Modeling of Non-Symmetrical Damage in Plate-Like Structures [1]

H.T. Banks and P. Emeric
Center for Research in Scientific Computation
North Carolina State University
Raleigh, NC

L. Plancke
Ecole Polytechnique
Palaiseau, France

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Abstract

A two-dimensional model for in-plane vibrations of a linear elastic hookean partially clamped plate-like structure with a non-symmetrical damage is proposed. The model includes both active and passive contributions of piezoelectric ceramic patches bonded to the surface of the structure. The strong form of the equations of motion is obtained by balancing the forces on an infinitesimal element of the structure and by using Hooke's constitutive law for the two-dimensional case. The weak form of the equations of motion is found by using Hamilton's principle for the undamped case and by integration by parts of the strong form if damping exists.
The problem is then formulated as a second order equation in time and the existence, uniqueness and continuity of solutions are discussed. The general input forces due to the patches allow well-posedness results for the weak form of the equations in the case of strong damping. A semigroup formulation is useful to find a solution to the problem in the case of weak internal damping.

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Contents

Introduction

Smart material technology is an active area of research, with promising applications that include, for instance, control systems [1, 5] and non-destructive evaluation [3, 5]. Among the different types of smart material structures currently studied, structures with bonded piezoceramic patches are of particular interest [2]. When an electric field is applied, piezoceramic patches induce strains in the material to which they are bonded and, conversely, they produce a voltage when a deformation occurs in the material [4, 5]. As a consequence, these patches can act both as actuators and sensors, providing the host structure with smart material capabilities.

The focus of this report is the study of a non-symmetrically damaged partially clamped plate-like structure with bonded piezoceramic patches. In the first part, a two-dimensional model for the in-plane vibrations of such a structure is proposed. Both strong and weak forms of the equations of motion are derived for the cases with strong damping. The strong forms of the equations of motion are obtained by balancing the forces on an infinitesimal element of the structure. The weak form for the undamped case is obtained by energy considerations and, for the damped case, by integration by parts of the strong form.

In the second part of this report, a mathematical formulation of the problem as a second order equation in time is obtained. This formulation allows, in the last three parts, the discussion of existence and uniqueness of the solutions and their continuity with respect to the data. The general input forces due to the patches allow well-posedness for the weak form of the equations in the case of strong damping. A semigroup formulation, presented in the third and fourth parts, is useful to obtain a solution to the problem in the case of weak internal damping. The last part of the report discusses the equivalence between the weak form and the strong form. An appendix contains detailed comparisons to known models for the undamped case.

1 Two-dimensional model of the damaged plate

1.1 Description

Fig. 1 depicts the geometry of the structure with a non-symmetrical damage. The overall structure is a partially clamped plate-like structure of length l (in the x direction), width b (in the z direction) and thickness h (in the y direction). Motion is assumed to be longitudinal (in the x direction) and transverse (in the y direction). The groove in the z direction represents the defect. Four piezoelectric patches, symmetrically bonded on the upper and lower surfaces, are also visible. The motion of the structure in the (x, y) plane cannot be modeled accurately by classical thin beam or thin plate models. Such models assume a shell coordinate system with an unperturbed middle surface as the reference surface. The presence of non-symmetrical damage introduces a coupling of the motions in the (x, y) plane which is not taken into account by these models. We model this coupled motion with a state of plane strain in the structure; that is, away from the edges, no displacement takes place in the z direction.
Fig. 2 represents an (x, y) cross section where a state of plane strain exists. On each side of the central section representing the damage are two regions with varying cross sections representing its edges. Ω is the interior of this structure, Ω_{pe} the patches and Γ the boundary with no displacement. The region $\Omega_{\mathcal{B}} = \Omega \cup \Omega_{pe}$ is the complete structure.

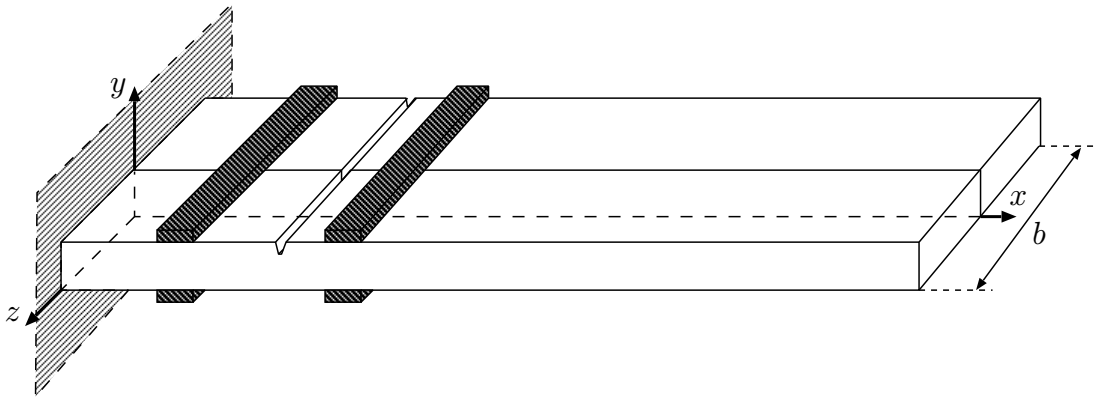

Figure 1: Damaged clamped structure with piezoceramic patches

1.2 Strong form of the equations of motion

We combine force balancing with constitutive hypotheses to obtain the equations of motion for the displacements U and V in the x and y directions, respectively.

1.2.1 Force resultants

For a plane strain as described for the structure in Fig. 2, the stresses are independent of z. A force resultant can be found for a corresponding component of the stress tensor by integration

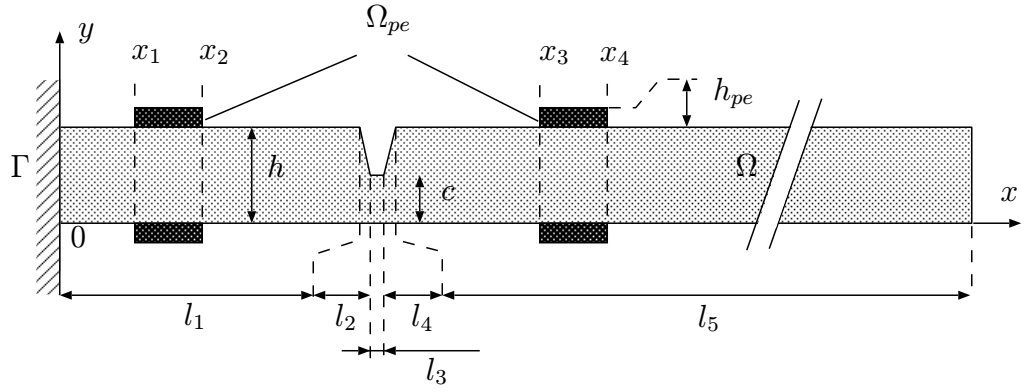

Figure 2: Plate-like cross section

along the z axis (see [5], Chapters 2 and 3 for the general modeling approach and principles we employ here). We obtain

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-b/2}^{b/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz = b \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{1}$$

where σ_x, σ_y and τ_{xy} are the horizontal, vertical and shear stresses and N_x, N_y and N_{xy} their corresponding force resultants. Care must be used with the notation throughout. Typically, subscripts such as those above do not represent derivatives; for partial derivatives, we shall use either $\frac{\partial U}{\partial x}$ or $U_{,x}$ whereas σ_x will denote the x component of the stress.

1.2.2 Strong form of the equations of motion

a. Force balancing

The equations of motion for the structure are found by considering an infinitesimal element of length dx, height dy and width b . Fig. 3 depicts the force resultants and body forces acting on such an element.

Balancing the forces along the x-axis gives:

$$\left(N_x + \frac{\partial N_x}{\partial x}dx - N_x\right)dy + \left(N_{xy} + \frac{\partial N_{xy}}{\partial y}dy - N_{xy}\right)dx + q_x dxdy = \hat{\rho}bdxdy\ddot{U}, \tag{2}$$

where q_x is the horizontal component of an external force/unit area acting on the element, $\ddot{U}$ the second time derivative of the displacement U and $\hat{\rho} > 0$ the mass per unit volume, a locally constant material property with definition:

$$\hat{\rho} = \rho\chi_\Omega + \rho_{pe}\chi_{\Omega_{pe}} \tag{3}$$

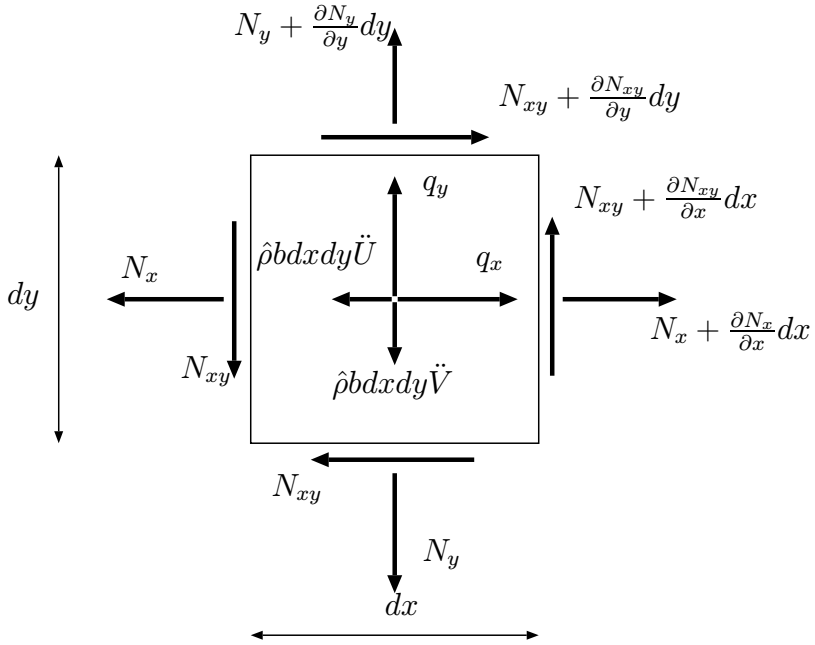

Figure 3: Force resultants on an element of the structure

where ρ and ρ_{pe} are the mass per unit volume for, respectively, the plate and the piezoelectric patches. χ_Ω and $\chi_{\Omega_{pe}}$ are characteristic functions with definition

$$\chi_\Omega = \begin{cases} 1 & in\, \Omega \\ 0 & otherwise \end{cases} \qquad and \quad \chi_{\Omega_{pe}} = \begin{cases} 1 & in\, \Omega_{pe} \\ 0 & otherwise \end{cases} \tag{4}$$

Eq. (2) reduces to (see, for comparison, Eq. (2.48) of [5]):

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x = \hat{\rho} b \ddot{U}. \tag{5}$$

Balancing the forces along the y-axis gives:

$$\left(N_y + \frac{\partial N_y}{\partial y} dy - N_y \right) dx + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x} dx - N_{xy} \right) dy + q_y dx dy = \hat{\rho} b dx dy \ddot{V} \tag{6}$$

where q_y the vertical component of a body force acting on the element and $\ddot{V}$ the second time derivative of the displacement V.

Eq. (6) reduces to:

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + q_y = \hat{\rho} b \ddot{V}. \tag{7}$$

b. Constitutive law

For a time dependent plane displacement field, $U = U(t, x, y)$ and $V = V(t, x, y)$, the linear strains are (see Eq. (5.3-7) p. 128 in [6]):

$$\epsilon_x = U_{,x}, \quad \epsilon_y = V_{,y}, \quad \gamma_{xy} = \frac{1}{2} \left(U_{,y} + V_{,x} \right). \tag{8}$$

For a two dimensional strain field that includes damping, the proposed Kelvin-Voigt stress-strain relationships assume that the stress is proportional to a linear combination of strain and strain rate (we also invoke the usual Kirchhoff assumption of negligible strain in the z direction):

$$\begin{aligned}
\hat{E}\epsilon_x + \hat{C}_D\dot{\epsilon}_x &= (\sigma_x - \hat{\nu}\,(\sigma_y + \sigma_z)) \\
\hat{E}\epsilon_y + \hat{C}_D\dot{\epsilon}_y &= (\sigma_y - \hat{\nu}\,(\sigma_x + \sigma_z)) \\
0 &= (\sigma_z - \hat{\nu}\,(\sigma_x + \sigma_y)) \\
\hat{E}\gamma_{xy} + \hat{C}_D\dot{\gamma}_{xy} &= (1+\hat{\nu})\tau_{xy}
\end{aligned} \tag{9}$$

where $\hat{C}_D > 0$ is a damping coefficient, $\hat{E} > 0$ is Young's modulus and $0 \leq \hat{\nu} < 1/2$ is Poisson's ratio. These are locally constant material properties with definitions:

$$\hat{C}_D = C_D\chi_\Omega + C_{Dpe}\chi_{\Omega_{pe}}, \quad \hat{E} = E\chi_\Omega + E_{pe}\chi_{\Omega_{pe}}, \quad \hat{\nu} = \nu\chi_\Omega + \nu_{pe}\chi_{\Omega_{pe}} \tag{10}$$

where the subscript "*pe*" indicates the piezoelectric patches.

The force resultants are:

$$\begin{aligned}
N_x &= \frac{b}{(1+\hat{\nu})(1-2\hat{\nu})}\left(\hat{E}(\epsilon_x - \hat{\nu}(\epsilon_x - \epsilon_y)\right) + \hat{C}_D(\dot{\epsilon}_x - \hat{\nu}(\dot{\epsilon}_x - \dot{\epsilon}_y))) \\
N_y &= \frac{b}{(1+\hat{\nu})\,(1-2\hat{\nu})}\left(\hat{E}(\epsilon_y + \hat{\nu}\,(\epsilon_x - \epsilon_y)) + \hat{C}_D(\dot{\epsilon}_y + \hat{\nu}\,(\dot{\epsilon}_x - \dot{\epsilon}_y))\right) \\
N_{xy} &= \frac{b}{1+\hat{\nu}}\left(\hat{E}\gamma_{xy} + \hat{C}_D\dot{\gamma}_{xy}\right)
\end{aligned} \tag{11}$$

c. Equations of motion

The equations of motion are found by replacing the expressions for the resultants obtained from Eq. (11) in Eq. (5) and (7). By replacing the strains by their expressions in Eq. (8), we obtain

$$\left\{\begin{aligned}
\hat{\rho}\ddot{U} = &\ \frac{\hat{E}}{1+\hat{\nu}}\left(\frac{1-\hat{\nu}}{1-2\hat{\nu}}U_{,xx} + \frac{1}{2}U_{,yy} + \frac{1}{2(1-2\hat{\nu})}V_{,xy}\right) \\
&+\frac{\hat{C}_D}{1+\hat{\nu}}\left(\frac{1-\hat{\nu}}{1-2\hat{\nu}}\dot{U}_{,xx} + \frac{1}{2}\dot{U}_{,yy} + \frac{1}{2(1-2\hat{\nu})}\dot{V}_{,xy}\right) + \frac{q_x}{b} \\
\hat{\rho}\ddot{V} = &\ \frac{\hat{E}}{1+\hat{\nu}}\left(\frac{1-\hat{\nu}}{1-2\hat{\nu}}V_{,yy} + \frac{1}{2}V_{,xx} + \frac{1}{2(1-2\hat{\nu})}U_{,xy}\right) \\
&+\frac{\hat{C}_D}{1+\hat{\nu}}\left(\frac{1-\hat{\nu}}{1-2\hat{\nu}}\dot{V}_{,yy} + \frac{1}{2}\dot{V}_{,xx} + \frac{1}{2(1-2\hat{\nu})}\dot{U}_{,xy}\right) + \frac{q_y}{b}
\end{aligned}\right. \tag{12}$$

d. Boundary conditions

The boundary conditions are expressed in terms of stresses or displacements depending on the boundaries considered. Nonetheless, they are all written here as conditions on the displacements.

(a) Clamped edge, $\{(x,y)|x=0, y\in[0,h]\}$

For a clamped end, there is no in-plane displacements at any time. These boundary conditions are written:

$$\begin{aligned} U(t,0,y) &= 0 \\ V(t,0,y) &= 0 \end{aligned} \tag{13}$$

(b) All other boundaries (traction free edges)

Except for the clamped end, all boundaries are free of surface traction. By using Cauchy's formula, this condition can be written as:

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix} = 0 \tag{14}$$

where n_x and n_y are the components of a unit outer normal surface vector along the boundary.

In terms of displacements, the free boundary conditions are given by:

$$\begin{cases} 0 = & \hat{E}\left(\left((1-\hat{\nu})U_{,x} + \hat{\nu}V_{,y}\right)n_x + \dfrac{(1-2\hat{\nu})}{2}(U_{,y}+V_{,x})n_y\right) \\ & +\hat{C}_D\left(\left((1-\hat{\nu})\dot{U}_{,x} + \hat{\nu}\dot{V}_{,y}\right)n_x + \dfrac{(1-2\hat{\nu})}{2}(\dot{U}_{,y}+\dot{V}_{,x})n_y\right) \\ 0 = & \hat{E}\left(\left((1-\hat{\nu})V_{,y} + \hat{\nu}U_{,x}\right)n_y + \dfrac{(1-2\hat{\nu})}{2}(V_{,x}+U_{,y})n_x\right) \\ & +\hat{C}_D\left(\left((1-\hat{\nu})\dot{V}_{,y} + \hat{\nu}\dot{U}_{,x}\right)n_y + \dfrac{(1-2\hat{\nu})}{2}(\dot{V}_{,x}+\dot{U}_{,y})n_x\right) \end{cases} \tag{15}$$

e. Initial conditions The initial conditions characterize the displacement and velocity at every point of the structure at $t=0$. These are assumed to be given by:

$$\begin{aligned} U(0,x,y) &= U_0(x,y), \quad V(0,x,y) = V_0(x,y), \\ \dot{U}(0,x,y) &= U_1(x,y), \quad \dot{V}(0,x,y) = V_1(x,y), \end{aligned} \quad \text{for } (x,y)\in\Omega_{\mathcal{B}} \tag{16}$$

1.2.3 Forces applied by the patches

For the structure depicted in Fig. 2, an applied voltage υ on a piezoceramic patch of thickness h_{pe} induces a strain in the x direction. The magnitude of the induced free strain is

$$(e_x)_{pe} = e_{pe} = \frac{\upsilon d_{31}}{h_{pe}} \tag{17}$$

where d_{31} is a piezoceramic constant (e.g, see p. 79, [5]).

The corresponding induced external stresses are obtained by replacing the induced free strain in Hooke's law (see Eq. (9) with $\hat{C}_D = 0$) and solving for the stresses. Using Eq. (11), we obtain the induced external forces terms:

$$\begin{cases} (N_x)_{pe} = -\dfrac{bE_{pe}(1-\nu_{pe})}{(1+\nu_{pe})(1-2\nu_{pe})} e_{pe} \\ (N_y)_{pe} = -\dfrac{bE_{pe}\nu_{pe}}{(1+\nu_{pe})(1-2\nu_{pe})} e_{pe}. \end{cases} \tag{18}$$

For a finite patch with edge coordinates x_1 and x_2 located on the upper surface of the structure, as depicted in Fig. 2, the induced external resultant forces are

$$\begin{cases} (N_x)_{pe1} = -\dfrac{bE_{pe}(1-\nu_{pe})}{(1+\nu_{pe})(1-2\nu_{pe})} e_{pe}\chi_{pe1}(x,y)S_{1,2}(x) \\ (N_y)_{pe1} = -\dfrac{bE_{pe}\nu_{pe}}{(1+\nu_{pe})(1-2\nu_{pe})} e_{pe}\chi_{pe1}(x,y)S_{1,2}(x), \end{cases} \tag{19}$$

where $\chi_{pe1}(x,y)$ is a characteristic function with definition

$$\chi_{pe1} = \begin{cases} 1 & x_1 \le x \le x_2, \quad h \le y \le h + h_{pe} \\ 0 & otherwise. \end{cases} \tag{20}$$

$S_{1,2}(x)$ is an indicator function that derives from the property that opposite but equal strains are induced about the patch midpoint. The definition is

$$S_{1,2}(x) = \begin{cases} 1 & , \, x < (x_1+x_2)/2 \\ 0 & , \, x = (x_1+x_2)/2 \\ -1 & , \, x > (x_1+x_2)/2. \end{cases} \tag{21}$$

In the case of interest, there are 2 sets of upper and lower patches. The resultant forces can be written with the appropriate characteristic and indicator functions.
The corresponding external forces/unit area are

$$\begin{cases} q_x = \dfrac{\partial (N_x)_{pe1}}{\partial x} \\ q_y = \dfrac{\partial (N_y)_{pe1}}{\partial y}. \end{cases} \tag{22}$$

1.3 Weak form of the equations of motion

1.3.1 Hamilton's principle applied to the undamped structure

In order to find the weak form of the equations for the structure depicted in Fig. 2, we need the expressions of the strain and kinetic energies. The strain energy $\mathbf{U}$ stored during deformation is given by

$$\mathbf{U} = \frac{1}{2}\int_{\Omega_{\mathcal{B}}} (\sigma_x\epsilon_x + \sigma_y\epsilon_y + \gamma_{xy}\tau_{xy}) d\Omega_{\mathcal{B}}. \tag{23}$$

Replacing the stresses and strains by their expressions in term of displacements obtained in Eq. (8) and (9) with $\hat{C}_D = 0$, we have

$$\begin{aligned} \mathbf{U} &= \frac{1}{2}\int_{\Omega_{\mathcal{B}}} \left(\frac{\hat{E}}{(1+\hat{\nu})(1-2\hat{\nu})}(U_{,x} - \hat{\nu}(U_{,x} - V_{,y}))U_{,x} + \right. \\ &\left. \frac{\hat{E}}{(1+\hat{\nu})(1-2\hat{\nu})}(V_{,y} + \hat{\nu}(U_{,x} - V_{,y}))V_{,y} + \frac{\hat{E}}{4(1+\hat{\nu})}(U_{,y} + V_{,x})(U_{,y} + V_{,x}) \right) d\Omega_{\mathcal{B}}, \end{aligned} \tag{24}$$

which reduces to

$$\begin{aligned} \mathbf{U} &= \frac{1}{2}\int_{\Omega_{\mathcal{B}}} \left(\frac{\hat{E}}{(1+\hat{\nu})(1-2\hat{\nu})} \left((1-\hat{\nu})({U_{,x}}^2 + {V_{,y}}^2) + 2\hat{\nu}V_{,y}U_{,x} \right) \right. \\ &\left. + \frac{\hat{E}}{4(1+\hat{\nu})} \left({U_{,y}}^2 + {V_{,x}}^2 + 2U_{,y}V_{,x} \right) \right) d\Omega_{\mathcal{B}}. \end{aligned} \tag{25}$$

The kinetic energy $\mathbf{T}$ of the structure is given by:

$$\mathbf{T} = \frac{1}{2}\int_{\Omega_{\mathcal{B}}} \hat{\rho}\left(\dot{U}^2 + \dot{V}^2\right) d\Omega_{\mathcal{B}} \tag{26}$$

For an arbitrary time interval $[t_0, t_1]$, we consider the action integral

$$A\left[\vec{U}\right] = \int_{t_0}^{t_1} (\mathbf{T} - \mathbf{U})\, dt \tag{27}$$

where $\vec{U} = [U, V]$ is considered in the space $\mathbf{V} = H_0^1(\Omega_{\mathcal{B}}) \times H_0^1(\Omega_{\mathcal{B}})$. We consider the variation of the form

$$\hat{\mathbf{U}} = \vec{\mathbf{U}} + \epsilon\vec{\Phi} = \begin{bmatrix} U(t,x,y) \\ V(t,x,y) \end{bmatrix} + \epsilon \begin{bmatrix} \eta_1(t)\phi_1(x,y) \\ \eta_2(t)\phi_2(x,y) \end{bmatrix} \tag{28}$$

where $\vec{\eta} = [\eta_1, \eta_2]$ and $\vec{\phi} = [\phi_1, \phi_2]$ are chosen so that

(i) $\hat{\mathbf{U}}(t,x,y) \in \mathbf{V}$

(ii) $\hat{\mathbf{U}}(t_0,x,y) = \hat{\mathbf{U}}(t_1,x,y)$

Hamilton's principle states that the motion of the structure must give a stationary value to the action integral when compared to variations in the motion. For all $\vec{\mathbf{\Phi}}$,

$$\frac{d}{d\epsilon} A\left[\vec{U} + \epsilon\vec{\Phi}\right]\Big|_{\epsilon=0} = 0. \tag{29}$$

The action integral takes the form

$$\begin{aligned} A\left[\hat{\mathbf{U}}\right] &= \int_{t_0}^{t_1} \left\{ \frac{1}{2} \int_{\Omega_{\mathcal{B}}} \left(\hat{\rho} \left((\frac{\partial}{\partial t}(U + \epsilon\eta_1\phi_1))^2 + ((\frac{\partial}{\partial t}(V + \epsilon\eta_2\phi_2))^2 \right) \right. \right. \\ & -\frac{\hat{E}}{(1+\hat{\nu})(1-2\hat{\nu})} \left((1-\hat{\nu})((\frac{\partial}{\partial x}(U + \epsilon\eta_1\phi_1))^2 + (\frac{\partial}{\partial y}(V + \epsilon\eta_2\phi_2))^2) \right. \\ & \left. +2\hat{\nu}\frac{\partial}{\partial y}(V + \epsilon\eta_2\phi_2)\frac{\partial}{\partial x}(U + \epsilon\eta_1\phi_1) \right) \\ & -\frac{\hat{E}}{4(1+\hat{\nu})} \left((\frac{\partial}{\partial y}(U + \epsilon\eta_1\phi_1))^2 + (\frac{\partial}{\partial x}(V + \epsilon\eta_2\phi_2))^2 \right. \\ & \left. \left. \left. +2\frac{\partial}{\partial y}(U + \epsilon\eta_1\phi_1)\frac{\partial}{\partial x}(V + \epsilon\eta_2\phi_2) \right) \right) d\Omega_{\mathcal{B}} \right\} dt. \end{aligned} \tag{30}$$

Hamilton's principle leads to the condition

$$\begin{aligned} 0 &= \frac{\partial}{\partial\epsilon} A\left[\hat{\mathbf{U}}\right]\Big|_{\epsilon=0} \\ &= \int_{t_0}^{t_1} \int_{\Omega_{\mathcal{B}}} \left\{ \hat{\rho} \left(\phi_1 \dot{U}\dot{\eta}_1 + \phi_2 \dot{V}\dot{\eta}_2 \right) \right. \\ & -\frac{\hat{E}}{(1+\hat{\nu})(1-2\hat{\nu})} \left((1-\hat{\nu}) \left(\eta_1 U_{,x}\phi_{1,x} + \eta_2 V_{,y}\phi_{2,y} \right) \right. \\ & \left. +\hat{\nu} \left(\eta_1 V_{,y}\phi_{1,x} + \eta_2 U_{,x}\phi_{2,y} \right) \right) \\ & \left. -\frac{\hat{E}}{2(1+\hat{\nu})} \left(\eta_1 U_{,y}\phi_{1,y} + \eta_2 V_{,x}\phi_{2,x} + \eta_1 V_{,x}\phi_{1,y} + \eta_2 U_{,y}\phi_{2,x} \right) \right\} d\Omega_{\mathcal{B}} dt. \end{aligned} \tag{31}$$

Since $\vec{\eta}(t_0) = \vec{\eta}(t_1)$, we can integrate by part the terms $\dot{U}$ and $\dot{V}$, and collect the η_1 and η_2 terms. We obtain

$$
\begin{cases}
\displaystyle\int_{t_0}^{t_1} \eta_1(t) \int_{\Omega_{\mathcal{B}}} \Big\{ \hat{\rho}\phi_1 \ddot{U} + \frac{\hat{E}(1-\hat{\nu})}{(1+\hat{\nu})(1-2\hat{\nu})} U_{,x}\phi_{1,x} + \frac{\hat{E}\hat{\nu}}{(1+\hat{\nu})(1-2\hat{\nu})} V_{,y}\phi_{1,x} \\
\qquad\qquad + \frac{\hat{E}}{2(1+\hat{\nu})} \left(U_{,y}\phi_{1,y} + V_{,x}\phi_{1,y}\right) \Big\} d\Omega_{\mathcal{B}} dt = 0 \\
\displaystyle\int_{t_0}^{t_1} \eta_2(t) \int_{\Omega_{\mathcal{B}}} \Big\{ \hat{\rho}\phi_2 \ddot{V} + \frac{\hat{E}(1-\hat{\nu})}{(1+\hat{\nu})(1-2\hat{\nu})} V_{,y}\phi_{2,y} + \frac{\hat{E}\hat{\nu}}{(1+\hat{\nu})(1-2\hat{\nu})} U_{,x}\phi_{2,y} \\
\qquad\qquad + \frac{\hat{E}}{2(1+\hat{\nu})} \left(V_{,x}\phi_{2,x} + U_{,y}\phi_{2,x}\right) \Big\} d\Omega_{\mathcal{B}} dt = 0.
\end{cases}
\tag{32}
$$

The weak form of the equations of motion for the free undamped structure is

$$
\begin{cases}
\displaystyle\int_{\Omega_{\mathcal{B}}} \Big\{ \frac{2(1+\hat{\nu})(1-2\hat{\nu})}{\hat{E}} \hat{\rho}\phi_1 \ddot{U} + (2-2\hat{\nu}) U_{,x}\phi_{1,x} + 2\hat{\nu} V_{,y}\phi_{1,x} \\
\qquad\qquad + (1-2\hat{\nu}) \left(U_{,y}\phi_{1,y} + V_x\phi_{1,y}\right) \Big\} d\Omega_{\mathcal{B}} = 0 \\
\displaystyle\int_{\Omega_{\mathcal{B}}} \Big\{ \frac{2(1+\hat{\nu})(1-2\hat{\nu})}{\hat{E}} \hat{\rho}\phi_2 \ddot{V} + (2-2\hat{\nu}) V_{,y}\phi_{2,y} + 2\hat{\nu} U_{,x}\phi_{2,y} \\
\qquad\qquad + (1-2\hat{\nu}) \left(V_{,x}\phi_{2,x} + U_{,y}\phi_{2,x}\right) \Big\} d\Omega_{\mathcal{B}} = 0.
\end{cases}
\tag{33}
$$

for all $\vec{\Phi} \in \mathbf{V}$.

1.3.2 Weak form of the equations for the damped structure

If the model includes Kelvin-Voigt damping, the system is non conservative and Hamilton's principle cannot be directly applied. However, the weak form of the equations of motion can be found by integration by parts of the strong form. Let Γ denote the portion of boundary of $\Omega_{\mathcal{B}}$ where $U = V = 0$ at all times and $\partial\Omega_{\mathcal{B}}$ be the remainder of the boundary with zero surface traction.

Defining $\tilde{E} = \dfrac{\hat{E}}{2(1+\hat{\nu})(1-2\hat{\nu})}$ and $\tilde{C}_D = \dfrac{\hat{C}_D}{2(1+\hat{\nu})(1-2\hat{\nu})}$, we may rewrite Eq. (12) as

$$
\begin{cases}
\hat{\rho}\ddot{U} = & \tilde{E}\left((2-2\hat{\nu})U_{,xx} + (1-2\hat{\nu})U_{,yy} + V_{,xy}\right) \\
& + \tilde{C}_D\left((2-2\hat{\nu})\dot{U}_{,xx} + (1-2\hat{\nu})\dot{U}_{,yy} + \dot{V}_{,xy}\right) + \dfrac{q_x}{b} \\
\hat{\rho}\ddot{V} = & \tilde{E}\left((2-2\hat{\nu})V_{,yy} + (1-2\hat{\nu})V_{,xx} + U_{,xy}\right) \\
& + \tilde{C}_D\left((2-2\hat{\nu})\dot{V}_{,yy} + (1-2\hat{\nu})\dot{V}_{,xx} + \dot{U}_{,xy}\right) + \dfrac{q_y}{b}
\end{cases}
\tag{34}
$$

Multiplying Eq. (34) by the spatial test functions ϕ_1 and ϕ_2, respectively, and integrating over $\Omega_{\mathcal{B}}$, we obtain

$$
\begin{cases}
\displaystyle\int_{\Omega_{\mathcal{B}}} \hat{\rho}\ddot{U}\phi_1 = & \displaystyle\int_{\Omega_{\mathcal{B}}} \tilde{E}(2-2\hat{\nu})U_{,xx}\phi_1 + \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})U_{,yy}\phi_1 + \int_{\Omega_{\mathcal{B}}} \tilde{E}V_{,xy}\phi_1 \\
& \displaystyle + \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(2-2\hat{\nu})\dot{U}_{,xx}\phi_1 + \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\hat{\nu})\dot{U}_{,yy}\phi_1 + \int_{\Omega_{\mathcal{B}}} \tilde{C}_D\dot{V}_{,xy}\phi_1 \\
& \displaystyle + \int_{\Omega_{\mathcal{B}}} \frac{q_x}{b}\phi_1 \\
\displaystyle\int_{\Omega_{\mathcal{B}}} \hat{\rho}\ddot{V}\phi_2 = & \displaystyle\int_{\Omega_{\mathcal{B}}} \tilde{E}(2-2\hat{\nu})V_{,yy}\phi_2 + \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})V_{,xx}\phi_2 + \int_{\Omega_{\mathcal{B}}} \tilde{E}U_{,xy}\phi_2 \\
& \displaystyle + \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(2-2\hat{\nu})\dot{V}_{,yy}\phi_2 + \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\nu)\dot{V}_{,xx}\phi_2 + \int_{\Omega_{\mathcal{B}}} \tilde{C}_D\dot{U}_{,xy}\phi_2 \\
& \displaystyle + \int_{\Omega_{\mathcal{B}}} \frac{q_y}{b}\phi_2
\end{cases}
\tag{35}
$$

for all sufficiently smooth ϕ_1 and ϕ_2 such that the integrals in Eq. (35) are defined and $\phi_1 = \phi_2 = 0$ on Γ.

Using the relationships

$$
\begin{aligned}
\int_{\Omega_{\mathcal{B}}} \tilde{E}V_{,xy}\phi_1 &= \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})\frac{\partial}{\partial y}\left(V_{,x}\right)\phi_1 + \int_{\Omega_{\mathcal{B}}} 2\tilde{E}\hat{\nu}\frac{\partial}{\partial x}\left(V_{,y}\right)\phi_1, \\
\int_{\Omega_{\mathcal{B}}} \tilde{E}U_{,xy}\phi_2 &= \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})\frac{\partial}{\partial x}\left(U_{,y}\right)\phi_2 + \int_{\Omega_{\mathcal{B}}} 2\tilde{E}\hat{\nu}\frac{\partial}{\partial y}\left(U_{,x}\right)\phi_2
\end{aligned}
\tag{36}
$$

and their equivalent for $\tilde{C}_D\dot{V}_{,xy}$ and $\tilde{C}_D\dot{U}_{,xy}$, we can integrate by parts in Eq. (35) (i.e., we use the divergence theorem on $\bar{F}\phi_i$ for $\bar{F}$ appropriately chosen and $\phi_i, \bar{F}$ sufficiently smooth) and obtain the system

$$
\begin{cases}
\displaystyle\int_{\Omega_{\mathcal{B}}} \hat{\rho}\ddot{U}\phi_1 = \int_{\Omega_{\mathcal{B}}} \frac{q_x}{b}\phi_1 \\
\displaystyle -\int_{\Omega_{\mathcal{B}}} \tilde{E}(2-2\hat{\nu})U_{,x}\phi_{1,x} - \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})(U_{,y}+V_{,x})\phi_{1,y} - 2\int_{\Omega_{\mathcal{B}}} \tilde{E}\hat{\nu}V_{,y}\phi_{1,x} \\
\displaystyle -\int_{\Omega_{\mathcal{B}}} \tilde{C}_D(2-2\hat{\nu})\dot{U}_{,x}\phi_{1,x} - \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\hat{\nu})(\dot{U}_{,y}+\dot{V}_{,x})\phi_{1,y} - 2\int_{\Omega_{\mathcal{B}}} \tilde{C}_D\hat{\nu}\dot{V}_{,y}\phi_{1,x} \\
\displaystyle +2\int_{\partial\Omega_{\mathcal{B}}\cup\Gamma} \left(\tilde{E}(((1-\hat{\nu})U_{,x}+\hat{\nu}V_{,y})n_x + \frac{1-2\hat{\nu}}{2}(U_{,y}+V_{,x})n_y) \right. \\
\displaystyle \qquad \left. + \tilde{C}_D(((1-\hat{\nu})\dot{U}_{,x}+\hat{\nu}\dot{V}_{,y})n_x + \frac{1-2\hat{\nu}}{2}(\dot{U}_{,y}+\dot{V}_{,x})n_y) \right)\phi_1 \\
\displaystyle\int_{\Omega_{\mathcal{B}}} \hat{\rho}\ddot{V}\phi_2 = \int_{\Omega_{\mathcal{B}}} \frac{q_y}{b}\phi_2 \\
\displaystyle -\int_{\Omega_{\mathcal{B}}} \tilde{E}(2-2\hat{\nu})V_{,y}\phi_{2,y} - \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})(U_{,y}+V_{,x})\phi_{2,x} - 2\int_{\Omega_{\mathcal{B}}} \tilde{E}\hat{\nu}U_{,x}\phi_{2,y} \\
\displaystyle -\int_{\Omega_{\mathcal{B}}} \tilde{C}_D(2-2\hat{\nu})\dot{V}_{,y}\phi_{2,y} - \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\hat{\nu})(\dot{U}_{,y}+\dot{V}_{,x})\phi_{2,x} - 2\int_{\Omega_{\mathcal{B}}} \tilde{C}_D\hat{\nu}\dot{U}_{,x}\phi_{2,y} \\
\displaystyle +2\int_{\partial\Omega_{\mathcal{B}}\cup\Gamma} \left(\tilde{E}(((1-\hat{\nu})V_{,y}+\hat{\nu}U_{,x})n_y + \frac{1-2\hat{\nu}}{2}(U_{,y}+V_{,x})n_x) \right. \\
\displaystyle \qquad \left. + \tilde{C}_D(((1-\hat{\nu})\dot{V}_{,y}+\hat{\nu}\dot{U}_{,x})n_y + \frac{1-2\hat{\nu}}{2}(\dot{U}_{,y}+\dot{V}_{,x})n_x) \right)\phi_2
\end{cases}
\tag{37}
$$

where the terms that correspond to the integration on the boundary $\partial\Omega$ have been grouped together. By using Eq. (15) and the fact that ϕ_1 and ϕ_2 vanish on Γ, these vanish. The remainder is the weak formulation:

$$
\begin{cases}
\displaystyle\int_{\Omega_{\mathcal{B}}} \hat{\rho}\ddot{U}\phi_1 = \int_{\Omega_{\mathcal{B}}} \frac{q_x}{b}\phi_1 \\
\displaystyle -\int_{\Omega_{\mathcal{B}}} \tilde{E}(2-2\hat{\nu})U_{,x}\phi_{1,x} - \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})(U_{,y}+V_{,x})\phi_{1,y} - 2\int_{\Omega_{\mathcal{B}}} \tilde{E}\hat{\nu}V_{,y}\phi_{1,x} \\
\displaystyle -\int_{\Omega_{\mathcal{B}}} \tilde{C}_D(2-2\hat{\nu})\dot{U}_{,x}\phi_{1,x} - \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\hat{\nu})(\dot{U}_{,y}+\dot{V}_{,x})\phi_{1,y} - 2\int_{\Omega_{\mathcal{B}}} \tilde{C}_D\hat{\nu}\dot{V}_{,y}\phi_{1,x} \\
\displaystyle\int_{\Omega_{\mathcal{B}}} \hat{\rho}\ddot{V}\phi_2 = \int_{\Omega_{\mathcal{B}}} \frac{q_y}{b}\phi_2 \\
\displaystyle -\int_{\Omega_{\mathcal{B}}} \tilde{E}(2-2\hat{\nu})V_{,y}\phi_{2,y} - \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})(U_{,y}+V_{,x})\phi_{2,x} - 2\int_{\Omega_{\mathcal{B}}} \tilde{E}\hat{\nu}U_{,x}\phi_{2,y} \\
\displaystyle -\int_{\Omega_{\mathcal{B}}} \tilde{C}_D(2-2\hat{\nu})\dot{V}_{,y}\phi_{2,y} - \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\hat{\nu})(\dot{U}_{,y}+\dot{V}_{,x})\phi_{2,x} - 2\int_{\Omega_{\mathcal{B}}} \tilde{C}_D\hat{\nu}\dot{U}_{,x}\phi_{2,y}
\end{cases}
\tag{38}
$$

for all sufficiently smooth ϕ_1 and ϕ_2 satisfying $\phi_1 = \phi_2 = 0$ on Γ.

2 Mathematical formulation

2.1 Formulation as a second-order equation in time

We define $H^1_L(\Omega_{\mathcal{B}}) = \{\varphi \in H^1(\Omega_{\mathcal{B}}) | \quad \varphi = 0 \text{ on } \Gamma\}$, a Hilbert space with the inner product

$$< \varphi, \psi >_1 = \int_{\Omega_{\mathcal{B}}} \left(\frac{\partial \varphi}{\partial x} \overline{\frac{\partial \psi}{\partial x}} + \frac{\partial \varphi}{\partial y} \overline{\frac{\partial \psi}{\partial y}} \right).$$

The associated norm is equivalent to the usual norm on $H^1(\Omega_{\mathcal{B}})$ (see Poincaré's inequality p. 116 in [10]). We define $V = H^1_L(\Omega_{\mathcal{B}}) \times H^1_L(\Omega_{\mathcal{B}})$ and $H = L^2(\Omega_{\mathcal{B}}) \times L^2(\Omega_{\mathcal{B}})$ with the $\hat{\rho}$ weighted inner product $< \varphi, \psi >= \int_{\Omega_{\mathcal{B}}} \hat{\rho} \varphi \psi$ on $L^2(\Omega_{\mathcal{B}})$. Since $\hat{\rho}$ is bounded above and below by a positive constant, this weighted inner product is equivalent to the usual L_2 inner product. Any $\varphi \in V$ or H is written $\varphi = (\varphi_1, \varphi_2)^T$. Both V and H are Hilbert spaces where V is continuously and densely embedded in H. We identify H with its dual H^* through the Riesz map. Then, V and H form the Gelfand triple $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ with duality pairing $< ., . >_{V^*,V}$, an extension by continuity of the inner product $< ., . >_1$ from $V \times H$ to $V^* \times H$ (see [10] for this standard construction).

The weak system given by Eq. (38) can be formulated in an initial value problem: For $0 < T < \infty$, find $W \in L^2((0,T); V)$ satisfying $\ddot{W} \in L^2((0,T); V^*)$ and

$$\begin{cases} < \ddot{W}(t), \psi >_{V^*,V} + \sigma_2(\dot{W}(t), \psi) + \sigma_1(W(t), \psi) =< F(t), \psi >_{V^*,V} \quad \forall \psi \in V, \\ W(0) = W_0, \qquad \dot{W}(0) = W_1. \end{cases} \tag{39}$$

Here we have

$$W = (U, V)^T \in L^2((0,T), V), \quad F = \left(\frac{q_x}{\hat{\rho} b}, \frac{q_y}{\hat{\rho} b} \right)^T \in L^2((0,T), V^*),$$
$$W_0 = (U_0, V_0)^T, \quad W_1 = (U_1, V_1)^T$$

and σ_1, σ_2 are the sesquilinear forms defined on V by: for all $\phi, \psi \in V$

$$\begin{aligned} \sigma_1(\phi, \psi) \;=\; & \int_{\Omega_{\mathcal{B}}} \tilde{E}(2 - 2\hat{\nu}) \frac{\partial \phi_1}{\partial x} \overline{\frac{\partial \psi_1}{\partial x}} + \int_{\Omega_{\mathcal{B}}} \tilde{E}(1 - 2\hat{\nu}) \frac{\partial \phi_1}{\partial y} \overline{\frac{\partial \psi_1}{\partial y}} \\ + & \int_{\Omega_{\mathcal{B}}} \tilde{E}(1 - 2\hat{\nu}) \frac{\partial \phi_2}{\partial x} \overline{\frac{\partial \psi_1}{\partial y}} + \int_{\Omega_{\mathcal{B}}} 2\tilde{E}\hat{\nu} \frac{\partial \phi_2}{\partial y} \overline{\frac{\partial \psi_1}{\partial x}} \\ + & \int_{\Omega_{\mathcal{B}}} \tilde{E}(1 - 2\hat{\nu}) \frac{\partial \phi_2}{\partial x} \overline{\frac{\partial \psi_2}{\partial x}} + \int_{\Omega_{\mathcal{B}}} \tilde{E}(2 - 2\hat{\nu}) \frac{\partial \phi_2}{\partial y} \overline{\frac{\partial \psi_2}{\partial y}} \\ + & \int_{\Omega_{\mathcal{B}}} \tilde{E}(1 - 2\hat{\nu}) \frac{\partial \phi_1}{\partial y} \overline{\frac{\partial \psi_2}{\partial x}} + \int_{\Omega_{\mathcal{B}}} 2\tilde{E}\hat{\nu} \frac{\partial \phi_1}{\partial x} \overline{\frac{\partial \psi_2}{\partial y}}. \end{aligned} \tag{40}$$

$$
\begin{aligned}
\sigma_2(\phi,\psi) \;=\; & \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(2-2\nu)\frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\psi_1}{\partial x}} \;+\; \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\nu)\frac{\partial\phi_1}{\partial y}\overline{\frac{\partial\psi_1}{\partial y}} \\
+ \; & \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\hat{\nu})\frac{\partial\phi_2}{\partial x}\overline{\frac{\partial\psi_1}{\partial y}} \;+\; \int_{\Omega_{\mathcal{B}}} \tilde{C}_D 2\hat{\nu}\frac{\partial\phi_2}{\partial y}\overline{\frac{\partial\psi_1}{\partial x}} \\
+ \; & \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\hat{\nu})\frac{\partial\phi_2}{\partial x}\overline{\frac{\partial\psi_2}{\partial x}} \;+\; \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(2-2\hat{\nu})\frac{\partial\phi_2}{\partial y}\overline{\frac{\partial\psi_2}{\partial y}} \\
+ \; & \int_{\Omega_{\mathcal{B}}} \tilde{C}_D(1-2\hat{\nu})\frac{\partial\phi_1}{\partial y}\overline{\frac{\partial\psi_2}{\partial x}} \;+\; \int_{\Omega_{\mathcal{B}}} \tilde{C}_D 2\hat{\nu}\frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\psi_2}{\partial y}}.
\end{aligned}
\tag{41}
$$

2.2 Properties of the sesquilinear forms σ_1 and σ_2

The well-posedness of the initial value problem relies on the properties of σ_1 and σ_2. We will focus on σ_1, noting that σ_2 has the same properties as long as C_D and C_{Dpe} are positive. The special case $\tilde{C}_D = 0$, i.e. the structure without damping, is discussed separately.

2.2.1 Properties of σ_1

Let $\check{E} = \dfrac{E}{2(1+\nu)(1-2\nu)}$ and $\check{E}_{pe} = \dfrac{E_{pe}}{2(1+\nu_{pe})(1-2\nu_{pe})}$. Then, σ_1 satisfies the following conditions:

1. σ_1 is symmetric:

   It is easy to verify that

   $$\sigma_1(\phi,\psi) = \overline{\sigma_1(\psi,\phi)} \qquad \text{for all } \phi,\psi \in V. \tag{42}$$

2. σ_1 is continuous:

   Indeed, the magnitude of each term in Eq. (40) is less than $2\max(\check{E},\check{E}_{pe})\|\phi\|_V\|\psi\|_V$, so that

   $$\sigma_1(\phi,\psi) \le 16\max\left(\check{E},\check{E}_{pe}\right)\|\phi\|_V\|\psi\|_V \quad \text{for all } \phi,\psi \in V. \tag{43}$$

3. σ_1 is elliptic:

   A straightforward proof of ellipticity for a completely clamped structure ($\Gamma = \partial\Omega_{\mathcal{B}}$) is outlined in the next section. For the case of interest to us, we wish to prove that for some $k > 0$,

   $$\sigma_1(\phi,\phi) \ge k\|\phi\|_V^2 \quad \forall\phi \in V. \tag{44}$$

The inequality (44) is not trivial to obtain, for the expression of $\sigma(\phi,\phi)$ only involves certain combinations of first derivatives of ϕ_1 and ϕ_2. Namely, grouping the appropriate terms, we have

$$\begin{aligned}\sigma_1(\phi,\phi) &= \int_{\Omega_{\mathcal{B}}} \tilde{E}(2-2\hat{\nu})\left|\frac{\partial\phi_1}{\partial x}\right|^2 + \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})\left|\frac{\partial\phi_1}{\partial y}\right|^2 \\ &\quad + \int_{\Omega_{\mathcal{B}}} \tilde{E}(1-2\hat{\nu})\left|\frac{\partial\phi_2}{\partial x}\right|^2 + \int_{\Omega_{\mathcal{B}}} \tilde{E}(2-2\hat{\nu})\left|\frac{\partial\phi_2}{\partial y}\right|^2 \\ &\quad + \int_{\Omega_{\mathcal{B}}} \tilde{E}(2-4\hat{\nu})\ \mathrm{Re}\left(\frac{\partial\phi_2}{\partial x}\overline{\frac{\partial\phi_1}{\partial y}}\right) + \int_{\Omega_{\mathcal{B}}} 4\tilde{E}\hat{\nu}\ \mathrm{Re}\left(\frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\phi_2}{\partial y}}\right).\end{aligned}$$

To establish inequality (44), we first split these integrals into integrals over Ω and Ω_{pe}, where $\tilde{E}$ and $\hat{\nu}$ are constant. Let $\phi=(\phi_1,\phi_2)\in V$, we then use Korn's inequality (see p. 110-111 in [8]) on Ω:
There exists $c_1>0$ such that for all $\phi_1,\phi_2\in H_0^1(\Omega)$,

$$\epsilon(\phi)+\int_\Omega|\phi_1|^2+\int_\Omega|\phi_2|^2\geq c_1\left(\|\phi_1\|^2_{H_0^1(\Omega)}+\|\phi_2\|^2_{H_0^1(\Omega)}\right), \tag{45}$$

where

$$\epsilon(\phi)=\int_\Omega\left|\frac{\partial\phi_1}{\partial x}\right|^2+\int_\Omega\left|\frac{\partial\phi_2}{\partial y}\right|^2+\frac{1}{2}\int_\Omega\left|\frac{\partial\phi_1}{\partial y}+\frac{\partial\phi_2}{\partial x}\right|^2.$$

Next, we use inequality (3.42) of p. 116 in [8]):
There exists $c_2>0$ such that for all $\phi_1,\phi_2\in H_0^1(\Omega)$,

$$\epsilon(\phi)\geq c_2\left(\|\phi_1\|^2_{L^2(\Omega)}+\|\phi_2\|^2_{L^2(\Omega)}\right). \tag{46}$$

Combining inequalities (45) and (46), we find that there exists $c>0$ such that for all $\phi_1,\phi_2\in H_0^1(\Omega)$,

$$\epsilon(\phi)=\int_\Omega\left|\frac{\partial\phi_1}{\partial x}\right|^2+\int_\Omega\left|\frac{\partial\phi_2}{\partial y}\right|^2+\frac{1}{2}\int_\Omega\left|\frac{\partial\phi_1}{\partial y}+\frac{\partial\phi_2}{\partial x}\right|^2\geq c\left(\|\phi_1\|^2_{H_0^1(\Omega)}+\|\phi_2\|^2_{H_0^1(\Omega)}\right). \tag{47}$$

Multiplying inequality (47) by the positive constant $(2-4\nu)$, adding

$$2\nu\int_\Omega\left|\frac{\partial\phi_1}{\partial x}\right|^2+2\nu\int_\Omega\left|\frac{\partial\phi_2}{\partial y}\right|^2+4\nu\int_\Omega\mathrm{Re}\ \left(\frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\phi_2}{\partial y}}\right)=2\nu\int_\Omega\left|\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_2}{\partial y}\right|^2\geq 0$$

to the left side, adding $(1-2\nu)$ times a similar term with ϕ_1,ϕ_2 interchanged and multiplying by $\check{E}$, we obtain

$$\int_\Omega \check{E}(2-2\nu)\left|\frac{\partial\phi_1}{\partial x}\right|^2 + \int_\Omega \check{E}(1-2\nu)\left|\frac{\partial\phi_1}{\partial y}\right|^2 + \int_\Omega \check{E}(1-2\nu)\left|\frac{\partial\phi_2}{\partial x}\right|^2 + \int_\Omega \check{E}(2-2\nu)\left|\frac{\partial\phi_2}{\partial y}\right|^2 + \int_\Omega \check{E}(2-4\nu)\ \mathrm{Re}\left(\frac{\partial\phi_2}{\partial x}\overline{\frac{\partial\phi_1}{\partial y}}\right) + \int_\Omega 4\check{E}\nu\ \mathrm{Re}\left(\frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\phi_2}{\partial y}}\right) \geq c(2-4\nu)\check{E}\left(\|\phi_1\|^2_{H^1_0(\Omega)} + \|\phi_2\|^2_{H^1_0(\Omega)}\right). \tag{48}$$

The same considerations for Ω_{pe} lead to:

$$\int_{\Omega_{pe}} \check{E}_{pe}(2-2\nu_{pe})\left|\frac{\partial\phi_1}{\partial x}\right|^2 + \int_{\Omega_{pe}} \check{E}_{pe}(1-2\nu_{pe})\left|\frac{\partial\phi_1}{\partial y}\right|^2 + \int_{\Omega_{pe}} \check{E}_{pe}(1-2\nu_{pe})\left|\frac{\partial\phi_2}{\partial x}\right|^2 + \int_{\Omega_{pe}} \check{E}_{pe}(2-2\nu_{pe})\left|\frac{\partial\phi_2}{\partial y}\right|^2 + \int_{\Omega_{pe}} \check{E}_{pe}(2-4\nu_{pe})\ \mathrm{Re}\left(\frac{\partial\phi_2}{\partial x}\overline{\frac{\partial\phi_1}{\partial y}}\right) + \int_{\Omega_{pe}} 4\check{E}_{pe}\nu_{pe}\ \mathrm{Re}\left(\frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\phi_2}{\partial y}}\right) \geq c'(2-4\nu)\check{E}_{pe}\left(\|\phi_1\|^2_{H^1_0(\Omega_{pe})} + \|\phi_2\|^2_{H^1_0(\Omega_{pe})}\right). \tag{49}$$

An estimate for σ_1 (see (44)) is then obtained by adding inequalities (48) and (49).

2.2.2 Proof of ellipticity for σ_1 for the case $\Gamma = \partial\Omega$

For the case of a completely clamped structure, $\Omega_{pe} = \emptyset$ and $\Gamma = \partial\Omega$, we have $V = H^1_0(\Omega)$. An easier proof for the ellipticity of σ is available. We have, for all $\phi_1, \phi_2 \in C^\infty_0(\Omega)$,

$$\int_\Omega \frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\phi_2}{\partial y}} = -\int_\Omega \frac{\partial^2\phi_1}{\partial x\partial y}\overline{\phi}_2 = \int_\Omega \frac{\partial\phi_1}{\partial y}\overline{\frac{\partial\phi_2}{\partial x}}$$

Since $(\phi_1,\phi_2) \mapsto \int_\Omega \frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\phi_2}{\partial y}}$ and $(\phi_1,\phi_2) \mapsto \int_\Omega \frac{\partial\phi_1}{\partial y}\overline{\frac{\partial\phi_2}{\partial x}}$ are continuous on $H^1(\Omega)^2$, we can conclude that, for all $\phi_1, \phi_2 \in \overline{C^\infty_0(\Omega)} = H^1_0(\Omega)$,

$$\int_\Omega \frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\phi_2}{\partial y}} = \int_\Omega \frac{\partial\phi_1}{\partial y}\overline{\frac{\partial\phi_2}{\partial x}}. \tag{50}$$

Using Eq. (50) in Eq. (40), we obtain

$$\begin{aligned}\sigma_1(\phi,\phi) \;=\; & \int_\Omega \check{E}(2-2\nu)\left|\frac{\partial\phi_1}{\partial x}\right|^2 + \int_\Omega \check{E}(1-2\nu)\left|\frac{\partial\phi_1}{\partial y}\right|^2 \\ & + \int_\Omega \check{E}(1-2\nu)\left|\frac{\partial\phi_2}{\partial x}\right|^2 + \int_\Omega \check{E}(2-2\nu)\left|\frac{\partial\phi_2}{\partial y}\right|^2 \\ & + \int_\Omega \check{E}(1-2\nu)\mathrm{Re}\left(\frac{\partial\phi_2}{\partial x}\overline{\frac{\partial\phi_1}{\partial y}}\right) + \int_\Omega \check{E}(1+2\nu)\mathrm{Re}\left(\frac{\partial\phi_1}{\partial x}\overline{\frac{\partial\phi_2}{\partial y}}\right).\end{aligned}$$

Using $\mathrm{Re}(a\overline{b}) \geq -\frac{|a|^2+|b|^2}{2}$, we obtain

$$\begin{aligned}\sigma_1(\phi,\phi) &\geq \int_\Omega 3\check{E}(\frac{1}{2}-\nu)\left|\frac{\partial\phi_1}{\partial x}\right|^2 + \int_\Omega \check{E}(\frac{1}{2}-\nu)\left|\frac{\partial\phi_1}{\partial y}\right|^2 \\ &\quad + \int_\Omega \check{E}(\frac{1}{2}-\nu)\left|\frac{\partial\phi_2}{\partial x}\right|^2 + \int_\Omega 3\check{E}(\frac{1}{2}-\nu)\left|\frac{\partial\phi_2}{\partial y}\right|^2 \\ &\geq \check{E}(\frac{1}{2}-\nu)\|\phi\|^2_{H^1_0(\Omega)}, \end{aligned} \tag{51}$$

which proves that σ_1 is elliptic, since $0 \leq \nu < \frac{1}{2}$.

This also applies to the case where $\Omega_{pe} \neq \emptyset$ and $\Gamma = \partial\Omega_{\mathcal{B}}$, but not to the most general case where $\partial\Omega_{\mathcal{B}} \setminus \Gamma \neq \emptyset$ because Eq. (50) does not hold in $H^1_L(\Omega)$.

We thus have proved that, for some $k > 0$ and $K > 0$,

$$k\|.\|^2_V \leq \sigma_1(.,.) \leq K\|.\|^2_V.$$

2.2.3 Properties of σ_2

When $\tilde{C}_D > 0$, we have

$$k_2\|.\|^2_V \leq \sigma_2(.,.) \leq K_2\|.\|^2_V$$

for some $k_2 > 0$ and $K_2 > 0$. Thus, σ_2 has exactly the properties of σ_1 detailed in 2.2.1 (1), (2), (3).

2.3 Generalization

A more general formulation of the problem is:

Let V and H be complex Hilbert spaces forming a Gelfand triple $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ with pivot space H and duality pairing $< .,. >_{V^*,V}$. That is, V is continuously and densely embedded in H with $\|\phi\|_H \leq C\|\phi\|_V$, and we identify H^* with H through the Riesz map. Here V^*, H^* are the dual spaces to V, H, respectively. The duality pairing $< .,. >_{V^*,V}$ is the extension by continuity of the inner product $< .,. >_H$ from $V \times H$ to $V^* \times H$. Hence, elements $v^* \in V^*$ have the representation $v^*(v) =< v^*, v >_{V^*,V}$.

Let σ_1 be a sesquilinear form $\sigma_1 : V \times V \to \mathbb{C}$ that has the following properties:

- Symmetry: $\forall\phi,\psi \in V, \quad \sigma_1(\phi,\psi) = \overline{\sigma_1(\psi,\phi)}$.
- Continuity: $\exists K > 0, \forall\phi,\psi \in V, \quad |\sigma_1(\phi,\psi)| \leq K\|\phi\|_V\|\psi\|_V$.

- V-ellipticity: $\exists k > 0, \forall \phi \in V, \quad \sigma_1(\phi, \phi) \geq k\|\phi\|_V^2$.

To allow a wide class of damping characteristics, we introduce a new Hilbert space V_2 on which the sesquilinear form σ_2 is defined. V_2 satisfies $V \subset V_2 \subset H$ and we assume that $V \hookrightarrow V_2 \hookrightarrow H \cong H^* \hookrightarrow V_2^* \hookrightarrow V^*$ is a Gelfand setting. The sesquilinear form $\sigma_2 : V_2 \times V_2 \to \mathbb{C}$ is assumed to have the following properties:

- Continuity: $\exists K_2 > 0, \forall \phi, \psi \in V_2, \quad |\sigma_2(\phi, \psi)| \leq K_2\|\phi\|_{V_2}\|\psi\|_{V_2}$.
- V_2-ellipticity: $\exists k_2 > 0, \lambda_0 \geq 0, \forall \phi \in V, \quad \sigma_1(\phi, \phi) + \lambda_0\|\phi\|_H^2 \geq k_2\|\phi\|_V^2$.

We then assume that the input function f satisfies $f \in L_2([0, T], V_2^*)$.
We consider the weak or variational form of the equation, given by

$$\begin{cases} < \ddot{y}(t), \phi > + \sigma_2(\dot{y}(t), \phi) + \sigma_1(y(t), \phi) =< f(t), \phi > \quad \forall \phi \in V, \\ y(0) = y_0, \quad \dot{y}(0) = y_1. \end{cases} \tag{52}$$

3 Existence and uniqueness of solutions

3.1 Resolution of the weak form

3.1.1 Statement of the theorem

We are now in a position to use known results to establish well-posedness of the general problem in section 2.3.

Theorem 3.1 *With the assumptions in section 2.3 and $y_0 \in V$, $y_1 \in H$, there exists a unique solution y to (52) with $y \in L_2([0,T],V)$, $\dot{y} \in L_2([0,T],V_2)$ and $\ddot{y} \in L_2([0,T],V^*)$. Moreover, the solutions depend continuously on the data (y_0, y_1, f) in the sense that the map $(y_0, y_1, f) \mapsto (y, \dot{y})$ is continuous from $V \times H \times L_2([0,T],V_2^*)$ to $L_2([0,T],V) \times L_2([0,T],V_2)$.*

A detailled proof of this result is given in Chapter 4 of [5].

3.1.2 Back to the damaged structure: application of the results

applying these results to our model of damaged structure, we obtain:

1. **Case $\hat{C}_D > 0$**

   In the case with damping, the sesquilinear forms σ_1 and σ_2 are both symmetric, continuous and V-elliptic. We can thus take $V_2 = V$ and we have the following result:

   > Suppose that $F \in L^2((0,T),V^*)$, $W_0 \in V$ and $W_1 \in H$.
   > The problem (38) has a unique solution W with $W \in L^2((0,T),V)$, $\dot{W} \in L^2((0,T),V)$ and $\ddot{W} \in L^2((0,T),V^*)$. Moreover, the solution depends continuously on the data (W_0, W_1, F) in that the map $(W_0, W_1, F) \mapsto (W, \dot{W})$ is continuous from $V \times H \times L^2((0,T),V^*)$ to $L^2((0,T),V) \times L^2((0,T),V)$.

2. **Case $\hat{C}_D = 0$: no damping**

   If there is no damping, the hypotheses on σ_2 force us to take $V_2 = H$. We thus only have the following weaker result:

   > Suppose that $F \in L^2((0,T),H)$, $W_0 \in V$ and $W_1 \in H$.
   > The problem (38) has a unique solution W with $W \in L^2((0,T),V)$, $\dot{W} \in L^2((0,T),H)$ and $\ddot{W} \in L^2((0,T),V^*)$. Moreover, the solution depends continuously on the data (W_0, W_1, F) in that the map $(W_0, W_1, F) \mapsto (W, \dot{W})$ is continuous from $V \times H \times L^2((0,T),H)$ to $L^2((0,T),V) \times L^2((0,T),H)$.

In both cases, the result can even be strengthened to conclude that $W \in C((0,T),V)$ and $\dot{W} \in C((0,T),H)$.

As seen in section 1, the forces applied by the patches on the structure can be very irregular, because they involve the derivatives of the characteristic function χ_{pe} of Ω_{pe} in $\Omega_{\mathcal{B}}$. In fact χ_{pe} is in $L_2(\Omega_{\mathcal{B}})$, so that the applied forces q_x and q_y are in $H^{-1}(\Omega_{\mathcal{B}})$. Hence, F is in $L_2([0,T],V^*)$. In particular, $F(t)$ is unbounded in H.

The result we have here is thus satisfactory in the case where there is damping, but not in the case without damping. In the following section we write a semigroup formulation for the problem, which allows us to give a solution to the problem without damping. Semigroup formulations are commonly used in control theory. Moreover, it will leads to interesting considerations about the energy of the system.

3.2 Semigroup formulation

We will now write a semigroup formulation for the solutions of Eq. (38). For simplicity, we assume that $\lambda_0 = 0$, which is the case of our example.

3.2.1 First order form

As a first step, we rewrite Eq. (38) in first order form. For this purpose, we define the Hilbert spaces $\mathcal{H} = V \times H$, $\mathcal{V} = V \times V$ and $\mathcal{V}^* = V \times V^*$. Note that σ_1 and σ_2 define the operators A_1 and A_2 in $\mathcal{L}(V,V^*)$ satisfying

$$\begin{array}{l} \sigma_1(\phi,\psi) =< A_1\phi,\psi >_{V^*,V} \\ \sigma_2(\phi,\psi) =< A_2\phi,\psi >_{V^*,V} \end{array} \quad \text{for all } \phi,\psi \in V.$$

With these notations, Eq. (38) can be rewritten as

$$\left\{ \begin{array}{ll} \ddot{W}(t) + A_2\dot{W}(t) + A_1 W(t) = F(t) & \text{in } V^*, \\ W(0) = W_0, \quad \dot{W}(0) = W_1, & \end{array} \right.$$

which can be reformulated as a first order differential equation problem:

$$\left\{ \begin{array}{ll} \dot{Z}(t) = \mathcal{A}Z(t) + \mathcal{F}(t) & \text{in } \mathcal{V}^*, \\ Z(0) = Z_0 & \end{array} \right. \tag{53}$$

where $Z = \begin{pmatrix} W \\ \dot{W} \end{pmatrix}$, $\mathcal{F} = \begin{pmatrix} 0 \\ F \end{pmatrix}$, $Z_0 = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix}$ and $\mathcal{A}$ is the operator given by

$$\text{dom}\mathcal{A} = \{(\phi,\psi) \in \mathcal{H} | \quad \psi \in V \text{ and } A_1\phi + A_2\psi \in H\} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 0 & I \\ -A_1 & -A_2 \end{pmatrix}.$$

Recalling that $k\|.\|_V^2 \leq \sigma_1(.,.) \leq K\|.\|_V^2$, we see that σ_1 and $< .,. >_V$ are equivalent norms. Hence, we can define V_1 as the set V with σ_1 as inner product: V_1 is then topologically equivalent to V. Therefore, it suffices to argue that $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}_1 = V_1 \times H$ to conclude that it generates a semigroup on $\mathcal{H}$.

To show that $\mathcal{A}$ is an infinitesimal generator in $\mathcal{H}_1$, we use Lumer-Phillips' theorem (see p. 14 in [7]).

3.2.2 $\mathcal{A}$ is dissipative

Indeed, for $\Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{H}_1$,

$$\begin{aligned} \mathrm{Re} < \mathcal{A}\Phi, \Phi >_{\mathcal{H}_1} &= \mathrm{Re}\,(\sigma_1(\psi,\phi) + < -A_1\phi - A_2\psi, \psi >_H) \\ &= \mathrm{Re}\,(\sigma_1(\psi,\phi) - \sigma_1(\phi,\psi) - \sigma_2(\psi,\psi)) \\ &= -\ \mathrm{Re}\ \sigma_2(\psi,\psi) \\ &\leq 0. \end{aligned} \tag{54}$$

Note that, in the case where there is no damping, $\mathrm{Re} < \mathcal{A}\Phi, \Phi >_{\mathcal{H}_1} = 0$.

3.2.3 The range of $\lambda I - \mathcal{A}$ is $\mathcal{H}_1$ for some $\lambda > 0$

Let $\lambda > 0$ and $\xi = \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \in \mathcal{H}_1$. We want to solve $(\lambda I - \mathcal{A})\Phi = \xi$ for $\Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathrm{dom}\mathcal{A}$. This equation is equivalent to

$$\begin{cases} \lambda\phi - \psi = \eta \\ A_1\phi + A_2\psi + \lambda\psi = \zeta. \end{cases}$$

We can solve the first equality in $\psi = \lambda\phi - \eta$ and substitute in the second to obtain

$$\lambda^2\phi + A_1\phi + \lambda A_2\phi = \zeta + \lambda\eta + A_2\eta. \tag{55}$$

Then, we introduce the sesquilinear form defined on $V \times V$:

$$\sigma_\lambda(\phi,\psi) = \lambda^2 < \phi, \psi >_H + \sigma_1(\phi,\psi) + \lambda\sigma_2(\phi,\psi),$$

which satisfies:

$$\begin{aligned} \mathrm{Re}\ \sigma_\lambda(\phi,\phi) &= \lambda^2\|\phi\|_H^2 + \sigma_1(\phi,\phi) + \lambda\sigma_2(\phi,\phi) \\ &\geq k\|\phi\|_V^2. \end{aligned} \tag{56}$$

Hence, σ_λ is V-elliptic and we can find $\phi \in V$ satisfying Eq. (55). The corresponding $\psi = \lambda\phi - \eta$ is also in V, and $A_1\phi + A_2\psi = \zeta - \lambda\psi \in H$, so that $(\phi,\psi) \in \mathrm{dom}\mathcal{A}$.

Thus, we have shown: $\mathrm{R}(\lambda I - \mathcal{A}) = \mathcal{H}_1$ for any $\lambda > 0$.

3.2.4 Conclusion: mild solution of the problem

Finally, we can conclude that the operator $\mathcal{A}$ generates a C_0-semigroup $\mathcal{T}(t)$ of contractions on $\mathcal{H}$. This enables us to define mild solutions for Eq. (53). For $Z_0 \in \mathcal{H}$ and $\mathcal{F} \in L^2((0,T),\mathcal{H})$, we let

$$Z(t) = \mathcal{T}(t)Z_0 + \int_0^t \mathcal{T}(t-s)\mathcal{F}(s)ds. \tag{57}$$

This mild solution is the unique strong solution of Eq. (53) when $Z_0 \in \mathrm{dom}\mathcal{A}$ and $\mathcal{F} \in C^1((0,T),\mathcal{H})$ [9].

Note that it is not possible to extend directly this formula to allow $\mathcal{F} \in L_2([0,T],\mathcal{V}^*)$. The next section shows how this extension is possible.

3.2.5 Extensions

1. **Case $\hat{C}_D > 0$**

   Under this assumption, we consider the operator $\tilde{\mathcal{A}}$ (which is an extension of $\mathcal{A}$ from $\mathrm{dom}\mathcal{A}$ to $\mathcal{V}_1 = V_1 \times V_1$) given by

   $$\begin{aligned} \sigma(\Phi,\Psi) &= -\sigma_1(\phi_2,\psi_1) + \sigma_1(\phi_1,\psi_2) + \sigma_2(\phi_2,\phi_2) \\ &= < \tilde{\mathcal{A}}\Phi, \Psi >_{\mathcal{V}_1^*,\mathcal{V}_1} . \end{aligned}$$

   The properties of σ_2 yield immediately:

   $$\begin{aligned} \mathrm{Re}\ \sigma(\Phi,\Phi) &\geq k_2\|\phi_2\|_V^2 \\ &\geq k_2(\|\phi_1\|_V^2 + \|\phi_2\|_V^2) - k_2\|\phi_1\|_V^2 \\ &\geq \tilde{k_2}\|\Phi\|_{\mathcal{V}_1}^2 - \tilde{\lambda}_0\|\Phi\|_{\mathcal{H}_1}^2 \end{aligned}$$

   where $\tilde{k}_2$ and $\tilde{\lambda}_0$ are positive constants. Hence, σ satisfies Garding's inequality and we can apply the result of the Riesz Theorem (see p. 25 and 76 in [9]):

   The infinitesimal generator $\tilde{\mathcal{A}}$ generates an analytic semigroup $\tilde{\mathcal{T}}(t)$ on both $\mathcal{H}_1$ and $\mathcal{V}_1^*$ (and hence $\mathcal{V}^*$). On $\mathcal{H}$, we have $\tilde{\mathcal{T}}(t) = \mathcal{T}(t)$; this allows us to extend the definition of the mild solution to the case $F \in L^2([0,T],V^*)$.

   This mild solution and the weak solution found in section 3.1 are equivalent (see section 4.2 and notes p. 258-259 in [7]).

2. **Case $\hat{C}_D = 0$: no damping**

   In the case without damping, σ does not satisfy Garding's inequality and $\mathcal{T}(t)$ cannot be extended to an analytic semigroup. It is possible, however, to extend $\mathcal{A}$ to an operator $\hat{\mathcal{A}}$ with domain $\mathcal{H}_1$, so that $\hat{\mathcal{A}}$ generates a C_0-semigroup $\hat{\mathcal{T}}(t)$ satisfying $\hat{\mathcal{T}}(t) = \mathcal{T}(t)$ on $\mathcal{H}$. This allows us to define a mild solution whenever $F \in L^2([0,T],V^*)$, even if the weak solution is not defined (see p. 113 in [5]).

   In this case, we have only defined a weak solution when $F \in L^2([0,T],H)$. This solution is then equivalent to the mild solution (see p. 114 in [5]).

We have now defined solutions for $F \in L_2([0,T],V^*)$ in all cases.

4 Particularities of the semigroup $\mathcal{T}(t)$ generated by $\mathcal{A}$: relations with the energy

4.1 When $\hat{C}_D = 0$, $\mathcal{T}(t)$ is a semigroup of isometries in $\mathcal{H}_1$ and the "internal energy" is constant in time

Let us assume that $\hat{C}_D = 0$. We already know that $\mathcal{T}(t)$ is a semigroup of contractions. In fact, we can even show a stronger result, namely, that $\mathcal{T}(t)$ is a semigroup of isometries:

$$\forall \phi \in \mathcal{H}_1, \qquad \|\mathcal{T}(t)\phi\|_{\mathcal{H}_1} = \|\phi\|_{\mathcal{H}_1}.$$

To prove this, we use (55) to see that, for $\phi \in \mathrm{dom}\mathcal{A}$, $\mathcal{T}(t)\phi \in \mathrm{dom}\mathcal{A}$ (see p. 4-5 in [7]) and

$$\begin{aligned} \frac{d}{dt}\|\mathcal{T}(t)\phi\|^2_{\mathcal{H}_1} &= 2\ \mathrm{Re} < \frac{d\mathcal{T}(t)\phi}{dt}, \mathcal{T}(t)\phi >_{\mathcal{H}_1} \\ &= 2\ \mathrm{Re} < \mathcal{A}\mathcal{T}(t)\phi, \mathcal{T}(t)\phi >_{\mathcal{H}_1} \\ &= 0 \end{aligned}$$

so that

$$\|\mathcal{T}(t)\phi\|_{\mathcal{H}_1} = \|\mathcal{T}(0)\phi\|_{\mathcal{H}_1} = \|\phi\|_{\mathcal{H}_1} \quad \text{for all } \phi \in \mathrm{dom}\mathcal{A}.$$

As $\mathcal{T}(t)$ is bounded, the map $\phi \mapsto \mathcal{T}(t)\phi$ is continuous. We can use the density of $\mathrm{dom}\mathcal{A}$ in $\mathcal{H}_1$ to conclude that

$$\|\mathcal{T}(t)\phi\|_{\mathcal{H}_1} = \|\phi\|_{\mathcal{H}_1} \quad \text{for all } \phi \in \mathcal{H}_1.$$

In particular, $|\mathcal{T}(t)|_{\mathcal{H}_1} = 1$.

Physically, this means that the system is neither intrinsically stable nor unstable: it is conservative. If we don't take into account the energy eventually given by $F(t)$, its total internal energy remains constant in time. More precisely, if $Z(t)$ is the solution of (53), we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} < Z(t), Z(t) >_{\mathcal{H}_1}\right) &= \mathrm{Re} < \dot{Z}(t), Z(t) >_{\mathcal{V}_1^*,\mathcal{V}_1} \\ &= \mathrm{Re} < \mathcal{A}Z(t), Z(t) >_{\mathcal{V}_1^*,\mathcal{V}_1} + \mathrm{Re} < \mathcal{F}(t), Z(t) >_{\mathcal{V}_1^*,\mathcal{V}_1} \\ &= \mathrm{Re} < \mathcal{F}(t), Z(t) >_{\mathcal{V}_1^*,\mathcal{V}_1} . \end{aligned}$$

$\left(\frac{1}{2} < Z(t), Z(t) >_{\mathcal{H}_1}\right)$ represents the "energy" of the system, which can be written

$$\mathcal{T} + \mathcal{U} = \frac{1}{2}\int_{\Omega_{\mathcal{B}}} \left[\hat{\rho}(\dot{U}^2 + \dot{V}^2) + \sigma_x\epsilon_x + \sigma_y\epsilon_y + \gamma_{xy}\tau_{xy}\right] =< \frac{1}{2\hat{\rho}}Z(t), Z(t) >_{\mathcal{H}_1} .$$

$\mathrm{Re} < \mathcal{F}(t), Z(t) >_{\mathcal{V}_1^*,\mathcal{V}_1}$ is the power given by $F(t)$ to the system.

If $F(t) = 0$, then $\|Z(t)\|_{\mathcal{H}_1} = \|\mathcal{T}(t)Z_0\|_{\mathcal{H}_1} = \|Z_0\|_{\mathcal{H}_1}$ for all t and the energy is constant in time.

Note that, in $\mathcal{H}_1$, the square of the distance to 0 is the energy of the system.

4.2 When $\hat{C}_D > 0$, the "internal energy" is exponentially decreasing

If $\tilde{C}_D \geq 0$, we only know that $|\mathcal{T}(t)|_{\mathcal{H}_1} \leq Me^{-\omega t}$ for some $M \geq 1$ and $\omega \geq 0$. It is possible, however, to show that $\omega > 0$ when $\tilde{C}_D > 0$; which means that the system is strictly dissipative if there is strong damping. As a first step, we note that we can strengthen the result obtained in Eq. (56):

$$\begin{aligned} \text{Re } \sigma_\lambda(\phi,\phi) &= (\text{Re } \lambda^2)\|\phi\|_H^2 + \sigma_1(\phi,\phi) + (\text{Re } \lambda)\sigma_2(\phi,\phi) \\ &\geq \left(-C^2|\lambda|^2 + k - k_2|\lambda|\right)\|\phi\|_V^2. \end{aligned}$$

It follows that σ_λ is V-elliptic for any λ satisfying $C^2|\lambda|^2 + k_2|\lambda| < k$, that is for

$$|\lambda| < \lambda_0 = \frac{\sqrt{k_2^2 + 4kC^2} - k_2}{2}.$$

For such values of λ, the range of $(\lambda I - \mathcal{A})$ is $\mathcal{H}_1$. Therefore, the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains the disk $\{|\lambda| < \lambda_0\}$. But, as $\mathcal{A}$ is an analytic semigroup, $0 \in \rho(\mathcal{A})$ and $\mathcal{T}(t)$ is a semigroup of contractions, we already know that (see p. 61 in [7])

$$\rho(\mathcal{A}) \supset \{\lambda,\ |\arg \lambda| < \frac{\pi}{2} + \delta\}.$$

It follows that

$$\sup\{\text{Re } \lambda,\ \lambda \in \sigma(\mathcal{A})\} > 0$$

where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$. Using that $\mathcal{A}$ is analytic, we can conclude (see p. 118 in [7]) that

$$\|\mathcal{T}(t)\| \leq Me^{-\omega t} \quad \forall t \geq 0$$

for some $M \geq 1$ and $\omega > 0$.

Therefore, the "energy" satisfies:

$$\frac{1}{2}\|Z(t)\|_{\mathcal{H}_1}^2 = \frac{1}{2}\|\mathcal{T}(t)Z_0\|_{\mathcal{H}_1}^2 \leq \frac{1}{2}M^2e^{-2\omega t}\|Z_0\|_{\mathcal{H}_1}^2.$$

It is exponentially decreasing.

5 Back to the strong formulation and the boundary conditions

The weak solution $W = \begin{pmatrix} U \\ V \end{pmatrix}$ found in section 3 is a priori in $C([0,T],V)$, where V is $H^1_L(\Omega_\mathcal{B}) \times H^1_L(\Omega_\mathcal{B})$. It readily satisfies the essential boundary conditions

$$U(t) = V(t) = 0 \qquad \text{on } \Gamma \quad \text{, for all } t.$$

It also satisfies the weak formulation of the problem (38). Taking the test functions ϕ_1, ϕ_2 in $C_0^\infty(\Omega)$ and in $C_0^\infty(\Omega_{pe})$ in (38), we see that (12) is satisfied in the sense of distributions. The natural boundary conditions in Eq. (15) are, however, more difficult to verify because we cannot give a sense (as a function) to the trace on $\partial\Omega_\mathcal{B}$ of the derivatives of a $H^1(\Omega_\mathcal{B})$ function. To give a sense to the boundary conditions in Eq. (15), we have to assume that the solution W is smoother, that is:

$$W \in C([0,T], H^2_b(\Omega_\mathcal{B}) \times H^2_b(\Omega_\mathcal{B})), \qquad \text{where } H^2_b(\Omega_\mathcal{B}) = H^1_L(\Omega_\mathcal{B}) \cap H^2(\Omega_\mathcal{B}). \tag{58}$$

Then, we can define the trace of the derivatives of $U(t)$ and $V(t)$ on $\partial\Omega_\mathcal{B}$ as elements of $H^{1/2}(\partial\Omega_\mathcal{B})$.

Assuming (58), the second space derivatives of $U(t)$ and $V(t)$ are now in $L^2(\Omega_\mathcal{B})$. As (34) is satisfied in the sense of distributions, $\hat{\rho}\ddot{U} - \tilde{q}_x$ and $\hat{\rho}\ddot{V} - \tilde{q}_y$ are both in $L^2(\Omega_\mathcal{B})$ and we have

$$\begin{cases} (2-2\nu)(\tilde{E}U_{,xx} + \tilde{C}_D\dot{U}_{,xx}) + (1-2\nu)(\tilde{E}U_{,yy} + \tilde{C}_D\dot{U}_{,yy}) + \tilde{E}V_{,xy} + \tilde{C}_D\dot{V}_{,xy} = \hat{\rho}\ddot{U} - q_x \\ (1-2\nu)(\tilde{E}V_{,xx} + \tilde{C}_D\dot{V}_{,xx}) + (2-2\nu)(\tilde{E}V_{,yy} + \tilde{C}_D\dot{V}_{,yy}) + \tilde{E}U_{,xy} + \tilde{C}_D\dot{V}_{,xy} = \hat{\rho}\ddot{V} - q_y \end{cases}$$

in the $L^2(\Omega)$ sense, that is, Eq. (34) is satisfied almost everywhere in $\Omega_\mathcal{B}$.

We can then multiply by $\phi_1, \phi_2 \in H^1_L(\Omega_\mathcal{B})$ and integrate by parts to obtain the same equations as (37).

Note that, in the most general case where $U(t)$ and $V(t)$ are only in $H^1_L(\Omega_\mathcal{B})$, a sense can be given to the natural boundary conditions in Eq. (15) as equalities in $(H^{1/2}_{00}(\Gamma_t))'$, that is, not in a sense of functions, but of linear continuous operators [11]. It can then be shown that Eq. (15) is satisfied by the weak solutions U and V in the $(H^{1/2}_{00}(\Gamma_t))'$ sense. This means that, as soon as one can give a sense to the natural boundary conditions, they will always be satisfied by the weak solutions.

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Conclusion

A two-dimensional model for a non-symmetrically damaged clamped plate-like structure was presented. The equations of motion were derived in both strong and weak forms for the cases with and without strong damping. The equations were modified to account for the addition of piezoelectric patches bonded to the structure.
A mathematical formulation of the problem as a second order equation in time allowed the discussion of the existence and uniqueness of solutions and their continuous dependence with respect to the data. For the case with strong damping, a satisfactory result is found in spite of the irregularity of the forces induced by the piezoelectric patches. For the theoretical case without damping, an equivalent semigroup formulation was necessary to obtain a solution.
Further studies will include numerical simulations and a comparison with experimental data.

A Verification of the equation of motion for the undamped case

A.1 Comparison of the model to a 2-D Navier displacement field

As a verification of the model, Eq. (12) can be expressed as a special case of Navier's equation, the basic field equation of the linearized theory of elasticity [6]:

$$G\left(\nabla^2 u_i + \frac{1}{1-2\nu}\frac{\partial e}{\partial x_i}\right) + X_i = \rho\frac{\partial^2 u_i}{\partial t^2} \tag{59}$$

where $G = \dfrac{E}{2(1+\nu)}$ is the shear modulus, $e = \dfrac{\partial u_j}{\partial x_j}$ the divergence of the displacement u_i or (U, V, W) and X_i a body force/unit volume.

In the present study, the displacements are specified. For a plane field,

$$U_{,z} = V_{,z} = W = 0. \tag{60}$$

Navier's equation can be written:

$$\begin{cases} G\left(U_{,xx} + U_{,yy}\right) + \dfrac{1}{1-2\nu}G\dfrac{\partial}{\partial x}\left(U_{,x} + V_{,y}\right) + X = \rho\ddot{U} \\ \\ G\left(V_{,xx} + V_{,yy}\right) + \dfrac{1}{1-2\nu}G\dfrac{\partial}{\partial y}\left(U_{,x} + V_{,y}\right) + Y = \rho\ddot{V} \end{cases} \tag{61}$$

After collecting the terms $U_{,xx}$,$U_{,yy}$ and $V_{,xy}$ in the first part of Eq. (61) and $V_{,xx}$,$V_{,yy}$ and $U_{,xy}$ in the second part, Eq. (12) is obtained.

A.2 Comparison of the model to Kirchhoff-Love plate theory

Love's basic assumption states that the transverse stresses in a plate or shell are small compared to the other normal stresses. Taking Love's assumption into account when writing Hooke's law leads to:

$$\begin{aligned} N_x &= \frac{Eb}{1-\nu^2}\left(\epsilon_x + \nu\epsilon_y\right) \\ N_y &= \frac{Eb}{1-\nu^2}\left(\epsilon_y + \nu\epsilon_x\right) \\ N_{xy} &= \frac{Eb}{1+\nu}\gamma_{xy} \end{aligned} \tag{62}$$

Using the force resultants from Eq. (62) into Eq. (5) and (7) provides:

$$\begin{cases} \dfrac{E}{1-\nu^2}\left(\epsilon_{x,x}+\nu\epsilon_{y,x}\right)+\dfrac{E}{1+\nu}\gamma_{xy,y}+\dfrac{q_x}{b}=\rho\ddot{U} \\ \dfrac{E}{1-\nu^2}\left(\epsilon_{y,y}+\nu\epsilon_{x,y}\right)+\dfrac{E}{1+\nu}\gamma_{xy,x}+\dfrac{q_y}{b}=\rho\ddot{V} \end{cases} \tag{63}$$

where the $,x$ and $,y$ refer to the derivatives with respect to x and y.

Replacing the strains in Eq. (63) by their expressions in Eq. (8) allows to obtain the equations of motion in terms of the displacements:

$$\begin{cases} \dfrac{E}{1-\nu^2}\left(U_{,xx}+\nu V_{,xy}\right)+\dfrac{E}{2(1+\nu)}\left(U_{,yy}+V_{,xy}\right)+\dfrac{q_x}{b}=\rho\ddot{U} \\ \dfrac{E}{1-\nu^2}\left(V_{,yy}+\nu U_{,xy}\right)+\dfrac{E}{2(1+\nu)}\left(U_{yx}+V_{,xx}\right)+\dfrac{q_y}{b}=\rho\ddot{V} \end{cases} \tag{64}$$

where the subscripts represent the derivatives of the displacements U and V.

Multiplying Eq. (64) by $\dfrac{1-\nu^2}{E}$ and rearranging the terms lead to the equations of in-plane vibration of a Kirchhoff-Love plate (see Eq. 2.51 p. 57 in [5]):

$$\begin{cases} U_{,xx}+\dfrac{(1-\nu)}{2}U_{,yy}+\dfrac{(1+\nu)}{2}V_{,xy}+\dfrac{(1-\nu^2)}{Eb}q_x=\rho\dfrac{(1-\nu^2)}{E}\ddot{U} \\ V_{,yy}+\dfrac{(1-\nu)}{2}V_{,xx}+\dfrac{(1+\nu)}{2}U_{,xy}+\dfrac{(1-\nu^2)}{Eb}q_y=\rho\dfrac{(1-\nu^2)}{E}\ddot{V} \end{cases} \tag{65}$$

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