SOME RESULTS IN THE THEORY
OF STOCHASTIC PROCESSES[1,*]

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ALAN J. LEE Some results in the theory of stochastic processes. (Under the direction of STAMATIS CAMBANIS.)

ABSTRACT

Consider a stochastic process $\{x(t), t \in T\}$ of random elements of a Hilbert space H, whose index set is a locally compact Hausdorff space. The results obtained in this work fall into two broad categories, first the study of weakly stationary processes and their representations, and secondly the study of the sample path properties of not necessarily stationary processes. In each case, we choose the index set T and the Hilbert space H to be spaces appropriate to the investigation in hand.

We first deal with stationary complex valued stochastic processes indexed by a locally compact topological group G, and consider their spectral representations. We show how the representation may be "inverted" to obtain an expression for the spectral measure; this result may be regarded as a weak law of large numbers. We give a decomposition of an arbitrary weakly stationary process into mean square continuous (m.s.c.) and "almost" uncorrelated components, and also show how the covariance function of a m.s.c. weakly stationary process may be estimated from the sample values via a strong law of large numbers.

We then turn to a discussion of path properties, and show that any measurable complex valued process indexed by an arbitrary locally compact Hausdorff space may be approximated as closely as we please in a certain sense by a measurable process with continuous paths. This result is the analog of the well known measure theoretic fact that any measurable

function is "almost" continuous (see e.g. Royden (1963) p. 57). Next we return to the group-indexed stationary processes, and find conditions on their spectral measures that are sufficient for path continuity with probability one. A special case of such processes are the so-called band limited processes - we show that group indexed band limited processes admit a sampling expansion, as do band limited processes over the real line. We also discuss band limited processes that are not necessarily stationary, extending an idea of Zakai (1965). We characterize such non-stationary band limited processes and show they admit a modified sampling expansion.

Finally we deal with the concept of group indexed second order processes having values in a separable Hilbert space. We prove a general representation theorem for such processes and from it derive a spectral representation for stationary processes. We prove a Hilbert space analog of the decomposition theorem mentioned above and lastly extend some of the results on path properties to the case of Hilbert space valued processes.

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CHAPTER I

INTRODUCTION

1.1 Introduction and a Summary of Results.

Traditionally, probabilists have treated stochastic processes as a collection $\{x(t), t \in T\}$ of real or complex random variables indexed by a set T which has usually been assumed to be either the real line $\mathbb{R}$ or the integers Z, or a subset of these sets. In this work we investigate a more general situation; we allow the random variables $x(t)$ to take values in vector spaces more general than the complex plane, and the "parameter" set T to be a topological space more general than $\mathbb{R}$ or Z.

More specifically, we will consider the case when the random variables of the process take values in a Hilbert space, and the parameter set T is a locally compact abelian (LCA) group. This choice of T is particularly natural when we are dealing with stationary processes, which depend for their definition on some binary operation on the parameter set. Moreover, the existing theory of abstract harmonic analysis for such groups enables us to utilize the methods of Fourier analysis so useful in the study of stationary processes whose parameter sets T are $\mathbb{R}$ or Z (briefly processes "over" $\mathbb{R}$ or Z). Finally, the fact that $\mathbb{R}$ and Z are themselves LCA groups indicates that the choice of T as an LCA group is a reasonable generalization of the known cases. In a similar way, it is reasonable to suppose that the random variables of our "generalized" process take values in a Hilbert space. Then we have at our disposal the theory of linear operators which will be used frequently in Chapter VI. This case includes the concept of a vector valued stochastic process, so again is a natural generalization of known cases.

The results obtained in this work fall into two broad categories: the first, the study of covariance functions of weakly stationary stochastic processes and their spectral representations; the second, the study of the properties of the sample paths of not necessarily stationary processes.

Chapter II is devoted to the first theme in the context of complex-valued weakly stationary processes over an LCA group G. Section 2.1 is concerned with the spectral representation of mean square continuous weakly stationary processes; the principal new result here is Theorem 2.1.1 which expresses the spectral measure of such processes in terms of a mean square stochastic integral. Section 2.2 gives a decomposition of an arbitrary weakly stationary process into a mean square continuous component and a component with "almost" uncorrelated values, (Theorem 2.2.2), a result derived from an elegant theorem in Hewitt and Ross (1970). The last section, 2.3, contains some results on the estimation of stationary covariances, together with a strong law of large numbers (Theorem 2.3.1)

Our second theme of path properties is pursued in Chapter III. In Section 3.1 we prove an analog for stochastic processes of the measure theoretic result that every measurable function is "almost" a continuous function. Our context here is measurable complex valued processes over a Hausdorff space T, the main result is Theorem 3.1.3.

In section 3.2 we extend certain criteria sufficient for path continuity for stationary processes over $\mathbb{R}$ to the case where G is a compactly generated LCA group (Theorem 3.2.3) in particular we show that "band limited" processes over such a group have continuous paths and admit a sampling expansion (Theorem 3.2.4). Section 3.3 also concerns band limited processes; we consider not necessarily stationary band-limited processes over $\mathbb{R}$ and extend some results of Zakai (1965).

Our final chapter, Chapter IV, deals with stationary Hilbert-space valued processes on an LCA group G. Section 4.1 contains some basic results on Hilbert space valued random elements (random variables), while section 4.2 introduces the concept of covariance operators for Hilbert space valued stationary processes. A Hilbert space analog of Theorem 2.2.2 is given in Theorem 4.2.4. Section 4.3 deals with spectral representations of weakly continuous stationary processes (theorem 4.3.5) while in our last section, 4.4, we give the Hilbert space analogs of some of the results of Chapter III concerning path properties and sampling.

More detailed descriptions of our results and bibliographic information are given at the beginning of each section.

1.2. Some Notations Used in the Sequel.

We will always denote the real line by $\mathbb{R}$, the integers by Z and the complex numbers by C. The binary operation in a group will be written as +, and the inverse of an element g of a group is written as -g. If U is a subset of a group, -U denotes the set

$\{-g : g \in U\}$ and $U + U$ the set $\{g+g' : g,g' \in U\}$. The set $U+U+\ldots+U$ (n summands) is written nU. If A is any set we denote the complement of A by CA, and we use the sumbol $\square$ to signal the end of each proof or definition.

CHAPTER II

STATIONARY PROCESSES ON GROUPS

This chapter contains some results from the theory of stationary processes indexed by locally compact abelian (LCA) groups. In section 2.1 we describe a spectral representation of second order weakly stationary processes due to Kampé de Feriet (1948), and show how this representation can be inverted to yield the spectral measure of singleton sets. We also make a few remarks on harmonizable processes which will be useful in Chapter III. The spectral representation applies only to mean square continuous (m.s.c.) processes. The decomposition of arbitrary weakly stationary processes into two uncorrelated components, one m.s.c. and the other with "almost" uncorrelated values is the subject of section 2.2. Finally, in section 2.3 we give a few results on the estimation of covariance functions. The results presented in this chapter are generalizations of well known results valid for the case when the index groups is the real line $\mathbb{R}$ or the integers Z, such results may be found in Doob (1953) or Rozanov (1967).

2.1. <u>Stationary Processes and their Spectral Representations</u>. Let $\{x(g) : g \in G\}$ be a weakly stationary second order stochastic process, i.e. a family of complex-valued random variables such that $E|x(g)|^2 < \infty$ for all $g \in G$ and $Ex(g+h)\overline{x(g)}$ is independent of g. We will initially suppose that the index set G is an LCA group though later we will assume that G has additional properties. Throughout section 2.1 we will assume that the function $g \to Ex(g+g_0)\overline{x(g_0)}$ is continuous, i.e. that x is m.s.c.; and that $Ex(g) = 0$ for all $g \in G$; indeed we assume throughout this work that all processes considered have zero means.

The function $R(g) = Ex(g+g_0)\overline{x(g_0)}$ is called the covariance function of the process. It is easy to see that R is positive definite and is continuous by our assumption of mean-square continuity. It follows by the well known Bochner theorem (see e.g. Rudin (1962) p. 19) that R has the representation

$$R(g) = \int_{\hat{G}} \langle \alpha, g \rangle \, \mu(d\alpha) \tag{2.1.1}$$

where μ is a positive measure in the class $M(G)$ of the Appendix. The measure μ is known as the spectral measure of the process. The corresponding representation of the process is due to Kampé de Feriet (1948), who showed that $x(g)$ has the representation

$$x(g) = \int_{\hat{G}} \langle \alpha, g \rangle Z(d\alpha) \tag{2.1.2}$$

where Z is an orthogonal stochastic measure on $\hat{G}$ with $E|Z(\Delta)|^2 = \mu(\Delta)$ for all Borel subsets Δ of $\hat{G}$. The representation is established analogously to the real line case. First we note that the characters $\{\langle \cdot, g \rangle : g \in G\}$ generate a dense linear manifold of $L_2(\hat{G},\mu)$ (Hewitt and Ross (1970) p. 211), so the correspondence $x(g) \longleftrightarrow \langle \cdot, g \rangle$ can be extended to an isometry between $H(x)$ (the Hilbert space spanned by the process) and $L_2(\hat{G},\mu)$. The integral (2.1.2) is just the element of

$H(x)$ corresponding to $\langle \circ, g \rangle$, in general a function $\phi \in L_2(\hat{G},\mu)$ corresponds to the random variable $\int_{\hat{G}} \phi(\alpha)Z(d\alpha)$ in $H(x)$.

In the case $G = Z$ or $G = \mathbb{R}$, it is possible to express the random measure $Z(\Delta)$ in terms of the process $x(t)$ by means of an integral representation which is an "inversion" of (2.1.2). For example, if $\{x(t), t \in \mathbb{R}\}$ has spectral representation

$$x(t) = \int_{-\infty}^{\infty} e^{2\pi i\lambda t} Z(d\lambda)$$

for a stochastic measure Z, then if $Z(\{a\}) = Z(\{b\}) = 0$ a.e. the Z-measure of the interval $[a,b]$ is given by

$$Z([a,b]) = \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-2\pi ibt} - e^{-2\pi iat}}{-it} x(t)dt . \tag{2.1.3}$$

If $Z(\{a\}) \neq 0$ then

$$Z(\{a\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-2\pi iat} x(t)dt . \tag{2.1.4}$$

Our aim in the sequel is to give an expression similar to (2.1.4) for the Z-measure of a singleton subset $\{\alpha\}$ of $\hat{G}$. We have not been successful in obtaining an expression similar to (2.1.3) for the Z-measure of more general sets.

We first make a few remarks as to the definition of a type of stochastic integral. We wish to define an integral of the form $\int_G f(g)x(g)m(dg)$ where $f \in L_1(G)$. Let us assume that $\{x(g), g \in G\}$ is a weakly stationary measurable process. Then

$$E\int |f(g)||x(g)|m(dg) = \int_G E|x(g)||f(g)|m(dg)$$

$$\leq \int (E|x(g)|^2)^{\frac{1}{2}}|f(g)|m(dg)$$

$$= ||f||_{L_1}(E|x(e)|^2)^{\frac{1}{2}} < \infty .$$

Thus by Fubini's theorem the function $f(g)x(g,\omega)$ is in $L_1(G,m)$ a.e. [P] where we assume all the random variables $\{x(g), g \in G\}$ are defined on the probability space $(\Omega,\mathcal{B},P)$. Hence the integral

$$\xi(\omega) = \int_G x(g,\omega)f(g)m(dg)$$

defines a random variable except on a set of P-measure zero.

Since we have assumed $x(g)$ to be measurable, it follows from Theorem 2.2.4 that $x(g)$ has a spectral representation (2.1.2). We will need the following lemma.

Lemma 2.1.1. Let $f \in L_1(G)$ and let $\{x(g), g \in G\}$ be a measurable (and hence m.s.c.) weakly stationary process on G, with spectral representation (2.1.2). Then

$$\text{(2.1.5)} \qquad \int_G x(g)f(g)m(dg) = \int_{\hat{G}} \hat{f}(-\alpha)Z(d\alpha) \quad \text{a.e. [P].}$$

Proof: The integral on the right exists since $\hat{f}(\alpha)$ is bounded and hence in $L_2(\hat{G},\mu)$ when $d\mu = E|dZ|^2$. Now

$$E\left|\int_G x(g)f(g)m(dg)\right|^2 \leqq E\int_G\int_G \left|x(g)x(h)f(g)\overline{f(h)}\right| m(dh)m(dg)$$

$$\leqq \int_G\int_G (E|x(g)|^2 E|x(h)|^2)^{\frac{1}{2}} |f(g)|\,|f(h)|\,m(dh)m(dg)$$

$$= E|x(e)|^2 ||f||^2_{L_1} < \infty$$

so the integral on the left side of (2.1.5) has a second absolute moment. Thus

$$\text{(2.1.6)}\quad \left\{\begin{aligned} & E\left|\int_G x(g)f(g)m(dg) - \int_{\hat G} \hat f(-\alpha)Z(d\alpha)\right|^2 \\ & = E\left|\int_G x(g)f(g)m(dg)\right|^2 - 2\mathrm{Re}\; E\int_G x(g)f(g)m(dg)\overline{\int_{\hat G}\hat f(-\alpha)Z(d\alpha)} \\ & \quad + E\left|\int_{\hat G}\hat f(-\alpha)Z(d\alpha)\right|^2 . \end{aligned}\right.$$

Let us now compute the second and third terms on the right hand side of (2.1.6).

Consider first the second term. Let us write $\xi(\omega)$ for $\int_{\hat G} \hat f(-\alpha)Z(d\alpha)$. Then since

$$E\int \left|x(g,\omega)\right|\left|f(g)\right|\left|\xi(\omega)\right| m(dg) \leqq \int \left|f(g)\right| (E\left|x(g,\omega)\right|^2 E\left|\xi(\omega)\right|^2)^{\frac{1}{2}} m(dg) < \infty$$

we can apply Fubini's theorem and obtain

$$E\int_G x(g)f(g)m(dg)\overline{\xi(\omega)} = \int_G E(x(g,\omega)\overline{\xi(\omega)}\; f(g)m(d\omega)$$

$$= \int_G\int_{\hat G} \langle\alpha,g\rangle\overline{\hat f(-\alpha)}\mu(d\alpha)f(g)m(dg)$$

$$= \int_G \int_{\hat{G}} \langle \alpha , g \rangle \int_G \overline{f(h)}\,\overline{\langle \alpha , h \rangle}\; m(dh)\mu(d\alpha)f(g)m(dg)$$

$$= \int_G \int_G R(g-h)f(g)\overline{f(h)}m(dh)m(dg)$$

by another application of Fubini's theorem, which is justified since $f \in L_1(G)$ and μ is a finite measure.

Consider the third term of the right side of (2.1.6). Using Fubini's theorem again, the third term is equal to

$$\int_{\hat{G}} \left| f(-\alpha) \right|^2 \mu(d\alpha) = \int_{\hat{G}} \int_G \langle \alpha , g \rangle f(g)m(dg) \int_G \overline{\langle \alpha , h \rangle}\,\overline{f(h)}m(dh)\mu(d\alpha)$$

$$= \int_G \int_G f(g)\overline{f(h)} \int_{\hat{G}} \langle \alpha , g-h \rangle \mu(d\alpha)m(dh)m(dg)$$

$$= \int_G \int_G R(g-h)f(g)\overline{f(u)}m(dg)m(du)$$

so (2.1.6) is equal to zero and the lemma is proved. □

Remark 2.1.1. If the process $\{x(g), g \in G\}$ is weakly stationary and measurable, then the closed linear subspace $H(X)$ of $L_2(\Omega,\mathcal{B},P)$ generated by the $x(g)$ is separable by Theorem 2.2.4, and we may define stochastic integrals of the form $\int_V x(g)f(g)m(dg)$ for f continuous and V relatively compact as is done in Rozanov (1967) pp. 9-12. These are mean-square integrals in contrast to the path integrals we have used in Lemma 2.1.1. The Rozanov integrals satisfy a modified form of Lemma 2.1.1 - if V is relatively compact then

$$\int_V x(g)f(g)m(dg) = \int_{\hat{G}} \left\{ \int_V \langle\alpha,g\rangle f(g)m(d\alpha)\right\} Z(d\alpha).$$

We use Lemma 2.1.1 in the proof of Theorem 2.1.1, it is immaterial for the purpose of the proof which definition of the integral we use.

With these facts at our disposal, we can proceed to the proof of the inversion theorem, which can also be regarded as a law of large numbers.

Theorem 2.1.1. Suppose that $\{x(g),\ g \in G\}$ is a wide sense stationary measurable stochastic process over the LCA group G. If G is σ-compact, then there exist an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of relatively compact open sets in G such that

$$(2.1.7)\qquad Z(\{\alpha\}) = \lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} x(g)\langle\alpha,g\rangle m(dg) \qquad \text{(limit in mean square)}$$

and

$$(2.1.8)\qquad \mu(\{\alpha\}) = \lim_{n\to\infty} \frac{1}{(m(E_n)} \int_{E_n} R(g)\langle\alpha,g\rangle m(dg)\ .$$

Proof: Let $U(G)$ denote the class of almost periodic functions on G. (See Hewitt and Ross (1963) p. 247). Then there exists a unique linear functional M on $U(G)$ with the properties

(i) $M(f_g) = M(f)$ $\forall f \in U(G)$, $\forall g \in G$ where f_g is the function $f_g(h) = f(h-g)$.

(ii) $M(f) > 0$ if $f > 0$.

(iii) $M(1) = 1$.

The functional M is known as an invariant mean. In their book, Hewitt and Ross show that the unique functional M can be expressed in the following way. If $f \in U(G)$ is fixed, then it can be shown that there is a unique complex number ξ such that given $\varepsilon > 0$ there exist group elements $g_1,\ldots,g_n \in G$ depending on ε such that

$$\sup_{g\in G}\left|\xi - \frac{1}{n}\sum_{k=1}^{n} f(g+g_k)\right| < \varepsilon . \tag{2.1.9}$$

The number ξ is $M(f)$.

Now if we consider the class $U_c(G)$ of continuous almost periodic functions, an even more explicit expression for $M(f)$ is obtained by Hewitt and Ross. If G is σ-compact, there exists an increasing sequence $\{E_n\}$ of relatively compact open sets of G such that

$$M(f) = \lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} f(g)m(dg) \tag{2.1.10}$$

for all continuous f in $U(G)$. Note that the integral exists since $\overline{E_n}$ is compact and hence $m(E_n) < \infty$ and f is bounded on E_n. We now apply these facts to prove a lemma essential to the proof of Theorem 2.1.1.

<u>Lemma 2.1.2.</u> Let $\{E_n\}$ be the increasing sequence of sets introduced above. Then if $f_n(g) = \frac{1}{m(E_n)} \chi_{E_n}(g)$, where χ_{E_n} is the indicator function of the set E_n, the sequence $\{\hat{f}_n(\alpha)\}_{n=1}^{\infty}$ converges to zero for all $\alpha \neq e_{\hat{G}}$ (the identity of $\hat{G}$), and equals 1 if $\alpha = e_{\hat{G}}$.

<u>Proof</u>:

$$\hat{f}_n(e_{\hat{G}}) = \frac{1}{m(E_n)} \int_G \chi_{E_n}(g)m(dg) = \frac{m(E_n)}{m(E_n)} = 1$$

so $\hat{f}_n(e_{\hat{G}}) \to 1$ as $n \to \infty$. Assume now that $\alpha \neq e_{\hat{G}}$, then

$$\hat{f}_n(\alpha) = \frac{1}{m(E_n)} \int_{E_n} \langle\overline{\alpha,g}\rangle \, m(dg) .$$

Now $\overline{\langle\alpha,g\rangle}$ is an almost periodic function, so $\lim \hat{f}_n(\alpha) = M(\overline{\langle\alpha,\cdot\rangle})$. Thus it remains to prove that $M(\overline{\langle\alpha,\cdot\rangle}) = 0$; we do this by showing thay for any $\varepsilon > 0$ we can find group elements $g_1,\ldots,g_n$ such that

$$\sup_{g\in G}\left|\frac{1}{n}\sum_{k=1}^{n}\langle\alpha,g+g_k\rangle\right| < \varepsilon .$$

We adopt a result of Jajte (1967) to this end. Let $N(K,\delta)$ be a neighborhood of $e_{\hat{G}}$ in $\hat{G}$ with $\alpha \notin N(K,\delta)$,

$$N(K,\delta) = \{\beta \in \hat{G} : |\langle\beta,g\rangle - 1| < \delta \quad \forall g \in \text{compact } K\} .$$

That $N(K,\delta)$ is a neighborhood of $e_{\hat{G}}$ is proved in Rudin (1962) p. 10. Since $\alpha \notin N(K,\delta)$ there exists an element $g_1 \in K$ with

$$|\langle\alpha,g_1\rangle - 1| \geq \delta .$$

Set $S_n = \frac{1}{n}\sum_{\nu=1}^{n}\langle\alpha,\nu g_1\rangle$, where $\nu g_1 = g_1+\ldots+g_1$ (ν summands). Then

$$|S_n| \leqq \frac{1}{n}\left|\sum_{\nu=1}^{n}\langle\alpha,g_1\rangle^{\nu}\right| = \frac{1}{n}\left|\frac{\langle\alpha,g_1\rangle(\langle\alpha,g_1\rangle^{n}-1)}{\langle\alpha,g_1\rangle - 1}\right| \leqq \frac{2}{n\delta}$$

so $S_n \to 0$ as $n \to \infty$. Let n be such that $|S_n| < \varepsilon < 1$, then if $g_\nu = \nu g_1$

$$|S_n| = \left|\frac{1}{n}\sum_{\nu=1}^{n}\langle\alpha,g_\nu\rangle\right| < \varepsilon$$

But $\sup_{g}\left|\frac{1}{n}\sum_{\nu=1}^{n}\langle\alpha,g+g_\nu\rangle\right| = |S_n| < \varepsilon$, so by (2.1.9) $M(\overline{\langle\alpha,\cdot\rangle}) = 0$.

□

We may now proceed to the completion of the proof of Theorem 2.1.1. In Lemma 2.1.1 set $f(g) = \frac{1}{m(E_n)}\chi_{E_n}(g)\overline{\langle\alpha,g\rangle}$. Then $f \in L_1(G)$

and
$$\hat{f}(-\beta) = \frac{1}{m(E_n)} \int_{E_n} \langle\beta,g\rangle\overline{\langle\alpha,g\rangle}m(dg) = \hat{f}_n(\beta-\alpha) \ ,$$

where f_n is the function of Lemma 2.1.2. Then by Lemma 2.1.1 we have

$$\frac{1}{m(E_n)} \int_{E_n} x(g)\langle\alpha,g\rangle m(dg) = \int_{\hat{G}} \hat{f}_n(\beta-\alpha)Z(d\beta)$$

and so

$$E\left|\frac{1}{m(E_n)} \int_{E_n} x(g)\overline{\langle\alpha,g\rangle}m(dg) - Z(\{\alpha\})\right|^2 = E\left|\int_{\hat{G}} \hat{f}_n(\beta-\alpha)Z(d\beta) - \int_{\hat{G}} \chi_{\{\alpha\}}(\beta)Z(d\beta)\right|^2$$

$$= \int\left|\hat{f}_n(\beta-\alpha) - \chi_{\{\alpha\}}(\beta)\right|^2\mu(d\beta) \ . \qquad (2.1.11)$$

Now by Lemma 2.1.2 $\hat{f}_n(\beta-\alpha) \to \chi_{\{\alpha\}}(\beta)$ $\forall\beta \in \hat{G}$ as $n \to \infty$, and $|\hat{f}_n(\beta-\alpha) - \chi_{\{\alpha\}}(\beta)| \leqq 2$ $\forall\beta \in \hat{G}$ so by dominated convergence (2.1.11) converges to zero, and (2.1.7) is proved. Also

$$\begin{aligned}\mu(\{\alpha\}) &= \int_{\hat{G}} \chi_{\{\alpha\}}(\beta)\mu(d\beta) \\ &= \lim_{n\to\infty} \int_{\hat{G}} \hat{f}_n(\beta-\alpha) \ \mu(d\beta) \\ &= \lim_{n\to\infty} \frac{1}{(m(E_n)} \int_{E_n} R(g)\langle\alpha,g\rangle m(dg)\end{aligned}$$

which proves (2.1.8). □

We conclude this section with a few remarks on harmonizable processes. If a second order process $\{x(t), t \in \mathbb{R}\}$ over the real line has covariance $R(t,s)$ which can be written as

(2.1.12) $$R(t,s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(2\pi i(tu-sv))\mu(du,dv)$$

where μ is a complex measure on the plane of bounded variation, then x is said to be a harmonizable process. Such processes have spectral representations

(2.1.13) $$x(t) = \int_{-\infty}^{\infty} e^{2\pi it\lambda} Z(d\lambda)$$

where Z is a random measure satisfying

$$E\, Z(\Delta)Z(\Delta') = \mu(\Delta\times\Delta').$$

The integral (2.1.13) is defined by means of an isometry between $H(X)$ and the Hilbert space $\Lambda_2(\mu)$ consisting of all functions ϕ for which the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(u)\overline{\phi(v)}\mu(du,dv)$$

exists; $\Lambda_2(\mu)$ is a Hilbert space with inner product

$$(\psi,\phi) = \int_{\infty}^{\infty}\int_{\infty}^{\infty} \psi(u)\overline{\phi(v)}\mu(du,dv) \ .$$

The isometry in question is given by the correspondence $x(t) \leftrightarrow e^{2\pi itu}$. These facts may be found in Cramèr (1951).

Similar definitions can be made in the case of a group parameter. Specifically, we will say that $\{x(g),\ g \in G\}$ is harmonizable if its covariance $R(g,h)$ can be written

$$R(g,h) = \int_{\hat{G}}\int_{\hat{G}} \langle\alpha,g\rangle\langle\overline{\beta,h}\rangle\mu(d\alpha,d\beta)$$

where $\mu \in M(G\times G)$ (see the appendix).

The spectral representation of a harmonizable process is

$$x(g) = \int_{\hat{G}} \langle \alpha, g \rangle Z(d\alpha)$$

where $E\, Z(\Delta)Z(\Delta') = \mu(\Delta\times\Delta')$ for all Borel subsets Δ,Δ' of $\hat{G}$. We omit proofs; they are similar to the real line case.

Harmonizable processes over $\mathbb{R}$ will be used in Chapter III, section 3.

2.2 Decomposition of Stationary Covariances and Stationary Processes.

The spectral representations described in the previous section apply only to mean square continuous weakly stationary processes. It is thus of interest to determine how an arbitrary weakly stationary process can be decomposed into m.s.c. and "other" components. In this section we describe how an arbitrary process can be split up into two orthogonal parts, one being m.s.c. and the other with "almost" uncorrelated values. As a consequence of this, we will show that every measurable stochastic process is m.s.c., and thus has a spectral representation. We begin with a theorem of Hewitt and Ross (1970) p. 260; we give a slightly modified version of their proof in some detail because it will be needed in Chapter IV.

Theorem 2.2.1. Let ϕ be a measurable, positive definite function on G. Then ϕ has a unique decomposition $\phi = \phi_1+\phi_2$ where ϕ_1 is continuous and ϕ_2 locally a.e. zero, and ϕ_1 and ϕ_2 are positive definite.

Proof: Let $\mathbb{E}$ denote the vector space of all complex valued functions on G that are zero except at finitely many points of G. If ε_h

denotes the function on G given by

$$\varepsilon_h(g) = \begin{cases} 1 & g = h \\ 0 & g \neq h \end{cases}$$

then every element u of $\mathbb{E}$ has a canonical expression

$$u(g) = \sum_{i=1}^{n} c_i \varepsilon_{g_i}(g)$$

where $c_1, \ldots, c_n$ are complex numbers. Now define the map $p : \mathbb{E} \times \mathbb{E} \to C$ by

$$p\Big(\sum_{i=1}^{n} c_i \varepsilon_{g_i}, \sum_{j=1}^{m} c_j' \varepsilon_{g_i'}\Big) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i c_j' \phi(g_i - g_j') .$$

Let $\mathbb{E}_\phi = \{u \in \mathbb{E} : p(u,u) = 0\}$. It can be shown that $\mathbb{E}_\phi$ is a subspace of $\mathbb{E}$, and that the map $\langle u+\mathbb{E}_\phi, v+\mathbb{E}_\phi\rangle \to p(u,v)$ is an inner product on $\mathbb{E}/\mathbb{E}_\phi$. Let H_ϕ denote the Hilbert space obtained by completing $\mathbb{E}/\mathbb{E}_\phi$ with respect to this inner product. Now define a map $V_g : \mathbb{E}/\mathbb{E}_\phi \to \mathbb{E}/\mathbb{E}_\phi$ by $V_g(u+\mathbb{E}_\phi) = u_g+\mathbb{E}_\phi$, where $u_g(h) = u(g+h)$. V_g is linear and onto, and

$$||V_g(u+\mathbb{E}_\phi)||^2 = p(u_g,u_g) = p(u,u) = ||u+\mathbb{E}_\phi||^2 .$$

Thus V_g is an isometry, and can be extended to an isometry $H_\phi \to H_\phi$. Moreover, $V_g^* = V_{-g}$ and $V_{g+h} = V_g V_h$, and for every $\xi \in H_\phi$ the map $g \to (V_g\xi,\xi)_{H_\phi}$ is positive definite since for scalars $c_1, \ldots, c_n$ and $g_1, \ldots, g_n \in G$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} (V_{g_i - g_j}\xi,\xi) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} (V_{g_i}\xi, V_{g_j}\xi)$$

$$= ||\sum_{i=1}^{n} c_i V_{g_i}\xi||^2 \geq 0 .$$

Also, if $\xi,\eta \in H_\phi$, by an approximation argument and using the fact that ϕ is measurable, it can be shown that the map $g \to (V_g\xi,\eta)$ is measurable. Now if $\in L_1(G)$

$$\int_G \left|f(g)(V_g\xi,\eta)\right| m(dg) \leqq \int_G \left|f(g)\right| \, ||V_g\xi|| \, ||\eta|| m(dg)$$

$$\leqq ||\xi|| \, ||\eta|| \int_G |f(g)| m(dg) < \infty$$

so we may define a bounded linear operator T_f on H_ϕ by

$$(T_f\xi,\eta) = \int_G f(g)(V_g\xi,\eta)m(dg) \ .$$

Consider the map $g \to (V_gT_f\xi,\eta)$. Then

$$|V_gT_f \, \xi,\eta) - (V_hT_f\xi,\eta)| = |(T_f\xi,V_{-g}\eta - V_{-h}\eta)|$$

$$= |\int_G (V_{g'}\xi,V_{-g}\eta - V_{-h}\eta)f(g')m(dg)|$$

$$= |\int_G (V_{g'}\xi,\eta)(f(g'-g) - f(g'-h))m(dg)|$$

$$\leqq ||\xi|| \, ||\eta|| \, ||f_g-f_h||_{L_1(G)} \quad \text{where} \quad f_g(g') = f(g'-g).$$

By a theorem in Rudin (1963) p. 3, given $\varepsilon > 0$ there exists a neighborhood V of e in G such that if $g-h \in V$ then $||f_g-f_h||_{L_1(G)} < \varepsilon$. Hence the map $g \to (V_gT_f\xi,\eta)$ is uniformly continuous. Now let S denote the closed linear span of $\{T_f\xi : f \in L_1(G), \xi \in H_\phi\}$. It can be shown that $V_gT_f = T_{f_g}$ so $V_g(S) \subseteq S \ \forall_g \in G$. Also $V_g^*(S) \subseteqq S$ so $V_g(S^\perp) \subseteqq S^\perp$ i.e. V_g reduces $S \ \forall_g \in G$. If $\xi \in S$, then the map

$g \to (V_g\xi,\eta)$ is continuous since it is the uniform limit of maps of the form

$$g \to \sum_{i=1}^{n} c_i (V_g T_{f_i} \xi_i,\eta)$$

which are continuous as proved above. If $\xi \in S^\perp$ and $\eta \in S$, then

$$(T_f\xi,\eta) = \int_G (V_g\xi,\eta)f(g)m(dg) = \int_G 0\ f(g)m(dg) = 0$$

so $T_f\xi \in S^\perp$. Thus $T_f\xi = 0$ since $T_f\xi \in S$. Thus for all $\xi \in S^\perp$, $\eta \in H_\phi$, $\forall f \in L_1(G)$

$$\int_{\hat{G}} (V_g\xi,\eta)f(g)m(dg) = 0$$

so $(V_g\xi,\eta) = 0$ locally a.e. [m].

Now let $\xi = \varepsilon_e + \mathbb{E}_\phi$. Then $(V_g\xi,\xi) = \phi(g)$. Let $\xi = \xi_1+\xi_2$ with $\xi_1 \in S$ and $\xi_2 \in S^\perp$. Then

$$\begin{aligned}\phi(g) = (V_g\xi,\xi) &= (V_g\xi_1,\xi_1) + (V_g\xi_2,\xi_2)\\ &= \phi_1(g) + \phi_2(g) \quad \text{say.}\end{aligned}$$

From the above remarks it is clear that ϕ_1 and ϕ_2 are positive definite, ϕ_1 is continuous and ϕ_2 is zero locally a.e. If $\phi = \phi_1'+\phi_2'$ is another representation with the desired properties, then $\phi_1-\phi_1' = \phi_2'-\phi_2$. The left side of this equation is continuous, the right side is zero locally a.e. so that both are zero i.e. the representation $\phi = \phi_1+\phi_2$ is unique. □

Corollary 2.2.1. If the Hilbert space H_ϕ has countable dimension, then $\phi_2 = 0$.

<u>Proof</u>: If H_ϕ has countable dimension, then $S^\perp$ has a countable dense set $\{\xi_n\}_{n=1}^\infty$. For each pair (m,n) there is a locally null set $A_{m,n}$ with $(V_g\xi_m,\xi_n) = 0$ for $g \notin A_{mn}$. Let $A = \underset{m,n}{\cup} A_{m,n}$. Then A is locally null and $(V_g\xi_m,\xi_n) = 0$ for all m,n, all $g \notin A$. Thus for all $\xi,\eta \in S^\perp$, $(V_g\xi,\eta) = 0$ $\forall_g \notin A$, so $V_g\xi \in S^\perp \cap S = \{0\}$ $\forall_g$ A. Thus $||\xi|| = ||V_g\xi|| = 0$ $\forall\xi \in S^\perp$ and so $S^\perp = \{0\}$ i.e. $S = H_\phi$ and $\phi_2 = 0$, and ϕ is continuous. □

We can now state and prove the corresponding decomposition for a weakly stationary process over an LCA group G. As before $H(x)$ denotes the closed subspace of $L_2(\Omega,\mathcal{B},P)$ spanned by $\{x(g) : g \in G\}$.

<u>Theorem 2.2.2.</u> Let $\{x(g), g \in G\}$ be a weakly stationary stochastic process over the LCA group G. If ϕ, the covariance function of $x(g)$, is measurable, then $x(g)$ can be expressed uniquely as the sum of two processes x_1 and x_2 such that

(a) $E\, x_1(g)\overline{x_2(h)} = 0$ $\forall g,h \in G$.

(b) x_1 is stationary with continuous covariance ϕ_1.

(c) x_2 is stationary with covariance ϕ_2 that is zero locally a.e. $[m]$.

(d) $H(x_1) \oplus H(x_2) = H(x)$.

If $H(x)$ is separable then $x_2 = 0$, i.e. x is m.s.c.

<u>Proof:</u> Consider a map Φ from the linear span of the r.v's $x(g)$ to H_ϕ defined by

$$\Phi\Big(\sum_{i=1}^{n} \alpha_i x(g_i)\Big) = \sum_{i=1}^{n} \alpha_i \varepsilon_{g_i} + \mathbb{E}_\phi .$$

Φ is well defined and onto $\mathbb{E}/\mathbb{E}_\phi$, moreover

$$E\left|\sum_{i=1}^{n}\alpha_i x(g_i)\right|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\phi(g_i-g_j) = p\left(\sum\alpha_i\varepsilon_{g_i},\sum\alpha_i\varepsilon_{g_i}\right)$$

$$= \left|\left|\sum\alpha_i\varepsilon_{g_i} + \mathbb{E}_\phi\right|\right|^2$$

so Φ preserves inner products. Thus Φ may be extended to an isometry between $H(x)$ and H_ϕ.

Define $x_i(g) = \Phi^{-1}V_g\xi_i$ where $\varepsilon_e + \mathbb{E}_\phi = \xi = \xi_1+\xi_2$ with $\xi_1 \in S$, $\xi_2 \in S^\perp$. Then

(i) $E\, x_1(g)\overline{x_2(h)} = (V_g\xi_1,V_h\xi_2) = 0$ since V_g reduces S, which proves (a).

(ii) $E\, x_i(g)\overline{x_i(h)} = (V_g\xi_i,V_h\xi_i) = \phi_i(g-h)$ which proves (b) and (c).

(iii)
$$\begin{aligned} x_1(g) + x_2(g) &= \Phi^{-1}(V_g\xi_1) + \Phi^{-1}(V_g\xi_1) \\ &= \Phi^{-1}(V_g\xi) \\ &= \Phi^{-1}(\varepsilon_g + \mathbb{E}_\phi) \\ &= x(g) \end{aligned}$$

which proves the decomposition.

(iv) (d) follows from (i) and (iii).

It is also true that $S = \overline{\text{sp}}\{V_g\xi_1 : g \in G\}$, $S^\perp = \overline{\text{sp}}\{V_g\xi_2 : g \in G\}$. By virtue of the isometry, if $H(x)$ has countable dimension so does H_ϕ and so $S^\perp = \{0\}$ whence $x_2 = 0$. Lastly, we show that the decomposition is unique. Suppose that $x_1'(g) + x_2'(g)$ is another decomposition of x with the properties (a) - (d). From the uniqueness of the covariance decomposition, the covariances of x_i' are just ϕ_i. Let U_1 be the map $V_g\xi_1 \to x_1'(g)$, U_2 the map $V_g\xi_2 \to x_2'(g)$ U_1 and U_2 can be extended to isometries between S and $H(x_1')$ and $S^\perp$ and $H(x_2')$. For $\eta \in H_\phi$, let U be the map given by $U(\eta) = U_1(\eta_1) + U_2(\eta_2)$ where η_1 and η_2 are the projections of η onto S and $S^\perp$. U is an isometry between H_ϕ and $H(x)$, $U|S = U_1$ and $U|S^\perp = U_2$. Now

$$U(V_g\xi) = U_1V_g\xi_1 + U_2V_g\xi_2 = x_1'(g) + x_2'(g) = x(g)$$

so $\Phi = U^{-1}$. Thus $x_1'(g) = U(V_g\xi_1) = \Phi^{-1}(V_g\xi_1) = x_1(g)$. Similarly $x_2'(g) = x_2(g)$. □

We now apply these results to prove that every measurable stationary process over G is mean square continuous. Let $\{x(g), g \in G\}$ be any second order process over G, not necessarily weakly stationary, with each random variable $x(g)$ defined on the same probability space $(\Omega,\mathcal{B},P)$. Let $\mathcal{M}$ denote the vector space of all complex random variables on Ω. $\mathcal{M}$ is a metric space with metric

$$d(X,Y) = E(|X-Y|/1+|X-Y|) ;$$

d metrizes the topology of convergence in probability.

We will need an unpublished result of Hoffmann-Jorgensen (see also Cohn (1972)). We give Hoffman-Jorgensen's proof.

Theorem 2.2.3. If the stochastic process $\{x(g),g \in G\}$ is measurable, then the set $M = \{x(g):g \in G\}$ is a d-separable subset of $\mathcal{M}$.

Proof: Let S denote the class of all stochastic processes over G such that M above is separable. We will show that every measurable process is in the class S. To this end, let x_1 and $x_2 \in S$, and $A \in \mathcal{B}(G) \times \mathcal{B}$ where $\mathcal{B}(G)$ denotes the Borel σ-field of G and $\mathcal{B}$ is the σ-field of the probability space $(\Omega,\mathcal{B},P)$ on which all r.v.'s are assumed to be defined. Set

$$Y_A(g,\omega) = \begin{matrix} x_1(g,\omega) & (g,\omega) \in A \\ x_2(g,\omega) & (g,\omega) \notin A . \end{matrix}$$

Let $\Sigma = \{A \in \mathcal{B}(G) \times \mathcal{B} : Y_A \in S\}$. Since S is closed under pointwise limits, Σ is a monotone class. Now suppose $A = \bigcup_{j=1}^{n} B_j \times C_j$ $B_j \in \mathcal{B}(G)$, $C_j \in \mathcal{B}$ where the rectangles $B_j \times C_j$ are disjoint. Choose a measurable partition $S_1,\dots,S_k$ of G and subsets $T_1,\dots,T_k \in \mathcal{B}$ such that

$$A = \bigcup_{k=1}^{m} S_k \times T_k \quad \text{and} \quad CA = \bigcup_{k=1}^{m} S_k \times CT_k .$$

Then

$$Y_A(g,\omega) = \begin{cases} x_1(g,\omega) & g \in S_k \text{ and } \omega \in T_k \quad k=1,\dots,m \\ x_2(g,\omega) & g \in S_k \text{ and } \omega \notin T_k \quad k=1,\dots,m \end{cases}$$

Suppose $g \in S_k$, $\varepsilon > 0$, then there are r.v.'s η_1 and η_2 belonging to the countable sets N_1 and N_2 which are dense in $\{x_1(g) : g \in G\}$ and $\{x_2(g) : g \in G\}$ such that

$$d(x_i(g),\eta_i) < \varepsilon/2 \qquad i=1,2 .$$

Define

$$\eta(\omega) = \begin{cases} \eta_1(\omega) & \omega \in \bigcup_{k=1}^{m} T_k \\ \eta_2(\omega) & \omega \notin \bigcup_{k=1}^{m} T_k \end{cases}$$

Then

$$(2.2.1) \qquad d(Y_A(g),\eta) \leq d(x_1(g),\eta_1) + d(x_2(g),\eta_2) < \varepsilon .$$

Since $\mathcal{N}_1$ and $\mathcal{N}_2$ are countable, the family of r.v.'s η of the type defined above is countable; (2.2.1) shows that the set of such r.v.'s η is dense, and so $Y_A \in \mathcal{S}$ whenever A is of the form $\bigcup_{j=1}^{n} B_j \times C_j$ with the rectangles $B_j \times C_j$ disjoint. Thus Σ is a monotone class containing the ring which generates $\mathcal{B}(G) \times \mathcal{B}$, it follows that $\mathcal{B}(G) \times \mathcal{B} \subseteq \Sigma$. By an induction argument it follows that if $x_1,\ldots,x_n$ are in $\mathcal{S}$ and $A_1,\ldots,A_n$ is a partition of $G \times \Omega$ with each $A_i \in \mathcal{B}(G) \times \mathcal{B}$, then the process y given by

$$y(g,\omega) = x_i(g,\omega) \quad \text{for } (g,\omega) \in A_i$$

is a process in the class $\mathcal{S}$ i.e. every $\mathcal{B}(G) \times \mathcal{B}$ simple function is in $\mathcal{S}$. Since $\mathcal{S}$ is closed under pointwise limits and every measurable process is the pointwise limit of $\mathcal{B}(G) \times \mathcal{B}$ simple functions, it follows that every measurable process in in the class $\mathcal{S}$. $\square$

We now apply this result to prove that every measurable weakly stationary process is m.s.c.

Theorem 2.2.4. Every measurable weakly stationary process $\{x(g), g \in G\}$ over an LCA group G is mean square continuous, and thus has a spectral representation.

Proof: By Theorem 2.2.2 it is enough to prove that $H(x)$ is separable, i.e. has countable dimension. By Theorem 2.2.3 there exists a countable subset G_0 of G such that the set $\{x(g) : g \in G_0\}$ is d-dense in set $\{x(g) : g \in G\}$. Thus if L denotes the linear manifold generated by $\{x(g) : g \in G_0\}$, $\overline{L}$ is separable. We will show that $\overline{L} = H(X)$. It is clear that $\overline{L} \subseteq H(X)$; to show the reverse inequality let $g \in G$. Then there exists a sequence $\{g_n\}_{n=1}^{\infty}$ in G_0 such that $x(g)$ is the limit in probability of the sequence $\{x(g_n)\}_{n=1}^{\infty}$, and so a subsequence $x(g_n')$ converges to $x(g)$ a.e. [P]. Moreover $E|x(g_n')|^2 = R(e)$ so the sequence $x(g_n')$ is uniformly norm-bounded in n. By Theorem 13.44 of Hewitt and Stromberg (1965) p. 207, $x(g_n')$ converges weakly to $x(g)$ in $H(x)$. By II 3.27 of Dunford and Schwarz (1958) p. 68, $x(g)$ is in the closed linear manifold generated by the $x(g_n')$ i.e. $x(g) \in \overline{L}$. It follows that $\overline{sp}(x(g) : g \in G) \subseteq \overline{L}$ i.e. $H(x) \subseteq \overline{L}$. Hence $H(x)$ is separable and $\{x(g) : g \in G\}$ is m.s.c. □

<u>An alternative proof to Theorem 2.2.4</u>: We present below a direct proof of Theorem 2.2.4 which is an adaptation of a result of M.M. Crum (1956).

Let ϕ be the covariance function of $x(g)$ and $\phi = \phi_1 + \phi_2$ its decomposition. Let $f \in L_2(G)$ be a positive function. Then since

$$E\int |f(g)|^2 |x(g,\omega)|^2 m(dg) = \int |f(g)|^2 \phi(e) m(dg) < \infty$$

it follows by Fubini's theorem that $f(g)x(g,\omega) \in L_2(G)$ a.e. [P].

Define random variables $\eta(h,\omega)$ by

$$\eta(h,\omega) = \int_G |f(g)x(g,\omega) - f(g+h)x(g+h,\omega)|^2 m(dg) .$$

If θ denotes the Haar measure on $\hat{G}$, then by Plancherel's theorem

$$\eta(h,\omega) = \int_{\hat{G}} |1-\langle\alpha,g\rangle|^2 |F(\alpha,\omega)|^2 \theta(d\alpha) ,$$

where $F(\alpha,\omega)$ is the L_2 Fourier transform of the function $f(g)x(g,\omega)$. Moreover,

$$\int_{\hat{G}} E|F(\alpha,\omega)|^2\theta(d\alpha) = E\int_{\hat{G}} |F(\alpha,\omega)|^2\theta(d\alpha) = E\int |f(g)x(g,\omega)|^2 m(dg)$$

$$= ||f||^2_{L_2(G)}\phi(e) < \infty \quad ,$$

so

$$E|F(\alpha,\omega)|^2 = H(\alpha) \quad \text{say,}$$

is a function in $L_1(G)$. Thus

$$E|\eta(h,\omega)|^2 = \int_{\hat{G}} |1-\langle\alpha,h\rangle|^2 H(\alpha)\theta(d\alpha).$$

Another expression for $E|\eta(h,\omega)|^2$ is

$$E|\eta(h,\omega)|^2 = E\ 2\int_{\wedge} \{|f(g)|^2|x(g,\omega)|^2$$

$$- \operatorname{Re}\ f(g+h)f(g)x(g+h,\omega)x(g,\omega)\}m(dg)$$

$$= C_1\ \phi(e) - C_2 \operatorname{Re}\ \phi(h)$$

$$= C_1\phi_1(e) - C_2\operatorname{Re}\phi_1(h) + C_1\phi_2(e) - C_2\operatorname{Re}\phi_2(h)$$

for positive constants C_1 and C_2. Now since $H(\alpha) \in L_1(\hat{G})$ it follows that $\int_{\hat{G}}|1-\langle\alpha,h\rangle|^2 H(\alpha)\theta(d\alpha)$ is a continuous function of h, and since ϕ_1 is continuous we deduce that $\operatorname{Re}\ \phi_2(h)$ is continuous. But by a result of Rudin (1962) p. 18

$$|\phi_2(h)-\phi_2(h')| \leq 2\ {}_2(e)(\operatorname{Re}\ \phi_2(e) - \operatorname{Re}\ \phi_2(h-h'))$$

so ϕ_2 is continuous. Since ϕ_2 is zero locally a.e., it follows that ϕ_2 is zero and hence ϕ is continuous. □

2.3 Estimation of the Covariance Function and the Strong Law of Large Numbers.

Our goal in this section is to discuss an estimator of the covariance function of a stationary process over an LCA group G of special type, and establish some of the properties of such an estimator. This work generalizes results known for the cases $G = \mathbb{R}$ and $G = Z$, which may be found, for example, in Doob (1953) Chapter X §7, Chapter XI §7. A key result is Theorem 2.3.1, which asserts that under certain conditions integral averages of the form (2.1.7) converge almost surely. Theorem 2.1.1 can be regarded as an ergodic theorem in mean square or a weak law of large numbers, while Theorem 2.3.1 is the corresponding strong law of large numbers. Theorem 2.3.1 is needed to establish the properties of our covariance estimator.

Throughout this section, we will assume that our indexing group G has the following property:

Property 2.3.1. There exists in G a relatively compact neighborhood U of e, and an element $g_0 \in G$ such that $m(U) = 1$ and if $U_n = ng_0 + U$, then $U_n \cap U_m = \phi$ $m \neq n$ and $\bigcup_{n=-\infty}^{\infty} U_n = G$.

For example $\mathbb{R}$ has this property: we may take $U = [-\frac{1}{2},\frac{1}{2})$ and get $g_0 = 1$. Also Z has the property: take $U = \{0\}$ and $g_0 = 1$.

Definition 2.3.1. A stochastic process $\{x(g) : g \in G\}$ is strictly stationary if for any finite subset $\{g_1,\ldots,g_n\}$ of G the joint distribution of $(x(g_1),\ldots,x(g_n))$ is the same as $(x(g_1+h),\ldots,x(g_n+h))$ for any $h \in G$.

Before we can formulate our strong law of large numbers for strictly stationary processes, we must first discuss the canonical form of a stochastic process and introduce a certain transformation. Suppose all the random variables of our process $\{x(g),\ g \in G\}$ are defined on a probability space $(\Omega,\mathcal{B},P)$. Consider the cartesian product space C^G of all functions $G \to C$. Let $\mathcal{B}(G^C)$ denote the σ-algebra of subsets of C^G generated by the "cylinder sets" of the form

$$\{\phi \in C^G : (\phi(g_1),\ldots,\phi(g_n)) \in B_n\}$$

where B_n is a Borel subset of C^n. It is well known (see e.g. Gikhman and Skorokhod (1969), p. 108) that there exists a probability measure Q on $(C^G,\mathcal{B}(G^C))$ and a stochastic process $\{\tilde{x}(g), g \in G\}$ on the probability space $\{C^G,\mathcal{B}(G^C),Q\}$ given by $\tilde{x}(g)(\phi) = \phi(g)$ such that the finite dimensional distributions of x coincide with those of $\tilde{x}$. $\tilde{x}$ is called the canonical representation of x. Moreoever, if $\Psi : \Omega \to C^G$ is the map $\omega \to x(\cdot,\omega)$ then Ψ is a measurable transformation and $Q = P\Psi^{-1}$. The following lemma will be useful.

<u>Lemma 2.3.1.</u> With the notation of the previous remarks, let f_n be a sequence of r.v.'s on $(\Omega,\mathcal{B},P)$ and let $\tilde{f}_n$ be r.v.'s on $\{C^G,\mathcal{B}(C^G),Q\}$ such that $\tilde{f}_n(\psi(\omega)) = f_n(\omega)$. Then $\tilde{f}_n \to 0$ a.e. [Q] iff $f_n \to 0$ a.e. [P].

<u>Proof</u>: Let $A = \{\phi \in C^G : \lim\limits_{n\to\infty} \tilde{f}_n(\phi) = 0\}$. Then

$$A = \bigcap_{\substack{r>0 \\ r \text{ rational}}} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{\phi : |\tilde{f}_k(\phi)| < r\}$$

so $A \in \mathcal{B}(C^G)$ since $\{\phi : -r < \tilde{f}_k(\phi) < r\}$ is a cylinder set. Hence

$$\begin{aligned} Q(A) = P\Psi^{-1}(A) &= P\{\omega \in \Omega : \Psi(\omega) \in A\} \\ &= P\{\omega \in \Omega : \tilde{f}_n\Psi(\omega) \to 0\} \\ &= P\{\omega \in \Omega : f_n(\omega) \to 0\} \qquad \square \end{aligned}$$

Now let T_g be the transformation $C^G \to C^G$ given by

$$T_g(\phi) = \phi_g \quad \text{where } \phi_g(h) = \phi(h+g).$$

T_g is a measurable transformation, and if $x(g)$ is a strictly stationary process, then $Q(A) = Q(T_g(A))$ for every $A \in \mathcal{B}(C^G)$. Note also that $T_{-g} = T_g^{-1}$.

Definition 2.3.2. A set $A \in \mathcal{B}(C^G)$ is said to be T_{g_0} -invariant if $Q(T_{g_0}A\Delta A) = 0$ where g_0 is the element of G in property 2.3.1. It is easy to see that the class of all T_{g_0} invariant sets is a sub-σ-algebra of $\mathcal{B}(C^G)$. This σ-algebra will be denoted by $\mathcal{A}$. We shall also denote by $\mathcal{A}^*$ the class of sets $\{\psi^{-1}(A) : A \in \mathcal{A}\}$; $\mathcal{A}^*$ is a sub-σ-algebra of $\mathcal{B}$. Note that A is T_{g_0} invariant $\Longleftrightarrow$ A is T_{-g_0} invariant. We can now state our strong law of large numbers.

Theorem 2.3.1. Let $\{x(g), g \in G\}$ be a strictly stationary measurable process over the LCA group G which has property 2.3.1. Let $E_n = \bigcup_{k=-n}^{n} U_n$; then if $E|x(e)| < \infty$

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} x(g,\omega)m(dg) = E(x(e)|\mathcal{A}^*) \quad \text{a.e. } [P]. \tag{2.3.1}$$

Proof: The proof will consist of two lemmas.

Lemma 2.3.2. Let $f \in L_1(C^G,\mathcal{B}(C^G),Q)$. Then there exists an $\mathcal{A}$-measurable function $f^* \in L_1(C^G,\mathcal{B}(C^G),Q)$ such that $f^*(T_{g_0}\phi) = f^*(T_{-g}\phi) = f^*(\phi)$ a.e. [Q], and

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} f(T_h(\phi))m(dh) = f^*(\phi) \quad \text{a.e. } [Q]$$

and

$$\int_A f(\phi)Q(d\phi) = \int_A f^*(\phi)Q(d\phi) \quad \forall A \in \mathcal{A} .$$

__Proof__: Let $F(\phi) = \int f(T_h(\phi))m(dh)$. Then

$$\int_{C^G} |F(\phi)|Q(d\phi) \leq \int_{C^G} \int_U |f(T_h(\phi))|m(dh)Q(d\phi)$$

$$= \int_U \int_{C^G} |f(\phi)|QT_h^{-1}(d\phi)m(dh) \quad \text{by change of variable and Fubini's theorem}$$

$$= \int_U \int_{C^G} |f(\phi)|Q(d\phi)m(dh)$$

$$= ||f||_{L_1} m(U) < \infty$$

so $F(\phi)$ exists and is in $L_1(C^G,\mathcal{B}(C^G),Q)$.

By a trivial corollary of the Birkhoff ergodic theorem (see e.g. Halmos (1956))

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} F(T^k_{g_0}\phi) = f^*(\phi) \quad \text{a.e.}[Q]$$

for some f^* in $L_1(C^G, (C^G),Q)$, such that $\forall A \in \mathcal{A}, \int_A F(\phi)Q(d\phi) = \int_A f^*(\phi)Q(d\phi)$ and $f^*(T_g\phi) = f^*(T_{-g}\phi) = f^*(\phi)$ a.e. [Q]. This last statement implies that f^* is $\mathcal{A}$ measurable. But

$$m(E_n) = \sum_{k=-n}^{n} m(U_k) = \sum_{k=-n}^{n} m(kg_0+U) = 2n+1$$

and

$$\sum_{k=-n}^{n} F(T^{k}_{-g_0}) = \sum_{k=-n}^{n}\int_U f(T_h T_{-kg_0}\phi)m(dh)$$

$$= \sum_{k=-n}^{n}\int_{kg_0+U} f(T_h\phi)m(dh)$$

$$= \int_{E_n} f(T_h\phi)m(dh) \ ,$$

so $\lim\limits_{n\to\infty} \frac{1}{m(E_n)}\int_{E_n} f(T_h\phi)m(dh) = f^*(\phi)$ a.e. [Q].

Also,

$$\int_A F(\phi)Q(d\phi) = \int_A\int_U f(T_h\phi)m(dh)Q(d\phi)$$

$$= \int_U\int_A f(T_h\phi)Q(d\phi)m(dh)$$

$$= \int_U\int_A f(\phi)QT_h^{-1}(d\phi)m(dh)$$

$$= \int_U\int_A f(\phi)Q(d\phi)m(dh)$$

$$= m(U)\int_A f(\phi)Q(d\phi)$$

$$= \int_A f(\phi)Q(d\phi)$$

so

$$\int_A f(\phi)Q(d\phi) = \int f^*(\phi)Q(d\phi) \quad \forall A \in \mathcal{A} \ . \qquad \Box$$

<u>Lemma 2.3.3.</u> With the notation of the preceeding page,

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} \tilde{x}(h,\phi)m(dh) = f^*(\phi) \quad \text{a.e. } [Q]$$

provided that $E|x(e)| < \infty$.

Proof: In Lemma 2.3.2, set $f(\phi) = \phi(e)$. Then

$$\int_{C^G} |f(\phi)|Q(d\phi) = \int_{\Omega} |f(\psi(\omega))|P(d\omega)$$

$$= \int_{\Omega} |x(e,\omega)|P(d\omega)$$

$$= E|x(e)| \; .$$

Thus if $E|x(e)| < \infty$ we can apply Lemma 2.3.2 with $f(\phi) = \phi(e)$ and obtain the result, since

$$f(T_g\phi) = T_g\phi(e) = \phi(e+g) = \phi(g) = \tilde{x}(g,\phi). \qquad \square$$

The proof of Theorem 2.3.1 is completed by setting

$$f_n(\omega) = \frac{1}{m(E_n)} \int_{E_n} x(h,\omega)m(dh) - \varepsilon(x(e)|\mathcal{A}^*) \;,$$

and

$$\tilde{f}_n(\phi) = \frac{1}{m(E_n)} \int_{E_n} \tilde{x}(h,\phi)m(dh) - f^*(\phi),$$

and noting that since the process $x(g,\omega)$ is measurable, $f_n(\omega)$ is measurable, also

$$\tilde{x}(h,\Psi(\omega)) = \Psi(\omega)(h) = x(h,\omega), \quad \text{and}$$

$$f^*(\Psi(\omega)) = \varepsilon(x(e)|\mathcal{A}^*) \quad \text{since for all } A \in \mathcal{A}$$

$$\begin{aligned}\int_{\Psi^{-1}(A)} f^*(\Psi(\omega))dP &= \int_A f^*(\phi)P\Psi^{-1}(d\phi)\\ &= \int_A f^*(\phi)Q(d\phi)\\ &= \int_A \phi(e)Q(d\phi)\\ &= \int_{\Psi^{-1}(A)} \Psi(\omega)(e)P(d\omega)\\ &= \int_{\Psi^{-1}(A)} x(e,\omega)P(d\omega)\ .\end{aligned}$$

Hence $\tilde{f}_n(\Psi(\omega)) = f_n(\omega)$. By Lemma 2.3.3 $\tilde{f}_n(\phi) \to 0$ a.e. [Q], and thus by Lemma 2.3.1 $f_n(\omega) \to 0$ a.e. [P], which proves (2.3.1). □

Definition 2.3.3. Let $\{x(g),\ g \in G\}$ be a stochastic process over the LCA group G. With the notation already established, the process x is said to be metrically transitive if for all $A \in \mathcal{A}$, either $Q(A) = 1$ or $Q(A) = 0$.

With this definition we can state the following corollary to Theorem 2.3.1.

Corollary 2.3.1. Let $\{x(g),\ g \in G\}$ be a strictly stationary measurable process with $E|x(e)| < \infty$. If in addition $\{x(g),\ g \in G\}$ is metrically transitive, then

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} x(g,\omega)m(dg) = Ex(e) \quad \text{a.e. } [P]. \tag{2.3.2}$$

Proof: Since f^* is A-measurable, if $x(g)$ is metrically transitive f^* is a constant a.e. [Q], and thus $\varepsilon(x(e)|A^*)$ is a constant a.e. [P], i.e. $\varepsilon(x(e)|A^*) = E(x(e))$ a.e. [P]. □

We now turn to the problem of covariance estimation. Suppose that $\{x(g), g \in G\}$ is a weakly stationary stochastic process over the LCA group G which has Property 3.2.1. By generalizing from the known results in the cases $G = Z$ or $\mathbb{R}$, a natural estimate for the covariance $R(g)$ of the process $x(g)$ is

$$\hat{R}_n(g') = \frac{1}{m(E_n)} \int_{E_n} x(g+g')x(g)m(dg) \quad . \tag{2.3.3}$$

We conclude this section by establishing a few properties of the estimator (2.3.3), generalizing results of Doob (1953) Ch. X §§7. our first result in this direction concerns the mean square limit of the sequence of estimators (2.3.3).

Theorem 2.3.2. Let $\{x(g), g \in G\}$ be a weakly stationary measurable process over the LCA group having Property 2.3.1. Suppose in addition that the sets E_n satisfy

$$\lim_{n\to\infty} \frac{m(E_n \cap (g+E_n))}{m(E_n)} = 1 \quad \forall g \in G \ . \tag{2.3.4}$$

Also suppose that if $X(g) = x(g+g')x(g)$ for fixed g' then the process $Y(g) = X(g) - E(X(g)) = X(g) - R(g')$ is a weakly stationary process. Then

$$\underset{n\to\infty}{\text{l.i.m.}} \frac{1}{m(E_n)} \int_{E_n} x(g'+g,\omega)x(g',\omega)m(dg) \tag{2.3.5}$$

exists in mean square and the limit (2.3.5) is equal to $R(g')$ with probability 1 iff

(2.3.6) $$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} E(X(g)\overline{X(e)})m(dg) = |R(g')|^2 .$$

<u>Proof</u>: The sets E_n are relatively compact, increasing and their union is G. If they satisfy (2.3.4) then they form a sequence such that equation (2.1.10) is satisfied. (Hewitt and Ross (1963) p. 255) Hence, if Z_Y is the orthogonal random measure in the spectral representation of the process $Y(g)$, ($Y(g)$ has a spectral representation since it is measurable and hence m.s.c.), by Theorem 2.1.1 we have

$$\underset{n\to\infty}{\text{l.i.m.}} \frac{1}{m(E_n)} \int_{E_n} Y(g)m(dg) = Z_Y(\{e\}) .$$

Hence $$\underset{n\to\infty}{\text{l.i.m.}} \frac{1}{m(E_n)} \int_{E_n} X(g)m(dg) = Z_Y(\{e\}) + R(g')$$

which proves (2.3.5). Also by Theorem 2.1.1

$$E|Z_Y(\{e\})|^2 = \mu_Y(\{e\}) = \lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} R_Y(g)m(dg)$$

where $$R_Y(g) = E(Y(g+e)\overline{Y(e)}) = E(X(g)\overline{X(e)}) - |R(g')|^2 .$$

Hence $$E|Z_Y(\{e\})|^2 = \lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} E(X(g)\overline{X(e)})m(dg) - |R(g')|^2 .$$

Thus if (2.3.6) is satisfied, $E|Z_Y(\{e\})|^2 = 0$ so the limit (2.3.5) is $R(g')$ with probability 1. Conversely, if the limit is $R(g')$ with probability 1, then (2.3.6) is true. □

If the process $x(g)$ is strictly stationary, we can say more.

Theorem 2.3.3. Let $\{x(g),\ g \in G\}$ satisfy the hypotheses of Theorem 2.3.2; then if $x(g)$ is strictly stationary and metrically transitive,

$$\lim \frac{1}{m(E_n)} \int_{E_n} x(g+g',\omega)x(g,\omega)m(dg) = R(g') \text{ a.e. } [P]. \tag{2.3.7}$$

Proof: In Lemma 2.3.2, set $f(\phi) = \phi(g')\phi(e)$ so that $f(T_g\Psi(\omega)) = X(g,\omega)$. Then

$$\begin{aligned}\int_{C^G} |f(\phi)|Q(d\phi) &= \int_\Omega |f(\Psi(\omega))|dP \\ &= \int_\Omega |X(e)|dP \\ &= \int_\Omega x(g',\omega)\overline{x(e,\omega)}dP \\ &\leq E|x(e)|^2 < \infty\end{aligned}$$

since x is weakly stationary so that $f \in L_1(C^G,\mathcal{B}(C^G),Q)$. Thus since $x(g)$ is metrically transitive

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} f(T_g\phi)m(dg) = f^*(\phi) \quad \text{a.e. } [Q]$$

where $f^*(\phi)$ is constant. It follows from Lemma 2.3.1 that

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} X(g,\omega)m(dg) = f^*(\Psi(\omega)) \quad \text{a.e. } [P].$$

But $f^*(\Psi(\omega)) = \varepsilon(X(e)|A^*)$; since $f^*\Psi$ is constant a.e. [P] it follows that $f^*(\Psi(\omega)) = EX(e) = R(g')$ which proves (2.3.7). □

Corollary 2.3.2. Let $\{x(g) : g \in G\}$ be a real-valued, zero mean measurable Gaussian process. If $x(g)$ is weakly stationary, and its spectral measure μ is absolutely continuous with respect to Haar measure and G is not compact, then the sequence

$$\text{(2.3.8)} \qquad \frac{1}{m(E_n)} \int_{m(E_n)} x(g+g')x(g)m(dg)$$

converges to $R(g')$ with probability 1 and also in mean square as $n \to \infty$.

Proof: By a result of Blum and Eisenberg (1972), if μ is absolutely continuous with respect to Haar measure on $\hat{G}$ then $x(g)$ is metrically transitive, hence the sequence (2.3.8) converges to $R(g')$ with probability 1 by Theorem 2.3.3.

Since $x(g)$ is Gaussian and real, with zero mean

$$\begin{aligned} EX(g)X(h) &= E(x(g+g')x(g)x(h+g')x(h)) \\ &= Ex(g+g')x(g)Ex(h+g')x(h) + Ex(g+g')x(h+g') \\ &\times Ex(g)x(h) + Ex(g+g')x(h)Ex(g)x(h+g') \\ &= R(g)^2 + R(g-h)^2 + R(g-h+g')R(g-h-g') \ . \end{aligned}$$

Hence $X(g)$ is stationary. Moreover,

$$\begin{aligned} \frac{1}{m(E_n)} \int_{E_n} EX(g)X(e)m(dg) &= \frac{1}{m(E_n)} \int_{E_n} (R(g')^2 + R(g)^2 + R(g+g')R(g-g'))m(dg) \\ &= R(g')^2 + \frac{1}{m(E_n)} \int_{E_n} (R(g)^2 + R(g+g')R(g-g'))m(dg) \ . \end{aligned}$$

Thus (2.3.6) is satisfied and the result proved by Theorem 2.3.2 if we can show that

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} (R(g)^2 + R(g+g')R(g-g'))m(dg) = 0 .$$

Now

$$(2.3.9) \quad \left\{ \begin{aligned} & \left| \int_{E_n} (R(g)^2 + R(g+g')R(g-g')m(dg) \right| \\ & \leqq \int_{E_n} R(g)^2 m(dg) + \int |R(g+g')R(g-g')|m(dg) \\ & \leqq \int_{E_n} R(g)^2 m(dg) + \left[\int_{E_n} R(g+g')^2 m(dg) \int_{E_n} R(g-g')^2 mdg \right]^{\frac{1}{2}} \end{aligned} \right.$$

Consider $\int_{E_n} R(g+g')^2 m(dg)$. Set $F_n = E_n + g'$. Then

$$m(F_n \cap F_n + g) = m(g' + E_n \cap g' + g + E_n)$$

$$= m((E_n \cap (E_n + g)) + g') = m(E_n \cap (E_n + g))$$

Hence

$$\lim_{n\to\infty} \frac{m(F_n \cap F_n + g)}{m(F_n)} = 1 \quad \forall g \in G .$$

Thus by Hewitt and Ross (1963) p. 255,

$$(2.3.10) \quad \lim_{n\to\infty} \frac{1}{m(F_n)} \int_{F_n} |R(g)|^2 m(dg) = \lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} |R(g)|^2 m(dg) .$$

Now let $f_n(g)$ be the sequence of functions defined in the statement of Lemma 2.1.2. Then

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} |R(g)|^2 m(dg)$$

$$= \lim_{n\to\infty} \int_G f_n(g) \int_{\hat{G}} \langle \alpha, g \rangle \mu(d\alpha) \int_{\hat{G}} \overline{\langle \beta, g \rangle} \mu(d\beta) m(dg)$$

$$= \lim_{n\to\infty} \int_{\hat{G}} \int_{\hat{G}} \int_G \langle \alpha - \beta, g \rangle f_n(g) m(dg) \mu(d\alpha) \mu(d\beta)$$

$$= \lim_{n\to\infty} \int_{\hat{G}} \int_{\hat{G}} \hat{f}_n(\alpha - \beta) \mu(d\alpha) \mu(d\beta)$$

$$(2.3.11) \qquad = \int_{\hat{G}} \int_{\hat{G}} \phi(\alpha, \beta) \mu(d\alpha) \mu(d\beta)$$

by the dominated convergence theorem where

$$\phi(\alpha,\beta) = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \quad \text{, since} \quad \hat{f}_n(\alpha) \to \chi_{\{e_{\hat{G}}\}}(\alpha) \text{ .}$$

Hence (2.3.11) is equal to

$$\int_{\hat{G}} \mu\{\beta\} \mu(d\beta) = \sum_{\beta \in \hat{G}} \mu(\{\beta\})^2$$

where at most countably many terms in the sum are non zero. But by hypothesis μ is absolutely continous with respect to Haar measure, so $\mu\{\beta\} = 0 \quad \forall \beta \in \hat{G}$. Thus $\lim\limits_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} |R(g)|^2 m(dg) = 0$. Thus by (2.3.0) and (2.3.10)

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} (R(g)^2 + R(g+g')R(g-g'))m(dg) = 0$$

which completes the proof of Corollary 2.3.2. □

CHAPTER III

SAMPLE PATH PROPERTIES OF STOCHASTIC PROCESSES

In this chapter we investigate questions relating to the sample paths of stochastic processes. In section 3.1 we discuss a stochastic analog of a well known result in measure theory, relating to the approximation of measurable functions by continuous functions. In section 3.2, we study the paths of a stationary second order process over a compactly generated group, and give conditions sufficient for the sample paths to be continuous with probability one, and also prove a sampling theorem for such processes. Finally, in section 3.3, we discuss a generalized notion of a "band limited process."

3.1 Continuous-Path Approximations to Measurable Stochastic Processes.

It is a well known fact in measure theory that any measurable function is "almost" continuous in the following sense: if f is a measurable function on a finite interval $[a,b]$ then given any $\varepsilon > 0$ there exists a continuous function g and a measurable set E with measure less than ε such that $\sup_{\substack{a\le x\le b \\ x\notin E}} |f(x)-g(x)| < \varepsilon$ (See e.g. Royden (1963) p. 57)

The aim of this section is to formulate similar results for measurable stochastic processes. We treat a class of processes that includes measurable processes and also processes having a finite p-th absolute moment.

The case $p=2$ is treated separately.

Let $(\Omega,\mathcal{B},P)$ be a probability space, and let M denote the vector space of all complex valued random variables on Ω. Let N be a real function defined on the positive reals such that N is continuous, increasing and $N(0) = 0$. Also, let N satisfy, for some fixed constant k

$$N(2u) \leqq kN(n) \qquad \forall u \geq 0 . \tag{3.1.1}$$

Let ρ_N be a function defined on M by

$$\rho_N(x) = \int_\Omega N(|x(\omega)|)dP .$$

Now let $N(L(\Omega,\mathcal{B},P))$ denote the space of all random variables x such that $\rho_N(kx)$ is finite for some constant k. If we identify r.v.'s that are equal a.e. [P], then $N(L(\Omega,\mathcal{B},P))$ is a modular space with modular ρ_N (see Rolewicz (1972) p. 18). The modular ρ_N induces a complete metric d_N on $N(L(\Omega,\mathcal{B},P))$ by

$$d_N(x,y) = \inf\{\varepsilon > 0, \rho_N(\frac{x-y}{\varepsilon}) < \varepsilon\} . \tag{3.1.2}$$

The spaces $N(L(\Omega,\mathcal{B},P))$ are thus complete metric linear spaces, with invariant metrics: they are thus F-spaces, in the sense of Rolewicz (1972).

<u>Remark 3.3.1</u> If $N(u) = u/1+u$, then N satisfies (3.1.1) and $N(L(\Omega,\mathcal{B},P))$ is just the space of all random variables, with the topology of convergence in probability, which is induced by the metric $d(x,y) = E\frac{|x-y|}{1+|x-y|}$. If $N(u) = u^P$ then $N(L(\Omega,\mathcal{B},P))$ is just the space of all r.v.'s x satisfying $E|x|^P < \infty$, which is an F-space for $0 < p \leq 1$ and a Banach space for $p \geq 1$ with F-norm $||x|| = E|x|^P$ in

the first case and $||x|| = \{E|x|^p\}^{1/p}$ in the second. □

Definition 3.1.1. If $(\Omega,\mathcal{B},P)$ is a probability space, we will say that the measure P is separable with respect to the σ-field $\mathcal{B}$ if there exists a countable subclass $\mathcal{A}$ of $\mathcal{B}$ such that for all $\varepsilon > 0$ and $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ with $P(A\Delta B) < \varepsilon$. □

Theorem 3.1.1. $N(L(\Omega,\mathcal{B},P)$ is separable as a metric space iff P is separable with respect to $\mathcal{B}$.

Proof: See Rolewicz (1972) p. 30. □

Definition 3.1.2. Let V be a metric linear space. A sequence $\{e_k\}_{k=1}^{\infty}$ of vectors in V is called a quasibasis if

(i) The subspace spanned by the e_k is dense in V,

(ii) If $\sum_{k=1}^{\infty} \lambda_k e_k = 0$ for scalars λ_k then each $\lambda_k = 0$.

According to Peck (1968) every separable F-space has a quasibasis.

□

We now formulate our first approximation theorem. We shall consider measurable stochastic processes $\{x(t) : t\in T\}$ defined on a probability space $(\Omega,\mathcal{B},P)$ where P is separable with respect to $\mathcal{B}$, and T is a locally compact normal space. The processes x will be measurable with respect to the σ-field $\mathcal{B}(T) \times \mathcal{B}$ where $\mathcal{B}(T)$ is the σ-field of Borel sets of T.

We need a result similar to that mentioned in the introduction.

Theorem 3.1.2. Let T be a locally compact normal space, μ a finite positive Borel measure on $\mathcal{B}(T)$. Then given $\varepsilon > 0$ and any measurable function f on T, there exists a compact set $C \subseteq T$ with $\mu(CC) < \varepsilon$ such that the restriction of f to C has a continuous extension to T.

Proof: By Lusin's theorem (Halmos (1950) p. 243) there exists a compact set C with $\mu(CC) < \varepsilon$ such that the restriction g of f to C is continuous. Since C is compact and T is locally compact, and normal, g can be extended to a continuous function of T.

□

We can now formulate our theorem.

Theorem 3.1.3. Let $\{x(t) : t \in T\}$ be a $\mathcal{B}(T) \times \mathcal{B}$ measurable process defined on the probability space $(\Omega, \mathcal{B}, P)$ where P is separable with respect to $\mathcal{B}$, and T is a locally compact normal space. Suppose also that $x(t) \in N(L(\Omega, \mathcal{B}, P))$ for each $t \in T$. Let μ be a finite positive Borel measure on T. Then given any $\varepsilon > 0$ there exists a compact set C with $\mu(CC) < \varepsilon$ and a measurable process $y(t)$ with continuous paths, and with $y(t) \in N(L(\Omega, \mathcal{B}, P))$ $\forall t \in T$ such that

$$\sup_{t \in C} \rho_N(x(t) - y(t)) < \varepsilon .$$

Proof: Since P is separable, $N(L(\Omega, \mathcal{B}, P))$ is separable by Theorem 3.1.1. and $N(L(\Omega, \mathcal{B}, P))$ has a quasibasis $\{e_k\}$. Let us say a process $z(t)$ is in the class $S(\{e_k\})$ if $z(t)$ can be expressed in the form $z(t) = \sum_{k=1}^{\infty} a_k(t) e_k$ where the sequence $\{a_k(t)\}$ is ultimately zero for each fixed t and each function $a_k(t)$ is $\mathcal{B}(T)$-measurable. No questions of convergence arise in this definition. The proof will be done in two parts: Lemma 3.1.1 which shows that every process in the class $S(\{e_k\})$ satisfies theorem 3.1.3, and Lemma 3.1.2 which shows that every measurable process $z(t)$ with $z(t) \in N(L(\Omega,\mathcal{B},P))$ $\forall t \in T$ can be approximated by a process in $\mathcal{S}(\{e_k\})$.

Lemma 3.1.1. Every process in $\mathcal{S}(\{e_k\})$ satisfies Theorem 3.1.3.

Proof: Let $z(t) = \sum_{k=1}^{\infty} a_k(t)e_k$ be a process in $\mathcal{S}(\{e_k\})$. Let C_k be the compact sets of Theorem 3.1.2 such that $a_k(t)$ is continuous on C_k and $\mu(CC_k) < 3\varepsilon/\pi^2k^2$ $k = 1,2,\ldots$. Let $b_k(t)$ be the continuous extension of a_k to T. Then if $C_0 = \bigcap_{k=1}^{\infty} C_k$,

$$\mu(CC_0) \leq \sum_{k=1}^{\infty} \mu(CC_k) < \sum_{k=1}^{\infty} \frac{3\varepsilon}{\pi^2k^2} = \varepsilon/2 ,$$

and C_0 is compact. Let $y_n(t) = \sum_{k=1}^{n} b_k(t)e_k$. Then clearly y_n is a measurable process with continuous paths, with $y_n(t) \in N(L(\Omega,\mathcal{B},P))$ for each t. It is clear that for each fixed $t \in C_0$ $y_n(t) = z(t)$ for n sufficiently large so $\rho_N(y_n(t) - z(t))$ converges to zero for all $t \in C_0$. Since $z(t)$ is also a measurable process (being the limit of measurable processes) it follows that $\rho_N(y_n(t) - z(t))$ is measurable by Fubini's theorem. Hence by Egoroff's theorem, there is a compact set $C_{00} \subseteq C_0$ such that $\mu(C_0 - C_{00}) < \varepsilon/2$ and

$$\lim_{n\to\infty} \sup_{t\in C_{00}} \rho_N(y_n(t) - z(t)) = 0 .$$

By choosing n large enough, and setting $y = y_n$

$$\sup_{t\in C_{00}} \rho_N(y(t) - z(t)) < \varepsilon$$

and $\mu(CC_{00}) \leqq \mu(C_0 - C_{00}) + \mu(CC_0) < \varepsilon$ □

The next lemma will complete the proof of Theorem 3.1.3.

Lemma 3.1.2. For every $\varepsilon > 0$ there exists a compact set C with

$\mu(CC) < \varepsilon$ and a process $z(t)$ in $S(\{e_k\})$ such that

$$\sup_{t \in C} \rho_N(x(t) - z(t)) < \varepsilon \ .$$

Proof: Let G denote the class of all $\mathcal{B}(T) \times \mathcal{B}$ measurable processes z, with $z(t) \in N(L(\Omega,\mathcal{B},P))$, that can be approximated by a process in $S(\{e_k\})$ in the sense of the lemma. If α,β are complex numbers and x,y are in G, then we will show that $\alpha x + \beta y$ is in G. For each $n = 1,2,\ldots$ let x_n', y_n' be processes such that

$\sup_{t \in C_n} \rho_N\big(x_n'(t) - x(t)\big) < \frac{3\varepsilon}{2n^2\pi^2}$, $\sup_{t \in D_n} \rho_N\big(y_n'(t) - y(t)\big) < \frac{3\varepsilon}{2\pi^2 n^2}$ where C_n and D_n are compact sets such that $\mu(CC_n) < \frac{3\varepsilon}{\pi^2 n^2}$, $\mu(CD_n) < \frac{3\varepsilon}{\pi^2 n^2}$.

Let $C = \bigcap_{n=1}^{\infty} C_n$, $D = \bigcap_{n=1}^{\infty} D_n$. Then for all $t \in C \cap D$, $\rho_N(x_n'(t) - x(t))$ and $\rho_N(y_n'(t) - y(t))$ converges to zero. By the properties of modulars, $\rho_N(\alpha(x_n'(t) - x(t)))$ and $\rho_N(\beta(y_n'(t) - y(t)))$ also converge to zero. If $||\circ||_N$ denotes the F-norm $||x||_N = d_N(x,0)$ where d_N is the metric associated with the modular ρ_N as in (3.1.1), then $||\alpha(x_n'(t) - x(t))||_N$ and $||\beta(y_n'(t) - y(t)||_N$ converge to zero. (Rolewicz (1972 p. 17). Thus for all $t \in C \cap D$

$$\overline{\lim_{n\to\infty}} \ ||\alpha \ x(t) + \beta \ y(t) - (\alpha x_n'(t) + \beta y_n'(t))||$$

$$\leqq \overline{\lim_{n\to\infty}} \ ||\alpha(x(t) - x_n'(t))|| + \overline{\lim_{n\to\infty}} ||\beta(y(t) - y_n'(t))||$$

so

$$\overline{\lim_{n\to\infty}} \ ||\alpha x(t) + \beta y(t) - (\alpha x_n'(t) + \beta y_n'(t)|| = 0 \quad \forall t \in C \cap D \ .$$

It follows (Rolewicz 1972 p. 17) that

$$\lim_{n\to\infty} \rho_N(\alpha x(t) + \beta y(t) - (\alpha x_n'(t) + \beta y_n'(t))) = 0 \ .$$

Thus by Egoroff's theorem there exists a compact set $C_0 \subseteq C \cap D$ such that $\mu(\complement C_0) < \varepsilon$ and an integer n such that

$$\sup_{t \in C_0} \rho_N(\alpha x(t) - \beta y(t) - (\alpha x_n'(t) + \beta y_n'(t))) < \varepsilon .$$

Since $\alpha x_n'(t) + \beta y_n'(t)$ is in $\mathcal{S}(\{e_k\})$, $\alpha x(t) + \beta y(t) \in \mathcal{G}$.

Next we show that every $\mathcal{B}(T) \times \mathcal{B}$ simple function is in $\mathcal{G}$. Let $A \in \mathcal{B}(T)$, $B \in \mathcal{B}$. Since

$$\rho_N(\chi_B) = \int_\Omega N(\chi_B)dP$$

$$= N(1)P(B) < \infty ,$$

it is clear that $\chi_B \in N(L(\Omega,\mathcal{B},P))$. Thus there exist complex numbers $a_1,\ldots,a_n$ such that

$$\rho_N(\chi_B - \sum_{k=1}^{n} a_k\, e_k) < \varepsilon .$$

Set $a_k(t) = a_k\, \chi_A(t)$. Then

$$\rho_N(\chi_{A\times B}(t,\circ) - \sum_{k=1}^{n} a_k(t)e_k) = \begin{cases} 0 & t \notin A \\ \rho_N(\chi_B - \sum_{k=1}^{n} a_k\, e_k) & t \in A \end{cases}$$

so $\sup_{t\in T} \rho_N(\chi_{A\times B}(t,\circ) - \sum_{k=1}^{n} a_k(t)e_k) < \varepsilon$, so that $\chi_{A\times B} \in \mathcal{G}$. Moreover, if $F = \bigcup_{i=1}^{m} A_i \times B_i$, we can suppose without loss of generality that the $A_i \times B_i$ are disjoint, so $\chi_F = \sum_{i=1}^{m} \chi_{A_i \times B_i}$ is in $\mathcal{G}$ by the previous paragraph. Sets of this type generate the σ-field $\mathcal{B}(T) \times \mathcal{B}$ so given any set $E \in \mathcal{B}(T) \times \mathcal{B}$, there is a sequence $\{F_n\}$ of sets of the above type such that $\mu \times P(E \,\Delta\, F_n) < 1/n$. It follows that $N(\chi_{E\Delta F_n}(t,\omega))$ is in $L_1(T \times \Omega,\ \mathcal{B}(T) \times \mathcal{B},\ \mu \times P)$ and that the sequence $N(\chi_{E\Delta F_n}(t,\omega))$ converges to 0 in this space. Thus $E\,N(\chi_{E\Delta F_n}(t,\cdot))$ converges to 0 in $L_1(T,\mathcal{B}(T),\mu)$ and a subsequence converges to 0 a.e.[μ], and so by Egoroff's theorem there is a compact subset C_1 with $\mu(CC_1) < \varepsilon/2$ such that $\sup_{t\in C_1} \rho_N(\chi_{E\Delta F_n}(t,\cdot)) < \varepsilon/2$ for some n. Now χ_{F_n} is in G so there is a compact subset C_2, $\mu(CC_2) < \varepsilon/2$ and a process $z(t)$ in $S(\{e_k\})$ with $z(t) \in N(L(\Omega,\mathcal{B},P))$ such that

$$\sup_{t\in C_2} \rho_N(\chi_{F_n}(t,\cdot) - z(t)) < \varepsilon/2 .$$

Set $C = C_1 \cap C_2$, then by noting that $\chi_{E\Delta F_n} = |\chi_{F_n} - \chi_E|$, we see that

$$\sup_{t\in C} \rho_N(\tfrac{1}{2}\chi_E(t,\cdot) - \tfrac{1}{2}z(t))$$

$$\leq \sup_{t\in C_1} \rho_N(\chi_E(t,\cdot) - \chi_{F_n}(t,\cdot)) + \sup_{t\ C_2} \rho_N(\chi_{F_n}(t,\cdot) - z(t))$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Hence χ_E and thus any simple function is in G. Now consider a measurable process $\{x(t) : t\in T\}$, $x(t) \in N(L(\Omega,\mathcal{B},P))$, that is positive. Let x_n be a sequence of simple functions such that $x_n(t,\omega)$ increases to $x(t,\omega)$ pointwise. Since N is increasing, $N(|x(t,\omega) - x_n(t,\omega)|) \leq N|x(t,\omega)|$. Since $E\,N(x(t)) < \infty$, $E\,N(|x(t,\omega) - x_n(t,\omega)|)$ converges to zero by dominated convergence. By Egoroff's theorem and the first part of the proof, $x \in G$. By expressing an arbitrary process in terms of its positive parts we see that every process x that is measurable and has $x(t) \in N(L(\Omega,\mathcal{B},P)$ for each $t \in T$ is in G, and so can be approximated by a process in $S(\{e_k\})$. □

<u>Corollary 3.1.1</u>. Let $\{x(t), t\in T\}$ be a measurable process, over the

probability space $(\Omega,\mathcal{B},P)$ where P is separable with respect to $\mathcal{B}$. Let T be a locally compact normal space, μ a finite positive Borel measure on T. Then given $\varepsilon > 0$, there exists a compact set C with $\mu(CC) < \varepsilon$ and a measurable process $y(t)$ with continuous paths such that

$$\sup_{t\in C} P[\omega: |x(t,\omega) - y(t,\omega)| > \varepsilon] < \varepsilon .$$

Proof: Set $N(t) = t/1+t$. Then, as mentioned in Remark 3.1.1, $N(L(\Omega,\mathcal{B},P))$ is the space of all r.v.'s on Ω with the topology of convergence in probability. The modular $\rho_N(x) = E\frac{|x|}{1+|x|}$ is an F norm. By Theorem 3.1.3 $\forall\delta > 0$, there exists a measurable process y with continuous paths and a compact set C with $\mu(CC) < \delta$ such that

$$\sup_{t\in C} E[|x(t) - y(t)|/(1+|x(t) - y(t)|)] < \delta .$$

Thus

$$\begin{aligned} &\sup_{t\in C} P[\omega: |x(t,\omega) - y(t,\omega)| > \varepsilon] \\ &\le \frac{1+\varepsilon}{\varepsilon} \sup_{t\in C} E[|x(t) - y(t)|/1+|x(t) - y(t)|] \\ &< \frac{1+\varepsilon}{\varepsilon}\,\delta . \end{aligned}$$

Choose δ so that $\frac{1+\varepsilon}{\varepsilon}\delta < \varepsilon$, then $\delta < \varepsilon$, the result follows. □

Corollary 3.1.2. Let $\{x(t) : t\in T\}$ be a measurable process as in Corollary 3.1.1 and suppose $E|x(t)|^p < \infty$ $\forall t \in T$ for some $p > 0$. Then given $\varepsilon > 0$ there exists a measurable process $y(t)$ with continuous paths and $E|y(t)|^p < \infty$ $\forall t$, and a compact set C with $\mu(CC) < \varepsilon$ such that $\sup_{t\in C} E|x(t) - y(t)|^p < \varepsilon$.

Proof: Take $N(u) = u^p$, then $\rho_N(x) = E|x|^p$, the result follows from Theorem 3.1.3. □

Remark 3.1.2. If we restrict T to be a real interval $I = [a,b]$, we can approximate a measurable process $\{x(t) t \in \mathbb{R}\}$ by a process $y(t)$ whose paths are infinitely differentiable, instead of merely continuous. In the proof of Theorem 3.1.3, the process y which approximates x was obtained by approximating measurable functions on T by continuous functions. In the case $T = [a,b]$ given $\varepsilon > 0$ and a measurable function f on $[a,b]$, we can find a C^∞ function h, and a compact set $C \subseteq [a,b]$, with the Lebesgue measure of $[a,b] - C$ less than ε, such that $g(x) = f(x)$ on C. This assertion may be proved as follows. Let f be a measurable simple function, then there is a step function g that coincides with f except on a set of small measure, and a C^∞ function h that coincides with g except on a set of small measure. (See Royden (1963) p. 58.) Thus the assertion is true for simple functions. Using the technique of Halmos in his proof of Lusin's theorem (Halmos (1950) p. 243), the assertion can be proved for general measurable functions. Thus the proof of Theorem 3.1.3 can be adapted to obtain an approximating process with C^∞ paths. $\square$

We now turn to the approximation of second order processes. Because every separable Hilbert space has a basis, and not just a quasibasis, we can prove a stronger result than Theorem 3.1.3.

Theorem 3.1.4. Let $\{x(t), t \in \mathbb{R}\}$ be a second order measurable process indexed by the real line $\mathbb{R}$, μ a positive finite measure on R that is absolutely continuous with respect to Lebesgue measure. Then given $\varepsilon > 0$ there exists a compact set C with $\mu(CC) < \varepsilon$ and a second order process $y(t)$ with continuous paths, with $y(t)$ in the Hilbert space spanned by $x(t)$, such that

$$\sup_{t\in C} E|x(t) - y(t)|^2 < \varepsilon .$$

Proof: Let R be the covariance function of x. Let ν be a measure on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} R(t,t)\nu(dt) < \infty$ and ν is equivalent to Lebesgue measure. Such a measure exists (Cambanis (1973)). Then according to Cambanis (1973) the process $x(t,\omega)$ has a representation of the form

$$x(t,\omega) = \sum_{k=1}^{\infty} a_k(t)\xi_k(\omega) + z(t,\omega) \tag{3.1.3}$$

where the ξ_k's are orthogonal random variables and the a_k's are measurable. The process $z(t)$ is orthogonal to ξ_k for each t and k, and $E|z(t)|^2 = 0$ a.e. [Lebesgue]. For each k, a_k is continuous on a compact set C_k with $\mu(CC_k) < 2\varepsilon/k^2\pi^2$. Set $C^{(0)} = \bigcap_{k=1}^{\infty} C_k$. Then $\mu(CC^{(0)}) < \varepsilon/3$. Let b_k be the continuous extension of a_k on C_k to $\mathbb{R}$ Set $y_N(t,) = \sum_{k=1}^{N} b_k(t)\xi_k(\omega)$. Then for all $t \in C^{(0)}$

$$E|y_N(t) - x(t)|^2 = E|z(t)|^2 + \sum_{k=N+1}^{\infty} |a_k(t)|^2 .$$

But $\sum_{k=1}^{\infty} |a_k(t)|^2 < \infty$ (Cambanis (1973)), so

$$\lim_{N\to\infty} E|y_N(t) - x(t)|^2 = E|z(t)|^2 \ \forall t \in C^{(0)} .$$

Let A denote the set on which $E|z(t)|^2 \neq 0$. Then there is a compact set $C^{(1)}$ with $A \subseteq CC^{(1)}$ and $\mu(C^{(1)} < \varepsilon/3$ since μ is assumed absolutely continuous with respect to Lebesgue measure. Thus on $C^{(1)} \cap C^{(0)}$

$$\lim_{N\to\infty} E|y_N(t) - x(t)|^2 = 0 . \tag{3.1.4}$$

By Egoroff's theorem, the result now follows from (3.1.4) by a familiar argument. □

Remark 3.1.3. This theorem is not a special case of Theorem 3.1.3 since we can assert that the approximating process $y(t)$ is in the space spanned by the process $x(t)$, and thus, in theory at least, $y(t)$ can be obtained by linear operations on $x(t)$. This is not the case in the general context. Also, we do not require P to be separable.

□

Remark 3.1.4. If the measurable process $\{x(t) t \in R\}$ is stationary and hence mean square continuous by Theorem 2.2.4, we can approximate $x(t)$ uniformly by processes with analytic paths. For if $x(t)$ has spectral representation

$$x(t) = \int_{-\infty}^{\infty} e^{2\pi i t\lambda} Z\{d\lambda\} , \quad E|Z(\Delta)|^2 = \mu(\Delta) ,$$

then the process $x_n(t) = \int_{-n}^{n} e^{2\pi i t\lambda} Z\{d\lambda\}$ satisfies

$$E|x(t) - x_n(t)|^2 = \mu(C[-n,n]) \quad \text{and so} \quad \lim_{n\to\infty} \sup_{t\in\mathbb{R}} E|x(t) - x_n(t)|^2 = 0 .$$

The process $x_n(t)$ is stationary and has analytic paths (Belayev (1959)). Such processes with spectral measures concentrated on compact sets are called "band-limited;" we shall return to them in the sequel. □

Remark 3.1.5. If the measurable process $x(t)$ is mean square continuous, then the process $z(t)$ in the expansion (3.1.3) vanishes, and the series (3.1.3) converges in the mean uniformly over compact subsets of $\mathbb{R}$ (Cambanis (1973)). It follows that for any compact subset C of $\mathbb{R}$

$$\lim_{n\to\infty} E|y_n(t) - x(t)|^2 = 0 \quad \text{uniformly on } C.$$

□

Remark 3.1.6. If the process $x(t)$ in Theorem 3.1.4 is Gaussian, then the random variables ξ_k in (3.1.3) can be chosen to be Gaussian (Cambanis and Rajput (1972)) and so the approximating process is Gaussian. □

3.2 Sufficient Conditions for Path Continuity of Stationary Processes

In this section we will discuss conditions on a stationary process that are sufficient for path continuity with probability one. We will generalize known results (Kawata (1969), Belayev (1959)) to weakly stationary second order processes indexed by a certain type of LCA group, and obtain conditions on the spectral measure of a mean square continuous (m.s.c.) stationary process sufficient for path continuity. We also discuss band limited processes indexed by a group and show that such processes satisfy a "sampling" theorem.

More specifically, many sufficient conditions for path continuity of a stationary process are known. For example, if $\{x(t)\ t \in \mathbb{R}\}$ is a m.s.c. stationary process on the real line $\mathbb{R}$, with spectral measure μ, then Kawata (1969) has proved that x has continuous paths with probability 1 if

$$(3.2.1) \qquad \int_{\infty}^{\infty} |x|(\log^{+}|x|)^{\beta}\mu(dx) < \infty$$

for $\beta > 1$. Kawata's proof depends heavily on the Borel-Cantelli lemma, and his approach fails even for the case of a process indexed by $\mathbb{R}_2$. A different approach, using weak convergence of probability measures, is used by Neuhaus (1972) to derive sufficient conditions for path continuity; conditions which depend on the behavior of first order differences of the process. We will follow his technique in the sequel.

As mentioned in section 3.1, band limited stationary processes are those whose spectral measures are concentrated on a compact set, i.e., have compact support. Such processes have analytic paths, and have the important property that the value of the process at any point can be obtained from its values at the "sampling points" $\frac{k\pi}{\alpha}$, $k = 0, \pm 1, \pm 2, \ldots$. More precisely, if $x(t)$ is a band limited process whose spectral measure μ has support in the interval $[-W,W]$ then

$$(3.2.2) \qquad x(t) = \sum_{k=-\infty}^{\infty} x(k\pi/\alpha) \frac{\sin \alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)} .$$

The series (3.2.2) converges uniformly on compact sets of $\mathbb{R}$ with probability 1 for all $\alpha > W$. (See e.g. Piranashvili (1967)).

We will consider m.s.c. stationary processes indexed by an LCA group that is compactly generated, and obtain a condition similar to (3.2.1) that is sufficient for path continuity with probability one. We will also discuss band limited processes on such groups and show that they can be expressed in a sampling series similar to (3.2.2).

Every compactly generated LCA group (i.e., an LCA group that has a compact neighborhood U of e such that $G = \bigcup_{n=1}^{\infty} n(U \cap (-U))$ can be expressed in the form

$$(3.2.3) \qquad G = \mathbb{R}^k \times Z^m \times F$$

where k and m are positive integers and F is a compact abelian group. (Hewitt and Ross (1963 p. 90). We first obtain results for $\mathbb{R}^k$ and then combine $\mathbb{R}^k$ with the other factors to get the corresponding result for G. We will use the following result of Neuhaus (1972):

Theorem 3.2.1. Let E_k denote the k-dimensional unit cube and let

$\{x(t) : t \in E_k\}$ be a stochastic process. If there exist constants $\gamma \geq 0$, $\alpha > 1$, $K \geq 0$ and continuous distribution functions F_i corresponding to finite measures on $E_{|i|}$ and satisfying

$$(3.2.4) \qquad P(|\Delta_{[t]_i}^{[t+h]_i} x^i| \geq \lambda) \leqq K \lambda^{-\gamma} |\Delta_{[t]_i}^{[t+h]_i} F_i|^{\alpha}$$

for all $i \in I$, $\forall t, t+h, h \in E_k$, $\forall \lambda > 0$, then $x(t)$ has continuous paths with probability 1.

Proof: See Newhaus (1972). □

A few remarks on the notation of the above theorem are necessary.

Let $I_p = \{i = (i_1,\dots,i_p) : i_1 \dots i_p$ are integers, $1 \leq i \leq \dots \leq i_p \leq k\}$, $I = \bigcup_{p=1}^{k} I_p$ and $|i| = p$. If f is a function $E_k \to \mathbb{R}$ then f^i is a function $E_{|i|} \to \mathbb{R}$ given by $f^i(u_1,\dots,u_p) = f(t)$ where $t_{i_\mu} = u_\mu$ for $\mu = 1,\dots,p$ and $t_\nu = 0$ if $\nu \notin i$. If $t = (t_1,\dots,t_k) \in \mathbb{R}^k$ and $i \in I$, $i = (i_1,\dots,i_p)$ then $[t]_i = (t_{i_1},\dots,t_{i_p})$. If f is a function $E_p \to R$, t, and $t, t' \in E_p$ then

$$\Delta_t^{t'} f = \sum_{\delta_1,\dots,\delta_p} (-1)^{\sum_\mu \delta_\mu} f(\delta_1 t_1 + (1-\delta_1)t_1', \dots, \delta_p t_p + (1-\delta_p)t_p')$$

where the summation taken over all $(\delta_1,\dots,\delta_p)$ with $\delta_\mu \in \{0,1\}$. In other words, Δ is a difference operator.

Remark 3.2.1. The theorem remains true with E_k replaced by $[-A,A]^k$ for $A > 0$.

Remark 3.2.2. By an application of the Tchebychev inequality, (3.2.4) can be replaced by

$$(3.2.5) \qquad E|\Delta_{[t]_i}^{[t+h]_i} x^i|^2 \leq K|\Delta_{[t]_i}^{[t+h]_i} F_i|^\alpha$$

in the case of second order processes. If we assume in addition that the process x is m.s.c. stationary, we get the following corollary:

<u>Corollary 3.2.1</u>. Let $\{x(t), t \in \mathbb{R}_k\}$ be a m.s.c. stationary stochastic process on $\mathbb{R}_k$ with spectral measure μ. Let $f(x_1,\ldots,x_k)$ be positive and integrable on $A_k = [-A,A]^k$. Then the paths of $x(t)$ are continuous with probability one on A_k if there exists $\alpha > 1$ $k > 0$ such that

$$(3.2.6) \qquad \int_{\mathbb{R}_k} \prod_{\mu=1}^{p} \sin^2 \pi\lambda_{i_\mu} h_{i_\mu} \, \mu(d\lambda_1,\ldots,d\lambda_k)$$

$$\leq K4^{-p}\left|\int_{\Lambda_i(t,h)} f(x_1,\ldots,x_k)dx_1,\ldots,dx_k\right|^\alpha$$

$$\forall i \in I, \ \forall t, \ t+h, h \in A_k$$

where $\Lambda_i(t,h)$ is the set $\{(x_1,\ldots,x_k): t_{i_\mu} \leq x_j \leq t_{i_\mu} + h_{i_\mu} \quad j \in i, \ j = i_\mu$

and $-A \leq x_j \leq 0$ if $j \notin i\}$.

<u>Proof</u>: Let $\phi(t_1,\ldots,t_k) = \exp 2\pi i \sum_{j=1}^{k} \lambda_j t_j$. Let $i = (i_1,\ldots,i_p)$ $i \in I$, $t = (t_1,\ldots,t_k)$ and $h = (h_1,\ldots,h_k)$. Then

(3.2.7)

$$\Delta_{[t]_i}^{[t+h]_i} \phi^i = \sum_{\delta_1,\ldots,\delta_p} (-1)^{\sum_\mu \delta_\mu} \prod_{\mu=1}^{p} \exp 2\pi i \ \lambda_{i_\mu}(\delta_\mu t_{i_\mu} + (1-\delta_\mu)(t_{i_\mu}+h_{i_\mu}))$$

$$= [\exp(2\pi i \ t_{i_p}\lambda_{i_p}) - \exp(2\pi i(t_{i_p}+h_{i_p})\lambda_{i_p}]$$

$$\times \sum_{\delta_1,\ldots,\delta_{p-1}} (-1)^{\sum_{\mu=1}^{p-1}\delta_\mu} \prod_{\mu=1}^{p-1} \exp 2\pi i \ \lambda_{i_\mu}(\delta_\mu t_{i_\mu} + 1-\delta_\mu(t_{i_\mu}+h_{i_\mu})) .$$

Proceeding in this way we see that the right hand side of (3.2.7) is equal to

$$\prod_{\mu=1}^{p}(\exp(2\pi i\ t_{i_\mu}\lambda_{i_\mu}) - \exp(2\pi i(t_{i_\mu}+h_{i_\mu})\lambda_{i_\mu}))$$

$$= \prod_{\mu=1}^{p} e^{2\pi i\ t_{i_\mu}\lambda_{i_\mu}}(1 - e^{2\pi i\ h_{i_\mu}\lambda_{i_\mu}})$$

so $$\left|\Delta_{[t_i]}^{[t+h]_i}\ \phi^i\right| = 2^p \prod_{\mu=1}^{p}\left|\ \sin^2\pi h_{i_\mu}\ \lambda_{i_\mu}\right| .$$

From the isomorphism theorem it follows that

$$E\left|\Delta_{[t_i]}^{[t+h]_i}\ x^i\right|^2 = 4^p \int_{R^k} \prod_{\mu=1}^{p} \sin^2 h_{i_\mu}\ \lambda_{i_\mu}\ \mu(d\lambda_1,\ldots,d\lambda_k).$$

In a similar manner it can be shown that if $F_i = F^i$, where

$$F(t_1,\ldots,t_k) = \int_{-A}^{t_1}\ldots\int_{-A}^{t_k} f(x_1,\ldots,x_k)\ \ dx_1,\ldots,dx_k\ ,$$

then
$$\Delta_{[t]_i}^{[t+h]_i}\ F^i = \int_{\Lambda_i} f(x_1,\ldots,x_k)\ dx_1,\ldots,dx_k\ .$$

Thus by 3.2.5 a sufficient condition for path continuity is (3.2.6).

□

If in Corollary 3.2.1 we set $f(x_1,\ldots,x_k) \equiv 1$, then (3.2.6) becomes

(3.2.8) $$\int_{R^k} \prod_{\mu=1}^{p} \sin^2 \pi\lambda_{i_\mu} h_{i_\mu}\ \mu(d\lambda_1,\ldots,d\lambda_k) \leqq K4^{-k}\left|A^{k-p}\prod_{\mu=1}^{p}\left|h_{i_\mu}\right|\right|^\alpha$$

$\forall i \in I$ and $h \in A_k$.

(3.2.8) is implied by

$$(3.2.9) \qquad \int_{R^k} \prod_{\mu=1}^{p} \sin^2 \frac{\lambda_{i_\mu} h_{i_\mu}}{2}\, \mu(d\lambda_1,\ldots,d\lambda_k) \le K' \prod_{\mu=1}^{p} \left|h_{i_\mu}\right|^\alpha$$

for some $K' \ge 0$. Since $\sin^2 \pi\lambda h \le |\pi\lambda h|^\alpha$ for $\alpha < 2$, it follows that (3.2.9) is true if

$$(3.2.10) \qquad \int_{R^k} \prod_{j=1}^{k} |\lambda_j|^\alpha\, \mu(d\lambda_1,\ldots,d\lambda_k) < \infty \quad \text{for } 1 < \alpha < 2\ .$$

We have proved

Theorem 3.2.2. If $\{x(t)\ t\in\mathbb{R}_k\}$ is a m.s.c. stationary stochastic process then $x(t)$ has continuous paths on every bounded set with probability one, and hence continuous paths with probability one, if (3.2.10) is satisfied.

Remark 3.2.3. This result is not as strong as Kawata's for the case $k = 1$. However, as remarked before, his proof fails if $k > 1$.

□

Remark 3.2.4. By a more judicious choice of the function f in Corollary 3.2.1 it might be possible to obtain a weaker condition on μ. However, we have not been able to find such an f. □

We now turn to the general case, where $\{x(g) : g\in G\}$ is a m.s.c. stationary process on G and G is a compactly generated LCA group. The dual $\hat{G}$ of G is of the form $\mathbb{R}^k \times D \times T^m$ where T is the circle group and D is discrete. Define for each $n = (n_1,\ldots,n_m)$ in Z^m a process y_n, indexed by $R^k\times F$ by $y_n(t,g;\omega)=x(t,g,n,\omega)$ where $t\in R^k$, $g\in F$. Since Z^m is a discrete group, x will have continuous paths with probability 1 iff y_n does for each n. Let us fix an n. Suppose x has the spectral representation

$$x(t,g,n) = \int_{R^k} \int_D \int_{T^m} e^{2\pi i\, t\cdot\lambda} \langle \alpha, g\rangle e^{2\pi i\, n\cdot s} Z\{d\lambda,\, d\alpha,\, ds\} \,.$$

y_n has covariance $E\, y_n(t,g)\, \overline{y_n(t',g')} = E(x(t,g,n)x(t',g',n))$ $= R(t-t', g-g', 0)$ where R is the covariance of x. Thus y_n is stationary with covariance $R_n(t,g) = R(t,g,0)$. The spectral measure of y_n is given by

$$\mu'(\Delta) = \mu(\Delta \times T^m)$$

where μ is the spectral measure of x, and Δ is a Borel set in the product σ-algebra of $\mathbb{R}^k \times D$. Also,

$$(3.2.11) \qquad y_n(t,g) = \int_{R^k \times D} e^{2\pi i\, t\circ\lambda} \langle \alpha, g\rangle\, Z_n'(d\lambda,\, d\alpha)$$

where $Z_n'(\Delta)$ $\displaystyle\int_{\Delta\times T^m} e^{2\pi i\, n\circ s} Z\{d\lambda,\, d\alpha,\, ds\}$, Δ a Borel subset of $R^k \times D$.

Now μ' is a finite measure; because D is discrete, it follows that $\mu'(R^k \times \{\alpha\}) = 0$ except for countably many α say $\{\alpha_j\}_{j=1}^{\infty}$. Thus if we set $Z_{j,n}'(E) = Z_n'(E \times \{\alpha_j\})$ for E a Borel set of R^k then $Z_{j,n}'$ is an orthogonal random measure on R^k and from (3.2.11)

$$(3.2.12) \qquad y_n(t,g) = \sum_{j=1}^{\infty} \langle \alpha_j, g\rangle \int_{R^k} e^{2\pi i\, t\circ\lambda}\, Z_{j,n}'\{d\lambda\} \,.$$

The series (3.2.12) converges in mean square since the series $\sum_{j=1}^{\infty} \mu'(R^k \times \{\alpha_j\})$ converges.

Write $x_{j,n}(t) = \int_{R^k} e^{2\pi i\, t\circ\lambda} Z'_{j,n}\{d\lambda\}$, then $x_{j,n}(t)$ is a stationary process on R^k and the $x_{j,n}(t)$ processes are mutually orthogonal, for $j_1 \neq j_2$ and fixed n. The next result shows that the series (3.2.12) converges uniformly with probability one under certain conditions.

Theorem 3.2.3. Let $\phi : D \to [1,\infty)$ be a function such that $\sum_{j=1}^{\infty} (\phi(\alpha_j))^{-1} < \infty$, where $\{\alpha_j\}_{j=1}^{\infty}$ is the set described above. If the spectral measure μ of the process x satisfies

(i) $\int_{\hat{G}} \phi(\alpha)\mu(d\lambda,\, d\alpha,\, ds) < \infty$ and

(ii) $\int_{\hat{G}} \prod_{\mu=1}^{k} |\lambda_\mu|^{\alpha}\, \mu(d\lambda,\, d\alpha,\, ds) < \infty$ for $1 < \alpha < 2$

then the process x has continuous paths with probability 1.

Proof: Because of (ii) the processes $x_{j,n}$ defined above have continuous paths with probability 1 by Theorem 3.2.2, since the spectral measure μ_j of $x_{j,n}$ is given by $\mu_j(0) = \mu'(\Delta \times \{\alpha_j\} \times T^m)$. Thus we only have to prove that the series (3.2.12) converges uniformly with probability one.

It is enough to show that $E\sum_{j=1}^{\infty} |x_{j,n}(t)|$ converges uniformly.

Now $E\sum_{j=1}^{\infty} |x_{j,n}(t)| = E\sum_{j=1}^{\infty} \phi(\alpha_j)^{-\frac{1}{2}}\, \phi(\alpha_j)^{\frac{1}{2}}\, |x_{j,n}(t)|$

$$\leq \Big(\sum_{j=1}^{n} \phi(\alpha_j)^{-1}\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{n} \phi(\alpha_j) E|x_{j,n}(t)|^2\Big)^{\frac{1}{2}}$$

$$= \Big(\sum_{j=1}^{n} \phi(\alpha_j)^{-1}\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{n} \phi(\alpha_j)\mu(R^k \times \{\alpha_j\} \times T^m\Big)^{\frac{1}{2}}$$

$$= \Big(\sum_{j=1}^{\infty} \phi(\alpha_j)^{-1}\Big)^{\frac{1}{2}}\Big(\int\phi(\alpha)\mu(d\lambda,\ d\alpha,\ ds)\Big)^{\frac{1}{2}} < \infty$$

so the series (3.2.12) converges uniformly with probability one. Hence y_n and thus x have continuous paths with probability 1. □

Corollary 3.2.2. If the spectral measure μ has compact support, the paths of x are continuous with probability 1.

Proof: If μ has compact support, then the conditions of Theorem 3.2.3 are trivially satisfied, since the support of μ is contained in a set of the form $[-A,A]^k \times D_0 \times T^m$ where D_0 is a finite subset of D. □

We now consider band limited processes indexed by G. The spectral measure of such a process is concentrated on a compact set $A_k \times D_0 \times T^m$ say. Since the set D_0 is finite, it follows that a band limited process is of the form $x(t,g,n) = \sum_{j=1}^{N} \langle\alpha_j, g\rangle x_{j,n}(t)$ where $D_0 = \{\alpha_1,\ldots,\alpha_N\}$. Since μ has compact support, the spectral measures μ_j of the $x_{j,n}$ also have compact support. We will show that a band limited process can be expressed in a sampling series similar to (3.2.2).

Since each $x_{j,n}$ has a spectral measure with compact support in A_k, it follows as in the case $k = 1$ that the series

$$x_{j,n}(t) = \sum_{\mu_1=-\infty}^{\infty} \cdots \sum_{\mu_1=-\infty}^{\infty} x_{j,n}\Big(\frac{\mu_1\pi}{\alpha},\ldots,\frac{\mu_k}{\alpha}\Big) \prod_{\ell=1}^{k} p_\ell \tag{3.2.13}$$

converges uniformly on compact sets of R_k with probability 1, where

$\alpha > A$ and $p_\ell = \dfrac{\sin \alpha(t_\ell - \mu_\ell\pi/\alpha)}{\alpha(t_\ell - \mu_\ell\pi/\alpha)}$. Thus

$$(3.2.14) \qquad x(t,g,n) = \sum_{\mu\in Z^k} \sum_{j=1}^{N} \langle \alpha_j, g\rangle \, x_{j,n}\left(\frac{\mu\pi}{\alpha}\right) \prod_{\ell=1}^{k} p_\ell$$

where $\mu = (\mu_1,\ldots,\mu_k) \in Z_k$.

Now let H be a discrete subgroup of F such that

(3.2.15) For $1\le i<j\le N$, $\alpha_1 - \alpha_j \notin \Lambda$, the annihilator of H in D.

Λ is discrete, since it is a subgroup of a discrete group. Choose a set $\Omega \subseteq D$ that contains exactly one element from each coset $\alpha + \Lambda$ in D/Λ. Without loss of generality we may assume that $\alpha_j \in \Omega$, $j = 1,\ldots,n$. Define $S(g) = \int_\Omega \langle \alpha, g\rangle \, \theta(d\alpha)$ where θ is the Haar measure on D, the integral exists (see Kluvanek 1965). Now let $f_j(g) = \langle \alpha_j, g\rangle$ $j=1,\ldots,n$. Then $\hat{f}_j$ vanishes off Ω, $\hat{f}_j \in L_2(D)$, $f_j \in L_2(F)$ and f_j is continuous. Hence by a theorem of Kluvanek (1965) the series expansion

$$(3.2.16) \qquad \sum_{h\in H} \langle \alpha_j, h\rangle \, S(g-h) = \langle \alpha_j, g\rangle$$

converges uniformly on F to the limit $\langle \alpha_j, g\rangle$. Hence by (3.2.14)

$$x(t,g,n) = \sum_\mu \sum_j \sum_h \langle \alpha_j, h\rangle \, S(g-h) x_{j,n}\left(\frac{\mu\pi}{\alpha}\right) \prod_{\ell=1}^{k} p_\ell$$

$$= \sum_\mu \sum_h x\left(\frac{\mu\pi}{\alpha}, h, n\right) S(g-h) \prod_{\ell=1}^{k} p_\ell .$$

We have proved

Theorem 3.2.4. If $\{x(g) : g \in G\}$ is a band limited m.s.c. stationary process indexed by a compactly generated group G, and H is a discrete subgroup of the compact factor of G with the property (3.2.15), then $x(g)$ has the sampling expansion

$$(3.2.17) \qquad x(g) = \sum_{\mu \in Z^n} \sum_{h \in H} x\left(\frac{\mu\pi}{\alpha}, h, n\right) S(f-h) \prod_{\ell=1}^{k} \frac{\sin\alpha\cdot(t_\ell - \mu_\ell \pi/\alpha)}{(t_\ell - \mu_\ell \pi/\alpha)}$$

where $g = (t,f,n)$, and the convergence is with probability 1.

Remark 3.2.5. The series (3.2.17) converges with probability one uniformly on compact sets of G. This follows from the uniform convergence of the series (3.2.13) and (3.2.16). □

3.3 General Band Limited Processes on IR and the Sampling Theorem.

In the previous two sections of this chapter we have alluded to the concept of a band limited m.s.c. stationary process, which on $\mathbb{R}$ is a process having a covariance R of the form

$$(3.3.1) \qquad R(t) = \int_{-W}^{W} e^{2\pi i\lambda t} \mu(d\lambda) \quad ,$$

and noted that the importance of these processes lies in the fact that almost all paths are analytic functions and that their paths may be reconstructed by a knowledge of the values of the process at the "sampling points" $k\pi/\alpha$, $k \in Z$, $\alpha > W$. It is of interest to consider processes that are not stationary, and to find conditions that such processes satisfy a sampling expansion, and have analytic paths.

Piranashvili (1967) proved that the sampling expansion (3.2.2) is satisfied by second order processes having covariance functions of the form

$$R(t,s) = \int_{\Lambda\times\Lambda} f(t,\lambda)\,\overline{f(s,\eta)}\,\mu(d\lambda,\,d\eta)\,.$$

where μ is of finite variation on the bounded set $\Lambda\times\Lambda$, and for each $\lambda\in\Lambda$, $f(\circ,\lambda)$ is a function of exponential type $\leq W$, say and $\sup_{t\in\mathbb{R}\,\lambda\in\Lambda} |f(t,\lambda)| < \infty$. In a different approach, Zakai (1965) considered second order processes $x(t)$ such that

$$\int_{-\infty}^{\infty} \frac{E|x(t)|^2}{1+t^2}\,dt < \infty \tag{3.3.2}$$

and found that such processes satisfy the sampling theorem (and have almost all paths analytic) if they can be reproduced without distortion when passed through a certain filter. Zakai's condition is "temporal" in contrast to the "spectral" condition of Piranashvili.

Below, we briefly describe Zakai's results and then use his methods to obtain similar results for more general processes than those satisfying (3.3.2).

Zakai considers functions f satisfying

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{1+t^2}\,dt < \infty \tag{3.3.3}$$

and defines such a function to be band limited (W,δ) if $f(t)$ is reproduced without distortion when passed through the filter h where h is the inverse Fourier transform of the function $H(W,\delta)$ in Fig. 3.3.1. That is to say, f is band limited (W,δ) if

$$f*h(t) = \int_{-\infty}^{\infty} f(s)h(t-s)ds = f(t) \quad \text{a.e.}$$

Zakai then defines a process to be band limited (W,δ) if almost all of

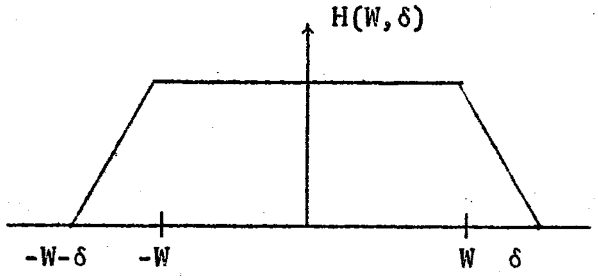

Fig. 3.3.1

its sample paths are band limited (W,δ) and it satisfies (3.3.2). He characterizes such functions and processes and shows that the sampling theorem is valid for them.

We will consider the class of all second order measurable processes $\{x(t), t \in \mathbb{R}\}$ such that there exists a polynomial p with

$$E|x(t)|^2 \leqq p(t) \quad \forall t \in \mathbb{R}. \tag{3.3.4}$$

If $R(t,s) = Ex(t)\overline{x(s)}$, then (3.3.4) implies that there exists a positive integer k such that

$$\int_{-\infty}^{\infty} \frac{R(t,t)}{(1+t^2)^k} dt < \infty . \tag{3.3.5}$$

We will denote by μ_k the measure with density $(1+t^2)^{-k}$ i.e., $d\mu_k = (1+t^2)^{-k} dt$ and by $L_2(\mu_k)$ the Hilbert space of all functions f with

$$||f||_k^2 = \int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)^k} dt < \infty .$$

We make the following definition:

Definition 3.3.1. Let $\phi(t) = \phi(t; W,\delta)$ be a C^∞ (i.e. infinitely differentiable) function whose Fourier transform $\hat{\phi}(x) = \hat{\phi}(x;W,\delta)$ exists and is a C^∞ function such that $\hat{\phi}(x;W,\delta) = 0$ for $x \notin [-W-\delta,W+\delta]$, $\hat{\phi}(x;W,\delta) = 1$ for $x \in [-W,W]$ and is real and symmetric about 0. We will say that a function f is band limited (W,δ) $(b\ell(W,\delta))$ if $f \in L_2(\mu_k)$ for some $k \geq 1$ and $f * \phi = f$ a.e. □

Remark 3.3.1. We should point out that if ψ is a C^∞ function with compact support $[-A,A]$ then there exists a rapidly decreasing function ϕ which is C^∞, and in $L_1 \cap L_2(\mathbb{R})$ such that $\hat{\phi} = \psi$ and ϕ can be extended to an entire function on $\mathbb{C}$ (complex numbers) that satisfies an inequality of the form (Donaghue (1969) p. 212):

(3.3.6) $$|\phi(z)| \leqq C_n(1+|z|)^{-n}e^{A|\mathrm{Im}\ z|}$$

for each positive integer n. We will consider ϕ to be such a function in the sequel. □

Lemma 3.3.1. For $f \in L_2(\mu_k)$ and ϕ as in Definition 3.3.1, $f * \phi$ exists, is in $L_2(\mu_k)$ and is continuous. $f * \phi$ may be extended to a continuous function u on $\mathbb{C}$ by

(3.3.7) $$u(z) = \int_{-\infty}^{\infty} f(t)\phi(z-t;W,\delta)dt \,.$$

u is entire if f is continuous and u satisfies

(3.3.8) $$|u(z)| \leqq C||f||_k(1+|z|)^k e^{(W+\delta)|\mathrm{Im}\ z|} \,.$$

Proof: From Remark 3.3.1 for every integer $n \geq 1$ there is a constant C_n such that

$$|\phi(z)| \leq C_n(1+|Z|)^{-n} e^{(W+\delta)|\text{Im } z|}.$$

Thus

$$\int_{-\infty}^{\infty} |f(t)|\ |\phi(x-t)|dt \leq C_{k+1} \int_{-\infty}^{\infty} |f(t)|(1+|x-t|)^{-k-1}\ dt$$

$$\leq C_{k+1} \int_{-\infty}^{\infty} \frac{|f(t)|}{(1+t^2)^{k/2}}\ \frac{(1+t^2)^{k/2}}{(1+|x-t|)^{k+1}}\ dt .$$

Now $(1+t^2)^{k/2}/(1+|x-t|)^{k+1}$ is in $L_2(\mathbb{R})$, so by the Cauchy-Schwartz inequality (3.3.9) is less than

(3.3.10)

$$C_{k+1}\left(\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)^k}\ dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \frac{(1+t^2)^k}{(1+|x-t|)^{2k}} \circ \frac{1}{(1+|x-t|)^2}\ dt\right)^{\frac{1}{2}}$$

But $(1+t^2) \leq 2(1+|x|)^2(1+|t-x|)^2$, so (3.3.10)is less than

$$C_{k+1}||f||_k\ 2^{k/2}(1+|x|)^k \int_{-\infty}^{\infty} \frac{1}{(1+|t|)^2}\ dt \quad < \infty$$

and so the convolution exists. To see that $f * \phi \in L_2(\mu_k)$, we proceed as follows:

$$\int_{-\infty}^{\infty} \frac{|f*\phi(x)|^2}{(1+x^2)^k}\ dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(t)\overline{f(s)}\phi(x-t)\phi(x-s)|}{(1+x^2)^k}\ dt\ ds\ dx$$

$$\leq (C_n)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t)||f(s)|(1+x^2)^{-k}(1+|x-t|)^{-n}(1+|x-s|)^{-n}\ dt\ ds\ dx$$

(3.3.11) $$\leq (C_n')^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t)|^2(1+x^2)^{-k}(1+|x-t|)^{-n}\ dt\ dx$$

since $2|a||b| \leqq |a|^2 + |b|^2$, where $(C_n')^2 = C_n^2 \int_{-\infty}^{\infty} (1+|s|)^{-n} ds$, $n > 1$.

Define $I_k(t) = \int_{-\infty}^{\infty} (1+x^2)^{-k}(1+(x-t)^2)^{-k} dx$.

Since $1 + t^2 \leqq 2(1+x^2)(1+(x-t)^2)$, it can be seen that

$$I_k(t) \leqq K_{(k)} I_{k-1}(t)/(1+t^2) \quad \text{for } k > 1 \text{ and some constant } K_{(k)}$$

and so $I_k(t) \leqq \frac{K_{(1)}}{(1+t^2)^{k-1}} I_1(t)$ for some constant $K_{(1)}$. But according to Zakai (1965), $I_1(t) \leqq \pi/1+t^2$. Thus $I_k(t) \leqq K_{(0)}(1+t^2)^{-k}$ for some constant $K_{(0)}$. Set $n = 2k$ in (3.3.11). Then (3.3.11) is less than

$$(C_{2k}')^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)|^2 (1+x^2)^{-k}(1+(x-t)^2)^{-k} \, dx \, dt$$

$$= (C_{2k}')^2 \int_{-\infty}^{\infty} |f(t)|^2 I_k(t) \, dt$$

$$\leqq (C_{2k}')^2 K_{(0)} \int_{-\infty}^{\infty} |f(t)|^2/(1+t^2)^k \, dt < \infty .$$

Hence

$$||f * \phi||^2 \leq (C_{2k}')^2 K_0 ||f||^2 . \tag{3.3.12}$$

To see that $U(x)$ is continuous, note that if $x_n \to x$ we have $f(t)\phi(x_n-t) \to f(t)\phi(x-t)$ for each t since ϕ is C^∞. Also

$$|f(t)\phi(x_n-t)| \leqq \frac{C_{2k}|f(t)|}{(1+t^2)^k} \; \frac{(1+t^2)^k}{(1+|x_n-t|)^{2k}}$$

$$\leqq C'(1+|x_n|)^{2k} |f(t)| (1+t^2)^{-k} .$$

Now $C'(1+|x_n|)^{2k} = 0(1)$ since x_n is a convergent sequence, and $f(t)(1+t^2)^{-k} \in L_1(\mathbb{R})$ since

$$\int_{-\infty}^{\infty} |f(t)|(1+t^2)^{-k}dt \le \left\{\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)^k}\,dt\right\}^{\frac{1}{2}} \left\{\int_{-\infty}^{\infty} \frac{1}{(1+t^2)^k}\,dt\right\}^{\frac{1}{2}} < \infty .$$

Thus we can apply the dominated convergence theorem to assert that

$$f * \phi(x_n) = \int_{-\infty}^{\infty} f(t)\phi(x_n-t)dt \quad \text{converges to} \quad \int_{-\infty}^{\infty} f(t)\phi(x-t)dt = f * \phi(x).$$

Thus $f * \phi = U$ is continuous. If f is continuous then U is entire by Theorem 2.84 of Titchmarsh (1939). Finally, (3.3.8) is proved in a similar way to the proof that the convolution exists; we omit the details.

□

<u>Remark 3.3.2</u>. If f is $b\ell(W,\delta)$ in our sense, then f is equal a.e. to a continuous function. Since f is defined only up to a set of measure zero, we take for f its continuous version $f * \phi$. Thus by the lemma, $f * \phi$ has an entire extension of $\mathbb{C}$. In the sequel, when we say that $f \in L_2(\mu_k)$ is $b\ell(W,\delta)$ we shall always mean this entire version of f. □

<u>Definition 3.3.2</u>. A function of the form

$$f(t) = \int_{-W}^{W} e^{2\pi i t x}\psi(x)\, dx$$

with $\psi \in L_2(-W,W)$ will be termed conventionally band limited (W) $(cb\ell(W))$. Such a function is bounded by a constant (polynomial), and is in $L_2(\mathbb{R})$. If ϕ is as in Definition 3.3.1 then $(f*\phi)\hat{}(s) = \hat{f}(s)\,\hat{\phi}(s) = \psi(s)$ since $\hat{\phi} = 1$ on $[-W,W]$ and $\psi(t) = 0$ off $[-W,W]$. Thus our definition contains the "conventional" definition as a special case.

□

Our next result shows how a $b\ell(W,\delta)$ function can be expressed in terms of $cb\ell(W+\delta)$ functions.

Theorem 3.3.1 (A) If $f \in L_2(\mu_k)$ is $b\ell(W,\delta)$ then its entire extension can be written

$$f(z) = \sum_{n=0}^{k-1} f^{(n)}(0)/n!/z^n + z^k g(z) \tag{3.3.13}$$

where $g(z)$ is the (entire) extension of a $cb\ell(W+\delta)$ function .

(B) Let $f(t) = \sum_{n=0}^{k-1} C_n t^n + t^k g(t)$ where g is $cb\ell(A)$. Then $f(t)$ is $b\ell(W,\delta)$ for $W > A$ and $\delta > 0$.

Proof: Define functions $g_0,\ldots,g_k$ recursively by

$$\begin{cases} g_0(z) = f(z) \\ g_{n+1}(z) = (g_n(z) - g_n(0))/z \qquad n=1,\ldots,k-1. \end{cases} \tag{3.3.14}$$

We claim that each g_n is entire, in $L_2(\mu_{k-n})$ and that

$$|g_n(z)| \leqq K_n(1+|z|)^{k-n}e^{(W+\delta)\mathrm{Im}|z|} \quad \text{for constants } K_n . \tag{3.3.15}$$

The claim is obviously true for $n = 0$ by Lemma 3.3.1. Suppose it is true for $n < k$. Expanding $g_n(z)$ in its Taylor series about 0 we obtain $g_n(z) = g_n(0) + z\psi(z)$ for some entire function ψ. But $\psi(z) = g_{n+1}(z)$ so g_{n+1} is entire.

Now consider $|z| \geq 1$. Then

$$\begin{aligned} |g_{n+1}(z)| &\leqq (|g_n(z)| + |g_n(0)|)/|z| \leqq \frac{K_n}{|z|}\{(1+|z|)^{k-n}e^{(W+\delta)|\mathrm{Im}\, z|}+1\} \\ &\leqq \frac{2K_n}{|z|}(1+|z|)^{k-n}e^{(W+\delta)|\mathrm{Im}\, z|} \\ &= 2K_n(1+|z|)^{k-n-1}\, e^{(W+\delta)|\mathrm{Im}\, z|}\left(\frac{1+|z|}{|z|}\right) \end{aligned}$$

$$\leq 4K_n(1+|z|)^{k-n-1}e^{(W+\delta)|\mathrm{Im}\ z|}.$$

For $|z| \leq 1$ $|g_{n+1}^{(z)}|$ is bounded by K'_{n+1} say, since g_{n+1} is entire. Set $K_{n+1} = \max(4K_n, K'_{n+1})$. Then (3.3.15) is satisfied for n + 1. To see that $g_{n+1} \in L_2(\mu_{k-n-1})$, it suffices to show that

$$\int\limits_{|t|\geq 1} |g_{n+1}(t)|^2 \mu_{k-n-1}(dt) < \infty$$

since g_{n+1} is continuous and hence integrable over every finite interval. Now

$$\int\limits_{|t|\geq 1} |g_{n+1}(t)|^2 d\mu_{k-n-1}(dt) \leq \int\limits_{|t|\geq 1} \frac{(1+t^2)}{|t|^2} \frac{(|g_n(t)|+|g_n(0)|)^2}{(1+t^2)^{k-n}}\, dt$$

$$\leq 2 \int\limits_{|t|\geq 1} \frac{|g_n(t)|^2+2|g_n(t)||g_n(0)|+|g_n(0)|^2}{(1+t^2)^{k-n}}\, dt$$

which is finite since $g_n \in L_2(\mu_{k-n})$. Thus the claim is satisfied for all n, n=0,...,k. From (3.3.14) it follows that

$$f(z) = \sum_{n=0}^{k-1} z^n g_n(0) + z^k g_k(z) . \tag{3.3.16}$$

Differentiating (3.3.16) successively and setting $z = 0$, we obtain

$$g_n(0) = f^{(n)}(0)/n! \quad n = 0,\dots,k-1$$

and so (3.3.13) is proved. Lastly, since $g_k \in L_2(\mathbb{R})$ and $|g_k(z)| \leq e^{(W+\delta)|\mathrm{Im}\ z|}$ it follows by the Paley-Weiner theorem (Boas (1954) p. 103) that g is cbℓ(W+δ).

B. Since f is not in general Lebesgue integrable, it is the Fourier transform not of a function but of a distribution. (See e.g. Donoghue 1969). Since f(z) is entire and

$$|f(z)| \leqq C(1+|z|)^k e^{A|\mathrm{Im}\, z|} \tag{3.3.17}$$

f is the Fourier transform of a distribution with compact support $[-A,A]$ (Donoghue (1969) p. 213). So $f = \hat{T}$ for some tempered distribution T, with support $[-A,A]$. Now if ϕ is as in Definition 3.3.1 with $W > A$ and $\delta > 0$, and ψ is any test function (i.e. C^∞ with compact support) set $\rho = (\hat{\phi}-1)\psi$, then ρ has compact support disjoint from $[-A,A]$. Then (for the symbols used see Donoghue (1969) §30)

$$\begin{aligned}(f*\phi)^\wedge(\psi) &= (\hat{T}*\phi)\psi \\ &= \hat{\hat{T}}(\hat{\phi}\psi) \\ &= \hat{\hat{T}}(\rho) + \hat{\hat{T}}(\psi) \\ &= \hat{\hat{T}}(\psi)\end{aligned}$$

since $\hat{\hat{T}} = \check{T}$, and so has the same support as T, hence $\hat{\hat{T}}(\rho) = 0$ as $\mathrm{supp}\, \hat{\hat{T}} \cap \mathrm{supp}\, \rho = \phi$.

Thus $(f*\phi)^\wedge = \hat{f}$ and so $f*\phi = f$ and f is bandlimited. □

We now turn to the sampling representation of a $b\ell(W,\delta)$ function f. Such a function is entire and satisfies (3.3.17) for some k, it follows that (Piranashvili (1967)) $f(z)$ has the sampling expansion

$$f(z) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\alpha}\right) \frac{\sin \alpha(z-n\pi/\alpha)}{\alpha(z-n\pi/\alpha)} \; \frac{\sin^k \beta(z-n\pi/\alpha)}{\beta^k (z-n\pi/\alpha)^k} \tag{3.3.18}$$

where $\alpha > W+\delta$ and $\beta < (\alpha-W-\delta)/k$.

If $S_N(z)$ denotes the partial sum of (3.3.18) obtained by summing from $-N$ to N then

$$|f(z) - S_N(z)| \leq \frac{K\, L(z)}{\beta^k(\alpha-W-\delta-k\beta)} \left(\left(\frac{\alpha}{N}\right)^{k+1} + \left(\frac{\alpha}{N}\right)\right)$$

where $L(z)$ is bounded on every bounded subset of $\mathbb{C}$. We will use this result to derive a sampling theorem for second order processes that are bounded in the following sense:

Definition 3.3.3. Let $\{x(t), t \in \mathbb{R}\}$ be a m.s.c. second order measurable process that satisfies (3.3.4) and hence satisfies (3.3.5) for some positive integer k. Since $x(t)$ is assumed measurable, almost all sample paths are in $L_2(\mu_k)$. We will say $x(t)$ is band limited (W,δ) if almost all of its paths are $b\ell(W,\delta)$.

In the same manner as Zakai (1965) we can prove the following result, whose proof is omitted.

Theorem 3.3.2. Let $\{x(t), t \in \mathbb{R}\}$ be a second order measurable process whose covariance R is continuous and satisfies (3.3.4). Then $x(t)$ is $b\ell(W,\delta)$ iff

$$(3.3.19) \qquad \int_{-\infty}^{\infty} R(t,v)\phi(s-v;W,\delta)dv = R(t,s)$$

for all t,s in $\mathbb{R}$. □

It is clear that a m.s.c. stationary process is $b\ell(W,\delta)$ if its spectral measure has compact support in $[-W,W]$, so that this definition of a band limited process extends the usual definition. From the definition it follows that if $x(t)$ is $b\ell(W,\delta)$ then

$$(3.3.20) \qquad x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n\pi}{\alpha}\right) \frac{\sin \alpha(t-n\pi/\alpha)}{\alpha(t-n\pi/\alpha)} \frac{\sin^k \beta(t-n\pi/\alpha)}{\beta^k(t-n\pi/\alpha)^k}$$

with probability one, for α,β as in (3.3.18) so our band limited processes admit a sampling expansion.

We now turn to the problem of characterizing the band limited processes in terms of simpler processes.

Theorem 3.3.3. Let $\{x(t) t \in \mathbb{R}\}$ be $b\ell(W,\delta)$ with covariance R satisfying (3.3.5). Then

$$x(t) = \sum_{n=0}^{k-1} \frac{x^{(n)}(0)}{n!} t^n + t^k y(t) , \tag{3.3.21}$$

where $x^{(n)}$ is the n-th mean square derivative of $x(t)$, $y(t)$ is a harmonizable process whose spectral measure is concentrated on $[-W-\delta, W+\delta]^2$ and $E|y(t)|^2 = R_y(t,t) \in L_1(\mathbb{R})$.

Proof: Since x is $b\ell(W,\delta)$ its covariance function R satisfies (3.3.19) and hence by another application of (3.3.19)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(u,v)\, \overline{\phi(t-u)}\, \phi(s-v)\, dv\, du = R(t,s) .$$

Using the methods of Lemma 3.3.1, R can be extended to an entire function $R(\xi,\eta)$ which satisfies

$$|R(\xi,\eta)| \leqq C(1+|\xi|)^k (1+|\eta|)^k e^{(W+\delta)(|\mathrm{Im}\, z|+|\mathrm{Im}\, \eta|)} .$$

Since the covariance R has partial derivatives of all orders, the mean square derivatives of the process $x(t)$ all exist.

Define new processes $v_n(t)$ recursively by

$$\begin{cases} v_0(t) = x(t) \\ v_{n+1}(t) = (v_n(t) - v_n(0)/t \quad n=1,\ldots,k-1 \end{cases} \tag{3.3.21}$$

whose covariances R_n are given by

$$\begin{cases} R_0(t,s) = R(t,s) \\ R_{n+1}(t,s) = (R_n(t,s) - R_n(t,0) - R_n(0,s) \\ \qquad\qquad + R_n(0,0))/ts \quad . \end{cases} \tag{3.3.22}$$

As in Theorem 3.3.1 we can show that for each $n = 0,\ldots,k$ R_n is analytic, $R_n(t,t) \in L_1(\mu_{k-n})$ and

$$|R_n(t,t)| \leqq C_n(1+|t|)^{2(k-n)}$$

for some constants C_n. Thus

$$x(t) = \sum_{n=0}^{k-1} \frac{x^{(n)}(0)}{n!} t^n + t^k y(t)$$

where $y(t) = v_k(t)$. It thus remains to prove that $y(t)$ is a process of the required type. Since $R_y = R_k$ is bounded, and $R_y(t,t) \in L_1(R)$, $R_y(t,s) \in L_2(\mathbb{R}^2)$ and hence its Fourier transform $\psi(x,y)$ exists. By copying the proof of Theorem 6 in Zakai (1965) we can see that $\psi(x,y)$ is zero off $[-(W+\delta),W+\delta]^2$ and so the result follows. □

Remark 3.3.3. A measurable harmonizable process $\{x(t),t\in\mathbb{R}\}$ whose spectral measure is concentrated on $[-A,A]^2$ is $b\ell(W,\delta)$ for $W > A$ and $\delta > 0$. To see this, consider

$$R(t,s) = \int_{-A}^{A}\int_{-A}^{A} e^{2\pi i(tu-sv)}\mu(du,dv) .$$

Then

$$\begin{aligned}
&\int_{-\infty}^{\infty} R(t,\lambda)\phi(s-\lambda,W,\delta)d\lambda \\
&= \int_{-\infty}^{\infty} \phi(s-\lambda) \int_{-A}^{A}\int_{-A}^{A} e^{2\pi i(tu-\lambda v)}\mu(du,dv)\ d\lambda \\
&= \int_{-A}^{A}\int_{-A}^{A} e^{2\pi itu} \int_{-\infty}^{\infty} e^{-2\pi i\lambda v}\phi(s-\lambda)d\lambda\ \mu(du,dv) \\
&= \int_{-A}^{A}\int_{-A}^{A} e^{2\pi i(tu-sv)}\hat{\phi}(-v)\ \mu(du,dv)
\end{aligned}$$

$$= \int_{-A}^{A}\int_{-A}^{A} e^{2\pi i(tu-sv)}\ \mu(du,dv) = R(t,s)$$

since $\hat{\phi}(-v) = \int_{-\infty}^{\infty} e^{2\pi i\lambda v}\phi(\lambda)d\lambda$ is equal to 1 on $[-A,A]$ for $W > A$. Thus by (3.3.19) $x(t)$ is $b\ell(W,\delta)$. If in addition $R(t,t) \in L_1(R)$ then almost all sample paths of $x(t)$ are in $L_1(R)$ and so by Theorem 3.3.1 almost all sample paths are $cb\ell(W')$ for $W' > A$.

Alternative proof to Theorem 3.3.3. An alternative proof which gives some insight into the form of the series expansion of a $b\ell(W,\delta)$ process is presented below. Since $x(t)$ is m.s.c. and satisfies 3.3.5, it can be expanded (Cambanis and Masry (1971)) in a series.

(3.3.23)
$$x(t,\omega) = \sum_{n=1}^{\infty} a_n(t)\xi_n(\omega)$$

where the series converges in mean square, the orthonormal random variables $\xi_n(\omega)$ are given by

(3.3.24)
$$\xi_n(\omega) = \int_{-\infty}^{\infty} x(t,\omega)\overline{f_n(t)}\mu_k(dt)$$

almost surely (the functions f_n are complete in $L_2(\mu_k)$) and the functions $a_n(t)$ are given by

(3.3.25)
$$a_n(t) = E\ x(t)\overline{\xi}_n = \int_{-\infty}^{\infty} R(t,s)\overline{f_n(s)}\mu_k(ds) .$$

We need the following lemma.

Lemma 3.3.2. If x is $b\ell(W,\delta)$ the functions $a_n(t)$ are also $b\ell(W,\delta)$.

Proof: The a_n's are all in $L_2(\mu_k)$ since $|a_n(t)|^2 \leq R(t,t)\ \forall t, \forall n$.

Also

$$(3.3.26) \qquad \int_{-\infty}^{\infty} a_n(s)\phi(t-s)ds = \int_{-\infty}^{\infty} E(x(s)\overline{\xi}_n)\phi(t-s)ds \;.$$

But $\displaystyle\int_{-\infty}^{\infty} E|x(s)||\overline{\xi}_n||\phi(t-s)|ds \leqq C \int_{-\infty}^{\infty} R(s,s)^{\frac{1}{2}}(1+|t-s|)^{-(k+1)}\,ds$

which is finite by a now familiar argument.

Thus we can apply Fubini's theorem to the right side of (3.3.26) to get

$$E(\overline{\xi}_n \int_{-\infty}^{\infty} x(s,\omega)\phi(t-s)ds) = E\;\overline{\xi}_n x(t) = a_n(t)$$

since $x(t)$ is band limited (W,δ). □

Returning to the alternate proof of Theorem 3.3.3: because the ξ_n's are a c.o.n.s (complete orthonormal system) for $H(x)$, the Hilbert space spanned by the $x(t)$, and because each derivative $x^{(j)}(t)$ of the process $x(t)$ is in $H(x)$, we can express $x^{(j)}(t)$ as a series converging in mean square, i.e., $x^{(j)}(t) = \sum_{n=1}^{\infty} E(x^{(j)}(t)\overline{\xi}_n)\xi_n$. It is easy to verify that $E\,x^{(j)}(t)\overline{\xi}_n = a^{(j)}(t)$, and so

$$(3.3.27) \qquad x^{(j)}(t) = \sum_{n=1}^{\infty} a_n^{(j)}(t)\;\overline{\xi}_n \;. \qquad j = 0,1,2,\ldots$$

Note that each a_n is $b\ell(W,\delta)$ and hence analytic, and in $L_2(\mu_k)$, so by Theorem 3.3.1,

$$(3.3.28) \qquad a_n(t) = \sum_{j=0}^{k-1} \frac{a_n^{(j)}(0)}{j!}\,t^j + t^k\,b_n(t)$$

where each b_n is $cb\ell(W+\delta)$. From (3.3.27) and (3.3.28) we get

$$x(t) = \sum_{j=0}^{k-1}\sum_{n=1}^{\infty} \frac{a_n^{(j)}(0)}{j!}\,t^j\xi_n + t^k \sum_{n=1}^{\infty} b_n(t)\xi_n$$

$$= \sum_{j=0}^{k-1} \frac{x^{(j)}(0)}{j!} t^j + t^k y(t) ,$$

where $y(t) = \sum_{n=1}^{\infty} b_n(t)\xi_n$. The convergence of this series is deduced by summing (3.3.28). As in the first proof we deduce that $E|y(t)|^2$ is Lebesgue integrable, hence,

$$(3.3.29) \qquad \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |b_n(t)|^2 dt = \int_{-\infty}^{\infty} E|y(t)|^2 dt < \infty .$$

Now each b_n is cbℓ(W+δ) so for each n there is a function $\psi_n(x) \in L_2(-W-\delta, W+\delta)$ such that

$$b_n(t) = \int_{-(W+\delta)}^{W+\delta} e^{2\pi ixt} \psi(x) dx .$$

By the Parseval theorem,

$$\int_{-\infty}^{\infty} |b_n(t)|^2 dt = \int_{-W-\delta}^{W+\delta} |\psi_n(x)|^2 \, dx .$$

Hence by (3.3.29)

$$(3.3.30) \qquad \sum_{n=1}^{\infty} \int_{-W-\delta}^{W+\delta} |\psi_n(x)|^2 dx < \infty .$$

Thus
$$R_y(t,s) = \sum_{n=1}^{\infty} b_n(t) \overline{b_n(s)}$$

$$= \sum_{n=1}^{\infty} \int_{-W-\delta}^{+W+\delta} \int_{-W-\delta}^{W+\delta} e^{2\pi i(tx-sy)} \psi_n(x) \overline{\psi_n(y)} \, dx \, dy .$$

Now
$$\sum_{n=1}^{\infty} \int_{-W-\delta}^{W+\delta} \int_{-W-\delta}^{W+\delta} |\psi_n(x)| |\psi_n(y)| \, dx \, dy$$

$$= \sum_{n=1}^{\infty} \left(\int_{-W-\delta}^{W+\delta} |\psi_n(x)|dx \right)^2$$

$$\leq 2(W+\delta) \sum_{n=1}^{\infty} \int_{-W-\delta}^{W+\delta} |\psi_n(x)|^2 dx < \infty$$

by the Cauchy-Schwartz inequality and (3.3.30).

Thus $\sum_{n=1}^{\infty} \psi_n(x)\overline{\psi_n(y)}$ converges to a function $\psi(x,y)$ integrable on $[-W-\delta,W+\delta]^2$ and

$$\sum_{n=1}^{\infty} \int_{-W-\delta}^{W+\delta} \int_{-W-\delta}^{W+\delta} e^{2\pi i(tx-sy)} \psi_n(x)\overline{\psi_n(y)}\, dx\, dy \text{ converges to}$$

$$\int_{-W-\delta}^{W+\delta} \int_{-W-\delta}^{W+\delta} e^{2\pi i(tx-sy)} \psi(x,y)\, dx\, dy .$$

Thus R_y is harmonizable with a a.c. spectral measure concentrated on $[-W-\delta,W+\delta]$.

□

The next result is a converse to Theorem 3.3.3.

Theorem 3.3.4. Let y be a harmonizable process whose spectral measure is absolutely continuous and concentrated on $[-A,A]^2$ and whose covariance R_y satisfies $R_y(t,t) \in L_1(\mathbb{R})$. If $C_0,\ldots,C_{k-1}$ are random variables with $E|C_n|^2 < \infty$, $n = 0,\ldots,k-1$ then the process

$$x(t) = \sum_{n=0}^{k-1} C_n t^n + t^k y(t)$$

is $b\ell(W,\delta)$ for all $W > A$ and $\delta > 0$.

Proof: By Remark 3.3.3, almost every sample function of y is $cb\ell(W)$. From Theorem 3.3.1(B) it follows that almost every sample path of x is $b\ell(W,\delta)$ and in $L_2(\mu_k)$. Thus $x(t)$ is $b\ell(W,\delta)$, $W > A$, $\delta > 0$. $\square$

Remark 3.3.4. If $x(t)$ is $b\ell(W,\delta)$ with covariance $R(t,s)$ satisfying (3.3.5), then the sample paths of x are $b\ell(W,\delta)$ and in $L_2(\mu_k)$. The same is true for the expansion coefficient functions a_n and also for the functions $R(\cdot,s)$ for each s. In fact we can say more; the closed subspaces of $L_2(\mu_k)$ generated by these three sets of functions coincide. This is a consequence of the following general theorem.

Theorem 3.3.5. Let $\{x(t), t \in \mathbb{R}\}$ be a measurable m.s.c. second order process whose covariance R satisfies

$$\int_{-\infty}^{\infty} R(t,t)\mu(dt) < \infty \tag{3.3.31}$$

for some finite measure μ, with μ mutually absolutely continuous with respect to Lebesgue measure and let $x(t)$ have the expansion (3.3.23). If $S(a)$, $S(x)$ and $S(R)$ denote the closed subspaces of $L_2(\mu)$ generated by the coefficient functions a_n, almost all the sample paths and the functions $\{R(\cdot,s) : s \in \mathbb{R}\}$ respectively, then $S(a) = S(x) = S(R)$.

Proof: First we show that $S(a) = S(R)$. For each fixed $s \in R$

$$R(t,s) = \sum_{n=1}^{\infty} a_n(t)\,\overline{a_n(s)} \;; \tag{3.3.32}$$

the series (3.3.32) converges absolutely. Let $A_N(t) = \sum_{n=1}^{N} a_n(t)\,\overline{a_n(s)}$.

Then $||R(\cdot,s) - A_N(\cdot)||^2_{L_2(\mu)} = \int_{-\infty}^{\infty} |R(t,s) - \sum_{n=1}^{N} a_n(t)\,\overline{a_n(s)}|^2 \mu(dt)$

the integrand of the right side converges pointwise everywhere to zero.

Also

$$
\begin{aligned}
\left|R(t,s) - \sum_{n=1}^{N} a_n(t)\,\overline{a_n(s)}\right| &\leq |R(t,s)| + \sum_{n=1}^{\infty} |a_n(t)|\,|a_n(s)| \\
&\leq (R(t,t)R(s,s))^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} |a_n(t)|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} |a_n(s)|^2\right)^{\frac{1}{2}} \\
&= (R(t,t)R(s,s))^{\frac{1}{2}}
\end{aligned}
$$

so $\left|R(t,s) - \sum_{n=1}^{N} a_n(t)\,\overline{a_n(s)}\right|^2 \leq 4R(t,t)R(s,s) \in L_1(\mu)$. So by the dominated convergence theorem it follows that $A_N(\cdot)$ converges to $R(\cdot,s)$ in $L_2(\mu)$ and thus $S(R) \subseteq S(a)$. To prove equality in this inclusion, let $a \in S(a)$ be orthogonal to each $R(\cdot,s)$. Then

$$\int_{-\infty}^{\infty} R(t,s)\,\overline{a(t)}\,\mu(dt) = 0 \qquad \forall s \in \mathbb{R}$$

which implies

$$(a_n,a) = \int_{-\infty}^{\infty} a_n(t)\,\overline{a(t)}\,\mu(dt)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(t,s)\overline{f_n(s)}\,\overline{a(t)}\,\mu(ds)\,\mu(dt)\ . \tag{3.3.33}$$

But $R(t,s) \in L_2(\mathbb{R} \times \mathbb{R},\ \mu \times \mu)$ since $R(t,t) \in L_1(\mathbb{R},\mu)$. Also $\overline{f_n(s)}\,\overline{a(t)} \in L_2(R \times R,\ \mu \times \mu)$ since each factor is in $L_2(\mu)$. Thus by Holder's inequality $R(t,s)\overline{\zeta_n(s)}\,\overline{a(t)} \in L_1(R \times R,\ \mu \times \mu)$ and we can appeal to Fubini's theorem to assert that (3.3.33) is equal to

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(t,s)\,\overline{a(t)}\,\mu(dt)\,\overline{f_n(s)}\,\mu(ds) = 0$. Thus $(a_n,a) = 0\ \forall n$ and

so $a = 0$, and $S(R) = S(a)$. Next we show that $S(x) = S(R)$. By theorem 8 of Cambanis and Masry (1971) the series (3.3.23) converges in $L_2(\mu)$ almost surely. Thus if Ω_0 denotes the set of probability zero where convergence fails, $x(t,\omega) \in S(a)$ $\forall\omega \notin \Omega_0$. Thus if $S(x) = \overline{sp}\{x(\cdot,\omega),\ \omega \in \Omega-\Omega_0\}$ $S(x) \subseteq S(a) = S(R)$. To prove the reverse inclusion, suppose that $f \in S(R)$ is orthogonal to each $x(\cdot,\omega)$ $\omega \in \Omega-\Omega_0$. Then

$$\int_{-\infty}^{\infty} f(t)\ x(t,\omega)\mu(dt) = 0 \quad \forall\omega \in \Omega-\Omega_0$$

so

$$0 = E\left|\int_{-\infty}^{\infty} \overline{f(t)}\ x(t,\omega)\ \mu(dt)\right|^2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(t)}\ R(t,s)f(s)\ \mu(dt)\ \mu(ds)\ . \tag{3.3.34}$$

Now let R be the operator defined on $L_2(\mu)$ by

$$Rf(t) = \int_{-\infty}^{\infty} f(s)R(t,s)\mu(ds). \tag{3.3.35}$$

R is a compact symmetric operator on $L_2(\mu)$ so every element f of $L_2(\mu)$ can be represented as

$$f = h + \sum_{n=1}^{\infty} (f,e_n)e_n \tag{3.3.36}$$

where h is the projection of f on the subspace $\{f\colon Rf = 0\}$ and the e_n's are eigenvectors forming a c.o.n.s. for the closure of the range of R, which is the closed subspace spanned by these eigenvectors of R corresponding to non-zero eigenvalues $\{\lambda_n\}$ of R. (See Reisz and Sz-Nagy (1955) p. 242. Then by (3.3.35) and (3.3.36) $(Rf,f) = \sum_{n=1}^{\infty} \lambda_n|(f,e_n)|^2$ and so by (3.3.34) and (3.3.35) $(f,e_n) = 0$. Thus f is in the null space of R. Also (Cambanis and Masry 1971)

$$R(t,s) = \sum_{n=1}^{\infty} \lambda_n e_n(t)\, \overline{e_n(s)} \tag{3.3.37}$$

the series (3.3.37) converging in $L_2(\mu_k)$ in each variable, so $R(\cdot,s)$ is in the closure of the range of R, and $S(R) \subseteq \overline{\text{Range } R}$. But $f \in \overline{\text{Range } R}^{\perp}$, $f \in S(R) \subseteq \overline{\text{Range } R}$ so $f = 0$. Hence $S(R) = S(x)$.

□

Remark 3.3.5. As noted in Remark 3.3.4, the three equal subspaces $S(R)$, $S(x)$ and $S(a)$ are generated by band limited functions if the process $x(t)$ is band limited. In fact, all the functions in these three subspaces are band limited. To prove this, we argue as follows. The linear manifold of finite linear combinations of the a_n's consists of band limited functions, since

$$\begin{aligned}(C_1 a_{n_1} + C_2 a_{n_2}) * \phi &= C_1(a_{n_1} * \phi) + C_2(a_{n_2} * \phi)\\ &= C_1 a_{n_1} + C_2 a_{n_2} .\end{aligned}$$

Now if $h \in S(a)$, there exists a sequence of functions h_n converging to h in $L_2(\mu_k)$ with each h_n a linear combination of the a_n's and hence each h_n is band limited, and

$$\begin{aligned}||h - h*\phi|| &\leqq ||h - h_n|| + ||h_n - h*\phi||\\ &= ||h - h_n|| + ||(h_n - h)*\phi||\\ &\leqq ||h - h_n|| + C'_{2k}(K_{(0)})^{\frac{1}{2}}||h_n - h|| \quad \text{by (3.3.12)}\\ &\leqq ||h - h_n||\,(1 + C'_{2k}(K_{(0)})^{\frac{1}{2}})\end{aligned}$$

Thus h is band limited and $S(a)$ contains only band limited functions.

CHAPTER IV

STOCHASTIC PROCESSES TAKING VALUES IN A HILBERT SPACE

In this chapter we consider a further generalization of a stochastic process. Instead of considering processes that are collections of complex-valued random variables, we now consider collections of random elements that take values in a separable Hilbert space H. These include as a special case the concept of a multivariate stochastic process, where the Hilbert space is the n-dimensional Euclidean space $\mathbb{R}_n$. Hilbert space valued processes have been considered by several authors. A basic source is the paper of Payen [1967], who defines second order Hilbert space valued processes, and shows how such a process may be realized as a family of operators. He defines the covariance operator of such a process and characterizes this class of operators. He also gives a definition of a stationary Hilbert space valued process and obtains a spectral representation and a moving average representation.

We will use several of Payen's results in the sequel. The integration of operator-valued functions will also play a role in what follows; for the basic facts of this theory, see e.g. Hille and Phillips (1957). Several results of Mandrekar and Salehi (1970) are also used.

In section 4.1 we introduce the concept of Hilbert space valued (H-valued) random elements, and show how they can be identified with a class of Hilbert-Schmidt operators in the case when they have second absolute moments. We have relied heavily on Payen (1967), and for the theory of Hilbert-Schmidt operators, on Shatten (1960). In section 4.2 we define a second order H-valued stochastic process, and show how it can be realized as a family of Hilbert-Schmidt operators. We define the covariance operator of such a process, and show that if the process is stationary in a sense to be defined, then the covariance operator can be decomposed into weakly continuous and locally a.e. zero components as in Theorem 2.2.2. The corresponding decomposition is given for processes. In section 4.3 we give the spectral representation of a stationary process. Our approach is based on Bochner's theorem, in contrast with that of Payen, who uses Stone's theorem to obtain his spectral representation. We prove a generalized Bochner's theorem, using a result of Mandrekar and Salehi (1970) to extend a theorem of Falb (1969). We rely heavily on the concept of the trace measure, as in Mandrekar and Salehi, to prove a spectral representation. Finally, in section 4.4, we take up the questions of path continuity and sampling considered in Chapter III, and give Hilbert space analogues of theorems proved there.

4.1. Random Elements Taking Values in a Hilbert Space

Much of this section is an elaboration of Payen (1967), and the material on tensor products is found in Schatten (1950).

Let H be a separable Hilbert space, H^* its dual. Define a map $\sim : H \to H^*$ by $\tilde{f}(h) = (h,f)$. $\sim$ is a conjugate linear isomorphism between H and H^* (Schatten (1950)). Let K be another Hilbert space, not necessarily separable. For $(f \in H, k \in K)$ denote by $f \otimes k$ the map $H^* \to K$ given by

$$f \otimes k(\tilde{h}) = (f,h)k \; . \tag{4.1.1}$$

It is easily seen that $f \otimes k$ is a finite rank bounded linear operator from H^* to K with norm $||f|| \; ||k||$ (see e.g. Schatten (1959)). Let $H \odot K$ denote the vector subspace of $L(H^*,K)$ (the space of bounded linear maps $H^* \to K$) generated by $f \otimes k$ for $f \in H$, $k \in K$. On $H \odot K$ we may define an inner product by

$$\left(\sum_{i=1}^{n} h_i \otimes k_i \; , \; \sum_{j=1}^{m} h_j' \otimes k_j' \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} (h_i, h_j')_H (k_i, k_j')_K \tag{4.1.2}$$

where the inner products on the right are those of the spaces H and K. The completion of $H \odot K$ with respect to this inner product is a Hilbert space which we denote by $H \otimes K$ - the tensor product of H and K. Now $H \odot K$ is a vector space of finite rank operators. Since H is reflexive, it follows (see Schatten (1950) p. 25) that $H \odot K$ is precisely the class of all finite rank operators from H^* to K. Let now $\{\phi_n\}_{n=1}^{\infty}$ be a complete orthonormal system (c.o.n.s.) for H. Then $\{\tilde{\phi}_n\}_{n=1}^{\infty}$ is a c.o.n.s. for H^*. Denote by $HS(H^*,K)$ the set of all Hilbert-Schmidt operators $H^* \to K$ i.e. the set of all bounded operators $A: H^* \to K$ for which $\sum_{n=1}^{\infty} ||A\tilde{\phi}_n||_K^2 < +\infty$. $HS(H^*,K)$ is a Hilbert space with inner product $(\circ,\circ)_{HS}$ given by

$$(A,B)_{HS} = \sum_{n=1}^{\infty} (A\tilde{\phi}_n, B\tilde{\phi}_n) \; . \tag{4.1.3}$$

The finite rank operators $H^* \to K$ are dense in $HS(H^*,K)$ in the topology induced by the Hilbert-Schmidt inner product $(\circ,\circ)_{HS}$. Moreover on

$H \otimes K$ the inner product (4.1.2) agrees with the inner product (4.1.3). It follows that $H \otimes K$ and $HS(H^*,K)$ are equal up to isomorphism. In the sequel we will identify $HS(H^*,K)$ with $H \otimes K$.

We now turn to a discussion of H-valued random elements.

Definition 4.1.1. Let $(\Omega,\mathcal{B},P)$ be a probability space. A function $x\colon \Omega \to H$ is weakly measurable if $(x(\omega),h)_H$ is measurable in the usual sense for all $h \in H$, strongly measurable if it is the limit a.e. [P] of a sequence of simple functions of the form $\sum_{i=1}^{n} h_i \chi_{B_i}$ where $h_i \in H$, $B_i \in \mathcal{B}$, $i = 1,\dots,n$, the limit being taken in the topology of H. Since H is assumed separable, the notions of strong and weak measurability coincide (see Hille and Phillips (1957) p. 73) so we will just speak of measurable maps. A measurable map $\Omega \to H$ is a random element.

□

Let $L_2(\Omega,H)$ be the set of all random elements x such that $\int_\Omega ||x(\omega)||^2 dP < +\infty$. $L_2(\Omega,H)$ is a Hilbert space under the inner product

$$(x,y) = \int_\Omega (x(\omega),y(\omega))\, dP\ .$$

The following theorem is fundamental (see Payen (1967)).

Theorem 4.1.1. Let x be a random element in $L_2(\Omega,H)$. Then there exists a unique operator X in $HS(H,L_2(\Omega,C))$ such that $Xh = (h,x)$. Conversely, if $X \in HS(H,L_2(\Omega,C))$ there exists a unique element x in $L_2(\Omega,H)$ such that $Xh = (h,x)$. This correspondence between $HS(H,L_2(\Omega,C))$ and $L_2(\Omega,H)$ is an isometry.

Proof: First we note that random elements of the form $\sum_{i=1}^{n} \xi_i(\omega)h_i$ with $\xi_i \in L_2(\Omega,C)$ are dense in $L_2(\Omega,H)$. Define a map $T : L_2(\Omega,H) \to H \otimes L_2(\Omega,C)$ ($L_2(\Omega,C)$ is the space of all complex r.v.'s ξ on Ω with $E|\xi|^2 < \infty$) by $T(\sum_{i=1}^{n} \xi_i(\omega)h_i) = \sum_{i=1}^{n} h_i \otimes \xi_i$. T is well defined and preserves inner products, so can be extended to an isometry between $L_2(\Omega,H)$ and $H \otimes L_2(\Omega,C)$. By the previous remarks of this section, we can regard Tx as a Hilbert Schmidt operator $H^* \to L_2(\Omega,H)$. Let X be the function $H \to L_2(\Omega,C)$ defined by $Xh = \overline{Tx(\tilde{h})}$ where the bar denotes complex conjugate. We now must prove that $Xh = (h,x)$ where (h,x) is the random variable $(h,x)(\omega) = (h,x(\omega))$, and that $X \in HS(H,L_2(\Omega,C))$. Since $x(\omega)$ is measurable, $(h,x(\omega))$ is measurable for each h, and is in $L_2(\Omega,C)$ since $E|(h,x(\omega))|^2 \leq E||x(\omega)||^2||h|| < \infty$.

First suppose that $x(\omega) = \sum_{i=1}^{n} \xi_i(\omega)h_i$, then

$$Tx(\tilde{h}) = (\sum_{i=1}^{n} h_i \otimes \xi_i)(\tilde{h}) = \sum_{i=1}^{n} (h_i,h)\xi_i = (\sum h_i\xi_i,h) = (x,h)$$

so $Xh = (h,x)$. Now let $x \in L_2(\Omega,H)$, then there exists a sequence x_n of random elements of the above form such that $x_n \to x$ in $L_2(\Omega,H)$. It follows that $Tx_n\tilde{h}$ converges to $Tx\,\tilde{h}$ in $L_2(\Omega,C)$ for each h since

$$\begin{aligned} ||Tx_n\tilde{h} - Tx\,\tilde{h}||_{L_2(\Omega,C)} &\leq ||Tx_n - Tx||\;||h|| \\ &= ||x_n - x||\;||h|| \to 0 . \end{aligned}$$

Also (x_n,h) converges to (x,h) in $L_2(\Omega,C)$. Hence

$$Tx\,\tilde{h} = \lim Tx_n\tilde{h} = \lim(x_n,h) = (x,h)$$

and so $Xh = (h,x)$ for all $x \in L_2(\Omega,H)$, $\forall h \in H$. This representation also shows that X is unique. Finally we must show that $X \in HS(H, L_2(\Omega, C))$. It is easy to see that X is linear, and $||X\phi_n||^2 = E|X(\phi_n)|^2 = E|Tx\tilde{\phi}|^2 = ||Tx\tilde{\phi}||^2$ so X is HS since since Tx is.

Conversely, if $X \in HS(H, L_2(\Omega, C))$ then the map

$$\tilde{X} : H^* \to L_2(\Omega, C) \quad \text{defined by} \quad \tilde{X}(\tilde{h}) = \overline{X(h)}$$

is in $HS(H^*, L_2(\Omega, C))$, and so there exists an $x \in L_2(\Omega, H)$ with $Tx = \tilde{X}$, and so $Xh = (h, x)$. □

4.2 *H*-valued Stochastic Processes

Definition 4.2.1. Let G be an LCA group and $\{x(g) : g \in G\}$ a family of random elements with values in the separable Hilbert space H. If for each $g \in G$ we have $E||x(g)||^2 < \infty$ we will say that $x(g)$ is a second order H-valued process.

Remark 4.2.1. For each $g \in G$, let X_g be the Hilbert-Schmidt operator associated with $x(g)$ as in Theorem 4.1.1. It is clear that the closed subspace of $L_2(\Omega, H)$ generated by the $x(g)$ and the closed subspace of $HS(H, L_2(\Omega, C))$ generated by the X_g are isometric. Then nothing is lost by identifying the process $x(g)$ with the family of operators X_g.

Definition 4.2.2. Let $\{x(g) : g \in G\}$ be a second order H-valued process, and $\{X_g : g \in G\}$ the corresponding family of operators. The covariance operator of the process $x(g)$ is the operator valued function $V(g,g') = X^*_{g'} X_g$, where $*$ denotes the adjoint. $V(g,g')$ is an operator $\mathcal{H} \to \mathcal{H}$ and since $X_{g'}$ and X_g are Hilbert-Schmidt, $V(g,g')$ is trace class. If the function $V(g,g')$ depends only on $g-g'$, the process $x(g)$ is said to be weakly stationary. For brevity, we will use the term "stationary" in the sequel, and we shall also refer to the operators X_g as a "stochastic process."

Definition 4.2.3. A map $V : G \times G \to \mathcal{L}(\mathcal{H},\mathcal{H})$ (the class of bounded linear operators $\mathcal{H} \to \mathcal{H}$) is of positive type if for any finite subset $\{h_1,\ldots,h_n\}$ of $\mathcal{H}$, and $\{g_1,\ldots,g_n\}$ of G

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (V(g_i,g_j)h_i,h_j) \geq 0 \ . \tag{4.2.1}$$

Theorem 4.2.1. (Payen) The following are equivalent.

(a) The map $V : G \times G \to \mathcal{L}(\mathcal{H},\mathcal{H})$ is of positive type.

(b) There exists a probability space $(\Omega,\mathcal{B},P)$ and a family of operators $\mathcal{H} \to L_2(\Omega,\mathcal{H})$ such that $X^*_{g'},X_g = V(g,g')$.

Definition 4.2.4. A map $V : G \to \mathcal{L}(\mathcal{H},\mathcal{H})$ is positive definite if for any subset $\{h_1,\ldots,h_n\}$ of $\mathcal{H}$ and $\{g_1,\ldots,g_n\}$ of G

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (V(g_i-g_j)h_i,h_j) \geq 0 \tag{4.2.2}$$

The following result is a corollary of Payen's theorem.

Theorem 4.2.2. The following are equivalent.

(a) V is a positive definite map $G \to T(\mathcal{H},\mathcal{H})$. ($T(\mathcal{H},\mathcal{H})$ denotes the set of trace class operators $\mathcal{H} \to \mathcal{H}$.)

(b) There exists a probability space $(\Omega,\mathcal{B},P)$ and a stationary process $\{X_g\}$ of Hilbert-Schmidt operators $\mathcal{H} \to L_2(\Omega,C)$ such that $X^*_{g'},X_g = V(g-g')$. □

Our next result is an analog of Theorem 2.2.2. We first need a definition.

Definition 4.2.5. Following Hille and Phillips (1957) p. 74, we say that a map $V : G \to L(H,H)$ is weakly measurable if for every $h, h' \in H$ the map $g \to (V(g)h,h')$ is measurable in the usual sense, and strongly measurable if $V(g)h$ is strongly measurable in the sense of Definition 4.1.1 for all $h \in H$. □

We can now state our theorem.

Theorem 4.2.3. Let $V : G \to T(H,H)$ be a weakly measurable trace class operator valued function on G. Then there exist unique weakly measurable trace class operator valued functions $V^{(1)}$ and $V^{(2)}$ on G such that

(i) $V(g) = V^{(1)}(g) + V^{(2)}(g)$.

(ii) The map $g \to V^{(1)}(g)$ is weakly continuous.

(iii) $V^{(2)}(g) = 0$ locally almost everywhere with respect to the Haar measure on G.

Proof: The proof is similar to that of Theorem 2.2.2. Let $\mathbb{E}$ denote the vector space of all maps $G \to H$ that are identically zero except at a finite number of points $g_1,\ldots,g_n$. Every element f of $\mathbb{E}$ has a canonical representation

$$f(g) = \sum_{i=1}^{n} h_i \varepsilon_{g_i}(g) \tag{4.2.1}$$

where

$$\varepsilon_{g_i}(g) = \begin{cases} 1 & g = g_i \\ 0 & g \neq g_i \end{cases}$$

and the g_i are distinct elements of G. Define a map $p : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$

by $$p\left(\sum_{i=1}^{n} h_i \varepsilon_{g_i}, \sum_{j=1}^{m} h'_j \varepsilon_{g'_j}\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} (V(g_i - g'_j)h_i, h'_j) \ .$$

As in the scalar case it can be shown that $\mathbb{E}_V = \{f \in \mathbb{E} : p(f,f) = 0\}$ is a subspace of $\mathbb{E}$, and that the equation

$$(f + \mathbb{E}_V, f' + \mathbb{E}_V) = p(f,f')$$

defines an inner product on the quotient space $\mathbb{E}/\mathbb{E}_V$. Let $\mathcal{H}_V$ denote the completion of $\mathbb{E}/\mathbb{E}_V$ with respect to this inner product. On $\mathbb{E}/\mathbb{E}_V$ the equation

$$U_g(f + \mathbb{E}_V) = f_g + \mathbb{E}_V$$

where $f_g(g') = f(g' - g)$ defines a unitary operator on $\mathbb{E}/\mathbb{E}_V$ which may be extended to a unitary operator on $\mathcal{H}_V$. We thus obtain a unitary group $\{U_g : g \in G\}$ of unitary operators on $\mathcal{H}_V$. As in the scalar case, since V is weakly measurable, we can find a closed subspace S of $\mathcal{H}_V$ such that

(i) $U_g S \subseteq S$, $U_g S^{\perp} \subseteq S^{\perp}$ $\quad \forall_g \quad G$

(ii) for all $\eta \in \mathcal{H}_V$ the map $g \to (U_g \xi, \eta)$ is continuous if $\xi \in S$ and locally a.e. zero if $\xi \in S^{\perp}$.

Let P, Q be the projection operators on S and $S^{\perp}$ so that $Q = I - P$, $PQ = QP = 0$. If e denotes the identity of G, and $h \in \mathcal{H}$, let $[h] = h\varepsilon_e + \mathbb{E}_V$, then $[h] \in \mathcal{H}_V$. Denote by ψ_g the map $\mathcal{H} \times \mathcal{H} \to C$ given by

$$\psi_g(h,h') = (U_g P[h], P[h'])_{\mathcal{H}_V} \ .$$

It is easy to see that ψ_g is a bilinear functional, moreover

$$|\psi_g(h,h')| \leqq ||U_g P[h]||_{H_v} ||P[h']||_{H_v}$$
$$\leqq ||[h]||_{H_v} ||[h']||_{H_v}$$

since $||P|| = 1$, $||U_g|| = 1$.

But $||[h]||^2_{H_v} = p(h\varepsilon_e, h\varepsilon_e) = (V(e)h,h) \leqq ||V_{(e)}||\ ||h||^2$.

Thus $|\psi_g(h,h')| \leqq ||V(e)||\ ||h||\ ||h'||$ so that ψ_g is a bounded bilinear functional. Thus there exists an operator $V^{(1)}(g)$ such that

$$(V^{(1)}(g)\ h,h') = (U_g P[h], P[h']).$$

That $V^{(1)}$ is weakly continuous follows from the properties of the subspace S. To see that $V^{(1)}$ is positive definite,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (V^{(1)}(g_i - g_j)h_i, h_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} (U_{g_i} P[h_i],\ U_{g_j} P[h_j])$$
$$= ||\sum_{i=1}^{n} U_{g_i} P[h_i]||^2 \geq 0 \ .$$

In a similar manner we can show that there is a positive definite operator function $V^{(2)}$ such that $V^{(2)} = 0$ locally a.e. and

$$(V^{(2)}(g)h,h') = (U_g Q[h],\ Q[h']) \quad .$$

Moreover
$$(V(g)h,h') = p(h\varepsilon_g, h'\varepsilon_e)$$
$$= (U_g[h],[h'])$$
$$= (U_g P[h], P[h']) + (U_g Q[h], Q[h'])$$
$$= (V^{(1)}(g)h,h') + (V^{(2)}(g)h,h') \ .$$

It only remains to prove that "$V^{(1)}$ and $V^{(2)}$ are trace class valued". and that the decomposition is unique. The latter is proved by the method used in the scalar case; the former is a consequence of the next theorem.

$\square$

Theorem 4.2.4. Let X_g be a stationary process of Hilbert-Schmidt operators corresponding to the stationary H-valued process $\{x(g), g \in G\}$. Suppose V, the covariance operator of the process is weakly measurable. If $V = V^{(1)} + V^{(2)}$ is the decomposition of V described in the previous theorem, then there exist stationary operator processes $X_g^{(1)}$, $X_g^{(2)}$ corresponding to stationary H-valued processes $x_1(g)$, $x_2(g)$ such that

(i) $X_g = X_g^{(1)} + X_g^{(2)}$ and $x(g) = x_1(g) + x_2(g)$.

(ii) $X_{g'}^{(i)*} X_g^{(i)} = V^{(i)}(g-g')$ where $V^{(i)}$ is the covariance operator of x_i, $i = 1,2$.

(iii) $X_{g'}^{(1)*} X_g^{(2)} = E(x_1(g'), x_2(g)) = 0 \quad \forall g, g' \in G$.

(iv) If $H(X)$, $H(X^{(1)}$, $H(X^{(2)})$ denote the closed subspaces of $HS(H, L_2(\Omega, C))$ generated by the processes X, $X^{(1)}$ and $X^{(2)}$ and $H(x)$, $H(x_1)$ $H(x_2)$ denote the closed subspaces of $L_2(\Omega, H)$ generated by the processes x, x_1 and x_2 then

$$H(X) = H(X^{(1)} \oplus H(X^{(2)}) \text{ and } H(x) = H(x_1) \oplus H(x_2).$$

Proof: Consider the map $L : \mathbb{E}/\mathbb{E}_V \to L_2(\Omega, C)$ given by $L(\sum_{i=1}^{n} h_i \varepsilon_{g_i} + \mathbb{E}_V) = \sum_{i=1}^{n} X_{g_i} h_i$. L is well defined and preserves inner products, and can be extended to an isometry from H_V to the closed subspace M of $L_2(\Omega, C)$ generated by $\{X_g h : g \in G, h \in H\}$. Let $\tilde{P} = LPL^{-1}$, where P is the projection onto S of the last theorem.

The $\tilde{P}$ is a projection.

Define $X_g^{(1)} = \tilde{P}X_g$. Then $X_g^{(1)}$ is Hilbert-Schmidt since it is the composition of a bounded operator with a Hilbert-Schmidt operator. Moreover

$$(X_{g'}^{(1)*}X_g^{(1)}h,h')_{H} = (PX_gh,\ PX_{g'}h')_{L_2(\Omega,C)}$$

$$= (LPL^{-1}X_gh,\ LPL^{-1}X_{g'}h)_{L_2(\Omega,C)}$$

(4.2.2) $$= (PU_g[h],\ PU_{g'}[h'])_{H_V}$$

Now S reduces U_g for each g, so P commutes with U_g and (4.2.2) equals

$$(U_{g-g'}P[h],\ P[h'])_{H_V}$$

$$= (V^{(1)}(g-g')h,h')_{H}$$

so $X_{g'}^{(1)*}X_g^{(1)} = V^{(1)}(g-g')$. Similarly let $\tilde{Q} = LQL^{-1}$ where Q is the projection onto $S^{\perp}$ of the last theorem. Define $X_g^{(2)} = \tilde{Q}X_g$ and $V^{(2)}(g-g') = X_{g'}^{(2)*}X_g^{(2)}$. Thus $X_g^{(1)}$ and $X_g^{(2)}$ are stationary, and $V^{(1)}$ and $V^{(2)}$ are trace-class valued. Since $\tilde{P}$ and $\tilde{Q}$ satisfy

$$\tilde{P} + \tilde{Q} = I,\ X_g = X_g^{(1)} + X_g^{(2)} \quad \forall_g \in G\ .$$

Also $X_{g'}^{(1)*}X_g^{(2)} = X_{g'}^{*}\tilde{P}\tilde{Q}X_g = 0$ since $PQ = 0$. The decomposition is unique since it is comprised of projections. The last statement follows as in the scalar case. The assertions for the process $x(g)$ follow from the above and Theorem 4.1.1. □

As in the scalar case we use these results to obtain sufficient conditions for a stationary H-valued process to be weakly continuous, i.e. for $(V(g)h,h')$ to be continuous. First we note that, exactly as in Chapter I, if H_V is separable, then $V^{(2)}(g) \equiv 0$ i.e. $V(g)$ is weakly continuous. In terms of the process, $V(g)$ is weakly continuous if $H(X)$ is separable.

Definition 4.2.6. An H-valued process x is measurable if for every $h \in H$, $(x(g,\omega),h)$ is measurable with respect to the product σ-field of subsets of $G \times \Omega$. $\square$

It is clear that if x is measurable and X_g is the "operator" process corresponding to it, then $X_g h$ (regarded as a complex stochastic process) is product measurable for each $h \in H$ and conversely, since $X_g h(\omega) = (h,x(g,\omega))_{L_2(\Omega,C)}$.

It follows from Theorem 2.2.4 that if $M(h)$ denotes the closed linear subspace of $L_2(\Omega,C)$ generated by $\{X_g h : g \in G\}$, then $M(h)$ is separable. Let D be a countable dense set in H , and let M be as in the last theorem. It is obvious that $M(h) \subseteq M$ for all $h \in H$, so if $\overline{[A]}$ is the closed linear manifold generated by a subset A of M we have

$$\overline{[\bigcup_{h\in D} M(h)]} \subseteq M$$

Conversely if $X_g \xi \in M$, let h_n be a sequence in D converging to $\xi \in H$. Then $X_g h_n$ converges to $X_g \xi$ in $L_2(\Omega,C)$ as $n \to \infty$. Since each $X_g h_n \in \bigcup_{h\in D} M(h)$, it follows that

$$X_g \xi \in \overline{[\bigcup_{h\in D} M(h)]} \text{ and so } M = \overline{[\bigcup_{h\in D} M(h)]} .$$

Each $M(h)$ is separable and D is countable, so M is separable. But M is isomorphic to H_v so H_v is separable, hence $V(g)$ is weakly continuous. We have proved:

Theorem 4.2.5. Let $\{X_g : g \in G\}$ be a stationary process of Hilbert-Schmidt operators. If X_g is weakly measurable (i.e. if the processes

$X_g h$ are measurable for all $h \in H$) then the covariance $V(g)$ of the process is weakly continuous.

4.3. The Spectral Representation of Weakly Continuous Stationary H-valued Processes.

In this section we derive the spectral representation of a weakly continuous process using methods different from those of Payen (1967). We use a verion of Bochner's theorem which is an extension of a result of Falb (1969), and draw on results of Mandrekar and Salehi (1970). We prove a general representation theorem to obtain the final result, that is more general than Payen's representation theorem. Our first result is a theorem of Mandrekar and Salehi; we give a proof based on a technique of Dincleanu (1967) p. 263 since this is not given in the paper of Mandrekar and Salehi.

Theorem 4.3.1. Let M be a set function on a measurable space $(S,\mathcal{S})$ that takes values in the set of positive trace class operators $T^+(H,H)$. Suppose also that M is weakly countably additive, i.e. if $\{\Delta_n\}_{n=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{S}$, then

$$(M(\bigcup_{n=1}^{\infty} \Delta_n)x,y) = \sum_{n=1}^{\infty} (M(\Delta_n)x,y) \tag{4.3.1}$$

for all $x,y \in H$. Then

(1) If τ is the set function defined on the sets in $\mathcal{S}$ by $\tau(\Delta) = \text{trace } M(\Delta)$ then τ is a positive finite measure on $\mathcal{S}$.

(2) There exists a strongly measurable operator valued function $M'(s)$ on S such that $M'(s)$ is positive and trace class a.e. $[\tau]$ and

$$M(\Delta) = \int_\Delta M'(s)\tau(ds) \; . \tag{4.3.2}$$

Remark 4.3.1. The integral (4.3.2) is interpreted in the following sense. Since $M'(s)$ is strongly measurable, $M'(s)h$ is a measurable H-valued function on S. τ is a positive measure, so the integral $\int_\Delta M'(s)h\tau(ds)$ exists as a Bochner integral (see e.g. Hille and Phillips (1957) p. 79) iff $\int_\Delta ||M'(s)h||_H\tau(ds) < \infty$. Moreover, if $\int_\Delta M'(s)h\tau(ds)$ exists as a Bochner integral for each $h \in H$ then (Hille and Phillips (1957) p. 85) the operator A given by

$$Ah = \int_\Delta M'(s)h\tau(ds)$$

is a bounded operator. The theorem asserts this operator is just $M(\Delta)$.

$\square$

Proof of Theorem 4.3.1: (1) Let $\{\Delta_n\}_{n=1}^\infty$ be a disjoint sequence in S and $\{\phi_k\}_{k=1}^\infty$ a c.o.n.s. in H. Then

$$\begin{aligned} \tau(\bigcup_{n=1}^\infty \Delta_n) &= \text{trace } M(\cup\Delta_n) \\ &= \sum_{k=1}^\infty (M(\bigcup_{n=1}^\infty \Delta_n)\phi_k,\phi_k) \\ &= \sum_{k=1}^\infty \sum_{n=1}^\infty (M(\Delta_n)\phi_k,\phi_k) \end{aligned} \tag{4.3.3}$$

But $M(\Delta_n)$ is a positive operator for each n so $(M(\Delta_n)\phi_k,\phi_k) \geq 0$ $\forall n$, $\forall k$. Thus we may rearrange the order of summation (4.3.3) and obtain

$$\sum_{n=1}^\infty \sum_{k=1}^\infty (M(\Delta_n)\phi_k,\phi_k) = \sum_{n=1}^\infty \text{trace } M(\Delta_n) = \sum_{n=1}^\infty \tau(\Delta_n)$$

so that τ is countably additive. That τ is positive and finite follows from the fact that $M(\Delta)$ is a positive trace class operator for each $\Delta \in S$.

(2) For each $x,y \in H$ let $\mu_{xy}(\Delta) = (M(\Delta)x,y)$. Since M is weakly countably additive, μ_{xy} is a complex measure. Let $|\mu_{xy}|$ denote the variation of μ_{xy}. Then

$$\begin{aligned} |\mu_{xy}|(\Delta) &= \sup\{\sum_{k=1}^{n} |\mu_{xy}(\Delta_k)| : \{\Delta_k\}_{k=1}^{n} \text{ is a partition of } \Delta\} \\ &\leqq \sup_{\{\Delta_k\}} \sum_{k=1}^{n} ||M(\Delta_k)||_u ||x|| \; ||y|| \\ &\leqq \sup_{\{\Delta_k\}} \sum_{k=1}^{\infty} \tau(\Delta_k) ||x|| \; ||y|| \\ &= \tau(\Delta) ||x|| \; ||y|| \end{aligned} \tag{4.3.4}$$

since $||A||_u$ (the uniform norm of A) is less than $\tau(A)$ for all positive trace class operators A. By (4.3.4) $|\mu_{xy}|$ is absolutely continuous with respect to τ. Hence by the Radon-Nikodym theorem there exists a complex function g_{xy} such that for all $\Delta \in S$

$$\mu_{xy}(\Delta) = \int_{\Delta} g_{xy}(s)\tau(ds) \qquad \text{and} \qquad |\mu_{xy}|(\Delta) = \int_{\Delta} |g_{xy}(s)|\tau(ds) .$$

Since $|\mu_{xy}|(\Delta) \leqq \tau(\Delta)||x|| \; ||y||$, it follows that

$$|g_{xy}(s)| \leqq ||x|| \; ||y|| \qquad \text{a.e. } [\tau]. \tag{4.3.5}$$

Without loss of generality we can modify g_{xy} on a set of zero τ-measure and assume that

$$|g_{xy}(s)| \leqq ||x|| \; ||y|| \quad \text{for all } s \in S. \tag{4.3.6}$$

It is easy to see that the map $\langle x,y\rangle \to g_{xy}(s)$ is a bilinear function on H for each fixed s; (4.3.6) shows that it is a bounded bilinear functional. Hence there is a bounded operator $M'(s)$ on H to H such that

$$(M'(s)x,y) = g_{xy}(s) \quad \forall s \in S, \ \forall x,y \in H.$$

(4.3.6) shows that $||M'(s)|| \leq 1$ $\forall s \in S$. Since g_{xy} is measurable, $M'(s)$ is weakly measurable and hence strongly measurable since H is separable. Now $||M'(s)x|| \leq ||M'(s)||\ ||x|| \leq ||x||$ and τ is finite so $\int_{\Delta}||M'(s)x||\tau(ds) < \infty$ $\forall x \in H$. Thus the equation $A_{\Delta}(x) = \int_{\Delta} M'(s)x\tau(ds)$ defines a bounded operator A_{Δ} for each $\Delta \in S$.

But

$$\begin{aligned}(A_{\Delta}x,y) &= \int_{\Delta}(M'(s)x,y)\tau(ds)\\ &= \int_{\Delta} g_{xy}(s)\tau(ds)\\ &= \mu_{xy}(\Delta)\\ &= (M(\Delta)x,y)\end{aligned}$$

so $M(\Delta) = \int_{\Delta} M'(s)\tau(ds)$ in the sense of Remark 4.3.1. It remains to prove that $M'(s)$ is positive and trace class.

Now $(M'(s)x,x) = g_{xx}(s) \geq 0$ a.e. $[\tau]$ since $\mu_{xx}(\Delta) = (M(\Delta)x,x) \geq 0$ for all $\Delta \in S$. Similarly

$$\sum_{k=1}^{\infty} (M'(s)\phi_k,\phi_k) = \sum_{k=1}^{\infty} g_{\phi_k,\phi_k}(s)$$

and

$$\int_{\Delta}\sum_{k=1}^{\infty} g_{\phi_k,\phi_k}(s)\tau(ds) = \sum_{k=1}^{\infty}\int_{\Delta} g_{\phi_k,\phi_k}(s)\tau(ds)$$

$$= \sum_{k=1}^{\infty}(M(\Delta)\phi_k,\phi_k)$$

$$= \text{trace } M(\Delta)$$

so $0 \leq \sum_{k=1}^{\infty} g_{\phi_k,\phi_k}(s) < \infty$ a.e.$[\tau]$ since $0 \leq \tau(\Delta) < \infty \; \forall\Delta$ and so $M'(s)$ is trace class a.e. $[\tau]$. □

We now turn to a discussion of orthogonal operator valued measures and integrals. We first make a definition.

Definition 4.3.1. A set function Z defined on the measurable space $(S,\mathcal{S})$ is an orthogonal Hilbert-Schmidt (H.-S.) measure if for every set $\Delta \in \mathcal{S}$, $Z(\Delta) \in HS(\mathcal{H},L_2(\Omega,C))$ and

(i) For any sequence $\{\Delta_n\}_{n=1}^{\infty}$ of disjoint sets of $\mathcal{S}$

$$\lim_{N\to\infty} \left|\left| \sum_{n=1}^{N} Z(\Delta_n) - Z(\bigcup_{n=1}^{\infty} \Delta_n)\right|\right|_{HS} = 0 \ .$$

(ii) If Δ and Δ' are disjoint sets in $\mathcal{S}$ then $Z(\Delta)^* Z(\Delta) = 0$. □

There exists a correspondence between orthogonal H.-S. measures and measures of the type considered in Theorem 4.3.1 which is described in the following theorem:

Theorem 4.3.2. Let Z be an orthogonal H.-S. measure on a measurable space $(S,\mathcal{S})$. Then the set function M defined on $(S,\mathcal{S})$ by $M(\Delta) = Z(\Delta)^* Z(\Delta)$ is a $T^+(\mathcal{H},\mathcal{H})$ valued set function satisfying the hypotheses of Theorem 4.3.1. Conversely if M is a $T^+(\mathcal{H},\mathcal{H})$ valued set function satisfying the hypotheses of Theorem 4.3.1, there exists an orthogonal H.-S. measure Z with $Z(\Delta)^*Z(\Delta') = M(\Delta \cap \Delta')$.

Proof: If M is a set function on $(S,\mathcal{S})$ defined by $M(\Delta) = Z(\Delta)^*Z(\Delta)$, it is clear that for each $\Delta \in \mathcal{S}$, $M(\Delta) \in T^+(\mathcal{H},\mathcal{H})$. Moreover, M is weakly countably additive. To see this, let $\{\Delta_n\}$ be a disjoint sequence of sets of $\mathcal{S}$. Since Z is strongly countably additive, for $x,y \in \mathcal{H}$ we have

$$\begin{aligned}
(M(\bigcup_{n=1}^{\infty} \Delta_n)x,y)_{\mathcal{H}} &= (Z(\bigcup_{n=1}^{\infty} \Delta_n)x,\ Z(\bigcup_{n=1}^{\infty} \Delta_n)y)_{\mathcal{H}} \\
&= \lim_{N\to\infty}(\sum_{n=1}^{N} Z(\Delta_n)x,\ \sum_{n=1}^{N} Z(\Delta_n)y) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{m=1}^{N} (Z(\Delta_n)x,\ Z(\Delta_m)y) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} (Z(\Delta_n)x,\ Z(\Delta_n)y) \\
&= \sum_{n=1}^{\infty} (M(\Delta_n)x,y) .
\end{aligned}$$

Thus the set function M satisfies the requirements of Theorem 4.3.1 and so $M(\Delta) = \int_{\Delta} M'(s)\tau(ds)$ for some strongly measurable $T^+(\mathcal{H},\mathcal{H})$ valued function M'.

Conversely, if M is a set function on $(S,\mathcal{S})$ satisfying the hypotheses of Theorem 4.3.1, define a map F on the Cartesian product of $\mathcal{S}$ with itself by $F(\Delta,\Delta') = M(\Delta \cap \Delta')$. F takes values in $\mathcal{L}(\mathcal{H},\mathcal{H})$, the space of all bounded linear operators $\mathcal{H} \to \mathcal{H}$. Moreover if $\Delta_1,\ldots,\Delta_n \in \mathcal{S}$ and $h_1,\ldots,h_n \in \mathcal{H}$ then

$$\sum_{i=1}^{n} \sum_{j=1}^{n}(F(\Delta_i,\Delta_j)h_i,h_j) = \sum_{i=1}^{n} \sum_{j=1}^{n}(M(\Delta_i \cap \Delta_j)h_i,h_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\Delta \cap \Delta_j} (M'(s)h_i, h_j)\tau(ds)$$

$$= \int_S (M'(s) \sum_{i=1}^{n} \chi_{\Delta_i}(s)h_i, \sum_{j=1}^{n} \chi_{\Delta_j}(s)h_j)\tau(ds) \geq 0$$

since $M'(s)$ is a positive operator. Thus we can apply a theorem of Payen (1967) and assert that there exists a set function Z on $(S,\mathcal{S})$ taking values in $\mathcal{L}(\mathcal{H}, L_2(\Omega, C))$ such that

$$Z(\Delta)^* Z(\Delta') = M(\Delta \cap \Delta') \quad \forall \Delta, \Delta' \in \mathcal{S}.$$

Let $\{\phi_k\}_{k=1}^{\infty}$ be a c.o.n.s. in $\mathcal{H}$, then

$$\sum_{k=1}^{\infty} ||Z(\Delta)\phi_k||_{\mathcal{H}}^2 = \sum_{k=1}^{\infty} (Z(\Delta)^* Z(\Delta)\phi_k, \phi_k)_{\mathcal{H}}$$

$$= \text{trace } M(\Delta) < \infty$$

so $Z(\Delta) \in HS(\mathcal{H}, L_2(\Omega, C)) \quad \forall \Delta \in \mathcal{S}$.

Using the property that $Z(\Delta)^* Z_0(\Delta') = M(\Delta \cap \Delta')$ it is easy to see that

$$||\sum_{k=1}^{N} Z(\Delta_k) - Z(\bigcup_{k=1}^{\infty} \Delta_k)||_{HS}^2 = \tau(\bigcup_{k=1}^{\infty} \Delta_k) - \sum_{k=1}^{N} \tau(\Delta_k)$$

so Z is countably additive in H.-S. norm, and $Z(\Delta)^* Z(\Delta') = 0$ if $\Delta \cap \Delta' = \emptyset$. Thus Z is an orthogonal H.-S. measure. □

Let us now define an operator valued integral of the form

$$\int_S f(s)Z(ds) \quad \text{for } f \in L_2(S, \mathcal{S}, \tau).$$

If f is a simple function, $f(s) = \sum_{i=1}^{n} c_i \chi_{\Delta_i}(s)$ say, define

$$\int f(s)Z(ds) = \sum_{i=1}^{n} c_i Z(\Delta_i) \ .$$

Let f' be another simple function $f'(s) = \sum_{j=1}^{m} c_j' \chi_{\Delta_j'}(s)$. Then

$$\left(\int_S f(s)Z(ds)\right)^* \left(\int_S f'(s)Z(ds)\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{c_i} c_j' Z(\Delta_i)^* Z(\Delta_j) \ . \tag{4.3.7}$$

It is easy to see that property (ii) of Definition 4.3.1 implies that $Z(\Delta)^* Z(\Delta') = Z(\Delta \cap \Delta')^* Z(\Delta \cap \Delta') = M(\Delta \cap \Delta')$ so that (4.3.7) is equal to

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \overline{c_i} c_j' \ M(\Delta_i \cap \Delta_j')$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \int_S c_i \ c_j' \chi_{\Delta_i}(s) \chi_{\Delta_j'}(s) \ M'(s)\tau(ds)$$

$$= \int_S \overline{f(s)} \ f'(s) \ M'(s)\tau(ds) \ .$$

Thus for all simple functions f and f'

$$\left(\int_S f(s)Z(ds)\right)^* \left(\int_S f'(s)Z(ds)\right) = \int_S \overline{f(s)} \ f'(s) \ M'(s)\tau(ds). \tag{4.3.8}$$

Let now Δ be any set in S. Then if ϕ_k is a c.o.n.s. in H

$$\int_\Delta \text{trace } M'(s) \ (ds) = \int_\Delta \sum_{k=1}^{\infty} (M'(s)\phi_k, \phi_k)\tau(ds)$$

$$= \sum_{k=1}^{\infty} \left(\int_\Delta M'(s)\tau(ds)\phi_k, \phi_k \right)$$

$$= \text{trace } M(\Delta)$$

$$= \tau(\Delta)$$

so that

$$\text{trace } M'(s) = 1 \quad \text{a.e. } [\tau]. \tag{4.3.9}$$

From (4.3.8) we obtain

$$\left(\int_S f(s)Z(ds), \int_S f'(s)Z(ds) \right)_{HS} = \text{trace}\left(\int_S f(s)Z(ds) \right)^* \left(\int_S f'(s)Z(ds) \right)$$

$$= \text{trace} \int_S \overline{f(s)}\, f'(s)\, M'(s)\tau(ds) \; . \tag{4.3.10}$$

Let now $\{\phi_k\}_{k=1}^{\infty}$ be a c.o.n.s. for H. Then (4.3.10) is equal to

$$\sum_{k=1}^{\infty} \int_S f(s)f'(s)(M'(s)\phi_k,\phi_k)\tau(ds). \tag{4.3.11}$$

Since $M'(s)$ is a positive operator for each $s \in S$ and since

$$\sum_{k=1}^{\infty} \int_S |f(s)||f'(s)|(M'(s)\phi_k,\phi_k)\tau(ds) = \int_S |f(s)||f'(s)| \sum_{k=1}^{\infty}(M'(s)\phi_k,\phi_k)\tau(ds)$$

$$= \int |f(s)||f'(s)| \text{trace } M'(s)\tau(ds)$$

$$= \int |f(s)||f'(s)|\tau(ds) < \infty$$

by (4.3.9), we may interchange sum and integral in (4.3.11) which is now equal to

$$\int_S \overline{f(s)}\, f'(s) \text{ trace } M'(s)\tau(ds)$$

$$= \int_S \overline{f(s)}\, f'(s)\tau(ds).$$

Thus

$$\left(\int_S f(s)Z(ds), \int_S f'(s)Z(ds) \right)_{HS} = \int_S \overline{f(s)}\, f'(s)\tau(ds) \; . \tag{4.3.12}$$

Now let f be any function in $L_2(S,\mathcal{S},\tau)$, then there is a sequence $\{f_n\}$ of simple functions such that $\{f_n\}$ converges to f in $L_2(S,\mathcal{S},\tau)$; (4.3.12) shows that the sequence $\{\int_S f_n(s)Z(ds)\}$ is a Cauchy sequence in $HS(\mathcal{H},L_2(\Omega,C))$. We define the integral $\int_S f(s)Z(ds)$ to be the limit of this Cauchy sequence; it can be shown that the limit is independent of the particular sequence chosen. The sequence also converges in the uniform norm, consequently if $f, f' \in L_2(S,\mathcal{S},\tau)$ and $\{f_n\},\{f'_n\}$ are sequences of simple functions converging in $L_2(S,\mathcal{S},\tau)$ to f and f', then $(\int_S f(s)Z(ds))^*(\int_S f'(s)Z(ds))$ is the limit uniform norm of the sequence $\{(\int_S f_n(s)Z(ds))^*(\int_S f'_n(s)Z(ds))\}_{n=1}^{\infty}$ which by (4.3.8) is the same sequence as $\{\int_S \overline{f_n(s)}f'(s)\ M'(s)\tau(ds)\}$. Now

$$\left|\left|\int_S \overline{f_n(s)}\ f'_n(s)M'(s)\tau(ds) - \int_S \overline{f(s)}\ f'(s)M'(s)\tau(ds)\right|\right|_u$$

$$\leqq \int \left|\overline{f_n(s)}f'_n(s) - \overline{f(s)}f'(s)\right|\ ||M'(s)||_u\tau(ds)$$

$$\leqq \int_S \left|\overline{f_n(s)}f'_n(s) - \overline{f(s)}f'(s)\right|\tau(ds)$$

which converges to zero as $n \to \infty$. Thus (4.3.8) and (4.3.12) are true for all functions in $L_2(S,\mathcal{S},\tau)$.

We now state and prove a general representation theorem from which we will derive a spectral representation of stationary processes.

Theorem 4.3.3. Let Z be an orthogonal H.-S. measure on a measurable space $(S,\mathcal{S})$ and let M, M' and τ be as above. Let $f(g,s)$ be a function on $G \times S$ such that for each $g \in G$, $f(g,\cdot) \in L_2(S,\mathcal{S},\tau)$. Then if

$$X_g = \int_S f(g,s)Z(ds)$$

is a process of Hilbert-Schmidt operators, the covariance operator of X_g is given by

$$(4.3.13) \qquad X^*_{g'}X_g = \int_S f(g,s)f(g',s)M'(s)\tau(ds) .$$

Conversely, if M is a measure on $(S,\mathcal{S})$ taking values in $T^+(\mathcal{H},\mathcal{H})$ and satisfying the hypotheses of theorem 4.3.1, and X_g is a process of Hilbert-Schmidt operators having covariance (4.3.13) then there exists an orthogonal H.-S. measure Z on $(S,\mathcal{S})$ such that $Z(\Delta')^*Z(\Delta) = M(\Delta' \cap \Delta)$ $\forall\Delta, \Delta' \in \mathcal{S}$ and $X_g = \int f(g,s)Z(ds)$.

Proof: The first assertion of the theorem follows from the proceeding remarks. For the converse, let Z_0 be an orthogonal H.-S. measure such that $Z_0(\Delta)^*Z_0(\Delta') = M(\Delta \cap \Delta')$. Such a measure exists by theorem 4.3.2. Define a process Y_g by $Y_g = \int_S f(g,s)Z_0(ds)$. Then by the first part of the theorem we see that $X^*_{g'}X_g = Y^*_{g'}Y_g = \int_S f(g,s)\overline{f(g',s)}M'(s)\tau(ds)$. Define now a map T from the manifold generated by the Y_g to that generated by the X_g by $T(\sum_{i=1}^{n} c_i Y_{g_i}) = \sum_{i=1}^{n} c_i X_{g_i}$. T is clearly linear and

$$\begin{aligned} ||T(\sum_{i=1}^{n} c_i Y_{g_i})||^2_{HS} &= ||\sum_{i=1}^{n} c_i X_{g_i}||^2_{HS} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j} \text{ trace } X_{g_j}{}^{*}X_{g_i} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \text{ trace } Y_{g_j}{}^{*}Y_{g_i} \\ &= ||\sum_{i=1}^{n} c_i Y_{g_i}||^2_{HS} . \end{aligned}$$

Thus T preserves norms and is onto, so T may be extended to an isometry between $H(Y)$ and $H(X)$, the closed subspaces generated by the Y_g's and X_g's. Define $Z(\Delta) = TZ_0(\Delta)$. Then $Z(\Delta) \in HS(\mathcal{H}, L_2(\Omega,C))$ $\forall \Delta \in \mathcal{S}$ and is countably additive in H.-S. norm since Z_0 is. Also

$$Z(\Delta')^* Z(\Delta) = Z_0(\Delta')^* T^* T Z_0(\Delta) = Z_0(\Delta')^* Z_0(\Delta) = M(\Delta \cap \Delta')$$

since $T^*T = I$ because T is an isometry; thus Z is an orthogonal H.-S. measure. Moreoever, if f is a simple function, it is easy to see that

$$T \int_S f(s) Z_0(ds) = \int_S f(s) Z(ds) \; . \tag{4.3.14}$$

A simple passage to the limit shows that (4.3.14) is true for all $f \in L_2(S,\mathcal{S},\tau)$, hence

$$\int f(g,s) Z(ds) = T \int f(g,s) Z_0(ds) = T(Y_g) = X_g .$$

□

Our next result is a Hilbert-space version of Bochner's theorem which is an extension of a theorem of Falb (1969).

Theorem 4.3.4. Let $V : G \to T(\mathcal{H},\mathcal{H})$ be weakly continuous. Then V is positive definite iff it has the representation

$$V(g) = \int_{\hat{G}} \langle \alpha, g \rangle M'(\alpha) \tau(d\alpha) \tag{4.3.15}$$

for some strongly measurable function $M' : \hat{G} \to T^+(\mathcal{H},\mathcal{H})$.

Proof: If $V(g) = \int_{\hat{G}} \langle \alpha, g \rangle M'(\alpha)\tau(d\alpha)$, then for elements $g_1, \ldots, g_n$ of G and $h_1, \ldots, h_n$ of $\mathcal{H}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(V(g_i-g_j)h_i,h_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\hat{G}}\langle\alpha,g_i-g_j\rangle(M'(\alpha)h_i,h_j)\tau(d\alpha)$$

$$= \int_{\hat{G}}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle\alpha,g_i\rangle\overline{\langle\alpha,g_j\rangle}\,(M'(\alpha)h_i,h_j)\tau(d\alpha)$$

$$= \int_{\hat{G}}\Big|\Big|\sum_{i=1}^{n}\langle\alpha,g_i\rangle B(\alpha)h_i\Big|\Big|^2\tau(d\alpha) \geq 0$$

where $B(\alpha)$ is the positive square root of the positive operator $M'(\alpha)$. Conversely, if $V(g)$ is positive definite, then the function $g \to (V(g)h,h)$ is positive definite. Also

$$(V(g)h,h') = \frac{1}{4}[\{(V(g)h+h',h+h') - (V(g)h-h',h-h')\}$$

$$+ i\{(V(g)h+ih',h+ih') - (V(g)h-ih,h-ih)\}]$$

so the function $(V(g)h,h')$ is a linear combination of positive definite functions. Thus there exists a complex measure $\mu_{h,h'}$ of finite variation such that

$$(V(g)h,h') = \int_{\hat{G}}\langle\alpha,g\rangle\mu_{hh'}(d\alpha) .$$

Now the map $\langle h,h'\rangle \to \mu_{hh'}(\Delta)$ is a bilinear functional on $H \times H$ for each fixed Borel set Δ and is bounded since μ_{hh} is a positive measure and

$$\mu_{hh}(\Delta) \leqq \mu_{hh}(\hat{G}) = (V(e)h,h) \leqq ||V(e)||\ ||h||^2 .$$

Thus there exists a bounded linear operator $M(\Delta)$ such that $(M(\Delta)h,h') = \mu_{hh'}(\Delta)$ for all Borel sets Δ and all $h,h' \in H$. Since $\mu_{hh'}$ is a measure, it is clear that $M(\Delta)$ is weakly countably additive, and

$M(\Delta)$ is a positive operator for each Δ since μ_{hh} is a positive measure. To see that $M(\Delta)$ is trace class for each Δ let $\{\phi_k\}$ be a c.o.n.s. for H. Then

$$\begin{aligned}\sum_{k=1}^{\infty}(M(\Delta)\phi_k,\phi_k) &= \sum_{k=1}^{\infty}\mu_{\phi_k,\phi_k}(\Delta)\\ &\leq \sum_{k=1}^{\infty}\mu_{\phi_k,\phi_k}(\hat{G})\\ &= \sum_{k=1}^{\infty}(V(e)\phi_k,\phi_k)\\ &= \text{trace } V(e) < \infty .\end{aligned}$$

Thus $M(\Delta)$ satisfies the requirements of Theorem 4.3.1,,and so there exists a function $M' : \hat{G} \to T^{+}(H,H)$ such that

$$M(\Delta) = \int_{\Delta} M'(\alpha)\tau(d\alpha) .$$

Hence $(V(g)h,h) = \int_{\hat{G}} \langle \alpha,g\rangle (M'(\alpha)h,h')\tau(d\alpha)$. But the integral $\int_{\hat{G}}\langle \alpha,g\rangle\, M'(\alpha)h\tau(d\alpha)$ exists as a Bochner integral for all $h \in H$ since

$$\int_{\hat{G}} ||\langle \alpha,g\rangle M'(\alpha)h||_u\tau(d\alpha) \leq \int_{\hat{G}} ||h||\tau(d\alpha) = ||h||\tau(\hat{G}) < \infty$$

so $V(g) = \int_{\hat{G}}\langle \alpha,g\rangle M'(\alpha)\tau(d\alpha)$ in the sense of Remark 4.3.1. □

With the aid of Theorems 4.3.2, 4.3.3 and 4.3.4, we can now easily prove the spectral representation of a stationary process with weakly continuous covariance.

Theorem 4.3.5. Let X_g be a weakly continuous stationary process of Hilbert-Schmidt operators. Then there exists an H.-S. orthogonal measure Z on the Borel sets of $\hat{G}$ such that

$$X_g = \int_{\hat{G}} \langle \alpha, g \rangle Z(d\alpha) \tag{4.3.16}$$

and $Z^*(\Delta)Z(\Delta) = M(\Delta)$ for all Borel sets Δ where M is the measure appearing in the Bochner representation of the covariance operator V of the process X_g.

Proof: By Theorem 4.3.4, the covariance operator V has the representation (4.3.15) and thus by Theorems 4.3.2 and 4.3.3 the process has the representation (4.3.16) and the measure Z satisfies $Z(\Delta)^*Z(\Delta) = M(\Delta)$.

$\square$

We conclude this section with a few remarks on Payen's approach to the spectral representation (4.3.16) which we will use in section 4.4.

Let M be the subspace of $L_2(\Omega,C)$ generated by the random variables $\{X_g h : g \in G, h \in H\}$. We may define an operator U_g by $U_g(X_{g'}h) = X_{g+g'}h$; U_g may be extended to a unitary operator on M. The family of operators $\{U_g : g \in G\}$ constitutes a unitary group of operators, which by Stone's theorem (see e.g. Hille and Phillips (1957) p. 598) admits a representation

$$U_g = \int_{\hat{G}} \langle \alpha, g \rangle E(d\alpha)$$

where E is a projection valued spectral measure on the Borel subsets of $\hat{G}$, and the integral is defined in the sense of Halmos (1951) p. 60. Hence $X_g = \int_G \langle \alpha, g \rangle Z(d\alpha)$ where $Z(\Delta) = E(\Delta)X_e$ since $X_g = U_g X_e$.

The measure M of Theorem 4.3.4 satisfies $M(\Delta) = X_e^* E(\Delta) X_e$. To see this, consider the function $(V(g)x,y) = (X_e^* X_g x, y) = (U_g X_e x, X_e y)$.

$(V(g)x,y)$ is the inverse Fourier transform of the measure μ_{xy} of Theorem 4.3.3, and by Stone's theorem

$$(U_g X_e x, X_e y) = \int_{\hat{G}} \langle \alpha, g \rangle \nu_{xy}(d\alpha)$$

where $\nu_{xy}(\Delta) = (E(\Delta)X_e x, X_e y)$. It follows that $\nu_{xy} = \mu_{xy}$ i.e. that $(E(\Delta)X_e x, X_e y) = (M(\Delta)x,y)$ $\forall x,\ y \in H$ and so $X_e{}^* E(\Delta)X_e = M(\Delta)$ for all Borel sets Δ.

Our final result is a lemma that will be useful in section 4.4.

Lemma 4.3.1. Let $f(\alpha)$ be a bounded measurable complex valued function on $\hat{G}$. Then

$$\text{Trace } X_e^* \int_{\hat{G}} f(\alpha)E(d\alpha)X_e = \int_{\hat{G}} f(\alpha)\tau(d\alpha).$$

Proof: It is enough to prove the lemma for the case $f \geq 0$, since we can then express a general f in terms of its positive and negative parts and apply the result for positive f to obtain the proof of the general case. First we prove the result for simple functions $f = \sum_{i=1}^{n} c_i \chi_{\Delta_i}$. Then

$$\begin{aligned} \text{trace } X_e^* \int_{\hat{G}} f(\alpha)E(d\alpha)X_e &= \text{trace} \sum_{i=1}^{n} c_i X_e^* E(\Delta_i)X_e \\ &= \sum_{i=1}^{n} c_i \text{ trace } M(\Delta_i) \\ &= \sum_{i=1}^{n} c_i \tau(\Delta_i) \\ &= \int_{\hat{G}} f(\alpha)\tau(d\alpha) . \end{aligned}$$

Now let f be a positive bounded measurable function. There exists a sequence f_n of simple functions such that f_n converges to f uniformly, and the sequence f_n is increasing. (See e.g. Halmos (1950) p. 86). Thus by monotone convergence

$$\int_{\hat{G}} f(\alpha)\tau(d\alpha) = \lim \int_{\hat{G}} f_n(\alpha)\tau(d\alpha)$$

$$= \lim \operatorname{trace} X_e^* \int_{\hat{G}} f_n(\alpha)E(d\alpha)X_e \ .$$

But

$$\left|\operatorname{trace} X_e^* \int_{\hat{G}} f(\alpha)E(d\alpha)X_e - \operatorname{trace} X_e^* \int_{\hat{G}} f_n(\alpha)E(d\alpha)X_e\right|$$

$$= \left|\operatorname{trace} X_e^* \int (f(\alpha)-f_n(\alpha))E(d\alpha)X_e\right|$$

$$= \left|(X_e, \int (f(\alpha)-f_n(\alpha))E(d\alpha)X_e)_{HS}\right|$$

$$\leqq ||X_e||_{HS}|| \int f(\alpha)-f_n(\alpha)E(d\alpha)X_e||_{HS}$$

$$\leqq ||X_e||_{HS}^2|| \int (f(\alpha)-f_n(\alpha))E(d\alpha)||_u$$

where $||A||_u$ denotes the uniform norm of the operator A on $L_2(\Omega,C)$.

But

$$|| \int_{\hat{G}} (f(\alpha)-f_n(\alpha))E(d\alpha)||_u = \sup_{\alpha\in\hat{G}}|f(\alpha)-f_n(\alpha)| \to 0 \text{ as } n \to \infty$$

(see e.g. Halmos (1951) p. 62), so

$$\lim_{n\to\infty} \operatorname{trace} X_e^* \int_{\hat{G}} f_n(\alpha)E(d\alpha)X_e = \operatorname{trace} X_e^* \int_{\hat{G}} f(\alpha)E(d\alpha)X_e$$

which proves the lemma. □

4.4. Path Properties and Sampling of H-valued Processes.

In this section we generalize some of the results proved for complex valued stationary processes in Chapter III relating to sampling and path continuity. We will assume throughout this section that the process in question is indexed by the real line $\mathbb{R}$, rather than an arbitrary LCA group G. We first give definitions of path continuity, and then generalize a result of Kawata (1969) to the H-valued case. Finally, we show that a band limited H-valued process satisfies a sampling theorem.

Definition 4.4.1. Let $\{x(t) : t \in \mathbb{R}\}$ be a weakly continuous H-valued stationary process, and $\{X_t : t \in \mathbb{R}\}$ the corresponding process of Hilbert-Schmidt operators. We will say that $x(t)$ has strongly continuous paths with probability 1 if for almost all ω, the function $x(\cdot,\omega)$ is continuous as a map from $\mathbb{R}$ to H. If $(x(\cdot,\omega),h)_H$ is continuous with probability 1 for all $h \in H$, then we say that $x(t)$ has weakly continuous paths with probability 1. Clearly $x(t)$ has weakly continuous paths with probability 1 iff the complex valued process $X_t h$ has continuous paths with probability 1 for each $h \in H$. □

We now give a sufficient condition on a weakly continuous stationary process $x(t)$ for strong path continuity. We rely heavily on a paper of Kawata (1969). We first define a "periodic" process based on $x(t)$. Let $I_n = [2n\pi/T, 2(n+1)\pi/T)$, and let E be the spectral measure in Stone representation of the unitary group U_g introduced at the end of section 4.3. Let $Q_n = E(I_n)$. Define

$$X_t(T) = \sum_{n=-\infty}^{\infty} e^{2\pi nit/T} Q_n X_0 \tag{4.4.1}$$

The series (4.4.1) converges in Hilbert-Schmidt norm since the Q_n are orthogonal, i.e. $Q_n Q_m = \delta_{nm} Q_n$, and if $\{\phi_k\}$ is a c.o.n.s. for H

$$\sum_{n=-\infty}^{\infty} ||e^{2\pi int/T}Q_nX_0||_{HS}^2 = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} (Q_nX_0\phi_k, Q_nX_0\phi_k)_{L_2(\Omega,C)}$$

$$= \sum_{k=1}^{\infty} \sum_{n=-\infty}^{\infty} ||Q_nX_o\phi_k||_{L_2(\Omega,C)}^2$$

$$= \sum_{k=1}^{\infty} ||\sum_{n=-\infty}^{\infty} Q_nX_0\phi_k||_{L_2(\Omega,C)}^2 \tag{4.4.2}$$

since the projections Q_n are orthogonal. Since $\sum_{n=-\infty}^{\infty} Q_n = I$, (4.4.2) equals $\sum_{k=1}^{\infty} ||X_0\phi_k||_{L_2(\Omega,C)}^2$ = trace $X_0^*X_0 < \infty$.

<u>Theorem 4.4.1.</u> $\hat{X}_t(T)$ converges to X_t in Hilbert-Schmidt norm as $T \to \infty$.

<u>Proof</u>: $||\hat{X}_t(t)-X_t||_{HS}^2 = ||\hat{X}_t(T)||_{HS}^2 + ||X_t||_{HS}^2 - 2\mathrm{Re}(\hat{X}_t(T),X_t)_{HS}$

and

$$||\hat{X}_t(T)||^2 = \left(\sum_{n=-\infty}^{\infty} e^{2\pi int/T}Q_nX_0, \sum_{m=-\infty}^{\infty} e^{2\pi int/T}Q_mX_0\right)$$

$$= \sum_{n=-\infty}^{\infty}(Q_nX_0,Q_nX_0)_{HS}$$

$$= \sum_{n=-\infty}^{\infty} \text{trace } X_0^*Q_nX_0$$

$$= \text{trace } X_0^*X_0 = ||X_t||_{HS}^2$$

$$(X_t,\hat{X}_t(T)) = \overline{\left(\sum_{n=-\infty}^{\infty} e^{2\pi int/T}Q_nX_0,U_tX_0\right)_{HS}}$$

$$= \sum_{n=-\infty}^{\infty} e^{-2\pi int/T}\text{trace } X_0^*Q_nU_tX_0$$

$$= \sum_{n=-\infty}^{\infty} e^{-2\pi int/T}\text{trace } X_0^* \int_{I_n} e^{it\lambda}E(d\lambda)X_0$$

$$= \sum_{n=-\infty}^{\infty} e^{-2\pi int/T} \int_{I_n} e^{it\lambda}\tau(d\lambda) \quad \text{(by Lemma 4.3.1)}$$

$$= \sum_{n=-\infty}^{\infty} \int_{I_n} e^{it(\lambda-2\pi n/T)}\tau(d\lambda) \ .$$

Thus $$||\hat{X}_t(T)-X_t||_{HS}^2 = 2\sum_{n=-\infty}^{\infty} \int_{I_n} (1-\cos t(\lambda-2\pi n/T))\tau(d\lambda)$$

$$\leq 2\sum_{n=-\infty}^{\infty} \int_{I_n} 2\sin^2\frac{\pi t}{T}\tau(d\lambda) = 4\tau(\mathbb{R})\sin^2\frac{\pi t}{T}$$

since $0 \leqq 1 - \cos t(x - \frac{2n\pi}{T}) \leqq 2\sin^2\frac{\pi t}{T}$ on I_n if $T > 2|t|$.

Thus $||X_t - \hat{X}_t(T)||_{HS}^2$ converges to 0 as $T \to \infty$. □

Now corresponding to the operators $\hat{X}_t(T)$ and $Q_n X_0$ we have H-valued second order random elements $\hat{x}(t,T)$ and ξ_n, given by $(\hat{x}(t,T),h)_H = \hat{X}_t(T)h$ and $(\xi_n,h)_H = Q_n X_0 h$ and the series

$$\hat{x}(t,T) = \sum_{n=-\infty}^{\infty} e^{2\pi int/T}\xi_n \tag{4.4.3}$$

converges in $L_2(\Omega,H)$ norm.

We wish to prove that the series (4.4.3) converges uniformly in H-norm with probability 1, under certain conditions. It is enough to prove that $E(\sum_{n=-\infty}^{\infty} ||\xi_n||_H) < \infty$. If there exists an even function $g : \mathbb{R} \to \mathbb{R}$ which increases for $t > 0$, satisfies $g(0) > 0$, $\sum_{n=0}^{\infty} g(n)^{-1} < \infty$ and $\int_{-\infty}^{\infty} g(\lambda)\tau(d\lambda) < \infty$ then

$$E\sum_{n=0}^{\infty} ||\xi_n|| \leqq \left(\sum_{n=0}^{\infty} g(\tfrac{2n\pi}{T})^{-1}\right) \left(\sum_{n=0}^{\infty} g(\tfrac{2n\pi}{T})E||\xi_n||^2\right)$$

$$= \Big(\sum_{n=0}^{\infty} g(\tfrac{2n\pi}{T})^{-1} \Big) \Big(\sum_{n=0}^{\infty} g(\tfrac{2n\pi}{T}) \text{ trace } X_0^* Q_n X_0 \Big)$$

$$= \Big(\sum_{n=0}^{\infty} g(\tfrac{2n\pi}{T})^{-1} \Big) \Big(\sum_{n=0}^{\infty} g(\tfrac{2n\pi}{T}) \tau(I_n) \Big)$$

$$\leqq \Big(\sum_{n=0}^{\infty} g(\tfrac{2n\pi}{T})^{-1} \Big) \int_0^{\infty} g(\lambda)\tau(d\lambda) < \infty .$$

Also

$$E\Big(\sum_{n=-\infty}^{-1} ||\xi_n|| \Big) \leqq \Big(\sum_{n=-\infty}^{-1} g(\tfrac{2(n+1)\pi}{T})^{-1} \Big) \Big(\sum_{n=-\infty}^{-1} g(\tfrac{2(n+1)\pi}{T}) \tau(I_n) \Big)$$

$$\leqq \Big(\sum_{n=-\infty}^{1} g(\tfrac{2(n+1)}{T})^{-1} \Big) \Big(\int_{\infty}^{0} g(\lambda)\tau(d\lambda) \Big) < \infty$$

so $E \sum_{n=-\infty}^{\infty} ||\xi_n||_H < \infty$.

Hence if such a g exists, $\hat{x}(t,T)$ has almost all paths strongly continuous.

The proof of the next result is the same as the corresponding result of Kawata (1969).

Theorem 4.4.2. Let $\{x(t) : t \in \mathbb{R}\}$ be a weakly continuous stationary H-valued process. If there exists a function $g(\lambda)$ that is positive, even and increasing for $\lambda \geq 0$ satisfying

(i) $\sum_{n=1}^{\infty} g(n)^{-1} < \infty$

and

(ii) $\int_{-\infty}^{\infty} g(\lambda)\tau(d\lambda) < \infty$

then $x(t)$ has almost all paths strongly continuous.

Proof: By the above remarks, the processes $\hat{x}(t,2^k)$ have strongly continuous paths. By copying the proof of Kawata's theorem 9 we see that as $k \to \infty$, $\hat{x}(t,2^k)$ tends uniformly to $x(t)$ on every compact interval with probability 1, and hence $x(t)$ has strongly continuous paths.

Definition 4.4.2. Let $\{x(t) : t \in \mathbb{R}\}$ be an H-valued weakly continuous stationary process. Let X_t be the corresponding operator process and U_t the unitary group associated with the X_t. If the spectral measure M of Theorem 4.3.2 is concentrated on a bounded set, we shall say that the process X_t is band-limited. Equivalently, if the spectral measure E in the Stone representation of the unitary group U_t is concentrated on a bounded set, then X_t is band limited.

$\square$

Our next theorem shows that the sampling expansion (3.2.2) is valid for H-valued processes.

Theorem 4.4.3. With the notation of Definition 4.4.2, if E is concentrated on the interval $[-W,W]$ then

$$X_t = \sum_{k=-\infty}^{\infty} X(k\pi/\alpha) \frac{\sin \alpha(t-k\pi/\alpha)}{(t-k\pi/\alpha)} \tag{4.4.4}$$

the series (4.4.4) converging uniformly in Hilbert-Schmidt norm for $\alpha > W$;

$$x(t) = \sum_{k=-\infty}^{\infty} x(k\pi/\alpha) \frac{\sin \alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)} \tag{4.4.5}$$

the series (4.4.5) converging in H-topology with probability 1 uniformly on compact sets for $\alpha > W$. Finally, the paths of $x(t)$ are analytic with probability 1.

Proof: Consider $\left|\left| X_t - \sum_{k=-n}^{n} X_{k\pi/\alpha} \frac{\sin \alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)} \right|\right|_{HS}$

$$\leq \left|\left|U_t - \sum_{k=-n}^{n} U_{k\pi/\alpha} \frac{\sin \alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)} \right|\right|_u \left|\left|X_0\right|\right|_{HS} \tag{4.4.6}$$

Now

$$U_t - \sum_{k=-n}^{n} U_{k\pi/\alpha} \frac{\sin \alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)} = \int_{-W}^{W} \left(e^{it\lambda} - \sum_{n=-n}^{n} e^{ik\pi\lambda/\alpha} \frac{\sin \alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)}\right) E(d\lambda)$$

so

$$\left|\left|U_t - \sum_{k=-n}^{n} U_{k\pi/\alpha} \frac{\sin \alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)}\right|\right|_u$$

$$= \sup_{-W\leq\lambda\leq W} \left| e^{it\lambda} - \sum_{k=-n}^{n} e^{ik\pi\lambda/\alpha} \frac{\sin \alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)} \right| \tag{4.4.7}$$

Now $e^{i\lambda z}$ is an entire function of exponential type $|\lambda|$ so by a result of Piranashvili (1967)

$$\left| e^{it\lambda} - \sum_{k=-n}^{n} e^{ik\pi\lambda/\alpha} \frac{\sin\alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)} \right| \leq \frac{K\alpha}{(\alpha-|\lambda|)n}$$

for all t and some constant K. Thus the left side of (4.4.7) is less than

$$\sup_{-W\leq\lambda\leq W} \frac{K\alpha}{(\alpha-|\lambda|)n} = \frac{K\alpha}{(\alpha-W)n} . \tag{4.4.8}$$

Thus by (4.4.6) and (4.4.8) the series (4.4.4) converges in H.-S. norm.

To prove (4.4.5), let $Y_n(t) = \sum_{k=-n}^{n} X_{k\pi/\alpha} \frac{\sin \alpha(t-k\pi/\alpha)}{\alpha(t-k\pi/\alpha)}$. Then

$$\sum_{n=1}^{\infty} \left|\left|Y_n(t) - X_t\right|\right|_{HS}^2 \leq \sum_{n=1}^{\infty} \left(\frac{K\alpha}{\alpha-W}\right)^2 \left|\left|X_0\right|\right|_{HS}^2 \frac{1}{n^2} < \infty \text{ , so}$$

$$E \sum_{n=1}^{\infty} \left|\left|x(t) - \sum_{k=-n}^{n} x(k\pi/\alpha) \frac{\sin \alpha(t-k\pi/\alpha)}{(t-k\pi/\alpha)} \right|\right|_{H}^2 = \sum_{n=1}^{\infty} \left|\left|X_t - Y_n(t)\right|\right|_{HS}^2 < \infty.$$

Hence the series (4.4.5) converges in H-topology with probability one. The convergence is uniform on compact sets since the convergence of the series $\sum_{n=1}^{\infty} ||X_t - Y_n(t)||_{HS}^2$ is uniform. Finally, the analyticity of the paths can be established by the methods of Belayev (1959) as in the case of complex valued processes. $\square$

APPENDIX

SOME FACTS FROM ABSTRACT HARMONIC ANALYSIS

A.1. LCA Groups. A topological group is a Hausdorff space G that is also a group, where the group structure and the topology are related by the requirement that the maps $\langle g,h\rangle \to g+h$ and $g \to -g$ be continuous. If G is locally compact as a topological space and abelian as a group, it is an LCA group. All groups considered in this work are LCA, the binary operation is always written $+$ and the inverse of $g \in G$ is written as $-g$. The symbol e will mean the identity of G.

A.2. Characters. Denote by $\hat{G}$ the set of all complex homomorphisms α of G such that α is continuous and $|\alpha(g)| = 1 \ \forall g \in G$. Following Rudin (1962) we have throughout denoted the value of α at g by $\langle\alpha,g\rangle$. It is clear that $\langle\alpha,e\rangle = 1 \ \forall\alpha \in \hat{G}$. If we define a binary operation $+$ on $\hat{G}$ by

$$\langle\alpha + \beta,g\rangle = \langle\alpha,g\rangle\langle\beta,g\rangle ,$$

$\hat{G}$ is an abelian group with identity the constant homomorphism 1, and the inverse $-\alpha$ of α is given by $\langle-\alpha,g\rangle = \overline{\langle\alpha,g\rangle}$.

$\hat{G}$ is called the character group (or dual group) of G, its elements are characters. It is possible to define a topology on $\hat{G}$ in such a way that $\hat{G}$ becomes an LCA group.

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