Dynamic Analysis of Viscoelastic Cracked Structures Using Incremental Finite Element Method

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ABSTRACT

This work presents an efficient and accurate incremental finite element procedure, without involving any integral transformations, to deal with viscoelastic cracked structures with complicated geometries subjected to dynamic presures and thermal loadings for which time effect and temperature effects for relaxation are considered.

To reduce data storage and computing time , the constitutive integral equations are expressed by an incremental recurrence formula which is developed for relaxation modulus represented by an exponential series. With the recurrence formula , the value of a hereditary integral at current time step depends only on that of previous time step . Hence , the viscoelastic problems can be solved in a fashion similar to the elastic problems for each time step.

The computation has been carried out for viscoelastic cracked rectangular plate which is initially loaded by uniform tractions along the boundary with a Heaviside function. The time variation of the mode I dynamic viscoelastic stress intensity factor is determined for the case with a center crack. As would be expected, the stress intensity factors of viscoelastic structure are smaller than those of the elastic solid due to the relaxation effect. Further, as time passes, the dynamic viscoelastic stress intensity factors will converge to their corresponding static values.

The technique developed here will be helpful for the safety evaluation of viscoelastic cracked structures serviced in severe loading and temperature environments.

1 INTRODUCTION

The fracture behaviour and dynamic effect estimation of viscoelastic cracked structure becomes important with the increasing utilization of polymeric or metal materials (Aoki et al. 1980; Schapery 1975). In rocketry, the fracture mechanics analysis of solid propellant requires viscoelastic analyses of which both time-dependent and temperature-dependent effects need to be taken into account (Williams et al. 1964). Other important engineering fracture problems, such as turbine blade design , jet engine design and building structural analysis in earthquakes, involve the same considerations.

As is known, the mechanical behaviour of viscoelastic materials depends not only upon the current stress at any given instant, but also upon the full history of stress. The finite element technique which has been demonstrated as an excellent analysis method has been introduced by numerous authors (Chen et al. 1982; White 1968) for analysing the viscoelastic or creep behaviour of non-cracked structures. Since the failure of structures is frequently attributed

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to the existence of crack, the knowledge for determining the stress and strain fields near the crack tip is necessary. Thus , it is the objective of this work to develop a hybrid finite element algorithm for determining the accurate stress and strain fields for thermoviscoelastic cracked solids subjected to dynamic pressure and thermal loadings.

2 INCREMENTAL FORM OF CONSTITUTIVE EQUATION

The hereditary integral of the isothermal viscoelastic constitutive relations can be formed as follow (Christensen 1982):

$$\begin{aligned} \sigma_{ij}(t) &= \int_{-\infty}^{t} G_{ijkl}(t-\tau)\frac{d\varepsilon_{kl}(\tau)}{d\tau}d\tau \\ \varepsilon_{ij}(t) &= \int_{-\infty}^{t} J_{ijkl}(t-\tau)\frac{d\sigma_{kl}(\tau)}{d\tau}d\tau \end{aligned}$$

where σ_{ij} represents the total stress, ε_{ij} represents the total strain; G_{ijkl} is called as relaxation function and J_{ijkl} is called as creep function. If there is a step discontinuity of strain history at t=0 , it is found that

$$\sigma_{ij}(t) = G_{ijkl}(t)\varepsilon_{kl}(0) + \int_{0}^{t} G_{ijkl}(t-\tau)\frac{d\varepsilon_{kl}(\tau)}{d\tau}d\tau$$

Since the material is considered as homogeneous, isotropic and viscoelastic, the most general isotropic representation of a fourth order tensor is given by

$$G_{ijkl}(t) = \frac{1}{3}\left[G_2(t) - G_1(t)\right]\delta_{ij}\delta_{kl} + \frac{1}{2}G_1(t)\left[\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right]$$

where $G_1(t)$ is the shear relaxation function and $G_1(t) = 2\mu(t)$. $G_2(t)$ denotes the dilatation relaxation function and $G_2(t) = 3\lambda(t) + 2\mu(t)$. The above equation becomes

$$\begin{aligned} \sigma_{ij}(t) = \ & 2\int_{0}^{t} \mu_{ijkl}(t-\tau)\frac{d\varepsilon_{kl}(\tau)}{d\tau}d\tau + \delta_{ij}\int_{0}^{t}\lambda(t-\tau)\frac{d\varepsilon_{kl}(\tau)}{d\tau}d\tau \\ + \ & 2\mu(t)\varepsilon_{ij}(0) + \delta_{ij}\lambda(t)\varepsilon_{kk}(0) \end{aligned}$$

To account for the temperature-effect, the so called WLF equation (Williams et al. 1964) is used as

$$\log a_T = \frac{-C_1(T-T_M)}{C_2 + (T-T_M)}$$

where $a_T(t)$ is the shift function and C_1 and C_2 are the constants to be measured. T_M is the reference temperature. According to the time-temperature correspondence principle , the hereditary integral can be reformed as the following equivalent equation with reduced time ξ

$$\begin{aligned} \sigma_{ij}(t) = \ & 2\int_{0}^{t} \mu_{ijkl}(\xi-\xi')\frac{d\varepsilon_{ij}(\tau)}{d\tau}d\tau + \delta_{ij}\int_{0}^{t}\lambda(\xi-\xi')\frac{d\varepsilon_{kk}(\tau)}{d\tau}d\tau \\ - \ & \delta_{ij}\int_{0}^{t} 3\alpha(\xi-\xi')\frac{\partial\theta}{d\tau}d\tau + 2\mu(\xi)\varepsilon_{ij}(0) + \delta_{ij}\lambda(\xi)\varepsilon_{kk}(0) \end{aligned} \qquad (1)$$

where $\xi = \int_0^t \frac{1}{a_T(\tau)} d\tau$, $\xi' = \int_0^\tau \frac{1}{a_T(u)} du$, α is thermal expansion coefficient, $K(\xi)$ is a bulk modulus and $\mu(\xi)$,$\lambda(\xi)$ are Lame constants.

In order to properly model the specific stress-strain-time characteristics in finite element analysis, the constitutive integral equations need be expressed by an incremental recurrence formula. Asume the incremental stress , strain and temperature during the time step $[t_{N-1}, t_N]$ as

$$\begin{aligned}
\Delta\sigma_{ij}(t_N) &= \sigma_{ij}(t_N) - \sigma_{ij}(t_{N-1}) \ , \\
\Delta\varepsilon_{ij}(t_N) &= \varepsilon_{ij}(t_N) - \varepsilon_{ij}(t_{N-1}) \ , \\
\Delta\theta(t_N) &= \theta(t_N) - \theta(t_{N-1})
\end{aligned}$$

After expressing the shear relaxation modulus and thermal expansion coefficient by a Prony series, the incremental form of thermo-viscoelastic constitutive equation (1) can be represented as the following equivalent elastic-like form

$$\begin{aligned}
\Delta\sigma_{ij}(t_N) = & \ 2 \ \overline{\mu}(t_N)\Delta\varepsilon_{ij}(t_N) + \delta_{ij}\overline{\lambda}(t_N)\Delta\varepsilon_{kk}(t_N) - \delta_{ij}\overline{\beta}(t_N)\Delta\theta(t_N) \\
& - \ \sigma_{ij}^0(t_N)
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
\overline{\mu}(t_N) &= \sum_{m=1}^{M} \left[A_m \left(1 - \frac{1}{2} a_m \Delta t_N B(t_N) \right) \right] \\
\overline{\lambda}(t_N) &= \sum_{m=1}^{M} \left[B_m \left(1 - \frac{1}{2} b_m \Delta t_N B(t_N) \right) \right] \\
\overline{\beta}(t_N) &= \sum_{m=1}^{M} \left[\beta_m \left(1 - \frac{1}{2} c_m \Delta t_N B(t_N) \right) \right] \\
\sigma_{ij}^0(t_N) &= 2 \sum_{m=1}^{M} \left(e^{-a_m \xi_N} - e^{-a_m \xi_{N-1}} \right) I_{N-1,m}^{(1)} \\
&+ \ \delta_{ij} \sum_{m=1}^{M} \left(e^{-b_m \xi_N} - e^{-b_m \xi_{N-1}} \right) I_{N-1,m}^{(2)} \\
&- \ \delta_{ij} \sum_{m=1}^{M} \left(e^{-c_m \xi_N} - e^{-c_m \xi_{N-1,m}} \right) I_{N-1,m}^{(3)} \\
&+ \ 2 \sum_{m=1}^{M} \left[A_m a_m \frac{1}{2} \Delta t_N \left(B(t_N) + e^{-a_m \Delta\xi_N} B(t_{N-1}) \right) \varepsilon_{ij}(t_{N-1}) \right] \\
&+ \ \delta_{ij} \sum_{m=1}^{M} \left[B_m b_m \frac{1}{2} \Delta t_N \left(B(t_N) + e^{-b_m \Delta\xi_N} B(t_{N-1}) \right) \varepsilon_{ij}(t_{N-1}) \right] \\
&- \ \delta_{ij} \sum_{m=1}^{M} \left[\beta_m c_m \frac{1}{2} \Delta t_N \left(B(t_N) + e^{-c_m \Delta\xi_N} B(t_{N-1}) \right) \theta(t_{N-1}) \right]
\end{aligned}$$

In the formula, $\Delta t_N = t_N - t_{N-1}$ is the time increment , $\Delta\xi_N = \xi_N - \xi_{N-1}$ is the reduced time increment , $B(t_N) = \frac{1}{a_T(t_N)}$ and $B(t_{N-1}) = \frac{1}{a_T(t_{N-1})}$. The recurrence formula is defined as follows :

$$I_{N-1,m}^{(1)} = A_m a_m \int_0^{t_{N-1}} e^{a_m \xi} \frac{1}{a_T(\tau)} \varepsilon_{ij}(\tau) d\tau \quad ,$$

$$I_{N-1,m}^{(2)} = B_m b_m \int_0^{t_{N-1}} e^{b_m \xi} \frac{1}{a_T(\tau)} \varepsilon_{kk}(\tau) d\tau \quad ,$$

$$I_{N-1,m}^{(3)} = \beta_m c_m \int_0^{t_{N-1}} e^{c_m \xi} \frac{1}{a_T(\tau)} \theta(\tau) d\tau$$

Then , equation (2) can be rewritten as

$$\Delta\sigma_{ij}(t_N) = C_{ijkl}^{*}(t_N)\Delta\varepsilon_{kl}(t_N) - \sigma_{ij}^{0}(t_N) - \overline{\beta}\Delta\theta(t_N)\delta_{ij} \tag{3}$$

where $C_{ijkl}^{*}(t_N)$ is the viscoelastic modulus matrix, $\sigma_{ij}^{0}(t_N)$ can be treated as the equivalent initial stress in each time increment .

3 INCREMENTAL HYBRID FINITE ELEMENT MODEL

Based on the modified Hamilton principle for dynamic thermoviscoelastic cracked problems, the functional of an incremental hybrid finite element model can be stated as

$$\begin{aligned} \pi_{HD} &= \int_{t_{N-1}}^{t_N} \Bigg[\sum_m \int_{\Omega_m} \left(\sigma_{ij}^{N-1} + \frac{1}{2} C_{ijkl}^{*} \Delta\varepsilon_{kl} - \sigma_{ij}^{0} - \overline{\beta}\Delta\theta\delta_{ij} \right) \Delta\varepsilon_{ij} d\Omega \\ &\quad - \frac{1}{2}\int_{\Omega_m} \rho_m \Delta\dot{u}_i \Delta\dot{u}_i d\Omega - \int_{\Omega_m} \left(\overline{F_i}^{N-1} + \Delta\overline{F_i} \right) \Delta u_i d\Omega \\ &\quad - \int_{s_{\sigma m}} \left(\overline{T_i}^{N-1} + \Delta\overline{T_i} \right) \Delta u_i ds + \int_{\partial\Omega_m} T_{Li}(\Delta v_i - \Delta u_i) ds \; \Bigg] dt \\ &= \text{stationary} \end{aligned} \tag{4}$$

where Ω_m is the mth element region and $\partial\Omega_m$ is the mth element boundary. Δv_i is the assumed incremental interelement displacement on $\partial\Omega_m$, Δu_i is the assumed incremental element interior displacement in Ω_m. T_{Li} is the Lagrangian multiplier which is nothing but the element boundary traction on $\partial\Omega_m$. ρ_m is the density function of the mth element. $\bar{F}_i$ and $\Delta\bar{F}_i$ represent the prescribed body force and incremental body force and $\bar{T}_i$ and $\Delta\bar{T}_i$ represent the prescribed surface traction and incremental surface traction,respectively.

Now , the incremental displacements $\Delta u_i(t)$ can be assumed in finite element version as (in matrix form) :

$$\{\Delta u\} = [U_1]\{\beta\} + [U_2]\{\overline{\beta}\} \tag{5}$$

where $[U_1]\{\beta\}$ is the pure strain displacement and $[U_2]\{\overline{\beta}\}$ is for rigid body motion (Chen et al. 1981). $\{\beta\}$ and $\{\overline{\beta}\}$ are the unknown independent parameters. From strain and displacement relation

$$\{\Delta\varepsilon\} = [B]\{\beta\} \tag{6}$$

Similarly, from equation (3)

$$\{\Delta\sigma\} = [C^*]\{\Delta\varepsilon\} - \{\Delta\sigma^0\} - \delta_{ij}\{\overline{\beta}\Delta\theta\} \tag{7}$$

The incremental interelement boundary displacement Δv_i and element boundary traction T_{Li} are

$$[\Delta v_i] = [L]\{\Delta q\} \tag{8}$$

$$\{T_{Li}\} = [R_1]\{\alpha\} + [R_2]\{\gamma\} + [R_3]\{\hat{\alpha}\} \tag{9}$$

where $\{\Delta q\}$ is the incremental nodal displacement for each time step. $[L]$ is interpolation function (Chen et al. 1981). $\{\alpha\}$ and $\{\hat{\alpha}\}$ are the unknown independent parameters. Substituting equations (5)$\sim$ (9) into equation (3), after tedious manipulations, one can obtain the final simultaneous ordinary differential equation for dynamic thermoviscoelastic problems:

$$[M^*]\{\Delta\ddot{q}^*\} + [K^*]\{\Delta q^*\} = \{\Delta Q^*\} \tag{10}$$

4 RESULTS AND DISCUSSION

For want of space, only the analysis for the isothermal viscoelastic rectangular plate with a center crack subjected to a Heaviside impact loading is presented. The geometry and finite element meshes of the first quadrant solved are illustrated in Fig.1. Fig.2 shows the time variation of the normalized mode I dynamic stress instensity factor $K_I(t)/\sigma\sqrt{\pi a_0}$ for the problem. The relaxation modulus used in the present estimation is

$$\mu(t) = 29.4\left(\frac{\alpha}{\beta} + \left(1.0 - \frac{\alpha}{\beta}\right)e^{-\beta t}\right) GN/m^2$$

Poisson's ratio is 0.286, density function ρ is $2.45 \times 10^3 kg/m^3$, α is $0.2 \times 10^5 s^{-1}$ and β is taken as $0.4 \times 10^5 s^{-1}$. Good agreement between the present computed results and available solutions (Aoki,1980) can be noted.

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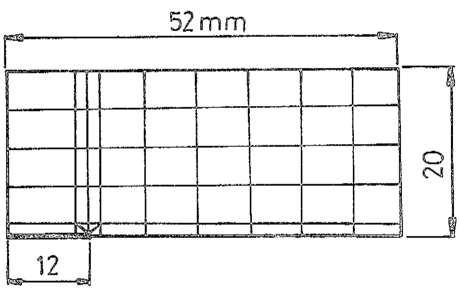

Fig.1 the geometry and finite element mesh of viscoelastic cracked plate

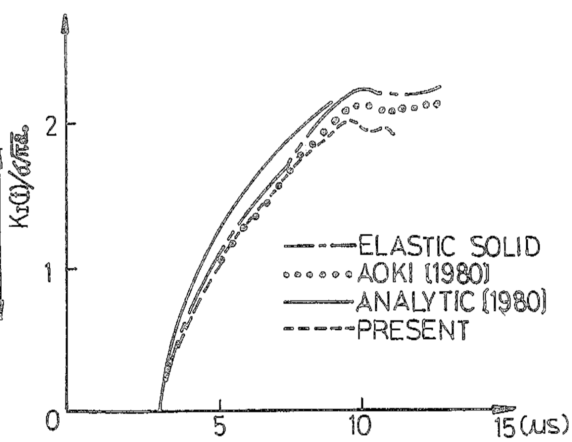

Fig.2 the time variation of normalized mode I dynamic stress intensity factor