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A VECTOR VALUED MULTIVARIATE HAZARD RATE - II

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1. INTRODUCTION

This paper is a continuation of [1], wherein can be found the definition of a vector valued multivariate hazard rate, some general properties and applications to specific distributions. In the present paper we first discuss some further general properties, and then consider some further specific distributions.

Throughout it is assumed that the variables are continuous and possess joint density functions. We use the notation

$$F_{\underline{X}}(\underline{x}) = \Pr\left[\bigcap_{j=1}^m (X_j \leq x_j)\right]; \quad G_{\underline{X}}(\underline{x}) = \Pr\left[\bigcap_{j=1}^m (X_j > x_j)\right].$$

2. RELATION BETWEEN HAZARD RATES AND ORTHANT DEPENDENCE

In section 3(i) of [1] it was shown that if X_1, \dots, X_m are mutually independent then

$$h_{\underline{X}}(\underline{x}) = (h_{X_1}(x_1), \dots, h_{X_m}(x_m))$$

where $h_{\underline{X}}(\underline{x})$ is the (vector) multivariate hazard rate of X_1, \dots, X_m and $h_{X_j}(x_j)$ is the (univariate) hazard rate of X_j . Another way of expressing this result is

$$h_{\underline{X}}(\underline{x})_j = h_{X_j}(x_j).$$

Lehmann [2] has defined *positive (negative) quadrant dependence* between two variables X_1, X_2 by

$$(1) \quad R_{F(X_1, X_2)} = F_{\underline{X}}(\underline{x}) / \left\{ \prod_{j=1}^2 F_{X_j}(x_j) \right\} \geq (\leq) 1$$

for all x_1, x_2 .

He has shown that these definitions are equivalent to those obtained by replacing $R_{F(X_1, X_2)}(x_1, x_2)$ by

$$(2) \quad R_{G(X_1, X_2)}(x_1, x_2) = G_{\underline{X}}(\underline{x}) / \left\{ \prod_{j=1}^2 G_{X_j}(x_j) \right\}.$$

Dykstra *et. al.* [3] have extended these definitions to define *positive (negative) orthant dependence* among $m(\geq 2)$ variables X_1, X_2, \dots, X_m to correspond to

$$(3) \quad R_{F(\underline{X})}(\underline{x}) = F_{\underline{X}}(\underline{x}) / \left\{ \prod_{j=1}^m F_{X_j}(x_j) \right\} \geq (\leq) 1$$

for all \underline{x} .

We, however, will work with the equally natural extended definition

$$(4) \quad R_{G(\underline{X})}(\underline{x}) = G_{\underline{X}}(\underline{x}) / \left\{ \prod_{j=1}^m G_{X_j}(x_j) \right\} \geq (\leq) 1$$

for all \underline{x} , which we will call *G-positive (negative) orthant dependence*.

For $m = 2$, the two definitions are equivalent; for $m > 2$ this is not so. (A counterexample is easily constructed (for $m = 3$) by noticing that (3) is satisfied if the conditional joint distribution of X_1 and X_2

given $(X_3 \leq x_3)$ has positive (negative) quadrant dependence, while (4) is satisfied if the joint distribution given $(X_3 > x_3)$ has positive (negative) quadrant dependence.)

We will now develop some relations among $R_{G(\underline{x})}(\underline{x})$ and various hazard rates. We first note that

$$(5) \quad \lim_{x_1 \rightarrow -\infty} R_{G(X_1, \dots, X_m)}(x_1, \dots, x_m) = R_{G(X_2, \dots, X_m)}(x_2, \dots, x_m)$$

In particular, for $m = 2$

$$(6) \quad \lim_{x_1 \rightarrow -\infty} R_{G(X_1, X_2)}(x_1, x_2) = R_{G(X_2)}(x_2) = G_{X_2}(x_2)/G_{X_2}(x_2)/G_{X_2}(x_2) = 1.$$

Since

$$(7) \quad \begin{aligned} \partial \log R_{G(\underline{x})}(\underline{x}) / \partial x_j &= \partial \log G_{\underline{x}}(\underline{x}) / \partial x_j - \partial \log G_{X_j}(x_j) / \partial x_j \\ &= h_{X_j}(x_j) - h_{\underline{x}}(\underline{x})_j \end{aligned}$$

we see that for bivariate distributions ($m=2$), if $h_{X_j}(x_j) > (<) h_{\underline{x}}(\underline{x})_j$ for all \underline{x} and $j = 1, 2$ then (remembering (6)), the distribution has *positive (negative) quadrant dependence*. The converse is not necessarily true.

We cannot immediately extend this result to cases $m > 2$. In view of (5), we do have results like:

"If X_1, \dots, X_{m-1} have G-positive (negative) orthant dependence and $h_{X_m}(x_m) > (<) h_{\underline{x}}(\underline{x})_m$ for all \underline{x} , then X_1, \dots, X_m have G-positive (negative) orthant dependence."

For the Morgenstern-Gumbel-Farlie bivariate distributions defined by

$$(8) \quad F_{X_1, X_2}(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)[1 + \alpha\{1 - F_{X_1}(x_1)\}\{1 - F_{X_2}(x_2)\}]$$

or equivalently

$$(8)' \quad G_{X_1, X_2}(x_1, x_2) = G_{X_1}(x_1)G_{X_2}(x_2)[1 + \alpha\{1 - G_{X_1}(x_1)\}\{1 - G_{X_2}(x_2)\}]$$

(see equation (6) in [1]), we have

$$(9) \quad R_{F(X_1, X_2)}(x_1, x_2) = 1 + \alpha\{1 - F_{X_1}(x_1)\}\{1 - F_{X_2}(x_2)\};$$

$$(10) \quad R_{G(X_1, X_2)}(x_1, x_2) = 1 + \alpha\{1 - G_{X_1}(x_1)\}\{1 - G_{X_2}(x_2)\}.$$

In this case $R_{G(X_1, X_2)}(x_1, x_2) \geq (\leq) 1$ according as $\alpha \geq (\leq) 0$. (The same is true of $R_{F(X_1, X_2)}(x_1, x_2)$). Also, in this case, from equation (7) of [1], $h_{X_j}(x_j) - h_{\underline{X}}(\underline{x})_j$ has the same sign as α , so in this case the converse result referred to above (that positive (negative) quadrant dependence implies $h_{X_j}(x_j) > (<) h_{\underline{X}}(\underline{x})_j$), does hold.

3. TRANSFORMATION OF VARIABLES

We first note that if \underline{X} have positive (negative) orthant dependence (in the sense of Dykstra *et. al.*) then $-\underline{X}$ have G-positive (negative) orthant dependence, and conversely.

If $Z_1 = -X_1$ then (remembering the variables are continuous)

$$G_{Z_1, X_2, \dots, X_m}(z_1, x_2, \dots, x_m) = G_{X_2, \dots, X_m}(x_2, \dots, x_m) \\ - G_{X_1, \dots, X_m}(-z_1, x_2, \dots, x_m)$$

and so

$$(11.1) \quad h_{Z_1, X_2, \dots, X_m} (z_1, x_2, \dots, x_m)_1 = h_{\underline{X}} (-z_1, x_2, \dots, x_m)_1$$

while for $j \geq 2$

$$(11.2) \quad h_{Z_1, X_2, \dots, X_m} (z_1, x_2, \dots, x_m)_j = h_{X_2, \dots, X_m} (x_2, \dots, x_m)_j - h_{\underline{X}} (-z_1, x_2, \dots, x_m)_j$$

Similar (though more complicated) results can be obtained when several of the X_j 's are changed in sign.

If Y_j is a continuous increasing monotonic function of X_j , so is X_j of Y_j and we can write

$$X_j = \xi_j(Y_j) \quad (j=1, \dots, m) .$$

Since the events $(Y_j > y_j)$ and $(X_j > \xi_j(y_j))$ are equivalent we have

$$G_{\underline{Y}}(\underline{y}) = G_{\underline{X}}(\underline{\xi}(\underline{y}))$$

and

$$(12) \quad h_{\underline{Y}}(\underline{y})_j = h_{\underline{X}}(\underline{\xi}(\underline{y}))_j (\partial \xi_j / \partial y_j)$$

If \underline{X} is IHR (DHR) and $\partial \xi_j / \partial y_j$ is a non-decreasing (non-increasing) function of y_j for $j = 1, 2, \dots, m$ then \underline{Y} also is IHR (DHR). In particular, if \underline{X} is IHR (DHR) then so is $(b_1 X_1 + a_1, \dots, b_m X_m + a_m)$ if $b_j > 0$. (See also section 3 (iv) of [1]).

It is known ([4] p. 37) that mixtures of univariate DHR distributions are also DHR. (A similar property is not valid for IHR distributions.) Since $h_{\underline{X}}(\underline{x})_1$ is the hazard rate of the conditional (univariate) distribution of X_1 given $\bigcap_{j=2}^m (X_j > x_j)$, (see Section 3(iv) of [1]), it follows

that mixtures of multivariate DHR distributions are also DHR.

4. SOME FURTHER SPECIAL DISTRIBUTIONS. Details of many of these distributions will be found in [5].

(i) Bivariate Logistic

We consider the joint cumulative distribution function:

$$F_{X_1, X_2}(x_1, x_2) = (1 + e^{-x_1} + e^{-x_2})^{-1}$$

for which

$$(13) \quad G_{X_1, X_2}(x_1, x_2) = 1 - (1 + e^{-x_1})^{-1} - (1 + e^{-x_2})^{-1} + (1 + e^{-x_1} + e^{-x_2})^{-1} \\ = e^{-x_1 - x_2} (1 + e^{-x_1})^{-1} (1 + e^{-x_2})^{-1} (1 + e^{-x_1} + e^{-x_2})^{-1} (2 + e^{-x_1} + e^{-x_2})$$

We find

$$h_{X_1}(x_1) = (1 + e^{-x_2})^{-1} (2 + e^{-x_1} + e^{-x_2})^{-1} \{ (1 + e^{-x_1} + e^{-x_2})^{-1} + (1 + e^{-x_1})^{-1} \}$$

which is an increasing function of x_1 . The joint distribution is IHR.

For general m with

$$F_{X_1, \dots, X_m}(x_1, \dots, x_m) = (1 + \sum_{j=1}^m e^{-x_j})^{-1}$$

it has not yet been possible to establish that the distribution is IHR, though it seems very likely.

(ii) Gumbel's Bivariate Exponential Distributions.

If the joint cumulative distribution function is

$$F_{X_1, X_2}(x_1, x_2) = 1 - e^{-x_1} - e^{-x_2} + e^{-(x_1 + x_2 + \theta x_1 x_2)} \\ (x_1 > 0, x_2 > 0; \theta > 0)$$

then

$$(15) \quad G_{X_1, X_2}(x_1, x_2) = \exp(-x_1 - x_2 - \theta x_1 x_2)$$

whence

$$(16) \quad h_{\underline{X}}(\underline{x}) = (1 + \theta x_2, 1 + \theta x_1) .$$

These components are constant with respect to variation in the corresponding variable (ie. $h_{\underline{X}}(\underline{x})_1$ does not depend on x_1 , nor $h_{\underline{X}}(\underline{x})_2$ on x_2) but not with respect to variation in the other variable. In section 3 (v) of [1] it was shown that if $h_{\underline{X}}(\underline{x}) = \underline{c}$, with \underline{c} constant (with respect to *all* variables) then X_1, X_2, \dots, X_m must have independent exponential distributions. For convenience we might term $h_{\underline{X}}(\underline{x}) = \underline{c}$ a *strictly* constant hazard rate; the distribution (10), on the other hand, may be regarded as having a *locally* constant hazard rate.

If we assume

$$h_{\underline{X}}(\underline{x}) = (f_1(x_2), f_2(x_1))$$

then

$$\log G_{\underline{X}}(\underline{x}) = -x_1 f_1(x_2) + A_1(x_1) = -x_2 f_2(x_1) + A_2(x_2)$$

where $A_1(\cdot)$, $A_2(\cdot)$ are arbitrary functions subject to the standard conditions on $G_{\underline{X}}(\underline{x})$.

Putting $x_1 = 0$ gives $A_1(0) = -x_2 f_2(0) + A_2(x_2)$, so $A_2(x_2)$ must be a linear function of x_2 . Similarly $A_1(x_1)$ must be a linear function of x_1 . Now putting $x_1 = a \neq 0$, we obtain

$$-a f_1(x_2) = -x_2 f_2(a) + (\text{linear function of } x_2)$$

so that $f_1(x_2)$ must be a linear function of x_2 . Similarly $f_2(x_1)$ must be a linear function of x_1 . It follows that the joint distribution of appropriate linear transforms of X_1, X_2 is of form (10).

To summarize, Gumbel's family of bivariate exponential distributions includes all distributions with locally constant bivariate hazard rates.

(iii) Bivariate Exponential Distributions of Marshall and Olkin.

These have joint survival functions of form

$$(17) \quad G_{X_1, X_2}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)\}$$

$$(\lambda_1, \lambda_2, \lambda_{12} > 0; x_1, x_2 > 0)$$

For these distributions

$$(18) \quad h_{\underline{X}}(\underline{x}) = \begin{cases} (\lambda_1, \lambda_2 + \lambda_{12}) & (x_1 < x_2) \\ (\lambda_1 + \lambda_{12}, \lambda_2) & (x_1 > x_2) \end{cases}.$$

They are not strictly IHR, but for $x_2(x_1)$ fixed the first (second) component is a non-decreasing function of $x_1(x_2)$.

The analysis for multivariate ($m > 2$) Marshall-Olkin distributions (see [5]) follows similar lines.

(iv) Freund's Bivariate Exponential Distribution.

For this distribution, the joint density is

$$P_{X_1, X_2}(x_1, x_2) = \begin{cases} \alpha\beta \exp\{-\beta x_2 - (\alpha + \beta - \beta')x_1\} & (0 \leq x_1 < x_2) \\ \alpha'\beta \exp\{-\alpha x_1 - (\alpha + \beta - \alpha')x_2\} & (0 \leq x_2 < x_1) \end{cases}$$

with $\alpha, \beta, \alpha', \beta' > 0$.

We will assume $\alpha + \beta \neq \alpha', \beta'$, and $\alpha' > \alpha, \beta' > \beta$. We find

$$(19) \quad G_{X_1, X_2}(x_1, x_2) = \begin{cases} (\alpha + \beta - \beta')^{-1} [\alpha \exp\{-\beta' x_2 - (\alpha + \beta - \beta') x_1\} + (\beta - \beta') \exp\{-(\alpha + \beta) x_2\}] & (0 \leq x_1 < x_2) \\ (\alpha + \beta - \beta')^{-1} [\beta \exp\{-\alpha' x_1 - (\alpha + \beta - \alpha') x_2\} + (\alpha - \alpha') \exp\{-(\alpha + \beta) x_1\}] & (0 \leq x_2 < x_1) \end{cases}$$

and

$$(20) \quad h_X(x)_1 = \begin{cases} \frac{\alpha(\alpha + \beta - \beta')}{\alpha - (\beta' - \beta) \exp\{-(\alpha + \beta - \beta')(x_2 - x_1)\}} & (0 \leq x_1 < x_2) \\ \frac{\beta\alpha' - (\alpha' - \alpha)(\alpha + \beta) \exp\{-(\alpha + \beta - \alpha')(x_1 - x_2)\}}{\beta - (\alpha' - \alpha) \exp\{-(\alpha + \beta - \alpha')(x_1 - x_2)\}} & (0 \leq x_2 < x_1) \end{cases}$$

Taking the case $\alpha + \beta > \alpha' > \alpha; \alpha + \beta > \beta' > \beta$ we find that for $x_1 < x_2$;

$h_X(x)_1$ is of form $c_1 \{c_2 - c_3 \exp(c_4 x_1)\}^{-1}$ where $c_j > 0$ ($j=1,2,3,4$), and so is an increasing function of x_1 .

For $x_1 > x_2$; $h_X(x)_1$ is of form

$$\frac{c_5 \exp(c_6 x_1) - c_7}{c_8 \exp(c_6 x_1) - c_9}$$

where $c_j > 0$, and is an increasing function of x_1 if

$$c_7 c_8 - c_5 c_9 > 0.$$

Now $c_7 c_8 - c_5 c_9 = [\beta(\alpha' - \alpha) \exp\{(\alpha + \beta - \alpha') x_2\}] [\alpha + \beta - \alpha'] > 0$. So

$h_X(x)_1$ is also an increasing function of x_1 for $x_1 > x_2$. Similarly,

$h_X(x)_2$ is an increasing function of x_2 for all x_2 . The joint

distribution is IHR.

The same result is obtained for other inequalities between $\alpha + \beta$, α' and β' (but always keeping $\alpha' > \alpha$, $\beta' > \beta$).

(v) A Multivariate F-Distribution.

We first consider the joint distribution of $G_j = V_j/V_0$ ($j=1, \dots, m$) where V_0, V_1, \dots, V_m are independently distributed as χ^2 's with $\nu, 2, 2, \dots, 2$ degrees of freedom respectively. The joint density is

$$p_G(\underline{g}) = \left\{ \Gamma\left(\frac{1}{2}\nu+m\right) / \Gamma\left(\frac{1}{2}\nu\right) \right\} \left(1 + \sum_{j=1}^m g_j \right)^{-\frac{1}{2}\nu-m} \quad (g_j > 0)$$

and the joint survivor function is

$$(21) \quad G_G(\underline{g}) = \left(1 + \sum_{j=1}^m g_j \right)^{-\frac{1}{2}\nu}$$

whence

$$(22) \quad h_G(\underline{g})_i = \frac{1}{2}\nu \left(1 + \sum_{j=1}^m g_j \right)^{-1} \quad \text{for all } i.$$

The joint distribution is DHR. In this case all elements of the vector multivariate hazard rate are equal.

These results also apply to the multivariate F-distribution which is the joint distribution of $F_j = \frac{1}{2}\nu G_j$ ($j=1, \dots, m$).

For the joint distribution of $T_j = \sqrt{G_j}$ we find (directly, or from (9))

$$(23) \quad h_T(\underline{t})_i = \nu t_i \left(1 + \sum_{j=1}^m t_j^2 \right)^{-1} \quad \begin{matrix} (i=1, \dots, m) \\ (t_j > 0) \end{matrix}$$

Recalling that $\frac{d}{dt} \left(\frac{t}{A+t} \right) = (A+t^2)^{-2} (A-t^2)$, we see that $h_T(\underline{t})$ is an

increasing function of t_i for $t_i^2 < 1 + \sum_{j=1}^m t_j^2 - t_i^2$, decreasing for $t_i^2 > 1 + \sum_{j=1}^m t_j^2 - t_i^2$. The joint distribution is neither IHR nor DHR.

(vi) Bivariate Fréchet Distributions

For given marginal cumulative distribution functions $F_{X_1}(x_1)$, $F_{X_2}(x_2)$ the Fréchet family of bivariate distributions has joint cumulative distribution function

$$F_{X_1, X_2}(x_1, x_2) = \omega \max\{0, F_{X_1}(x_1) + F_{X_2}(x_2) - 1\} + (1-\omega) \min\{F_{X_1}(x_1), F_{X_2}(x_2)\} \\ (0 < \omega < 1).$$

Hence

$$(24) \quad G_{X_1, X_2}(x_1, x_2) = \begin{cases} 1 - \max\{F_{X_1}(x_1), F_{X_2}(x_2)\} - \omega \min\{F_{X_1}(x_1), F_{X_2}(x_2)\} \\ \quad \text{for } F_{X_1}(x_1) + F_{X_2}(x_2) < 1 \\ (1-\omega) [1 - \max\{F_{X_1}(x_1), F_{X_2}(x_2)\}] \\ \quad \text{for } F_{X_1}(x_1) + F_{X_2}(x_2) > 1 \end{cases}$$

and

$$(25) \quad G_{X_1}(x_1) h_{X_2}(x_2) = \begin{cases} (f_{X_1}(x_1), \omega f_{X_2}(x_2)) & \text{for } F_{X_1}(x_1) > F_{X_2}(x_2); \\ & F_{X_1}(x_1) + F_{X_2}(x_2) < 1 \\ (\omega f_{X_1}(x_1), f_{X_2}(x_2)) & \text{for } F_{X_1}(x_1) < F_{X_2}(x_2); \\ & F_{X_1}(x_1) + F_{X_2}(x_2) < 1 \\ ((1-\omega) f_{X_1}(x_1), 0) & \text{for } F_{X_1}(x_1) > F_{X_2}(x_2); \\ & F_{X_1}(x_1) + F_{X_2}(x_2) > 1 \\ (0, (1-\omega) f_{X_2}(x_2)) & \text{for } F_{X_1}(x_1) < F_{X_2}(x_2); \\ & F_{X_1}(x_1) + F_{X_2}(x_2) > 1 \end{cases}$$

(with $f_{X_j}(x_j) = dF_{X_j}(x_j)/dx_j$).

Note that in general there are discontinuities in the components of $h_{\underline{X}}(\underline{x})$ even when $h_{X_1}(x_1)$ and $h_{X_2}(x_2)$ are continuous.

The x_j component of $G_{\underline{X}}(\underline{x})h_{\underline{X}}(\underline{x})$ can be

$$0, \omega f_{X_j}(x_j), (1-\omega)f_{X_j}(x_j), \text{ or } f_{X_j}(x_j) .$$

For x_2 fixed, as x_1 increases $G_{\underline{X}}(\underline{x})h_{\underline{X}}(\underline{x})$ takes the values

$$\omega f_{X_1}(x_1), f_{X_1}(x_1), (1-\omega)f_{X_1}(x_1)$$

over successive intervals of x_1 if $F_{X_2}(x_2) < \frac{1}{2}$.

If $F_{X_2}(x_2) > \frac{1}{2}$, the sequence is

$$\omega f_{X_1}(x_1), 0, (1-\omega)f_{X_1}(x_1) .$$

If $G_{\underline{X}}(\underline{x})$ is a continuous function of \underline{x} , it is clear these cannot be increasing or decreasing functions over the whole range of x_1 , for any x_2 , if $f_{X_1}(x_1) \neq 0$. So, generally, Fréchet distributions cannot be IHR or DHR, using the vector multivariate hazard rate we have defined.

(vii) Multinormal Distributions

Numerical evidence in Section 4.1 indicates that bivariate normal distributions are IHR, at least when the correlation coefficient is not negative. Dr. J. Galambos (private communication) has shown that this is so.

Here we note a few points in connection with general m -variate multinormal distributions.

If X_1, X_2, \dots, X_m have a joint multinormal distribution, the conditional distribution of X_1 , given X_2, \dots, X_m is normal with a fixed

standard deviation (which we can arrange to be 1) and expected value equal to a linear function of X_2, \dots, X_m ($Z = \sum_{j=2}^m b_j X_{j1} + b_1$, say).

The first element of the vector multivariate hazard rate is therefore

$$(26) \quad h_{\underline{X}}(\underline{x})_1 = \frac{E[\phi(x_1 - Z)]}{E[1 - \phi(x_1 - Z)]}$$

where $\phi(t) = (\sqrt{2\pi})^{-1} e^{-\frac{1}{2}t^2}$; $\Phi(y) = \int_{-\infty}^y \phi(t) dt$ and expectations are taken with respect to Z , distributed conditionally on $\bigcap_{j=2}^m (X_j > x_j)$.

Although

$$\phi(x_1 - Z) / \{1 - \phi(x_1 - Z)\}$$

is an increasing function of x_1 , for any Z it does not follow that

$h_{\underline{X}}(\underline{x})_1$ is an increasing function of x_1 for all \underline{x} .

5. CONCLUDING REMARKS

We feel that enough results have been presented, in [1] and the present paper, to provide evidence of the usefulness of our proposed vector definition of multivariate hazard rate.

Using the definitions of IHR and DHR based on this concept we are able to classify a number of important multivariate distributions, comparable with those so classified by other definitions of IHR and DHR.

We feel however that the criteria of overall IHR and DHR are too sweeping to be of general usefulness. Even for univariate distributions, the DHR property implies that (if continuous) the density must be a decreasing function of x over its support (because

$$\frac{dh_X(x)}{dx} = \frac{G_X(x)f'_X(x) + \{f_X(x)\}^2}{\{G_X(x)\}^2}$$

where $f_X(x) = -dG_X(x)/dx$ is the density function of X .

Rather, the hazard rate function itself is of value as a description of the distribution - in particular of the intervals in which hazard rate is increasing or decreasing.

From this point of view, the desirability of some form of vector hazard rate for multivariate distributions is evident. Whether the particular one we have chosen is, in any sense, optimal, is an open question. We believe that we have shown it to be useful.

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MIMEO SERIES NO. 873 - CORRIGENDA

The following corrections and changes should be made to Mimeo Series No. 873 (A Vector Valued Multivariate Hazard Rate - I by N.L. Johnson and S. Kotz).

| Page | Line | |
|-----------------|------|--|
| 1 | 2 | $h_X(x)$ should be $h_X(x)$ |
| 2 | 6 | Insert "of" between "rate" and " X_1 ," |
| 2 | 11 | $g(x)$ should be $g(X)$ |
| 2 | 6f | Should read: x_1, x_2, \dots, x_m for all x_1', \dots, x_m' (including $x_j' < x_j$) |
| 4 | 11 | Replace "monotonic" by "increasing" |
| 4 | 14 | $g^{-1}(x_1)$ should be $g^{-1}(y_1)$ |
| 4 | 4f | X should be \underline{X} |
| 8 | 7f | Insert "bivariate" between "joint" and "distributions" |
| 10, lower table | | Interchange x_1 and x_2 |
| 13 | 6 | $\theta_j^{-1} X_j$ should be $\theta_j^{-1} x_j$ |
| 13 | 8 | Should read: |
| | | $-\partial \log G_X(x) / \partial x_r = a \theta_r \left(\sum_{j=1}^m \theta_j^{-1} x_j^{-m+1} \right)^{-1}$ |
| 16 | 3-2f | Insert "Since, for all $g > 2$, $2 < K_j(g) < \log g$, the upper bound is, in fact $[\exp\{K_j(g_0)\} - 1]^{-1}$ " |
| 17 | 1-3 | Delete |