INELASTIC ASYMMETRIC BUCKLING OF RING-STIFFENED CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE

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SUMMARY

In this paper, a new analysis on the effects of prebuckling deformation on the inelastic asymmetric buckling of a ring-stiffened cylindrical shell under external pressure is made. The shell is made of a work hardening material having incremental type constitutive relationships and a yield surface with vertices of varying sizes. A variational principle involving Kirchhoff stress rates, $\dot{S}^{ij}$, and velocity fields, $\dot{u}_i$, in Lagrangian descriptions is employed to solve the problem.

Consider a body of a continuum occupying a region V bounded by a surface $A=A_u+A_T$ in its natural underformed state. It may be shown that among all possible velocity fields satisfying prescribed velocities over A_u, the true velocity field caused by prescribed Lagrangian traction rates, $\dot{T}^i$, over A_T is distinguished by a stationary value of the functional

$$I = \frac{1}{2}\int_V (\dot{S}^{ij}\dot{\varepsilon}_{ij} + S^{ij}\dot{u}^k_{;i}\dot{u}_{k;j})\,\mathrm{d}V - \int_{A_T} \dot{T}^i\dot{u}_i\,\mathrm{d}A \tag{1}$$

where $\dot{\varepsilon}_{ij}$ is the Lagrangian strain rate. It may further be shown that the solution is unique when

$$F(\Delta\dot{u}_i) = \int_V (\Delta\dot{S}^{ij}\Delta\dot{\varepsilon}_{ij} + S^{ij}\Delta\dot{u}^k_{;i}\Delta\dot{u}_{k;j})\,\mathrm{d}V > 0 \tag{2}$$

where $\Delta\dot{u}_i=\dot{u}_i^{(1)}-u_i^{(2)}$ is the difference between any two kinematically admissible velocities. When inequality (2) is satisfied, the value of the functional I for the true velocity field is a minimum. The bifurcation phenomenon may occur when $F(\Delta\dot{u}_i)=0$. The foregoing results are essentially identical to that obtained by R. Hill, "A General Theory of Uniqueness and Stability of Elastic Plastic Solids," *J. Mech. Phys. Solids*, *6*, 1958, pp. 236-249, except that they differ in form. Hill uses the current geometry as a reference. Here, the natural configuration is used as a reference for the convenience of analyzing the deformation process.

The variational principle in conjunction with the Rayleigh-Ritz method is used to determine the prebuckling axi-symmetrical deformation caused by the pressure and the fixed edges at the ends and rings. The pressure is normal to the deformed surface. A layered shell idealization and an incremental procedure are used to determine the non-linear deformation process. For each pressure increment, the condition for asymmetric bifurcation is evaluated until the critical pressure is reached.

A comparison of the theoretical predictions obtained numerically by an IBM 370 computer with available experimental results is made. The numerical results indicate a significant fact that the buckling pressure is sensitive to the size of the vertices of the yield surface. Good agreements between the numerical and experimental results are found.

1. Introduction

In the field of inelastic buckling of structures, substantial advances have been made since 1947 when Shanley [2] introduced his significant concept that the first possible bifurcation of an inelastic structure occurs under increasing compressive load. The concept has been further clarified by Hill [1, 3]. In spite of the progress there remain many perplexing phenomena, especially in the area of inelastic buckling of shells. A well known paradox in plastic buckling is that experimental buckling loads have been markedly lower than the theoretical bifurcation load obtained from employing idealized geometry, Mises smooth yield surface and stress-strain relations by the incremental theories of plasticity [4, 5]. Onat and Drucker [4] examined this paradox. They showed that for a compressed cruciform section, which fails by twisting, very small geometrical imperfections may account for the difference. On the other hand, Murphy and Lee [6] found that for an axially compressed cylindrical shell buckled in an asymmetric mode, only by an assumed initial imperfection of a large amplitude, the incremental theory will yield an analytical buckling load agreeable with the experimental results. It appears that the interaction between geometric change and material response is complex and may be different in different problems.

In most of existing inelastic buckling analyses, Mises yield surface, which has been experimentally verified [7], is employed. However, this does not rule out the existence of yield surface corners, at least obtuse or rounded corners [8, 9, 10]. Sewell [11] has shown for a simply supported plate in uniaxial compression, the buckling load obtained at a vertex of the yield surface, which is locally similar to that of Tresca, is substantially lower than that of Mises. The corner essentially reduces the effective shear stiffness of the material. However, available experimental evidence on the existence of corners and the associated effective moduli is not conclusive [9, 10].

In many existing inelastic bifurcation analyses, a shell just prior to buckling is assumed to have a perfect geometry and a uniform compressive stress state. In reality the edge constraints may cause bending waves to develop near the edges. It has been shown [6], that for a cylindrical shell under axial compression, the maximum load obtained by Mises yield surface, incremental theory and incorporating the edge constraints agrees well with experimental results and is substantially lower than that obtained by the simple bifurcation approach. In essence, the edge constraints induce a change in geometry and a region of relatively high stress accompanied by reduced material stiffness. Therefore, it is necessary to determine the prebuckling deformation.

In Hill's general theory of uniqueness and bifurcation [1], the non-symmetric Lagrangian stress rate tensor is employed and the current deformed state of a structure is used as a reference which requires continuous updating involving coordinate transformations. In this paper, a variational principle and the associated bifurcation and stability criteria essentially identical to that of Hill, except expressed in terms of Lagrangian strain and Kirchhoff stress rate tensors, are introduced. The variational principle is next employed to treat the problem of general elastic-plastic stability of a ring-stiffened cylindrical shell under external pressure. The problem has been formerly studied by Gerard [12] and by Lunchick [13]. They employed the deformation theory of plasticity and the classical bifurcation approach.

2. Variational Principle

Consider a body of continuum occupying a region v bounded by $a = a_u + a_t$ in the deformed state, which corresponds to a region V bounded by $A=A_u+A_T$ in the natural state. Let the initial position of a particle be at X_i in a Cartesian space and its current position be at x_i in the same space. Let u_i be the displacement of the particle, i.e., $u_i=x_i-X_i$. The body is currently subjected to specified body force (per unit mass) f_i, surface traction t_i over a_t and displacements u_i over a_u. The following relationships may be defined

$$\rho_o \, dV = \rho \, dv, \qquad F_i \, \rho_o \, dV = f_i \, \rho \, dv, \qquad T_i \, dA = t_i \, da, \tag{1}$$

where ρ_o and ρ are the natural and current mass densities, respectively. F_i and T_i are the conceptual body force and surface traction associated with the natural state. The orientation of the area dA is denoted by its unit normal N_i and that of the corresponding da by n_i. It may be shown that the areas have the following relationship

$$\rho \, n_i \, da = \rho_o \frac{\partial X_k}{\partial x_i} N_k \, dA \tag{2}$$

where the repetition of an index in a term denotes a summation with respect to that index over its range. The Kirchhoff stress tensor, S_{ij}, has the following relationship with the surface traction

$$N_i S_{ij} x_{k,j} = T_k \text{ over } A_T \tag{3}$$

where as well as in the sequel, an index j following a comma indicates partial differentiation with respect to X_j. It may be shown that the quasi-static equilibrium of any portion of the body requires that

$$\left\{\frac{\partial}{\partial \tau}\left[S_{jk}(\delta_{ik} + u_{k,k})\right]\right\}_{,j} + \rho_o \dot{F}_i = 0, \text{ where } \tau \text{ is time and } \dot{F}_i = \frac{DF_i}{D\tau}. \tag{4}$$

Consider a class of arbitrary virtual velocities $\delta \dot{u}_i$ which are continuous, triply differentiable over V, and which vanish on A_u. Multiplying eq. (4) by $\delta \dot{u}_i$ and using Gauss theorem, the following variational equation of equilibrium is obtained.

$$\int_V [\dot{S}_{ij} \delta \dot{\epsilon}_{ij} + S_{ij} \dot{u}_{k,i} \delta \dot{u}_{k,j}] \, dV = \int_V \rho_o \dot{F}_i \delta \dot{u}_i \, dV + \int_{A_T} \dot{T}_i \, \delta \dot{u}_i \, dA \tag{5}$$

where ϵ_{ij} is the Lagrangian strain tensor, or

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}) \tag{6}$$

It is to be noted that eq. (5) is valid for stable or unstable equilibrium as well as for a continuum having any material properties.

3. Material Properties

It has been shown [14] that the finite Lagrangian strain tensor can be expressed as the sum of elastic, ϵ_{ij}^e, and plastic, ϵ_{ij}^p, parts. Consider an isothermal plastic state at which the loading (yield) function vanishes,

$$f(S_{ij}, \epsilon_{ij}^p, \varkappa) = 0 \tag{7}$$

where $\varkappa$ is the strain-hardening parameter which depends on the states of stress and strain and their histories. Drucker's postulate [8] establishes two requirements: (a) the loading surface is convex and (b) at a smooth point of f=0, the plastic strain rate vector is always directed along the normal to the loading surface or

$$\dot{\epsilon}_{ij}^p = \begin{cases} G \dfrac{\partial f}{\partial S_{ij}} \dfrac{\partial f}{\partial S_{kl}} \dot{S}_{kl} & \text{for } f = 0 \text{ and } \dfrac{\partial f}{\partial S_{kl}} \dot{S}_{kl} \geq 0 \\ 0 & \text{for } f < 0 \text{ or } \dfrac{\partial f}{\partial S_{kl}} S_{kl} \leq 0 \end{cases} \tag{8}$$

where G is a scalar function of the state of the material. If the recoverable elastic strains are infinitesimal, the isothermal relations between the stress rates and elastic strain rates may be taken to be homogeneous and linear, or

$$\dot{S}_{ij} = C_{ijkl}\dot{\epsilon}_{kl}^{e} = C_{ijkl}(\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^{p}) \tag{9}$$

where the coefficients C_{ijkl} may be functions of the current state of stress and the orientation of the axes of anisotropy or constants.

When a state of stress is at a vertex of the yield surface, Drucker's postulate permits the plastic strain-rate vector to have a direction pointed within a cone shape region. For the convenience of treating vertices in an analysis, the following notion is suggested. A vertex may be considered as a small region of a smooth yield surface with the radius of curvature approaching to zero. In that case, the plastic strain-rate vector may be described by an expression suggested by Hill [1], but in a somewhat different sense, or

$$\dot{\epsilon}_{ij}^{p} = \begin{cases} \frac{1}{\eta} m_{kl}\dot{S}_{kl} m_{ij} & \text{when } f = 0 \text{ and } m_{kl}\dot{S}_{kl} \ge 0 \\ 0 & \text{when } f < 0 \text{ or } m_{kl}\dot{S}_{kl} \le 0 \end{cases} \tag{10}$$

where η is a positive scalar measure of the current rate of work hardening and m_{ij} is a unit vector in the stress space at the vertex.

Analogous to the elastic strain-energy functions, strain-rate potential functions of the form

$$\dot{S}_{ij} = \frac{\partial U(\dot{\epsilon}_{kl})}{\partial \dot{\epsilon}_{ij}} \quad \text{and } \delta U = \dot{S}_{ij}\,\delta\,\dot{\epsilon}_{ij} \tag{11}$$

may be established. The strain-rate potential for an elastic element (f<0) is

$$U^{e}(\dot{\epsilon}_{ij}) = \tfrac{1}{2} C_{ijkl}\dot{\epsilon}_{ij}\dot{\epsilon}_{kl} \tag{12}$$

The strain-rate potential, $U^{p}(\dot{C}_{ij})$, for a plastic element (f=0) has two branches:

$$U^{pl}(\dot{\epsilon}_{ij}) = \tfrac{1}{2} C_{ijkl}\dot{\epsilon}_{ij}\dot{\epsilon}_{kl} - \tfrac{1}{2}\frac{(C_{pqrs}m_{pq}\dot{\epsilon}_{rs})^2}{C_{pqrs}m_{pq}m_{rs}+\eta} \quad \text{when } C_{pqrs}m_{pq}\dot{\epsilon}_{rs} \ge 0 \tag{13}$$

$$U^{pe}(\dot{\epsilon}_{ij}) = \tfrac{1}{2} C_{ijkl}\dot{\epsilon}_{ij}\dot{\epsilon}_{kl} \quad \text{when } C_{pqrs}m_{pq}\dot{\epsilon}_{rs} \le 0 \tag{14}$$

It is to be noted that, in a plastic element, the strain-rate potential, U^{pl}, for the loading case is always less than or equal to that for the unloading case, U^{pe}.

4. Bifurcation and Stability Criteria

For an elastic-plastic solid which has predetermined current states of stress, strain and material properties, the velocity field caused by prescribed boundary traction-rates and velocities must satisfy eqs. (3, 4, 5, 10). By substituting the strain-rate potentials, eqs. (12-14) into eq. (5), the velocity field is characterized by having a stationary value of the following functional

$$\delta I(\dot{u}_i) = \delta\left[\int_{V^e} U^e\,dV + \int_{V^p} U^p\,dV + \int_V \tfrac{1}{2} S_{ij}\dot{u}_{k,i}\dot{u}_{k,j}\,dV - \int_V \rho_o \dot{F}_i \dot{u}_i\,dV - \int_{A_T} \dot{T}_i \dot{u}_i\,dA\right] = 0 \tag{15}$$

where V^e is the region where f<0 and V^p is the region where f=0. It is assumed here that $\dot{F}_i$ and $\dot{T}_i$ are independent of $\dot{u}_i$.

Assume that there could be two distinct velocity fields, $\dot{u}_i^1$ and $\dot{u}_i^2$, satisfying eq. (4), the kinematic and traction boundary conditions. Let the prefix Δ denote the difference of corresponding quantities in the two fields. Using the divergence theorem and eq. (4), it is found that

$$H(\Delta \dot{u}_i, s) = \tfrac{1}{2}\int_V (\Delta \dot{S}_{ij}\Delta \dot{\epsilon}_{ij} + S_{ij}\,\Delta \dot{u}_{k,i}\,\Delta \dot{u}_{k,j})\,dV = 0 \qquad (16)$$

where s indicates the current state of S_{ij}. Thus, a sufficient condition for uniqueness is

$$H(\Delta \dot{u}_i, s) > 0 \qquad (17)$$

for all pairs of velocity fields satisfying only the kinematic boundary conditions. It may further be shown that when the solution is unique, the functional I for the true velocity field has an absolute minimum value as compared with that of any other kinematically admissible velocity field. It has been shown by Hill [1] that in a plastic element,

$$\tfrac{1}{2}\Delta \dot{S}_{ij}\,\Delta \dot{\epsilon}_{ij} \geq U^{pl}(\Delta \dot{\epsilon}_{ij}) \qquad (18)$$

for any pair of kinematically admissible velocity fields. Therefore,

$$H(\Delta \dot{u}_i, s) \geq F(\Delta \dot{u}_i, s) = \int_{V^e} U^e(\Delta \dot{\epsilon}_{ij})\,dV + \int_{V^p} U^{pl}(\Delta \dot{\epsilon}_{ij})\,dV + \tfrac{1}{2}\int S_{ij}\,\Delta \dot{u}_{k,i}\,\Delta \dot{u}_{k,j}\,dV \qquad (19)$$

The state of stress, s_c, at which a bifurcation of solutions occurs, may be characterized by

$$F(\Delta \dot{u}_i^c, s_c) = 0, \qquad (20)$$

where $\Delta \dot{u}_i^c$ is the eigenmode solution which vanishes on A_u.

Consider that a body is in a state of equilibrium. A sufficient condition for its stability is that in any possible infinitesimal displacement from the position of equilibrium, the internal energy stored or dissipated should exceed the work done on the body by the constant acting forces. Consider an arbitrary, compatible displacement δu_i which occurs fictitiously in the time interval $\delta\tau$. The above condition leads to

$$\int \tfrac{1}{2}\left[\delta S_{ij}\,\delta \epsilon_{ij} + S^o_{ij}\,\delta u_{k,i}\,\delta u_{k,j}\right] dV > 0 \qquad (21)$$

Inequalities (17) and (21) appear to be similar and yet they are distinct criteria. Criterion (17) is for any pair of kinematically admissible velocity fields whereas criterion (21) is for any admissible velocity field. Therefore, inequality (21) is always satisfied when inequality (17) is (but not vice versa); for (17) reduces to (21) when one field of a pair is null. Thus, it is possible that at a certain state, the solution of the incremental boundary value problem is not unique yet the state may be stable. An example is the bifurcation phenomenon at the tangent modulus load of an inelastic column as illustrated by Shanley [2, 3].

5. Bifurcation Under Hydrostatic Pressure

The hydrostatic pressure loading is always perpendicular to the current area and may be expressed by eq. (1), or

$$T_i\,dA = -\,p\,n_i\,da \qquad (22)$$

where p is the magnitude of the pressure, a function of τ. When the displacements are relatively small, the traction rate, by eqs. (2, 22), may be simplified to

$$\dot{T}_i = -\,\dot{p}\,N_i - p\,(N_i\,\dot{u}_{j,j} - N_j\,\dot{u}_{j,i}) \qquad (23)$$

which shows that $\dot{T}_i$ is dependent on $\dot{u}_i$. Substituting Eq. (23) into eq. (5), the variational equation of equilibrium for the body under hydrostatic pressure may be expressed as

$$\delta\Big[\int_{V^e} U^e(\dot{\epsilon}_{ij})dV + \int_{V^p} U^{pl}(\dot{\epsilon}_{ij})dV + \int_V \tfrac{1}{2} S_{ij}\dot{u}_{k,i}\dot{u}_{k,j}\,dV$$
$$+\,\dot{p}\int_{A_T} N_i\dot{u}_i\,dA\Big] + \int_{A_T} p\,(N_i\dot{u}_{j,j} - N_j\dot{u}_{j,i})\,\delta\dot{u}_i\,dA = 0 \qquad (24)$$

Here, the body force is assumed to be negligible. Furthermore, the bifurcation criterion may be specialized to the following form

$$2F(\Delta\dot{u}_i^c, s_c) + \int_{A_T} p_c (N_i \Delta\dot{u}_{j,j}^c - N_j \Delta\dot{u}_{j,i}^c)\, \Delta\dot{u}_i^c \, dA = 0 \qquad (25)$$

where p_c is the critical pressure at which bifurcation occurs. It is to be noted that all the terms in eqs. (24, 25) are invariant under coordinate transformations and they can be conveniently expressed in appropriate curvilinear coordinates for any shell problem.

6. Shell Geometry

Consider a ring-stiffened cylindrical shell with built-in edges and of length L, mean radius R and thickness h as shown in Fig. 1. There are n equally spaced ring stiffeners, each of width b and depth R_2-R_1. Let $\bar{x}$, θ and $\bar{z}$ be the axial, circumferential and radial coordinates; and let $\bar{u}$, $\bar{v}$, $\bar{w}$ be the corresponding displacements of a point at the middle surface of the shell, respectively, $\bar{z}$ and $\bar{w}$ are positive radially inward. To simplify the subsequent computations, the following non-dimensional quantities are introduced:

$$x = \frac{\bar{x}}{R},\ z = \frac{\bar{z}}{R},\ u = \frac{\bar{u}}{R},\ v = \frac{\bar{v}}{R},\ w = \frac{\bar{w}}{R},\ w,_x = \frac{\partial w}{\partial x} \text{ and } w,_\theta = \frac{\partial w}{\partial \theta} \qquad (26)$$

The deformation process of the shell may be divided into two periods: prebuckling and bifurcation periods. Utilizing Love-Kirchhoff assumptions for thin shells and eq. (6), the axial and circumferential Lagrangian strain rates, $\dot{\epsilon}_x$ & $\dot{\epsilon}_\theta$, in the axisymmetric prebuckling period, when v=0 and u<<w, may be expressed as follows:

$$\dot{\epsilon}_x = \dot{u}_{a,x} - z\,\dot{w},_{xx}, \qquad \dot{\epsilon}_\theta = -\dot{w}/(1-z), \qquad (27)$$

where $$u_a = u + u_b, \text{ and } u_b = \tfrac{1}{2}\int_0^x w,_x^2 \, dx \qquad (28)$$

are introduced for computational convenience. Eq. (27_2) is equally applicable to the shell and rings, in which the axial stresses do not exist.

In the bifurcation period, the differences of strain rates in two admissible velocity fields are required. Employing the fact that $u,_\theta = v = w,_\theta = 0$ in the axisymmetric deformation prior to bifurcation, the strain rate differences may be written as

$$\underline{\dot{\epsilon}}_x = \underline{\dot{u}},_x + w,_x \underline{\dot{w}},_x - z\,\underline{\dot{w}},_{xx}, \quad \underline{\dot{\epsilon}}_\theta = \underline{\dot{v}},_\theta - \underline{\dot{w}} + w(\underline{\dot{w}} - \underline{\dot{v}},_\theta) - z\,(\underline{\dot{w}},_{\theta\theta} + \underline{\dot{v}},_\theta)$$

$$\underline{\dot{\gamma}}_{x\theta} = 2\underline{\dot{\epsilon}}_{x\theta} = \underline{\dot{u}},_\theta + \underline{\dot{v}},_x(1-z-w) + w,_x(\underline{\dot{w}},_\theta + \underline{\dot{v}}) - 2z\underline{\dot{w}},_{x\theta} \qquad (29)$$

where $\underline{\dot{\epsilon}}_x = \Delta\dot{\epsilon}_x$ and $\underline{\dot{w}} = \Delta\dot{w}$. With the build-in edge at x=0 and the plane of symmetry at x=L/2R, the boundary conditions may be written in terms of the displacement rates as follows:

$$\dot{u} = \dot{u}_a = \dot{u},_\theta = \dot{u}_{a,\theta} = \dot{v} = \dot{v},_x = \dot{v},_\theta = \dot{w} = \dot{w},_x = \dot{w},_\theta = 0 \quad \text{at } x = 0,$$

$$\dot{u},_\theta = \dot{v},_x = \dot{w},_x = 0 \quad \text{at } x = \frac{L}{2R}. \qquad (30)$$

Having defined the strain-displacement relationships, the stress-strain relationships may now be reviewed.

7. Stress-Strain Relationships

The uniaxial Kirchhoff stress (S) versue Lagrangian strain (ε) curve of a metal may be described by the following Ranberg-Osgood description [15]:

$$\frac{\epsilon E}{F_{.7}} = \frac{S}{F_{.7}}\left[1 + \frac{3}{7}\left(\frac{S}{F_{.7}}\right)^{N-1}\right], \qquad (31)$$

where E is Young's modulus, $F_{.7}$ is the secant stress interested by the lines S=0.7Eε, and N is a shape factor. The magnitude of the maximum elastic strain of a metal used for a vessel is usually small. Therefore, for an isotropic and homogeneous material, the elastic stress-strain relationship may be reasonably expressed by

$$\dot{\epsilon}^e_{ij} = \frac{1}{E}[(1+\nu)\dot{S}_{ij} - \nu\ \dot{S}_{kk}\delta_{ij}] \tag{32}$$

where ν is Poisson's ratio. It is assumed that the yield surface has a general shape of Mises' function except that there may be corners. Mises' yield function for an axisymmetric state of stress may be written as

$$f = J_2 - \kappa^2 = \frac{1}{3}(S_x^2 - S_x S_\theta + S_\theta^2) - \kappa^2 = 0 \tag{33}$$

where S_x and S_θ are the axial and circumferential Kirchhoff stress components, respectively. κ^2 is the hardening parameters which has a value equal to the maximum value of J_2 previously attained. Assuming that the direction of the plastic strain rate, m_{ij}, at a corner makes a small angle α (less than $\pi/6$) with the normal to Mises' surface, the explicit stress-strain relationship for the shell may be written, by eq. (10) as follows:

$$\dot{S}_x = f_{11}\dot{\epsilon}_x + f_{12}\dot{\epsilon}_\theta + f_{13}\dot{\gamma}_{x\theta},\quad \dot{S}_\theta = f_{12}\dot{\epsilon}_x + f_{22}\dot{\epsilon}_\theta + f_{23}\dot{\gamma}_{x\theta},\quad \dot{S}_{x\theta} = f_{13}\dot{\epsilon}_x + f_{23}\dot{\epsilon}_\theta + f_{33}\dot{\gamma}_{x\theta} \tag{34}$$

where the symmetric coefficients f_{ij} may be expressed in terms of the inverse of the matrix c_{ij} as

$$f_{ij} = c_{ij}^{-1},\ i,j = 1,2,3 \tag{35}$$

where

$$\begin{aligned} c_{11} &= \frac{1}{E} + \frac{G'}{9}(2S_x - S_\theta)^2, & c_{22} &= \frac{1}{E} + \frac{G'}{9}(2S_\theta - S_x)^2, \\ c_{12} &= -\frac{\nu}{E} + \frac{G'}{9}(2S_x - S_\theta)(2S_\theta - S_x), & c_{23} &= \frac{2G'}{3}(2S_\theta - S_x)\alpha\sqrt{J_2}, \\ c_{13} &= \frac{2G'}{3}(2S_x - S_\theta)\alpha\sqrt{J_2}, & c_{33} &= \frac{2(1+\nu)}{E} + 4G'\alpha^2 J_2. \end{aligned} \tag{36}$$

The value of G' is determined by the concept of isotropic hardening and given by

$$G' = \frac{3}{4(1+\alpha^2)J_2}\left(\frac{1}{E_t} - \frac{1}{E}\right) \text{ for } f=0 \text{ and } (2S_x - S_\theta)\dot{S}_x + (2S_\theta - S_x)\dot{S}_\theta + 3\alpha\sqrt{J_2}\ \dot{S}_{x\theta} \geq 0,$$

$$G' = 0 \text{ for } f<0 \text{ or } (2S_x - S_\theta)\dot{S}_x + (2S_\theta - S_x)\dot{S}_\theta + 3\alpha\sqrt{J_2}\ \dot{S}_{x\theta} \leq 0, \tag{37}$$

where E_t is the tangent modulus which depends on the value of J_2 in a general stress state in the same way that E_t depends on J_2 in a uniaxial tension test.

In the prebuckling period, the deformation of the shell remains axisymmetric only if $\alpha = 0$ everywhere. If there are no geometric or material initial imperfections, the deformation is naturally axisymmetric. A number of values for α are to be used in the bifurcation period. It is to be noted that the maximum value of α at a corner of Treseca's yield surface and measured with repsect to the unit normal to Mises' surface is $\pm\pi/6$.

In the bifurcation period, the rings may be subjected to circumferential stresses and shear stresses, $S_{\theta z}$. The stress-strain relationships may be written as follows:

$$(\dot{S}_\theta)_{ring} = E_R\,\dot{\epsilon}_\theta \tag{38}$$

$$E_R = \begin{cases} E_t\dfrac{1+\alpha^2}{1+\alpha^2 E_t/E} & \text{for } S_\theta = S_\theta^* \text{ and } S_\theta\dot{S}_\theta + \dfrac{\sqrt{3}}{2}\alpha S_\theta\dot{S}_{\theta z} \geq 0 \\ E \text{ for } |S_\theta| < |S_\theta^*| & \text{or } S_\theta\dot{S}_\theta + \dfrac{\sqrt{3}}{2}\alpha\, S_\theta\dot{S}_{\theta z} \leq 0 \end{cases} \tag{39}$$

and

$$(\dot{S}_{\theta z})_{ring} = G_R\ \dot{\gamma}_{\theta z} \tag{40}$$

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$$G_R = \begin{cases} \dfrac{1}{\dfrac{2(1+\nu)}{E} + \dfrac{3\alpha^2}{1+\alpha^2}\left(\dfrac{1}{E_t} - \dfrac{1}{E}\right)} & \text{for } S_\theta = S_\theta^* \text{ and } S_\theta \dot{S}_\theta + \dfrac{\sqrt{3}}{2}\alpha S_\theta \dot{S}_{\theta z} \geq 0 \\ \dfrac{E}{2(1+\nu)} & \text{for } |S_\theta| < |\dot{S}_\theta^*| \text{ or } S_\theta \dot{S}_\theta + \dfrac{\sqrt{3}}{2}\alpha S_\theta \dot{S}_{\theta z} \leq 0. \end{cases} \quad (41)$$

where S_θ^* is the largest compressive stress that the material has ever experienced.

8. Method of Solution

The prebuckling axisymmetric deformation of the ring-stiffened cylindrical shell may be determined by eq. (24) which may now be specialized to the form

$$\delta I + 4\pi R^3 p \int_0^{\frac{L}{2R}} [\dot{u},_x \delta \dot{w} - \dot{w},_x \delta\dot{u} - \dot{w}\delta \dot{w}](1+\frac{h}{2R})\, dx = 0 \quad (42)$$

where

$$I = 2\pi R^3 \left\{ \int_0^{\frac{L}{2R}} \int_{-\frac{h}{2R}}^{\frac{h}{2R}} (f_{11}\dot{\epsilon}_x^2 + 2 f_{12}\dot{\epsilon}_x \epsilon_\theta + f_{22}\dot{\epsilon}_\theta^2 + S_x \dot{w},_x^2 + S_\theta \dot{w}^2)\, dz\, dx \right. +$$

$$\left. \frac{b}{R}\sum_{Rings} \int_{-(R_2/R-1)}^{-(R_1/R-1} (E_R \dot{\epsilon}_\theta^2 + S_\theta \dot{w}^2)(1-z)dz - 2\dot{p}\int_0^{\frac{L}{2R}} \dot{w}(1+\frac{h}{2R})dx + (1+\frac{h}{2R})^2 \dot{p}\dot{u}(\frac{L}{2R}) \right\} \quad (43)$$

In the subsequent numerical calculations, the second term in eq. (42) is found to be negligible and may be omitted. A solution of the problem may be obtained by using Rayleigh-Ritz method of variational calculus and a numerical procedure. The velocities $\dot{u}$ and $\dot{w}$ may be represented by a number of chosen admissible coordinate functions with unknown coefficients. When these functions are substituted into the functional I, the latter becomes a function of the coefficients. These coefficients may then be determined so that the functional I has an extremum. After the velocities are obtained, the displacements, stress and material states of the shell at a small time interval later may be determined. Then the variational principle may be applied again. By repeating this procedure, the deformation process of the shell is obtained.

The coordinate functions are so chosen that each satisfies independently the boundary conditions by eq. (30). The velocities in the prebuckling period are represented by

$$\dot{w} = \sum_{i=1}^{4} a_i (1 - \cos\frac{2iR\pi x}{L}), \quad \dot{u}_a = a_5 x + \sum_{i=6}^{8} \sin\frac{2(i-5)R\pi x}{L} \quad (44)$$

Substituting eq. (44) into eqs. (27, 43), the functional I may be written in the form

$$I = \sum_{i=1}^{8}\sum_{j=1}^{8} a_i a_j (I_{ij} + I_{R_{ij}}) + \sum_{i=1}^{8} a_i \dot{p} I_{\dot{p}_i} \quad (45)$$

where I_{ij}, I_{Rij} and $I_{\dot{p}_i}$ are the consequent definite integrals for the shell, rings and surface, respectively. In eq. (45), the coefficients a_i have such values that the functional I has an extremum, i.e.

$$\frac{\partial I}{\partial a_i} = 0, \qquad i = 1, \cdots, 8 \quad (46)$$

For a given pressure rate $\dot{p}$, the previous system of equations may be solved for the values of a_i. However, the shell may fail either in an asymmetric buckling mode or in an axisymmetric mode depending on the geometry and material properties. In the axisymmetric mode, $\dot{p}$ will be zero when p reaches a maximum and the solution of eq. (46) may not be unique. To circumvent this difficulty, the average radial velocity, $\dot{w}_{avg}$, is used as an additional constraint. Instead of specifying the pressure rate, the value of $\dot{w}_{avg}$ is assigned and the corresponding $\dot{p}$ may be determined.

The asymmetric bifurcation occurs at a critical pressure, p_c, satisfying eq. (25). In view of the axisymmetric deformation just prior to bifurcation, eq. (25) may be specialized to the following form:

$$\bar{F}(\underline{u}_i^c, p_c) = R^3 \int_0^{L/2R}\int_0^{2\pi/k}\int_{-h/2R}^{h/2R}\Big\{f_{11}\dot{\underline{\epsilon}}_x^2 + f_{22}\dot{\underline{\epsilon}}_\theta^2 + f_{33}\dot{\underline{\gamma}}_{x\theta}^2 + 2f_{12}\dot{\underline{\epsilon}}_x\dot{\underline{\epsilon}}_\theta + 2f_{13}\dot{\underline{\epsilon}}_x\dot{\underline{\gamma}}_{x\theta}$$
$$+ 2f_{23}\dot{\underline{\epsilon}}_\theta\dot{\underline{\gamma}}_{x\theta} + S_x(\dot{\underline{u}},_x^2 + \dot{\underline{v}},_x^2 + \dot{\underline{w}},_x^2) + S_\theta[\dot{\underline{u}},_\theta^2 + (\dot{\underline{v}},_\theta - \dot{\underline{w}})^2 + (\dot{\underline{w}},_\theta + \dot{\underline{v}})^2]\Big\}\, dz\, d\theta\, dx$$
$$+ R^2 b \sum_{Rings}\int_0^{2\pi/k}\int_{-(R_2/R-1)}^{-(R_1/R-1)}\Big\{E_R\dot{\underline{\epsilon}}_\theta^2 + S_\theta[\dot{\underline{u}},_\theta^2 + (\dot{\underline{v}},_\theta - \dot{\underline{w}})^2 + (\dot{\underline{w}},_\theta + \dot{\underline{v}})^2]\Big\}\, dz\, d\theta$$
$$+ k^* \frac{b^3(R_2 - R_1)}{R}\sum_{Rings}\int_0^{2\pi/k} G_R \dot{\underline{w}},_{x\theta}^2\, d\theta - R^3\int_0^{L/2R}\int_0^{2\pi/k} p_c(\dot{\underline{u}},_x\dot{\underline{w}} - \dot{\underline{w}},_x\dot{\underline{u}}$$
$$- \dot{\underline{w}}^2 - \dot{\underline{v}}^2 + \dot{\underline{v}},_\theta\dot{\underline{w}} - \dot{\underline{w}},_\theta\dot{\underline{v}})\, d\theta\, dx = 0 \qquad (47)$$

where k is the number of circumferential waves and k^* is a constant defining the torsional property of the rings. In the term containing k^* in eq. (47), an average value of G_R of a ring is employed. It is also to be noted that S_x and S_θ are functions of the critical pressure and the history of deformation.

An approximate solution of the bifurcation problem may be obtained by employing the following admissible field of velocity differences:

$$\dot{\underline{u}} = B_1 \sin\frac{2\pi Rx}{L}\sin k\theta,\quad \dot{\underline{v}} = B_2(\cos\frac{2\pi Rx}{L} - 1)\cos k\theta,\quad \dot{\underline{w}} = B_3(\cos\frac{2\pi Rx}{L} - 1)\sin k\theta \qquad (48)$$

Substituting eq. (48) into eq. (47), the bifurcation criterion becomes

$$\bar{F}(B_i, k, p_c) = 0, \qquad (49)$$

which contains quadratic terms in B_i. The eigenmode solution may be determined by solving the following simultaneous equations

$$\frac{\partial\bar{F}}{\partial B_i} = 0 \text{ or } \sum_{j=1}^{3}(p_c\, a_{ij} + b_{ij})\, B_j = 0, \qquad i = 1, 2, 3 \qquad (50)$$

The nontrivial solution exists only if the determinant of eq. (50) vanishes, or

$$|p_c\, a_{ij} + b_{ij}| = 0 \qquad (51)$$

The coefficients a_{ij} and b_{ij} are functions of k. The critical value of k, an integer, is such that eq. (51) yields a minimum positive value other than zero for p_c and may be determined by a numerical searching procedure. The stresses S_x and S_θ and the displacement w in eq. (47) should correspond to the critical pressure p_c just prior to bifurcation. However, the critical pressure and the corresponding stress state are both unknowns which may not be expressed explicitly in terms of each other, because the changes in material properties and geometry are non-linear. Therefore, the critical pressure may have to be determined by a trial and error procedure. At an arbitrary pressure p, the quantities a_{ij}, b_{ij} and a critical pressure p_c may be calculated. If p_c is greater than p, the calculation may be repeated at a higher value of p until $p = p_c$.

Knowing p_c, the corresponding eigenmode can be determined and the corresponding velocity, strain rate and stress rate differences can be calculated. It has been indicated by eq. (20) that the asymmetric bifurcation occurs under increasing pressure and that the material is subjected to the loading condition everywhere. In the absence of a yield surface corner, i. e. $\alpha = 0$, the condition implies $\dot{f} > 0$ everywhere. Consequently, the bifurcation path, for $\alpha = 0$, may be determined by the condition that

$$\dot{f}_{min} = (2S_x - S_\theta)(\dot{S}_x + \dot{\underline{S}}_x/\lambda) + (2S_\theta - S_x)(\dot{S}_\theta + \dot{\underline{S}}_\theta/\lambda) \geq 0 \quad (52)$$

where $\dot{S}_x$ and $\dot{S}_\theta$ are the stress rates corresponding to a pressure rate, $\dot{p}$, in the axisymmetric deformation path just prior to the bifurcation. λ is a proportionality factor scaling the admissible stress rate differences, $\dot{\underline{S}}_x$ and $\dot{\underline{S}}_\theta$. It may be shown that the unloading condition, $\dot{f} < 0$, may occur somewhere in the shell immediately after the occurrence of the bifurcation. Therefore, λ can be determined by the condition that $\dot{f}_{min} = 0$. This can be accomplished by a numerical searching procedure.

In the asymmetric buckling, part of the shell buckles radially inward and the other part outward. For comparison, the average radial velocity difference, $\dot{\underline{w}}_{avg}$ over a half-wave length, $0 < \theta < \pi/k$, may be calculated. Furthermore, the asymmetric bifurcation path or gradient, β, may be defined as

$$\beta = \frac{\dot{p}}{\dot{w}_{avg} + \dot{\underline{w}}_{avg}} \quad (53)$$

where $\dot{w}_{avg}$ is the average radial velocity in the axisymmetric deformation path corresponding to the specific pressure rate, $\dot{p}$.

Finally, the problem of calculating the values of the integrals such as in eqs. (43, 47) is now discussed. There is the difficulty of keeping track of the material response at each point of the shell. To alleviate the difficulty, the uniform shell is conceptually replaced by a structure composed of six thin load carrying cylindrical sheets which are separated by a fictitious material whose only function is to maintain a fixed spacing between any two adjacent sheets. In addition, each sheet is divided axially into q-1 segments separated by q axial stations. An integration may then be approximated by a summation of the discrete values at the nodal points and by using Simpson's rule. Each ring is also conceptually replaced by six thin concentric bands and treated in a similar manner.

9. Comparison with Experimental Results

Numerical results have been obtained by the foregoing procedure and a computing program on a IBM 370 digital computer for nine ring-stiffened cylindrical shell models made of 7075-T6 aluminum alloy and employed in the tests by Boichot, et al [16]. Table I gives the geometric properties, $F_{.7}$ yield stresses, experimental collapse pressures, p_{ex}, calculated asymmetric bifurcation pressures, p_c, and circumferential buckling wave numbers, k, of the nine cylinders for $\alpha = 0$ (Mises surface). $F_{.7}$ is practically identical to the corresponding 0.2% offset nominal yield stress. In the numerical calculations, constant values of $E = 10.3 \times 10^6$ psi, $\nu = 0.3$ and (shape factor) N = 20 are used. Each of the models has six ring stiffeners which are not quite evenly spaced. It is believed that the calculated bifurcation pressure is not sensitive to the slight unevenness of the ring spacings.

The theoretical deformation processes of the cylinders are typified by that shown in Figures 2, 3, 4 and 5. Fig. 2 shows the variation of the average radial deflection versus pressure and Fig. 3 shows the radial deflection profile just prior to bifurcation for model 15-82. Similar curves for Model 20-84 are shown in Figures 4 and 5. Fig. 2 shows that model 15-82, with flexible ring stiffeners, has a flat bifurcation gradient. Fig. 4 shows that model 20-84, with comparatively stiff ring stiffeners, has a steeper bifurcation gradient which indicates the cylinder may sustain more pressure beyond the bifurcation pressure.

The comparison demonstrates a favorable agreement between the present theoretical and the experimental results for the range of parameters investigated.

The effects of a yield surface corner (α) on the asymmetric bifurcation pressures of model 10-82 have been numerically determined as illustrated by Table II, which shows that a yield surface corner could lower the asymmetric bifurcation pressure of a ring-stiffened cylindrical shell.

In the numerical calculations for the models in Table I, the pressure is considered to be normal to the deformed surface of a shell. The effect of normality of pressure on the bifurcation pressure of a shell is illustrated by the following numerical results obtained for model 10-82. The bifurcation pressure is 7697 psi when it is normal to the initial surface, which is about 2.3% higher than (7521 psi) that normal to the deformed surface.

Finally, a few words may be said on the numerical accuracy of the approach. The piece-wise linear extrapolation procedure has a tendency to raise the stress increments or pressure rate and consequently reduce the stiffness in a shell. It is essential to use small increments when the stresses in the shell approach the yield stress. In the foregoing numerical calculations, $\dot{w}_{avg} = 2 \times 10^{-4}$ is used for the elastic deformation and $\dot{w}_{avg} = 2 \times 10^{-5}$ is used for the plastic deformation. The six-layer shell idealization, which may reduce the bending stiffness of a shell by no more than two percent everywhere, is believed to be reasonably adequate. The accuracy is also influenced by the number of axial segments used in the summation procedure. In the present computation, one half span of a shell is divided into 56 segments. In a trial computation, 84 segments have been employed for an integration. The difference between the results and that of 56 segments was negligible.

10. Conclusions

The results of this investigation illustrate the relative ease of employing a variational principle expressed in terms of Lagrangian variables to determine the non-linear, continuous deformation process of an elastic-plastic solid. By using the variational principle and the incremental Rayleigh-Ritz technique, the inelastic buckling behavior of ring stiffened cylindrical shells under hydrostatic pressure can be reasonably predicted. It is found that the asymmetric bifurcation pressure of a shell may be substantially influenced by many factors, among which are the prebuckling deformation, the condition of normality of pressure and the presence of a yield surface corner. It is also found that the stress-strain relationship by the incremental theory and Mises' yield function, when it is employed concurrently with more realistic treatments of the boundary conditions and prebuckling finite deformation, yields a fairly accurate prediction of the buckling behavior of a ring stiffened cylindrical shell.

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Table I

Experimental Data [16] and Numerical Results

Model	L in	h in	b in	R_2 in	$F_{.7}$ psi	p_{ex} psi	p_c psi	k
10-52	1.311	0.0518	0.027	1.1298	84,100	4,700	4,670	2
10-54	1.389	0.0516	0.040	1.1656	83,800	5,625	5,915	2
10-82	1.715	0.0833	0.039	1.1983	83,200	7,650	7,521	1
10-88	2.002	0.0833	0.084	1.3313	83,500	12,100	10,440	4
15-82	2.502	0.0830	0.046	1.2230	83,300	7,650	7,647	1
15-84	2.638	0.0830	0.068	1.2850	83,000	9,100	8,331	8
15-88	2.843	0.0830	0.100	1.3800	82,700	11,600	10,987	3
20-82	3,280	0.0832	0.054	1.2412	82,000	7,400	7,687	1
20-84	3.430	0.0830	0.079	1.3130	81,400	8,800	8,131	6

$$R = 1.000 + \frac{h}{2}, \quad R_1 = 1.000 + h, \quad k^* = 0.262$$

TABLE II

Effects of a Yield Surface Corner

Model 10-82, $p_{ex} = 7650$ psi

α	0	$\pi/18$	$\pi/6$
$p_c(\alpha)$ psi	7521	7472	7159
$\frac{p_c(\alpha) - p_c(0)}{p_c(0)}$	0	0.65 %	4.82 %

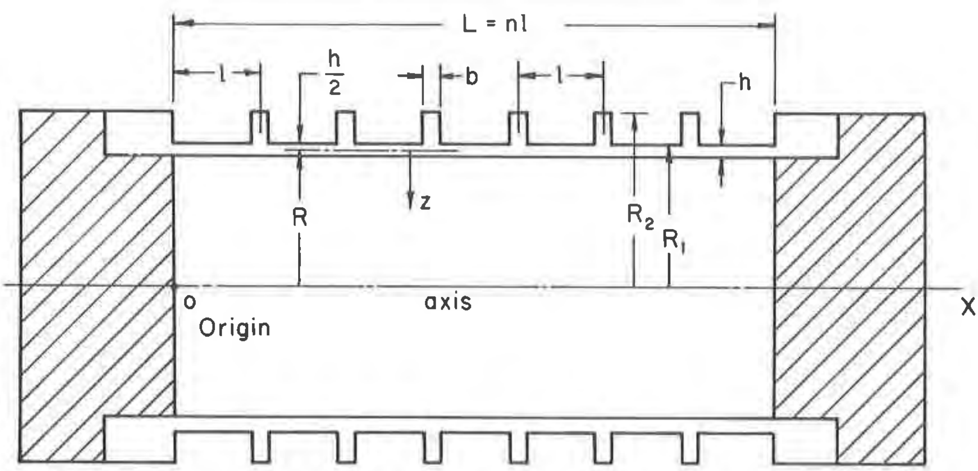

Fig. 1 Shell Geometry.

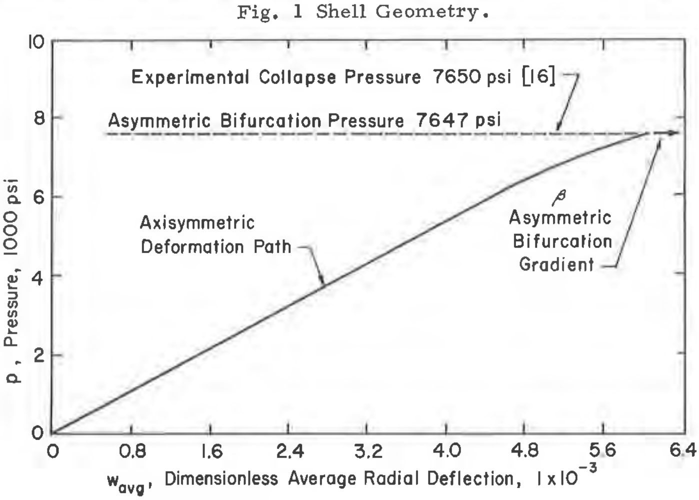

Fig. 2 Pressure vs. Average Radial Deflection for Model 15-82.

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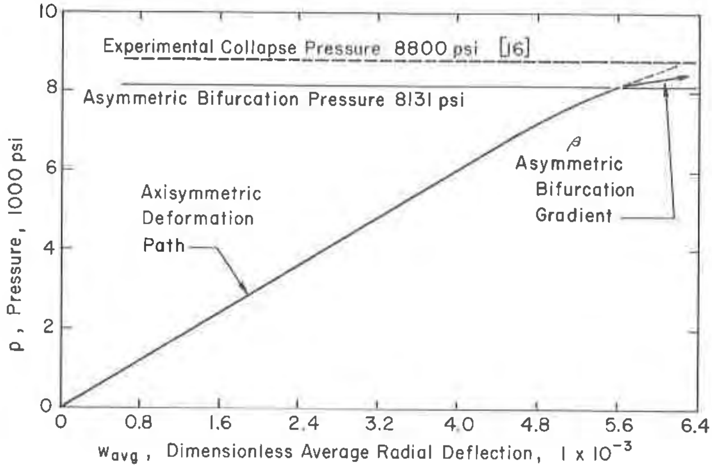

Fig. 4 Pressure vs. Average Radial Deflection for Model 20-84.

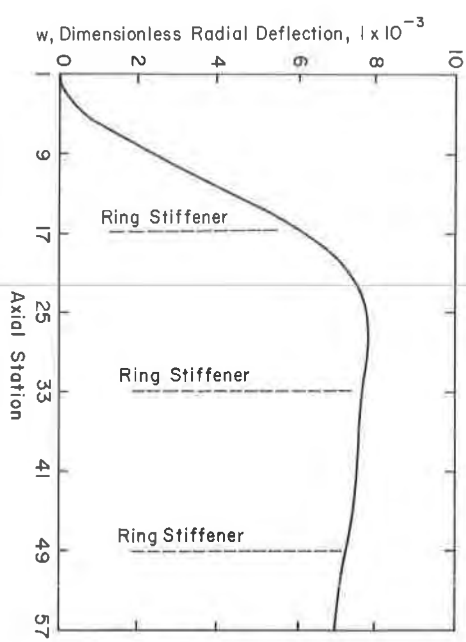

Fig. 3 Axisymmetric Deflection Profile Just Prior to Asymmetric Bifurcation for Model 15-82.

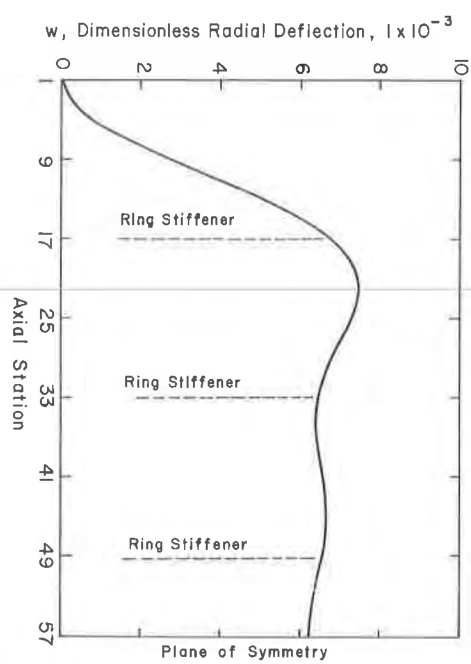

Fig. 5 Axisymmetric Deflection Profile Just Prior to Asymmetric Bifurcation for Model 20-84.