / \partial y^2) + (1 / R_y) (\partial^2 \Phi / \partial x^2) - 2 / (R_{xy} \partial^2 \Phi / \partial x \partial y] dx dy = 0 \quad ....(4)$$

Where, $R = - D (t)^{1..5}$

$F = - D / (\sqrt{t}) [3 u,_{xx} u,_x{}^2 +3 u,_{yy} u,_y{}^2 + u,_{xx} u,_y +u,_{yy} u,_x 2\ 4\ u,_{xy} u,_x u,_y]$

$G = - D / (t)^{1..5} [u,_{xxx} u,_x{}^3 + u,_{yyy} u,_y{}^3 + (2 - \nu) (u,_{xxx} u,_x u,_y{}^2 + u,_{xy} u,_y{}^3 + u,_{yyy} u,_y u,_x{}^2 + u,_{xyy} u,_x{}^3) + (2 \nu - 1) (u,_{xyy} u,_x u,_y{}^2 + u,_{xxy} u,_x{}^2 u,_y) + (1 - \nu) (u,_{xx} - u,yy) (u,_{xx} u,_y{}^2 - u,_{yy} u,_x{}^2)$
$-2 (1 - \nu) u,_{xy} (u,_x u,_y u,_{xx} - u,_y{}^2 u,_{xy} - u,_x 2 u,_y + u,_x u,_y u,_{yy})] + 2 D (1 - \nu)/ (t)^{2.5} [u,_{xy} (u,_x{}^2 - u,_y{}^2) - u,_x u,_y (u,_{xx} - u,_{yy})]^2$

$\sqrt{t} = (u,_x{}^2 + u,_y{}^2)$ and $D = (E h^3 / 12 (1 - \nu^2)$,is the flexural rigidity.

Here, $\Omega = 0$ when $e \leq 1$, the region is elastic ; when $e > 1$ the region is plastic. Also,

$$\Omega = \lambda [1 - (3 / 2e) + (1 /2e^3) \quad ....(4)$$

$$\text{and } e^2 = (h^2/3e_s{}^2) [(\partial^2 w/\partial x^2)+(\partial^2 w/\partial y^2)+(\partial^2 w/\partial x\partial y)+(\partial^2 w/\partial x^2) (\partial^2 w/\partial y^2) \quad .... \quad (5)$$
$$= (h^2/3e_s{}^2) [M (\partial w/\partial u)^2 + N(\partial w/\partial u) (\partial^2 w/\partial u^2) (\partial w/\partial u) + t\ 2 (\partial^2 w/\partial u^2)$$

in which e_s is the yield strain, ν is the poisson`s ratio, D is the flexural rigidity of the plate material, λ is a material constant .

Here , $M =[u,_{xx}{}^2 + u,_{yy}{}^2 + u,_{xx} u,_{yy} + u,_{xy}{}^2]$

$N = [2 u,_x{}^2 u,_{xx} + 2 u,_y{}^2 u,_{yy} + u,_{xx} u,_y{}^2 +u,_x u,_{yy} + 2 u,_x u,_y u,_{xy}]$

$t^2 = (u,_x{}^2 + u,_y{}^2)$

Considering only the transverse vibration, we assume that $w = W(x,y) f (t)$ ....(9)

$\Phi = \Phi (x,y) f(t)$......................... (10)

Equation (4) will now reduce to

$$\Big[(\partial^3 W/\partial u^3) \int (1 - \Omega) R ds + (\partial^2 W/\partial u^2) \int(1 - \Omega) F ds + (\partial W/\partial u) \int(1 - \Omega) G ds + (\partial^2 W/\partial u^2) \int D [(\partial \Omega /\partial x) (\partial u /\partial x) + (\partial \Omega /\partial y) (\partial u /\partial y)] \sqrt{t} ds + (\partial W/\partial u) \int (D / \sqrt{t}) [K(\partial \Omega /\partial x) + L (\partial \Omega /\partial y)] ds \Big] f(t) + \iint \Big[\rho h W f``(t) +\{ (1/ R_x)(\partial^2 \Phi / \partial y^2) + (1 / R_y) (\partial^2 \Phi / \partial x^2) -2/(R_{xy} \partial^2 \Phi/\partial x\partial y\} f(t)\Big] dx dy = 0 \quad ....(11)$$

Consequently, the condition for continuity of deformation reduces to

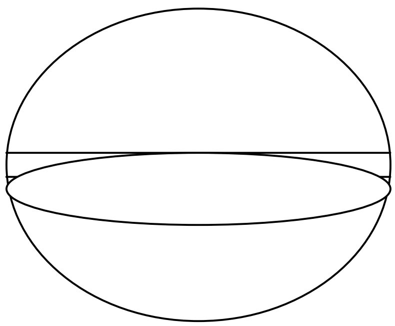
$$\nabla^4 \Phi = \{ 12D (1 - \nu^2) \} / h^2 (1 - \Omega) [(1/ R_x)(\partial^2 \Phi / \partial y^2) + (1 / R_y) (\partial^2 \Phi / \partial x^2) - 2 / (R_x \quad ....(12)$$

This equation must hold over all points in the interior of the shell. After integration over the area and application of Greens theorem one obtains :

$(d^3\Phi/du^3) \int Rds +(d^2 \Phi/du^2) \int Fds +(d\Phi/du) \int Gds * 12D^2 (1 - \nu^2) / h^2 (1 - \Omega)(dW/du)$

$$\int K_x (\partial u / \partial y)^2 + K_y (\partial u / \partial x)^2 / t^{1/2} \, ds = 0 \quad \text{...(13)}$$

where K_x and K_y denote curvatures at a point and K_{xy} has been assumed to be zero in accordance with the shallowshell theory. Equations (11) and (13) are now the two basic equations for large amplitude vibration of shallow shell.



Elliptical shell

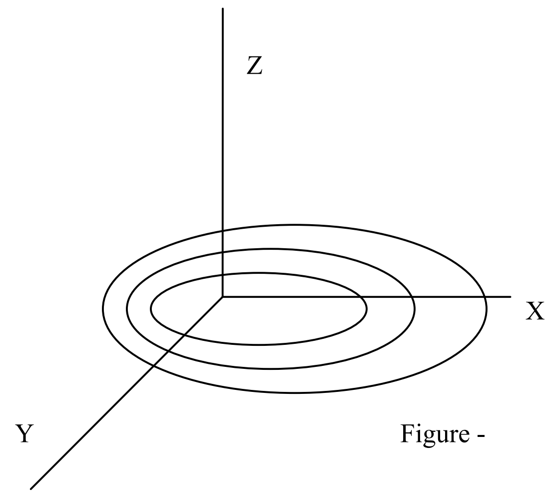


Figure -

ILLUSTRATION

Let us now consider a clamped dome of non-zero curvature upon an elliptic base. For the first approximation under symmetry consideration we may write

$$u = 1 - x^2/a^2 - y^2/b^2 \quad \text{..............................(14)}$$

Performing the contour integrations taken around the closed contour $u = 1 - x^2/a^2 - y^2/b^2$

And the double integration extending over the ellipse $x^2/a^2 + y^2/b^2 = 1 - u$ ....(15)

Equation (11) in nondimensional form becomes

$$(1 - \Omega)(1 - u)(d^3W/du^3) - 2(1 - \Omega)(d^2W/du^2) - (d\Omega/du)\left[(1 - u)(d^2W/du^2) - 2P\{(1/a^4) + (1/b^4) + 2\nu/a^2b^2\}(dW/du)\right] + (\rho h^2 \omega^2 P)/(2 D e_s a^2) - \{(E h \gamma)/D\}(d\Phi/du) = 0$$

where $P = (a^4 b^4)/(3a^4 + 2a^2b^2 + 3b^4)$, while equation (13) in nondimensional form will reduce to

$$(1 - u)(d^3\Phi/du^3) - 2(d^2\Phi/d^2u) + (1 - \Omega)\gamma(dW/du) = 0 \quad \text{..............(17)}$$

with $\gamma = p(k_x/b^2 + k_y/a^2)$; $W = wh/e_s a^2$; $\Phi = \varphi/E e_s a^2$ ....(18)

METHOD OF SOLUTION

On substitution of the value of Ω into equations (16) & (17), one obtains

$$\left[(1 - u)(d^3W/du^3) - 2(d^2W/du^2)\right] Q_1 f(t) - \left[2M(d^2W/du^2)(dW/du) + N(d^2W/du^2)^2 \;\; N(d^3W/du^3)(dW/du) + 2t^2(d^3W/du^3)(dW/du)\right] \left[(1 - u)(d^2W/du^2) - 2P_1(dW/du) Q_2 f^3(t) - (Eh\gamma/D) f(t)(d\Phi/du) + (\rho h/(2 D e_s a^2)) f`(t) = 0 \right.$$

..............................(19)

and $(1-u)(d^3\Phi/du^3) - 2(d^2\Phi/d^2u) + Q_1\gamma(dW/du) = 0$ ....(20)
Where, $Q_1 = [2e^3(1-\lambda) + \lambda(3e^2-1)]/2e^3$; $Q_2 = (\lambda/4e^5)(e^2-1)$

$P_1 = P(1/a^4 + 1/b^4 + 2\nu/a^2b^2)$
Also, e^2 is given by $e^2 = ⅓\left[M(dW/du)\,2 + N(dW/du)(d^2W/du^2) + t\,2(d^2W/du)^2\right]$ ....(21)

Suppose the shell is completely clamped along the boundary.
The boundary conditions are given by

$$W\Big|_{u=0} = 0 = (dW/du)\Big|_{u=0}$$

$$\Phi\Big|_{u=0} = 0 = (d\Phi/du)\Big|_{u=0}$$

To find an approximate solution, we assume the following trial solutions:

$$W = \Sigma a_j u^j \; ; \quad \Phi = \Sigma b_j u^j \quad \text{...(22)}$$

On substitution of these trial solutions in equations (17) & (19), we get the residuals R_1 & R_2 which, after the application of Galerkin`s procedure, yield the following results.

$(\rho h^2 P)/(6 D e_s a^2) f`(t) = [(4/3) Q_1 a_2 + (Eh\gamma/2D) b_2] f(t) + [(4/5) M + (2/3) N - (32/5) M P_1 - 4 N P_1] Q_2 a_2^3 f^3(t)$ .... (23)
and $b_2 = (3/8) Q_1 \gamma a_2$ ....(24)

while the average value of `e` happens to be $e = a_2 \sqrt{[(1/a^4)+(1/b^4)+(2/a^2b^2)]\,(40/9)}$ ... (25)
From equations (22) & (23), one obtains the following time differential equations in f(t):

$$f`(t) + \mu f(t) + \zeta f^3(t) = 0 \quad \text{...(26)}$$

where, $\mu = -(6 D e_s a_2/\rho h^2 P)\left[(4/5) M + (2/3) N - (32/5) M P_1 - 4 N P_1\right] Q\,2\,a_2^3$
Since the series is rapidly converging hence considering the first few terms one may obtain the approximate value of the central deflection w* as

$$w^* = \Sigma_j a_j$$

The solution of equation (25) is given by

$$f(t) = a_0 \sin[\mu t\{1 + (3/8) a_0^2(\zeta/\mu)\} + \theta_0] \quad \text{......................................(27)}$$

The time periods of the non-linear and linear oscillations are
$T^* = 2\pi/[\mu\{1 + (3/8) a_0^2(\zeta/\mu)\}]$ and $T = 2\pi/\mu$.
Thus $[T/T^*] = 1/\{1 + (3/8) a_0^2(\zeta/\mu)\}$ ....(28)

NUMERICAL RESULTS

Numerical results are computed both for circular and elliptic plates & shallow domes upon the circular & elliptic bases in the elastic and elastic plastic regions and these are presented in tables (1 – 4). The computations are made with different values of the shallowness parameter ($2\gamma/h$) and material constant $\nu = 0.3$. Dynamic responses of the elasto-plastic shells for moderately large amplitude are obtained from the same differential equations . This is no doubt advantageous as such both static and dynamic behaviers are obatined simultaneously with least effort.

T A B L E - 1

Free vibrations of clamped plastic shallow shell with circular planform. $e > 1$, $a = b$, $\nu = 0.3$,$\lambda = 1$

T*/T→

$W^*a_0\rightarrow$	0	0.5	1.0	1.5	2.0
$2\gamma/=0$	1.0000	1.0256	1.1111	1.2903	1.6666
Ref./5	1.0000	1.0233	1.1100	1.2885	1.6545
$2\gamma/h=1$	1.0000	1.0184	1.0778	1.1940	1.4063
$2\gamma/h=2$	1.0000	1.0099	1.0411	1.0975	1.1874
$2\gamma/h=3$	1.0000	1.0056	1.0230	1.0532	1.0985
$2\gamma/h=4$	1.0000	1.0037	1.0149	1.0345	1.0621
$2\gamma/h=5$	1.0000	1.0024	1.0095	1.0215	1.0390

T A B L E -2

Free vibrations of clamped plastic shallow shell with elliptic planform . $e > 1$, $a = 2b$, $\nu = 0.3$,$\lambda = 1$

T*/T→

$W^*a_0\rightarrow$	0	0.5	1.0	1.5	2.0
$2\gamma/h=0$	1.0000	1.0866	1.4682	3.5398	26.6241
Ref.5	1.0000	1.1004	1.4568	3.5433	28.9876
$2\gamma/h=1$	1.0000	1.0611	1.2995	2.0768	14.2857
$2\gamma/h=2$	1.0000	1.0325	1.1439	1.3947	2.0125
$2\gamma/h=3$	1.0000	1.0182	10771	1.1919	1.4009
$2\gamma/h=4$	1.0000	1.0133	1.0468	1.1117	1.2176
$2\gamma/h=5$	1.0000	1.0075	1.0310	1.0726	1.1367

T A B L E - 3

Free vibrations of clamped elastic shallow shell with circular planform . $e > 1$, $a = b$, $\nu = 0.3$,$\lambda = 1$

T*/T→

$W^*a_0\rightarrow$	0	0.5	1.0	1.5	2.0
$2\gamma/=0$	1.0000	1.9582	0.8513	0.7179	0.5887
	1.0000	1.9665	0.8876	0.7211	0.6021
$2\gamma/h=1$	1.0000	1.9695	0.8880	0.7789	0.6646
$2\gamma/h=2$	1.0000	1.9831	0.9355	0.8657	0.7840
$2\gamma/h=3$	1.0000	1.9903	0.9623	0.9189	0.8645
$2\gamma/h=4$	1.0000	1.9936	0.9751	0.9457	0.9074
$2\gamma/h=5$	1.0000	1.9960	0.9838	0.9645	0.9385

T A B L E - 4

Free vibrations of clamped elastic shallow shell with elliptic planform . e >1 , a = 2b, ν = 0.3 ,λ = 1

T*/T→

W^*a_0→	0	0.5	1.0	1.5	2.0
2γ/=0	1.0000	0.8778	0.6423	0.4436	0.3098
2γ/h=1	1.0000	0.9085	0.7130	0.5247	0.3883
2γ/h=2	1.0000	0.9479	0.8200	0.6692	0.5323
2γ/h=3	1.0000	0.9697	0.8888	0.7805	0.6667
2γ/h=4	1.0000	0.9808	0.9276	0.8507	0.7622
2γ/h=5	1.0000	0.9865	0.9480	0.8902	0.8203

O B S E R V A T I O N A N D C O N C L U S I O N S

It is observed that the results obtained for elastic shallow shells are in excellent agreement with those obtained in [4, 5].Also, the results for plastic shells based on circular and elliptic domes are in good agreement with those obtained in [4]. It is found in both the cases that the ratio of nonlinear to linear time periods become larger in case of plastic shell in comparison to those obtained in case of elastic shells and which is expected. This actually supports the application of the isodeflection contour lines method to analyse such problems. Inspite of that the method has some limitations. It heavily relies on the accuraqcy of the choice of the iso-deflection contour function u(x,y). The main advantage of the method lies in the fact that once the form of the function u (x,y) is chosen suitably the remaining task can be tackled with ease and accuracy.

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