THERMAL STRESSES IN A SPHERICAL SHELL WITH A CIRCULAR ELASTIC INCLUSION

P. PRAKASH and K.P. RAO

Department of Aeronautical Engineering,
Indian Institute of Science, Bangalore — 560012, India

SUMMARY

Spherical shells are widely used in nuclear, aerospace and chemical industries. Cutouts are introduced into them for access or visibility. These cutouts may be covered by a lid which may be made of a material having mechanical and thermal properties different from that of the shell. For an efficient design of shells containing such inclusions, a study of stresses that are generated due to static and thermal loadings is essential.

In this paper, thermal stress analysis of a spherical shell with a circular elastic inclusion is presented. Using Reissner's theory for thin shallow spherical shells, displacements and stresses are obtained in the shell and the inclusion in terms of Hankel and Bessel functions. The thermal loading is assumed to be given by a temperature which is uniform on the inner and outer surfaces of the shell and the inclusion but may vary linearly across the thickness. The thermal stress problem is converted into an equivalent static loading problem. The satisfaction of the boundary conditions at the shell inclusion junction leads to the evaluation of the unknowns in the solutions.

Results obtained show the inadequacy of flat plate solutions and the effect of curvature. The effect of various parameters of the problem such as elastic moduli, coefficients of linear expansion, curvature parameters (defining the dimensions of the inclusion relative to the radius and thickness of the shell) is studied. Limiting cases of a rigid circular inclusion either free to move with the shell or clamped are also considered and the results are compared with the existing analytical solutions. Stress concentrations that are developed are presented in graphical form which is useful in the design of spherical shells containing a rigid or an elastic inclusion. It is found that a clamped rigid inclusion results in a more severe stress concentration than a rigid inclusion free to move with the shell. The maximum principal stress in a uniformly heated spherical shell with a circular elastic inclusion is found to be tensile in the shell and compressive in the inclusion. An increase in E_c/E_s or α_c/α_s (E_c, α_c, E_s, α_s are the elastic modulus and coefficient of linear expansion of the inclusion and the shell material, respectively) causes an increase in the shell stresses and a reduction in the inclusion stresses.

1. Introduction:

Thermal stresses play an important role in the fields of nuclear and chemical engineering. Thermal stresses develop whenever free thermal expansion is restrained or when the heat flow is disturbed by the presence of a discontinuity such as an inclusion in a structure. A rigorous understanding of the stress field under such conditions is essential for an efficient design of structural components.

Extensive studies have been made by several investigators on thermoelastic problems of flat plates with discontinuities [1 through 8]. The information available on thermal stresses in shells with discontinuities seems to be inadequate. Ariman [9] presented an analysis for obtaining thermal stresses and deformations in a thin cylindrical shell representing a nuclear reactor vessel with an insulated cutout of arbitrary shape. Here we present the thermal stress analysis of a spherical shell containing a circular elastic inclusion.

2. Method of solution:

The usual procedure adopted to solve a class of thermal stress problems is to express the thermal loading as an equivalent static loading. The solution is thus obtained by superposing the following two components (i) stresses and displacements caused by a thermal loading in a similar shell without any discontinuity ; (ii) stresses and displacements caused by edge loads and displacements along the boundary of the discontinuity which is otherwise free of any thermal loads.

The thermal loading is assumed to be given by a uniform temperature on the inner and outer surfaces of the shell and the inclusion but it may vary linearly across the thickness. Such a loading case can be treated as a super position of a uniform temperature every where and a linearly varying temperature with the middle surface experiencing no change in temperature. If T_1 and T_2 are the temperatures on the inner and outer surfaces of the shell and the inclusion, the temperature at any point across the thickness is given by

$$T_s = T_o + \tau_s Z \text{ (for the shell) and } T_c = T_o + \tau_c Z \text{ (for the inclusion)},$$

where

$$T_o = \frac{T_1 + T_2}{2}, \quad \tau_s = \frac{T_1 - T_2}{t_s}, \quad \tau_c = \frac{T_1 - T_2}{t_c}$$

and Z is measured normal to the middle surface.

3. Governing differential equations and their solutions:

Consider a spherical shell with a circular inclusion (Figure 1). For convenience we assume that the shell and the inclusion have the same curvature. Assigning suffixes 's' and 'c' to the shell and inclusion respectively, the differential equations governing the behaviour of a thin shallow spherical shell with an elastic inclusion can be written from Reissner's formulation [10] in the dimensional form as

$$\nabla^4 w_s' + \frac{1}{D_s R} \nabla^2 F_s' = 0 \; ; \quad \nabla^4 F_s' - \frac{E_s t_s}{R} \nabla^2 w_s' = 0 \quad \text{(for shell)}$$
$$\nabla^4 w_c' + \frac{1}{D_c R} \nabla^2 F_c' = 0 \; ; \quad \nabla^4 F_c' - \frac{E_c t_c}{R} \nabla^2 w_c' = 0 \quad \text{(for inclusion)} \qquad ..(1)$$

where the primes indicate that the quantities are dimensional, ∇^2 is the Laplace operator, w_s', F_s', D_s, E_s, t_s and w_c', F_c', D_c, E_c, t_c are the deflection, Airy's stress function, flexural rigidity, elastic modulus and thickness of the shell and the inclusion respectively.

Defining,

$$w_s = \frac{E_s t_s^2 w_s'}{a^2 N_{ref}\left[12(1-\nu_s^2)\right]^{1/2}} \; ; \quad F_s = \frac{F_s'}{a^2 N_{ref}} \; ; \quad \phi_s = w_s - iF_s \; ; \quad \beta_s^2 = \frac{a^2}{Rt_s}\sqrt{12(1-\nu_s^2)}$$
$$w_c = \frac{E_c t_c^2 w_c'}{a^2 N_{ref}\left[12(1-\nu_c^2)\right]^{1/2}} \; ; \quad F_c = \frac{F_c'}{a^2 N_{ref}} \; ; \quad \phi_c = w_c - iF_c ; \quad \beta_c^2 = \frac{a^2}{Rt_c}\left[12(1-\nu_c^2)\right]^{1/2}$$

the set of eqs.(1) can be reduced to,

$$\nabla^4 \phi_s + i\,\beta_s^2 \nabla^2 \phi_s = 0 \; ; \quad \nabla^4 \phi_c + i\,\beta_c^2 \nabla^2 \phi_c = 0 \qquad ..(2)$$

As the inclusion causes only local disturbance, the stresses and displacements can be expressed as,

$$F_s = \bar{F}_s + F_s^* \; ; \quad w_s = \bar{w}_s + w_s^* \; ; \quad F_c = \bar{F}_c + F_c^* \; ; \quad w_c = \bar{w}_c + w_c^* \qquad ..(3)$$

F_s^*, w_s^*, F_c^* and w_c^* are the perturbation components due to the interaction between the shell and the inclusion and $\bar{F}_s$, $\bar{w}_s$, $\bar{F}_c$ and $\bar{w}_c$ are the values far away from the junction. The solutions relevant to the problem under consideration are,

$$\phi_s^* = i\,C_1 \ln r + (A_1 + iA_2)\, H_0^1(\beta_s r\sqrt{i})$$
$$\phi_c^* = C_2 + (A_3 + iA_4)\, J_0(\beta_c r\sqrt{i}) \qquad ..(4)$$

where C_1, C_2, A_1, A_2, A_3 and A_4 are constants which can be evaluated by satisfying the boundary conditions at the junction and $r = r'/a$.

4. Boundary conditions:

Equilibrium of forces and moments at the junction implies,

$$(N_{rr})_s = (N_{rr})_c \quad \text{i.e.,} \quad (\bar{N}_{rr})_s + (N_{rr}^*)_s = (\bar{N}_{rr})_c + (N_{rr}^*)_c$$
$$(M_{rr})_s = (M_{rr})_c \quad \text{i.e.,} \quad (\bar{M}_{rr})_s + (M_{rr}^*)_s = (\bar{M}_{rr})_c + (M_{rr}^*)_c \qquad ..(5)$$

$$(Q_r)_s = (Q_r)_c \quad \text{i.e.,} \quad (\overline{Q}_r)_s + (Q_r^*)_s = (\overline{Q}_r)_c + (Q_r^*)_c$$

compatibility of displacements and slopes at the junction implies

$$\begin{aligned} w_s' &= w_c' && \text{i.e.,} \quad \overline{w}_s' + w_s^{*\prime} = \overline{w}_c' + w_c^{*\prime} \\ \frac{dw_s'}{dr} &= \frac{dw_c'}{dr} && \text{i.e.,} \quad \frac{d\overline{w}_s'}{dr} + \frac{dw_s^{*\prime}}{dr} = \frac{d\overline{w}_c'}{dr} + \frac{dw_c^{*\prime}}{dr} \end{aligned} \qquad ..(6)$$

$$(\varepsilon_\theta)_s = (\varepsilon_\theta)_c \quad \text{i.e.,} \quad (\overline{\varepsilon_\theta})_s + (\varepsilon_\theta^*)_s = (\overline{\varepsilon_\theta})_c + (\varepsilon_\theta^*)_c$$

using Eqs.(5) and (6) the unknowns in ϕ_s^* and ϕ_c^* can be evaluated and hence the stresses and displacements in the shell and the inclusion. Moment-Curvature relationships in the case of shallow shells are the same as those in the case of a plate.

4.1 Uniform temperature: (T_o)

$$\begin{aligned} &\overline{F}_s = \overline{F}_c = 0 \;; \quad \overline{w}_s' = \alpha_s T_o R \;; \quad \overline{w}_c' = \alpha_c T_o R \\ &(\overline{\varepsilon_\theta})_s = \alpha_s T_o \;; \quad (\overline{\varepsilon_\theta})_c = \alpha_c T_o \end{aligned} \qquad ..(7)$$

where α_s and α_c are the coefficients of linear expansion of shell and inclusion material respectively.

Reference stress in this case is chosen as: $N_{ref} = E_s t_s \alpha_s T_o$.

4.2 Linearly varying temperature across the thickness (middle surface experiencing no change in temperature):

Following the lines of reference [11]

$$\begin{aligned} &\overline{F}_s = \overline{F}_c = \overline{w}_s = \overline{w}_c = (\overline{\varepsilon_\theta})_s = (\overline{\varepsilon_\theta})_c = 0 \\ &(\overline{M}_{rr})_s = (\overline{M}_{\theta\theta})_s = \alpha_s D_s (1 + \nu_s) \tau_s \\ &(\overline{M}_{rr})_c = (\overline{M}_{\theta\theta})_c = \alpha_c D_c (1 + \nu_c) \tau_c \end{aligned} \qquad ..(8)$$

Reference stress in this case is chosen as: $N_{ref} = \dfrac{6\,\alpha_s D_s (1 + \nu_s)\tau_s}{t_s}$

4.3 Rigid inclusion:

A linearly varying temperature distribution with the middle surface experiencing no change in temperature causes no stress concentration. Hence we consider only the case of uniform heating.

4.3.1 Free rigid inclusion:

Conditions are:

$$(Q_r^*)_s = (N_{rr}^*)_s \frac{a}{R} \;; \quad \frac{dw_s^*}{dr} = 0 \;; \quad \alpha_s T_o + (\varepsilon_\theta^*)_s = 0 \qquad ..(9)$$

4.3.2 Clamped rigid inclusion:

Conditions are:

$$\alpha_s T_o R + w_s^{*\prime} = 0 \; ; \quad \frac{dw_s^*}{dr} = 0 \quad ; \quad \alpha_s T_o + (\epsilon_\theta^*)_s = 0 \qquad ..(10)$$

5. Results and Discussion:

The variation of membrane and bending stresses along the meridian in a uniformly heated spherical shell containing a rigid circular inclusion is shown in Fig.2 and Fig.3 shows the maximum principal stresses for the range of β_s in $0 < \beta_s < 10$. It is seen that the stress concentrations in the case of a clamped rigid inclusion are more severe than in the case of a free rigid inclusion. Table I gives the values of the shell membrane and bending stresses in the radial and tangential directions in the case of a clamped or free rigid inclusion for $0 < \beta_s < 10$. The membrane stresses in the case of a clamped rigid circular inclusion can be shown to be $(N_{rr}/N_{ref}) = 0$ and $(N_{\theta\theta}/N_{ref}) = -1$ irrespective of the value of β_s. As β_s tends to zero, the results can be seen to tend to those of a flat plate.

The maximum principal stresses in the uniformly heated shell with an elastic inclusion are presented in Figs. 4 and 5 for curvature parameters $\beta_s = \beta_c = 2$ and 4 as a function of E_c/E_s and α_c/α_s. The stresses in the shell vary linearly with respect to α_c/α_s for a given E_c/E_s ; the stresses reduce in the range $0 < \alpha_c/\alpha_s < 1$ and increase in the range $1 < \alpha_c/\alpha_s$, with increasing α_c/α_s. Also for a given α_c/α_s, the stresses increase with E_c/E_s. Generally it is found that tensile stresses are predominant in the shell whereas compressive stresses are predominant in the inclusion. It is appropriate to present the maximum compressive principal stresses that occur in the inclusion. Figs. 6 and 7 give the compressive stresses as a function of β_c, E_c/E_s and α_c/α_s. The results obtained by the present analysis for the limiting cases of $\beta_s \to 0$ compare very well with flat plate results.

The results obtained for the case of a spherical containing a circular elastic inclusion and subjected to a linearly varying temperature across the thickness such that the middle surface experiences no change in temperature, indicate that the bending stresses are predominant as compared to the membrane stresses. Typical plots showing the variation of the maximum principal stresses in the shell and the inclusion as a function of α_c/α_s and E_c/E_s for a value of $\beta_s = \beta_c = 2$ are given in Figs. 8 and 9. It is seen that the stress concentration factor increases as E_c/E_s or α_c/α_s increases. Whenever $\alpha_c D_c(1 + \nu_c)\tau_c = \alpha_s D_s(1 + \nu_s)\tau_s$, a state of constant bending moment prevails in the shell and the inclusion.

In all the computations $\nu_s = \nu_c = 1/3$ was chosen.

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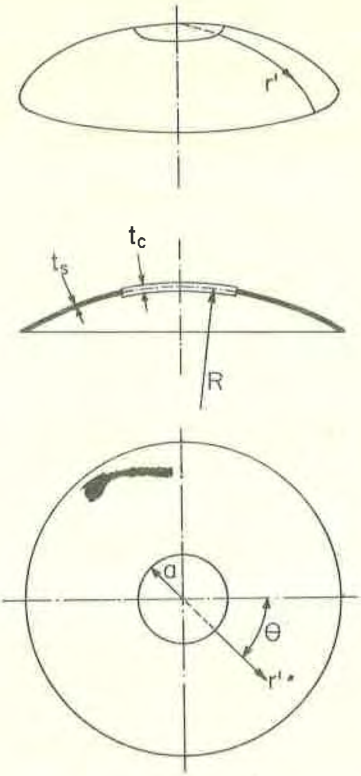
Table I : Stresses in a uniformly heated spherical shell with a rigid circular inclusion

β_s	Rigid circular inclusion free to move with the shell				Clamped rigid circular inclusion $N_{rr}/N_{ref} = 0$, $N_{\theta\theta}/N_{ref} = -1$	
	$\frac{N_{rr}}{N_{ref}}$	$\frac{N_{\theta\theta}}{N_{ref}}$	$\frac{N_{rb}}{N_{ref}}$	$\frac{N_{\theta b}}{N_{ref}}$	$\frac{N_{rb}}{N_{ref}}$	$\frac{N_{\theta b}}{N_{ref}}$
0.1	0.746	-0.751	-0.033	-0.011	-6.026	-2.009
0.5	0.680	-0.773	-0.325	-0.108	-3.493	-1.164
1.0	0.577	-0.808	-0.653	-0.218	-2.827	-0.942
1.5	0.490	-0.837	-0.882	-0.294	-2.549	-0.850
2.0	0.423	-0.859	-1.043	-0.348	-2.394	-0.798
2.5	0.371	-0.876	-1.160	-0.387	-2.295	-0.765
3.0	0.329	-0.890	-1.249	-0.416	-2.226	-0.742
3.5	0.296	-0.901	-1.317	-0.439	-2.175	-0.725
4.0	0.268	-0.911	-1.371	-0.457	-2.136	-0.712
4.5	0.246	-0.918	-1.416	-0.472	-2.105	-0.702
5.0	0.226	-0.925	-1.453	-0.484	-2.080	-0.693
6.0	0.195	-0.935	-1.510	-0.503	-2.042	-0.681
7.0	0.172	-0.943	-1.553	-0.518	-2.014	-0.671
8.0	0.153	-0.949	-1.585	-0.528	-1.933	-0.664
9.0	0.138	-0.954	-1.611	-0.537	-1.976	-0.659
10.0	0.126	-0.958	-1.633	-0.544	-1.963	-0.654



1. Figure 1 Spherical shell with a circular elastic inclusion

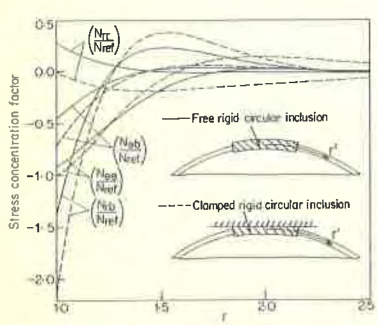


Figure 2 Membrane and bending stresses in a uniformly heated spherical shell with a rigid circular inclusion ($\beta_s = 4$, $1 < r < 2.5$)

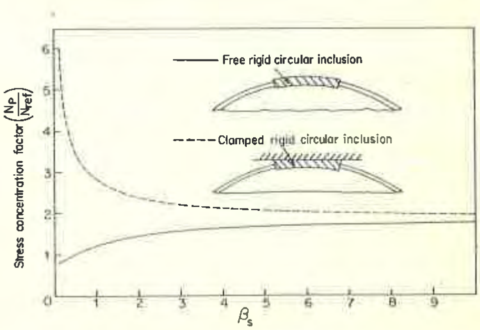


Figure 3 Maximum principal stress in a uniformly heated spherical shell with a rigid circular inclusion ($0 < \beta_s < 10$, $r = 1$)

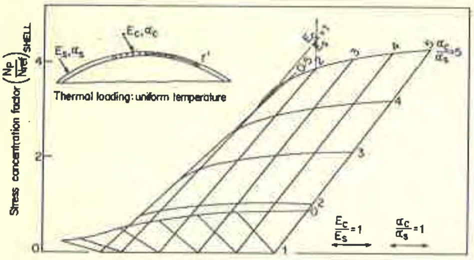


Figure 4 Maximum principal stress in a uniformly heated spherical shell with a circular elastic inclusion ($\beta_s = \beta_c = 2$, $t_s = t_c$; $r = 1$; $0 < E_c/E_s < 5$, $0 < \alpha_c/\alpha_s < 5$)

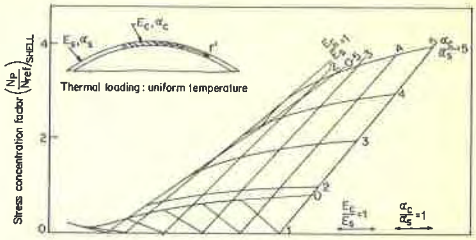


Figure 5 Maximum principal stress in a uniformly heated spherical shell with a circular elastic inclusion ($\beta_s = \beta_c = 4$, $t_s = t_c$, $r = 1$, $0 < E_c/E_s < 5$, $0 < \alpha_c/\alpha_s < 5$).

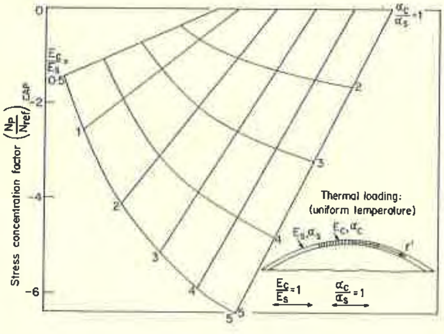


Figure 6 Maximum compressive principal stress in the circular elastic inclusion in a uniformly heated spherical shell ($\beta_s = \beta_c = 2$, $t_s = t_c$, $r = 1$, $0 < E_c/E_s < 5$, $0 < \alpha_c/\alpha_s < 5$)

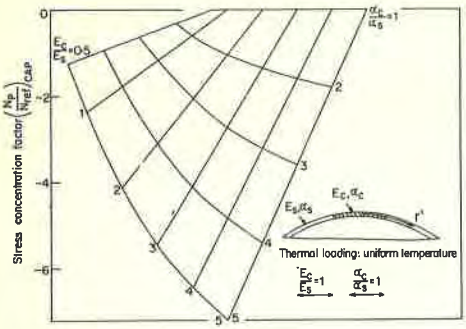


Figure 7 Maximum compressive principal stress in the circular elastic inclusion in a uniformly heated spherical shell ($\beta_s = \beta_c = 4$, $t_s = t_c$, $r = 1$, $0 < E_c/E_s < 5$, $0 < \alpha_c/\alpha_s < 5$)

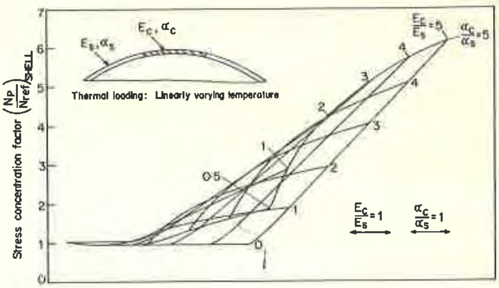


Figure 8 Maximum principal stress in a spherical shell with a circular elastic inclusion, with temperature varying linearly across the thickness. ($\beta_s = \beta_c = 2$, $t_s = t_c$, $r = 1$, $0 < E_c/E_s < 5$, $0 < \alpha_c/\alpha_s < 5$)

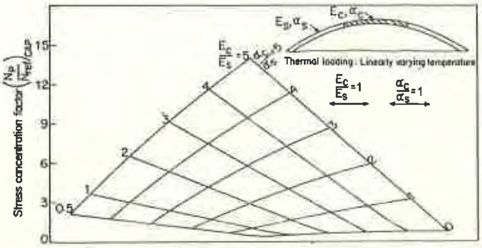


Figure 9 Maximum principal stress in the circular elastic inclusion in a spherical shell with temperature varying linearly across the thickness ($\beta_s = \beta_c = 2$, $t_s = t_c$, $r = 1$, $0 < E_c/E_s < 5$, $0 < \alpha_c/\alpha_s < 5$)