ACCURATE DETERMINATION OF STRESS CONCENTRATIONS AROUND ELLIPTICAL HOLES IN CYLINDRICAL SHELLS

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SUMMARY

The problem of stress concentrations around cutouts in thin shell structures is of considerable importance in reactor pressure vessels. While sufficient data are available in literature on circular cutouts, information on various types of noncircular openings is rather scarce.

A theoretical analysis is presented for evaluating stresses around an elliptical hole in a cylindrical shell under axial tension with the major axis of the hole parallel to shell axis. Solution for the Donnell equations for a circular cylindrical shell is expressed in elliptic coordinates as an infinite series in terms of Mathieu and Modified Mathieu functions by the method of separation of variables. The solution is then reduced to a Fourier series and the boundary conditions are satisfied numerically term by term in the Fourier series. An earlier analysis by Venkitapathy for the same problem is found to be erroneous due to the omission of a part of the series solution. Results obtained for the complete range of $k(0<k<1)$ and for $0<\beta\leqslant 2.5$ where: $k = b/a$, a = length of semi-major axis of hole, b = length of semi-minor axis of hole, β = curvature parameter, $\beta^2 = a^2(12(1-\nu^2)^{\frac{1}{2}}/8Rt$, R, t = radius and thickness of shell, respectively, ν = Poisson's ratio.

A chart is presented from which the stress concentration factor can be directly read over the range of values of k and β mentioned. An interesting feature of the problem is that for each value of k, there exists a certain value of curvature parameter β at which the usual series solutions become numerically unstable, because the Mathieu equation has an infinite number of branching points on the complex plane. Such critical combinations of k and β need a separate treatment and this aspect is also discussed in the paper.

The limiting case of an axial crack is a trivial one for axial loading case but is of great practical significance for internal pressure loading. E.S. Folias used the method of integral equations to predict singular stresses and the present authors gave a perturbation method, both applicable for small cracks. A much simpler and direct method is now offered which can be applied to small and large cracks. A chart is presented from which the stress intensity factor can be read for a wide range of values of Numerical results from the present work are also used to establish the range of validity of perturbation solutions for these problems.

1 INTRODUCTION

Openings in reactor pressure vessels and high pressure ducts become invariably necessary for a variety of functional requirements. The existing codes for design of such type of structures lay down procedures which are mainly empirical or semi-empirical. For satisfactory design of reinforcement around such openings, there is a need to establish methods of accurate stress analysis for various configurations of the opening and for different loading conditions.

The earliest attempt at the analytical solution for the problem of stresses around a cutout in a shell is due to Lurie [1]. Lurie considered the problem of a small circular hole in a cylindrical shell under combined axial tension and internal pressure. He used Donnel's cylindrical shell equations and by using the method of separation of variables, obtained a perturbation solution valid small holes. The method was later extended to large holes by Withum [2] , Lekkerkerker [3] , Eringen et. al.[4] and Van Dyke [5] by numerical methods. The boundary conditions cannot, in general, be satisfied term by term in the series solution and the number of terms to be taken in the series increases with increasing value of a curvature parameter β , defining the size of the hole with respect to dimensions of the shell. This seems to be a limitation of the separation of variables method but this has been partly overcome by Van Dyke [5] . He defined the reciprocal of β as a small parameter and obtained an asymptotic solution by carrying out a boundary layer analysis. Other limitations of the separation of variables methods are ; (i) the analysis may have to be restricted to shallow shell theory in order to avoid analytical difficulties and (ii) the method can be successfully used only in cases where the solution can be expressed in terms of known mathematical functions whose behaviour and properties are well-understood. With these restrictions, the separation of variables method seems to be limited in application only to circular and elliptic cutouts.

While the work on circular cutouts is quite exhaustive, the available data on cutouts of various non-circular shapes is not so complete. Savin and Guz have done considerable amount of work on the stresses in thin shells weakened by curvilinear holes of any shape. A comprehensive report of these investigations can be found in reference [6] . The hole is assumed to be small in size compared to the shell, and boundary perturbation method is used to reduce the problem to a series of boundary value problems in polar coordinates. The first approximation corresponds to the problem of a circular hole. The solutions are accurate enough only if the hole does not depart much from a circle. The method has been extended later to holes of large size but it still suffers from the drawback that the accuracy is limited by the smallness of ϵ , which is a

measure of 'out-of-roundness'. For instance, although solutions can be obtained for an elliptic hole, results for a very slender elliptic hole would be far from accurate. Further, effects of sharp discontinuities such as corners and cracks cannot be studied. However, on the credit side, it should be mentioned that it is possible to analyze the stress problem of holes of any shape as long as one can find a suitable mapping function. Secondly, one could go further in the ϵ series and minimize the errors due to significant 'out-of-roundness'.

Finite difference [7] and variational methods [8] of solving the cutout problems have been attempted, but are not found very successful for many reasons. The only other successful method is the finite element technique which is now gaining importance in view of its versatile nature and ease of application in terms of manual effort. Fully conforming, high precision triangular shell elements of displacement type have been developed and applied successfully to certain cutout problems with better than one per cent accuracy in the results [9, 10]. But the main drawback seems to be, as in all finite element analyses, the difficulty in dealing with steep stress gradients in the vicinity of re-entrant corners. This difficulty has been overcome in part by Rao et.al. [11] by using hybrid techniques.

Another technique in cutout problems, which is recently being developed, is the reduction of the boundary value problem to integral equations by means of integral representations of general solutions, a technique widely used in problems in potential theory. This method relies on the computing machine to do most of the work. While the mathematical machinery necessary for application of the method has been assembled [12], successful application of the method to practical problems has not been sufficiently demonstrated except in certain crack problems [13, 14]. One of the limitations of the method is that it is applicable only to shells for which singular solutions can be obtained in a suitable form. Further, there may be unforeseen difficulties with numerical analysis and programming for cutout problems. The main advantages claimed are: (i) No particular difficulties are involved with cutouts of non-circular shapes and (ii) The method is expected to be not so limited with respect to the curvature parameter as the separation of variables method.

The various methods of analyzing cutout problems available in the literature has been briefly outlined above. It is not possible to generalize that a particular method is best suited to all problems. It should, however, be admitted that the finite element method is ideally suited to handle practical problems with arbitrary loading and boundary conditions. The problem of steep stress gradients can be handled by using continuum mechanics methods in conjunction with finite element technique [11]. The continuum mechanics approach, thus, has a great

potential as a method of stress analysis around sharp corners, cracks etc. In the present paper , the problem of stresses around an elliptic hole in a cylindrical shell is analyzed by a continuum mechanics approach and the limiting case of a crack is investigated. It has been demonstrated that even the singular stresses around the crack tip can be successfully determined by this method. The method of separation of variables is used to obtain a series solution for the governing equations in terms of Mathieu and Modified Mathieu functions. In spite of the drawbacks of separation of variables method as already enunciated, the authors feel that it should still be preferred wherever the solution can be expressed in terms of known functions whose properties are well established, as in the present problem.

Two specific problems are investigated in this paper.

Problem No.1 - Stresses around an elliptic hole in a cylindrical shell under axial tension with major axis parallel to shell axis (Fig.1).

Problem No.2 - As the limiting case of crack for the above problem is a trivial one, the problem of stresses around an axial crack is examined for internal pressure loading case (Fig.2). The problem is of great practical significance in the determination of residual strength of pressurized cylindrical shells.

We define a curvature parameter β as follows:

$$\beta^2 = a^2 \left[12(1-\nu^2)\right]^{\frac{1}{2}}/8Rt$$

where, a = length of semi-major axis of shell, ν = Poisson's ratio, R,t = radius and thickness of shell respectively.

Perturbation solutions for both Problem No.1 [15] and problem No.2 [16, 17] , valid for small values of the curvature parameter β , are available. These solutions, as well as the available perturbation solutions for all other elliptic hole problems [18] - [21] , are limited in their practical application due to restriction on the value of β . Problem No.1 was investigated by Venkitapathy [22] for large values of β , but the analysis is found to be erroneous due to omission of a part of the series solution. The same problem was analyzed by Tingleff [23] but by taking major axis perpendicular to shell axis. Tingleff's analysis was limited to $\beta < 1.75$, a restriction resulting from his method of determining characteristic numbers of the Mathieu equation. The present analysis is carried out upto $\beta = 2.5$ and, in principle, the method does not impose any restriction on the value of β . Numerical solution for problem No.2, valid for large β , were presented by Copley and Sanders [13]. This solution, as well as similar solution for a large circumferential crack [24] by Erdogan and Ratwani are both based on the reduction of the

boundary value problem to integral equations as mentioned earlier. The analysis presented in this paper for problem No.2 is based on differential equation approach and appears to be the first analysis of its kind for the problem of a large crack in a shell. The authors feel that this alternative procedure would be of interest.

2 ANALYSIS

For the sake of convenience, distances, stresses, membrane forces, moments, stress function and transverse shear forces are nondimensionalized with respect to a, σ, σt, $\sigma t^2/[12(1-\nu^2)]^{\frac{1}{2}}$, $\sigma t a^2$ and $\sigma t^2/a[12(1-\nu^2)]^{\frac{1}{2}}$ respectively, where σ is a reference stress and, in this case, denotes the uniform major principal stress at infinity.

We define an elliptic coordinate system (ξ,η) as follows:

$$x = h \cosh\xi \cos\eta \quad , \quad y = h \sinh\xi \sin\eta \qquad (1)$$

where (x, y) represent a nondimensional cartesian coordinate system (Fig.1) and h is the nondimensional distance between the focii of the ellipse.

Donnel's governing differential equation for a cylindrical shell, in the absence of external surface loads, can be written [15] as

$$\nabla^4 F + 8i\beta^2 \frac{\partial^2 F}{\partial x^2} = 0 \qquad (2)$$

where F = a complex stress displacement fuction, $W - i\,\phi$
W = nondimensional normal displacement
ϕ = nondimensional Airy's stress function

The membrane forces N_ξ, N_η and $N_{\xi\eta}$, moments M_ξ, M_η and $M_{\xi\eta}$ and transverse shear forces Q_ξ and Q_η can be obtained [15] from the complex stress-displacement function F.

2.1 Problem No.1 (Fig.1)

It can be shown [15] that the state of stress in the shell is the sum of uniform state of stress in the absence of hole and the stress system in an edge load problem for which the conditions at the boundary of the hole $(\xi = \xi_0)$ are

$$(N_\xi)_{\xi=\xi_0} = -\frac{1}{2}\left[1 + \frac{(\cosh 2\xi_0 \cos 2\eta - 1)}{(\cosh 2\xi_0 - \cos 2\eta)}\right]$$

$$\left(N_{\xi\eta}\right)_{\xi=\xi_0} = \frac{1}{2}\,\frac{\sinh 2\xi_0 \sin 2\eta}{(\cosh 2\xi_0 - \cos 2\eta)}$$

$$\left(M_{\xi}\right)_{\xi=\xi_0} = 0$$

$$\left(\tilde{Q}_{\xi}\right)_{\xi=\xi_0} = 0 \qquad \text{(3a)-(3d)}$$

where

$\tilde{Q}_{\xi}$ = Kirchhoff shear, $Q_{\xi} + \frac{1}{K}\frac{\partial M_{\xi\eta}}{\partial \eta}$

K = nondimensional scale factor for elliptic coordinates.

The edge load system defined above is a self-equilibrant one and for such a system, the stresses should vanish at infinity. Solution for the governing equation (2) satisfying this condition and also the condition of periodicity in η and symmetry about x and y axes can be written in terms of Mathieu and Modified Mathieu functions as follows [15] :

$$F = \sum_{n=0,1,2,3,\cdots}^{\infty} (C_n + i D_n) F_n \qquad (4)$$

where

$$F_{2j} = \cos[(1+i)\beta x]\, Me^{(1)}_{2j}(\xi, q)\, ce_{2j}(\eta, q)$$

$$F_{2j+1} = \sin[(1+i)\beta x]\, Me^{(1)}_{2j+1}(\xi, q)\, ce_{2j+1}(\eta, q)$$

$$q = i\beta^2 h^2/2 \qquad \text{(5a)-(5c)}$$

Expansions for the Mathieu functions $ce_n(\eta, q)$ as Fourier series in η are available [25]. The hyperbolic functions in equations (5a) and (5b), with the help of (1), can also be expressed as Fourier series in η. Thus each term in the series solution (4) can be expressed in the form of a Fourier series. The arbitrary constants in equation (4) are determined by satisfying the boundary conditions (3a)-(3d) term by term in the Fourier series. The number of terms taken in equation (4) is increased until the maximum total stress on the boundary determined from two successive approximations does not differ by more than 0.1%. The Fourier series method has been used in preference to other methods, because it can be shown that for a given number of terms in the series solution, the Fourier series method of satisfying the boundary conditions is, in effect, equivalent to minimizing the error integral over the boundary.

The maximum total stress occurs always at the ends of minor axis of the hole and on the inner surface of the shell. Fig. 3 shows a plot of the total stress concentration factor as a function of β and κ, where κ is minor to major axis ratio. The plot extends over $0 < \beta \leq 2.5$ and over the entire range of κ ($0 \leq \kappa \leq 1$).

2.2 Problem No.2 (Fig.2)

Boundary conditions for the edge load problem in this case are

$$N_\xi = -1 \,, \quad N_{\xi\eta} = 0 \,, \quad M_\xi = 0 \quad \text{and} \quad \tilde{Q}_\xi = 0 \quad \text{at} \quad \xi = 0$$

The arbitrary constants are solved exactly as in the previous problem by satisfying the boundary conditions term by term in the Fourier series. The tests of convergence, however, is applied not on the stress concentration factor (which is infinity) but on the total stress intensity factor. In order to determine the singular stresses at the crack tip, the solution has to be transformed to local polar coordinates (r,θ) (Fig.2) by using the following relations

For $r << 1$

$$\xi = (2r)^{1/2} \cos\frac{\theta}{2} - \frac{1}{6\sqrt{2}} r^{3/2} \cos\frac{3\theta}{2} + O(r^2)$$

$$\eta = (2r)^{1/2} \sin\frac{\theta}{2} - \frac{1}{6\sqrt{2}} r^{3/2} \sin\frac{3\theta}{2} + O(r^2) \qquad \text{(6a)-(6b)}$$

By using Taylor's series expansion, the Mathieu and Modified Mathieu functions can be written as

$$ce_n(\eta,q) = ce_n(0,q) + \frac{\eta}{\lfloor 1} ce_n'(0,q) + \frac{\eta^2}{\lfloor 2} ce_n''(0,q) + \frac{\eta^3}{\lfloor 3} ce_n'''(0,q) + O(\eta^4)$$

$$Me_n(\xi,q) = Me_n(0,q) + \frac{\xi}{\lfloor 1} Me_n'(0,q) + \frac{\xi^2}{\lfloor 2} Me_n''(0,q) + \frac{\xi^3}{\lfloor 3} Me_n'''(0,q) + O(\xi^4) \qquad \text{(7a)-(7b)}$$

where the dot ($\cdot$) and dash ($'$) denote differentiation with respect to η and ξ respectively.

Using equations (6a), (6b), (7a) and (7b), the solution can be expressed in polar coordinate (r,θ) and the singular stresses can be

determined.

We define membrane and bending stress intensity factor ratios $B^{(m)}$ and $B^{(b)}$ as follows:

$$B^{(m)} = \left[\sigma_{\theta\,(membrane)}\right]_{\theta=0}^{\text{in shell}} \Big/ (\sigma_\theta)_{\theta=0}^{\text{in plate}}, \quad r \ll 1$$

$$B^{(b)} = \left[\sigma_{\theta\,(bending)}\right]_{\theta=0}^{\text{in shell}} \Big/ (\sigma_\theta)_{\theta=0}^{\text{in plate}}, \quad r \ll 1$$

The membrane and bending stress intensity factor ratios $B^{(m)}$ and $B^{(b)}$ are plotted in Figs.4 and 5 against β for $\nu = 0.3$. It can be seen from Fig.4 that membrane stress intensity factor ratio $B^{(m)}$ is almost linear with respect to β, for $\beta > 0.5$. The singular component of bending stress near the crack tip changes sign at a particular value of β, as it can be seen from Fig.5. At this value of β, the singular stresses are purely membrane in nature.

2.3 SPECIAL SOLUTION FOR A CRITICAL VALUE OF q.

For q = 1.46876852, two of the eigenvalues of the Mathieu equation are equal and, as a result, the first and the third terms in the solution eq.(4) become identical. The boundary conditions cannot be completely satisfied by dropping one of these terms. However, for this particular value of q, there exists a different type of solution, also a series, but not of the variable-separable type and this replaces one of the two identical terms. Thus a final solution can be obtained for this special case also. Contrary to normal anticipation, the standard solution has not shown any numerical instability nor even noticeable changes in convergence in the neighbourhood of the above critical value of q.

3. CONCLUSION

The various methods of stress analysis of cut outs in shells are reviewed. The authors feel that if one can obtain solution in terms of well known functions by separation of variables, it is advantageous to use this method. This is demonstrated by application of the method to the elliptic hole problem. It is shown that the limiting case of a crack can be successfully analyzed, a problem which eludes satisfactory analysis by idealisation procedures in numerical methods such as finite elements, finite differences etc. Integration of this method with finite element methods on the lines of hybrid techniques [11] is likely to prove very useful in solving practical stress analysis problems of shell structures.

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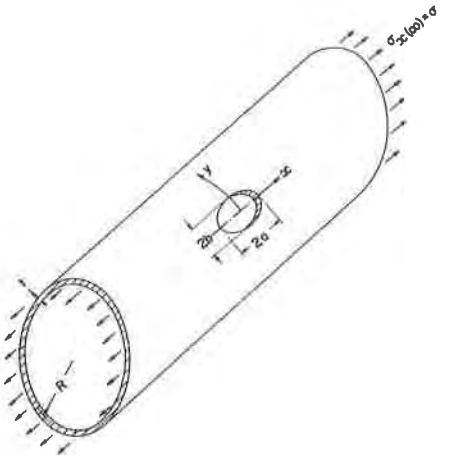

Fig. 1 Cylindrical Shell with an elliptic hole under axial tension

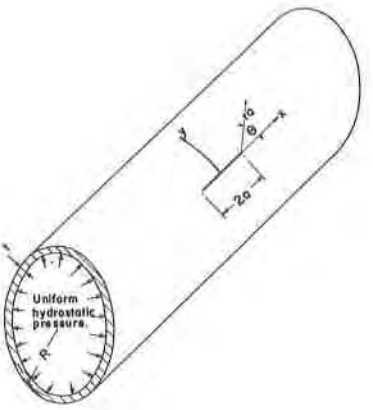

Fig. 2 An axial crack in a pressurized cylindrical shell

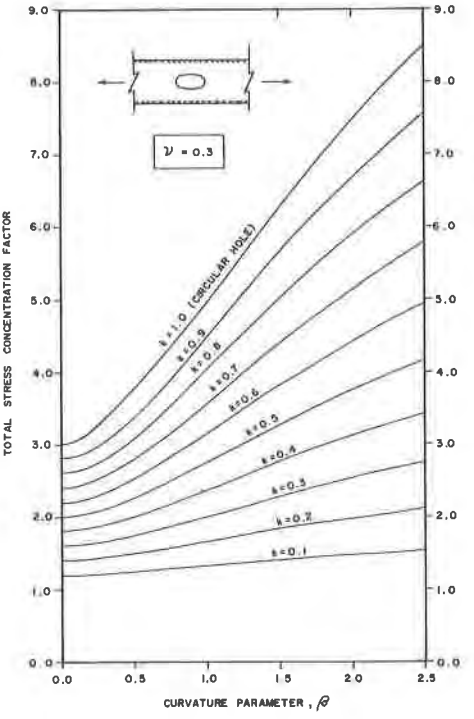

Fig. 3 Plot of total stress concentration factor versus β and κ

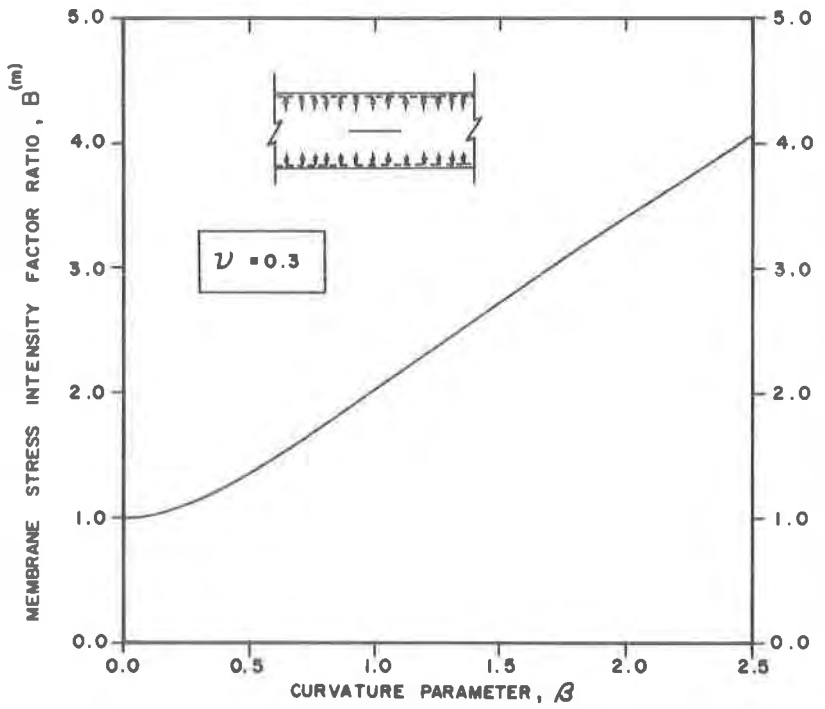

Fig. 4 Plot of membrane stress intensity factor ratio versus curvature parameter

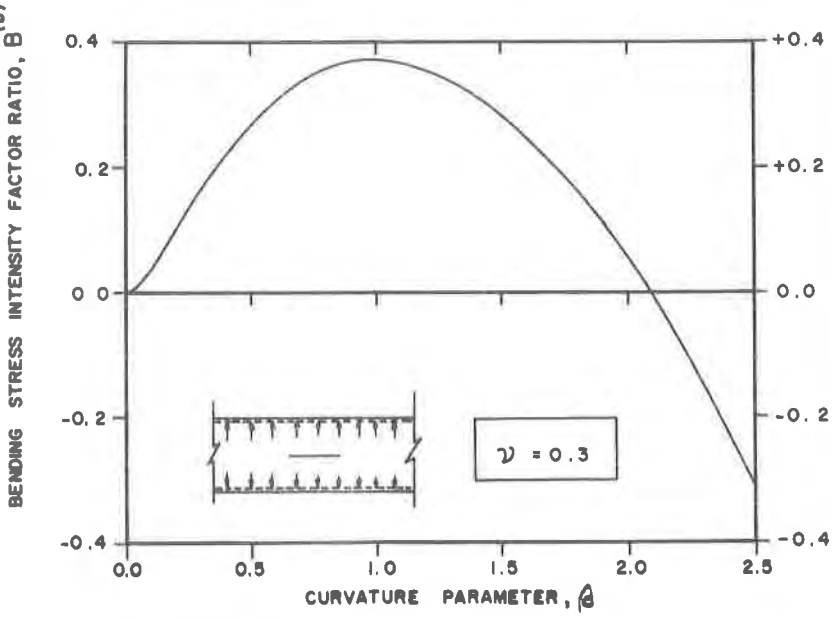

Fig. 5 Plot of bending stress intensity factor ratio versus curvature parameter