COMBINATORIAL MATHEMATICS YEAR

February 1969 - June 1970

SPERNER TYPE THEOREMS[1]

by

G. KATONA[2]

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

Department of Statistics, University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 600.17

JULY 1969

[1] This research was partially sponsored by the National Science Foundation under Grant GU-2059.

[2] This work was done while the author was at the Department of Statistics, University of North Carolina at Chapel Hill.

Sperner Type Theorems[1]

by

G. Katona[2]
Mathematical Institute of the
Hungarian Academy of Sciences, Budapest

First, we should like to list four problems from several branches of mathematics.

1. A *logical* or *truth function* is an n-dimensional function defined on the n-dimensional 0-1 vectors and taking on the values 0, 1. A truth function f is said to be *monotonically increasing* if $f(a_1,a_2,\dots,a_n) = 1$ and $a_1 \le b_1,\dots,a_n \le b_n$ imply $f(b_1,b_2,\dots,b_n) = 1$.

If we fix the values of the variables $a_{i_1},a_{i_2},\dots,a_{i_r}$, say, $a_{i_1}^*,a_{i_2}^*,\dots,a_{i_r}^*$, then we call the set $\{a_{i_j}^*\}$ an *implicant* of f if $f(a_1,a_2,\dots,a_{i_1}^*,\dots,a_{i_r}^*,\dots,a_n) = 1$ for all the values of a_j's ($j \ne i_\ell$, $\ell = 1,2,\dots,r$).

For example, let us consider a ballot with n people voting, and if at least k of them vote "yes", suppose that the proposal is accepted. It is obvious that this vote function is monotonically increasing, and we obtain an implicant if we fix the votes of at least k people as "yes" and the votes of some other people as "no".

From this example, it is obvious that an important class of implicants are those which cease to be implicants when some of their fixed values are

---

1 This research was partially sponsored by the National Science Foundation under Grant GU-2059.

2 This work was done while the author was at the Department of Statistics, University of North Carolina at Chapel Hill.

omitted. These are called *prime-implicants*. In our example, if we fix the votes of exactly k people as "yes", we obtain a prime-implicant.

Returning to the general case, a prime-implicant can not have a fixed value equal to zero (e.g., $a_{i_r}* = 0$) if the function is monotonically increasing. For by changing this zero to 1 by the monotonicity $f(a_1,\ldots,a_{i_1}*,\ldots,a_{i_r}*+1,\ldots,a_n)$ also equals 1 for all values of the a_i's. That is, $a_{i_1}*,\ldots,a_{i_{r-1}}*$ would also be an implicant, contradicting the definition of prime implicants. That means we may consider the prime implicants as subsets of the variables (the elements of which are fixed to be 1), and such a subset can not be contained in any other such subset. The question arises: *What is the maximum number of prime implicants of a monotonically increasing truth function?* It is then clear that the problem is equivalent to the following:

(P) *Given a set of n elements, what is the size of a maximal family of subsets with the property that none of its members contains any other one?*

2. [2]. *Given a positive square-free integer m, what is the size of a maximal subset of divisors of m with the property that none of these divisors is a multiple of any other one.* E.g., if $m=30=2\cdot3\cdot5$, $\{2\cdot3,\ 2\cdot5,\ 3\cdot5\}$ is such a set. In general, we can associate with every divisor of m a subset of its prime factors, and, therefore, if a set of divisors has the above property then the family of the associated subsets has the property described under (P). Thus, this problem also leads to problem (P).

3. [8]. *Given a set of real numbers,* $a_1, a_2, \ldots, a_n$, *with the property* $a_i \geq 1$ $(1 \leq i \leq n)$, *and an interval* $I = (b, b+1)$. *How many of the* 2^n *numbers* $\Sigma_{i=1}^n \varepsilon_i a_i$ *lie in* I, *where* $\varepsilon_i = 0$ *or* $+1$?

Let $\Sigma_{i=1}^n \varepsilon_i a_i$ and $\Sigma_{i=1}^n \varepsilon_i' a_i$ be two numbers lying in I, where $\varepsilon_i \neq \varepsilon_i'$ for some i. Denote by A and A' the set of indices for which $\varepsilon_i = 1$ and $\varepsilon_i' = 1$, respectively. $A \supset A'$ cannot hold, since this would imply $\Sigma_{i=1}^n \varepsilon_i a_i - \Sigma_{i=1}^n \varepsilon_i' a_i = \Sigma_{i \in A} a_i - \Sigma_{i \in A'} a_i = \Sigma_{i \in A-A'} a_i \geq 1$, which is a contradiction, for two numbers of difference at least 1 could not lie in an interval $I = (b, b+1)$.

That means, if we take all the numbers $\Sigma_{i=1}^n \varepsilon_i a_i$ lying in I and if we consider the subsets of indices for which $\varepsilon_i = 1$, the family of these subsets possesses the property described under (P). Thus, this problem also leads to problem (P).

3A. We can formulate this problem in the language of the probability theory, too. Let ε_i $(1 \leq i \leq n)$ be a random variable taking on the values 0 and 1 with probability $\frac{1}{2}$. What is the maximum value of $P(b < \Sigma_{i=1}^n \varepsilon_i a_i < b+1)$?

4. Finally, we give a problem connected with finite search theory.

Let a set S of n elements be given and suppose we are looking for an unknown element of S, having information as to whether certain subsets contain the unknown element or not. It is reasonable to assume that any two of these subsets are such that if we received the information concerning the first one, we can never predict the information concerning the second one. In other words, any two subsets divide the set into four non-void parts.

(Following Rényi, two such subsets are called qualitatively independent.)

The problem is the following: What is the size of a maximal family $A_1, A_2, \ldots, A_m$ of subsets of S having the property that A_i and A_j $(i \neq j)$ are qualitatively independent?

Clearly, A_i and A_j are qualitatively independent if and only if none of the subsets $A_i, \overline{A}_i, A_j, \overline{A}_j$ ($\overline{A}$ means $S-A$) contains any other one. That means, $A_1, \overline{A}_1, A_2, \overline{A}_2, \ldots, A_m, \overline{A}_m$ must have this property, and we are again lead to problem (P).

The answer to problem (P) is contained in the following theorem:

1. SPERNERS THEOREM [2]. *Let S be a set of n elements, and let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a family of subsets of S with the Sperner-property: $A_i \not\supseteq A_j$ $(i \neq j)$. Then*

$$m \leq \binom{n}{[n/2]}.$$

We now give three proofs of this theorem.

1. PROOF (Sperner, [2], 1928). Let $\mathcal{B} = \{B_1 . B_2, \ldots, B_r\}$ be a family of subsets of S such that $|B_i| = v$ $(1 \leq i \leq r)$. Then $c(\mathcal{B})$ denotes the family of all subsets C satisfying $|C| = v-1$ and $C \subset B_i$ for some i. We need a simple lemma:

LEMMA 1. *If $\mathcal{B} = \{B_1, \ldots, B_r\}$ is a family of subsets of a set S of n elements, and if $|B_i| = v$ $(1 \leq i \leq r)$ then*

$$|c(\mathcal{B})| \geq r \cdot \frac{v}{n-v+1}.$$

PROOF: Count the number of the pairs B_i, C where $C \subset B_i$ and $|C| = v-1$; then

$$r \cdot v \le |c(\mathcal{B})| \cdot (n-v+1)$$

because a set C can be contained in at most $n-v+1$ of the B_i's. The proof is complete.

Let us now return to the proof of the theorem. Put $v = \max_{1 \le i \le m} |A_i|$, and let $i_1, i_2, \ldots, i_r$ be those indices for which the maximum is attained, that is, $\mathcal{B} = \{A_{i_1}, \ldots, A_{i_r}\}$ is the family of maximal elements of $\mathcal{A}$.

Now we prove that if $\mathcal{A}$ is an optimal family and n is even, then

$$v \le \frac{n}{2}, \tag{1}$$

Assume the opposite, $v > \frac{n}{2}$. Then $|c(\mathcal{B})| \ge r \cdot v/(n-v+1) \ge r((n/2 + 1)/(n/2)) > r$.

Replacing $A_{i_1}, \ldots, A_{i_r}$ by the elements of $c(\mathcal{B})$, we obtain a new family having more than m elements and possessing the Sperner property (or briefly, which is a Sperner-family). This contradicts the maximality of $\mathcal{A}$.

In the case when n is odd, similar considerations lead only to $v \le (n+1)/2$, however, if $v = (n+1)/2$, then $v/(n-v+1) = 1$ and $|c(\mathcal{B})| \ge r$, that is, there always exists an optimal family with $v \le (n-1)/2$.

If $\mathcal{A} = \{A_1, \ldots, A_m\}$ possesses the Sperner-property, then obviously so does $\mathcal{A}^* = \{\overline{A}_1, \ldots, \overline{A}_m\}$.

Thus, for an optimal family $\mathcal{A}$, $|\overline{A}_i| \le \frac{n}{2}$ must hold if n is even. Comparing this with (1), we obtain $|A_i| = \frac{n}{2}$ $(1 \le i \le m)$. If n is odd, we have $|\overline{A}_i| \le (n+1)/2$ and thus, there is an optimal family with $|A_i| = (n-1)/2 = [n/2]$. It is trivial that the family of all $[n/2]$ tuples of S is a Sperner-family.

The proof is finished, but the uniqueness of the optimal family was proved only for even n's. If n is odd, we proved that $m \le \binom{n}{[n/2]}$ and in an optimal family all the subsets have $(n-1)/2$ or $(n+1)/2$ elements. We will prove later, that there exist only two optimal families: the family of all the $(n-1)/2$ tuples and the family of all the $(n+1)/2$ tuples.

2. PROOF (De Bruijn, [1], 1952). We say that a family $\mathcal{B} = \{B_1, B_2, \ldots, B_r\}$ is a *symmetrical chain* if

1. $B_1 \subset B_2 \subset \ldots \subset B_r$
2. $|B_{i+1}| - |B_i| = 1$
3. $|B_1| + |B_r| = n$ (that is, the chain is symmetrical with respect to $n/2$).

LEMMA 2. *It is possible to divide all the subsets of S into disjoint symmetrical chains.*

PROOF We shall use induction on n. For $n=1$, the statement is trivial. Assume now, that for $n-1$ we have constructed disjoint symmetrical chains $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_u$. From $\mathcal{B}_i = \{B_{i1}, B_{i2}, \ldots, B_{ir_i}\}$ we construct two (if $r_i = 1$, only one) new chains:

$$\mathcal{B}_i^{*} = \{B_{i1}, B_{i2}, \ldots, B_{ir_i}, B_{ir_i} \cup \{x_n\}\}$$

$$\mathcal{B}_i^{**} = \{B_{i1} \cup \{x_n\}, \ldots, B_{i,r_i-1} \cup \{x_n\}\} ,$$

where X_n is the new, n-th element. It is obvious that the new chains are symmetrical, further that every subset is contained in at least one new chain and that the new chains are disjoint.

Now we return to the proof of the theorem. Every $[n/2]$-tuple must be contained in at least one symmetrical chain. On the other hand, every symmetrical chain contains exactly one $[n/2]$-tuple because of the symmetry. Thus, the number of symmetrical chains is exactly $\binom{n}{[n/2]}$. Finally, a symmetrical chain can contain at most one element from a Sperner-family and the theorem follows.

3. PROOF (Lubell, [18], 1966). A *maximal chain* is a chain of $n+1$ subsets. Count all the maximal chains of a set of n elements. A maximal chain is determined uniquely by a permutation of the n elements (in which order we add them to the preceding subset). Thus the number of maximal chains is $n!$ Now fix a subset A and count the maximal chains containing A as an element. We obtain $|A|!(n-|A|)!$ If $\mathcal{A} = \{A_1,\ldots,A_m\}$ is a Sperner-family, then no maximal chain can contain more than one A_i. Thus,

$$\sum_{i=1}^{m} |A_i|!(n-|A_i|)! \le n!.$$

Hence, we obtain

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \le 1$$

or

$$(2) \qquad \sum_{k=0}^{n} \frac{d_k}{\binom{n}{k}} \leq 1,$$

where d_k is the number of A_i's with $|A_i| = k$. (2) is already a generalization of the Sperner theorem. Indeed, by $\binom{n}{k} \leq \binom{n}{[n/2]}$ we obtain

$$\frac{m}{\binom{n}{[n/2]}} \leq 1.$$

and the proof is complete.

REMARK. The inequality (2) was first observed by L.D. Meshalkin [4] (1963).

Now we will consider several generalizations of Sperner's theorem. The four problems mentioned at the beginning will serve as a basis for new generalizations. Let us first return to Problem 1. For practical purposes, it is very important to write a truth function in a well-determined form, for example, in a disjunctive-normal form. This is the basis of the synthesis of a truth function in electrical engineering. The *disjunctive-normal form* is a disjunction of different terms, where a term is a conjunction of the variables a_i's and $\bar{a}_i$'s such that at most one of a_i and $\bar{a}_i$ occurs in the same term:

$$f(a_1,a_2,\ldots,a_n)=(a_{i_{11}}\&\bar{a}_{i_{12}}\&\ldots\&a_{i_{1r}})\vee\ldots\vee(\bar{a}_{i_{t1}}\&a_{i_{t2}}\&\ldots\&\bar{a}_{i_{ts}}).$$

$(0\vee 0 = 0,\ \ 0\vee 1 = 1\vee 0 = 1\vee 1 = 1,\ \ 0\&0 = 0\&1 = 1\&0 = 0,\ \ 1\&1 = 1)$.

It is a well-known fact in the theory of truth functions that the disjunction of all the prime-implicants has minimal length (minimal number of operations) among the disjunctive-normal forms of a monotonically increasing truth function. That means the number of needed electrical components will be minimal if we make the synthesis on a basis of this disjunctive-normal form instead of others. Thus, the real problem, in this case, is not to determine the maximum number of prime-implicants but the maximum of the sum of variables of the prime-implicants. Or, in the language of subsets, we have to determine $\max \Sigma_{i=1}^{m} |A_i|$, where $A = \{A_1,\ldots,A_m\}$ is a Sperner-family. Here we can use (2): What is the maximum of $\Sigma_{k=0}^{n} kd_k$ if $\Sigma_{k=0}^{n}(d_k/\binom{n}{k}) \leq 1$?

We can write

$$(3) \qquad \sum_{k=0}^{n} d_k\{n/2\}\binom{n}{\{n/2\}}/\binom{n}{k} \leq \{n/2\}\binom{n}{\{n/2\}},$$

where $\{x\}$ denotes the least integer not less than x. Here, if $k \geq \{\frac{n}{2}\}$ then

$$(4) \qquad \{n/2\}\binom{n}{\{n/2\}}/\binom{n}{k} = \frac{n(n-1)\ldots(n-\{n/2\}+1)}{1\ldots(\{n/2\}-1)} \cdot \frac{1 \cdot 2\ldots k}{n\ldots(n-k+1)}$$

$$= \frac{\{n/2\}}{n-k+1} \cdot \frac{\{n/2\}+1}{n-k+2} \cdots \frac{k-1}{n-\{n/2\}} \cdot k \geq k$$

and if $k < \{n/2\}$, then $n-k \geq \{n/2\}$, thus

$$(5) \qquad \{n/2\}\binom{n}{\{n/2\}}/\binom{n}{k} = \{n/2\}\binom{n}{\{n/2\}}/\binom{n}{n-k} \geq \{n/2\} > k.$$

Using (4) and (5) in (3), we obtain

$$\sum_{k=0}^{n} kd_k \leq \{n/2\}\binom{n}{\{n/2\}}.$$

Thus we can state:

THEOREM 2. *If $A = \{A_1,\ldots,A_m\}$ is a Sperner-family, then*

$$\sum_{i=1}^{m} |A_i| \leq \{n/2\}\binom{n}{\{n/2\}}.$$

Let us now consider Problem 3. Previously we proved that the maximum number of sums $\Sigma_{i=1}^{n}\varepsilon_i a_i$ lying in an open interval $(b,b+1)$ is less than or equal to the solution of the Sperner-problem, that is, $\le \binom{n}{[n/2]}$. If we choose $a_i = 1$ $(1 \le i \le n)$, then $\binom{n}{[n/2]}$ sums have the value $[n/2]$, that is, our estimation obtained by Sperner's theorem is sharp.

Now let us generalize the problem. It is clear that if we use $\varepsilon_i = -\frac{1}{2},\frac{1}{2}$ instead of $\varepsilon_i = 0,1$, then the sums $\Sigma_{i=1}^{n}\varepsilon_i a_i$ decrease uniquely by $\frac{1}{2}\Sigma_{i=1}^{n} a_i$, thus the maximum number of sums lying in an open interval of length 1 does not change. Multiplying by 2, we obtain $\varepsilon_i = -1,1$ and an interval of length 2. Here we can already allow negative a_i's with $|a_i| \ge 1$ (instead of $a_i \ge 1$). Indeed, in this case, the set of values of sums $\Sigma_{i=1}^{n}\varepsilon_i a_i$ is unchanged if we replace some of the a_i's by $-a_i$. Hence we obtain the following result:

If $a_1,\ldots,a_n$ are real numbers with $|a_i| \ge 1$, and $\varepsilon_i = -1,+1$, then the maximum number of sums $\Sigma_{i=1}^{n}\varepsilon_i a_i$ lying in an open interval of length 2 is $\binom{n}{[n/2]}$.

But what can we say about longer intervals, say, of length $2h$?

Let us again associate with each sum $\Sigma_{i=1}^{n}\varepsilon_i a_i$ lying in an interval $(b,b+2h)$ the subset of indices with $\varepsilon_i = 1$. Clearly, among these subsets there does not exist a pair A, A' for which $A \supset A'$ and $|A-A'| \ge h$ because of

$$\sum_{i=1}^{n} \varepsilon_i a_i - \sum_{i=1}^{n} \varepsilon_i' a_i = \Big(\sum_{i\in A} a_i - \sum_{i\notin A} a_i\Big) - \Big(\sum_{i\in A'} a_i - \sum_{i\notin A'} a_i\Big)$$

$$= 2 \sum_{i\in A-A'} a_i \ge 2h.$$

Thus, the following theorem will be useful.

THEOREM 3. [8]. *Let* $A = \{A_1, A_2, \ldots, A_m\}$ *be a family of subsets of a set* S *of* n *elements, and* h *a natural number. If*

(6) $$A_i \supset A_j \quad |A_i - A_j| \geq h \qquad (1 \leq i,\ j \leq m)$$

never holds, then m *is less than or equal to the sum of the* h *largest binomial coefficients of order* n.

This theorem will be a special case of the following theorem:

THEOREM 3A. (Erdős, 1945, [8]). *Let* $A = \{A_1, \ldots, A_m\}$ *be a family of different subsets of a set* S *of* n *elements and* h *a natural number. If the family has the property that a sequence of different subsets*

(7) $$A_{i_1} \supset A_{i_2} \supset \ldots \supset A_{i_{h+1}}$$

does not exist, then m *is less than or equal to the sum of the largest* h *binomial coefficients of order* n, *and this estimate is the best possible.*

PROOF OF THEOREM 3A. The proof is based on Lemma 2. In a symmetrical chain at most h of the above subsets can occur. If the length ℓ of the chain is less than h, then this number is obviously at most ℓ. We only have to count the number of chains having length at least ℓ.

Consider a subset of $[(n+\ell)/2]$ elements. Because of the symmetricity, it is contained in a chain of length at least ℓ. On the other hand, each chain of length at least ℓ must contain a subset of $[(n+\ell)/2]$ elements. Thus, the number of chains of length at least ℓ is exactly $\binom{n}{[(n+\ell)/2]}$. Thus, the desired estimate is:

$$\left(\sum_{\ell=1}^{h-1} \ell \cdot (\text{the number of symmetrical chains with length } \ell)\right)$$

$$+ (h \cdot (\text{the number of symmetrical chains with length at least } h))$$

$$= \sum_{\ell=1}^{h} (\text{the number of symmetrical chains with length at least } \ell)$$

$$= \sum_{\ell=1}^{h} \binom{n}{[(n+\ell)/2]} = \sum_{i=[(n-h+1)/2]}^{[(n+h-1)/2]} \binom{n}{i} .$$

This latter quantity is the sum of the h largest binomial coefficients.

The family of all the $[(n-h+1)/2]$-tuples, $[(n-h+1)/2]+1$-tuples, ..., $[(n+h-1)/2]$-tuples shows that the estimate of the theorem is the best possible.

PROOF OF THEOREM 3. (Erdős' original proof followed the proof of Sperner. Here we follow De Bruijn's proof.) If a family has no subsets of type (6), then it cannot have subsets of type (7), thus the same estimate is valid. The family of all $[(n-h+1)/2]$-tuples, $[(n-h+1)/2]+1$-tuples, ..., $[(n+h-1)/2]$-tuples shows again that the estimate is the best possible.

We can formulate Theorem 3A in another form, too:

THEOREM 3B. *Let* $\mathsf{A}_1 = \{A_{11},\dots,A_{1m_1}\}$, $\mathsf{A}_2 = \{A_{21},\dots,A_{2m_2}\}$, $\dots$, $\mathsf{A}_h = \{A_{h1},\dots,A_{hm_h}\}$ *be disjoint families of subsets of a set* S *of* n *elements. If each* A_i $(1 \le i \le h)$ *is a Sperner-family, then*

$$\sum_{i=1}^{h} m_i$$

is less than or equal to the sum of the h *largest binomial coefficients of order* n, *and this estimate is the best possible.*

PROOF. The family $\mathsf{A} = \cup_{i=1}^{h} \mathsf{A}_i$ satisfy the conditions of Theorem 3A, thus the same estimate is valid.

Returning now to Problem 3A, can we generalize it to two-dimensional vectors a_i with $|a_i| \ge 1$? The answer is yes, but not trivial. It is clear that in this case we can reduce the problem to the case when the first coordinate of the a_i's is non-negative. However, the Sperner-theorem does not help, because the sum of vectors a_i in the half plane with $|a_i| \ge 1$ can have absolute value less than 1. (In the half-line this cannot happen, which is the reason for the applicability of the Sperner-theorem in the one-dimensional case.) But, in a quadrant, the sum of the a_i's have again absolute value greater than 1. Thus, instead of the Sperner-theorem, we can use a theorem where the elements are divided into two disjoint parts:

THEOREM 4. (Kleitman [11] and Katona [19], independently (1965)). *Let* S_1 *and* S_2 *be disjoint sets of* n_1 *and* n_2 *elements respectively* $n_1 \le n_2$. *If* $\mathsf{A} = \{A_1,\dots,A_m\}$ *is a family of subsets of* $S_1 \cup S_2$ *and if neither*

(8) $$A_i \cap S_1 = A_j \cap S_1, \quad A_i \cap S_2 \supset A_j \cap S_2 \quad (i \neq j)$$

nor

(9) $$A_i \cap S_1 \supset A_j \cap S_1, \quad A_i \cap S_2 = A_j \cap S_2 \quad (i \neq j)$$

can occur, then

$$m \leq \binom{n}{[n/2]}, \quad \text{where} \quad n = n_1 + n_2.$$

PROOF. By Lemma 2, we can divide the subsets of S_1 into disjoint symmetrical chains. Let $B_1, B_2, \ldots, B_h$ be such a chain. Denote by $\mathcal{A}_\ell$ the following family of intersections of S_2 with those A_j which have a common intersection B_ℓ with S_1:

$$\mathcal{A}_\ell = \{A: \text{ for some } j, \ A = A_j \cap S_2 \text{ and } A_j \cap S_1 = B_\ell\}.$$

$\mathcal{A}_\ell$ and $\mathcal{A}_k$ $(\ell \neq k)$ are disjoint because of (9). Namely, if $A \in \mathcal{A}_\ell$ and $A \in \mathcal{A}_k$ were true, then the sets $B_\ell \cup A$ and $B_k \cup A$ would satisfy (9), since $B_\ell \supset B_k$. Moreover, $\mathcal{A}_\ell$ $(1 \leq i \leq h)$ is a Sperner-family because of (8). Thus, we can use Theorem 3B:

$$\sum_{\ell=1}^{h} |\mathcal{A}_\ell| \leq \sum_{\ell=[(n-h+1)/2]}^{[(n+h-1)/2]} \binom{n_2}{\ell}.$$

Hence we obtain

(10) $$m \leq \sum \sum_{\ell=[(n_2-h+1)/2]}^{[(n_2+h-1)/2]} \binom{n_2}{\ell},$$

where the first sum runs over all the symmetrical chains given in the decomposition of S_1.

From the proof of Theorem 3A, we know that the number of symmetrical chains of length at least h in S_1 is $\binom{n_1}{[(n_1+h)/2]}$. It follows from (10) that

$$m \le \sum_{h=1}^{n_1+1} \binom{n_1}{[(n_1+h)/2]}\binom{n_2}{[(n_2-h+1)/2]} = \sum_{i=0}^{n_1} \binom{n_1}{i}\binom{n_2}{[(n_1+n_2)/2]-i}$$

$$= \binom{n_1+n_2}{[(n_1+n_2)/2]}.$$

The proof is finished.

This theorem is a stronger form of Sperner's theorem since the condition is weaker, but the result is the same. The only difference is that the optimal family is not unique. An example of an optimal family is the family of different A's satisfying $|A\cap S_1| = u$ and $|A\cap S_2| = (n_2-n_1)/2 + u$ (n_1+n_2 even, $0 \le u \le n_1$).

Using Theorem 4, it is not difficult to verify the statement of Problem 3A in the two-dimensional case. The statement for the three-dimensional case is not yet proved. The generalization of Theorem 4 for 3 parts does not help. There are two possible generalizations. If we exclude

(i) $A_i\cap S_1 = A_j\cap S_1 \quad A_i\cap S_2\supset A_j\cap S_2 \quad A_i\cap S_3\supset A_j\cap S_3$,

(ii) $A_i\cap S_1\supset A_j\cap S_1 \quad A_i\cap S_2 = A_j\cap S_2 \quad A_i\cap S_3\supset A_j\cap S_3$,

(iii) $A_i\cap S_1\supset A_j\cap S_1 \quad A_i\cap S_2\supset A_j\cap S_2 \quad A_i\cap S_3 = A_j\cap S_3$,

then for S_1 and $S_2\cup S_3$, the conditions of Theorem 4 are satisfied, and the same result is valid, but it is not useful.

If we exclude

$$(11)\quad \begin{array}{llll} \text{(i)} & A_i\cap S_1 = A_j\cap S_1 & A_i\cap S_2 = A_j\cap S_2 & A_i\cap S_3\supset A_j\cap S_3, \\ \text{(ii)} & A_i\cap S_1 = A_j\cap S_1 & A_i\cap S_2\supset A_j\cap S_2 & A_i\cap S_3 = A_j\cap S_3, \\ \text{(iii)} & A_i\cap S_1\supset A_j\cap S_1 & A_i\cap S_2 = A_j\cap S_2 & A_i\cap S_3 = A_j\cap S_3, \end{array}$$

it would be useful, but the estimate of Theorem 4 is not true in this case. For example, if $|S_1| = n-2$, $|S_2| = |S_3| = 1$ (n is odd), the family of all subsets A satisfying one of

$$\begin{array}{llll} \text{(i)} & |A\cap S_1| = \frac{n-3}{2} & |A\cap S_2| = 0 & |A\cap S_3| = 0, \\ \text{(ii)} & |A\cap S_1| = \frac{n-3}{2} & |A\cap S_2| = 1 & |A\cap S_3| = 1, \\ \text{(iii)} & |A\cap S_1| = \frac{n-1}{2} & |A\cap S_2| = 0 & |A\cap S_3| = 1, \\ \text{(iv)} & |A\cap S_1| = \frac{n-1}{2} & |A\cap S_2| = 1 & |A\cap S_3| = 0, \end{array}$$

does not contain subsets of type (11). Nevertheless, it has $4\binom{n-2}{(n-3)/2} = 2\binom{n-1}{(n-1)/2}$ $\left(> \binom{n}{(n-1)/2}\right)$ elements.

OPEN PROBLEM. What is the maximum number of elements of a family A of subsets having the property that none of (11) can hold?

Let us consider again Problem 2. Is its statement true for an arbitrary non-square-free natural number m, too? Obviously, instead of subsets (which are 0-1 valued functions), we have to consider non-negative integer-valued functions which have an upper bound (the exponent of p_i in m) for the values at each point.

THEOREM 5. (De Bruijn, Tengbergen, Kruyswijk [1], 1952). *Let $f_1, f_2, \dots, f_m$ be integer-valued functions defined on a set $S = \{x_1, \dots, x_n\}$,* *satisfying the condition*

$$(12) \qquad 0 \le f_i(x_k) \le \alpha_k, \quad (1 \le i \le m, \quad 1 \le k \le n)$$

where the α_k*'s age given positive integers. If*

$$f_i \le f_j$$

does not hold for any pair $(i \neq j)$ *then*

$$m \le M$$

where M *is the number of functions satisfying* $\Sigma_{k=1}^{n} f(x_k) = [(\Sigma_{i=1}^{n} \alpha_i)/2] = [\alpha/2]$.

In the case of $\alpha_j = 1$ $(1 \le j \le n)$, the theorem reduces to the Sperner-theorem.

PROOF. First we prove a generalization of Lemma 2, and give the generalizations of the needed concepts. We say that a set $F = \{f_1, \ldots, f_r\}$ of functions satisfying (12) is a *chain*, if $f_1 \le f_2 \le \ldots \le f_r$ and each f_i $(r \ge i \ge 1)$ differs from f_{i-1} only in one place by 1, that is there exists an x_{k_i} for every f_i such that

$$f_i(x_{k_i}) = f_{i-1}(x_{k_i}) + 1 \quad \text{and}$$

$$f_i(x) = f_{i-1}(x) \quad \text{if} \quad x \neq x_{k_i}.$$

We say that a chain $F = \{f_1, \ldots, f_r\}$ is *symmetrical* if

$$\sum_{k=1}^{n} f_1(x_k) + \sum_{k=1}^{n} f_r(x_k) = \alpha.$$

LEMMA 3. [1]. *It is possible to divide all the functions satisfying (12) into disjoint symmetrical chains.*

PROOF. We shall use induction over n. For $n=1$, the statement is trivial. Assume now that for $n-1$, we have the disjoint symmetrical chains $F_1, F_2, \ldots, F_u$. If f is a function defined on the first $n-1$ x's, f^y denotes its extension with $f^y(x_n) = y$. We construct for a chain $F_i = \{f_{i1}, \ldots, f_{ir_i}\}$ new chains in the following manner:

$$\begin{array}{ccccc}
f^0_{i1} & f^1_{i1} & f^2_{i1} & \cdots\cdots & f^{\alpha_n}_{i1} \\
f^0_{i2} & f^1_{i2} & f^2_{i2} & \cdots\cdots & f^{\alpha_n}_{i2} \\
\vdots & \vdots & \vdots & & \\
f^0_{i,r_i-2} & f^1_{i,r_i-2} & f^2_{i,r_i-2} & & f^{\alpha_n}_{i,r_i-2} \\
f^0_{i,r_i-1} & f^1_{i,r_i-1} & f^2_{i,r_i-1} & & f^{\alpha_n}_{i,r_i-1} \\
f^0_{i,r_i} & f^1_{i,r_i} & f^2_{i,r_i} & & f^{\alpha_n}_{i,r_i}
\end{array}$$

Clearly, $F^j_i = \{f^j_{i1}, f^j_{i2}, \ldots, f^j_{i,r_i-j}, f^{j+1}_{i,r_i-j}, \ldots, f^{\alpha_n}_{i,r_i-j}\}$ $(0 \le j \le \min(\alpha_n, r_i-1))$ will be a symmetrical chain again.

Indeed,

$$\sum_{k=1}^{n} f^j_{i1}(x_k) + \sum_{k=1}^{n} f^{\alpha_n}_{i,r_i-j}(x_k) = \sum_{k=1}^{n-1} f_{i1}(x_k) + \sum_{k=1}^{n-1} f_{i,r_i-j}(x_k) + j + \alpha_n$$

$$= \sum_{k=1}^{n-1} f_{i1}(x_k) + \left(\sum_{k=1}^{n-1} f_{i,r_i}(x_k) - j\right) + j + \alpha_n = \sum_{k=1}^{n} \alpha_k .$$

Further, it is easy to see that:

1. Every function is contained in at least one new chain.

2. The new chains are disjoint.

Now, we can return to the proof of the theorem. Every function having the value-sum

$$(13) \qquad \sum_{k=1}^{n} f(x_k) = [\alpha/2]$$

must be contained in a symmetrical chain. On the other hand, every symmetrical chain contains exactly one function of type (13). Thus, the number of symmetrical chains is exactly the same as the number of functions satisfying (13), that is, M.

The set of all functions satisfying (13) shows that our estimate is the best possible. The theorem is proved.

Recently, Schönheim [9] proved a common generalization of Theorems 3A and 5, further of Theorems 4 and 5.

Now we give a generalization of all three theorems in a more general language.

We will say that a directed graph G is a *symmetrical chain-graph* if

1. There exists a partition of its vertices into disjoint subsets $K_0, K_1, \dots, K_n$ (they are called *levels*) of $k_0, k_1, \dots, k_n$ elements, and all the directed edges connect a vertex of K_i with a vertex of K_{i+1} $(0 \le i < n)$.
2. $k_0 \le k_1 \le \dots \le k_{[n/2]}$
   $k_i = k_{n-i} \qquad (0 \le i \le n)$

3. There exists a partition of its vertices into disjoint symmetrical chains, where a symmetrical chain is a set of vertices such that the vertices are connected by a directed path, and further, if the starting point of this path is in K_i, then the endpoint is in K_{n-i}.

Consider now a set S of n elements. Let its subsets be the vertices of a graph G and connect two vertices A and B if $B \supset A$ and $|B-A| = 1$. It is easy to see that this graph satisfies the conditions 1 and 2, and by Lemma 2, condition 3 also.

Similarly, if we consider the graph of functions of type (12) and we connect two vertices f and g if $f = g$ except in one place x where $f(x) + 1 = g(x)$, then by Lemma 3, it will also be a symmetrical chain-graph.

The direct sum $G+H$ of two symmetrical chain-graphs G and H will be the following. Its vertices will be the ordered pairs (g,h) $g \in G$, $h \in H$, and (g_1,h_1) is joined to (g_2,h_2) only if $g_1=g_2$ and h_1 and h_2 are joined or $h_1=h_2$ and g_1 and g_2 are joined.

If G is the graph of the subsets of a set S_1 and H is the graph of the subsets of a (disjoint) set S_2, then $G+H$ is the graph of the subsets of $S_1 \cup S_2$. The situation is similar in the case of finite functions of type (12); the direct sum produces a larger graph of similar type.

The generalization of the Sperner (and also of De Bruijn-Tenbergen-Kruyswijk) theorem in this language is the following. If we have a set $a_1, a_2, \ldots, a_n$ of vertices of a symmetrical chain-graph and no two of them are connected by a directed path, then

$$m \leq k_{[n/2]} .$$

The generalization of Theorem 3A would then be:

(14) If no $h+1$ of $a_1, a_2, \ldots, a_m$ $(a_i \neq a_j)$ lie in a directed chain, then $m \leq$ the sum of the h largest numbers of type k_i $(0 \leq i \leq n)$.

In a direct sum graph, we will use a weaker condition instead of (14).

THEOREM 6. *Let G and H be symmetrical chain-graphs with levels $K_0, K_1, \ldots, K_n$ (of $k_0, \ldots, k_n$ elements) and $L_0, L_1, \ldots, L_p$ (of $l_0, \ldots, l_p$ elements), respectively. If we have a set $(g_1, h_1), \ldots, (g_m, h_m)$ of vertices of the direct sum graph $G+H$ such that no $h+1$ different ones of then satisfy the condition*

$$g_{i_1} = g_{i_2} = g_{i_3} = \cdots = g_{i_w}$$

$$(15) \quad h_{i_1}, \ldots, h_{i_w} \quad \textit{lie in a directed path in } H \textit{ (in this order)}$$

$$h_{i_w} = \cdots = h_{i_{h+1}}$$

$$g_{i_w}, \ldots, g_{i_{h+1}} \quad \textit{lie in a directed path in } G \textit{ (in this order)}$$

for some w $(1 \leq w \leq h+1)$, then $m \leq$ the sum of the h largest numbers of type $\Sigma_{i=0}^{\alpha} k_i l_{\alpha-i}$.

REMARK 1. If G and H are subsets-graphs of sets S_1 and S_2 of n and p elements, respectively, then

$$k_i = \binom{n}{i}, \; \ell_i = \binom{p}{i} \quad \text{and} \quad \sum_{i=0}^{\alpha} k_i \, \ell_{\alpha-i} = \binom{n+p}{\alpha}.$$

On the other hand, (15) means that there are no $h+1$ different subsets

$A_1, A_2, \ldots, A_{h+1}$ in the given family of subsets of $S_1 \cup S_2$ such that

$$(16) \qquad \begin{aligned} &A_1 \cap S_1 = A_2 \cap S_1 = \ldots = A_w \cap S_1, \quad A_1 \cap S_2 \subset A_2 \cap S_2 \subset \ldots \subset A_w \cap S_1 \\ &A_w \cap S_1 \subset \ldots \subset A_{h+1} \cap S_1, \quad A_w \cap S_2 = \ldots = A_{h+1} \cap S_2 \end{aligned}$$

holds for some w $(1 \le w \le h+1)$.

It is clear if (16) holds, then

$$A_1 \subset A_2 \subset \ldots \subset A_w \subset \ldots \subset A_{h+1}$$

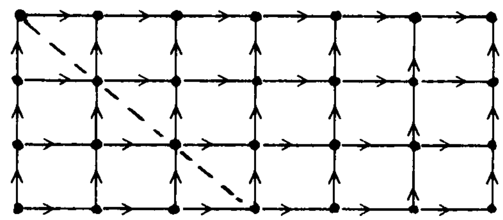
also holds, that is, in this case we have a weaker condition than the one in Theorem 3A, but we obtain the same result. The connection between this case of Theorem 6 and Theorem 3 is the same as the connection between Theorem 4 and Sperner's theorem.

REMARK 2. If we put $h=1$ in the above example, we obtain Theorem 4.

REMARK 3. Let us consider now another important special case. Let S_1 be a one-element set, and let the vertices of G be the "functions" f defined on S_1, such that $0 \le f \le n$ and f is an integer. There is a directed edge from f to g only if $g = f+1$. Thus G will be a directed path of length $n+1$.

Let H be the same with p instead of n. $G+H$ is in this case a rectangular $(n+1) \times (p+1)$ lattice.

H p+1



n+1

G

In this special case, Theorem 6 would state that if we have a set of points of this rectangle, no two of which are connected by a directed path, then the maximal number of these points is the length of a maximal diagonal (a diagonal is a set of vertices with the same coordinate-sums), that is $\min(n+1,p+1)$.

Schönheim's generalization of Theorem 3A in this special case would state that if we have a set of points in this rectangle with the property that

(17) no $h+1$ different members of this set lie in a directed path,

then the maximal number of these points is the sum of the lengths of the h largest different diagonals.

Theorem 6 says that if we exclude the existence of $h+1$ different points lying in a directed path which consists of two straight lines (instead of (17)), we obtain the same maximum.

First we prove this special case.

LEMMA 4. *Let* R *be a graph with vertices* (i,j) $(0 \le i \le a;$ $0 \le j \le b;$ i,j *are integers) where there are directed edges from* (i,j) *only to* $(i, j+1)$ *and* $(i+1, j)$*. If we have a set of* m *vertices* *such that there are no $h+1$ different vertices $(i_1,j_1),\ldots,(i_{h+1},j_{h+1})$ with the property*

$$(18) \qquad \begin{array}{ll} i_1 = i_2 = \ldots = i_w & j_1 < j_2 < \ldots < j_w \\ i_w < i_{w+1} < \ldots < i_{h+1} & j_w = j_{w+1} = \ldots = j_{h+1} \end{array}$$

for some w $(1 \le w \le h+1)$, then $m \le$ sum of the lengths of the h largest different diagonals.

PROOF. The set of vertices (i_0,j) where i_0 is fixed and $0 \le j \le b$ is called the i_0-th *column*. The *rows* are similarly defined. Let V be a set of vertices satisfying the conditions of the lemma and denote by c_t the number of columns with exactly t vertices from V. Obviously, by (18) $c_t = 0$ if $t > h$. Thus

$$(19) \qquad \sum_{t=0}^{h} c_t = a + 1.$$

Let us count in two different ways the number of points which are at least u-th elements of V in any column starting from below. In a column where the number of elements of V is less than u, we have to count 0, so counting column by column we obtain

$$c_u + 2c_{u+1} + 3c_{u+2} + \ldots + (h-u+1)c_h \ .$$

On the other hand, counting row by row, we obtain that this number is at most $(h-u+1)(b-u+2)$ because we do not have to consider the first $u-1$ rows, and in the other rows we can have at most $h-u+1$ such points by condition (18). Thus we obtain the inequality

$$(20) \qquad c_u + 2c_{u+1} + 3c_{u+2} + \ldots + (h-u+1)c_h \le (h-u+1)(b-u+2), \quad (1 \le u \le h).$$

We have to maximize $\Sigma_{i=0}^{h} ic_i$ under the conditions (19) and (20). If $a+1 \le b-h+2$, then obviously the optimal solution is $c_h = a+1$, $c_{h-1} = \dots = c_1 = 0$. Assume now that $a+1 > b-h+2$. First we will show that there is an optimal solution with $c_h = b-h+2$. If we have an optimal solution $c_h', c_{h-1}', \dots, c_1'$, with $c_h' < b-h+2$, then $c_h = b-h+2$, $c_{h-1} = c_{h-1}' - 2(b-h+2-c_h')$, $c_{h-2} = c_{h-2}'+b-h+2-c_h'$, $c_{h-3} = c_{h-3}', \dots$, $c_1 = c_1'$, is also an optimal solution because $\Sigma_{i=1}^{h} ic_i = \Sigma_{i=1}^{h} ic_i'$, and since the c_i''s satisfy (20), it follows that (20) holds for the c_i's, too.

$$u = h-1: \quad c_{h-1} + 2c_h = c'_{h-1} - 2(b-h+2-c'_h) + 2(b-h+2)$$
$$= c'_{h-1} + 2c'_h \le 2(b-h+2)$$

$$u \le h-2: \quad c_u + \dots + (h-u-1)c_{h-2} + (h-u)c_{h-1} + (h-u+1)c_h$$
$$= c_u + \dots + (h-u-1)(c'_{h-2}+b-h+1-c'_h)$$
$$+ (h-u)(c'_{h-1} - 2(b-h+2-c'_h)) + (h-u+1)(b-h+2)$$
$$= c'_u + \dots + (h-u-1)c'_{h-2} + (h-u)c'_{h-1} + (h-u+1)c'_h$$
$$\le (h-u+1)(b-u+2).$$

Thus, for this special optimal solution we have

$$c_u + 2c_{u+1} + \dots + (h-u)c_{h-1} \le (h-u+1)(h-u) \qquad (1 \le u \le h-1) \tag{21}$$

instead of (20). If $a+1 \le b-h+4$, then the optimal solution is: $c_{h-1} = 2$ or 1, according to whether $a+1 = b-h+4$ or $b-h+3$, $c_{h-2} = \dots = c_1 = 0$. Let us assume that $a+1 > b-h+4$. It is easy to see that there exists an optimal solution for which $c_{h-1} = 2$ (naturally $c_h = b-h+2$ is fixed). If we have another optimal solution $c'_{h-1}, \dots$, c'_1, with $c'_{h-1} < 2$, then we can change to another one through $c_{h-1} = 2$, $c_{h-2} = c'_{h-2} - 2(2-c'_{h-1})$, $c_{h-3} = c'_{h-3} + (2-c'_{h-1})$, $c_{h-4} = c'_{h-4}, \dots, c_1 = c'_1$, which satisfies (21). Following this procedure, we obtain that there exists an optimal solution of the form $c_h^0 = b-h+2$, $c_{h-1}^0 = 2$, $c_{h-2}^0 = 2, \ldots, c_{v+1}^0 = 2$, $c_v^0 = 1$ or 0, $c_{v-1}^0 = \ldots = c_1^0 = 0$, where v is determined by (19).

The set of points (i,j) satisfying $i+j = k$ is called the *k-th diagonal* of the rectangle and is denoted by D_k. The h *middle diagonals* are $D_y, D_{y+1}, \ldots, D_{y+h-1}$, where $y = [(a+b-h+1)/2]$. It is easy to see that the number of points of the h middle diagonals is just $\Sigma i c_i^0$, and hence maximal. This means that they are also the h largest diagonals because any h arbitrary diagonals satisfy the conditions of the lemma. The lemma is proved.

PROOF OF THEOREM 6. By part 3 of the definition of symmetrical chain graphs, the vertices of G and H can be partitioned into symmetrical chains. Denote by G' and H' the graphs which have edges only along these chains. Thus, G' and H' are subgraphs of G and H respectively. It follows that $G'+H'$ is a subgraph of $G+H$. So it is sufficient to prove the theorem for $G'+H'$ instead of $G+H$. However, $G'+H'$ consists of rectangular lattices, and condition (17) of the theorem simply means condition (18) in every such rectangular lattice. We know that an optimal set of points in every rectangle is the union of the h middle diagonals. Define the levels of $G'+H'$ in the following manner:

$$(g,h) \in M_j \quad \text{iff} \quad g \in K_i,\ h \in L_{j-i} \quad \text{for some } i.$$

By the definition of the direct sum, it is easy to see that the M_j's satisfy the first part of the definition of a symmetrical chain-graph. The h *middle levels* of $G'+H'$ are $M_z, M_{z+1}, \ldots, M_{z+h-1}$, where $z = [(n+p-h+1)/2]$.

We will show that the union of the h middle diagonals for all the rectangles is just the union of the h middle levels in $G'+H'$.

First we verify that an element of the h middle diagonals in a rectangle is an element of the h middle levels in $G'+H'$. Let us consider a fixed rectangle which is a direct sum of two symmetrical chains from G' and H' with vertices $g_0, g_1, \ldots, g_a$ and $h_0, h_1, \ldots, h_b$ respectively. If $g_0 \in K_i$, then by the symmetry $g_a \in K_{n-i}$ and thus $i+a = n-i$, or

$$(22) \qquad i = \frac{n-a}{2}.$$

(Obviously, n and a have the same parity.) Similarly, if $h_0 \in L_j$, then

$$(23) \qquad j = \frac{p-b}{2}.$$

If a point (g_k, h_ℓ) is in D_r, one of the h middle diagonals of the rectangle, then

$$(24) \qquad k + \ell = r$$

$g_k \in K_{i+k}$, $h_\ell \in L_{j+\ell}$, thus $(g_k, h_\ell) \in M_{i+j+k+\ell}$, or using (22), (23),(24)

$$(25) \qquad (g_k,\, h_\ell) \in M_{\frac{n+p-a-b}{2}+r}.$$

Since $y = [(a+b-h+1)/2] \le r \le y + h - 1$, we have $z = [(n+p-h)/2] \le (n+p-a-b)/2 + r \le z + h - 1$, and (25) means that (g_k, h_ℓ) is in one of the h middle levels of $G'+H'$.

Conversely, let (g,h) be an element of M_s, where $z \le s \le z+h-1$. Then (g,h) is contained in a rectangle which is a direct sum of two symmetrical chains, say, $g_0, g_1, \ldots, g_a$ and $h_0, h_1, \ldots, h_b$. Then by (22) and (23)

$$g_0 \in K_{\frac{n-a}{2}} \qquad h_0 \in L_{\frac{p-b}{2}} .$$

If $(g,h) = (g_k, h_\ell)$, then $(n-a)/2 + (p-b)/2 + k + \ell = s$, that is for $r = k+\ell = s - (n-a)/2 - (p-b)/2$ the following inequality holds:

$$\left[\frac{a+b-h+1}{2}\right] = z - \frac{n+p-a-b}{2} \le r \le \left[\frac{a+b-h+1}{2}\right] + h - 1,$$

and thus (g,h) is an element of one of the h middle diagonals.

Thus we have proved that the points of the h middle levels form an optimal set. For the union of h arbitrarily chosen levels of $G+H$, the conditions of the theorem are satisfied; so the h middle levels must be the h largest. The number of elements in M_α is obviously $\Sigma_{i=0}^{\alpha} k_i \ell_{\alpha-i}$; thus the optimal number is the sum of the h largest of these numbers. The proof is complete.

Now we will prove another generalization of a theorem of Erdős. We intend to give a class of problems where the maximum is a sum of binomial coefficients. We will state the theorem only for subsets of a finite set, but everything is the same for any symmetrical chain graph.

Let S be a set of n elements and let $A_1, A_2, \ldots, A_h$ be families of subsets of S, where $A_1, A_2, \ldots, A_h$ satisfy a finite system $\sum$ of axioms σ_i $(1 \le i \le s)$, where the σ_i's satisfy several conditions:

1. σ_i $(1 \le i \le s)$ has the form $\neg \exists (C_1, C_2, \ldots, C_{k_i}) F_i$, where F is a finite formula containing the variables $C_1, C_2, \ldots, C_{k_i}$, $A_1, \ldots, A_h$ and the signs $\in$ (is an element of), $\subset$ (is contained by), $\not\subset$, $|\ |$ (number of elements), $=$, $\neq$, $<$, $\le$.

2. If $C_1, \ldots, C_{k_i}$, $A_1, \ldots, A_h$ satisfy F, then for every pair C_j, C_k, either $C_j \subset C_k$ or $C_k \subset C_j$ holds (or both).

Let $D_0 \subset D_1 \subset \dots \subset D_n$ be a fixed sequence of subsets of S with the property $|D_i| = i$ $(0 \le i \le n)$. If $A_1^{n+1}, A_2^{n+1}, \dots, A_h^{n+1}$ is a system of families satisfying $\sum$ but having elements only among $D_0, D_1, \dots, D_n$, then denote by $A_1^{\ell}, A_2^{\ell}, \dots, A_h^{\ell}$ $(\ell = n+1, n-1, n-3, \dots;\ \ell \ge 1)$ the families defined by

$$A_i^{\ell} = \{D_j:\ D_j \in A_i^{n+1},\ \frac{n-\ell+1}{2} \le j \le \frac{n+\ell-1}{2}\} .$$

3. There exists a system $A_1^{n+1}, \dots, A_h^{n+1}$ of families satisfying $\sum$ and having elements only from $D_0, D_1, \dots, D_n$, such that among the systems of families satisfying $\sum$ and having elements only from $D_{(n-\ell+1)/2}, \dots, D_{(n+\ell-1)/2}$,

$$\sum_{i=1}^{h} |A_i^{\ell}| \quad \text{is maximal for all } \ell \quad (\ell = n+1, n-1, n-3, \dots;\ \ \ell \ge 1).$$

THEOREM 7. *If $A_1, A_2, \dots, A_h$ are families of subsets of a set S of n elements, $\sum$ is a system of axioms satisfying 1, 2, and 3, and $A_1, A_2, \dots, A_h$ satisfy $\sum$, then*

$$\sum_{i=1}^{h} |A_i| \le \sum \binom{n}{j} ,$$

where j runs over the number of elements of the subsets lying in $A_1^{n+1}, \dots, A_h^{n+1}$ which are defined under 3. This estimate is best possible.

PROOF. By Lemma 2, we can divide the set of all subsets into disjoint symmetrical chains. Consider a fixed chain $E_{(n-\ell+1)/2}, \dots, E_{(n+\ell-1)/2}$. We will show that the maximal number of elements that

$A_1, \ldots, A_h$ can have in $E_{(n-\ell+1)/2}, \ldots, E_{(n+\ell-1)/2}$ is

$$\sum_{i=1}^{h} |A_i^\ell| .$$

In order to show this, it is sufficient to verify that if we have a system of families satisfying $\sum$, then the system obtained by permutation of the elements of S will also satisfy $\sum$. Indeed, we can obtain $E_{(n-\ell+1)/2}, \ldots, E_{(n+\ell-1)/2}$ by a permutation of S from $D_{(n-\ell+1)/2}, \ldots, D_{(n+\ell-1)/2}$. However, the above property follows from 1. The proof of the above statement is trivial. If after the permutation we would have $C_1, C_2, \ldots, C_{k_i}$ satisfying F_i, then the re-permuted sets $C_1', C_2', \ldots, C_{k_i}'$ would satisfy F_i (in contradiction to our hypothesis) because our signs $\in$, $\subset$, $\cup$, $\cap$, $|\ |$, $=$, $\le$ are invariant relative to the permutation. A more detailed proof would use induction over the number of signs in F_i.

So, in every chain of length ℓ we determined an upper bound

$$\sum_{i=1}^{h} |A_i^\ell| . \tag{26}$$

In order to obtain an upper bound for $\Sigma_{i=1}^{h} |A_i|$, we only have to sum (26) for all chains. The best method to accomplish this is to form an optimal system in every chain and to number the sets in all the chains. Let $A_1^{\ell *}, A_2^{\ell *}, \ldots, A_h^{\ell *}$ be an optimal system in a chain of length ℓ, where the star denotes the appropriate permutation. The permutation does not change the number of elements of a set in $A_1^\ell, \ldots, A_h^\ell$. If $D_j \in A_k^{n+1}$ then an arbitrary set D of j elements is an element of a chain of length ℓ $((n-\ell+1)/2 \le j \le (n+\ell-1)/2)$, and by definition of A_k^ℓ and $A_k^{\ell *}$, $D \in A_k^{\ell *}$. Similarly, if $D_j \notin A_k^{n+1}$ then no set D of j elements can be an element of the optimal system.

The upper bound is established. We only have to prove that the optimal systems defined in this way satisfy $\sum$. If this were false then we would have an axiom σ_i and sets $C_1, C_2, \ldots, C_{k_i}$ satisfying F_i. However, by property 2 it follows that $C_1, C_2, \ldots, C_{k_i}$ form a chain (perhaps with equal terms) which is part of a chain of length $n+1$. From that we can form, by an appropriate permutation of elements of S, the chain $D_0 \subset D_1 \subset \ldots \subset D_n$. Thus, the sets $C_1', C_2', \ldots, C_{k_i}'$ thus obtained would be elements of this chain $D_0 \subset D_1 \subset \ldots \subset D_n$. The permutation does not change the number of elements, so $C_1', C_2', \ldots, C_{k_i}'$ would remain elements of the corresponding families $\mathcal{A}_1^{n+1}, \ldots, \mathcal{A}_h^{n+1}$ and they would satisfy F_i which contradicts our hypothesis. The proof is finished.

EXAMPLES.

1. SPERNER'S THEOREM. Let $\sum$ contain only one axiom:

$\neg\exists\,(C_1, C_2)$: $(C_1 \in \mathcal{A}_1,\ C_2 \in \mathcal{A}_1,\ C_1 \neq C_2,\ C_1 \subset C_2)$. In any chain $D_{(n-\ell+1)/2}, \ldots, D_{(n+\ell-1)/2}$ we can only choose a family $\mathcal{A}_1$ with a single element, and if we choose $D_{[n/2]}$, it satisfies the condition 3, and so, by Theorem 7

$$\max|\mathcal{A}_1| = \binom{n}{[n/2]}.$$

2. ERDÖS' THEOREM (Theorem 3A).

$\neg\exists\,(C_1, C_2, \ldots, C_{h+1})$: $(C_1 \in \mathcal{A}_1, \ldots, C_{h+1} \in \mathcal{A}_1,\ C_1 \neq C_2, \ldots, C_1 \neq C_{h+1}, \ldots, C_h \neq C_{h+1},$ $C_1 \subset C_2 \subset \ldots \subset C_{h+1})$, where $h \leq n+1$. In a chain $D_{(n-\ell+1)/2}, \ldots, D_{(n+\ell-1)/2}$ we can choose a family $\mathcal{A}_1$ with at most $\min(\ell, h)$ elements, and if we choose for $\ell = n+1$ the family $D_{[(n-h+1)/2]}, \ldots, D_{[(n+h-1)/2]}$ it will be maximal for every ℓ. Thus, by Theorem 7:

$$\max |A_1| = \sum_{i=[(n-h+1)/2]}^{[(n+h-1)/2]} \binom{n}{i} .$$

3. $\neg \exists (C_1,C_2,\ldots,C_{h+1})$: $(C_1 \in A_1,\ldots,C_{h+1} \in A_1,\ C_1 \neq C_2,\ldots,C_1 \neq C_{h+1},$ $\ldots,\ C_h \neq C_{h+1},\ C_1 \subset C_2 \subset \ldots \subset C_{h+1})$

$$\neg \exists\ C_1\ (C_1 \in A_1,\ |C_1| \le k)$$

where $h \le n+1$ and $[(n-h+1)/2] < k < [(n+h-1)/2]$. In a chain $D_{(n-\ell+1)/2},\ldots,D_{(n+\ell-1)/2}$ we can choose a family A_1 with at most $\min(((n+\ell-1)/2) - k,\ h)$ elements, and if we choose for $\ell = n+1$ the family $D_{k+1},D_{k+2},\ldots,D_{k+h}$, it will be maximal for every ℓ. Thus, by Theorem 7:

$$\max |A_1| = \sum_{i=k+1}^{k+h} \binom{n}{i} .$$

4. The previous example with $|C_1|=k$ instead of $|C_1| \le k$.

$$\max |A_1| = \sum_{\substack{i=[(n-h)/2] \\ i \neq k}}^{[(n+h)/2]} \binom{n}{i} .$$

5. $\neg \exists (C_1,C_2)$: $(C_1 \in A_1, C_2 \in A_1,\ C_1 \neq C_2,\ C_1 \subset C_2,\ |C_2 - C_1| \le 1)$

$$\neg \exists\ (C_1):\ (C_1 \in A_1,\ |C_1| = [n/2] + 1) .$$

In a chain $D_{(n-\ell+1)/2},\ldots,D_{n+\ell-1)/2}$ we can form a maximal family A_1 choosing every other one, leaving out $D_{[n/2]+1}$. Thus, if for $\ell = n+1$ we choose $\ldots D_{[n/2]-2},D_{[n/2]},D_{[n/2]+2},\ldots,$ this family will be maximal for every ℓ. By Theorem 7,

$$\max |A_1| = \sum_{(-1)^i=(-1)^{[n/2]}} \binom{n}{i} = 2^{n-1}.$$

6. A version of Erdös' theorem (Theorem 3B)

$$\neg \exists (C_1,C_2): \quad (C_1 \subset C_2,\ C_1 \in A_1,\ C_2 \in A_1)$$
$$\vdots$$
$$\neg \exists (C_1,C_2): \quad (C_1 \subset C_2,\ C_1 \in A_H,\ C_2 \in A_H)$$
$$\neg \exists C_1: \quad C_1 \in A_1,\ C_1 \in A_2$$
$$\neg \exists C_1: \quad C_1 \in A_1,\ C_1 \in A_3$$
$$\vdots$$
$$\neg \exists C_1: \quad C_1 \in A_{H-1},\ C_1 \in A_H.$$

The solution is the same as in the example 2.

7. $$\neg \exists (C_1,C_2,C_3)\ (C_1 \in A_1,\ C_2 \in A_1,\ C_3 \in A_1,\ C_1 \neq C_2,\ C_1 \neq C_3,\ C_2 \neq C_3,\ C_1 \subset C_2 \subset C_3)$$

$$\neg \exists (C_1,C_2,C_3)\ (C_1 \in A_2,\ C_2 \in A_2,\ C_3 \in A_2,\ C_1 \neq C_2,\ C_1 \neq C_3.\ C_2 \neq C_3,\ C_1 \subset C_2 \subset C_3)$$

$$\neg \exists (C_1,C_2)\ (C_1 \in A_1,\ C_2 \in A_2,\ C_2 \neq C_1,\ C_2 \subset C_1)$$

For $\ell = n+1$ an optimal solution is $A_1 = \{D_{[n/2]-1}, D_{[n/2]}\}$, $A_2 = \{D_{[n/2]}, D_{[n/2]+1}\}$, and it is an optimal solution for each ℓ, since the optimum for $\ell=2$ is 3 and for $\ell=1$ is 2.

$$\max(|A_1|+|A_2|) = \binom{n}{[n/2]-1} + 2\binom{n}{[n/2]} + \binom{n}{[n/2]+1} = \binom{n+2}{[n/2]+1}.$$

8. Let h and H be integers with the property $h+H-1 \geq n+1$.

$$\neg \exists (C_1, C_2, \ldots, C_{h+2-w}) : \ (C_1 \in A_{i_1}, C_1 \in A_{i_2}, \ldots, C_1 \in A_{i_w}, C_2 \in A_{i_w},$$

(27) $$C_2 \in A_{i_w}, \ldots, C_{h+2-w} \in A_{i_w}, C_1 \subset C_2 \subset \ldots \subset C_{h+2-w}, C_1 \neq C_2, C_1 \neq C_3, \ldots,$$

$$C_{h+1-w} \neq C_{h+2-w})$$

for every w $(1 \leq w \leq h+1)$ and $1 \leq i_1 < i_2 < \ldots < i_w \leq H$.

Let us construct a graph with vertices (i,j) where $(n-\ell+1)/2 \leq i \leq (n+\ell-1)/2$ and $1 \leq j \leq H$. There are directed edges from (i,j) only to $(i,j+1)$ and $(i+1,j)$. Let D_i, A_j correspond to the vertex (i,j), the condition (27) is just (18), so we can use Lemma 4: the maximal number of vertices is the sum of lengths of the h largest diagonals of this rectangle. So if for $\ell=n+1$ we choose the h largest diagonals in the "middle":

$$\begin{aligned}
A_1^{n+1} &= \{D_{[(n+H-h)/2]}, \ldots, D_{[(n+H-h)/2]+h-1}\} \\
A_2^{n+1} &= \{D_{[(n+H-h)/2]-1}, \ldots, D_{[(n+H-h)/2]+h-2}\} \\
&\vdots \\
A_H^{n+1} &= \{D_{[(n+H-h)/2]-(H-1)}, \ldots, D_{[(n+H-h)/2]+h-H}\},
\end{aligned}$$

then $A_1^{\ell}, \ldots, A_H^{\ell}$ will have the optimal property in all the smaller rectangles, since the corresponding vertices will also form the h "middle" diagonals. Thus, we may apply Theorem 7:

$$\max \sum_{i=1}^{H} |A_i| = \sum_{j=0}^{H-1} \sum_{i=[(n+H-h)/2]-j}^{[(n+H-h)/2]-j+h+1} \binom{n}{i}$$

$$= \binom{n}{[(n+H-h)/2]-H+1} + 2\binom{n}{[(n+H-h)/2]-H+2} + 3\binom{n}{[(n+H-h)/2]-H+3} + \cdots$$

$$+ 2\binom{n}{[(n+H-h)/2]+h-2} + \binom{n}{[(n+H-h)/2]+h-1} .$$

Obviously, conditions 2 and 3 are strong. For instance, if in example 5 we omit the second axiom and n is even, then for $\ell=n+1$ the only optimal solution is $A_1^{n+1} = \{D_0,D_2,D_4,\ldots,D_h\}$. In this case, $\{D_2,D_4,\ldots,D_{n-2}\}$ does not form an optimal solution for $\ell=n-1$ because $\{D_1,D_3,\ldots,D_{n-1}\}$ has more elements. Thus, in this case Theorem 7 does not work, although it is easy to see that in this case

$$\max |A_i| = 2^{n-1}$$

holds, too. Let e be a fixed element of S. If $A\subset S-\{e\}$, then among A and $A\cup\{e\}$ A_1 can contain at most one. That means A_1 can contain only half of the subsets of S; $|A_1| \le (2^n/2) = 2^{n-1}$. The construction of example 5 shows that this estimate is best possible.

Now, we prove a generalization of this statement.

THEOREM 8. *If* $A = \{A_1,\ldots,A_m\}$ *is a family of subsets of a set* S *of* n *elements and no two different subsets satisfy the condition*

$$A_i \subset A_j, \qquad |A_j-A_i| \le k-1 \qquad (k \ge 2)$$

then

$$m \le \sum_{i\equiv[n/2] \pmod k} \binom{n}{i}$$

and this estimate is best possible.

First we prove the following lemma:

Lemma 5. *If $k \geq 2$, then it is possible to divide the subsets of a set S of n elements into disjoint chains which are*

(a) of length k

or (b) of length less than k and symmetrical.

Proof of Lemma 5. We use induction on n. For $n=1$, we have one chain of length 2 which is symmetrical. Assume we have constructed appropriate chains for $n-1$. From these chains, we will constuct new chains using the n-th element, x_n. From a chain $\{B_1,\ldots,B_k\}$ of length k we form two chains $\{B_1,\ldots,B_k\}$ and $\{B_1 \cup \{x_n\},\ldots,B_k \cup \{x_n\}\}$. They have property (a) again. From a chain $\{B_1,\ldots,B_i\}$ $(i < k)$ we form the following two chains:

$$\{B_1, \ldots, B_i, B_i \cup \{x_n\}\}, \qquad \{B_1 \cup \{x_n\}, \ldots, B_{i-1} \cup \{x_n\}\}.$$

The lengths of the new chains of this type will be $\leq k$, and they will be symmetrical; $|B_1|+|B_i \cup \{x_n\}| = |B_1|+|B_i| + 1 = n-1+1 = n$ and $|B_1 \cup \{x_n\}| + |B_{i-1} \cup \{x_n\}| = |B_1|+|B_{i-1}| + 2 = n$. It is clear, further, that the new chains are disjoint and that they contain every subset of S. The proof of the lemma is finished.

Proof of Theorem 8. Let us consider a set of disjoint chains given by Lemma 5. A chain of length at most k cannot have two elements of A by the condition of the theorem. Thus,

$$|A| = m \leq \text{number of chains}. \tag{28}$$

By symmetry, every chain of type (b) has an element of size $[n/2]$, and every chain of type (a) has an element of size $\equiv [n/2] \pmod{k}$. Since a chain cannot contain two elements of size $\equiv [n/2] \pmod{k}$, the number of chains is exactly

$$\sum_{i \equiv [n/2] \pmod{k}} \binom{n}{i},$$

which by (28) yields the desired inequality. The family of all sets having $i \equiv [n/2] \pmod{k}$ elements shows that the estimate is best possible. The proof is complete.

Brief List of Further Results.

We have seen in Lemma 1 that $|c(B)| \geq r \cdot v/(n-v+1)$. An interesting question is the following: "What is the minimum of $|c(B)|$ if $B = \{B_1, \ldots, B_r\}$ and r is fixed?" The natural conjecture that if $r = \binom{N}{v}$, then $\min|c(B)| = \binom{N}{v-1}$ is correct. More generally, every $r>0$

$$r = \binom{a_v}{v} + \binom{a_{v-1}}{v-1} + \ldots + \binom{a_t}{t}$$

in a uniquely determined manner if v is fixed and $a_v > \ldots > a_t$, $t \geq 1$, $a_j \geq j$ $(t \leq j \leq v)$. J. B. Kruskal [15] proved that

$$\min|c(B)| = \binom{a_v}{v-1} + \binom{a_{v-1}}{v-2} + \ldots + \binom{a_t}{t-1}.$$

A simpler proof is given by Katona [16].

A similar problem is the following. If $B = \{B_1, B_2, \ldots, B_r\}$ is a family of subsets of S $(|S|=n)$ and $|B_i|=v$, $|B_i \cup B_j| \geq k$ $(1 \leq i,j \leq r;$ $1 \leq k \leq v \leq n)$, then we know [14]

$$|c(B)| \geq \frac{v}{v-k+1} \cdot r.$$

The exact minimum is still an open problem.

Using the result mentioned above, Milner [13] proved the following theorem:

If $A = \{A_1,\ldots,A_m\}$ is a family of subsets of S such that no two different ones are in the relation $A_i \subset A_j$, and such that for all pairs $|A_i \cup A_k| \geq k$, then

$$m \leq \binom{n}{[(n+k+1)/2]}$$

and the estimate is best possible.

Sperner's Theorem says if we have a family $A = \{A_1,\ldots,A_m\}$, where $m > \binom{n}{[n/2]}$ there is a pair satisfying $A_i \subset A_j$, $i \neq j$. Kleitman [7] considered the question of the least number of such pairs we must have: if $m = \binom{n}{[n/2]}+x$, then the number of such pairs is at least $x([n/2]+1)$. This is exact if $x \leq \binom{n}{[n/2]+1}$, but for larger x the exact estimate is not yet known.

The most interesting application of Sperner's theorem is given by Kolmogorov [17]. The concentration function of a random variable ξ is defined as follows:

$$Q^{\xi}(v) = \sup_x P(x < \xi \leq x+v) .$$

If $\xi_1,\xi_2,\ldots,\xi_n$ are independent, identically distributed random variables, $Q_n^{\xi}(v)$ will denote the concentration function of the sum $\xi_1 + \xi_2 + \ldots + \xi_n$. Kolmogorov proved

$$Q_n^{\xi}(v) \leq c(v) \frac{1}{\sqrt{n}}$$

In the proof, he used Sperner's theorem through the sums $\Sigma\varepsilon_i a_i$ which were mentioned in the third example in the introduction.

For the more-dimensional generalization of this probability theory theorem, Sazonov [5] needed a new generalization of Sperner's theorem. This generalization is given by Meshalkin [4]: Let $\mathcal{A} = \{A_1,\ldots,A_m\}$ be a family of partitions of a set S of n elements, where A_i denotes the partition $A_{i1},A_{i2},\ldots,A_{ir}$ (r is fixed). Assume no two different partitions A_i, A_j have subsets A_{ik}, A_{jk} with the property $A_{ik} \subset A_{jk}$. Then

$$m \le \max_{\Sigma n_i = n} \frac{n!}{n_1! \ \ldots \ n_r!}$$

and this estimate is best possible. If we exclude two partitions A_i, A_j if $A_{ik} \subset A_{jk}$ or $A_{ik} \supset A_{jk}$ for each k, then the problem is still open.

Rényi [3] asked how many subsets we can choose having the Sperner property if we choose every subset independently and with probability $1/2^n$. The answer is $(2/\sqrt{3})^n$ which is definitely less than $\binom{n}{[n/2]} \sim 2^n/\sqrt{n} \ \sqrt{2/\pi}$. The other generalizations of Sperner's theorem are not yet investigated in the random case.

An old problem is: how many Sperner families are there? It is easy to give a lower bound for the number $\psi(n)$ of Sperner families on a set S of n elements. The family of all $[n/2]$-tuples is a Sperner family. Similarly, an arbitrary subfamily of this is again a Sperner family. So

$$2^{\binom{n}{[n/2]}} \le \psi(n).$$

The newest result was obtained by Kleitman [20]:

$$2^{\binom{n}{[n/2]}\left(1+\frac{c_1}{\psi\sqrt{e^n}}\right)} \le \psi(n) \le 2^{\binom{n}{[n/2]}\left(1+\frac{c_2 \log n}{\sqrt{n}}\right)} .$$

References.

[1]. De Bruijn, N.G., Van Ebbenhorst Tengbergen, C.A., and Kruyswijk, D.: On the set of divisors of a number. Nieuw Archief voor Wiskunde, 2, 23 (1952), 191-193.

[2]. Sperner, E.: Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928), 544-548.

[3]. Rényi, A.: On random subsets of a finite set, Mathematica, 3 (26) (1961), 355-362.

[4]. Meshalkin, L.D.: Generalization of Sperner's theorem on the number of subsets of a finite set, Theory of Probability and its Application, 8(1963), 203-204. (English translation)

[5]. Sazonov, V.V.: On multi-dimensional concentrations, Theory of Probability and its Applications, 11 (1966) (English translation).

[6]. Sevast'yanov, V.A.: On multi-dimensional concentrations, Theory of Prob. Applications, 8 (1963) 116 (English translation).

[7]. Kleitman, D.: On a conjecture of Erdös-Katona on commensurable pairs among subsets of an n-set, Theory of Graphs, Proc. Colloq. held at Tihany, 215-218.

[8]. Erdös, P.: On a lemma of Littlewood and Offord, Bulletin of the Amer. Math. Soc. 51 (1945), 898-902.

[9]. Schönheim, J.: A generalization of results of P. Erdös and G. Katona concerning Sperner's theorem, J. of Combinatorial Theory, 1970 (to appear).

[10]. Anderson, I.: On the divisors of a number, J. London Math. Soc. 43 (1968), 410-418.

[11]. Kleitman, D.: On a lemma of Littlewood and Offord on the distribution of certain sums, Math. Z. 90 (1965), 251-259.

[12]. Hilton, A.J. and Milner, E.C.: Some intersection theorems for systems of finite sets, Quart. J. Math. 18 (1967) 369-384.

[13]. Milner, E.C.: A combinatorial theorem on systems of sets, J. London Math. Soc. 43 (1968), 204-206.

[14]. Katona, G: Intersection theorems for systems of finite sets, Acta Math., Acad. Sci. Hung. 15 (1964), 329-337.

[15]. Kruskal, F.B.: The number of simplices in a comples, Mathematical Optimization Techniques, Edited by R. Bellman, 1963, 251-278.

[16]. Katona, G.: A theorem on finite sets, Theory of Graphs, Proc. Colloq. held at Tihany, Hungary. 1966, Akadémiai Kiadó, 1968, 187-207.

[17]. Kolmogorov, A.: Sur les propriétés des fonctions de concentrations de M. P. Lévy, Ann. Ins. II. Poincaré 16, 1 (1958) 27-34.

[18]. Lubell, D.: A short proof of Sperner's Lemma, J. Combinatorial Theory 1 (1966), 299.

[19]. Katona, G.: On a conjecture of Erdős and a stronger form of Sperner's Theorem, Studia Sci. Math. Hungary 1 (1966), 59-63.

[20]. Kleitman, D.: On Dedekind's problem: The number of monotone Boolean functions, (to appear in Proc. Amer. Math. Soc.).