Fluid Structure Interaction with Sloshing

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Abstract

In this paper, three different formulations for fluid-structure interaction with sloshing are discussed. When the surface displacements are large, the problems are nonlinear, and Arbitrary Lagrangian Eulerian (ALE) methods and direct time integration are most appropriate. Explicit direct time integration has the disadvantage of a limited time-step whereas implicit method has the disadvantage of nonconvergence and high computational cost. A mixed time method which employs E-mE (explicit-multiple explicit) integration for obtaining the velocity and free surface displacement and I-mI (implicit-multiple implicit) integration for obtaining the pressure is described. An iterative solution procedure is used to enhance the efficiency of the implicit solution procedure as well as to reduce the computer storage. For linear problems, the surface wave effects can be approximated by a perturbation method on the body force term if the surface displacements are small. Furthermore, if the fluid can be idealized as inviscid, incompressible and irrotational, the pressure, velocity, and free surface displacement variables can be eliminated via a velocity potential formulation.

Finite Element Formulation of the Arbitrary Lagrangian Eulerian (ALE) Method

The basic conservation equations used in the ALE formulation parallel that developed in [1]. The fluid B occupies a region R_X. We describe the motion of B by a mapping $\underset{\sim}{\phi}$ such that the image of R_X at time t is denoted by R_x, and the image of $\underset{\sim}{X}\epsilon R_X$ is denoted by $\underset{\sim}{x}$, i.e.,

$$\underset{\sim}{x} = \underset{\sim}{\phi}(\underset{\sim}{X},t) = \underset{\sim}{X} + \underset{\sim}{u}(\underset{\sim}{X},t) \tag{1}$$

Here R_X is considered to be the region occupied at $t=t_o$ by the "material particles" which occupy R_x at time t, so R_x is the "spatial" region. With this definition, material displacement, velocity and acceleration can be defined as follows:

$$\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{X},t) = \underset{\sim}{x} - \underset{\sim}{X} \qquad \text{displacement} \tag{2}$$

$$\underset{\sim}{v} = \left.\frac{\partial \underset{\sim}{x}}{\partial t}\right|_{[X]} = \underset{\sim}{x}_{,t[X]} = \underset{\sim}{u}_{,t[X]} = \overset{\circ}{\underset{\sim}{u}} \qquad \text{velocity} \tag{3}$$

$$\underset{\sim}{a} = \left.\frac{\partial \underset{\sim}{v}}{\partial t}\right|_{[X]} = \underset{\sim}{v}_{,t[X]} = \overset{\circ}{\underset{\sim}{v}} = \overset{\circ\circ}{\underset{\sim}{u}} \qquad \text{acceleration} \tag{4}$$

In eqs. (3-4), the symbol [X] designates the partial derivative holding "X" fixed. For a given scalar function $f=f(\underset{\sim}{X},t)$, the "material time derivative," which appears in continuum conservation laws, is defined as:

$$\mathring{f} = \frac{d}{dt}(f) = \left.\frac{\partial f}{\partial t}\right|_{[X]} = f_{,t[X]} \qquad \begin{matrix}\text{material time}\\ \text{derivative}\end{matrix} \tag{5}$$

The "spatial derivative", which also appears in conservation laws, is:

$$\frac{\partial f}{\partial x_i} \overset{def}{=} f_{,i} \tag{6}$$

These two important shorthand notations will be used subsequently.

In order to complete the "referential" description, it is necessary to define another motion - the motion of the referential domain (we call this motion, the "mesh" motion, in our finite element formulation). We recall that this referential domain, R_χ, is fixed throughout. Its motion is defined by the mapping $\hat{\underset{\sim}{\phi}}$ such that the image of R_χ at time t is denoted by R_x, and so $\underset{\sim}{x}$ is the image of $\underset{\sim}{\chi}\epsilon R_\chi$. Therefore,

$$\underset{\sim}{x} = \hat{\underset{\sim}{\phi}}(\underset{\sim}{\chi},t) = \underset{\sim}{\chi} + \hat{\underset{\sim}{u}}(\underset{\sim}{\chi},t) \tag{7}$$

$\hat{\underset{\sim}{u}}(\underset{\sim}{\chi},t)$ is the "displacement" (we refer this displacement as "mesh" displacement in our finite element formulation). The corresponding velocity ("mesh" velocity) and acceleration ("mesh" acceleration) are defined as follows:

$$\hat{\underset{\sim}{v}} = \left.\frac{\partial \underset{\sim}{x}}{\partial t}\right|_{[\chi]} = \underset{\sim}{x}_{,t[\chi]} = \hat{\underset{\sim}{u}}_{,t[\chi]} \qquad \text{velocity} \tag{8}$$

$$\hat{\underset{\sim}{a}} = \left.\frac{\partial \underset{\sim}{v}}{\partial t}\right|_{[\chi]} = \hat{\underset{\sim}{v}}_{,t[\chi]} \qquad \text{acceleration} \tag{9}$$

In general this motion, $\hat{\underset{\sim}{\phi}}$, can be arbitrary and the usefulness of the referential description will depend on how we pick this motion. For our purpose, we fix this motion, $\hat{\underset{\sim}{\phi}}$, such that the mapping between $\underset{\sim}{x}$ and $\underset{\sim}{\chi}$ is one to one. With this assumption and by the composition of the mappings, we can define a third mapping such that

$$\underset{\sim}{\chi} = \underset{\sim}{\psi}(\underset{\sim}{X},t) = \hat{\underset{\sim}{\phi}}^{-1} \circ \underset{\sim}{\phi}(\underset{\sim}{X},t) \quad . \tag{10}$$

Similarly, we can define "displacement", "velocity" and "acceleration" for this motion. However, this is not necessary. These displacement, velocity and acceleration variables can be defined with the help of the chain rule and the mappings.

Material Time Derivative and Convective Velocity

In material balance laws, the material time derivative of a function f is required. This function f can be "mass", "linear momentum", "angular momentum", "total energy", and "entropy", for example; f is usually defined as a function of $\underset{\sim}{x}$ and t. In our formulation, since $\underset{\sim}{\chi}$ is fixed throughout, so it is convenient to express these quantities in terms of $\underset{\sim}{\chi}$ and t. In general, by mapping, f can be a function of $\underset{\sim}{\chi}$ and t or a function of $\underset{\sim}{X}$ and t. With this understanding we can write f as:

$$f = f(\underset{\sim}{x},t) = f(\underset{\sim}{X},t) = f(\underset{\sim}{\chi},t) \tag{11}$$

Therefore the material time derivative of f can be written as:

$$\frac{df}{dt} = \mathring{f} = \text{material time derivative} \qquad (\underset{\sim}{X},t) \tag{12a}$$

$$= f_{,t[x]} + \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t}\Big|_{[X]} = f_{,t[x]} + f_{,i}v_i \qquad (\underset{\sim}{x},t) \tag{12b}$$

$$= f_{,t[\chi]} + \frac{\partial f}{\partial \chi_i}\frac{\partial \chi_i}{\partial t}\Big|_{[X]} \qquad (\underset{\sim}{\chi},t) \tag{12c}$$

In eq. (12c), the variable $\underset{\sim}{\chi}$ is not defined explicitly in terms of $\underset{\sim}{X}$ and t. Since,

$$x_j = \phi_j(\underset{\sim}{X},t) = \hat{\phi}_j(\underset{\sim}{\chi},t) \tag{13}$$

differentiate above equations with respect to time t holding $\underset{\sim}{X}$ fixed, eq. (13) implies:

$$v_j = \frac{\partial \hat{\phi}_j}{\partial t}\Big|_{[\chi]} + \frac{\partial \hat{\phi}_j}{\partial \chi_i}\frac{\partial \chi_i}{\partial t}\Big|_{[X]} = \hat{v}_j + \frac{\partial x_j}{\partial \chi_i}\frac{\partial \chi_i}{\partial t}\Big|_{[X]} \tag{14}$$

or we have

$$\frac{\partial x_j}{\partial \chi_i}\frac{\partial \chi_i}{\partial t}\Big|_{[X]} = v_j - \hat{v}_j \overset{\text{def}}{=} c_j = \text{convective velocity} \tag{15}$$

Employing eq. (15) together with chain rule, eq. (12c) reduces to:

$$\frac{df}{dt} = \mathring{f} = f_{,t[\chi]} + f_{,i}c_i \tag{16}$$

It can be shown that eq. (16) reduces to eqs. (12a) and (12b) by picking $\underset{\sim}{\chi}=\underset{\sim}{X}$ and $\underset{\sim}{\chi}=\underset{\sim}{x}$ respectively. The former is usually called a Lagrangian description, whereas the latter is usually referred as Eulerian description. Eq. (16) is the material time derivative in referential description.

Balance Laws in Referential Description

Conservation of Mass (Equation of Continuity)

The principle of mass conservation can be written as:

$$\frac{d}{dt}\int_{R_x} \rho(\underset{\sim}{x},t)dR_x = \int_{R_\chi} \frac{d}{dt}(\rho J)dR_\chi = 0 \tag{17}$$

In eq. (17), ρ is the mass density defined in terms of $\underset{\sim}{x}$ and t, and J is the Jacobian defined by

$$J = \det\left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{\chi}}\right) \tag{18}$$

We call $\rho_o(\underset{\sim}{\chi})$ the initial mass density, viz.,

$$\rho_o(\chi) = \rho(\hat{\underset{\sim}{x}}(\underset{\sim}{\chi}, t_o), t_o) \tag{19}$$

Eq. (17) implies

$$\mathring{\rho} J + \rho \mathring{J} = 0 \tag{20}$$

As

$$\mathring{J} = J v_{k,k} \text{ and } \mathring{\rho} = \rho_{,t[\chi]} + \rho_{,i} c_i \tag{21}$$

The continuity equation in nonconservation form is then:

$$\rho_{,t[\chi]} + \rho_{,i} c_i + \rho v_{k,k} = 0 \tag{22}$$

Using the relationship

$$J_{,t[\chi]} = J \hat{v}_{k,k} \tag{23}$$

the continuity equation (22) can be put into conservation form as:

$$(\rho J)_{,t[\chi]} + J(\rho c_i)_{,i} = 0 \tag{24}$$

Principles of Linear and Angular Momentum

Newton's Law states that the time rate of change of linear momentum of a body, L_i, is equal to the net force exerted on it, i.e.;

$$L_i = \int_{R_\chi} b_i(\underset{\sim}{x}, t) dR_x + \int_{\partial R_x} t_i d\Gamma_x \tag{25}$$

where

$$L_i = \frac{d}{dt} \int_{R_x} \rho(\underset{\sim}{x}, t) v_i(\underset{\sim}{x}, t) dR_x = \int_{R_\chi} \rho J \frac{dv_i}{dt} dR_\chi + \int_{R_\chi} \frac{d(\rho J)}{dt} v_i dR_\chi = \int_{R_x} \rho \frac{dv_i}{dt} dR_x \tag{26}$$

and

$$t_i = \tau_{ij} n_j \tag{27}$$

In the above, τ_{ij} is the component of the Cauchy stress tensor; n_i is the component of the outward unit normal vector to ∂R_x; b_i is the body force per unit; and t_i is the Cauchy surface traction vector. Employing the divergence theorem, eq. (25) yields

$$\rho(v_{i,t[\chi]} + v_{i,j} c_j) = \tau_{ij,j} + b_i \tag{28}$$

or

$$(\rho v_i)_{,t[\chi]} + (\rho v_i c_j)_{,j} + \rho v_i \hat{v}_{j,j} = \tau_{ij,j} + b_i \tag{29}$$

Eqs. (28) and (29) are the momentum equations in nonconservation form and conservation form, respectively. The principle of angular momentum simply implies the symmetry of the Cauchy stress τ_{ij}.

Energy Equation

Energy conservation is expressed as:

$$\frac{d}{dt} \int_{R_x} \rho E dR_x = \int_{\partial R_x} \tau_{ij} n_j v_i d\Gamma_x + \int_{R_x} b_i v_i dR_x - \int_{\partial R_x} q_i n_i d\Gamma_x \tag{30}$$

In eq. (30) q_i is the heat flux going out the boundary ∂R_x, and E is the total energy density. The total energy is related to the specific internal energy, e, by:

$$E = e + \frac{V^2}{2} \tag{31}$$

in which $e=e(T,\rho)$, $V^2=v_i v_i$ and T is the temperature. Replacing $q_i=-k_{ij}T_{,j}$ (Fourier Law of heat conduction and k_{ij} is the conductivity matrix) in eq. (30) and employing the divergence theorem, the energy equations in nonconservation form and conservation form are:

$$\rho(E_{,t[\chi]} + E_{,i}c_i) = (\tau_{ij}v_i)_{,j} + b_i v_i + (k_{ij}T_{,j})_{,i} \tag{32}$$

and

$$(\rho E)_{,t[\chi]} + (\rho E c_i)_{,i} + \rho E \hat{v}_{i,i} = (\tau_{ij}v_j)_{,i} + b_i v_i + (k_{ij}T_{,j})_{,i} \tag{33}$$

respectively.

Constitutive Equation and Equation of State

In order to describe this referential description completely, the constitutive equation and the equation of state are required. To this end, we restrict the fluid under consideration is compressible, isotropic, and viscous. The constitutive equation and equation of state are:

$$\tau_{ij} = -p\delta_{ij} + s_{ij} \tag{34}$$

where

$$s_{ij} = \mu(v_{i,j}+v_{j,i} - \frac{2}{3} v_{k,k}\delta_{ij}) \tag{35}$$

and

$$e = e(T,p) \tag{36}$$

respectively. In particular, if an isothermal process is assumed, the energy equation can be dropped and eq. (36) reduces to

$$\mathring{p} = B(p,\rho)\mathring{\rho}/\rho \tag{37}$$

In above, p is the thermodynamic pressure, B is the bulk modulus, s_{ij} is the component of the shear stress tensor, μ is the dynamic viscosity and δ_{ij} is the Kronecker delta.

If we focus on fluid structure interaction with sloshing, the $\rho_{,i}c_i$ term appearing in eq. (21) is small as compared to the term $\rho_{,t[\chi]}$. Substitute $\mathring{\rho}=\rho_{,t[\chi]}$ into (37) to obtain

$$\mathring{p} = -Bv_{k,k} \tag{38}$$

with similar argument

$$\mathring{p} \tilde{=} p_{,t[\chi]} = -Bv_{k,k} \text{ hence } p = p_o - \int_{t_o}^{t} Bv_{k,k}dt \tag{39}$$

where p_o is the pressure at $t=t_o$.

Governing Equations

The governing equations for the fluid structure system are:

$$\frac{1}{B} p_{,t[\chi]} + v_{k,k} = 0 \qquad \text{in } R_x \tag{40a}$$

$$\rho(v_{i,t[\chi]} + v_{i,j}c_j) = -p_{,i} + \mu(v_{i,j}+v_{j,i})_{,j} + b_i \qquad \text{in } R_x \tag{40b}$$

where

$$v_i = g_i \qquad \text{on } \partial R_x^g \tag{40c}$$

$$\tau_{ij}n_j = h_i \qquad \text{on } \partial R_x^h \tag{40d}$$

$$\hat{v}_i = \text{a given representation depending on } \underset{\sim}{\phi}. \qquad \text{in } R_x \tag{40e}$$

and g_i and h_i are prescribed boundary velocity and traction respectively.

Finite Element Matrix Equations

The weak of variational equations corresponding to eq. (40) are:

$$\int_{R_x} \frac{1}{B} q p_{,t[\chi]} dR_x + \int_{R_x} q v_{k,k} dR_x = 0 \tag{41a}$$

and

$$\int_{R_x} \rho w_i v_{i,t[\chi]} dR_x + \int_{R_x} \rho w_i v_{i,j} c_j dR_x - \int_{R_x} w_{i,i} p dR_x$$

$$+ \int_{R_x} 2\mu w_{(i,j)} v_{(i,j)} dR_x - \int_{R_x} w_i b_i dR_x - \int_{\partial R_x} w_i h_i d\Gamma_x = 0 \tag{41b}$$

In eq. (41a), the test function is denoted by q, and is assumed to be discontinuous across interelement boundaries. In particular, we employ the constant pressure element so that the test function, q, for each element is simply a constant. In eq. (41b), the test function is denoted by w_i, and it satisfies $w_i=0$ on ∂R_x^g.

The corresponding matrix equations are:

$$\underset{\sim}{M}^p \overset{*}{\underset{\sim}{p}}_{n+j} + \underset{\sim}{G}^T \underset{\sim}{v}_{n+j} = \underset{\sim}{0} \tag{42a}$$

and

$$\overset{*}{\underset{\sim}{M}} \underset{\sim}{v}_{n+j} + \underset{\sim}{\eta}(\tilde{\underset{\sim}{v}}_{n+j}) + \underset{\sim}{K}_\mu \tilde{\underset{\sim}{v}}_{n+j} - \underset{\sim}{G} p_{n+j} = \underset{\sim}{f}_{n+j} \tag{42b}$$

where $\underset{\sim}{M}^p$ is the ALE pressure-mass matrix; $\underset{\sim}{M}$ is the ALE fluid-mass matrix; $\overset{*}{p}$ and $\overset{*}{v}$ are $p_{,t[\chi]}$ and $v_{,t[\chi]}$ respectively; η is the convective nodal force; $\underset{\sim}{K}_\mu$ is the viscosity matrix; $\underset{\sim}{G}$ is the gradient operator matrix and $\underset{\sim}{f}$ is the external nodal force vector which consists of both the body force and surface traction contributions.

The mixed time integration procedures [2-3] is used to obtain $\underset{\sim}{v}_{n+j}$ and p_{n+j}. (n is the time step number and j=1,...,m is the sub-step incremental counter). The pressure, p_{n+j}, is integrated with I-mI (i.e. implicit-m implicit); whereas the velocity, $\underset{\sim}{v}_{n+j}$, is integrated with E-mE (i.e. explicit-m explicit). $\tilde{\underset{\sim}{v}}_{n+j}$ is a suitable extrapolator. In order to reduce the computer storage (i.e. the implicit pressure operator, $\underset{\sim}{G}^T \underset{\sim}{M}^{-1} \underset{\sim}{G}$) an iterative solution procedure, the conjugate gradient method [4] is employed. In order to avoid Guassian quadrature numerical integration, one point integration (therefore explicit expressions can be obtained) is applied to all the terms in eq. (42). The stabilization procedures [5,6] are applied to $\underset{\sim}{K}_\mu$ if $\mu \neq 0$. The trapezoidal rule, with the nodes as integration points, is employed to obtain the ALE mass matrix $\underset{\sim}{M}$, so that the matrix is diagonal.

Remarks:

1. The equations of motion derived here can also be used directly in structural analysis. However, it is more convenient to employ the Lagrangian description (i.e. $R_\chi = R_X$). The standard structural dynamic equation:

$$\underset{\sim}{M}^s \underset{\sim}{a}_{n+j} = \underset{\sim}{f}^{ext}_{n+j} - \underset{\sim}{f}^{int}_{n+j} \tag{42c}$$

can be coupled to the ALE fluid equations (eqs. 42a and 42b) directly using the

standard finite element assembly procedures.

2. At the fluid-structure interface, only normal compatibility is required.
3. The E-mE direct time integration is employed for eq. 42c.
4. The matrix $\underset{\sim}{G}^T\underset{\sim}{M}^{-1}\underset{\sim}{G}$ is symmetric and profiled since $\underset{\sim}{M}=\underset{\sim}{M}^{ALE}+\underset{\sim}{M}^{s}$ is diagonal. It is quite different from the conventional added mass concept in which the added mass is a full but symmetric matrix.

Linear Fluid Structure Systems

If the motion of the fluid is small, the convective term, $\underset{\sim}{\eta}$, can be neglected as compared to the impulsive term, $\overset{*}{\underset{\sim}{M}}\underset{\sim}{v}$, and hence there is no distinction between $\underset{\sim}{x}$ and $\underset{\sim}{X}$. With this assumption, the wave effects can be approximated by a perturbation method on the body force term b_i; i.e.,

$$b_i = b_{oi} + b_{1i} \tag{43}$$

where b_{oi} is the body force per unit original volume and b_{1i} is the fluctuation of the body force per unit volume about the "undeformed force surface". Let the Cartesian coordinates be chosen such that x_1 corresponds with the direction of gravity. So

$$b_i = 0 \quad \text{for} \quad i = 2,\ldots,NSD \tag{44}$$

and

$$b_1 = b_{o1} + \gamma u_1 A_F \tag{45}$$

where $\gamma=\rho g$ is the specific weight of the fluid, g is the gravity acceleration, A_F is the area of the free surface and u_1 is the free surface displacement about the "undeformed free surface" in the x_1 direction.

With the above assumption, eq. (42) reduces to:

$$\underset{\sim}{M}^{p}\overset{*}{\underset{\sim}{p}}_{n+j} + \underset{\sim}{G}^T\underset{\sim}{v}_{n+j} = \underset{\sim}{0} \tag{46a}$$

and

$$\overset{*}{\underset{\sim}{M}}\underset{\sim}{v}_{n+j} + \underset{\sim}{K}_{\mu}\tilde{\underset{\sim}{v}}_{n+j} - \underset{\sim}{G}\underset{\sim}{p}_{n+j} + \underset{\sim}{K}_{FS}\hat{\underset{\sim}{u}}_{n+j} = \hat{\underset{\sim}{f}}_{n+j} \tag{46b}$$

where $\underset{\sim}{K}_{FS}$ is the free-surface matrix, $\hat{\underset{\sim}{u}}$ is the free surface displacement and $\hat{\underset{\sim}{f}}_{n+j}=\tilde{\underset{\sim}{f}}_{n+j}-\underset{\sim}{K}_{FS}\hat{\underset{\sim}{u}}_{n+j}$.

Linear Incompressible Irrotational Fluid

If we further assume the fluid is incompressible and irrotational, then there exists a velocity potential, ϕ, such that

$$v_i = \phi_{,i} \tag{47}$$

The continuity equation, eq. (21), implies:

$$\phi_{,ii} = 0 \qquad \text{in } R_x \tag{48}$$

The linearized kinetic boundary condition combined with the linearized momentum equation on the free surface boundary, ∂R_x^{FS}, yield:

$$\phi_{,i}n_i = -\frac{\ddot{\phi}}{g} \qquad \text{on } \partial R_x^{FS} \tag{49}$$

the pressure and the free surface displacement can be determined from

$$p = -\rho\dot{\phi} \qquad \text{in } R_x \tag{50}$$

and

$$u_1 = -\frac{\dot{\phi}}{g} \qquad \text{on } \partial R_x^{FS} \tag{51}$$

respectively. The weak or variational equation is:

$$\int_{\partial R_x^{FS}} \frac{1}{g} q\ddot{\phi} d\Gamma_x + \int_{R_x} q_{,i}\phi_{,i} dR_x = -\int_{\partial R_x^{I}} \rho q v_{ns} d\Gamma_x \tag{52}$$

where q is the test function, v_{ns} is the prescribed inward normal velocity on the fluid-structure boundary, ∂R_x^{I}. The corresponding matrix equation is:

$$\underset{\sim}{M}^{FS}\ddot{\underset{\sim}{\phi}} + \underset{\sim}{K}_{\phi}\underset{\sim}{\phi} = \underset{\sim}{f}_{\phi} \tag{53}$$

Concluding Comments

In this paper, three different finite element fluid models have been described for fluid-structure interaction with sloshing. All these methodologies have been integrated into working finite element computer codes [7,8]. Because of the space limitation, numerical examples will not be presented here. They can be found in [8-10] and references cited therein.

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