ABSTRACT

RUBTSOV, ALEXEY VLADIMIROVICH. Stochastic Control in Financial Models of Investment and Consumption. (Under the direction of Dr. Min Kang.)

An extension of the classical Merton's model of optimal investment and consumption is considered. We consider a problem of optimal portfolio management under uncertainty in utility function. In some research, it is claimed that the utility of goods depends not only on the goods themselves, but also on quality of the goods. However, the quality of the goods is subject to random changes (for example, the technological progress). Although there is much debate on the dynamics of technological progress, it is not uncommon in the literature that the progress in technology in some areas grows exponentially. This implies that it makes sense to model these random changes by a Geometric Brownian motion. Thus, it is a natural problem of interest to find out how the optimal policy changes when some uncertainty in the utility function is introduced.

The definitions and theoretical results used in the research are provided and reviewed. Once the problem of stochastic control is defined, the classical Merton's model of optimal investment and consumption is presented. The importance of the proposed uncertainty in utility is also discussed.

Once the required theoretical results are stated, the problem of expected utility maximization with fully observed uncertainty in utility is solved for a specific utility function of hyberbolic absolute risk aversion class. To obtain the optimal solution, the Hamilton-Jacobi-Bellman (HJB) equation is derived and the solution is obtained. To verify that the viscosity solution to the HJB equation is the value function, a so-called Verification Theorem is proved. As a result, the optimal investment and consumption are found for the problems of maximizing the expected utility of consumption and final wealth, only the expected utility of consumption, and only the expected utility of final wealth.

Having obtained the solution to the fully observed case, the problem of expected utility maximization is solved under the assumption that uncertainty in utility is not fully observed. After the corresponding HJB equation is derived and solved, the Verification Theorem is used to verify that the solution is the value function. Thus, the optimal investment and consumption under partial observations are obtained for the problems of maximizing the expected utility of consumption and final wealth, only the expected utility of consumption, and only the expected utility of final wealth.

Stochastic Control in Financial Models of Investment and Consumption

by
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DEDICATION

To my parents.

BIOGRAPHY

Alexey Rubtsov was born on January 8, 1983 in Saratov, Russia. After five years of studies at Saratov State Social-Economic University, he received a diploma in Statistics, which is usually considered to be equivalent to Master's Degree in Statistics. In 2008 he received his Masters Degree in Financial Mathematics at North Carolina State University. From year 2008 to 2011, he has been pursuing the Ph.D. degree in Operations Research at North Carolina State University.

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- $\mathbb{R}^k$ - k-dimensional Euclidean space.
- $C([0,T];\mathbb{R}^k)$ - the space of all continuous functions defined on $[0,T]$ and taking values in $\mathbb{R}^k$.
- $|x| = (|x_1| \ ... \ |x_k|)$ - a vector made of absolute values of each component of vector $x \in \mathbb{R}^k$.
- $||x||$ - norm of x, (unless otherwise indicated, if $x \in \mathbb{R}^k$ then $||x|| = \Big(\sum_{i=1}^{k} x_i^2\Big)^{1/2}$, if $x \in \mathbb{R}^{k\times m}$ then $||x|| = \Big(\sum_{i=1}^{k}\sum_{j=1}^{m} x_{ij}^2\Big)^{1/2}$).
- $\mathcal{B}(U)$ - the Borel sigma-algebra generated by all the open sets in a metric space U (the smallest sigma-algebra containing all the open sets of U).
- $\mathcal{F}_1 \otimes \mathcal{F}_2$ - direct product of sigma-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ (the sigma-algebra generated by sets $A \times B, \ \forall A \in \mathcal{F}_1, \ \forall B \in \mathcal{F}_2$).
- $\mathbf{B}^m[0,T] \triangleq C([0,T];\mathbb{R}^m)$ - the space of all continuous $\mathbb{R}^m$-valued functions defined on $[0,T]$.
- $\mathbf{B}^m \triangleq C([0,\infty);\mathbb{R}^m)$ - the space of all continuous $\mathbb{R}^m$-valued functions defined on $[0,\infty)$ with metric $\hat{\rho}(b_1,b_2) = \sum_{j\geq 1} 2^{-j}\Big(||b_1 - b_2||_{C([0,j];\mathbb{R}^m)} \wedge 1\Big), \forall b_1, b_2 \in \mathbf{B}^m$. Under the metric $\hat{\rho}$ the space $\mathbf{B}^m$ is a Polish space (complete separable metric space).
- $L^p_{\mathcal{F}}(0,T;\mathbb{R}^k)$ - the space of all $\{\mathcal{F}_s\}_{s\in[0,T]}$-adapted, $\mathbb{R}^k$-valued processes X such that $E\int_0^T ||X_t||^p dt < \infty$, where $||x||^p = |x_1|^p + ... + |x_k|^p, \ x \in \mathbb{R}^k, \ p \geq 1$.
- $\mathcal{A}^n_T(\mathbb{U})$ - the space of all $\{\mathcal{B}_{t+}(C([0,T];\mathbb{R}^k))\}_{t\geq 0}$-progressively measurable processes $\psi : [0,T] \times C([0,T];\mathbb{R}^k) \to \mathbb{U}$, where

$$\begin{cases} \mathbf{B}^m_t[0,T] = \{X(\cdot \wedge t) | X(\cdot) \in \mathbf{B}^m[0,T]\}, & \forall t \in [0,T], \\ \mathcal{B}_t(\mathbf{B}^m[0,T]) = \sigma(\mathcal{B}(\mathbf{B}^m_t[0,T])), & \forall t \in [0,T], \\ \mathcal{B}_{t+}(\mathbf{B}^m[0,T]) = \bigcap_{s>t} \mathcal{B}_s(\mathbf{B}^m([0,T]), & \forall t \in [0,T). \end{cases}$$

Note that $\mathcal{B}_t(\mathbf{B}^m[0,T])$ is the sigma-algebra in $\mathbf{B}^m[0,T]$ generated by $\mathcal{B}(\mathbf{B}_t^m[0,T])$, and thus it contains $\mathbf{B}^m[0,T]$. The fact that $\mathcal{B}_{t+}(\mathbf{B}^m[0,T]) \neq \mathcal{B}_t(\mathbf{B}^m[0,T])$ is proved in [10], p.122. If the interval $[0,\infty)$ is considered then we write $\mathcal{A}^k(\mathbb{U})$.

- $X_t^{U_t}$ - portfolio (wealth) process when the control U_t is used.
- S_t - n-dimensional column vector of stock prices (risky assets).
- N_t - the value of the riskless asset.
- Z_t - utility randomness process.
- L_t - natural logarithm of the utility randomness process.
- P_t - the observed process (the utility randomness process is not fully observed).
- Π_t - n-dimensional row vector that represents the fractions of wealth invested in the risky assets.
- C_t - consumption per unit time.
- U_t - the control process (in this dissertation the controls are Π_t and C_t).
- $\mathbb{U}$ - the space of control values.
- $B_{t,1}$ - n-dimensional Brownian motion (column vector).
- $B_{t,2}$ - one-dimensional Brownian motion.
- $B_{t,3}$ - one-dimensional Brownian motion.

Chapter 1

Introduction

1.1 Stochastic Optimization and its Applications

The model of optimal investment and consumption under uncertainty in utility function is considered in this dissertation. The maximized criterion in the model is the expected utility. To maximize the expected utility, the methods of stochastic optimization are used to find the optimal solution.

Stochastic optimization plays an important role in the design, analysis, and operation of modern systems. Stochastic optimization methods are used in models that are inappropriate for classical deterministic methods of optimization. Algorithms that take advantage of stochastic optimization techniques find their applications in problems in statistics, science, engineering, and business.

Classical deterministic optimization is based on the assumption of perfect information about the minimized (maximized) function (and its derivatives, if necessary). This information is then used to determine the direction of search in a deterministic manner at every step of the algorithm. However, in many practical problems, this information is not available.

In contrast to deterministic optimization, stochastic optimization methods are used when randomness appears in the formulation of the optimization problem itself (random objective function, random constraints, etc.). Stochastic optimization also includes methods with random iterates. Some stochastic optimization methods use random iterates in solving stochastic problems.

Random real data arise in such problems as real-time estimation and control, simulation based optimization where Monte Carlo simulations are run as estimates of an actual system, and problems where there is experimental (random) error in the measurements of the criterion. In such cases, knowledge that the function values contain random noise leads naturally to algorithms that use statistical inference tools in estimation of the true values of the function and/or make statistically optimal decisions about the next steps of the optimization algorithm. Methods of this class include: stochastic approximation, stochastic gradient descent, finite-difference stochastic approximation, simultaneous perturbation stochastic approximation, etc.

On the other hand, even when the data set consists of exact measurements, some methods introduce randomness into the search-process to accelerate progress. Such randomness can also make the method less sensitive to modeling errors. Further, the injected randomness may enable the method to escape a local minimum and eventually to approach a global optimum. Indeed, this randomization principle is known to be a simple and effective way to obtain algorithms with almost certain good performance uniformly across many data sets, for many sorts of problems. Stochastic optimization methods of this kind include: simulated annealing, reactive search optimization, cross-entropy method, random search, etc.

Stochastic optimization is closely connected to stochastic control. The theory of stochastic control provides a vast array of theoretical and computational tools that find their applications in many areas dealing with decision-making under uncertainty. These areas include industrial processes, robotics, insurance, economics, and finance. A common feature of stochastic control problems is that a controlled dynamical system is subject to random perturbations and the goal is to optimize some performance criterion. One of the most interesting applications of the theory of stochastic control in finance is portfolio optimization problem in which an agent invests wealth into risky and riskless assets and chooses a rate of consumption with the goal of maximizing the expected utility of consumption.

Under some assumptions, in [15] Merton solved the problem of expected utility maximization. In that work it was assumed that the utility function is a power function and the market includes the riskless asset with constant rate of return and a risky asset with constant mean rate of return and volatility parameter. Some of the assumptions were relaxed later. The restriction to power utility functions was removed in [11], and in [12, 18], the market coefficients were allowed to be non-constant. After that initial paper, the Merton's model was generalized in many directions.

One generalization is the introduction of transaction costs [22] which are the costs incurred when wealth is moved from one asset to the other. The presence of transaction costs makes the model more realistic.

Another extension is including past stock prices in the decision-making process. In the Merton's model the investor makes investment decisions based on current information and does not consider the past stock prices. However, in the real world, investors take into account the historic performance of the risky assets. This approach is called stochastic portfolio optimization with memory and is treated in [19].

In many cases, the investor does not possess complete information about the stock prices, model parameters' values, etc. This implies that the optimal investment and consumption should be obtained under the assumption that some of the required information is partially observed. The models with partial observations are discussed in [3, 4].

These generalizations are not the only ones and there are many other possible extensions which make the model more realistic, and this dissertation thesis suggests one more way of extending the model to include such factor as technological progress in investment decision-making under uncertainty.

In some research [14, 20], it is claimed that the utility of goods depends not only on the goods themselves but also on qualities of the goods. That is why it makes sense to extend the classical model of optimal investment and consumption [7, 16, 21] to incorporate this feature. When the suggested uncertainty in utility is introduced, it is interesting to find the difference in the optimal policies of the new model and the classical model.

It is natural to assume that when we buy different things we are not buying just objects, we are buying the qualities that those objects possess and we need those qualities. For example, if we consider a diamond that costs as much as a house then clearly it has utility which is different from that of the house. However, using the price of a merchandise as the only argument to the utility function, the utilities of the diamond and the house are the same.

Since the technology is changing, new products keep coming out and substitute the old ones giving the increase in utility. For example, having a computer today gives much more opportunities to its user compared to the computers and technologies available 30 years ago. Therefore, preferences might change because of better characteristics of new goods. It is important to note that this change does not have to entail the change of prices.

On the other hand, preference change might also be due to worsened quality of the products or some other reason (for example, buying the same product over and over might decrease its utility and ends up in satiation with the product). The described process is not deterministic because it is not known how the market will change. Therefore, it makes sense to model the uncertainty in utility by a stochastic process.

Although there is much debate on the dynamics of technological progress [1, 5], it is not new in literature that in some areas it is growing exponentially [8]. A model that assumes exponential growth can also be used to model linear behavior because of the representation $e^x = 1 + x + o(x^2)$ and, thus, changing the parameters of the model accordingly, will help analyze the results when the growth is close to linear. Apart from big technological advancements there are minor improvements in products that people use every day. Therefore, the Geometric Brownian motion can be used to model the uncertainty in utility.

1.2 Mathematical Preliminaries

Let Ω be a nonempty set and $\mathcal{F}$ be a sigma-algebra on Ω, then $(\Omega, \mathcal{F})$ is called a *measurable space*. Let $\mathbb{P}$ be a probability measure defined on the sets from $\mathcal{F}$, then $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*. The probability space is said to be *complete* if for any set $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$ (null set) a set $B \subseteq A$ is also in $\mathcal{F}$, i.e. $B \in \mathcal{F}$.

Definition 1. *Let a measurable space $(\Omega, \mathcal{F})$ be given. A monotone family of sub-sigma-algebras $\mathcal{F}_t \subseteq \mathcal{F},\ t \in [0, \infty)$ is called a* ***filtration*** *if $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2},\ \forall t_1, t_2,\ 0 \le t_1 \le t_2 < \infty$.*

Definition 2. *A filtration is called* ***right (left)*** *continuous if $\mathcal{F}_t = \mathcal{F}_{t+} \triangleq \bigcap_{s>t} \mathcal{F}_s$ ($\mathcal{F}_t = \mathcal{F}_{t-} \triangleq \sigma(\bigcup_{s<t} \mathcal{F}_s)$), $0 \le t < \infty$.*

Definition 3. *A* ***filtered probability space*** *$(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$ is said to satisfy the* ***usual condition*** *if it is complete[1], $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathcal{F}$[2], and the filtration $\{\mathcal{F}_s\}_{s\in[0,\infty)}$ is right continuous[3].*

[1] It is necessary to require the filtered probability space be complete because, for example, if ξ is a random variable ($\mathcal{F}$-measurable function) and $\eta \equiv \xi$ almost everywhere then η is not necessarily measurable. Thus, by completing the probability space we extend the space of measurable (and, therefore, integrable) functions.

[2] The Ito's stochastic integral with finite upper limit might not be a martingale if the filtration is incomplete in the sense that all $\mathbb{P}$-null sets of $\mathcal{F}$ should be included into $\mathcal{F}_0$. See [13], p.93.

[3] In Lemma 2 (see below) that plays an important role in appropriately formulating stochastic optimal control problems, the obtained process ϕ is only $\{\mathcal{B}_{t+}(\mathbf{B}^m[0,T])\}_{t\in[0,T]}$-progressively measurable, not necessarily $\{\mathcal{B}_t(\mathbf{B}^m[0,T])\}_{t\in[0,T]}$-progressively measurable, see [9], p.20.

Definition 4. *Let* $(\Omega, \mathcal{F})$ *and* $(E, \mathcal{E})$ *be two measurable spaces. A function* $X : \Omega \to E$ *is an* $\mathcal{F}/\mathcal{E}$ ***measurable function****, or* ***random element****, or* E*-valued* ***random variable*** *if*

$$\{\omega : X(\omega) \in B\} \in \mathcal{F},\ \forall B \in \mathcal{E}.$$

Definition 5. *If* Ω *is a topological space then the smallest sigma-algebra* $\mathcal{B}(\Omega)$ *containing all open sets of* Ω *is called the* ***Borel sigma-algebra*** *of* Ω*.*

Definition 6. *Let* $\mathcal{I}$ *be a subset of the real line. A family of random variables* $\{X_s, s \in \mathcal{I}\}$ *from* $(\Omega, \mathcal{F}, \mathbb{P})$ *to* $\mathbb{R}^k$ *is called a* ***stochastic process****. For any* $\omega \in \Omega$*, the map* $t \mapsto X_t(\omega)$[1] *is called a sample path.*

Definition 7. *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)})$ *be a filtered measurable space and* X_t *be a stochastic process taking values in a metric space* (Q, d)*.*
(1) The process X_t *is said to be* ***measurable*** *if the map* $(t, \omega) \mapsto X_t(\omega)$ *is* $(\mathcal{B}([0, \infty)) \otimes \mathcal{F})/\mathcal{B}(Q)$*-measurable.*
(2) The process X_t *is said to be* $\mathcal{F}_t$*-****adapted*** *if for all* t *in* $[0, \infty)$ *the map* $\omega \mapsto X_t(\omega)$ *is* $\mathcal{F}_t/\mathcal{B}(Q)$*-measurable.*
(3) The process X_t *is* $\mathcal{F}_t$*-****progressively measurable*** *if for all* t *in* $[0, \infty)$ *the map* $(s, \omega) \mapsto X_s(\omega)$ *is* $(\mathcal{B}([0, t]) \otimes \mathcal{F}_t)/\mathcal{B}(Q)$*-measurable or* $\{(s, \omega) : 0 \le s \le t,\ \omega \in \Omega,\ X_s \in A\} \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t$*, for all* $A \in \mathcal{B}(Q)$*.*

Remark 1. *(a) Notice that in the above definition* $3 \implies 1, 2$*. (b) If the process* X_t *is* $\mathcal{F}_t$*-adapted it does not mean the process* $Y_t = \int_0^t X_s ds$ *is* $\mathcal{F}_t$*-adapted. However, in many cases it is required that the process* Y_t *be adapted (for example, in proving the Bellman's Principle of Optimality). By Fubini's theorem (see for example, [13], p.23), the process* Y_t *is adapted if the process* X_t *is* $\mathcal{F}_t$*-progressively measurable.*

Definition 8. *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$ *be a filtered probability space. An* $\mathcal{F}_t$*-adapted* $\mathbb{R}^m$*-valued process* B_t *is called an* m*-dimensional* $\mathcal{F}_t$*-****Brownian motion*** *over* $[0, \infty)$ *if for all* $0 \le s < t$*,* $B_t - B_s$ *is independent of* $\mathcal{F}_s$ *and is normally distributed with mean* 0 *and covariance* $(t - s)I$*, where* I *is the* $m \times m$ *identity matrix. If* $\mathbb{P}(B_0 = 0) = 1$ *then* B_t *is called an* m*-dimensional standard* $\mathcal{F}_t$*-Brownian motion over* $[0, \infty)$*.*

Let $a \in \mathcal{A}^k(\mathbb{R}^k)$, $s_1 \in \mathcal{A}^k(\mathbb{R}^{k\times m})$ then we have the following

[1]The notations $X_t(\omega)$ and X_t will be used interchangeably.

Definition 9. *An equation of the form*

$$\begin{cases} dX_t = a(t, X)dt + s_1(t, X)dB_t, \\ X_0 = \xi \end{cases} \tag{1.1}$$

*is called a **stochastic differential equation** with the initial condition.*

Remark 2. *Notice that the coefficients a, s_1 are not random and depend on $\omega \in \Omega$ through X.*

There are different notions of solutions to (1.1) depending on different roles that the underlying filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$ and the Brownian motion B_t are playing.

Definition 10. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$, m-dimensional standard $\mathcal{F}_t$-Brownian motion B_t be given, and ξ is $\mathcal{F}_0$-measurable. An $\mathcal{F}_t$-adapted continuous process $X_t,\ t \geq 0$ is called a **strong solution** of (1.1) if*

1. $X_0 = \xi,\ \mathbb{P}$*-a.s.,*

2. $\displaystyle\int_0^t \Big(||a(s, X)|| + ||s_1(s, X)||^2\Big)ds < \infty, \quad \forall t \geq 0,\ \mathbb{P}$*-a.s.,*

3. $\displaystyle X_t = X_0 + \int_0^t a(s, X)ds + \int_0^t s_1(s, X)dB_s, \quad \forall t \geq 0,\ \mathbb{P}$*-a.s.*

Definition 11. *If for any two strong solutions X and Y of equation (1.1) defined on any given $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$ along with any given standard $\mathcal{F}_t$-Brownian motion, we have $\mathbb{P}(X_t = Y_t,\ t \geq 0) = 1$ then we say that the strong solution is **unique**[1].*

Definition 12. *A 6-tuple $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P}, B, X)$ is called a **weak solution** of (1.1) if*

1. $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$ is a filtered probability space satisfying the usual condition,

2. B is an m-dimensional standard $\mathcal{F}_t$-Brownian motion and X is $\mathcal{F}_t$-adapted and continuous,

3. X_0 and ξ have the same distribution,

4. 2 and 3 of the definition 10 hold.

Remark 3. *For the strong solution the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$ and the $\mathcal{F}_t$-Brownian motion B on it are fixed a priori. For the weak solution, $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$ and B are parts of the solution.*

[1]Therefore, the solution is unique if X_t, Y_t are *indistinguishable* processes.

Definition 13. *If for any two weak solutions* $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P}, B, X)$ *and* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_s\}_{s\in[0,\infty)}, \tilde{\mathbb{P}}, \tilde{B}, \tilde{X})$ *of (1.1) with*

$$\mathbb{P}(X_0 \in D) = \tilde{\mathbb{P}}(\tilde{X}_0 \in D), \quad \forall D \in \mathcal{B}(\mathbb{R}^k),$$

we have

$$\mathbb{P}(X \in A) = \tilde{\mathbb{P}}(\tilde{X} \in A), \quad \forall A \in \mathcal{B}(\mathbf{B}^k),$$

then we say that the weak solution is ***unique****.*

Now we define another type of SDE which will be used in stochastic control problems.

Definition 14. *An equation of the form*

$$\begin{cases} dX_t = a(t, X, \omega)dt + s_1(t, X, \omega)dB_t, \\ X_0 = \xi \end{cases} \tag{1.2}$$

is called a ***stochastic differential equation*** *with random coefficients* $a : [0,\infty) \times \mathbf{B}^k \times \Omega \to \mathbb{R}^k,\ s_1 : [0,\infty) \times \mathbf{B}^k \times \Omega \to \mathbb{R}^{k\times m}$ *which explicitly depend on* $\omega \in \Omega$ *(*X_t, B_t, ξ *also depend on* ω *but it is suppressed to simplify the notation).*

Remark 4. *Notice that the case when the coefficients* $a,\ s_1$ *depend on* X_t *instead of* X *(i.e.* $a : [0,\infty) \times \mathbb{R}^k \times \Omega \to \mathbb{R}^k,\ s_1 : [0,\infty) \times \mathbb{R}^k \times \Omega \to \mathbb{R}^{k\times m}$*) is a special case of the equation in the above definition.*

Next, we define what we mean by a *solution* to (1.2).

Definition 15. *Let the maps* $a : [0,\infty) \times \mathbf{B}^k \times \Omega \to \mathbb{R}^k$ *and* $s_1 : [0,\infty) \times \mathbf{B}^k \times \Omega \to \mathbb{R}^{k\times m}$ *and an* m*-dimensional standard* $\mathcal{F}_t$*-Brownian motion* B_t *be given on a given filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$*. Let* ξ *be* $\mathcal{F}_0$*-measurable. An* $\mathcal{F}_t$*-adapted continuous process* $X_t,\ t \geq 0$*, is called a* ***solution*** *of (1.2) if*

1. $X_0 = \xi,\ \mathbb{P} - a.s.;$
2. $\int_0^t \Big(||a(s, X, \omega)|| + ||s_1(s, X, \omega)||^2\Big) ds < \infty,\ t \geq 0,\ \mathbb{P} - a.s.;$
3. $X_t = \xi + \int_0^t a(s, X, \omega)ds + \int_0^t s_1(s, X, \omega)dB_s,\ t \geq 0,\ \mathbb{P} - a.s..$

Definition 16. *If* $\mathbb{P}(X_t = Y_t,\ t \geq 0) = 1$ *holds for any two solutions* $X_t,\ Y_t$ *of (1.2) defined on the given filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$ *along with the given* $\mathcal{F}_t$*-Brownian motion* B_t *then we say that the solution is* ***unique****.*

Remark 5. *Since the coefficients* $a,\ s_1$ *should be given a priori, equation (1.2) should be defined on a given probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *where* $a,\ s_1$ *are defined. Therefore, for an SDE with random coefficients it does not make sense to talk about weak solutions.*

Now we state the conditions under which the SDE (1.2) admits a unique solution.

Theorem 1. *Assume that for any* $\omega \in \Omega$, $a(\cdot,\cdot,\omega) \in \mathcal{A}^k(\mathbb{R}^k)$ *and* $s_1(\cdot,\cdot,\omega) \in \mathcal{A}^k(\mathbb{R}^{k\times m})$ *and for any* $x \in \mathbf{B}^k$, $a(\cdot,x,\cdot)$ *and* $s_1(\cdot,x,\cdot)$ *are both* $\{\mathcal{F}_t\}$*-adapted processes. Moreover, there exists a constant* $L > 0$ *such that for all* $t \in [0,\infty)$, $x, y \in \mathbf{B}^k$*, and* $\omega \in \Omega$*,*

$$\begin{cases} ||a(t,x,\omega) - a(t,y,\omega)|| \leq L||x-y||_{\mathbf{B}^k}, \\ ||s_1(t,x,\omega) - s_1(t,y,\omega)|| \leq L||x-y||_{\mathbf{B}^k}, \\ |a(\cdot,0,\cdot)| + |s_1(\cdot,0,\cdot)| \in L^2_{\mathcal{F}}(0,T;\mathbb{R}),\ \forall T > 0. \end{cases} \tag{1.3}$$

Then for any $\xi \in L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^k)$ $(p \geq 1)$*, (1.2) admits a unique solution* X_t *such that*[1] *for any* $T > 0$

$$\begin{cases} E\Big(\max_{0\leq s\leq T} ||X_s||^p\Big) \leq K(1 + E||\xi||^p), \\ E||X_t - X_s||^p \leq K(1 + E||\xi||^p)|t-s|^{p/2},\ \forall s,t \in [0,T]. \end{cases} \tag{1.4}$$

Moreover, if $\eta \in L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^k)$ *is another random variable and* Y_t *is the corresponding solution of (1.2), then for any* $T > 0$*, there exists a* $K > 0$ *such that*

$$E\Big(\max_{0\leq s\leq T} ||X_s - Y_s||^p\Big) \leq KE||\xi - \eta||^p. \tag{1.5}$$

The proof of the theorem is given in appendix A.1.
The following theorem will be used in defining the problem of stochastic control.

Theorem 2. *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty)}, \mathbb{P})$ *be a filtered probability space and let* X_t *be* $\mathcal{F}_t$*-adapted, and left or right continuous. Then* X_t *is* $\mathcal{F}_t$*-progressively measurable.*

The proof of the theorem is given in appendix A.2. Therefore, if the assumptions of Theorem 1 are satisfied then the solution X_t of (1.2) is $\mathcal{F}_t$-progressively measurable.

[1] Although the constant K depends on T, the notation K is used.

These are the basic terminology and results that will be used in formulation of stochastic control problems discussed in the next section.

1.3 The Problem of Stochastic Control

Let $0 < T < \infty$ be given. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,T]}, \mathbb{P})$ be a *given* filtered probability space satisfying the usual condition (see Definition 3), on which is defined an m-dimensional standard Brownian motion B_t. Let $\mathbb{U}$ be a Polish space (complete separable metric space). Consider the following stochastic controlled system[1]

$$\begin{cases} dX_t^{U_t} = a(t, X_t^{U_t}, U_t)dt + s_1(t, X_t^{U_t}, U_t)dB_t, \ t \in [0,T], \\ X_0 = x_0 \end{cases} \tag{1.6}$$

with the reward functional[2]

$$w(x_0, \{U_s\}_{s\in[0,T]}) = E_{x_0}\Big[\int_0^T e^{-\zeta t} f(t, X_t^{U_t}, U_t)dt + e^{-\zeta T} g(X_T^{U_T})\Big],$$

where $\zeta > 0$ is a parameter. We define the space of *feasible* controls[3]

$$\mathcal{U}^s \triangleq \{U : [0,T] \times \Omega \to \mathbb{U} | U_t \text{ is } \mathcal{F}_t\text{-progressively measurable}\}.$$

The problem of stochastic control is to find U that maximizes $w(x_0, U)$ over the set $\mathcal{U}^s$[4]. One of the approaches to solve the problem is to use the *dynamic programming* [9].

To guarantee that the reward functional is well-defined (measurability and integrability of $\int_0^T e^{-\zeta t} f(t, X_t^{U_t}, U_t)dt + e^{-\zeta T} g(X_T^{U_T})$, uniqueness of X_t) we will make some assumptions specified later. Notice that the coefficients $a(t, X_t^{U_t}(\omega), U_t(\omega)) = \bar{a}(t, X_t^{U_t}(\omega), \omega)$, $s_1(t, X_t^{U_t}(\omega), U_t(\omega)) = \bar{s}_1(t, X_t^{U_t}(\omega), \omega)$ of equation (1.6) depend on ω not only through X_t but also through U_t and, thus, the theory for an SDE with random coefficients can be applied.

This problem is stated in a strong form (the probability space is given and fixed) and it is the problem that we want to solve eventually (the initial time is 0 and the

[1]Notation $X_t^{U_t}$ means that X_t depends on U_t.
[2]$E_{x_0}[\cdot] \triangleq E[\cdot | X_0 = x_0]$.
[3]To simplify the notation, U will be used instead of $\{U_s\}_{s\in[0,T]}$.
[4]The superscript 's' in $\mathcal{U}^s$ means that the strong formulation is being considered. See Remark 3.

initial state x_0 is deterministic). In order to apply the dynamic programming technique we need to consider a *family* of problems with different initial times and states. In the deterministic case we can change the initial time and state without changing the mathematical framework of the problem. However, in the stochastic case the states along a given trajectory become random variables on the *original* probability space. More specifically, if X is a state trajectory starting from x_0 at time 0 in a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,T]}, \mathbb{P})$, then for any time $t > 0$, X_t is a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$ rather than a deterministic point in $\mathbb{R}^k$. Since the control U is $\mathcal{F}_t$-progressively measurable then at any time instant t the controller knows all the relevant past information of the system up to time t and in particular about X_t. This implies that X_t is actually *not* uncertain for the controller at time t. In mathematical terms, X_t is almost surely deterministic under a *different* probability measure $\mathbb{P}(\cdot|\mathcal{F}_t)$. Indeed, this can be made precise and we have the following proposition.

Proposition 1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,T]}, \mathbb{P})$ be a filtered probability space. Let X_t be an $\mathcal{F}_t$-adapted process. Then for any $s \in [0,T]$*

$$\mathbb{P}(\{\omega'|X_t(\omega') = X_t(\omega)\}|\mathcal{F}_s)(\omega) = 1, \quad \mathbb{P} - a.s., \ \forall t \in [0,s].$$

Proof.

$$\begin{aligned}\mathbb{P}(\{\omega'|X_t(\omega') = X_t(\omega)\}|\mathcal{F}_s)(\omega) &= E[\mathbf{I}_{\{\omega'|X_t(\omega')=X_t(\omega)\}}|\mathcal{F}_s](\omega) \\ &= \mathbf{I}_{\{\omega'|X_t(\omega')=X_t(\omega)\}}(\omega) \\ &= 1, \ \mathbb{P} - a.s..\end{aligned}$$

□

Therefore, under the probability measure $\mathbb{P}(\cdot|\mathcal{F}_s)(\omega)$ where ω is fixed, the random variable X_t is almost surely a deterministic constant equal to $X_t(\omega)$ for any $t \in [0,s]$.

Thus, the above idea requires us to vary the probability spaces as well in order to employ dynamic programming. Therefore, we consider the *weak* formulation[1].

[1]The word 'weak' means that when solving the problem the probability space is allowed to vary which is similar to the case with obtaining a weak solution to (1.1). The weak formulation does not make sense if the coefficients a, s_1 depend on ω explicitly. See Remark 5.

For any $(t,x) \in [0,T) \times \mathbb{R}^k$ consider the state equation

$$\begin{cases} dX_s^{U_s} = a(s, X_s^{U_s}, U_s)ds + s_1(s, X_s^{U_s}, U_s)dB_s, \ s \in [t,T], \\ X_t = x, \end{cases} \tag{1.7}$$

with the reward functional[1]

$$w(t,x,U) = E_{t,x}\Big[\int_t^T e^{-\zeta(s-t)} f(s, X_s^{U_s}, U_s)ds + e^{-\zeta(T-t)} g(X_T^{U_T})\Big]. \tag{1.8}$$

Next, we define the space of admissible controls $\mathcal{U}^w[t,T]$[2] on $[t,T]$ as the set of 5-tuples $(\Omega, \mathcal{F}, \mathbb{P}, B, U)$ satisfying the following assumption.
Assumption **(B)**:

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space[3];
2. $\{B_s\}_{s\in[t,T]}$ is an m-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ over $[t,T]$ (with $B_t = 0$ almost surely) and $\mathcal{F}_{s,t}$ is the sigma-algebra generated by $B_r, \ t \le r \le s$ augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$;
3. $U : [t,T] \times \Omega \to \mathbb{U}$ is an $\mathcal{F}_{s,t}$-progressively measurable process on $(\Omega, \mathcal{F}, \mathbb{P})$;
4. Under U for any $x \in \mathbb{R}^k$ equation (1.7) admits a unique solution (in the sense of definitions 15, 16) on $(\Omega, \mathcal{F}, \{\mathcal{F}_{s,t}\}_{s\in[t,T]}, \mathbb{P})$;
5. The function $f(\cdot, X^U, U)$ is in $L^1_{\mathcal{F}}(0,T;\mathbb{R})$ and the function $g(X_T^{U_T})$ is in $L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$. Here the spaces $L^1_{\mathcal{F}}(0,T;\mathbb{R})$ and $L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$ are defined on the given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{s,t}\}_{s\in[t,T]}, \mathbb{P})$.

Under this restriction on the space of controls the reward functional defined by (1.8) is well-defined. Also, note that the expectation in formula (1.8) is taken with respect to the probability measure $\mathbb{P}$.

The problem of stochastic control is to maximize $w(t,x,U)$ over the space of admissible controls $\mathcal{U}^w[t,T]$.

[1] $E_{t,x}[\cdot] \triangleq E[\cdot|X_t = x]$.
[2] The superscript 'w' means that the weak formulation is considered.
[3] Although denoted similarly, this space should be distinguished from the original filtered probability space on which (1.6) is defined.

To guarantee the uniqueness of solutions to (1.7) we have to impose some assumptions. Let d be the dimension of $\mathbb{U} \subseteq \mathbb{R}^d$, the space of control values. We make the following assumption.

Assumption **(A)**:

The maps $a : [0,T] \times \mathbb{R}^k \times \mathbb{U} \to \mathbb{R}^k$, $s_1 : [0,T] \times \mathbb{R}^k \times \mathbb{U} \to \mathbb{R}^{k\times m}$, $f : [0,T] \times \mathbb{R}^k \times \mathbb{U} \to \mathbb{R}$, $g : \mathbb{R}^k \to \mathbb{R}$ are continuous. There exist concave, increasing in each independent variable, continuous functions $\varphi_1 : \mathbb{R}^{k+d} \to [0,\infty)$ and $\varphi_2 : \mathbb{R}^k \to [0,\infty)$ such that $\varphi_i = 0,\ i = 1,2,$ if any of the independent variables is equal to 0, and there exists a constant $L > 0$ such that we have[1]

$$\begin{cases} (1) \quad ||a(t,x_1,u) - a(t,x_2,u)|| \le L||x_1 - x_2||, & \forall t \in [0,T],\ x_1,x_2 \in \mathbb{R}^k,\ u \in \mathbb{U}, \\ (2) \quad ||s_1(t,x_1,u) - s_1(t,x_2,u)|| \le L||x_1 - x_2||, & \forall t \in [0,T],\ x_1,x_2 \in \mathbb{R}^k,\ u \in \mathbb{U}, \\ (3) \quad |f(t,x_1,u) - f(t,x_2,u)| \le L\varphi_1(|u|, |x_1 - x_2|), & \forall t \in [0,T],\ x_1,x_2 \in \mathbb{R}^k,\ u \in \mathbb{U}, \\ (4) \quad |g(x_1) - g(x_2)| \le L\varphi_2(|x_1 - x_2|), & x_1,x_2 \in \mathbb{R}^k, \\ (5) \quad ||a(t,0,u)|| + ||s_1(t,0,u)|| \le L, & \forall (t,u) \in [0,T] \times \mathbb{U}, \\ (6) \quad E \sup_{s\in[t,T]} ||U_s|| \le L, & U \in \mathcal{U}^w[t,T],\ t \in [0,T]. \end{cases}$$

Under the assumptions **A(1),(2),(5)**, equation (1.7) admits a unique solution by Theorem 1 (the solution is continuous, see Definition 15). Indeed, if in **A(1),(2),(5)** we consider $a,\ s_1$ as maps $\bar{a} : [0,T] \times \mathbf{B}^k \times \Omega,\ \bar{s}_1 : [0,T] \times \mathbf{B}^k \times \Omega$ then the Theorem is applicable. Also, since U_t is assumed to be $\mathcal{F}_{s,t}$-progressively measurable, X_t is $\mathcal{F}_{s,t}$-progressively measurable (see Theorem 2), and $a,\ s_1$ are continuous, we have that $\bar{a}(\cdot,\cdot,\omega) \in \mathcal{A}^k(\mathbb{R}^k),\ \bar{s}_1(\cdot,\cdot,\omega) \in \mathcal{A}^k(\mathbb{R}^{k\times m}),\ \omega \in \Omega$.

Since $f,\ g$ are continuous[2], the reward functional (1.8) is well-defined. Thus, we can define the *value function*:

$$\begin{cases} v(t,x) = \sup_{U\in\mathcal{U}^w[t,T]} w(t,x,U), & \forall (t,x) \in [0,T) \times \mathbb{R}^k, \\ v(T,x) = g(x), & \forall x \in \mathbb{R}^k. \end{cases}$$

Next we derive some properties of the value function that will be used in proving Bellman's Principle of Optimality.

[1]Notice the difference between $||\cdot||$ and $|\cdot|$. If $x = (x_1\ x_2),\ x_1,x_2 \in \mathbb{R}$ then $||x|| = \sqrt{x_1^2 + x_2^2}$ but $|x| = (|x_1|\ |x_2|)$.

[2]Since X and U are $\mathcal{F}_{s,t}$-progressively measurable and f is continuous, the process $\int_t^T f(s, X_s^{U_s}, U_s)ds$ is $\mathcal{F}_{T,t}$-measurable (see Remark 1). Since g is continuous then $g(X_T^{U_T})$ is also $\mathcal{F}_{T,t}$-measurable.

Lemma 1. *Let the assumption **(A)** hold. Then for any* $\varepsilon > 0$ *and* $t \in [0, T]$ *there exists* $\delta(\varepsilon) > 0$ *such that*

$$|w(t,x,U) - w(t,y,U)| \leq \varepsilon, \quad U \in \mathcal{U}^w[t,T],$$
$$|v(t,x) - v(t,y)| \leq \varepsilon,$$

if $||x - y|| \leq \delta(\varepsilon)$.

Proof. Let $0 \leq t \leq T,\ x, y \in \mathbb{R}^k$. For any admissible control U let $X^U,\ Y^U$ represent the states starting at time t with values x and y, respectively. Then by Theorem 1 (1.4 and 1.5) we have $E\Big(\sup_{s\in[t,T]} ||X_s - Y_s||\Big) \leq K||x - y||$. Using this result, assumption **(A)**, and Jensen's inequality we obtain

$$\begin{aligned}
&|w(t,x,U) - w(t,y,U)| \\
&= \Big|E\Big[\int_t^T e^{-\zeta(s-t)}(f(s, X_s^{U_s}, U_s)ds - f(s, Y_s^{U_s}, U_s))ds + e^{-\zeta(T-t)}(g(X_T^{U_T}) - g(Y_T^{U_T}))\Big]\Big| \\
&\leq E\Big[\int_t^T L\varphi_1(|U_s|, |X_s^{U_s} - Y_s^{U_s}|)ds + L\varphi_2(|X_T^{U_T} - Y_T^{U_T}|)\Big] \\
&\leq E[LT\varphi_1(\sup_{s\in[t,T]} |U_s|, \sup_{s\in[t,T]} |X_s^{U_s} - Y_s^{U_s}|) + L\varphi_2(|X_T^{U_T} - Y_T^{U_T}|)] \\
&\leq E[LT\varphi_1(\sup_{s\in[t,T]} ||U_s||\mathbf{1}^d, \sup_{s\in[t,T]} ||X_s^{U_s} - Y_s^{U_s}||\mathbf{1}^k) + L\varphi_2(||X_T^{U_T} - Y_T^{U_T}||\mathbf{1}^k)] \qquad (1.9) \\
&\leq LT\varphi_1(E\sup_{s\in[t,T]} ||U_s||\mathbf{1}^d, E\sup_{s\in[t,T]} ||X_s^{U_s} - Y_s^{U_s}||\mathbf{1}^k) + L\varphi_2(E||X_T^{U_T} - Y_T^{U_T}||\mathbf{1}^k) \\
&\leq LT\varphi_1(E\sup_{s\in[t,T]} ||U_s||\mathbf{1}^d, E\sup_{s\in[t,T]} ||X_s^{U_s} - Y_s^{U_s}||\mathbf{1}^k) + L\varphi_2(E\sup_{s\in[t,T]} ||X_s^{U_s} - Y_s^{U_s}||\mathbf{1}^k) \\
&\leq LT\varphi_1(L\mathbf{1}^d, K||x - y||\mathbf{1}^k) + L\varphi_2(K||x - y||\mathbf{1}^k),
\end{aligned}$$

where $\mathbf{1}^d = (1, ..., 1)^\intercal$ is a d-dimensional vector and $\mathbf{1}^k = (1, ..., 1)^\intercal$ is a k-dimensional vector. Notice that in (1.9) value of each independent variable in $\varphi_i,\ i = 1, 2$ was changed from $|\cdot|$ to $||\cdot||$ and the direction of the inequality follows from the fact that $|x_i| \leq ||x|| = \sqrt{x_1^2 + ... + x_k^2},\ i = 1, ..., k,\ x \in \mathbb{R}^k$.

Since functions $\varphi_i,\ i = 1, 2$ are continuous, increasing in each variable, and $\varphi_i = 0,\ i = 1, 2$, if any of their independent variables is equal to 0, we have that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|w(t,x,U) - w(t,y,U)| \leq \varepsilon$, if $||x - y|| \leq \delta(\varepsilon)$.

Taking the supremum over $U \in \mathcal{U}^w[t,T]$ we obtain

$$\begin{aligned}\sup_{U\in\mathcal{U}^w[t,T]} |w(t,x,U) - w(t,y,U)| &\geq |\sup_{U\in\mathcal{U}^w[t,T]} (w(t,x,U) - w(t,y,U))| \\ &\geq |\sup_{U\in\mathcal{U}^w[t,T]} w(t,x,U) - \sup_{U\in\mathcal{U}^w[t,T]} w(t,y,U))| \\ &= |v(t,x) - v(t,y)|.\end{aligned}$$

Therefore, $|v(t,x) - v(t,y)| \leq \varepsilon$. □

Lemma 2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathbb{U}, d)$ be a Polish space. Let $B : [0,T] \times \Omega \to \mathbb{R}^m$ be a continuous process and $\mathcal{F}_t^B$ be the sigma-algebra generated by $B_s,\ 0 \leq s \leq t$. Then $U : [0,T] \times \Omega \to \mathbb{U}$ is $\{\mathcal{F}_t^B\}_{t\in[0,T]}$-adapted if and only if there exists $\phi \in \mathcal{A}_T^m(\mathbb{U})$ such that*

$$U_t(\omega) = \phi(t, B_{\cdot\wedge t}(\omega))^1, \quad \mathbb{P} - a.s.,\ \forall t \in [0,T].$$

The proof of the lemma is given in appendix A.3.

Let $t \in [0,T), \theta \in [t,T)$ and ξ be an $\mathcal{F}_{\theta,t}$-measurable random variable and X be a solution of

$$\begin{cases} dX_s^{U_s} = a(s, X_s^{U_s}, U_s)ds + s_1(s, X_s^{U_s}, U_s)dB_s,\ s \in [\theta, T], \\ X_\theta(\omega) = \xi(\omega). \end{cases} \tag{1.10}$$

Now we are ready for the next lemma.

Lemma 3. *Let $t \in [0,T)$ and $U \in \mathcal{U}^w[t,T]$. Then for any $\theta \in [t,T)$ and $\mathcal{F}_{\theta,t}$-measurable random variable ξ*

$$w(\theta, \xi, U) = E_{\theta,\xi}\Big[\int_\theta^T e^{-\zeta(s-\theta)} f(s, X_s^{U_s}, U_s)ds + e^{-\zeta(T-\theta)} g(X_T^{U_T})|\mathcal{F}_{\theta,t}\Big],\ \mathbb{P} - a.s.$$

Proof. Since the control U is $\mathcal{F}_{s,t}$-adapted (see the assumption **B**) and $\mathcal{F}_{s,t}$ is the sigma-algebra generated by the Brownian motion $B_r,\ t \leq r \leq s$, then by Lemma 2 there exists a function $\phi \in \mathcal{A}_T^m(\mathbb{U})$ such that $U_s(\omega) = \phi(s, B_{\cdot\wedge s}(\omega)),\ \mathbb{P} - \text{a.s.},\ \omega \in \Omega,\ s \in [t,T]$.

[1]The process $B_{\cdot\wedge t}$ is the process B_s if $s \leq t$ with values equal to B_t if $s > t$.

Therefore, (1.10) can also be written as

$$\begin{cases} dX^{\phi}_s = a(s, X^{\phi}_s, \phi(s, B_{\cdot\wedge s}))ds + s_1(s, X^{\phi}_s, \phi(s, B_{\cdot\wedge s}))dB_s, \ s \in [\theta, T], \\ X_\theta(\omega) = \xi(\omega). \end{cases} \tag{1.11}$$

This equation has a unique strong solution because the equation

$$\begin{cases} dX_s = \bar{a}(s, X_s, B_{\cdot\wedge s})ds + \bar{s}_1(s, X_s, B_{\cdot\wedge s})dB_s, \ s \in [\theta, T], \\ X_\theta(\omega) = \xi(\omega). \end{cases} \tag{1.12}$$

is a special case of (1.1). Indeed, if we write $dY_s = dB_s$ and consider Y as a component of X, then we have (1.1) and thus, under the assumptions **A(1), (2), (5)** the equation (1.12) has a unique strong solution. On the other hand, from Proposition 1 it follows that $\mathbb{P}(\omega' : \xi(\omega') = \xi(\omega)|\mathcal{F}_{\theta,t})(\omega) = 1$, $\mathbb{P}$-a.s. This means that there is an $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega_0 \in \Omega_0$, ξ becomes a deterministic constant $\xi(\omega_0)$ under the new probability space $(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_{\theta,t})(\omega_0))$. In addition, for any $s \geq \theta$, we have that $U_s(\omega) = \phi(s, B_{\cdot\wedge s}(\omega)) = \phi(s, \tilde{B}_{\cdot\wedge s}(\omega) + B_\theta(\omega))$, where $\tilde{B}_s = B_s - B_\theta$ is a standard Brownian motion. Note that B_θ almost surely equals a constant $B_\theta(\omega_0)$ under the probability measure $\mathbb{P}(\cdot|\mathcal{F}_{\theta,t})(\omega_0)$. It follows then that U_s is adapted to the filtration generated by the standard Brownian motion $\tilde{B}_s$ for $s \geq \theta$. Hence by the definition of admissible controls $(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_{\theta,t})(\omega_0)), \tilde{B}_s, U_s) \in \mathcal{U}^w[\theta, T]$.

Thus, if we work under the probability space $(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_{\theta,t})(\omega_0))$ and notice the weak uniqueness (see Definition 13) of (1.10) and (1.11) we obtain the result. □

Now we derive Bellman's Principle of Optimality that will be very important in deriving the Hamilton-Jacobi-Bellman equation used in solving the optimal control problem presented in this dissertation.

Theorem 3. *Let assumption **(A)** hold. Then for any* $(t, x) \in [0, T) \times \mathbb{R}^k$ *and for all* t, θ *satisfying* $0 \leq t \leq \theta \leq T$ *we have*

$$v(t, x) = \sup_{U \in \mathcal{U}^w[t,T]} E_{t,x}\Big[\int_t^\theta e^{-\zeta(s-t)} f(s, X^{U_s}_s, U_s)ds + e^{-\zeta(\theta-t)} v(\theta, X^{U_\theta}_\theta)\Big]. \tag{1.13}$$

Proof. By definition of supremum, for any $\varepsilon > 0$ there exists an admissible control U (there exists $(\Omega, \mathcal{F}, \mathbb{P}, B, U)$) such that (using Lemma 3)

$$
\begin{aligned}
&v(t,x)-\varepsilon < w(t,x,U)\\
&= E_{t,x}\Big[\int_t^T e^{-\zeta(s-t)} f(s,X_s^{U_s},U_s)ds + e^{-\zeta(T-t)} g(X_T^{U_T})\Big]\\
&= E_{t,x}\Big[\int_t^\theta e^{-\zeta(s-t)} f(s,X_s^{U_s},U_s)ds\\
&+ E_{t,x}\Big[\int_\theta^T e^{-\zeta(s-t)} f(s,X_s^{U_s},U_s)ds + e^{-\zeta(T-t)} g(X_T^{U_T})|\mathcal{F}_{\theta,t}\Big]\Big]\\
&= E_{t,x}\Big[\int_t^\theta e^{-\zeta(s-t)} f(s,X_s^{U_s},U_s)ds\\
&+ e^{-\zeta(\theta-t)} E_{\theta,X_\theta^{U_\theta}}\Big[\int_\theta^T e^{-\zeta(s-\theta)} f(s,X_s^{U_s},U_s)ds + e^{-\zeta(T-\theta)} g(X_T^{U_T})|\mathcal{F}_{\theta,t}\Big]\Big]\\
&= E_{t,x}\Big[\int_t^\theta e^{-\zeta(s-t)} f(s,X_s^{U_s},U_s)ds + e^{-\zeta(\theta-t)} w(\theta,X_\theta^{U_\theta},U)\Big]\\
&\le E_{t,x}\Big[\int_t^\theta e^{-\zeta(s-t)} f(s,X_s^{U_s},U_s)ds + e^{-\zeta(\theta-t)} v(\theta,X_\theta^{U_\theta})\Big]\\
&\le \sup_{U\in\mathcal{U}^w[t,T]} E_{t,x}\Big[\int_t^\theta e^{-\zeta(s-t)} f(s,X_s^{U_s},U_s)ds + e^{-\zeta(\theta-t)} v(\theta,X_\theta^{U_\theta})\Big].
\end{aligned}
$$

Conversely, by Lemma 1 for any $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that if $||x-y|| \le \delta(\varepsilon)$ then

$$
e^{-\zeta(\theta-t)}\Big(|w(\theta,x,U)-w(\theta,y,U)| + |v(\theta,x)-v(\theta,y)|\Big) \le \frac{\varepsilon}{3},\ \forall U \in \mathcal{U}^w[\theta,T]. \tag{1.14}
$$

Let $\{D_j\}_{j\ge1}$ be a Borel partition of $\mathbb{R}^k$ ($D_j \in \mathcal{B}(\mathbb{R}^k),\ \bigcup_{j=1}^{\infty} D_j = \mathbb{R}^k,\ D_i\cap D_j = \emptyset, i\ne j$) with diameter $diam(D_j) < \delta(\varepsilon)$. Choose $x^j \in D_j$. For each j there is an admissible control U^j (there exists $(\Omega^j,\mathcal{F}^j,\mathbb{P}^j,B^j,U^j)$) such that

$$
e^{-\zeta(\theta-t)} w(\theta,x^j,U^j) \ge e^{-\zeta(\theta-t)} v(\theta,x^j) - \frac{\varepsilon}{3}. \tag{1.15}
$$

Hence for any $x \in D_j$, combining (1.14) and (1.15), we have

$$
\begin{aligned}
e^{-\zeta(\theta-t)} w(\theta,x,U^j) &\ge e^{-\zeta(\theta-t)} w(\theta,x^j,U^j) - \frac{\varepsilon}{3}\\
&\ge e^{-\zeta(\theta-t)} v(\theta,x^j) - \frac{2\varepsilon}{3}\\
&\ge e^{-\zeta(\theta-t)} v(\theta,x) - \varepsilon.
\end{aligned}
$$

By the definition of admissible control $(\Omega^j, \mathcal{F}^j, \mathbb{P}^j, B^j, U^j)$ (see assumption **(B)**) and Lemma 2 there is a function $\phi^j \in \mathcal{A}_T^m(\mathbb{U})$ such that $U_s^j(\omega) = \phi^j(s, B^j_{\cdot\wedge s}(\omega)),\ \mathbb{P}^j - a.s.$, for all $s \in [\theta, T]$.

For any admissible $(\Omega, \mathcal{F}, \mathbb{P}, B, U)$ define a new control

$$\bar{U}_s(\omega) = \begin{cases} U_s(\omega), & s \in [t, \theta), \\ \phi^j(s, B_{\cdot\wedge s}(\omega) - B_\theta(\omega)), & s \in [\theta, T] \text{ and } X_s \in D_j. \end{cases}$$

Clearly, $(\Omega, \mathcal{F}, \mathbb{P}, B, \bar{U})$ is admissible. Therefore,

$$v(t,x) \geq w(t,x,\bar{U}) = E_{t,x}\Big[\int_t^T e^{-\zeta(s-t)} f(s, X_s^{\bar{U}_s}, \bar{U}_s)ds + e^{-\zeta(T-t)} g(X_T^{\bar{U}_T})\Big]$$

$$\begin{aligned}
&= E_{t,x}\Big[\int_t^\theta e^{-\zeta(s-t)} f(s, X_s^{U_s}, U_s)ds \\
&+ e^{-\zeta(\theta-t)} E_{\theta, X_\theta^{\bar{U}_\theta}}\Big[\int_\theta^T e^{-\zeta(s-\theta)} f(s, X_s^{\bar{U}_s}, \bar{U}_s)ds + e^{-\zeta(T-\theta)} g(X_T^{\bar{U}_T}) | \mathcal{F}_{\theta,t}\Big]\Big] \\
&= E_{t,x}\Big[\int_t^\theta e^{-\zeta(s-t)} f(s, X_s^{U_s}, U_s)ds + e^{-\zeta(\theta-t)} w(\theta, X_\theta^{\bar{U}_\theta}, \bar{U})\Big] \\
&\geq E_{t,x}\Big[\int_t^\theta e^{-\zeta(s-t)} f(s, X_s^{U_s}, U_s)ds + e^{-\zeta(\theta-t)} v(\theta, X_\theta^{U_\theta}) - \varepsilon\Big].
\end{aligned}$$

Taking the supremum over $U \in \mathcal{U}^w[t,T]$ we obtain (1.13). □

Equation (1.13) is very difficult to solve directly. One of the techniques allowing to obtain the value function $v(t,x),\ t \in [0,T],\ x \in \mathbb{R}^k$ is to use the Bellman's Principle of Optimality to derive the second-order partial differential equation that this function should satisfy. This equation is called the **Hamilton-Jacobi-Bellman** (HJB) equation. Although the equation can be obtained in general form, it will be derived for the specific stochastic control problems considered in the next sections.

1.4 Classical Model of Optimal Investment and Consumption

In this section we consider the model of optimal investment and consumption originated by Merton [15]. Consider an investor who at each time t has a portfolio valued at X_t. This portfolio invests in a money market account (riskless asset) paying rate of interest $r(t)$ and in n stocks (risky assets) modeled by Geometric Brownian motion[1]. Suppose at each time t, the agent holds H_t shares of the risky assets, H_t^0 shares of the riskless asset, and consumes at a rate C_t per unit time. We define the corresponding processes below[2]

- *Riskless asset*: $dN_t = r(t)N_t dt$, where $r(t)$ is a continuous deterministic function. Let us denote $r \triangleq r(t)$.

- *Risky assets* : $dS_t^i = S_t^i(\mu^i(t)dt + \sum_{j=1}^{n} \sigma_1^{i,j}(t)dB_t^j)$, where $\sigma_1(t) = (\sigma_1^{i,j}(t))_{i,j=1\ldots n}$ is continuous deterministic volatility matrix which is invertible for each t and, thus, the market is complete, $\mu^i(t)$ is continuous deterministic expected return. Let us denote $\sigma_1 \triangleq \sigma_1(t),\ \mu^i \triangleq \mu^i(t),\ i = 1, ..., n$.

- *Portfolio value* : $X_t^{U_t} = H_t^0 N_t + H_t S_t$, where H_t^0 is the number of shares of the riskless asset, H_t is a row vector of numbers of shares of the risky assets, $S_t = (S_t^1, ..., S_t^n)^\intercal$ is a vector of assets' prices.

- *Portfolio process*: $dX_t^{U_t} = H_t^0 dN_t + H_t dS_t - C_t dt$, where C_t denotes consumption per unit time at time t. Expanding the expression for the portfolio process we obtain

$$
\begin{aligned}
dX_t^{U_t} &= rH_t^0 N_t dt + \sum_{i=1}^{n} H_t^i S_t^i(\mu^i dt + \sum_{j=1}^{n} \sigma_1^{i,j} dB_t^j) - C_t dt \\
&= (1 - \Pi_t \mathbf{1})rX_t^{U_t} dt + \Pi_t X_t^{U_t}(\mu dt + \sigma_1 dB_t) - C_t dt \\
&= \Big((1 - \Pi_t \mathbf{1})rX_t^{U_t} + \Pi_t X_t^{U_t}\mu - C_t\Big)dt + \Pi_t \sigma_1 X_t^{U_t} dB_t, \qquad (1.16)
\end{aligned}
$$

[1]See [23], p.147.

[2]This model is more general than the model in Merton's paper [15] where, for example, the rate of interest was considered to be a constant. This generalization is necessary for comparing the results of this research with the results in the classical model.

where $\Pi_t = \left(\frac{diag(H_t)S_t}{X_t^{U_t}}\right)^\top$ represents the fractions of wealth invested in the risky assets, $1 - \Pi_t \mathbf{1} = \frac{H_t^0 N_t}{X_t^{U_t}}$ is the fraction of wealth invested in the riskless asset (borrowing and shortselling (borrowing and selling assets) are allowed), $\mathbf{1} = (1, ..., 1)^\top$, $\mu = (\mu^1, ..., \mu^n)^\top$ is a vector of expected returns[1], and $B_t = (B_t^1, ..., B_t^n)^\top$ is an n-dimensional Brownian motion.

Using the previous notation $a \equiv (1 - \Pi_t \mathbf{1}) r X_t^{U_t} + \Pi_t X_t^{U_t} \mu - C_t,\ s_1 \equiv \Pi_t \sigma_1 X_t^{U_t},\ U_t = (\Pi_t\ C_t),\ \mathbb{U} = \mathbb{R}^n \times [0, \infty)$.

Once the controlled system (1.16) has been defined, we set up the reward functional and the value function. The problem of stochastic control is to find the consumption and investment strategy that maximizes the investor's expected utility. Let $\zeta > 0$ denote the utility discount rate which may be different from the risk-free rate r.

Remark 6. *The reward functionals, value functions, and optimal strategies obtained in the following sections will be denoted by the same notation even though they are not necessarily the same and, thus, should be interpreted in the context of each section only.*

1.4.1 Maximizing the Utility of Consumption and Final Wealth

Let $f(s, X_s^{U_s}, U_s) = \frac{(C_s)^\gamma}{\gamma}$, $g(X_T^{U_T}) = \frac{(X_T^{U_T})^\gamma}{\gamma}$, $\gamma \in (0, 1)$ be the hyperbolic absolute risk aversion (HARA) utility functions for consumption and final wealth, respectively. These utility functions are very general and are often used because of their mathematical simplicity. The reward functional

$$w(t, x, U) = \frac{1}{\gamma} E_{t,x} \Big[\int_t^T e^{-\zeta(s-t)} (C_s)^\gamma ds + e^{\zeta(T-t)} (X_T^{U_T})^\gamma \Big],$$

and the value function

$$v(t, x) = \sup_{U \in \mathcal{U}^w[t,T]} w(t, x, U).$$

The corresponding HJB equation for $t \in (0, T)$ and $x > 0$ is

$$p_t - \zeta p + \sup_{(\pi,\ c) \in \mathbb{R}^n \times [0,\infty)} \left(\Big((1 - \pi \mathbf{1}) r x + \pi x \mu - c \Big) p_x + \frac{1}{2} ||\pi \sigma_1 x||^2 p_{xx} + \frac{c^\gamma}{\gamma} \right) = 0.$$

[1]In this section, $\mathbf{1}$ is an n-dimensional vector.

The terminal and boundary conditions are

$$\begin{cases} p(T,x) = \frac{1}{\gamma}x^\gamma, & x > 0, \\ p(t,0) = 0, & t \in (0,T). \end{cases}$$

The meaning of the terminal condition is that if the investor starts at time T then there is no time for trading and the utility is equal to the utility of the wealth he starts with. The boundary condition means that if the investor has no money then there in nothing to invest and the utility is zero.

The solution to this boundary value problem is

$$p(t,x) = \frac{x^\gamma}{\gamma}\Big(e^{\int_t^T q(\tau)d\tau} + \int_t^T e^{\int_t^\tau q(y)dy}d\tau\Big)^{1-\gamma}.$$

where

$$q(t) = -\frac{\zeta}{1-\gamma} + \frac{\gamma r}{1-\gamma} + \frac{||(r-\mu^\intercal)(\sigma_1^\intercal)^{-1}||^2\gamma}{2(\gamma-1)^2}. \tag{1.17}$$

It has been verified [15] that $p \equiv v$. Therefore, the optimal control[1] is

$$\Pi_t^* = \frac{(r-\mu^\intercal)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)},\ C_t^* = \frac{X_t^{U_t}}{e^{\int_t^T q(\tau)d\tau} + \int_t^T e^{\int_t^\tau q(y)dy}d\tau}.$$

1.4.2 Maximizing the Utility of Consumption

Let $f(s, X_s^{U_s}, U_s) = \frac{(C_s)^\gamma}{\gamma},\ g(X_T^{U_T}) \equiv 0,\ \gamma \in (0,1)$ be the utility functions for consumption and final wealth, respectively. The reward functional

$$w(t,x,U) = \frac{1}{\gamma}E_{t,x}\Big[\int_t^T e^{-\zeta(s-t)}(C_s)^\gamma ds\Big],$$

and the value function

$$v(t,x) = \sup_{U\in\mathcal{U}^w[t,T]} w(t,x,U).$$

[1]If some of the entries of the vector Π_t^* are negative (positive), then the investor should shortsell (buy) the corresponding assets.

The corresponding HJB equation for $t \in (0, T)$ and $x > 0$ is

$$p_t - \zeta p + \sup_{(\pi,\ c)\in\mathbb{R}^n\times[0,\infty)} \left(\left((1-\pi\mathbf{1})rx + \pi x\mu - c\right)p_x + \frac{1}{2}||\pi\sigma_1 x||^2 p_{xx} + \frac{c^\gamma}{\gamma}\right) = 0.$$

The terminal and boundary conditions are

$$\begin{cases} p(T, x) = 0, & x > 0, \\ p(t, 0) = 0, & t \in (0, T). \end{cases}$$

The solution to this boundary value problem is

$$p(t, x) = \frac{x^\gamma}{\gamma}\Big(\int_t^T e^{\int_t^\tau q(y)dy} d\tau\Big)^{1-\gamma},$$

where q is defined in (1.17). It has been verified [15] that $p \equiv v$. Therefore, the optimal control is

$$\Pi_t^* = \frac{(r-\mu^\intercal)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)},\ C_t^* = \frac{X_t^{U_t}}{\int_t^T e^{\int_t^\tau q(y)dy} d\tau}.$$

1.4.3 Maximizing the Utility of Final Wealth

Let $f(s, X_s^{U_s}, U_s) = 0,\ g(X_T^{U_T}) = \frac{(X_T^{U_T})^\gamma}{\gamma},\ \gamma \in (0, 1)$ be the utility functions for consumption and final wealth, respectively. The reward functional

$$w(t, x, U) = \frac{1}{\gamma} E_{t,x}\Big[e^{\zeta(T-t)}(X_T^{U_T})^\gamma\Big],$$

and the value function

$$v(t, x) = \sup_{U\in\mathcal{U}^w[t,T]} w(t, x, U).$$

The corresponding HJB equation for $t \in (0, T)$ and $x > 0$ is

$$p_t - \zeta p + \sup_{(\pi,\ c)\in\mathbb{R}^n\times[0,\infty)} \left(\left((1-\pi\mathbf{1})rx + \pi x\mu - c\right)p_x + \frac{1}{2}||\pi\sigma_1 x||^2 p_{xx}\right) = 0.$$

The terminal and boundary conditions are

$$\begin{cases} p(T, x) = \frac{1}{\gamma}x^\gamma, & x > 0, \\ p(t, 0) = 0, & t \in (0, T). \end{cases}$$

The solution to this boundary value problem is

$$p(t,x) = \frac{x^\gamma}{\gamma}\left(e^{\int_t^T q(\tau)d\tau}\right)^{1-\gamma},$$

where q is defined in (1.17). It has been verified [15] that $p \equiv v$. Therefore, the optimal control is

$$\Pi_t^* = \frac{(r-\mu^\mathsf{T})(\sigma_1\sigma_1^\mathsf{T})^{-1}}{(\gamma-1)},\ C_t^* = 0.$$

1.5 Uncertainty in the Utility Function

In some papers [14, 20], it is claimed that the utility of goods depends not only on the goods themselves but also on qualities of the goods. The classical models of optimal investment and consumption [7, 16, 21] can be extended to incorporate this feature. Thus, it is of interest to find out how the optimal policy changes when the uncertainty in utility is introduced.

Indeed, when we buy different things we are not buying just objects, we are buying the qualities that those objects possess and we need those qualities. Clearly, a diamond that costs as much as a house has utility which is different from that of the house. However, if the only argument to the utility function is the price of a merchandise then the utilities of the diamond and the house are the same.

> The good, per se, does not give utility to the consumer; it possesses characteristics, and these characteristics give rise to utility.[1]

Since the technology is changing, new products keep coming out and substitute the old ones giving the increase in utility. For example, having a computer today gives much more opportunities to its user compared to the computers and technologies available 30 years ago. Therefore, preferences might change because of better characteristics of new goods. It is important to note that this change does not have to entail the change of prices.

> The inflationary increase in prices being relative to the decrease caused by technological progress means there is no variation in the price index.[2]

[1]Lancaster, K.J. (1966) *A New Approach to Consumer Theory*, Journal of Political Economy, Vol.74, N.2, p.134.

[2]Cencini, A. (1996) *Inflation and Unemployment: Contributions to a New Macroeconomic Approach*, Routledge studies in the modern world economy, Routledge, p.23.

On the other hand, preference change might also be due to worsened quality of the products or some other reason (for example, buying the same product over and over might decrease its utility and ends up in satiation with the product). The described process is not deterministic because it is not known how the market will change. Therefore, it makes sense to model the uncertainty in utility by a stochastic process.

Although there is much debate on the dynamics of technological progress [1, 5], it is not new in literature that in some areas it is growing exponentially [8]. A model that assumes exponential growth can also be used to model linear behavior because of the representation $e^x = 1 + x + o(x^2)$ and, thus, changing the parameters of the model accordingly, will help analyze the results when the growth is close to linear. Apart from big technological advancements there are minor improvements in products that people use every day. Therefore, the Geometric Brownian motion can be used to model the uncertainty in utility.

The role played by the utility discount rate (ζ in the notation of this paper) used in classical models should not be confused with the proposed utility randomness. The utility discount rate accounts for the preference to obtain something now instead of waiting and getting it later. For example, it is preferable to have a laptop today rather than tomorrow. However, this only works if the implied computer has the same characteristics. In general, it is not true because computer in the future can be more advanced.

Chapter 2

Fully Observed Case

2.1 Formulation of the Problem

The problem of optimal portfolio management under uncertainty in utility function is considered in this chapter. We begin by assuming that we have a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,T]}, \mathbb{P})$ satisfying the usual condition and there are two stochastic processes defined on this space, $X_t^{U_t},\ Z_t$ which represent the agent's wealth and *utility randomness process* at time t, respectively (here we use the same notation as in chapter 1 and $X_t^{U_t}$ means that X_t depends on the control U_t). The two processes are defined by the following stochastic differential equations:

$$dX_t^{U_t} = a(X_t^{U_t}, U_t)dt + s_1(X_t^{U_t}, U_t)dB_{t,1}, \tag{2.1}$$
$$dZ_t = b(Z_t)dt + s_2(Z_t)dB_{t,2},$$

where $B_{t,1} = (B_{t,1}^1, ..., B_{t,1}^n)^\intercal$ is an n-dimensional Brownian motion, $B_{t,2}$ is a one dimensional Brownian motion, $B_{t,1},\ B_{t,2}$ are correlated and $dB_{t,1}^i dB_{t,2} = \rho_i(t)dt$, where function $\rho(t) = (\rho_1(t), ..., \rho_n(t))^\intercal$ is continuous and deterministic $(||\rho(t)||^2 < 1$ for all $t \in [0,T])$. To shorten the notation, let $a \triangleq a(X_t^{U_t}, U_t),\ b \triangleq b(Z_t),\ s_1 \triangleq s_1(X_t^{U_t}, U_t),\ s_2 \triangleq s_2(Z_t)$, and $\rho \triangleq \rho(t)$.

Note that the utility randomness process Z_t does not depend on the control $U_t = (\Pi_t, C_t)$. However, the randomness of the stock prices represented by the Brownian motions $B_{t,1}$ can be correlated with that of the utility randomness process specified by $B_{t,2}$. This assumption makes sense because it is reasonable to assume that technological progress might influence the stock prices.

Denote the utility discounting factor by $q(s) = e^{-\zeta(s-t)}$, where $\zeta > 0$ is the utility discount rate. Define the *reward functional*[1] as

$$w(t,x,z,U) = E_{t,x,z}\Big[\int_t^T q(s)f(s,X_s^{U_s},C_s,Z_s)ds + q(T)g(X_T^{U_T},Z_T)\Big], \qquad (2.2)$$

and the *value function* as

$$v(t,x,z) = \sup_{U\in\mathcal{U}^w[t,T]} w(t,x,z,U). \qquad (2.3)$$

Therefore, the goal is to find a feasible control process $\{U_s^*\}_{s\in[t,T]}$ that gives the supremum of the reward functional.

The system (2.1) can be put in a form (1.6) and, thus, under certain assumptions (Lipschitz continuity in space variable and boundedness at zero) has a unique strong solution. Indeed, changing the Brownian motions $dB_{t,1},\ dB_{t,2}$ into independent Brownian motions $d\tilde{B}_{t,1} = dB_{t,1},\ d\tilde{B}_{t,2} = \frac{dB_{t,2}-\rho^\intercal dB_{t,1}}{\sqrt{1-||\rho||^2}}$, the system (2.1) can be written as

$$d\tilde{X}_t = \tilde{a}(t,\tilde{X}_t^{U_t},U_t)dt + \tilde{s}_1(t,\tilde{X}_t^{U_t},U_t)d\tilde{B}_t,$$

where $\tilde{X}_t = (X_t^{U_t},\ Z_t)^\intercal,\ \tilde{B}_t = (\tilde{B}_{t,1},\ \tilde{B}_{t,2})^\intercal$ and the functions $a,\ b$ and $s_1,\ s_2$ are combined into $\tilde{a}$ and $\tilde{s}_1$, respectively. This suggests the assumptions on $a,\ s_1,\ b,\ s_2$ required for the results of sections 1.2 and 1.3 in chapter 1 be applicable, namely (defining $\mathbb{U}$ as the space of control values), the functions $a:[0,\infty)\times\mathbb{U}\to\mathbb{R},\ s_1:[0,\infty)\times\mathbb{U}\to\mathbb{R}^{1\times n},\ b:[0,\infty)\to\mathbb{R}$, and $s_2:[0,\infty)\to\mathbb{R}$ are continuous and satisfy[2] the assumptions **A(1),(2)**, and **(5)**. Similarly, functions f, and g should satisfy assumption **A(3)**, and **(4)**.

2.2 Fully Observed Utility Randomness Process

In this section the problem (2.3) for certain class of functions $f,\ g,\ a,\ b,\ s_1$, and s_2 is solved under the assumption that the utility randomness process Z_t is fully observed. In addition to the processes defined in section 1.4, we also define

[1] $E_{t,x,z}[\cdot] \triangleq E[\cdot|X_t^{U_t}=x, Z_t=z]$.

[2] It should be taken into account that these functions do not depend on t explicitly, and b, and s_2 do not depend on the control.

Utility randomness process:

$$dZ_t = \beta Z_t dt + \sigma_2 Z_t dB_{t,2},\ Z_0 = z_0.$$

As it was mentioned in section 1.5, a Geometric Brownian motion is an appropriate process to model the uncertainty in the utility function. Let us use the previous notation $b \equiv \beta Z_t,\ s_2 \equiv \sigma_2 Z_t$. The parameter β represents the expected instantaneous growth in utility and $\sigma_2 > 0$ is the utility growth volatility. Since technological achievements usually tend to raise the utility, it is only natural to assume that $\beta > 0$. We use the utility functions of hyperbolic absolute risk aversion (HARA) type, which is $U(C) = \frac{C^\gamma}{\gamma}$ with $\gamma \in (0, 1)$. To model the uncertainty in the utility, we multiply the utility function by the utility randomness process Z_t. Therefore, the functions f and g are $f(t, x, c, z) = \frac{c^\gamma z}{\gamma}$ and $g(x, z) = \frac{x^\gamma z}{\gamma}$, respectively.

As it was mentioned in section 1.3, the problem (2.3) is difficult to solve directly. One way to solve it is to derive a corresponding second-order partial differential equation (more precisely, HJB equation) that the value function should satisfy. It is assumed that[1] the value function $v \in C^{1,2,2}(\mathbb{D}) \cap C(\bar{\mathbb{D}}),\ \mathbb{D} = \{(t, x, z) : t \in [0, T),\ x > 0,\ z > 0\}$ and the HJB equation admits a classical solution. If the solution has been obtained and it has been verified that the solution to the HJB equation is the value function then the optimal control can be found.

2.3 Derivation of the HJB Equation

In this section we derive the HJB equation using the Bellman's equation defined in (1.13). Consider the times $t, \theta \in [0, T),\ \theta > t$ and a constant control $U \equiv u \in \mathcal{U}^w[t, T]$ then from Bellman's Principle of Optimality (1.13), we have the following inequality for the value function

$$v(t, x, z) \geq E_{t,x,z}\Big[\int_t^\theta q(s) f(s, X_s^u, C_s, Z_s)\ ds + q(\theta) v(\theta, X_\theta^u, Z_\theta)\Big]. \tag{2.4}$$

[1]Notation $\bar{\mathbb{D}}$ means the completion of $\mathbb{D}$, and $C^{1,2,2}(\mathbb{D})$ is the space of functions continuously differentiable in first independent variable and twice continuously differentiable in the second and third independent variables.

Since $v \in C^{1,2,2}(\mathbb{D}) \cap C(\bar{\mathbb{D}})$, Ito's formula (see [23], p.167) yields

$$\begin{aligned} d\big(q(s)v(s, X_s^u, Z_s)\big) =\ & q(s)\big((v_s - \zeta v)ds + v_x dX_s^u + v_z dZ_s + v_{xz} d[X_s^u, Z_s] \\ & + \frac{1}{2} v_{xx} d[X_s^u, X_s^u] + \frac{1}{2} v_{zz} d[Z_s, Z_s]\big) \\ =\ & q(s)\big((v_s - \zeta v)ds + v_x a ds + v_x s_1 dB_{s,1} + v_z b ds + v_z s_2 dB_{s,2} \\ & + \frac{1}{2} v_{xx} s_1 s_1^{\intercal} ds + \frac{1}{2} v_{zz} s_2^2 ds + v_{xz} s_1 \rho s_2 ds\big). \end{aligned}$$

Integrating from t to θ, and noting that $q(t) = 1$, we get

$$\begin{aligned} q(\theta)v(\theta, X_\theta^u, Z_\theta) =\ & v(t, X_t^u, Z_t) \\ & + \int_t^\theta q(s)\Big(v_s - \zeta v + v_x a + v_z b + \frac{1}{2} v_{xx} s_1 s_1^{\intercal} + \frac{1}{2} v_{zz} s_2^2 + v_{xz} s_1 \rho s_2\Big) ds \\ & + \int_t^\theta q(s) v_x s_1 dB_{s,1} + \int_t^\theta q(s) v_z s_2 dB_{s,2}. \end{aligned}$$

The stochastic integrals in the above expression are local martingales (see [10], p.36).

Consider a sequence of stopping times $\tau_n = \inf\{h \geq t \ : \ \int_t^h \Big(||q(s)v_x s_1||^2 + ||q(s)v_z s_2||^2\Big) ds \geq n\}$. Notice that τ_n diverges to infinity almost surely as n goes to infinity. Let $\tau = \theta \wedge \tau_n$, then the stochastic integrals $\int_t^\tau q(s) v_x s_1 dB_{s,1}$ and $\int_t^\tau q(s) v_z s_2 dB_{s,2}$ are martingales. Plugging $q(\tau)v(\tau, X_\tau^u, Z_\tau)$ into equation (2.4) we obtain

$$\begin{aligned} v(t, x, z) \geq E_{t,x,z}\Big[& \int_t^\tau q(s)\Big(f(s, X_s^u, C_s, Z_s) + v_s - \zeta v + v_x a + v_z b \\ & + \frac{1}{2} v_{xx} s_1 s_1^{\intercal} + \frac{1}{2} v_{zz} s_2^2 + v_{xz} s_1 \rho s_2\Big) ds \\ & + \int_t^\tau q(s) v_x s_1 dB_{s,1} + \int_t^\tau q(s) v_z s_2 dB_{s,2} + v(t, X_t^u, Z_t)\Big]. \end{aligned}$$

Since $E_{t,x,z}\Big[v(t, X_t^u, Z_t)\Big] = v(t, x, z)$, we have

$$\begin{aligned} E_{t,x,z}\Big[& \int_t^\tau q(s)\Big(f(s, X_s^u, C_s, Z_s) + v_s - \zeta v + v_x a + v_z b \\ & + \frac{1}{2} v_{xx} s_1 s_1^{\intercal} + \frac{1}{2} v_{zz} s_2^2 + v_{xz} s_1 \rho s_2\Big) ds\Big] \leq 0, \end{aligned}$$

in other words,

$$E_{t,x,z}\Big[\int_t^{\tau} q(s)\Big(f(s,X_s^u,C_s,Z_s)ds+\mathcal{L}v(s,X_s^u,Z_s)\Big)ds\Big]\leq 0, \tag{2.5}$$

where $\mathcal{L}=\frac{\partial}{\partial s}-\zeta+a\frac{\partial}{\partial x}+b\frac{\partial}{\partial z}+\frac{1}{2}s_1 s_1^{\intercal}\frac{\partial^2}{\partial x^2}+\frac{1}{2}s_2^2\frac{\partial^2}{\partial z^2}+s_1\rho s_2\frac{\partial^2}{\partial z\partial x}$ is a differential operator.

Assume that $E_{t,x,z}\Big[\int_t^{T} q(s)\Big|f(s,X_s^u,C_s,Z_s)ds+\mathcal{L}v(s,X_s^u,Z_s)\Big|ds\Big]<\infty$ if $t<T<\infty$. Then taking the limit as n goes to infinity of (2.5) and using the Dominated Convergence Theorem (see [23], p.27), we obtain, for any $t<\theta<T$,

$$E_{t,x,z}\Big[\int_t^{\theta} q(s)\Big(f(s,X_s^u,C_s,Z_s)ds+\mathcal{L}v(s,X_s^u,Z_s)\Big)ds\Big]\leq 0. \tag{2.6}$$

Note that f is continuous, $v\in C^{1,2,2}(\mathbb{D})\cap C(\bar{\mathbb{D}})$, and we divide equation (2.6) by $\theta-t$ and take the limit as θ decreases to t to get

$$f(t,x^u,c,z)+\mathcal{L}v(t,x^u,z)\leq 0, \qquad \forall (t,x,z)\in\mathbb{D}.$$

Also notice that this is true for any constant control $u\in\mathcal{U}^w[t,T]$ for all $t\in[0,T)$, then we reach

$$\sup_{u\in\mathbb{U}}\Big(f(t,x^u,c,z)+\mathcal{L}v(t,x^u,z)\Big)\leq 0, \qquad \forall (t,x,z)\in\mathbb{D}. \tag{2.7}$$

On the other hand, suppose that U^* is an optimal control then by definition of the value function

$$v(t,x,z)=E_{t,x,z}\Big[\int_t^{\theta} q(s)f(s,X_s^{U_s^*},C_s,Z_s)ds+q(\theta)v(\theta,X_\theta^{U_\theta^*},Z_\theta)\Big].$$

Using the same approach as before, we obtain

$$f(t,x^{U_t^*},c,z)+\mathcal{L}v(t,x^{U_t^*},z)=0, \qquad \forall (t,x,z)\in\mathbb{D}. \tag{2.8}$$

Thus, equations (2.7) and (2.8) suggest that the function v should satisfy the following equation

$$\sup_{u\in\mathbb{U}}\Big(f(t,x^u,c,z)+\mathcal{L}v(t,x^u,z)\Big)=0, \qquad \forall(t,x,z)\in\mathbb{D}.$$

Therefore, v satisfies (solves) the HJB equation with the boundary condition given below.

$$\begin{cases} \sup_{u\in\mathbb{U}}\Big(f(t,x^u,c,z)+\mathcal{L}v(t,x^u,z)\Big)=0, & \forall(t,x,z)\in\mathbb{D}, \\ v(T,x,z)=g(x,z), & \forall x>0,\ \forall z>0. \end{cases} \tag{2.9}$$

where $\mathcal{L}=\frac{\partial}{\partial t}-\zeta+a\frac{\partial}{\partial x}+b\frac{\partial}{\partial z}+\frac{1}{2}s_1s_1^{\mathsf{T}}\frac{\partial^2}{\partial x^2}+\frac{1}{2}s_2^2\frac{\partial^2}{\partial z^2}+s_1\rho s_2\frac{\partial^2}{\partial z\partial x}$ is a differential operator.

2.4 Verification Theorem

Once the HJB equation has been solved and we have a solution $v(t,x,z)$ we need to check that this solution is indeed the function defined by (2.3). The next theorem gives sufficient conditions that the solution to the HJB equation should satisfy to be the value function. The notation p and v are used to distinguish a solution of the HJB equation from the value function, respectively.

Theorem 4. ***(Verification Theorem)*** *Let a function $p\in C^{1,2,2}(\mathbb{D})\cap C(\bar{\mathbb{D}})$ and satisfy a quadratic growth condition such as $|p(t,x,z)|\le C(1+|x|^2+|z|^2),\ \forall(t,x,z)\in\bar{\mathbb{D}}$ for some constant $C>0$.*
(1) Suppose that

$$-\sup_{u\in\mathbb{U}}\Big(f(t,x^u,c,z)+\mathcal{L}p(t,x^u,z)\Big)\ge 0, \qquad \forall(t,x,z)\in\mathbb{D}, \tag{2.10}$$

$$p(T,x,z)\ge g(x,z), \qquad \forall x>0,\ \forall z>0, \tag{2.11}$$

where $\mathcal{L}=\frac{\partial}{\partial t}-\zeta+a\frac{\partial}{\partial x}+b\frac{\partial}{\partial z}+\frac{1}{2}s_1s_1^{\mathsf{T}}\frac{\partial^2}{\partial x^2}+\frac{1}{2}s_2^2\frac{\partial^2}{\partial z^2}+s_1\rho s_2\frac{\partial^2}{\partial z\partial x}$ is a differential operator. Then $p\ge v$ on $\bar{\mathbb{D}}$.
(2) Suppose that $p(T,x,z)=g(x,z)$ and there exists a measurable function $U^*(t,x,z)$,

where $(t, x, z) \in \mathbb{D}$*, and taking values in* $\mathbb{U}$ *such that*

$$-\sup_{u \in \mathbb{U}} \Big(f(t, x^u, c, z) + \mathcal{L}p(t, x^u, z) \Big) = f(t, x^{U_t^*}, c, z) + \mathcal{L}p(t, x^{U_t^*}, z) = 0. \tag{2.12}$$

Also assume that the stochastic differential equation

$$dX_s = a(X_s, U^*(s, X_s, Z_s))ds + s_1(X_s, U^*(s, X_s, Z_s))dB_{s,1}$$

admits a unique strong solution denoted by X_t^**, given an initial condition* $X_t = x$ *and the process* $U^* \in \mathcal{U}^w[0, T]$*. Then the function* p *is the value function* v *given by (2.3), in other words* $p = v$ *on* $\bar{\mathbb{D}}$*, and* U^* *is an optimal Markov control (at time* t *depends only on* X_t *and* Z_t*).*

Proof. (1) Assume that the function p satisfies the stated in the theorem assumptions. Let $\tau_n = \inf\{h \geq t \ : \ \int_t^h \Big(||q(s)p_x s_1||^2 + ||q(s)p_z s_2||^2\Big) ds \geq n\}$ be a sequence of stopping times diverging to infinity almost surely as n goes to infinity and let $\tau = \tau_n \wedge \theta$, then by the integral form of Ito's formula, we have for all U in $\mathcal{U}^w[t, \tau]$

$$q(\tau)p(\tau, X_\tau^{U_\tau}, Z_\tau) = p(t, X_t^{U_t}, Z_t) + \int_t^\tau q(s)\mathcal{L}p ds + \int_t^\tau q(s)p_x s_1 dB_{s,1} + \int_t^\tau q(s)p_z s_2 dB_{s,2}.$$

Taking expectations of both sides and using the fact that the stochastic integrals are martingales, we obtain

$$E_{t,x,z}[q(\tau)p(\tau, X_\tau^{U_\tau}, Z_\tau^{U_\tau})] = p(t, x, z) + E_{t,x,z}\Big[\int_t^\tau q(s)\mathcal{L}p ds\Big], \ \forall U \in \mathcal{U}^w[t, \tau].$$

Since p satisfies inequality (2.10), we have $q(s)f(s, X_s^{U_s}, C_s, Z_s) + q(s)\mathcal{L}p(s, X_s^{U_s}, Z_s) \leq 0$, $\forall U_s \in \mathbb{U}, \ s \in [t, T]$. Hence for all U in $\mathcal{U}^w[t, \tau]$,

$$E_{t,x,z}[q(\tau)p(\tau, X_\tau^{U_\tau}, Z_\tau^{U_\tau})] \leq p(t, x, z) - E_{t,x,z}\Big[\int_t^\tau q(s)f(s, X_s^{U_s}, C_s, Z_s)ds\Big]. \tag{2.13}$$

Note that $\Big|\int_t^\tau q(s)f(s, X_s^{U_s}, C_s, Z_s)ds\Big| \leq \int_t^T \Big|q(s)f(s, X_s^{U_s}, C_s, Z_s)\Big| ds$ and also recall that $|p(\tau, X_\tau^{U_\tau}, Z_\tau)| \leq C(1 + \sup_{s \in [t,T]} |X_s^{U_s}|^2 + \sup_{s \in [t,T]} |Z_s|^2)$ and the right hand sides of these inequalities are integrable (see Theorem 1). If we apply the Dominated Convergence

Theorem to take the limit in (2.13) as n goes to infinity then one can obtain

$$E_{t,x,z}[q(\theta)p(\theta, X_\theta^{U_\theta}, Z_\theta^{U_\theta})] \leq p(t,x,z) - E_{t,x,z}\Big[\int_t^\theta q(s)f(s, X_s^{U_s}, C_s, Z_s)ds\Big], \ \forall U \in \mathcal{U}^w[t,\theta].$$

By the assumption that the function p is continuous in $\bar{\mathbb{D}}$, by the condition (2.11) and by the Dominated Convergence Theorem, as θ goes to T, we get that for all $U \in \mathcal{U}^w[t,T]$ the following inequality holds

$$\begin{aligned} E_{t,x,z}[q(T)g(X_T^{U_T}, Z_T)] &\leq E_{t,x,z}[q(T)p(T, X_T^{U_T}, Z_T^{U_T})] \\ &\leq p(t,x,z) - E_{t,x,z}\Big[\int_t^T q(s)f(s, X_s^{U_s}, C_s, Z_s)ds\Big], \end{aligned}$$

or equivalently,

$$p(t,x,z) \geq E_{t,x,z}\Big[\int_t^T q(s)f(s, X_s^{U_s}, C_s, Z_s)ds + q(T)g(X_T^{U_T}, Z_T)\Big]. \tag{2.14}$$

Since (2.14) holds for all $U \in \mathcal{U}^w[t,T]$, we can conclude $p \geq v$.
*(2)*Adopting a way similar to the proof of part *(1)*, we consider a sequence of stopping times τ_n defined as in *(1)*, $\tau = \tau_n \wedge \theta$, and apply the Ito's formula to the function $q(\tau)p(\tau, X_\tau^{U^*_\tau}, Z_\tau)$. Then we take expectation on both sides and take limit as n tends to infinity. Then it is easy to see

$$E_{t,x,z}[q(\theta)p(\theta, X_\theta^{U^*_\tau}, Z_\theta)] = p(t,x,z) + E_{t,x,z}\Big[\int_t^\theta q(s)\mathcal{L}p(s, X_s^{U^*_s}, Z_s)ds\Big].$$

By (2.12) we have that

$$E_{t,x,z}[q(\theta)p(\theta, X_\theta^{U^*_\theta}, Z_\theta)] = p(t,x,z) - E_{t,x,z}\Big[\int_t^\theta q(s)f(s, X_s^{U^*_s}, C_s, Z_s)ds\Big]$$

and taking the limit as θ goes to T we obtain

$$p(t,x,z) = E_{t,x,z}\Big[\int_t^T q(s)f(s, X_s^{U^*_s}, C_s, Z_s)ds + q(T)g(X_T^{U^*_T}, Z_T)\Big] = w(t,x,z,U^*),$$

which means that $p(t,x,z) = w(t,x,z,U^*) \leq v(t,x,z)$. This, together with the result in part *(1)* implies that $p = v$ with U^* as an optimal Markov control. □

2.5 Maximizing the Utility of Consumption and Final Wealth

In this section we consider the problem of maximizing the utility of consumption and final wealth. The value function (assuming $\gamma \in (0,1)$) is

$$v_1(t,x,z) = \frac{1}{\gamma} \sup_{U \in \mathcal{U}^w[t,T]} E_{t,x,z}\Big[\int_t^T q(s)(C_s)^\gamma Z_s ds + q(T)(X_T^U)^\gamma Z_T\Big].$$

The corresponding HJB equation (2.9) for $t \in (0,T)$, $x > 0$, and $z > 0$ is

$$p_t - \zeta p + \beta z p_z + \frac{1}{2}(\sigma_2 z)^2 p_{zz} + \sup_{(\pi,\ c) \in \mathbb{R}^n \times [0,\infty)} \Big(\Big((1-\pi\mathbf{1})rx + \pi x\mu - c\Big)p_x + \frac{1}{2}||\pi\sigma_1 x||^2 p_{xx} + \pi\sigma_1\rho\sigma_2 xzp_{zx} + \frac{c^\gamma z}{\gamma}\Big) = 0 \qquad (2.15)$$

where for each $t \in (0,T)$ we have $\pi \in \mathbb{R}^n$ is a row vector that represents the fractions of wealth invested in the risky assets, $\mathbf{1} = (1,...,1)^\intercal \in \mathbb{R}^n$, $\mu \in \mathbb{R}^n$ is a column vector of expected returns, c is a scalar-valued consumption rate, and $\rho \in \mathbb{R}^n$ is the correlation column vector defined in section 2.1. The terminal and boundary conditions are

$$\begin{cases} p(T,x,z) = \frac{1}{\gamma}x^\gamma z, & x > 0,\ z > 0, \\ p(t,0,z) = 0, & t \in (0,T),\ z > 0, \\ p(t,x,0) = 0, & t \in (0,T),\ x > 0. \end{cases}$$

We now discuss the meaning of the terminal and boundary conditions. The terminal condition $p(T,x,z) = \frac{1}{\gamma}x^\gamma z$ means that if the investor starts trading at time T then there is no time for investment and the utility of his wealth is equal to the utility of the wealth he starts with. The boundary condition $p(t,0,z) = 0$ says that if the initial capital is zero then the value function is zero. The third condition $p(t,x,0) = 0$ implies that the investor is not interested in trading because the quality of goods he can buy is zero and therefore the value function is zero regardless of the amount of wealth he has. This case seems unrealistic and the probability of this happening is zero.

2.5.1 Solution to the HJB equation

We look for a solution in the form of $p(t,x,z) = \frac{1}{\gamma}x^{\gamma}zh(t)^{1-\gamma}$ where $h(t)$ is some positive function. This form of solution is suggested by the functions $f(t,x,c,z) = \frac{c^{\gamma}z}{\gamma}$ and $g(x,z) = \frac{x^{\gamma}z}{\gamma}$, $\gamma \in (0,1)$ defined in section 2.2. First, we evaluate the supremum

$$\begin{aligned}
&\sup_{(\pi,\ c)\in\mathbb{R}^n\times[0,\infty)} \bigg(\Big((1-\pi\mathbf{1})rx + \pi x\mu - c\Big)x^{\gamma-1}zh(t)^{1-\gamma} \\
&+ \frac{1}{2}||\pi\sigma_1 x||^2(\gamma-1)x^{\gamma-2}zh(t)^{1-\gamma} + \pi\sigma_1\rho\sigma_2 z x^{\gamma}h(t)^{1-\gamma} + \frac{c^{\gamma}z}{\gamma}\bigg) \\
&= x^{\gamma}zh(t)^{1-\gamma} \\
&\sup_{(\pi,\ c)\in\mathbb{R}^n\times[0,\infty)} \Big((1-\pi\mathbf{1})r + \pi\mu - \frac{c}{x} + \frac{1}{2}||\pi\sigma_1||^2(\gamma-1) + \pi\sigma_1\rho\sigma_2 + \frac{c^{\gamma}}{x^{\gamma}h(t)^{1-\gamma}\gamma}\Big) \\
&= x^{\gamma}zh(t)^{1-\gamma} \\
&\Big(\sup_{\pi\in\mathbb{R}^n}\Big((1-\pi\mathbf{1})r + \pi\mu + \frac{1}{2}||\pi\sigma_1||^2(\gamma-1) + \pi\sigma_1\rho\sigma_2\Big) + \sup_{c\in[0,\infty)}\Big(\frac{c^{\gamma}}{x^{\gamma}h(t)^{1-\gamma}\gamma} - \frac{c}{x}\Big)\Big).
\end{aligned}$$

Consider the functions

$$\begin{aligned}
g_1(\pi) &= (1-\pi\mathbf{1})r + \pi\mu + \frac{1}{2}||\pi\sigma_1||^2(\gamma-1) + \pi\sigma_1\rho\sigma_2, \\
g_2(c) &= \frac{c^{\gamma}}{x^{\gamma}h(t)^{1-\gamma}\gamma} - \frac{c}{x}.
\end{aligned}$$

The Hessian of the function $g_1(\pi)$ is $H(\pi) = \sigma_1\sigma_1^{\mathsf{T}}(\gamma-1)$ and it is negative definite. Also, $\frac{d^2g_2}{dc^2} = \frac{(\gamma-1)c^{\gamma-2}}{x^{\gamma}h(t)^{1-\gamma}} < 0$. Therefore, the maximum (π^*, c^*) is obtained from

$$\begin{aligned}
\nabla g_1(\pi) &= (\mu - r) + \sigma_1\sigma_1^{\mathsf{T}}\pi^{\mathsf{T}}(\gamma-1) + \sigma_1\rho\sigma_2 = 0, \\
\frac{dg_2}{dc} &= -\frac{1}{x} + \frac{c^{\gamma-1}}{x^{\gamma}h(t)^{1-\gamma}} = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\pi^* &= \frac{(r - \mu^{\mathsf{T}} - \rho^{\mathsf{T}}\sigma_1^{\mathsf{T}}\sigma_2)(\sigma_1\sigma_1^{\mathsf{T}})^{-1}}{(\gamma-1)}, \\
c^* &= \frac{x}{h(t)}.
\end{aligned}$$

Substituting π^* into the supremum over π we obtain

$$\Big((1 - \frac{(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\mathbf{1})r + \frac{(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\mu$$
$$+ \frac{1}{2}\Big|\Big|\frac{(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\sigma_1\Big|\Big|^2(\gamma-1) + \Big(\frac{(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\Big)\sigma_1\rho\sigma_2\Big)$$
$$= \Big(r - \frac{(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\mathbf{1}r + \frac{(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\mu$$
$$+ \frac{1}{2}\Big|\Big|\frac{(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\sigma_1\Big|\Big|^2(\gamma-1) + \Big(\frac{(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\Big)\sigma_1\rho\sigma_2$$
$$= \Big(r - \frac{||(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1^\intercal)^{-1}||^2}{2(\gamma-1)}\Big).$$

Now we substitute c^* into the supremum over c we have

$$-\frac{\frac{x}{h(t)}}{x} + \frac{\big(\frac{x}{h(t)}\big)^\gamma}{x^\gamma h(t)^{1-\gamma}\gamma} = -\frac{1}{h(t)} + \frac{1}{h(t)\gamma}.$$

Therefore, the HJB equation (2.15) becomes

$$\frac{(1-\gamma)}{\gamma}x^\gamma z h(t)^{-\gamma}h'(t) - \frac{\zeta}{\gamma}x^\gamma z h(t)^{1-\gamma} + \frac{\beta}{\gamma}z x^\gamma h(t)^{1-\gamma}$$
$$+ x^\gamma z h(t)^{1-\gamma}\Big(r - \frac{||(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1^\intercal)^{-1}||^2}{2(\gamma-1)} - \frac{1}{h(t)} + \frac{1}{h(t)\gamma}\Big) = 0. \tag{2.16}$$

Dividing both sides of the equation (2.16) by $\frac{(1-\gamma)}{\gamma}x^\gamma z h(t)^{-\gamma}$ we obtain

$$h'(t) - \frac{\zeta}{1-\gamma}h(t) + \frac{\beta}{1-\gamma}h(t)$$
$$+ \frac{\gamma}{1-\gamma}h(t)\Big(r - \frac{||(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1^\intercal)^{-1}||^2}{2(\gamma-1)} - \frac{1}{h(t)} + \frac{1}{h(t)\gamma}\Big) = 0$$

which is equivalent to

$$h'(t) - \frac{\zeta}{1-\gamma}h(t) + \frac{\beta}{1-\gamma}h(t)$$
$$+ \frac{\gamma r}{1-\gamma}h(t) + \frac{||(r-\mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1^\intercal)^{-1}||^2\gamma}{2(\gamma-1)^2}h(t) + \frac{\gamma}{\gamma-1} + \frac{1}{1-\gamma} = 0.$$

In other words,

$$h'(t) + y(t)h(t) + 1 = 0, \tag{2.17}$$

where $y(t)$ is defined as

$$y(t) = \frac{-\zeta + \beta + \gamma r}{1-\gamma} + \frac{||(r - \mu^\intercal - \rho^\intercal \sigma_1^\intercal \sigma_2)(\sigma_1^\intercal)^{-1}||^2 \gamma}{2(\gamma-1)^2}. \tag{2.18}$$

Since $y(t)$ is continuous, the equation (2.17) with the terminal condition $h(T) = 1$ admits the unique solution

$$h_1(t) = e^{\int_t^T y(\tau)d\tau} + \int_t^T e^{\int_t^\tau y(q)dq} d\tau.$$

The function $h_1(t)$ is in $C^1([0,T])$ and is positive. Therefore, the solution to the HJB equation (2.15) is

$$p_1(t,x,z) = \frac{x^\gamma z}{\gamma}\Big(e^{\int_t^T y(\tau)d\tau} + \int_t^T e^{\int_t^\tau y(q)dq} d\tau\Big)^{1-\gamma}. \tag{2.19}$$

2.5.2 Verification

The solution p_1 to equation (2.15) is a $C^{1,2,2}(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ function. The quadratic growth condition is also satisfied as one can see below. We see that the function $h_1(s) > 0$ is bounded on the interval $[0,T]$ and since $0 < \gamma < 1,\ x \geq 0,\ z \geq 0$ we have $x^\gamma < 1 + x$ which implies

$$x^\gamma z \leq z + xz < 1 + xz + z^2 < 1 + 2x^2 + 2z^2 + z^2 < 3(1 + x^2 + z^2).$$

It should also be verified that the obtained control is admissible. The wealth process (1.16) when the control $\{\Pi_s^*, C_s^*\}_{s\in[t,T]}$ is used, satisfies the following stochastic differential equation

$$dX_s^* = \Big((1-\Pi_s^*)rX_s^* + \Pi_s^* X_s^* \mu - C_s^*\Big)ds + \Pi_s^* \sigma_1 X_s^* dB_{s,1}$$

$$
\begin{aligned}
=&\Big((1-\frac{(r-\mu^\intercal-\rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\mathbf{1})r+\frac{(r-\mu^\intercal-\rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\mu\\
&-\frac{1}{h_1(s)}\Big)X_s^*ds+\frac{(r-\mu^\intercal-\rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)}\sigma_1X_s^*dB_{s,1}\\
=&\Big(r-\frac{(r-\mu^\intercal-\rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}(r-\mu)}{(\gamma-1)}-\frac{1}{h_1(s)}\Big)X_s^*ds\\
&+\frac{(r-\mu^\intercal-\rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1^\intercal)^{-1}}{(\gamma-1)}X_s^*dB_{s,1}
\end{aligned}
$$

which is a Geometric Brownian motion (assumption **B(4)** in Section 1.3 is satisfied). Since X_s^* is continuous then by Theorem 2 X_s^* is $\mathcal{F}_{s,t}$-progressively measurable and **B(3)** in section 1.3 is fulfilled. From the fact that $h_1(s)>K>0$ for all $s\in[0,T]$ for some constant $K>0$ and $f(t,x,z,c)=\frac{c^\gamma z}{\gamma}<\frac{3}{\gamma}(1+c^2+z^2)$, we have $\frac{1}{\gamma}(C_s)^\gamma Z_s=\frac{1}{\gamma}\left(\frac{X_s}{h_1(s)}\right)^\gamma Z_s<\frac{3}{\gamma}(1+\frac{1}{K^2}(X_s)^2+(Z_s)^2)$ and, thus, by Theorem 1 function f is integrable. Similarly, function g is also integrable. This implies that **B(5)** in section 1.3 is satisfied. Therefore, the control is admissible.

Since the solution $p_1(t,x,z)$ to (2.15) satisfies the assumptions of the verification theorem then $v_1=p_1$ and the optimal control is

$$
\Pi_s^*=\frac{(r-\mu^\intercal-\rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)},\ C_{s,1}^*=\frac{X_s}{h_1(s)}. \tag{2.20}
$$

2.6 Maximizing the Utility of Consumption

In this section, we will consider a problem of maximizing the utility of consumption. The value function is

$$
v_2(t,x,z)=\frac{1}{\gamma}\sup_{U\in\mathcal{U}^w[t,T]}E_{t,x,z}\Big[\int_t^T q(s)(C_s)^\gamma Z_s ds\Big].
$$

The corresponding HJB equation (2.9) for $t\in(0,T)$, $x>0$ and $z>0$ is

$$
\begin{aligned}
p_t-\zeta p+\beta zp_z+\frac{1}{2}(\sigma_2 z)^2p_{zz}+\sup_{(\pi,\ c)\in\mathbb{R}^n\times[0,\infty)}&\Big(\Big((1-\pi\mathbf{1})rx+\pi x\mu-c\Big)p_x\\
&+\frac{1}{2}||\pi\sigma_1x||^2p_{xx}+\pi\sigma_1\rho\sigma_2xzp_{zx}+\frac{c^\gamma z}{\gamma}\Big)=0.
\end{aligned} \tag{2.21}
$$

The dimensions and meaning of all the variables and parameters is the same as in section 2.5. The terminal and boundary conditions are

$$\begin{cases} p(T,x,z) = 0, & x > 0,\ z > 0, \\ p(t,0,z) = 0, & t \in (0,T),\ z > 0, \\ p(t,x,0) = 0, & t \in (0,T),\ x > 0. \end{cases}$$

The meaning of the terminal and boundary conditions is the same as in section 2.5.

2.6.1 Solution to the HJB equation

The calculations are the same as in section 2.5 and the only difference is that the terminal condition for the function $h(t)$ is given as $h(T) = 0$. Therefore, we have

$$h'(t) + y(t)h(t) + 1 = 0, \tag{2.22}$$

where $y(t)$ is defined by (2.18).
It is easy to verify that the solution to equation (2.22) with the terminal condition $h(T) = 0$ is $h_2(t) = \displaystyle\int_t^T e^{\int_t^\tau y(q)dq} d\tau$.
The function $h_2(t)$ is in $C^1([0,T])$ and is positive. Therefore, the solution to the HJB equation (2.21) is given as

$$p_2(t,x,z) = \frac{x^\gamma z}{\gamma}\Big(\int_t^T e^{\int_t^\tau y(q)dq} d\tau\Big)^{1-\gamma}. \tag{2.23}$$

2.6.2 Verification

The verification is mostly identical to the one in section 2.5 but the function $h_2(s)$ goes to 0 as s goes to T due to continuity. Hence the condition $h_2(s) > K > 0$ is not satisfied. To verify that the function f is integrable, let us denote $A_1 \triangleq r - \frac{(r-\mu^\intercal - \rho^\intercal \sigma_1^\intercal \sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}(r-\mu)}{(\gamma-1)}$ and $A_2 \triangleq \frac{(r-\mu^\intercal - \rho^\intercal \sigma_1^\intercal \sigma_2)(\sigma_1^\intercal)^{-1}}{(\gamma-1)}$, then the equation for the wealth process (assuming $s < T$) becomes

$$dX_s^* = \Big(A_1 - \frac{1}{h_2(s)}\Big)X_s^* ds + A_2 X_s^* dB_{s,1}.$$

The solution to this stochastic differential equation with the initial condition $X_0 = x$ is

$$\begin{aligned} X_s^* =& xe^{\int_0^s A_2 dB_{\tau,1} + \int_0^s \left((A_1 - \frac{1}{h_2(\tau)}) - \frac{1}{2}A_2^2 \right) d\tau} \\ =& xe^{\int_0^s A_2 dB_{\tau,1} + \int_0^s \left(A_1 - \frac{1}{2}A_2^2 \right) d\tau} \cdot e^{-\int_0^s \frac{1}{h_2(\tau)} d\tau}. \end{aligned}$$

Thus,

$$C_s^* = \frac{X_s^*}{h_2(s)} = xe^{\int_0^s A_2 dB_{\tau,1} + \int_0^s \left(A_1 - \frac{1}{2}A_2^2 \right) d\tau} \cdot \frac{e^{-\int_0^s \frac{1}{h_2(\tau)} d\tau}}{h_2(s)} = Y_s \; \frac{e^{-\int_0^s \frac{1}{h_2(\tau)} d\tau}}{h_2(s)}$$

where $Y_s = xe^{\int_0^s A_2 dB_{\tau,1} + \int_0^s \left(A_1 - \frac{1}{2}A_2^2 \right) d\tau}$ is a Geometric Brownian motion with the initial condition $Y_0 = x$. We will show that the term $\frac{1}{h_2(s)} \cdot e^{-\int_0^s \frac{1}{h_2(\tau)} d\tau}$ is bounded on $[0, T]$.

Since the function $y(s),\ s \in [0, T]$ is continuous, it attains its minimum and maximum. That is why, for some $\varepsilon > 0$ we may denote $m = \min(\min_{s\in[0,T]} y(s), -\varepsilon)$ and $M = \max(\max_{s\in[0,T]} y(s), \varepsilon)$. The following inequalities hold

$$\int_s^T e^{m(\tau - s)} d\tau \le h_2(s) = \int_s^T e^{\int_s^\tau y(q)dq} d\tau \le \int_s^T e^{M(\tau - s)} d\tau$$

and, hence,

$$\frac{1}{m}(e^{m(T-s)} - 1) \le h_2(s) \le \frac{1}{M}(e^{M(T-s)} - 1).$$

Using the above inequalities, we obtain

$$\frac{e^{-\int_0^s \frac{1}{h_2(\tau)} d\tau}}{h_2(s)} \le \frac{e^{-\int_0^s \frac{1}{h_2(\tau)} d\tau}}{\frac{1}{m}(e^{m(T-s)} - 1)} \le \frac{e^{-\int_0^s \frac{M}{e^{M(T-l)} - 1} dl}}{\frac{1}{m}(e^{m(T-s)} - 1)} = \frac{e^{Ms + \ln(\frac{1 - e^{M(T-s)}}{1 - e^{MT}})}}{\frac{1}{m}(e^{m(T-s)} - 1)}$$

and after simlification, the last term in the above inequality equals

$$me^{Ms} \frac{\frac{1 - e^{M(T-s)}}{1 - e^{MT}}}{e^{m(T-s)} - 1} = \frac{me^{Ms}}{1 - e^{MT}} \frac{1 - e^{M(T-s)}}{e^{m(T-s)} - 1} = \frac{me^{Ms}}{e^{MT} - 1} \frac{1 - e^{M(T-s)}}{1 - e^{m(T-s)}}$$

which goes to $\frac{me^{MT}}{e^{MT}-1}\frac{M}{m}$, or after simplification $\frac{Me^{MT}}{e^{MT}-1}$ as s approaches T from the left.

From the fact that function f satisfies $f(t,x,z,c)=\frac{c^\gamma z}{\gamma}<\frac{3}{\gamma}(1+c^2+z^2)$, we have the inequality $\frac{1}{\gamma}(C_s)^\gamma\cdot Z_s=\frac{1}{\gamma}\left(\frac{X_s}{h_2(s)}\right)^\gamma Z_s<\frac{3}{\gamma}(1+K^2(Y_s)^2+(Z_s)^2)$, where $K=\frac{Me^{MT}}{e^{MT}-1}$. Thus, by Theorem 1 the function f is integrable. Therefore, $v_2=p_2$. The optimal control is

$$\Pi_s^*=\frac{(r-\mu^\intercal-\rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{(\gamma-1)},\ C_{s,2}^*=\frac{X_s}{h_2(s)}. \tag{2.24}$$

2.7 Maximizing the Utility of Final Wealth

In this section, we look at the problem of maximizing the utility of final wealth. The value function is

$$v_3(t,x,z)=\frac{1}{\gamma}\sup_{U\in\mathcal{U}^w[t,T]}E_{t,x,z}[q(T)(X_T^U)^\gamma Z_T].$$

The corresponding HJB equation (2.9) for $t\in(0,T)$, $x>0$ and $z>0$ is

$$\begin{aligned}p_t-\zeta p+\beta z p_z+\frac{1}{2}(\sigma_2 z)^2p_{zz}+\sup_{(\pi,\ c)\in\mathbb{R}^n\times[0,\infty)}\Big(\Big((1-\pi\mathbf{1})rx+\pi x\mu-c\Big)p_x\\ +\frac{1}{2}||\pi\sigma_1 x||^2p_{xx}+\pi\sigma_1\rho\sigma_2 xzp_{zx}\Big)=0.\end{aligned} \tag{2.25}$$

The dimensions and meaning of all the variables and parameters is the same as in section 2.5. The terminal and boundary conditions are

$$\begin{cases}p(T,x,z)=\frac{1}{\gamma}x^\gamma z, & x>0,\ z>0,\\ p(t,0,z)=0, & t\in(0,T),\ z>0,\\ p(t,x,0)=0, & t\in(0,T),\ x>0.\end{cases}$$

The meaning of the terminal and boundary conditions is the same as in section 2.5.

2.7.1 Solution to the HJB equation

We may look for a solution in the form of $p(t,x,z)=\frac{1}{\gamma}x^\gamma zh(t)^{1-\gamma}$ where $h(t)$ is some positive function. This form of solution is suggested by the functions $f(t,x,c,z)=\frac{c^\gamma z}{\gamma}$ and $g(x,z)=\frac{x^\gamma z}{\gamma}$, $\gamma\in(0,1)$ defined in section 2.2. First we evaluate the supremum in the equation (2.25)

$$
\begin{aligned}
&\sup_{(\pi,\ c)\in\mathbb{R}^n\times[0,\infty)} \Big(\Big((1-\pi\mathbf{1})rx+\pi x\mu - c\Big)x^{\gamma-1}zh(t)^{1-\gamma} \\
&+\frac{1}{2}||\pi\sigma_1 x||^2(\gamma-1)x^{\gamma-2}zh(t)^{1-\gamma}+\pi\sigma_1\rho\sigma_2 z x^{\gamma}h(t)^{1-\gamma}\Big) \\
&= x^{\gamma}zh(t)^{1-\gamma}\sup_{(\pi,\ c)\in\mathbb{R}^n\times[0,\infty)}\Big((1-\pi\mathbf{1})r+\pi\mu-\frac{c}{x}+\frac{1}{2}||\pi\sigma_1||^2(\gamma-1)+\pi\sigma_1\rho\sigma_2\Big) \\
&= x^{\gamma}zh(t)^{1-\gamma}\Big(\sup_{\pi\in\mathbb{R}^n}\Big((1-\pi\mathbf{1})r+\pi\mu+\frac{1}{2}||\pi\sigma_1||^2(\gamma-1)+\pi\sigma_1\rho\sigma_2\Big)+\sup_{c\in[0,\infty)}\Big(-\frac{c}{x}\Big)\Big).
\end{aligned}
$$

Consider the functions

$$
\begin{aligned}
g_1(\pi) &= (1-\pi\mathbf{1})r+\pi\mu+\frac{1}{2}||\pi\sigma_1||^2(\gamma-1)+\pi\sigma_1\rho\sigma_2, \\
g_2(c) &= -\frac{c}{x}.
\end{aligned}
$$

The Hessian of $g_1(\pi)$ is $H(\pi)=\sigma_1\sigma_1^{\intercal}(\gamma-1)$ and is negative definite. Also, the maximum of $g_2(c)$ is reached when $c=0$. Therefore, the maximum of $g_1(\pi)$ is obtained from

$$\nabla g_1(\pi)=(\mu-r)+\sigma_1\sigma_1^{\intercal}\pi^{\intercal}(\gamma-1)+\sigma_1\rho\sigma_2=0.$$

Thus,

$$
\begin{aligned}
\pi^* &= \frac{(r-\mu^{\intercal}-\rho^{\intercal}\sigma_1^{\intercal}\sigma_2)(\sigma_1\sigma_1^{\intercal})^{-1}}{(\gamma-1)}, \\
c^* &= 0.
\end{aligned}
$$

Substituting (π^*, c^*) into the suprema we obtain

$$
\begin{aligned}
&\Big((1-\frac{(r-\mu^{\intercal}-\rho^{\intercal}\sigma_1^{\intercal}\sigma_2)(\sigma_1\sigma_1^{\intercal})^{-1}}{(\gamma-1)}\mathbf{1})r+\frac{(r-\mu^{\intercal}-\rho^{\intercal}\sigma_1^{\intercal}\sigma_2)(\sigma_1\sigma_1^{\intercal})^{-1}}{(\gamma-1)}\mu \\
&+\frac{1}{2}\Big|\Big|\frac{(r-\mu^{\intercal}-\rho^{\intercal}\sigma_1^{\intercal}\sigma_2)(\sigma_1\sigma_1^{\intercal})^{-1}}{(\gamma-1)}\sigma_1\Big|\Big|^2(\gamma-1)+\Big(\frac{(r-\mu^{\intercal}-\rho^{\intercal}\sigma_1^{\intercal}\sigma_2)(\sigma_1\sigma_1^{\intercal})^{-1}}{(\gamma-1)}\Big)\sigma_1\rho\sigma_2\Big)
\end{aligned}
$$

$$= \Big(r - \frac{(r-\mu^\top - \rho^\top\sigma_1^\top\sigma_2)(\sigma_1\sigma_1^\top)^{-1}}{(\gamma-1)}\mathbf{1}r + \frac{(r-\mu^\top - \rho^\top\sigma_1^\top\sigma_2)(\sigma_1\sigma_1^\top)^{-1}}{(\gamma-1)}\mu$$

$$+ \frac{1}{2}\Big|\Big|\frac{(r-\mu^\top - \rho^\top\sigma_1^\top\sigma_2)(\sigma_1\sigma_1^\top)^{-1}}{(\gamma-1)}\sigma_1\Big|\Big|^2(\gamma-1) + \Big(\frac{(r-\mu^\top - \rho^\top\sigma_1^\top\sigma_2)(\sigma_1\sigma_1^\top)^{-1}}{(\gamma-1)}\Big)\sigma_1\rho\sigma_2$$

$$= \Big(r - \frac{||(r-\mu^\top - \rho^\top\sigma_1^\top\sigma_2)(\sigma_1^\top)^{-1}||^2}{2(\gamma-1)}\Big).$$

Therefore, the HJB equation (2.25) becomes

$$\frac{(1-\gamma)}{\gamma}x^\gamma z h(t)^{-\gamma}h'(t) - \frac{\zeta}{\gamma}x^\gamma z h(t)^{1-\gamma} + \frac{\beta}{\gamma}zx^\gamma h(t)^{1-\gamma}$$
$$+ x^\gamma z h(t)^{1-\gamma}\Big(r - \frac{||(r-\mu^\top - \rho^\top\sigma_1^\top\sigma_2)(\sigma_1^\top)^{-1}||^2}{2(\gamma-1)}\Big) = 0. \tag{2.26}$$

Dividing both sides of the equation (2.26) by $\frac{(1-\gamma)}{\gamma}x^\gamma z h(t)^{-\gamma}$, we obtain

$$h'(t) - \frac{\zeta}{1-\gamma}h(t) + \frac{\beta}{1-\gamma}h(t)$$
$$+ \frac{\gamma}{1-\gamma}h(t)\Big(r - \frac{||(r-\mu^\top - \rho^\top\sigma_1^\top\sigma_2)(\sigma_1^\top)^{-1}||^2}{2(\gamma-1)}\Big) = 0$$

equivalently,

$$h'(t) - \frac{\zeta}{1-\gamma}h(t) + \frac{\beta}{1-\gamma}h(t)$$
$$+ \frac{\gamma r}{1-\gamma}h(t) + \frac{||(r-\mu^\top - \rho^\top\sigma_1^\top\sigma_2)(\sigma_1^\top)^{-1}||^2\gamma}{2(\gamma-1)^2}h(t) = 0.$$

Hence

$$h'(t) + y(t)h(t) = 0, \tag{2.27}$$

where $y(t)$ is defined by (2.18).

The solution to (2.27) with the terminal condition $h(T) = 1$ is $h_3(t) = e^{\int_t^T y(\tau)d\tau}$. The function $h_3(t)$ is in $C^1([0,T])$ and is positive. Therefore, the solution to the HJB equation (2.25) is

$$p_3(t,x,z) = \frac{x^\gamma z}{\gamma}\Big(e^{\int_t^T y(\tau)d\tau}\Big)^{1-\gamma}. \tag{2.28}$$

2.7.2 Verification

For this problem, the verification is quite similar to the one already done in section 2.5 and $v_3 = p_3$. The optimal control here is

$$\Pi_s^* = \frac{(r - \mu^\intercal - \rho^\intercal \sigma_1^\intercal \sigma_2)(\sigma_1 \sigma_1^\intercal)^{-1}}{(\gamma - 1)}, \; C_{s,3}^* = 0. \tag{2.29}$$

2.8 Analysis of the Results and Numerical Experiments

Here we analyse and interpret the results obtained in sections 2.5, 2.6, and 2.7. Throughout this section, we assume that all the parameters are constant, $t < T$, and the investor has the utility function $U(C) = \frac{C^\gamma}{\gamma}, \; \gamma \in (0,1)$ (the utulity function assumed in sections 2.5, 2.6, and 2.7). From (2.20), (2.24), and (2.29) we have the following claim.

Proposition 2. *The optimal portfolio is*

$$\Pi_t^* = \frac{(r-\mu)^\intercal (\sigma_1 \sigma_1^\intercal)^{-1} v_x}{X_t v_{xx}} - \frac{\rho^\intercal \sigma_1^{-1} \sigma_2 Z_t v_{zx}}{X_t v_{xx}}$$

regardless of the problem (maximizing both the utility of consumption and final wealth, only the utility of consumption, or only the utility of final wealth).

There are a few things to discuss about this proposition. By comparison with the solution of the classical model (section 1.4), one should notice that the second fraction is the effect of the randomness in the utility. For $n = 1$ (there is only one risky asset), if $\mu < r$, then the investor will short (borrow and sell) some of the risky assets, and if $\mu > r$, then he will invest in the risky assets.

We next show that the Mutual Fund Theorem holds.

Theorem 5. *The optimal portfolio is made of three mutual funds. First fund $\Phi_1(t)$ consists of the risk-free asset and the other two funds $\Phi_2(t) = (r-\mu)^\intercal (\sigma_1 \sigma_1^\intercal)^{-1}$, $\Phi_3(t) = \rho^\intercal \sigma_1^{-1}$ include risky assets. The vectors $\Phi_2(t), \Phi_3(t)$ represent the second and third portfolio's weights of the risky assets at time t. The optimal allocation of the wealth in each fund is given by $\lambda_2 = \frac{v_x}{X_t v_{xx}}, \lambda_3 = \frac{\sigma_2 Z_t v_{zx}}{X_t v_{xx}}$, and $\lambda_1 = 1 - \lambda_2 - \lambda_3$.*

Proof. The proof follows immediately from the obtained optimal portfolio (see Proposition 2). □

The mutual fund theorem above states that the investor who wants to maximize his expected utility (2.2) will be indifferent between choosing from the linear combination of $n+1$ assets or a linear combination of the three mutual funds. The first and second funds are the same as those in the classical problem whereas the third fund arises from the correlation between $B_{t,1}$ and $B_{t,2}$.

Let $n = 1$, and we realize the optimal portfolio is $\Pi_t^* = \frac{(r-\mu)}{\sigma_1^2(\gamma-1)} - \frac{\rho\sigma_2}{\sigma_1(\gamma-1)}$. Since $\frac{dZ_t}{Z_t}\frac{dS_t}{S_t} = \rho\sigma_1\sigma_2 dt$, consider Π_t^* as a function of ρ, then $\frac{d\Pi_t^*}{d\rho} = \frac{\sigma_2}{\sigma_1(1-\gamma)} > 0$ and we see that the higher the correlation between relative changes in asset price and utility randomness process is, the more the investor should invest in the asset. Also, comparing the behaviors of the investor in the classical case and the investor who takes into account technological progress, product improvements and other factors, the latter is investing more in the risky asset when $\rho > 0$ and less when $\rho < 0$.

The optimal portfolio does not include the drift parameter β of the utility randomness process. This means that the portion of the wealth invested in the risky asset does not depend on how fast new products come into the market. It only depends on how volatile these changes are which is characterized by the variance parameter σ_2. Also if $\rho > 0$ ($\rho < 0$), then the larger the value of σ_2 is, the more (less) shares of the risky asset should be included in the optimal portfolio.

Proposition 3. *The highest satisfaction the investor can acquire is from maximizing both the utility of consumption and final wealth. In other words,* $v_1(t,x,z) \geq v_2(t,x,z)$, *and* $v_1(t,x,z) \geq v_3(t,x,z)$, *for all* $(t,x,z) \in \mathbb{D}$.

To see why proposition 3 is true one needs to compare (2.19), (2.23), and (2.28).

Remark 7. *In general, similar conclusions comparing* v_2 *and* v_3 *(value functions in the problems of maximizing the utility of consumption and final wealth, respectively) cannot be made because* $e^{\int_t^T y(\tau)d\tau} - \int_t^T e^{\int_t^\tau y(q)dq} d\tau$ *can be positive or negative depending on the function* y. *For example, if* $y = 0$, *then* $e^{\int_t^T y(\tau)d\tau} = 1$ *and* $\int_t^T e^{\int_t^\tau y(q)dq} d\tau = T - t$ *and thus we can clearly see that* $v_2 > v_3$, *if* $T - t > 1$ *and* $v_3 > v_2$, *if* $T - t < 1$.

Proposition 4. *If the utility uncertainty and the market risk are uncorrelated, then the investor who takes into account the technological progress invests as much in the risky* *asset as the investor who does not consider that. The only difference is in their optimal consumption and the value function.*

This result readily follows from the definition of $y(t)$ in (2.18) and the obtained optimal portfolio. One should also notice that the utility discount rate ζ plays the role opposite to that of the expected instantaneous growth rate β in the utility.

Next we state a rather obvious observation about the dependence of the value functions on the technological progress.

Proposition 5. *The more rapid the technological progress is, the higher the value functions $v_1, v_2,$ and v_3 are.*

To verify this proposition we need to find how sensitive the value functions are to the change in the parameter β (the higher the value of β is, the faster products imporve in the market). Thus, we may consider the value functions as functions of β. Since the signs of $\frac{dv_1}{d\beta}$, $\frac{dv_2}{d\beta}$, and $\frac{dv_3}{d\beta}$ are the same as the signs of $\frac{dh_1}{d\beta}$, $\frac{dh_2}{d\beta}$, and $\frac{dh_3}{d\beta}$ respectively, we consider $h_1,\ h_2$, and h_3 as functions of β and recall that $h_2 = \frac{e^{y(T-t)}-1}{y},\ h_3 = e^{y(T-t)}$, and $\frac{dy}{d\beta} = \frac{1}{1-\gamma}$. Then we reach

$$\frac{dh_2}{d\beta} = \frac{dh_2}{dy}\frac{dy}{d\beta} = \frac{(T-t)e^{y(T-t)}y - (e^{y(T-t)}-1)}{y^2}\frac{1}{1-\gamma} = \frac{e^{y(T-t)}(y(T-t)-1)+1}{(1-\gamma)y^2},$$
$$\frac{dh_3}{d\beta} = \frac{dh_3}{dy}\frac{dy}{d\beta} = \frac{T-t}{1-\gamma}e^{y(T-t)}.$$

From the above expressions, we see that $\frac{dh_2}{d\beta} \geq 0$ and $\frac{dh_3}{d\beta} > 0$. Indeed, to show that $\frac{dh_2}{d\beta} \geq 0$, it is enough to check that the minimum of $f(x,y) = e^{xy}(xy-1)+1,\ y > 0$ is equal to 0. We have $\frac{\partial f}{\partial x} = xy^2e^{xy} = 0$ if $x = 0$. Let $z = xy$ then $f(x,y) = g(z) = e^z(z-1)+1$. Since $\frac{dg}{dz} = ze^z = 0$ if $z = 0$ and $\frac{dg}{dz}$ is positive when $z > 0$ and negative when $z < 0$. So we have that the minimum of $g(z)$ is reached at $z = 0$. Therefore, the minimum of $f(x,y)$ is equal to zero. Since $h_1 = h_2 + h_3$ we have $\frac{dh_1}{d\beta} > 0$ as well.

Although the signs of the second derivatives for the functions $v_1,\ v_2$ are quite difficult to determine in general, for the function v_3 it is fairly easy. Indeed, from the above expression for the derivative of h_3 with respect to β and from (2.28), we have

$$\frac{dv_3}{d\beta} = \frac{x^\gamma z}{\gamma}(T-t)\left(e^{y(T-t)}\right)^{1-\gamma}$$

and after differentiating again, we obtain

$$\frac{d^2 v_3}{d\beta^2} = \frac{x^\gamma z}{\gamma}(T-t)^2\left(e^{y(T-t)}\right)^{1-\gamma}.$$

Since the second derivative with respect to β is positive, the function v_3 is convex in β. Therefore, we have the following proposition.

Proposition 6. *When the objective is to maximize the expected utility of final wealth, the corresponding value function v_3 is convex in β.*

This result means that the value function increases at a higher rate as the parameter β increases. For example, to double the agent's expected utility, the expected instanteneous growth rate β has to increase by less then its current value. This makes sense because, as it was mentioned in the introduction, the technological progress is assumed to increase the utility exponentially.

Proposition 7. *If $y > 0$, then as the terminal time increases, the value functions grow at an exponential rate.*

To verify this proposition, one needs to consider the value functions as functions of the final time T. Since $h_1 = h_2 + h_3$, we find that the derivatives are positive

$$\frac{dh_2}{dT} = e^{y(T-t)},$$
$$\frac{dh_3}{dT} = ye^{y(T-t)}.$$

For example, if the agent wants to double the value of his value function obtained over time interval $(0, T)$, then he needs to increase T by the amount smaller than T.

Remark 8. *If $y \leq 0$, the proposition 7 is not true in general.*

Next, to simplify the analysis, we assume that $n = 1$, then we have

Proposition 8. *If $\mu - r \geq 0$ and $\rho > 0$, then the value functions are increasing in the volatility constant σ_2 and correlation ρ.*

For the same reason as above, it is enough to determine the signs of the derivatives of $h_1,\ h_2$, and h_3 with respect to σ_2 and ρ. Recall that

$$\frac{dy}{d\sigma_2} = \frac{(\mu - r + \rho\sigma_1\sigma_2)\rho\gamma}{(\gamma-1)^2\sigma_1},$$
$$\frac{dy}{d\rho} = \frac{(\mu - r + \rho\sigma_1\sigma_2)\sigma_2\gamma}{(\gamma-1)^2\sigma_1}.$$

Then

$$\frac{dh_2}{d\sigma_2} = \frac{dh_2}{dy}\frac{dy}{d\sigma_2} = \frac{(T-t)e^{y(T-t)}y - (e^{y(T-t)}-1)}{y^2} \cdot \frac{(\mu - r + \rho\sigma_1\sigma_2)\rho\gamma}{(\gamma-1)^2\sigma_1},$$
$$\frac{dh_3}{d\sigma_2} = \frac{dh_3}{dy}\frac{dy}{d\sigma_2} = \frac{(T-t)(\mu - r + \rho\sigma_1\sigma_2)\rho\gamma}{(\gamma-1)^2\sigma_1} \cdot e^{y(T-t)},$$
$$\frac{dh_2}{d\rho} = \frac{dh_2}{dy}\frac{dy}{d\rho} = \frac{(T-t)e^{y(T-t)}y - (e^{y(T-t)}-1)}{y^2} \cdot \frac{(\mu - r + \rho\sigma_1\sigma_2)\sigma_2\gamma}{(\gamma-1)^2\sigma_1},$$
$$\frac{dh_3}{d\rho} = \frac{dh_3}{dy}\frac{dy}{d\rho} = \frac{(T-t)(\mu - r + \rho\sigma_1\sigma_2)\sigma_2\gamma}{(\gamma-1)^2\sigma_1} \cdot e^{y(T-t)}.$$

Since $h_1 = h_2 + h_3$, we also have $\frac{dh_1}{d\sigma_2} > 0$ and $\frac{dh_1}{d\rho} > 0$. The assumption that $\mu - r \geq 0$ is not unrealistic, because this is usually the case (the expected return on a risky asset is usually higher than the riskless interest rate). The fact that under the assumptions that were made the value functions are increasing in σ_2 makes sense, because in this case there is a high probability that the products are to be improved significantly and this, in turn, will increase the agent's satisfaction. It is worth noting that this interpretation is in accordance with that of an option price sensitivity to the change in the underlying stock volatility ('vega' in greeks). It is well known that the larger the volatility of the underlying is, the higher the option price becomes because in this case the probability that the option will be in-the-money is bigger than for the option written on a stock with small volatility.

Remark 9. *If $\rho < 0$, the derivatives can take on positive or negative values depending on the values of the other parameters. As can be easily verified, the necessary and sufficient condition for the value functions to be increasing in σ_2 is $\rho^2\sigma_1\sigma_2 > (r-\mu)\rho$ and to be increasing in ρ is $\rho\sigma_1\sigma_2 > r - \mu$.*

Proposition 9. *For a given wealth process X_t, the optimal rates of the consumption per unit time are decreasing functions of β. Furthermore, if $\mu - r \geq 0$ and $\rho > 0$, then they are also decreasing both in σ_2 and ρ.*

As it was found in (2.20), (2.24) the optimal rates of the consumption per unit time are $C^*_{t,i} = \frac{X_t}{h_i(t)}$, $i = 1, 2$ depending on the objective. We can use the evaluated derivatives of the functions h_i, $\frac{dh_i}{d\beta}$, $\frac{dh_i}{d\sigma_2}$, $i = 1, 2$ to find the derivatives of the functions $C^*_{t,i}$, $\frac{dC^*_{t,i}}{d\beta}$, $\frac{dC^*_{t,i}}{d\sigma_2}$. Indeed,

$$\frac{dC^*_{t,i}}{d\beta} = -\frac{X_t}{(h_i(t))^2}\frac{dh_i}{d\beta},$$
$$\frac{dC^*_{t,i}}{d\sigma_2} = -\frac{X_t}{(h_i(t))^2}\frac{dh_i}{d\sigma_2},$$
$$\frac{dC^*_{t,i}}{d\rho} = -\frac{X_t}{(h_i(t))^2}\frac{dh_i}{d\rho}.$$

Therefore, under the assumptions that were made, the derivatives above are negative. This proposition agrees with the propositions 2, 5, and 6, because it says that when the products improve fast and the relative change in the asset's price is positively correlated with that of the technological improvements, the investor should invest more in the risky asset and, thus, decrease the consumption.

To help the readers' understanding we consider the following numerical example with parameters' values chosen only for demonstration. Let the correlation be $\rho = 0.4$ and the risky asset volatility be $\sigma_1 = 0.1$. The risk-free interest rate is $r = 0.05$. The risky asset's instantaneous mean rate of return is $\mu = 0.2$. The utility discount rate is $\zeta = 0.05$. The investor's initial wealth is $x = 1$. The relative risk aversion is $\gamma = 0.5$, and the terminal time is $T = 1$. The parameters β and σ_2 vary in the interval $[0, 0.5]$. The consumption per unit wealth as a function of time and β, time and σ_2, when the utility of consumption and final wealth is maximized, is shown in Figure 2.1.

As one can see from Figure 2.1, the consumption rate is increasing over time. The consumption per unit wealth as a function of time and β, time and σ_2, when only the utility of consumption is maximized, is shown in Figure 2.2.

One can observe that as t approaches terminal time T, the optimal rate of consumption $C^*_{t,2}$ per unit wealth approaches infinity. However, it shouldn't be interpreted as an infinite rate of consumption. Rather more proper explanation may be the following. Since there is no utility associated with the wealth for $t > T$, the consumption rate should increase, thus, making X_t approach 0 as t approaches T. A similar explanation of this behavior in the classical model can be found in [16].

The graphs of the optimal consumptions $C^*_{t,1}$ and $C^*_{t,2}$ per unit wealth as functions

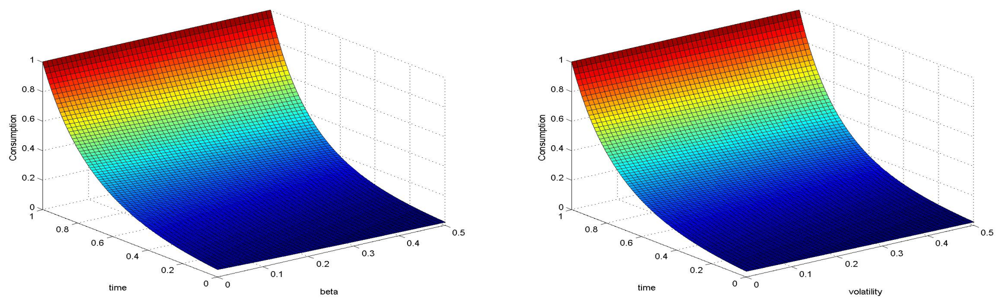

Figure 2.1: *Consumption $C^*_{t,1}$ per unit wealth as (a) function of time and β ($\sigma_2 = 0.4$); (b) function of time and σ_2 ($\beta = 0.4$).*

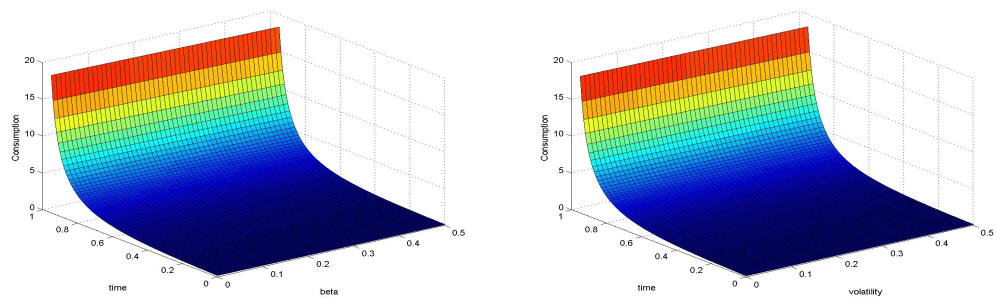

Figure 2.2: *Consumption $C^*_{t,2}$ per unit wealth as (a) function of time and β ($\sigma_2 = 0.4$); (b) function of time and σ_2 ($\beta = 0.4$).*

of time and correlation ρ look very similar to the graphs in Figure 2.1 and Figure 2.2, respectively.

2.9 Conclusions

In this chapter so far we mainly extended the classical model of optimal investment and consumption by adding uncertainty in the utility function. It was shown that the Bellman's Principle of Optimality also holds for this new randomized model and as a result, we derived the Hamilton-Jacobi-Bellman equation associated with the value function. Since not all the solutions of the HJB equation are the value functions, we proved the theorem that can be used to verify that the obtained solution becomes the value function.

The problem was solved explicitly for some specific utility function of HARA type. The optimal consumption and investment were obtained for the problems of maximizing the expected utility of: consumption and final wealth, only consumption, and only final wealth. Although the obtained optimal portfolios are the same for these problems, the optimal consumption rates are different. As in the classical model, if the parameters are constant, the optimal portion of wealth to be invested in the risky asset is constant but depends on the volatility of the utility randomness process and its correlation with the assets' prices.

It was also shown that the so-called Mutual Fund Theorem holds and the optimal portfolio consists of three funds: one includes the riskless asset and the other two contain only the risky assets. The third fund arises from the correlation of the utility uncertainty with the market risk. If the correlation is zero then the agent who takes into account the uncertainty in the utility invests as much in the risky asset as the agent who does not consider it. The investor who is maximizing the utility of his consumption and final wealth gets the highest satisfaction compared with the other investors who maximize either the expected utility of consumption only or the expected utility of final wealth only.

Another quite natural and expected result is that more rapid technological growth yields higher satisfaction. In the particular case when the objective is to maximize the utility of final wealth, the agent's happiness grows at increasing rates with the parameter that defines how fast the products improve. Furthermore, the optimal consumption is decreasing when either the correlation or the volatility of the utility randomness process are increasing provided the risk premium for investing in the stock is non-negative and the correlation is positive. On the other hand, the satisfaction in this case is actually getting higher.

Chapter 3

Partially Observed Case

3.1 Partially Observed Utility Randomness Process

In general, technological progress and other factors modeled by the utility randomness process Z_t are difficult to measure exactly. That is why it makes sense to consider the case when the process Z_t is not fully observed and the observed process is a noisy version of Z_t. For convenience, we will work with the process $L_t = \ln Z_t$ where

$$dL_t = \Big(\beta - \frac{1}{2}\sigma_2^2\Big)dt + \sigma_2 dB_{t,2}.$$

Instead of the process L_t, the investor observes its noisy version, that is,
Observed process:

$$dP_t = L_t dt + \sigma_3 dB_{t,3}$$

with the initial condition $P_0 = 0$. The constant $\sigma_3 > 0$ represents the observed process volatility. The one-dimensional Brownian motion $B_{t,3}$ is $\mathcal{F}_t$-adapted[1] and is independent of the driving force behind the randomness, namely, Brownian motions, $B_{t,1},\ B_{t,2}$. Since the matrix σ_1 is invertible the Brownian motion $B_{t,1}$ can be obtained from observing the asset price process S_t. Thus, the investor observes both processes $B_{t,1}$ and P_t. This means that the optimal controls should be progressively measurable with respect to the filtration $\mathcal{G}_t = \sigma\{B_{s,1}, P_s, |\ s \leq t\}$. It is immediate that $\mathcal{G}_t \subset \mathcal{F}_t$, and the objective is to

[1]For example, $\mathcal{F}_t$ can be defined as $\mathcal{F}_t = \sigma\{B_{s,1}, B_{s,2}, B_{s,3}, |\ s \leq t\}$.

maximize the expected utility conditional on $\mathcal{G}_t$ (partial observation), namely,

$$E\Big[\int_t^T q(s)f(s,X_s^{U_s},C_s,L_s)ds+q(T)g(X_T^{U_T},L_T)\ \Big|\ \mathcal{G}_t\Big]$$

Since $X_s^{U_s},\ C_s$ are measurable with respect to $\mathcal{G}_s$ for any $s\in[t,T]$, to evaluate the expectation, we need to find the conditional distribution of L_t given $\mathcal{G}_t$.

For convenience, we rewrite the equation of wealth (1.16) in the form below

$$dX_t^{U_t}=(rX_t^{U_t}+\Pi_t\sigma_1\theta X_t^{U_t}-C_t)dt+\Pi_t\sigma_1X_t^{U_t}dB_{t,1} \tag{3.1}$$

where $\mu_i-r=\sum\limits_{j=1}^{n}\sigma_1^{i,j}\theta_j,\ i=1,...,n.$ The meaning of the variables in equation (3.1) is the same as in section 1.4. Since the market is complete (in other words σ_1 is invertible), a unique column vector θ exists which is called the market price of risk, the ratio of the reward (from investing in stocks) and the risk (associated with the investment). We also assume that the correlation ρ between $B_{t,1}$ and $B_{t,2}$ is constant.

3.2 Conditional Distribution

Here we obtain the conditional distribution of L_t given $\mathcal{G}_t$. The proofs of the stated results are given in the appendix to organize this chapter 3 better.

Following the nonlinear filtering theory [2], we first change the processes $B_{t,1},\ B_{t,2}$, and P_t into the processes which are independent Brownian motions.

Lemma 4. *Under the probability measure $\tilde{\mathbb{P}}$ given by $d\tilde{\mathbb{P}}=M_td\mathbb{P}$, where*

$$M_t=\exp\Big(-\int_0^t\Big(\theta^{\mathsf{T}}dB_{s,1}+\frac{L_s}{\sigma_3}dB_{s,3}\Big)-\frac{1}{2}\int_0^t\Big(||\theta||^2+\frac{L_s^2}{\sigma_3^2}\Big)ds\Big),$$

such that $E\Big[\int_0^T||\theta M_s||^2ds\Big]<\infty$, and $E\Big[\int_0^T\Big|\frac{L_s}{\sigma_3}M_s\Big|^2ds\Big]<\infty$, the underlying Gaussian processes

$$d\tilde{B}_{t,1}=dB_{t,1}+\theta dt,\ d\tilde{B}_{t,2}=\frac{dB_{t,2}-\rho^{\mathsf{T}}dB_{t,1}}{\sqrt{1-||\rho||^2}},\ d\tilde{P}_t=\frac{1}{\sigma_3}dP_t, \tag{3.2}$$

are independent standard Brownian motions.

Proof. Note that M_t is a $\mathbb{P}-$martingale. In fact,

$$dM_t = -M_t\Big(\theta_t^{\mathsf{T}} dB_{t,1} + \frac{L_t}{\sigma_3} dB_{t,3}\Big),\ M_0 = 1,$$

or in the integral form

$$M_t = M_0 - \int_0^t M_s\Big(\theta_s^{\mathsf{T}} dB_{s,1} + \frac{L_s}{\sigma_3} dB_{s,3}\Big).$$

Since Ito's integrals are martingales, M_t is a martingale such that $M_0 = 1$.

It is easy to check, using the Lévy's theorem, that the process defined by

$$d\tilde{B}_{t,2} = \frac{dB_{t,2} - \rho^{\mathsf{T}} dB_{t,1}}{\sqrt{1 - ||\rho||^2}}$$

is a Brownian motion independent of $B_{t,1}$ under this measure $\mathbb{P}$. Consider the process Y_t

$$dY_t = \begin{pmatrix} \theta_t dt \\ 0 \\ \frac{L_t}{\sigma_3} dt \end{pmatrix} + \begin{pmatrix} dB_{t,1} \\ d\tilde{B}_{t,2} \\ dB_{t,3} \end{pmatrix}.$$

Since $B_{t,1}$, $\tilde{B}_{t,2}$, and $B_{t,3}$ are independent Brownian motions, according to Girsanov theorem (see [17], p.162), the process Y_t is an $(n+2)$-dimensional Brownian motion, and thus, the processes $d\tilde{B}_{t,1}$, $d\tilde{B}_{t,2}$, and $d\tilde{P}_t$ are independent Brownian motions under the measure $\tilde{\mathbb{P}}$. □

To compute the conditional probability of L_t given $\mathcal{G}_t$, let us define a linear operator defined as

$$\Delta_t(\psi) = E[\psi(L_t, t)|\mathcal{G}_t] = \int_{-\infty}^{\infty} p(l, t)\psi(l, t) dl$$

where $\psi(\cdot, \cdot)$ is a smooth bounded function with compact support and $p(l, t)$ is a conditional probability density with respect to probability measure $\mathbb{P}$. The operator is a solution to the Kushner-Stratonovich equation (see [2], chapter 4). It can also be written as $\Delta_t(\psi) = \frac{\tilde{p}_t(\psi)}{\tilde{p}_t(1)}$ where $\tilde{p}_t(\psi) = \int_{-\infty}^{\infty} \tilde{p}(l, t)\psi(l, t) dl$ and $\tilde{p}$ is called the un-normalized conditional probability and is a solution to the Zakai equation (see [2], chapter 4).

Lemma 5. *The process of the un-normalized probability density $\tilde{p}(l,t)$ is given by*

$$d\tilde{p} = \Big[-\tilde{p}_l\Big(\beta - \frac{1}{2}\sigma_2^2\Big) + \frac{1}{2}\sigma_2^2\tilde{p}_{ll}\Big]dt + (\tilde{p}\theta - \tilde{p}_l\sigma_2\rho)^\intercal d\tilde{B}_{t,1} + \tilde{p}\frac{l}{\sigma_3}d\tilde{P}_t, \tag{3.3}$$

where $\tilde{p}(l,0) = p_0(l)$ is the initial distribution and $\displaystyle\int_{-\infty}^{\infty} p(l,t)\psi(l,t)dl = \frac{\int_{-\infty}^{\infty}\tilde{p}(l,t)\psi(l,t)dl}{\int_{-\infty}^{\infty}\tilde{p}(l,t)dl}$ *for any given test function $\psi \in C^{2,1}(\mathbb{R},[0,T])$ with bounded support.*

The proof of the lemma 5 is given in appendix B.1.

Theorem 6. *Let the initial probability density be that of a normal distribution, namely,*

$$p_0(l) = \frac{1}{\sqrt{2\pi m_0}}e^{-\frac{(l-l_0)^2}{2m_0}}.$$

Then the solution to (3.3) is

$$\tilde{p}(l,t) = \frac{K_t}{\sqrt{2\pi m(t)}}e^{-\frac{(l-\hat{L}_t)^2}{2m(t)}}, \tag{3.4}$$

where $\hat{L}_t = E[L_t|\mathcal{G}_t]$ and variance $m(t) = E[(L_t - \hat{L}_t)^2|\mathcal{G}_t]$ is deterministic and given by

$$m(t) = \begin{cases} \sigma_3\lambda_1\frac{\lambda_2\exp(2\lambda_1 t/\sigma_3)-1}{\lambda_2\exp(2\lambda_1 t/\sigma_3)+1} & if\ m_0 < \sigma_3\lambda_1, \\ \sigma_3\lambda_1 & if\ m_0 = \sigma_3\lambda_1, \\ \sigma_3\lambda_1\frac{\lambda_2\exp(2\lambda_1 t/\sigma_3)+1}{\lambda_2\exp(2\lambda_1 t/\sigma_3)-1} & if\ m_0 > \sigma_3\lambda_1, \end{cases} \tag{3.5}$$

where $\lambda_1 = \sigma_2\sqrt{1-||\rho||^2}$ and $\lambda_2 = \left|\frac{\sigma_3\lambda_1+m_0}{\sigma_3\lambda_1-m_0}\right|$. Furthermore, the process $\hat{L}_t$ is the solution to the Kalman filter (see [17], p.99)

$$d\hat{L}_t = (\beta - \frac{1}{2}\sigma_2^2 - \sigma_2\rho^\intercal\theta)dt + \sigma_2\rho^\intercal d\tilde{B}_{t,1} + \frac{m(t)}{\sigma_3}\Big(d\tilde{P}_t - \frac{\hat{L}_t}{\sigma_3}dt\Big) \tag{3.6}$$

where $\hat{L}_0 = l_0$. The variable K_t in (3.4) is adapted to $\mathcal{G}_t$ and is given by

$$K_t = \exp\Big(-\frac{1}{2}\int_0^t\Big(\frac{\hat{L}_s^2}{\sigma_3^2} + ||\theta||^2\Big)ds + \int_0^t\frac{\hat{L}_s}{\sigma_3}d\tilde{P}_s + \int_0^t\theta^\intercal d\tilde{B}_{s,1}\Big). \tag{3.7}$$

The proof of the theorem 6 is given in appendix B.2.

From lemma 5 we have

$$\int_{-\infty}^{\infty} p(l,t)\psi(l,t)dl = \frac{\displaystyle\int_{-\infty}^{\infty} \tilde{p}(l,t)\psi(l,t)dl}{\displaystyle\int_{-\infty}^{\infty} \tilde{p}(l,t)dl} = \Big(\int_{-\infty}^{\infty} e^{-\frac{(l-\hat{L}_t)^2}{2m(t)}}dl\Big)^{-1}\int_{-\infty}^{\infty} e^{-\frac{(l-\hat{L}_t)^2}{2m(t)}}\psi(l,t)dl.$$

Therefore, the conditional distribution of L_t given $\mathcal{G}_t$ is normal with mean $\hat{L}_t$ and variance $m(t)$. The differential $d\hat{L}_t$ can be written in a more convenient form using the following lemma.

Lemma 6. *Let the innovation process $\tilde{B}_{t,3}$ be defined by*

$$d\tilde{B}_{t,3} = \frac{1}{\sigma_3}\Big(dP_t - \hat{L}_t dt\Big),\ \ \tilde{B}_{0,3} = 0. \tag{3.8}$$

Then $\tilde{B}_{t,3}$ and $B_{t,1}$ together form an (n+1)-dimensional $\mathbb{P}$-Brownian motion adapted to the filtration $\mathcal{G}_t$.

The proof of the lemma 6 is given in appendix B.3.

From (3.2), (3.6) and (3.8) we have

$$\begin{aligned} d\hat{L}_t &= (\beta - \frac{1}{2}\sigma_2^2 - \sigma_2\rho^\top\theta)dt + \sigma_2\rho^\top d\tilde{B}_{t,1} + \frac{m(t)}{\sigma_3}\Big(d\tilde{P}_t - \frac{\hat{L}_t}{\sigma_3}dt\Big) \\ &= (\beta - \frac{1}{2}\sigma_2^2 - \sigma_2\rho^\top\theta)dt + \sigma_2\rho^\top dB_{t,1} + \sigma_2\rho^\top\theta dt + \frac{m(t)}{\sigma_3^2}\Big(dP_t - \hat{L}_t dt\Big) \\ &= (\beta - \frac{1}{2}\sigma_2^2)dt + \sigma_2\rho^\top dB_{t,1} + \frac{m(t)}{\sigma_3}d\tilde{B}_{t,3}. \end{aligned} \tag{3.9}$$

The process $\hat{L}_t$ is driven by two independent Brownian motions: $B_{t,1}$ and $\tilde{B}_{t,3}$.

3.3 Reward Functional and Value Function

Since L_t is not fully observable, the objective is to maximize the expected utility conditional on $\mathcal{G}_t$. Therefore, the initial conditions at time t are $\hat{L}_t = \hat{l},\ X_t = x$, and the reward functional[1] is

[1]Although denoted by the same notation, functions f and g may not be the same as in chapter 2.

$$
\begin{aligned}
&\tilde{w}(t,x,\hat{l},U)\\
&= E_{t,x,\hat{l}}\Big[\int_t^T q(s)f(s,X_s^{U_s},C_s,L_s)ds + q(T)g(X_T^{U_T},L_T)\ \Big|\ \mathcal{G}_t\Big]\\
&= E_{t,x,\hat{l}}\Big[\int_t^T q(s)E[f(s,X_s^{U_s},C_s,L_s)\mid\mathcal{G}_s]ds + q(T)E[g(X_T^{U_T},L_T)\mid\mathcal{G}_T]\ \Big|\ \mathcal{G}_t\Big].
\end{aligned}
$$

Since $X_t^{U_t}, C_t$ are $\mathcal{G}_t$-measurable and the conditional distribution of L_t given $\mathcal{G}_t$ is known, we can evaluate the conditional expectations $E[f(s,X_s^{U_s},C_s,L_s)|\mathcal{G}_s]$ and $E[g(X_T^{U_T},L_T)|\mathcal{G}_T]$.

$$
\begin{aligned}
\tilde{f}(s,X_s^{U_s},C_s,\hat{L}_s) &\triangleq E[f(s,X_s^{U_s},C_s,L_s)|\mathcal{G}_s]\\
&= \frac{1}{\sqrt{2\pi m(s)}}\int_{-\infty}^{\infty} f(s,X_s^{U_s},C_s,l)e^{-\frac{(l-\hat{L}_s)^2}{2m(s)}}dl\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s,X_s^{U_s},C_s,\hat{L}_s+l\sqrt{m(s)})e^{-\frac{l^2}{2}}dl.
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\tilde{g}(X_T^{U_T},\hat{L}_T) &\triangleq E[g(X_T^{U_T},L_T)|\mathcal{G}_T]\\
&= \frac{1}{\sqrt{2\pi m(T)}}\int_{-\infty}^{\infty} g(X_T^{U_T},l)e^{-\frac{(l-\hat{L}_T)^2}{2m(T)}}dl\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(X_T^{U_T},\hat{L}_T+l\sqrt{m(T)})e^{-\frac{l^2}{2}}dl.
\end{aligned}
\tag{3.11}
$$

The reward functional is

$$
\tilde{w}(t,x,\hat{l},U) = E_{t,x,\hat{l}}\Big[\int_t^T q(s)\tilde{f}(s,X_s^{U_s},C_s,\hat{L}_s)ds + q(T)\tilde{g}(X_T^{U_T},\hat{L}_T)\Big]. \tag{3.12}
$$

And the value function is

$$
\tilde{v}(t,x,\hat{l}) = \sup_{U\in\mathcal{U}^w[t,T]} \tilde{w}(t,x,\hat{l},U). \tag{3.13}
$$

Thus, the problem with partial observations has been put in the form (3.12), (3.13) similar to the fully observed case (2.2), (2.3), respectively, with functions $\tilde{f},\ \tilde{g}$ instead of $f,\ g$ and the process $\hat{L}_t$ instead of Z_t. This implies that a similar approach to that used to solve the problem with full observations can also be employed to obtain the solution to the partially observed case.

3.4 Derivation of the HJB Equation

In this section we derive the HJB equation for the value function $\tilde{v}$ using Bellman's equation (1.13). Assume that $\tilde{v} \in C^{1,2,2}(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ where $\mathbb{D} = \{(t, x, \hat{l}) : t \in (0, T),\ x > 0,\ \hat{l} \in \mathbb{R}\}$. To keep the notation simple for the processes (3.1), (3.9), let us denote the coefficients of dt and $dB_{t,1}$ in (3.1) as $a \triangleq rX_t^{U_t} + \Pi_t\sigma_1\theta X_t^{U_t} - C_t$ and $s_1 \triangleq \Pi_t\sigma_1 X_t^{U_t}$, respectively, and the coefficients of $dt, dB_{t,1}$, and $d\tilde{B}_{t,3}$ in (3.9) as $\hat{b} \triangleq \beta - \frac{1}{2}\sigma_2^2,\ \hat{s}_2 \triangleq \sigma_2\rho^\mathsf{T}$, and $\hat{s}_3 \triangleq \frac{m(t)}{\sigma_3}$, respectively.

Consider the times $t, \kappa \in [0, T),\ \kappa > t$ and a constant control $U \equiv u \in \mathcal{U}^w[t, T]$ then from the Bellman's Principle of Optimality (1.13)

$$\tilde{v}(t, x, \hat{l}) \geq E_{t,x,\hat{l}}\Big[\int_t^\kappa q(s)\tilde{f}(s, X_s^u, C_s, \hat{L}_s)ds + q(\kappa)\tilde{v}(\kappa, X_\kappa^u, \hat{L}_\kappa)\Big]. \tag{3.14}$$

From the fact that $\tilde{v} \in C^{1,2,2}(\mathbb{D}) \cap C(\bar{\mathbb{D}})$, Ito's formula (see [23], p.167) yields

$$\begin{aligned} d\big(q(s)\tilde{v}(s, X_s^u, \hat{L}_s)\big) =&\ q(s)\big((\tilde{v}_s - \zeta\tilde{v})ds + \tilde{v}_x dX_s^u + \tilde{v}_{\hat{l}} d\hat{L}_s + \tilde{v}_{x\hat{l}} d[X_s^u, \hat{L}_s] \\ &+ \frac{1}{2}\tilde{v}_{xx} d[X_s^u, X_s^u] + \frac{1}{2}\tilde{v}_{\hat{l}\hat{l}} d[\hat{L}_s, \hat{L}_s]\big) \\ =&\ q(s)\big((\tilde{v}_s - \zeta\tilde{v})ds + \tilde{v}_x a ds + \tilde{v}_x s_1 dB_{s,1} + \tilde{v}_{\hat{l}}\hat{b}ds + \tilde{v}_{\hat{l}}\hat{s}_2 dB_{s,1} \\ &+ \tilde{v}_{\hat{l}}\hat{s}_3 d\tilde{B}_{s,3} + \frac{1}{2}\tilde{v}_{xx}s_1 s_1^\mathsf{T} ds + \frac{1}{2}\tilde{v}_{\hat{l}\hat{l}}(\hat{s}_2\hat{s}_2^\mathsf{T} + \hat{s}_3^2)ds + \tilde{v}_{x\hat{l}} s_1 \hat{s}_2^\mathsf{T} ds\big). \end{aligned}$$

Integrating from t to κ on both sides, we get

$$\begin{aligned} &q(\kappa)\tilde{v}(\kappa, X_\kappa^u, \hat{L}_\kappa) \\ &= \tilde{v}(t, X_t^u, \hat{L}_t) \\ &+ \int_t^\kappa q(s)\Big(\tilde{v}_s - \zeta\tilde{v} + \tilde{v}_x a + \tilde{v}_{\hat{l}}\hat{b} + \frac{1}{2}\tilde{v}_{xx}s_1 s_1^\mathsf{T} + \frac{1}{2}\tilde{v}_{\hat{l}\hat{l}}(\hat{s}_2\hat{s}_2^\mathsf{T} + \hat{s}_3^2) + \tilde{v}_{x\hat{l}}s_1\hat{s}_2^\mathsf{T}\Big)ds \\ &+ \int_t^\kappa q(s)\tilde{v}_x s_1 dB_{s,1} + \int_t^\kappa q(s)\tilde{v}_{\hat{l}}\hat{s}_2 dB_{s,1} + \int_t^\kappa q(s)\tilde{v}_{\hat{l}}\hat{s}_3 d\tilde{B}_{s,3}. \end{aligned}$$

The stochastic integrals from the last line in the above expression are local martingales (see [10], p.36). Consider a sequence of stopping times

$$\tau_n = \inf\{h \geq t\ :\ \int_t^h \Big(||q(s)\tilde{v}_x s_1||^2 + ||q(s)\tilde{v}_{\hat{l}}\hat{s}_2||^2 + ||q(s)\tilde{v}_{\hat{l}}\hat{s}_3||^2\Big)ds \geq n\}.$$

Notice that τ_n diverges to infinity almost surely as n goes to infinity. Let $\tau = \kappa \wedge \tau_n$, then the stochastic integrals $\int_t^\tau q(s)\tilde{v}_x s_1 dB_{s,1}$, $\int_t^\tau q(s)\tilde{v}_{\hat{l}}\hat{s}_2 dB_{s,1}$, and $\int_t^\tau q(s)\tilde{v}_{\hat{l}}\hat{s}_3 d\tilde{B}_{s,3}$ are martingales. Plugging $q(\tau)\tilde{v}(\tau, X_\tau^u, \hat{L}_\tau)$ into (3.14), we obtain

$$\begin{aligned}
&\tilde{v}(t,x,\hat{l}) \\
&\geq E_{t,x,\hat{l}}\Big[\int_t^\tau q(s)\Big(\tilde{f}(s,X_s^u,C_s,\hat{L}_s) + \tilde{v}_s - \zeta\tilde{v} + \tilde{v}_x a + \tilde{v}_{\hat{l}}\hat{b} \\
&+ \frac{1}{2}\tilde{v}_{xx}s_1 s_1^\intercal + \frac{1}{2}\tilde{v}_{\hat{l}\hat{l}}(\hat{s}_2\hat{s}_2^\intercal + \hat{s}_3^2) + \tilde{v}_{x\hat{l}}s_1\hat{s}_2^\intercal\Big)ds \\
&+ \int_t^\tau q(s)\tilde{v}_x s_1 dB_{s,1} + \int_t^\tau q(s)\tilde{v}_{\hat{l}}\hat{s}_2 dB_{s,1} + \int_t^\tau q(s)\tilde{v}_{\hat{l}}\hat{s}_3 d\tilde{B}_{s,3} + \tilde{v}(t,X_t^u,\hat{L}_t)\Big].
\end{aligned}$$

Since $E_{t,x,\hat{l}}\Big[\tilde{v}(t,X_t^u,\hat{L}_t)\Big] = \tilde{v}(t,x,\hat{l})$, we have

$$\begin{aligned}
&E_{t,x,\hat{l}}\Big[\int_t^\tau q(s)\Big(\tilde{f}(s,X_s^u,C_s,\hat{L}_s) + \tilde{v}_s - \zeta\tilde{v} + \tilde{v}_x a + \tilde{v}_{\hat{l}}\hat{b} \\
&+ \frac{1}{2}\tilde{v}_{xx}s_1 s_1^\intercal + \frac{1}{2}\tilde{v}_{\hat{l}\hat{l}}(\hat{s}_2\hat{s}_2^\intercal + \hat{s}_3^2) + \tilde{v}_{x\hat{l}}s_1\hat{s}_2^\intercal\Big)ds\Big] \leq 0,
\end{aligned}$$

or

$$E_{t,x,\hat{l}}\Big[\int_t^\tau q(s)\Big(\tilde{f}(s,X_s^u,C_s,\hat{L}_s)ds + \tilde{\mathcal{L}}\tilde{v}(s,X_s^u,\hat{L}_s)\Big)ds\Big] \leq 0, \tag{3.15}$$

where $\tilde{\mathcal{L}} = \frac{\partial}{\partial s} - \zeta + a\frac{\partial}{\partial x} + \hat{b}\frac{\partial}{\partial \hat{l}} + \frac{1}{2}s_1 s_1^\intercal\frac{\partial^2}{\partial x^2} + \frac{1}{2}(\hat{s}_2\hat{s}_2^\intercal + \hat{s}_3^2)\frac{\partial^2}{\partial \hat{l}^2} + s_1\hat{s}_2^\intercal\frac{\partial^2}{\partial \hat{l}\partial x}$ is an operator. Assume that $E_{t,x,\hat{l}}\Big[\int_t^T q(s)\Big|\tilde{f}(s,X_s^u,C_s,\hat{L}_s)ds + \tilde{\mathcal{L}}\tilde{v}(s,X_s^u,\hat{L}_s)\Big|ds\Big] < \infty$. Then by the Dominated Convergence Theorem (see [23], p.27), if we take the limit of (3.15) as n goes to infinity, the inequality in (3.15) becomes

$$E_{t,x,\hat{l}}\Big[\int_t^\kappa q(s)\Big(\tilde{f}(s,X_s^u,C_s,\hat{L}_s)ds + \tilde{\mathcal{L}}\tilde{v}(s,X_s^u,\hat{L}_s)\Big)ds\Big] \leq 0. \tag{3.16}$$

Recall that we assume that $\tilde{f}$ is continuous and that $\tilde{v} \in C^{1,2,2}(\mathbb{D}) \cap C(\bar{\mathbb{D}}))$, if we divide (3.16) by $\kappa - t$ and then take the limit as κ decreases to t, then we see that

$$\tilde{f}(t,x^u,c,\hat{l}) + \tilde{\mathcal{L}}\tilde{v}(t,x^u,\hat{l}) \leq 0, \qquad \forall (t,x,\hat{l}) \in \mathbb{D}.$$

Since this is true for any constant control $u \in \mathcal{U}^w[t,T]$ for all $t \in [0,T)$, one can see

$$\sup_{u\in\mathbb{U}} \Big(\tilde{f}(t, x^u, c, \hat{l}) + \tilde{\mathcal{L}}\tilde{v}(t, x^u, \hat{l}) \Big) \leq 0, \qquad \forall (t, x, \hat{l}) \in \mathbb{D}. \tag{3.17}$$

On the other hand, suppose that U^* is an optimal control, then by definition of the value function

$$\tilde{v}(t, x, \hat{l}) = E_{t,x,\hat{l}}\Big[\int_t^{\kappa} q(s)\tilde{f}(s, X_s^{U_s^*}, C_s, \hat{L}_s)ds + q(\kappa)\tilde{v}(\kappa, X_\kappa^{U_\kappa^*}, \hat{L}_\kappa)\Big].$$

Using the same approach as the one just performed for the inequality (3.14), we obtain

$$\tilde{f}(t, x^{U_t^*}, c, \hat{l}) + \tilde{\mathcal{L}}\tilde{v}(t, x^{U_t^*}, \hat{l}) = 0, \qquad \forall (t, x, \hat{l}) \in \mathbb{D}. \tag{3.18}$$

Therefore, (3.17) and (3.18) suggest that $\tilde{v}$ should satisfy

$$\sup_{u\in\mathbb{U}} \Big(\tilde{f}(t, x^u, c, \hat{l}) + \tilde{\mathcal{L}}\tilde{v}(t, x^u, \hat{l}) \Big) = 0, \qquad \forall (t, x, \hat{l}) \in \mathbb{D}.$$

As a result, the HJB equation with the boundary condition is

$$\begin{cases} \sup_{u\in\mathbb{U}} \Big(\tilde{f}(t, x^u, c, \hat{l}) + \tilde{\mathcal{L}}\tilde{v}(t, x^u, \hat{l}) \Big) = 0, & \forall (t, x, \hat{l}) \in \mathbb{D}, \\ \tilde{v}(T, x, \hat{l}) = \tilde{g}(x, \hat{l}), & \forall x > 0,\ \forall \hat{l} \in \mathbb{R}. \end{cases} \tag{3.19}$$

where $\tilde{\mathcal{L}} = \frac{\partial}{\partial s} - \zeta + a\frac{\partial}{\partial x} + \hat{b}\frac{\partial}{\partial \hat{l}} + \frac{1}{2}s_1 s_1^\intercal \frac{\partial^2}{\partial x^2} + \frac{1}{2}(\hat{s}_2\hat{s}_2^\intercal + \hat{s}_3^2)\frac{\partial^2}{\partial \hat{l}^2} + s_1\hat{s}_2^\intercal \frac{\partial^2}{\partial \hat{l}\partial x}$ is a differential operator.

3.5 Maximizing the Utility of Consumption and Final Wealth

Here we use the same utility function as in section 2.2, namely, $U(C) = \frac{C^\gamma}{\gamma}$ with $\gamma \in (0,1)$. To model the uncertainty in the utility, we multiply the utility function by the utility randomness process Z_t which is the same as multiplying by e^{L_t} because $L_t = \ln Z_t$. Therefore, the functions f and g are $f(t, x^u, c, l) = \frac{c^\gamma e^l}{\gamma}$ and $g(x, l) = \frac{x^\gamma e^l}{\gamma}$, respectively.

We evaluate the conditional expectations (3.10) and (3.11), respectively

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s, X_s^{U_s}, C_s, \hat{L}_s + l\sqrt{m(s)})e^{-\frac{l^2}{2}}dl &= \frac{(C_s)^\gamma}{\sqrt{2\pi}\gamma}\int_{-\infty}^{\infty} e^{\hat{L}_s + l\sqrt{m(s)}}e^{-\frac{l^2}{2}}dl \\ &= \frac{1}{\gamma}(C_s)^\gamma e^{\hat{L}_s + \frac{m(s)}{2}}. \end{aligned} \tag{3.20}$$

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(X_T^{U_T}, \hat{L}_T + l\sqrt{m(T)})e^{-\frac{l^2}{2}}dl &= \frac{(X_T^{U_T})^\gamma}{\sqrt{2\pi}\gamma}\int_{-\infty}^{\infty} e^{\hat{L}_T + l\sqrt{m(T)}}e^{-\frac{l^2}{2}}dl \\ &= \frac{1}{\gamma}(X_T^{U_T})^\gamma e^{\hat{L}_T + \frac{m(T)}{2}}. \end{aligned} \tag{3.21}$$

The value function is

$$\tilde{v}_1(t, x, \hat{l}) = \frac{1}{\gamma}\sup_{U \in \mathcal{U}^w[t,T]} E_{t,x,\hat{l}}\Big[\int_t^T q(s)(C_s)^\gamma e^{\hat{L}_s + \frac{m(s)}{2}}ds + q(T)(X_T^U)^\gamma e^{\hat{L}_T + \frac{m(T)}{2}}\Big].$$

The corresponding HJB equation (3.19) for $t \in (0, T)$, $x > 0$, and $\hat{l} \in \mathbb{R}$ is

$$\begin{aligned}&\tilde{p}_t - \zeta\tilde{p} + rx\tilde{p}_x + \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\tilde{p}_{\hat{l}} + \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\tilde{p}_{\hat{l}\hat{l}} \\ &+ \sup_{c \in [0,\infty)}\Big(\frac{c^\gamma e^{\hat{l} + \frac{m(t)}{2}}}{\gamma} - c\tilde{p}_x\Big) + \sup_{\pi \in \mathbb{R}^n}\Big(x\pi\sigma_1(\theta\tilde{p}_x + \rho\sigma_2\tilde{p}_{\hat{l}x}) + \frac{||x\pi\sigma_1||^2}{2}\tilde{p}_{xx}\Big) = 0. \end{aligned} \tag{3.22}$$

The terminal and boundary conditions are

$$\begin{cases} \tilde{p}(T, x, \hat{l}) = \frac{x^\gamma e^{\hat{l} + \frac{m(T)}{2}}}{\gamma}, & x > 0,\ \hat{l} \in \mathbb{R}, \\ \tilde{p}(t, 0, \hat{l}) = 0, & t \in (0, T),\ \hat{l} \in \mathbb{R}. \end{cases}$$

We discuss the meaning of the terminal and boundary conditions. The terminal condition $\tilde{p}(T, x, \hat{l}) = \frac{x^\gamma e^{\hat{l} + \frac{m(T)}{2}}}{\gamma}$ means that if the investor starts trading at time T then there is no time for investment and the utility (conditional utility evaluated in (3.21)) of his wealth is equal to the utility of the wealth he starts with. The boundary condition $\tilde{p}(t, 0, \hat{l}) = 0$ says that if the initial capital is zero then the value function is zero.

3.5.1 Solution to the HJB equation

We consider a solution in the form of $\tilde{p}(t,x,\hat{l}) = \frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}(t)^{1-\gamma}$. This form of solution is suggested by the functions $f(t,x,c,l) = \frac{c^\gamma e^l}{\gamma}$ and $g(x,l) = \frac{x^\gamma e^l}{\gamma}$, $\gamma \in (0,1)$. Substituting into the equation (3.22) we get

$$\begin{aligned}
&\frac{(1-\gamma)x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{-\gamma}\tilde{h}' - \zeta\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + rx^\gamma e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma} + \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma}\\
&+ \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + \sup_{c\in[0,\infty)}\Big(\frac{c^\gamma e^{\hat{l}+\frac{m(t)}{2}}}{\gamma} - cx^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\Big)\\
&+ \sup_{\pi\in\mathbb{R}^n}\Big(x\pi\sigma_1(\theta x^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma} + \rho\sigma_2 x^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma})\\
&+ \frac{||x\pi\sigma_1||^2}{2}(\gamma-1)x^{\gamma-2}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\Big) = 0.
\end{aligned}$$

in other words,

$$\begin{aligned}
&\frac{(1-\gamma)x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{-\gamma}\tilde{h}' - \zeta\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + rx^\gamma e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma} + \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma}\\
&+ \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + \sup_{c\in[0,\infty)}\Big(\frac{c^\gamma e^{\hat{l}+\frac{m(t)}{2}}}{\gamma} - cx^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\Big)\\
&+ x^\gamma e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\sup_{\pi\in\mathbb{R}^n}\Big(\pi\sigma_1(\theta+\rho\sigma_2) + \frac{||\pi\sigma_1||^2}{2}(\gamma-1)\Big) = 0. \qquad (3.23)
\end{aligned}$$

Consider the following two functions

$$\begin{aligned}
g_1(c) &= \frac{c^\gamma e^{\hat{l}+\frac{m(t)}{2}}}{\gamma} - cx^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma},\\
g_2(\pi) &= \pi\sigma_1\theta + \pi\sigma_1\sigma_2\rho + \frac{||\pi\sigma_1||^2}{2}(\gamma-1).
\end{aligned}$$

Since $\frac{d^2 g_1}{dc^2} = (\gamma-1)c^{\gamma-2}e^{\hat{l}+\frac{m(T)}{2}} < 0$ and the Hessian $H(\pi) = \sigma_1\sigma_1^\mathsf{T}(\gamma-1)$ is negative definite, the functions g_1 and g_2 are concave. The maxima may be obtained from the equations

$$\nabla g_2(\pi) = \sigma_1\theta + \sigma_1\sigma_2\rho + \sigma_1\sigma_1^\mathsf{T}\pi^\mathsf{T}(\gamma-1) = 0,$$

$$\frac{dg_1}{dc} = c^{\gamma-1}e^{\hat{l}+\frac{m(t)}{2}} - x^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma} = 0.$$

The maxima are achieved at

$$\begin{aligned}
\pi^* &= \frac{(-\theta^\top - \sigma_2\rho^\top)\sigma_1^{-1}}{(\gamma-1)},\\
c^* &= \frac{x}{\tilde{h}}e^{\frac{m(T)-m(t)}{2(\gamma-1)}}.
\end{aligned}$$

Substituting (π^*, c^*) into the functions $g_1(c),\ g_2(\pi)$, we obtain

$$\begin{aligned}
g_1(c^*) &= \frac{1}{\gamma}\Big(\frac{x}{\tilde{h}}e^{\frac{m(T)-m(t)}{2(\gamma-1)}}\Big)^\gamma e^{\hat{l}+\frac{m(t)}{2}} - \frac{x}{\tilde{h}}e^{\frac{m(T)-m(t)}{2(\gamma-1)}}x^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\\
&= \frac{1-\gamma}{\gamma}x^\gamma\tilde{h}^{-\gamma}e^{\hat{l}+\frac{\gamma m(T)-m(t)}{2(\gamma-1)}},
\end{aligned}$$

$$\begin{aligned}
&g_2(\pi^*)\\
&= \frac{(-\theta^\top - \sigma_2\rho^\top)\sigma_1^{-1}}{(\gamma-1)}\sigma_1\theta + \frac{(-\theta^\top - \sigma_2\rho^\top)\sigma_1^{-1}}{(\gamma-1)}\sigma_1\sigma_2\rho + \frac{1}{2}\Big|\Big|\frac{(-\theta^\top - \sigma_2\rho^\top)\sigma_1^{-1}}{(\gamma-1)}\sigma_1\Big|\Big|^2(\gamma-1)\\
&= \frac{(-\theta^\top - \sigma_2\rho^\top)}{(\gamma-1)}\theta + \frac{(-\theta^\top - \sigma_2\rho^\top)}{(\gamma-1)}\sigma_2\rho + \frac{1}{2}\Big|\Big|\frac{(-\theta^\top - \sigma_2\rho^\top)}{(\gamma-1)}\Big|\Big|^2(\gamma-1)\\
&= \frac{(-\theta^\top - \sigma_2\rho^\top)}{(\gamma-1)}\Big(\theta + \sigma_2\rho + \frac{1}{2}(-\theta - \sigma_2\rho)\Big)\\
&= -\frac{||\theta + \sigma_2\rho||^2}{2(\gamma-1)}.
\end{aligned}$$

Substituting into the equation (3.23) we get

$$\begin{aligned}
&\frac{(1-\gamma)x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{-\gamma}\tilde{h}' - \zeta\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + rx^\gamma e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma} + \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma}\\
&+ \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma}\\
&+ \frac{1-\gamma}{\gamma}x^\gamma\tilde{h}^{-\gamma}\exp\Big(\hat{l} + \frac{\gamma m(T) - m(t)}{2(\gamma-1)}\Big)\\
&- x^\gamma e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\frac{||\theta + \sigma_2\rho||^2}{2(\gamma-1)} = 0.
\end{aligned}$$

Dividing both sides by $\frac{1-\gamma}{\gamma}x^{\gamma}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{-\gamma}$, we have

$$\tilde{h}' - \frac{\zeta}{1-\gamma}\tilde{h} + \frac{r\gamma}{1-\gamma}\tilde{h} + \Big(\frac{\beta - \frac{1}{2}\sigma_2^2}{1-\gamma}\Big)\tilde{h} + \frac{1}{2(1-\gamma)}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\tilde{h}$$
$$+ \exp\Big(\frac{m(T)-m(t)}{2(\gamma-1)}\Big) + \frac{\gamma||\theta+\sigma_2\rho||^2}{2(1-\gamma)^2}\tilde{h} = 0,$$

which can be rewritten as

$$\tilde{h}' + \tilde{y}(t)\tilde{h} + e^{\frac{m(T)-m(t)}{2(\gamma-1)}} = 0, \tag{3.24}$$

where $\tilde{y}(t) = \frac{1}{1-\gamma}\Big(-\zeta + r\gamma + \beta - \frac{1}{2}\sigma_2^2 + \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\Big) + \frac{\gamma||\theta+\sigma_2\rho||^2}{2(1-\gamma)^2}$.
The solution of the equation (3.24) that satisfies the condition $\tilde{h}(T) = 1$ is

$$\tilde{h}_1(t) = e^{\int_t^T \tilde{y}(\tau)d\tau} + \int_t^T e^{\int_t^\tau \tilde{y}(q)dq + \frac{m(T)-m(\tau)}{2(\gamma-1)}}d\tau.$$

We simplify the expression for $\tilde{h}_1(t)$. Since the function $m(t)$ satisfies the equation (see appendix B.2)

$$m'(t) = -\frac{1}{\sigma_3^2}m^2(t) + \sigma_2^2(1-||\rho||^2),$$

we obtain

$$\int_t^T \frac{m^2(\tau)}{\sigma_3^2}d\tau = \sigma_2^2(1-||\rho||^2)(T-t) - (m(T)-m(t)).$$

Thus, we have

$$\begin{aligned}
\tilde{h}_1(t) &= \exp\Big(\int_t^T y(\tau)d\tau + \frac{1}{2(1-\gamma)}\int_t^T \frac{m^2(\tau)}{\sigma_3^2}d\tau - \frac{\sigma_2^2(1-||\rho||^2)}{2(1-\gamma)}(T-t)\Big) \\
&\quad + \int_t^T e^{\int_t^\tau y(q)dq + \frac{1}{2(1-\gamma)}\int_t^\tau \frac{m^2(q)}{\sigma_3^2}dq - \frac{\sigma_2^2(1-||\rho||^2)}{2(1-\gamma)}(\tau-t) + \frac{(m(T)-m(\tau))}{2(\gamma-1)}}d\tau \\
&= \exp\Big(\int_t^T y(\tau)d\tau - \frac{(m(T)-m(t))}{2(1-\gamma)}\Big) + \int_t^T e^{\int_t^\tau y(q)dq - \frac{(m(\tau)-m(t))}{2(1-\gamma)} + \frac{(m(T)-m(\tau))}{2(\gamma-1)}}d\tau \\
&= \exp\Big(\int_t^T y(\tau)d\tau - \frac{(m(T)-m(t))}{2(1-\gamma)}\Big) + \int_t^T e^{\int_t^\tau y(q)dq - \frac{(m(T)-m(t))}{2(1-\gamma)}}d\tau
\end{aligned}$$

$$= e^{-\frac{(m(T)-m(t))}{2(1-\gamma)}} \left(e^{\int_t^T y(\tau)d\tau} + \int_t^T e^{\int_t^\tau y(q)dq} d\tau \right)$$

$$= e^{-\frac{(m(T)-m(t))}{2(1-\gamma)}} h_1(t) \tag{3.25}$$

Therefore, the solution to the HJB equation (3.22) is

$$\tilde{p}_1(t, x, \hat{l}) = \frac{x^\gamma e^{\hat{l} + \frac{m(t)}{2}}}{\gamma} \left(e^{\int_t^T y(\tau)d\tau} + \int_t^T e^{\int_t^\tau y(q)dq} d\tau \right)^{1-\gamma}.$$

3.5.2 Verification

The solution $\tilde{p}_1$ is a function in $C^{1,2,2}(\mathbb{D}) \cap C(\bar{\mathbb{D}})$, but the quadratic growth condition is not satisfied. However, in the Verification Theorem this condition is required to be able to take advantage of the Dominated Convergence Theorem. From section 2.5.2 we have $xe^{\hat{l}} \leq 3(1 + x^2 + (e^{\hat{l}})^2)$. Since $\hat{L}_s = E[L_s|\mathcal{G}_s]$,

$$\sup_{s\in[t,T]} (\exp(\hat{L}_s))^2 = (\exp(\sup_{s\in[t,T]} \hat{L}_s))^2 \leq (\exp(\sup_{s\in[t,T]} L_s))^2 = \sup_{s\in[t,T]} (\exp(L_s))^2 = \sup_{s\in[t,T]} (Z_s)^2$$

where Z_s is the Geometric Brownian motion. Taking into account that function $\tilde{h}_1$ is bounded and the term $e^{\frac{m(t)}{2}}$ is bounded (see the formula (3.5)), there exists a constant $\tilde{K} > 0$ such that

$$\tilde{p}_1(\tau, X_\tau^{U_\tau}, \hat{L}_\tau) \leq K(1 + \sup_{s\in[t,T]} |X_s^{U_s}|^2 + \sup_{s\in[t,T]} (e^{\hat{L}_s})^2) \leq \tilde{K}(1 + \sup_{s\in[t,T]} |X_s^{U_s}|^2 + \sup_{s\in[t,T]} |Z_s|^2)$$

which is integrable.

Since the function $m(t)$ is continuous, the rest of the verification is analogous to that of the section 2.5.2. Therefore, using (3.25), the optimal controls are

$$\tilde{\Pi}_s^* = \frac{(-\theta^\intercal - \sigma_2 \rho^\intercal)\sigma_1^{-1}}{(\gamma - 1)},$$

$$\tilde{C}_{s,1}^* = \frac{X_s}{\tilde{h}_1(s)} e^{\frac{m(T)-m(s)}{2(\gamma-1)}} = \frac{X_s}{h_1(s)}.$$

3.6 Maximizing the Utility of Consumption

We already evaluated in (3.20) the conditional expectation (3.10), and thus, the value function is

$$\tilde{v}_2(t,x,\hat{l}) = \frac{1}{\gamma} \sup_{U\in\mathcal{U}^w[t,T]} E_{t,x,\hat{l}}\Big[\int_t^T q(s)(C_s)^\gamma e^{\hat{L}_s+\frac{m(s)}{2}}ds\Big].$$

The corresponding HJB equation (3.19) for $t\in(0,T)$, $x>0$, and $\hat{l}\in\mathbb{R}$ is

$$\begin{aligned}&\tilde{p}_t - \zeta\tilde{p} + rx\tilde{p}_x + \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\tilde{p}_{\hat{l}} + \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\tilde{p}_{\hat{l}\hat{l}} \\ &+ \sup_{c\in[0,\infty)}\Big(\frac{c^\gamma e^{\hat{l}+\frac{m(t)}{2}}}{\gamma} - c\tilde{p}_x\Big) + \sup_{\pi\in\mathbb{R}^n}\Big(x\pi\sigma_1(\theta\tilde{p}_x + \rho\sigma_2\tilde{p}_{\hat{l}x}) + \frac{||x\pi\sigma_1||^2}{2}\tilde{p}_{xx}\Big) = 0. \qquad (3.26)\end{aligned}$$

The terminal and boundary conditions are

$$\begin{cases}\tilde{p}(T,x,\hat{l}) = 0, & x>0,\ \hat{l}\in\mathbb{R}, \\ \tilde{p}(t,0,\hat{l}) = 0, & t\in(0,T),\ \hat{l}\in\mathbb{R}.\end{cases}$$

The meaning of the terminal and boundary conditions is the same as in the section 3.5.

3.6.1 Solution to the HJB equation

The calculations are analogous to those in the previous section 3.5.1 and the only difference is the condition $\tilde{h}(T)=0$ for the equation

$$\tilde{h}' + \tilde{y}(t)\tilde{h} + e^{\frac{m(T)-m(t)}{2(\gamma-1)}} = 0, \qquad (3.27)$$

where $\tilde{y}(t) = \frac{1}{1-\gamma}\Big(-\zeta + r\gamma + \beta - \frac{1}{2}\sigma_2^2 + \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\Big) + \frac{\gamma||\theta+\sigma_2\rho||^2}{2(1-\gamma)^2}$.
The solution of (3.27) that satisfies the condition $\tilde{h}(T)=0$ is

$$\tilde{h}_2(t) = \int_t^T e^{\int_t^\tau \tilde{y}(q)dq + \frac{m(T)-m(\tau)}{2(\gamma-1)}}d\tau.$$

Taking into account (3.25), we have

$$\tilde{h}_2(t) = e^{-\frac{(m(T)-m(t))}{2(1-\gamma)}}\Big(\int_t^T e^{\int_t^\tau y(q)dq} d\tau\Big) = e^{-\frac{(m(T)-m(t))}{2(1-\gamma)}} h_2(t).$$

Therefore, the solution to the HJB equation (3.26) is

$$\tilde{p}_2(t,x,\hat{l}) = \frac{x^\gamma e^{\hat{l}+\frac{m(t)}{2}}}{\gamma}\Big(\int_t^T e^{\int_t^\tau y(q)dq} d\tau\Big)^{1-\gamma}.$$

3.6.2 Verification

The verification that the obtained solution is the value function is identical to the verification in the previous section 3.5.2 and section 2.6.2. Therefore, the optimal controls are

$$\tilde{\Pi}_s^* = \frac{(-\theta^\intercal - \sigma_2\rho^\intercal)\sigma_1^{-1}}{(\gamma-1)},$$
$$\tilde{C}_{s,2}^* = \frac{X_s}{\tilde{h}_2(s)} e^{\frac{m(T)-m(s)}{2(\gamma-1)}} = \frac{X_s}{h_2(s)}.$$

3.7 Maximizing the Utility of Final Wealth

We already evaluated in (3.21) the conditional expectation (3.11), and thus, the value function is

$$\tilde{v}_3(t,x,\hat{l}) = \frac{1}{\gamma}\sup_{U\in\mathcal{U}^w[t,T]} E_{t,x,\hat{l}}\Big[q(T)(X_T^U)^\gamma e^{\hat{L}_T+\frac{m(T)}{2}}\Big].$$

The corresponding HJB equation (3.19) for $t \in (0,T)$, $x > 0$, and $\hat{l} \in \mathbb{R}$ is

$$\tilde{p}_t - \zeta\tilde{p} + rx\tilde{p}_x + \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\tilde{p}_{\hat{l}} + \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\tilde{p}_{\hat{l}\hat{l}}$$
$$+ \sup_{c\in[0,\infty)}\Big(-c\tilde{p}_x\Big) + \sup_{\pi\in\mathbb{R}^n}\Big(x\pi\sigma_1(\theta\tilde{p}_x + \rho\sigma_2\tilde{p}_{\hat{l}x}) + \frac{||x\pi\sigma_1||^2}{2}\tilde{p}_{xx}\Big) = 0. \qquad (3.28)$$

The terminal and boundary conditions are

$$\begin{cases} \tilde{p}(T,x,\hat{l}) = \frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}, & x>0,\ \hat{l}\in\mathbb{R}, \\ \tilde{p}(t,0,\hat{l}) = 0, & t\in(0,T),\ \hat{l}\in\mathbb{R}. \end{cases}$$

3.7.1 Solution to the HJB equation

We start by looking for a solution in the form $\tilde{p}(t,x,\hat{l}) = \frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}(t)^{1-\gamma}$. Substituting into the equation (3.28), we get

$$\begin{aligned}
&\frac{(1-\gamma)x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{-\gamma}\tilde{h}' - \zeta\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + rx^\gamma e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma} + \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} \\
&+ \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + \sup_{c\in[0,\infty)}\Big(-cx^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\Big) \\
&+ \sup_{\pi\in\mathbb{R}^n}\Big(x\pi\sigma_1(\theta x^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma} + \rho\sigma_2 x^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}) \\
&+ \frac{||x\pi\sigma_1||^2}{2}(\gamma-1)x^{\gamma-2}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\Big) = 0.
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{(1-\gamma)x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{-\gamma}\tilde{h}' - \zeta\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + rx^\gamma e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma} + \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} \\
&+ \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\frac{x^\gamma e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + \sup_{c\in[0,\infty)}\Big(-cx^{\gamma-1}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\Big) \\
&+ x^\gamma e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\sup_{\pi\in\mathbb{R}^n}\Big(\pi\sigma_1(\theta+\rho\sigma_2) + \frac{||\pi\sigma_1||^2}{2}(\gamma-1)\Big) = 0. \qquad (3.29)
\end{aligned}$$

Similarly to calculations in section 3.5, the suprema are achieved at

$$\begin{aligned}
\pi^* &= \frac{(-\theta^\top - \sigma_2\rho^\top)\sigma_1^{-1}}{(\gamma-1)}, \\
c^* &= 0.
\end{aligned}$$

Therefore, substituting into the equation (3.29), we obtain

$$\frac{(1-\gamma)x^{\gamma}e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{-\gamma}\tilde{h}' - \zeta\frac{x^{\gamma}e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} + rx^{\gamma}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma} + \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\frac{x^{\gamma}e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma}$$
$$+\frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\frac{x^{\gamma}e^{\hat{l}+\frac{m(T)}{2}}}{\gamma}\tilde{h}^{1-\gamma} - x^{\gamma}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{1-\gamma}\frac{||\theta+\sigma_2\rho||^2}{2(\gamma-1)} = 0.$$

Dividing the above equation by $\frac{1-\gamma}{\gamma}x^{\gamma}e^{\hat{l}+\frac{m(T)}{2}}\tilde{h}^{-\gamma}$, we obtain

$$\tilde{h}' - \frac{\zeta}{1-\gamma}\tilde{h} + \frac{r\gamma}{1-\gamma}\tilde{h} + \Big(\frac{\beta-\frac{1}{2}\sigma_2^2}{1-\gamma}\Big)\tilde{h} + \frac{1}{2(1-\gamma)}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\tilde{h}$$
$$+\frac{\gamma||\theta+\sigma_2\rho||^2}{2(1-\gamma)^2}\tilde{h} = 0.$$

We simplify this to

$$\tilde{h}' + \tilde{y}(t)\tilde{h} = 0, \tag{3.30}$$

where $\tilde{y}(t) = \frac{1}{1-\gamma}\Big(-\zeta + r\gamma + \beta - \frac{1}{2}\sigma_2^2 + \frac{1}{2}\Big(\sigma_2^2||\rho||^2 + \frac{m^2(t)}{\sigma_3^2}\Big)\Big) + \frac{\gamma||\theta+\sigma_2\rho||^2}{2(1-\gamma)^2}$.
The solution of the equation (3.30) that satisfies the condition $\tilde{h}(T) = 1$ is

$$\tilde{h}_3(t) = e^{\int_t^T \tilde{y}(\tau)d\tau}.$$

Taking into account the equation (3.25), we obtain

$$\tilde{h}_3(t) = e^{-\frac{(m(T)-m(t))}{2(1-\gamma)}}\Big(e^{\int_t^T y(\tau)d\tau}\Big) = e^{-\frac{(m(T)-m(t))}{2(1-\gamma)}}h_3(t)$$

Therefore, the solution to the HJB equation (3.28) is

$$\tilde{p}_3(t,x,\hat{l}) = \frac{x^{\gamma}e^{\hat{l}+\frac{m(t)}{2}}}{\gamma}\Big(e^{\int_t^T y(\tau)d\tau}\Big)^{1-\gamma}.$$

3.7.2 Verification

The verification that the obtained solution is the value function is very similar to the verification when the utility of consumption and final wealth is maximized. It can be easily seen that all the inequalities obtained in section 3.5.2 hold for the function $\tilde{p}_3(t,x,\hat{l})$.

This means that the obtained solution $\tilde{p}_3(t, x, \hat{l})$ is the value function $\tilde{v}(t, x, \hat{l})$. Therefore, the optimal controls are

$$\tilde{\Pi}_s^* = \frac{(-\theta^\intercal - \sigma_2\rho^\intercal)\sigma_1^{-1}}{(\gamma - 1)},$$
$$\tilde{C}_{s,3}^* = 0.$$

3.8 Analysis of the Results and Numerical Experiments.

Here we analyze the obtained results in the partially observed case. Throughout this section, we assume that all the parameters (such as $\zeta, r, \beta, \sigma_2, \sigma_3, \theta$, and ρ) are constant, $t < T$, and the investor has the utility function $U(C) = \frac{C^\gamma}{\gamma}, \ \gamma \in (0, 1)$.

Proposition 10. *The optimal portfolio ($\tilde{\Pi}_s^*$) and optimal consumption ($\tilde{C}_{s,i}^*, \ i = 1, 2, 3$) in the partially observed case are the same as the optimal portfolio (Π_s^*) and the optimal consumption ($C_{s,i}^*, \ i = 1, 2, 3$) in the fully observed case, respectively.*

Proof. Indeed, comparing the formulae for optimal portfolios Π_s^* and $\tilde{\Pi}_s^*$, and noticing that by definition $\theta = \sigma_1^{-1}(\mu - r)$ we have

$$\Pi_s^* = \frac{(r - \mu^\intercal - \rho^\intercal\sigma_1^\intercal\sigma_2)(\sigma_1\sigma_1^\intercal)^{-1}}{\gamma - 1} = \frac{((r - \mu^\intercal)(\sigma_1^\intercal)^{-1} - \rho^\intercal\sigma_2)(\sigma_1)^{-1}}{\gamma - 1}$$
$$= \frac{(-\theta^\intercal - \rho^\intercal\sigma_2)(\sigma_1)^{-1}}{\gamma - 1} = \tilde{\Pi}_s^*.$$

Comparing the formulae for optimal consumption we see that $\tilde{C}_{s,i} = C_{s,i}, \ i = 1, 2, 3$. □

Therefore, the uncertainty in the knowledge of the utility randomness process Z_t does not influence the investor's optimal portfolio and optimal consumption. It means that most of the results of the section 2.8 obtained in the fully observed case are also applicable in the partially observed case. Indeed, we have the following relation

$$\tilde{v}_i = \frac{e^{\hat{l} + \frac{m(t)}{2}}}{z} v_i, \ i = 1, 2, 3$$

where the function $m(t)$ depends on the parameters σ_2, σ_3, ρ. Therefore, only the results (Proposition 8, Proposition 9) of section 2.8 involving these parameters don't have to hold.

Proposition 11. *The variance $m(t)$ is decreasing if $m(0) > \sigma_2\sigma_3\sqrt{1-||\rho||^2}$, increasing if $m(0) < \sigma_2\sigma_3\sqrt{1-||\rho||^2}$, and it approaches value $\sigma_2\sigma_3\sqrt{1-||\rho||^2}$ as t approaches infinity.*

Proof. The proof follows easily from the formula for $m(t)$ given in (3.5). □

From Theorem 6 it follows that if the initial variance $m(0)$ for the conditional distribution of L_0 is higher than $\sigma_2\sigma_3\sqrt{1-||\rho||^2}$, then the variance $m(t)$ for the conditional distribution of L_t decreases over time to $\sigma_2\sigma_3\sqrt{1-||\rho||^2}$ and vice versa.

3.9 Conclusions

We consider the model of optimal investment and consumption described in the previous chapter 2 under the assumption that the utility randomness process is partially observed. It was shown that the Bellman's Principle of Optimality also holds and as a result we derived the Hamilton-Jacobi-Bellman equation associated with the value function. The Verification Theorem from the chapter 2 can be used in checking that the obtained solution is the value function.

The problem was solved explicitly for a specific utility function of HARA type. The optimal consumption and investment were obtained for the problems of maximizing the expected utility of: consumption and final wealth, only consumption, and only final wealth. The obtained optimal portfolios and consumptions are the same for these problems. It was shown that the solutions to these problems are the same as when the utility randomness process is fully observed. One of the differences from the fully observed case is the value function. Therefore, some of the results obtained in the previous chapter also hold for partially observed utility randomness process.

Chapter 4

Summary and Future Research

4.1 Summary

The research done in this dissertation extended the Merton's model [15] to include technological progress. Since advancements in technology influence consumers' satisfaction (utility), we include them in the model by means of the utility function. It is reasonable to assume that the future technological progress is not fully known to investors. On the other hand, in some reasearch [8] it is claimed that in some areas the technological progress exhibits exponential growth. Therefore, a Geometric Brownian motion is used to model the proposed uncertainty in the utility function.

As the criterion to maximize (the reward function), the expected utility is chosen. Since the technological progress is a characteristic that is difficult to measure exactly, the cases of fully observed and partially observed utility randomness process are considered and solved for a specific utility function of hyperbolic absolute risk aversion type. The three problems solved in the two cases are the problems of maximizing both the expected utility of consumption and final wealth, the expected utility of consumption only, and the expected utility of final wealth only.

The problems were solved via second-order partial differential equations, also known as Hamilton-Jacobi-Bellman equations. The Verification Theorem, necessary to show that the obtained solutions are optimal solutions to the original problem of expeceted utility maximization, was also proved.

For the case of fully observed utility randomness process, it was shown that the so-called Mutual Fund Theorem holds and the optimal portfolio consists of three funds:

one includes the riskless asset and the other two contain only the risky assets. The third fund arises from the correlation of the utility uncertainty with the market risk. If the correlation is zero then the agent who takes into account the uncertainty in the utility invests as much in the risky asset as the agent who does not consider it. In other words, when the correlation is zero, the optimal portfolio is the same as in the classical Merton's model. It was also shown, that the investor who is maximizing the utility of his consumption and final wealth gets the highest satisfaction compared with the other investors who maximize either the expected utility of consumption only or the expected utility of final wealth only.

Another quite natural and expected result is that more rapid technological growth yields higher satisfaction. In the particular case when the objective is to maximize the expected utility of final wealth, the agent's happiness grows at increasing rates with the parameter that defines how fast the products improve. Furthermore, the optimal consumption is decreasing when either the correlation or the volatility of the utility randomness process are increasing provided the risk premium for investing in the stock is non-negative and the correlation is positive. On the other hand, the satisfaction in this case is actually getting higher.

In case of partially observed utility randomness process, the optimal portfolios and consumptions for the considered three problems are the same as for the case of full observations. One of the differences from the fully observed case is the value functions. Therefore, most of the results obtained for the fully observed case also hold for partially observed utility randomness process.

4.2 Future Research

The classical model generalization proposed in this dissertation can be further extended. It is common to assume that the stock prices follow a Geometric Brownian Motion. However, a financial portfolio can include assets that are usually described by different stochastic processes (for example, arithmetic Brownian Motion). This gives a problem of finding the optimal portfolio under the assumption that the assets in the portfolio are modeled by different stochastic processes.

Liquidity risk is one the most important factors that should be taken into account when modelling financial markets. Thus, the problem of expected utility maximization can also be considered under the assumption that the risky assets are traded in illiquid markets. In this setting the assets' prices, modeled by a jump-diffusion process, are observed and traded at random times. The goal is to find the optimal portfolio and consumption that yield the maximum of expected utility from consumption. Once the problem is formulated, different stochastic control approaches (HJB equations, probabilistic methods, etc.) can be used to solve it.

Portfolio optimization for various risk measures is another interesting area of research. In this regard, a proper joint distribution of financial data becomes an issue, which can be resolved by using, for example, copula functions. This, in turn, implies the problem of choice of individual data distributions. Once these issues are resolved, the problem of portfolio optimization can be set up and and attempted to be solved.

It is very common to use Brownian motion as the only source of randomness in financial models. For example, Brownian motion appears in the stochastic process that models the dynamics of stock prices (Geometric Brownian motion). However, Brownian motion has normally distributed increments and all the models that use it are based on this assumption. Thus, considering other sources of randomness which do not include normal distribution could be a further extension of the classical model of investment and consumption.

Assuming there are many investment opportunities (not only stocks), one can consider the problem of diversifying the portfolio among the opportunities. The issue is to find appropriate stochastic processes that model the risks accociated with different investement venues. If we can construct the cost related to the degree of diversification of the portfolio, then we can pose a problem of ideal optimization of reaching a portfolio that maximizes the expected return.

In the model of optimal investment and consumption under partial observations the observed process was assumed to be a noisy version of the actual process. The noise was chosen to be a Brownian motion. This was one of the reasons that the obtained optimal solution is the same as when the utility randomness process is fully observed. Therefore, it would be interesting to find the optimal solution if the noise has different distribution.

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APPENDICES

Appendix A

Proofs for Chapter 1

A.1 Theorem 1

Let $T > 0$. To prove that (1.2) admits a strong solution on $[0, T]$, for any $0 \le \tau \le T$, we denote

$$\begin{aligned}\mathcal{X}_l[0,\tau] &\triangleq L^l_{\mathcal{F}}(\Omega; C([0,\tau];\mathbb{R}^n)) \\ &= \{x : [0,\tau]\times\Omega \to \mathbb{R}^n | x(\cdot) \text{ is } \mathcal{F}_t- \text{ adapted, continuous, and } E[\sup_{0\le t\le\tau} |x(t)|^l] < \infty\}.\end{aligned}$$

Clearly, $\mathcal{X}_l[0, \tau]$ is a Banach space with the norm

$$|x(\cdot)|_{\mathcal{X}_l[0,\tau]} \triangleq \Big(E[\sup_{0\le t\le\tau} |x(t)|^l]\Big)^{\frac{1}{l}}. \tag{A.1}$$

For any $x(\cdot), y(\cdot) \in \mathcal{X}_l[0, \tau]$, define for $t \in [0, \tau]$,

$$\begin{cases} X_t = \xi + \int_0^t a(s,x,\omega)ds + \int_0^t s_1(s,x,\omega)dB_s, \\ Y_t = \xi + \int_0^t a(s,y,\omega)ds + \int_0^t s_1(s,y,\omega)dB_s. \end{cases} \tag{A.2}$$

Where the functions a and s_1 are assumed to satisfy the following conditions. For any $\omega \in \Omega$, $a(\cdot,\cdot,\omega) \in \mathcal{A}^k(\mathbb{R}^k)$ and $s_1(\cdot,\cdot,\omega) \in \mathcal{A}^k(\mathbb{R}^{k\times m})$ and for any $x \in \mathbf{B}^k$, $a(\cdot, x, \cdot)$ and $s_1(\cdot, x, \cdot)$ are both $\{\mathcal{F}_t\}$-adapted processes. Moreover, there exists a constant $L > 0$ such that for all $t \in [0,\infty)$, $x, y \in \mathbf{B}^k$, and $\omega \in \Omega$,

$$\begin{cases} ||a(t,x,\omega) - a(t,y,\omega)|| \leq L||x-y||_{\mathbf{B}^k}, \\ ||s_1(t,x,\omega) - s_1(t,y,\omega)|| \leq L||x-y||_{\mathbf{B}^k}, \\ |a(\cdot,0,\cdot)| + |s_1(\cdot,0,\cdot)| \in L^2_{\mathcal{F}}(0,T;\mathbb{R}),\ \forall T > 0. \end{cases}$$

By (1.3) and the Burkholder-Davis-Gundy inequality (see, for example, [9]), we have

$$\begin{cases} |X_\cdot|^l_{\mathcal{X}_l[0,\tau]} \leq K, \\ |X_\cdot - Y_\cdot|^l_{\mathcal{X}_l[0,\tau]} \leq K\Big(\tau^{\frac{l}{2}}|x(\cdot) - y(\cdot)|^l_{\mathcal{X}_l[0,\tau]}\Big). \end{cases} \tag{A.3}$$

Here the constant K is independent of τ, ξ, $x(\cdot)$, and $y(\cdot)$.

We let $\tau \in [0,T]$ be a given deterministic constant such that $K\tau^{\frac{l}{2}} < 1$, where K is in (A.3). From the equation (A.3), it follows that for any $\xi \in L^l_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)$, the map $x(\cdot) \mapsto X_\cdot$ defined via (A.2) is from space $\mathcal{X}_l[0,\tau]$ to itself (with the norm (A.1)) and is contractive. Thus, there exists a unique fixed point, which gives a strong solution $X_\cdot$ to (1.2) on $[0,\tau]$. Next, repeating the procedure on $[\tau, 2\tau]$, etc., we are able to get the unique strong solution on $[0,T]$. Since $T > 0$ is arbitrary, we obtain the strong solution on $[0,\infty)$. The proof of the remaining conclusions follow easily from the Burkholder-Davis-Gundy inequality.

The proof of the theorem follows the approach taken in [9].

A.2 Theorem 2

We treat the case of right-continuity. With $t > 0$, $n \geq 1$, $k = 0, 1, ..., 2^n - 1$, and $0 \leq s \leq t$, we define

$$X^{(n)}_s(\omega) = X_{(k+1)t/2^n}(\omega) \text{ for } \frac{kt}{2^n} < s \leq \frac{(k+1)t}{2^n},$$

as well as $X^{(n)}_0(\omega) = X_0(\omega)$. The so-constructed map $(s,\omega) \mapsto X^{(n)}_s(\omega)$ from $[0,t] \times \Omega$ into $\mathbb{R}^k$ is demonstrably $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable. Besides, by right-continuity of the process X_t we have $\lim_{n\to\infty} X^{(n)}_s(\omega) = X_s(\omega)$, $\forall (s,\omega) \in [0,t] \times \Omega$. Therefore, the (limit) map $(s,\omega) \mapsto X_s(\omega)$ is also $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.

The proof of the theorem follows the approach taken in [10].

A.3 Lemma 2

In order to prove the Lemma, we need two propositions. A set $B \subseteq \mathbf{B}^m[0,T]$ is called a Borel cylinder, if there exists $0 \leq t_1 < t_2 < ... < t_j \leq T$ and $E \in \mathcal{B}(\mathbb{R}^{jm})$ such that

$$B = \{\zeta \in \mathbf{B}^m[0,T] \mid (\zeta(t_1), \zeta(t_2), ..., \zeta(t_j)) \in E\}. \tag{A.4}$$

We let $\mathbf{C}_s$ be the set of all Borel cylinders in $\mathbf{B}_s^m[0,T]$ of the form (A.4) with $t_1, ..., t_j \in [0,s]$.

Proposition 12. *The sigma-algebra $\sigma(\mathbf{C}_T)$ generated by $\mathbf{C}_T$ coincides with the Borel sigma-algebra $\mathcal{B}(\mathbf{B}^m[0,T])$ of $\mathbf{B}^m[0,T]$.*

Proof. Let $0 \leq t_1 < t_2 < ... < t_j \leq T$ be given. We define a map $\mathcal{T} : \mathbf{B}^m[0,T] \to \mathbb{R}^{jm}$ as follows

$$\mathcal{T}(\zeta) = (\zeta(t_1), \zeta(t_2), ..., \zeta(t_j)), \ \forall \zeta \in \mathbf{B}^m[0,T].$$

Clearly, $\mathcal{T}$ is continuous. Consequently, for any $E \in \mathcal{B}(\mathbb{R}^{jm})$, it follows that $\mathcal{T}^{-1}(E) \in \mathcal{B}(\mathbf{B}^m[0,T])$. This implies

$$\mathbf{C}_T \subseteq \mathcal{B}(\mathbf{B}^m[0,T]). \tag{A.5}$$

Next, for any $\zeta_0 \in \mathbf{B}^m[0,T]$ and $\varepsilon > 0$, we have

$$\begin{aligned} &\{\zeta \in \mathbf{B}^m[0,T] \mid ||\zeta - \zeta_0||_{\mathbf{B}^m[0,T]} \leq \varepsilon\} \\ &= \bigcap_{r \in \mathbb{Q}, r \in [0,T]} \{\zeta \in \mathbf{B}^m[0,T] \mid ||\zeta(r) - \zeta_0(r)|| \leq \varepsilon\} \in \sigma(\mathbf{C}_T), \end{aligned} \tag{A.6}$$

since $\{\zeta \in \mathbf{B}^m[0,T] \mid ||\zeta(r) - \zeta_0(r)|| \leq \varepsilon\}$ is a Borel cylinder, and $\mathbb{Q}$ is the set of all rational numbers (which is countable). Because the set of all sets in the form of the left-hand side of (A.6) is a basis of the open sets in $\mathbf{B}^m[0,T]$, we have

$$\mathcal{B}(\mathbf{B}^m[0,T]) \subseteq \sigma(\mathbf{C}_T). \tag{A.7}$$

Combining (A.5) and (A.7), we obtain our result. □

Proposition 13. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and* $\xi : [0,T] \times \Omega \to \mathbb{R}^m$ *be a continuous process. Then there exists an* $\Omega_0 \in \mathcal{F}$ *with* $\mathbb{P}(\Omega_0) = 1$ *such that* $\xi : \Omega_0 \to \mathbf{B}^m[0,T]$ *and for any* $s \in [0,T]$,

$$\Omega_0 \bigcap \mathcal{F}_s^\xi = \Omega_0 \bigcap \xi^{-1}(\mathcal{B}_s(\mathbf{B}^m[0,T])).$$

Proof. Let $t \in [0,s]$ and a set $E \in \mathcal{B}(\mathbb{R}^m)$ be fixed. Then for simplicity denote by $E_t \triangleq \{\zeta \in \mathbf{B}^m[0,T] \mid \zeta(t) \in E\} \in \mathbf{C}_s$. Note that

$$\omega \in \xi^{-1}(E_t) \Longleftrightarrow \xi(\cdot,\omega) \in E_t \Longleftrightarrow \xi(t,\omega) \in E \Longleftrightarrow \omega \in \xi(t,\cdot)^{-1}(E).$$

Thus, $\xi(t,\cdot)^{-1}(E) = \xi^{-1}(E_t)$. We obtain the result by the previous proposition 12. □

Proof. Proof of the Lemma 2.

We prove only the 'only if' part. The 'if' part is clear.

For any $s \in [0,T]$, we consider a mapping

$$\theta^s(t,\omega) \triangleq (t \wedge s, \xi(\cdot \wedge s, \omega)) \ : \ [0,T] \times \Omega \to [0,s] \times \mathbf{B}_s^m[0,T].$$

By Proposition 13, we have $\mathcal{B}[0,s] \otimes \mathcal{F}_s^\xi = \sigma(\theta^s)$. On the other hand, we have that $(t,\omega) \mapsto \varphi(t \wedge s, \omega)$ is $(\mathcal{B}[0,s] \otimes \mathcal{F}_s^\xi)/\mathcal{B}(U)$-measurable (see definition 4 in chapter 1 for the meaning of the notation). Thus, there exists a measurable map which is given by $\eta_s : ([0,T] \times \mathbf{B}_s^m[0,T], \mathcal{B}[0,s] \times \mathcal{B}_s(\mathbf{B}^m[0,T])) \to U$ such that

$$\varphi(t \wedge s, \omega) = \eta_s(t \wedge s, \xi(\cdot \wedge s, \omega)), \ \forall \omega \in \Omega, \ t \in [0,T].$$

Now, for any $i \geq 0$, let $0 = t_0^i < t_1^i < ...$ be a partition of $[0,T]$ with $\max_{j \geq 1}(t_j^i - t_{j-1}^i) \to 0$ as $i \to \infty$, and define

$$\eta^i(t,\zeta) = \eta_0(0, \zeta(\cdot \wedge 0)) I_{\{0\}}(t) + \sum_{j \geq 1} \eta_{t_j^i}(t, \zeta(\cdot \wedge t_j^i)) I_{(t_{j-1}^i, t_j^i]}(t), \ \forall (t,\zeta) \in [0,T] \times \mathbf{B}^m[0,T].$$

For any $t \in [0,T]$, there exists a uniquely determined index j such that $t_{j-1}^i < t \leq t_j^i$. Then

$$\eta^i(t, \xi(\cdot \wedge t_j^i, \omega)) = \eta_{t_j^i}(t, \xi(\cdot \wedge t_j^i, \omega)) = \varphi(t,\omega).$$

Now, in case U is either $\mathbb{R}$ or $\mathbb{N}$, we may define

$$\eta(t, \zeta) = \limsup_{i \to \infty} \eta^i(t, \zeta)$$

to get the desired result. $\square$

The proof of the lemma follows the approach taken in [9].

Appendix B

Proofs for Chapter 3

B.1 Lemma 5

Step 1: Un-Normalized Conditional Probability. Let us introduce a new filtration defined by $\tilde{\mathcal{G}}_t = \sigma\{\tilde{B}_{s,1},\ \tilde{P}_s \mid s \leq t\}$. Obviously, $\mathcal{G}_t = \tilde{\mathcal{G}}_t$ and, therefore, $\Delta_t(\psi) = E[\psi(L_t, t)|\mathcal{G}_t] = E[\psi(L_t, t)|\tilde{\mathcal{G}}_t]$, where $\psi \in C^{2,1}(\mathbb{R}, [0, T])$ is a test function with a bounded support.

It is convenient to use probability measure $\tilde{\mathbb{P}}$ instead of $\mathbb{P}$ since the observation processes under $\tilde{\mathbb{P}}$ are a Brownian motions. Therefore, we also need the Radon-Nikodym derivative

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \frac{1}{M_t} = Q_t$$

on $\mathcal{F}_t$, and we have

$$\Delta_t(\psi) = E[\psi(L_t, t)|\tilde{\mathcal{G}}_t] = \frac{\tilde{E}[\psi(L_t, t)Q_t|\tilde{\mathcal{G}}_t]}{\tilde{E}[Q_t|\tilde{\mathcal{G}}_t]}$$

where $\tilde{E}$ is the expectation with respect to $\tilde{\mathbb{P}}$. This formula leads to the introduction of the un-normalized conditional probability defined by

$$p_t(\psi) = \tilde{E}[\psi(L_t, t)Q_t|\tilde{\mathcal{G}}_t].$$

Step 2: Zakai Equation. We note that

$$\begin{aligned} dB_{t,2} &= \sqrt{1-||\rho||^2}d\tilde{B}_{t,2} + \rho^\intercal dB_{t,1}, \\ dB_{t,1} &= d\tilde{B}_{t,1} - \theta dt, \\ dB_{t,3} &= d\tilde{P}_t - \tfrac{1}{\sigma_3}L_t dt, \end{aligned}$$

we have that

$$\begin{aligned} dL_t &= \Big(\beta - \frac{1}{2}\sigma_2^2\Big)dt + \sigma_2 dB_{t,2} \\ &= \Big(\beta - \frac{1}{2}\sigma_2^2 - \sigma_2\rho^\intercal\theta\Big)dt + \sigma_2\Big(\sqrt{1-||\rho||^2}d\tilde{B}_{t,2} + \rho^\intercal d\tilde{B}_{t,1}\Big) \end{aligned}$$

and

$$\begin{aligned} dQ_t &= Q_t\Big(\theta^\intercal dB_{t,1} + \frac{L_t}{\sigma_3}dB_{t,3}\Big) + Q_t\Big(\theta^\intercal\theta + \frac{L_t^2}{\sigma_3^2}\Big)dt \\ &= Q_t\Big(\theta^\intercal d\tilde{B}_{t,1} + \frac{L_t}{\sigma_3}d\tilde{P}_t\Big). \end{aligned}$$

Therefore, by Ito's formula,

$$\begin{aligned} d(Q_t\psi(L_t,t)) &= Q_t\Big(\frac{\partial\psi}{\partial t} + \Big(\beta - \frac{1}{2}\sigma_2^2 - \sigma_2\rho^\intercal\theta\Big)\frac{\partial\psi}{\partial l} + \frac{1}{2}\sigma_2^2\frac{\partial^2\psi}{\partial l^2}\Big)dt \\ &\quad + Q_t\sigma_2\frac{\partial\psi}{\partial l}\Big(\sqrt{1-||\rho||^2}d\tilde{B}_{t,2} + \rho^\intercal d\tilde{B}_{t,1}\Big) \\ &\quad + Q_t\psi\Big(\theta^\intercal d\tilde{B}_{t,1} + \frac{L_t}{\sigma_3}d\tilde{P}_t\Big) + Q_t\sigma_2\frac{\partial\psi}{\partial l}\rho^\intercal\theta dt \\ &= Q_t\Big(\Big(\frac{\partial\psi}{\partial t} - \mathcal{A}\psi\Big)dt + \sigma_2\frac{\partial\psi}{\partial l}\sqrt{1-||\rho||^2}d\tilde{B}_{t,2} \\ &\quad + \Big(\psi\theta + \frac{\partial\psi}{\partial l}\sigma_2\rho\Big)^\intercal d\tilde{B}_{t,1} + \psi\frac{L_t}{\sigma_3}d\tilde{P}_t\Big), \end{aligned}$$

where the second-order differential operator $\mathcal{A}$ is given by

$$\mathcal{A} = -\Big(\beta - \frac{1}{2}\sigma_2^2\Big)\frac{\partial}{\partial l} - \frac{1}{2}\sigma_2^2\frac{\partial^2}{\partial l^2}.$$

In the integral form

$$Q_t\psi(L_t,t) = \psi(L_0,0) + \int_0^t Q_s\Big(\frac{\partial\psi}{\partial s} - \mathcal{A}\psi\Big)ds + \int_0^t Q_s\sigma_2\frac{\partial\psi}{\partial l}\sqrt{1-||\rho||^2}d\tilde{B}_{s,2}$$
$$+ \int_0^t Q_s\Big(\psi\theta + \frac{\partial\psi}{\partial l}\sigma_2\rho\Big)^{\mathsf{T}}d\tilde{B}_{s,1} + \int_0^t Q_s\psi\frac{L_s}{\sigma_3}d\tilde{P}_s.$$

To compute the conditional expectation $p_t(\psi_t) = \tilde{E}[\psi(L_t,t)Q_t|\tilde{\mathcal{G}}_t]$, we use test functions which are $\tilde{\mathcal{G}}_t$-measurable. Because the generating processes are Brownian motions, it is sufficient to test with stochastic processes of the form

$$dJ_t = iJ_t(\xi_1(t)^{\mathsf{T}}d\tilde{B}_{t,1} + \xi_2(t)d\tilde{P}_t),\ \kappa_0 = 1,$$

where $i = \sqrt{-1}$, and $\xi_1(t) \in \mathbb{R}^n$ and $\xi_2(t) \in \mathbb{R}$ are arbitrarily chosen deterministic bounded functions. Recall that by the process $Q_t\psi(L_t,t)$, the definition of $p_t(\psi)$ and $\tilde{E}[\tilde{B}_{t,2}|\tilde{\mathcal{G}}_t] = 0$, we have ($\tilde{B}_{t,1},\ \tilde{B}_{t,2},\ \tilde{P}_t$ are independent Brownian motions under $\tilde{\mathbb{P}}$),

$$\tilde{E}[J_tp_t(\psi)] = \tilde{E}[J_t\Delta_0(\psi)] + \tilde{E}\Big[J_t\Big(\int_0^t p_s\Big(\frac{\partial\psi}{\partial s} - \mathcal{A}\psi\Big)ds$$
$$+ \int_0^t p_s\Big(\psi\theta + \frac{\partial\psi}{\partial l}\sigma_2\rho\Big)^{\mathsf{T}}d\tilde{B}_{s,1} + \int_0^t p_s\Big(\psi\frac{L_s}{\sigma_3}\Big)d\tilde{P}_s\Big)\Big].$$

Because this equality holds for all J_t, we obtain the Zakai equation

$$p_t(\psi) = \Delta_0(\psi) + \int_0^t p_s\Big(\frac{\partial\psi}{\partial s} - \mathcal{A}\psi\Big)ds + \int_0^t p_s\Big(\psi\theta + \frac{\partial\psi}{\partial l}\sigma_2\rho\Big)^{\mathsf{T}}d\tilde{B}_{s,1} + \int_0^t p_s\Big(\psi\frac{L_s}{\sigma_3}\Big)d\tilde{P}_s.$$

Step 3. Un-normalized Density. We look for a density that solves the Zakai equation, that is $\tilde{p}(l,t)$ such that

$$p_t(\psi) = \int_{-\infty}^{\infty}\tilde{p}(l,t)\psi(l,t)dl.$$

Substituting into the Zakai equation we obtain[1]

$$\int_{-\infty}^{\infty}\tilde{p}(l,t)\psi(l,t)dl = \int_{-\infty}^{\infty}\tilde{p}(l,0)\psi(l,0)dl + \int_0^t\int_{-\infty}^{\infty}\tilde{p}(l,s)\Big(\frac{\partial\psi}{\partial s} - \mathcal{A}\psi\Big)dl\ ds$$
$$+ \int_0^t\int_{-\infty}^{\infty}\tilde{p}(l,s)\Big(\psi\theta + \frac{\partial\psi}{\partial l}\sigma_2\rho\Big)^{\mathsf{T}}dl\ d\tilde{B}_{s,1}$$
$$+ \int_0^t\int_{-\infty}^{\infty}\tilde{p}(l,s)\Big(\psi\frac{l}{\sigma_3}\Big)dl\ d\tilde{P}_s.$$

Using integration by parts in t and l, we get

$$\int_{-\infty}^{\infty}\Big(d\tilde{p} + \mathcal{A}^*\tilde{p}dt - (\tilde{p}\theta - \tilde{p}_l\sigma_2\rho)^{\mathsf{T}}d\tilde{B}_{t,1} - \tilde{p}\frac{l}{\sigma_3}d\tilde{P}_t\Big)\psi dl = 0,$$

where $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$ given by

$$\mathcal{A}^* = \Big(\beta - \frac{1}{2}\sigma_2^2\Big)\frac{\partial}{\partial l} - \frac{1}{2}\sigma_2^2\frac{\partial^2}{\partial l^2}.$$

This gives the stochastic partial differential equation for the density.

The proof of this lemma follows the approach taken in [3].

B.2 Theorem 6

The Theorem 6 is proved by showing that (3.4)-(3.7) give (3.3).

Step 1: Un-normalized Density Process. We look for a solution in the form

$$\tilde{p}(l,t) = e^{-\frac{1}{2}(\phi(t)l^2 - 2F_t l + G_t)}, \tag{B.1}$$

where $\phi(t)$ is deterministic, and F_t and G_t are Ito processes such that F_t satisfies

$$dF_t = F_{t,0}dt + F_{t,1}^{\mathsf{T}}d\tilde{B}_{t,1} + F_{t,2}d\tilde{P}_t, \tag{B.2}$$

and G_t satisfies

$$dG_t = G_{t,0}dt + G_{t,1}^{\mathsf{T}}d\tilde{B}_{t,1} + G_{t,2}d\tilde{P}_t. \tag{B.3}$$

[1]The functions under the integral involving ψ (for example, $\frac{\partial\psi}{\partial s} - \mathcal{A}\psi$), are functions of l and s, not L_s and s as before.

Considering function $\tilde{p}(l,t)$ as a function $\tilde{p}(t,F_t,G_t)$, and using Ito's lemma we obtain

$$d\tilde{p} = \tilde{p}\Big(-\frac{1}{2}\phi' l^2 dt + l dF_t - \frac{1}{2} dG_t + \frac{1}{2}||lF_{t,1} - \frac{1}{2}G_{t,1}||^2 dt + \frac{1}{2}(lF_{t,2} - \frac{1}{2}G_{t,2})^2 dt \Big). \quad \text{(B.4)}$$

We also note that $\tilde{p}_l = \tilde{p}(-l\phi + F_t)$ and $\tilde{p}_{ll} = \tilde{p}(-l\phi + F_t)^2 - \tilde{p}\phi$, and, thus, equating the diffusion terms of (3.3) and (B.4), we obtain

$$\begin{aligned} &\tilde{p}(lF_{t,1}^{\mathsf{T}} - \frac{1}{2}G_{t,1}^{\mathsf{T}}) = (\tilde{p}\theta - \tilde{p}_l \sigma_2 \rho)^{\mathsf{T}}, \\ &\tilde{p}\frac{l}{\sigma_3} = \tilde{p}(lF_{t,2} - \frac{1}{2}G_{t,2}). \end{aligned}$$

This implies that

$$F_{t,1} = \sigma_2 \phi \rho, \ -\frac{1}{2}G_{t,1} = \theta - \sigma_2 F_t \rho, \ F_{t,2} = \frac{1}{\sigma_3}, \ G_{t,2} = 0. \quad \text{(B.5)}$$

Equating the drift terms of (3.3) and (B.4) we obtain

$$\begin{aligned} -\frac{1}{2}\phi' l^2 + lF_{t,0} - \frac{1}{2}G_{t,0} + \frac{1}{2}\Big(\frac{l^2}{\sigma_3^2} + ||(l\phi - F_t)\sigma_2\rho + \theta||^2\Big) \quad \text{(B.6)} \\ = (l\phi - F_t)(\beta - \frac{1}{2}\sigma_2^2) + \frac{1}{2}\sigma_2^2((l\phi - F_t)^2 - \phi). \end{aligned}$$

We select the parameters so that (B.6) holds.

Step 2. Variance. Equating the coefficients of l^2 in (B.6) we obtain

$$-\phi' + \frac{1}{\sigma_3^2} + \phi^2 \sigma_2^2 (||\rho||^2 - 1) = 0. \quad \text{(B.7)}$$

By Lemma 2 and Theorem 6, $\tilde{p}(l,0) = p_0(l) = \frac{1}{\sqrt{2\pi m_0}} e^{-\frac{(l-l_0)^2}{2m_0}}$, and, thus, equation (B.1) implies that $\phi(0) = \frac{1}{m_0}$. Therefore, by setting $m(t) = \frac{1}{\phi(t)}$ we obtain the following Riccati equation for the variance

$$m'(t) = -\frac{m^2(t)}{\sigma_3^2} + \sigma_2^2(1 - ||\rho||^2), \ m(0) = m_0, \quad \text{(B.8)}$$

The solution to this equation is (3.5).

Step 3: Kalman Filter. Equating the coefficients of l in (B.6), we can obtain the coefficient $F_{t,0}$ of l to be

$$F_{t,0} = \sigma_2^2 F_t \phi(||\rho||^2 - 1) + \phi(\beta - \frac{1}{2}\sigma_2^2 - \sigma_2 \rho^\mathsf{T} \theta),$$

which with (B.2) and (B.5) give

$$dF_t + (1 - ||\rho||^2)\sigma_2^2 \phi F_t \; dt = \phi(\beta - \frac{1}{2}\sigma_2^2 - \sigma_2 \rho^\mathsf{T} \theta)dt + \sigma_2 \phi \rho^\mathsf{T} d\tilde{B}_{t,1} + \frac{1}{\sigma_3} d\tilde{P}_t.$$

Let $\hat{L}_t = F_t m(t)$, then by Ito's lemma, we obtain the Kalman filter (3.6). Because $\tilde{p}(l, 0) = p_0(l)$ and (B.1) imply $F_0 = \frac{l_0}{m_0}$, we get the initial condition $\hat{L}_0 = l_0$.

Step 4: Conditional Probability Density. Equating the terms independent of l in (B.6) permits us to compute $G_{t,0}$. This gives

$$G_{t,0} = ||\theta||^2 + F_t^2 \sigma_2^2 (||\rho||^2 - 1) + F_t(2\beta - \sigma_2^2) + \sigma_2^2 \phi - 2F_t \sigma_2 \theta^\mathsf{T} \rho.$$

From equations (B.3) and (B.5) we obtain

$$\begin{aligned} dG_t = &\Big(||\theta||^2 + F_t^2 \sigma_2^2 (||\rho||^2 - 1) + F_t(2\beta - \sigma_2^2) + \sigma_2^2 \phi - 2F_t \sigma_2 \theta^\mathsf{T} \rho \Big) dt \\ &+ 2(\sigma_2 F_t \rho - \theta)^\mathsf{T} d\tilde{B}_{t,1}. \end{aligned} \tag{B.9}$$

Using $\tilde{p}(l, 0) = p_0(l)$ and (B.1), we have the initial condition

$$e^{-\frac{G_0}{2}} = \frac{1}{\sqrt{2\pi m_0}} e^{-\frac{l_0^2}{2m_0}}. \tag{B.10}$$

Using $\phi(t) = \frac{1}{m(t)}$ and $F_t = \frac{\hat{L}_t}{m(t)}$, we can write (B.1) as follows

$$\tilde{p}(l, t) = \frac{K_t}{\sqrt{2\pi m(t)}} e^{-\frac{(l - \hat{L}_t)^2}{2m(t)}}, \tag{B.11}$$

where $K_t = \sqrt{2\pi m(t)} e^{\frac{1}{2}\left(-G_t + \frac{\hat{L}_t^2}{m(t)}\right)} \triangleq e^{\Phi_t}$ and $\Phi_t = \frac{1}{2}\Big(- G_t + \frac{\hat{L}_t^2}{m(t)}\Big) + \ln(\sqrt{2\pi m(t)})$. From (B.10) and $\hat{L}_0 = l_0$, we get $\Phi_0 = 0$. By Ito's lemma and equations (3.6), (B.8), and

(B.9), we obtain

$$d\Phi_t = -\frac{1}{2}\frac{\hat{L}_t^2}{\sigma_3^2}dt - \frac{1}{2}\theta^\top\theta dt + \theta^\top d\tilde{B}_{t,1} + \frac{\hat{L}_t}{\sigma_3}d\tilde{P}_t.$$

which gives (3.7). Thus, the un-normalized density in Theorem 6 equals (B.1) and (B.11). Then by $\phi(t),\ F_t,\ G_t$, equation (3.3) holds. Using Lemma 2 we get that the density solves the Zakai equation.

The proof of this theorem follows the approach taken in [3].

B.3 Lemma 6

Since the observed process is given by $dP_t = L_t dt + \sigma_3 dB_{t,3}$, then combining with (3.8) we obtain $d\tilde{B}_{t,3} = \frac{\varepsilon_t}{\sigma_3}dt + dB_{t,3}$, where $\varepsilon_t = L_t - \hat{L}_t$. In order to solve the distribution of $(B_{t,3}, B_{t,1})$ under $\mathbb{P}$, we analyze characteristic function

$$\begin{aligned}\varphi(t) &= E\Big[\exp\Big(i\int_0^t(\xi_2(s)d\tilde{B}_{s,3} + \xi_1(s)dB_{s,1})\Big)|\mathcal{G}_0\Big]\\ &= E\Big[\exp\Big(i\int_0^t\Big(\frac{\xi_2(s)\varepsilon_s}{\sigma_3}ds + \xi_2(s)dB_{s,3} + \xi_1(s)dB_{s,1}\Big)\Big)|\mathcal{G}_0\Big],\end{aligned}$$

where $i = \sqrt{-1}$, and $\xi_1 \in \mathbb{R}^n$ and $\xi_2 \in \mathbb{R}$ are arbitrarily chosen deterministic bounded functions. Let us define

$$H_t = \exp\Big(i\int_0^t\Big(\frac{\xi_2(s)\varepsilon_s}{\sigma_3}ds + \xi_2(s)dB_{s,3} + \xi_1(s)dB_{s,1}\Big)\Big).$$

By Ito's lemma, iterated expectation, and the fact that $\varepsilon(s)$ is independent of $\mathcal{G}_s$, we have

$$\begin{aligned}\varphi(t) &= E[H_0|\mathcal{G}_0] + E\Big[\int_0^t dH_s|\mathcal{G}_0\Big]\\ &= \varphi(0) + i\int_0^t \xi_2(s)E\Big[\frac{H_s\varepsilon_s}{\sigma_3}|\mathcal{G}_0\Big]ds + i\int_0^t \xi_1(s)E[H_s dB_{s,1}|\mathcal{G}_0]\\ &- \frac{1}{2}\int_0^t(\xi_2^2(s) + ||\xi_1(s)||^2)E[H_s|\mathcal{G}_0]ds\end{aligned}$$

$$= 1 + i\int_0^t \xi_2(s)E\Big[\frac{H_s}{\sigma_3}E[\varepsilon_s|\mathcal{G}_s]|\mathcal{G}_0\Big]ds - \frac{1}{2}\int_0^t (\xi_2^2(s) + ||\xi_1(s)||^2)E[H_s|\mathcal{G}_0]ds$$

$$= 1 - \frac{1}{2}\int_0^t (\xi_2^2(s) + ||\xi_1(s)||^2)E[H_s|\mathcal{G}_0]ds = \exp\Big(-\frac{1}{2}\int_0^t (\xi_2^2(s) + ||\xi_1(s)||^2)ds\Big).$$

A comparison of this with the characteristic function of the standard $(n+1)$-dimensional Brownian motion completes the proof.

The proof of this lemma follows the approach taken in [3].