

ABSTRACT

RIGGS, BRITTANY ANNE. An Improved Degree Bound on Exactifying Multipliers for Descartes' Rule of Signs. (Under the direction of Hoon Hong.)

Determining the number of roots of a univariate polynomial has been a fundamental mathematical question for hundreds of years. The number of complex roots has been resolved elegantly since Gauss proved the Fundamental Theorem of Algebra in 1799. While methods for determining the number of real roots of a polynomial have been found, the pursuit of better methods of counting is an active and non-trivial area of research.

One of the most used and simplest results was stated by Descartes in 1637: the number of positive real roots of a polynomial f is bounded by the number of sign variations in the consecutive non-zero coefficients of f . There followed a number of significant results, by Budan, Fourier, Hermite and Sturm. While Hermite and Sturm provided an exact count of the number of real roots, their methods were complex and costly to implement. Despite this later work, Descartes' Rule of Signs is still most commonly used, due to its simplicity. However, Descartes only provided a bound on the number of positive real roots, not an exact count.

In 1888, Poincaré proved that Descartes' Rule of Signs is exact, in some sense. He proved that for a non-zero univariate polynomial f , there exists a multiplier g such that the number of sign variations in the consecutive non-zero coefficients of gf is exactly the number of positive real roots of f . Studying this multiplier can provide an avenue to a faster way to count the exact number of positive real roots of a polynomial. In particular, an effective bound on the degree of this multiplier is essential to progress.

This research focuses on the fundamental case in which the polynomial f has no positive real roots but still has sign variation. We improve upon the previous bound on the degree of the multiplier, by Powers and Reznick, and then extend the bound to arbitrary polynomials with any number of positive real roots.

The previous bound arose from a careful study of a multiplier given by Pólya as a certificate of positivity in a generalized multivariate case. The new bound is derived from a reformatting of the problem using linear algebra, resulting in a bound based on the geometry of the complex roots of the polynomial. In general, the new bound is optimal in some cases and appears to be exponentially smaller in the degree of f than its predecessor, a significant overall improvement.

A summary of results is as follows:

1. We provide a new bound on the degree of an exatifying multiplier for Descartes' Rule of Signs in the special case of polynomials with no positive real root.
2. We prove the new bound is optimal for quadratic polynomials with no positive real root.
3. We provide a witness for the multiplier for quadratic polynomials with no positive real root.
4. We discuss optimality in higher degrees.
5. We prove an extension of the bound to polynomials with any number of positive real roots.
6. We provide a detailed comparison of the previous bound by Powers and Reznick to the new bound.

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An Improved Degree Bound on Exactifying Multipliers for Descartes' Rule of Signs

by
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DEDICATION

To my husband Aaron, for putting up with me. To my friends and family, for supporting me. To my dog and cats, for vital snugs.

BIOGRAPHY

Brittany Anne Riggs was born in the suburbs of Chicago and grew up all across the United States, from Florida to Nevada. She completed a Bachelor's in Mathematics at Butler University in 2005 and a Master's in Ethics, Peace and Global Affairs at American University in 2007. Brittany married Aaron in 2010 and then began a 5 year position at Northern Virginia Community College. In 2015, she returned to school to pursue her doctorate in Mathematics under the advisement of Dr. Hoon Hong. In Fall 2020, she will begin teaching at Elon University.

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CHAPTER

1

INTRODUCTION

Determining the number of roots of a univariate polynomial has been a fundamental mathematical question for hundreds of years. The number of complex roots has been resolved elegantly since Gauss proved the Fundamental Theorem of Algebra in 1799. While methods for determining the number of real roots of a polynomial have been found, the pursuit of better methods of counting is an active and non-trivial area of research.

Chapter 2 will include several such results. One of the most used and simplest results was stated by Descartes in 1637: the number of positive real roots of a polynomial f is bounded by the number of sign variations in the consecutive non-zero coefficients of f . There followed a number of significant results, by Budan, Fourier, Hermite and Sturm. While Hermite and Sturm provided an exact count of the number of real roots, their methods were complex and costly to implement. Despite this later work, Descartes' Rule of Signs is still most commonly used, due to its simplicity.

In the following table, we will list some research from the last century based on Descartes' Rule of Signs.

Table 1.1 A Sample of Research Citing Descartes' Rule of Signs

Credit	Title
Curtiss 1918 [10]	Recent Extentions of Descartes' Rule of Signs
Schoenberg 1955 [30]	A Note on Multiply Positive Sequences and the Descartes Rule of Signs
Carnicer-Peña 1998 [9]	Characterizations of the Optimal Descartes' Rule of Signs
Eigenwillig 2008 [13]	Real Root Isolation for Exact and Approximate Polynomials using Descartes' Rule of Signs
Sagraloff 2014 [28]	A Near-Optimal Algorithm for Computing Real Roots of Sparse Polynomials
Kostov 2020 [18]	Descartes' Rule of Signs and Moduli of Roots

However, Descartes only provided a bound on the number of positive real roots, not an exact count. In 1888, Poincaré proved that Descartes' Rule of Signs is exact, in some sense. He proved that for a non-zero univariate polynomial f , there exists a multiplier g such that the number of sign variations in the consecutive non-zero coefficients of gf is exactly the number of positive real roots of f . Studying this multiplier can provide an avenue to a faster way to count the exact number of positive real roots of a polynomial. In particular, an effective bound on the degree of this multiplier is essential to progress.

Chapter 2 will detail a formulation for this multiplier by Avendaño and a bound on its degree in a foundational case, by Powers and Reznick. The previous bound arose from a careful study of a multiplier given by Pólya as a certificate of positivity in a generalized multivariate scenario and a subsequent specialization to univariate polynomials. In addition to its mathematical importance, this work by Avendaño and Powers-Reznick is cited in applications throughout applied math and the sciences.

Table 1.2 Current Research Citing Avendaño

Field Credit	Title
Modeling Duong-Han 2019 [12]	On Equilibrium Properties of the Replicator-Mutator Equation in Deterministic and Random Games
Quantum Information Science Starke Chrysosthemos-Maziero 2018 [32]	Quantum Coherence - Classical Uncertainty Tradeoff Relations
Modeling Ble-Castellanos-Dela Rosa 2018 [5]	Coexistence of Species in a Tritrophic Food Chain Model with Holling Functional Response Type IV

Table 1.3 Current Research Citing Powers-Reznick

Field Credit	Title
Modeling Duong-Han 2019 [12]	On Equilibrium Properties of the Replicator-Mutator Equation in Deterministic and Random Games
Electrical Engineering Toth-Cox-Weiland 2018 [27]	Affine Parameter-Dependent Lyapunov Functions for LPV Systems with Affine Dependence
Controls Fetzer-Scherer 2017 [14]	Full-block Multipliers for Repeated, Slope-Restricted Scalar Nonlinearities
Artificial Intelligence Ariño-Querol-Sala 2017 [1]	Shape-independent Model Predictive Control for Takagi-Sugeno Fuzzy Systems
Physics Brandão-Harrow 2017 [8]	Quantum de Finetti Theorems Under Local Measurements with Applications

This dissertation, like the work by Powers-Reznick, focuses on the fundamental case in which the polynomial f has no positive real roots but still has sign variation. We improve upon the previous bound on the degree of the multiplier by Powers and Reznick and then extend the bound to arbitrary polynomials with any number of positive real roots. The new bound is derived from a reformatting of the problem using linear algebra, resulting in a bound based on the geometry of the complex roots of the polynomial. In general, the new bound is optimal in some cases and appears to be exponentially smaller in the degree of f than its predecessor, a significant overall improvement.

A summary of results is as follows:

1. Chapter 3: We provide a new bound on the degree of an exactifying multiplier for Descartes' Rule of Signs in the special case of polynomials with no positive real root.
2. Chapter 3: We prove the new bound is optimal for quadratic polynomials with no positive real root.
3. Chapter 3: We provide a witness for the multiplier for quadratic polynomials with no positive real root.
4. Chapter 3: We discuss optimality in higher degrees.
5. Chapter 4: We prove an extension of the bound to polynomials with any number of positive real roots.
6. Chapter 5: We provide a detailed comparison of the previous bound by Powers and Reznick to the new bound.

CHAPTER

2

PREVIOUS WORK

This dissertation will provide an improved degree bound on exactifying multipliers for Descartes' Rule of Signs. This chapter will serve to define this concept, detail the work previously done on the existence of such multipliers, and show the derivation of the previous degree bound.

2.1 Descartes' Rule of Signs

First, we will describe the foundational result, Descartes' Rule of Signs. Let

$p(f)$ = the number of positive real roots of f (counting multiplicity)

$v(f)$ = the number of sign differences between consecutive non-zero coefficients.

Theorem 2.1.1 (Descartes 1637 [11]). *Let $f \in \mathbb{R}[x]$ be non-zero. Then $p(f) \leq v(f)$ and $p(f) \equiv_2 v(f)$.*

Descartes provided an elegant upper bound on the number of positive real roots of a given polynomial. While computationally simple, there is no guarantee this is a particularly tight bound.

Example 2.1.1. *Let $f = x^2 - 2x + 4$. Then $p(f) = 0$ and $v(f) = 2$. Clearly, $p(f) < v(f)$ and $p(f) \equiv_2 v(f)$, since $v(f) - p(f) = 2$.*

Example 2.1.2. *Let $f = (x^2 - 2x + 4)(x^2 - 2x + 2) = x^4 - 4x^3 + 10x^2 - 12x + 8$. Then $p(f) = 0$ and $v(f) = 4$. Clearly, $p(f) < v(f)$ and $p(f) \equiv_2 v(f)$, since $v(f) - p(f) = 4$.*

2.2 Efforts to Improve upon Descartes' Rule of Signs

For the following theorems, we define

$$\begin{aligned} r_I(f) &= \text{the number of real roots of } f \text{ (counting multiplicity) in the interval } I \\ r_I^*(f) &= \text{the number of distinct real roots of } f \text{ in the interval } I \\ v(S) &= \text{the number of sign differences in the sequence } S \end{aligned}$$

Budan and Fourier independently proved a generalization of Descartes' Rule of Signs.

Theorem 2.2.1 (Budan 1807 [6] Fourier 1820 [15]). *Let $f \in \mathbb{R}[x]$ be non-zero. Define $S_a = \{f(a), f'(a), \dots, f^{(n)}(a)\}$. Then for any interval (a, b) ,*

1. $v(S_a) \geq v(S_b)$
2. $v(S_a) - v(S_b) - r_{(a,b)}(f)$ is a non-negative even integer

and hence $r_{(a,b)}(f) \leq v(S_a) - v(S_b)$.

Note that this is still an upper bound of the count of real roots of f and Descartes' Rule of Signs is a special case of the above theorem. Sturm defined a new sequence in order to improve upon this counting technique.

Definition 2.2.1. The Sturm sequence of a polynomial $f \in \mathbb{R}[x]$ is the sequence given by

1. $f_0 = f$
2. $f_1 = f'$ (the derivative of f)
3. $f_{i+1} = -\text{rem}(f_{i-1}, f_i)$

for $i \geq 1$ and where $\text{rem}(f_{i-1}, f_i)$ is the remainder in the Euclidean division of f_{i-1} by f_i .

Theorem 2.2.2 (Sturm 1829 [34]). *Let $f \in \mathbb{R}[x]$ and $a < b$ such that $f(a), f(b) \neq 0$. Let S_a be the Sturm sequence for f evaluated at $x = a$. Then $r_{(a,b)}^*(f) = v(S_a) - v(S_b)$.*

Sturm's theorem provides an exact count of the number of distinct real roots of a polynomial in a given interval, but is computationally costly to implement. Hermite provided another such exact result, but several definitions are needed first.

Definition 2.2.2. Let $f = (x - z_1) \cdots (x - z_n)$, where $z_i \in \mathbb{C}$. Then the discriminant matrix Q of f is defined by $Q = V V^T$ where

$$V = \begin{bmatrix} z_1^0 & \cdots & z_n^0 \\ \vdots & & \vdots \\ z_1^{n-1} & \cdots & z_n^{n-1} \end{bmatrix}.$$

Definition 2.2.3. Let M be a real symmetric matrix. The signature of M , $\sigma(M)$, is the number of positive eigenvalues of M minus the number of negative eigenvalues of M , counting multiplicity.

Theorem 2.2.3 (Hermite 1853 [16]). *Let $f \in \mathbb{R}[x]$. Then $r_{\mathbb{R}}^*(f) = \sigma(Q)$, where Q is the discriminant matrix of f .*

While this is an exact count of the distinct real roots of f , it does require knowing the roots of f in order to count them.

2.3 Exactifying Multipliers for Descartes' Rule of Signs

In spite of the exact root counts detailed in the previous section, Descartes' Rule of Signs remains frequently used. As such, research has been conducted to address the "inexactness" of the bound. Poincaré proved that Descartes' Rule of Signs is exact in some sense.

Theorem 2.3.1 (Poincaré 1888 [23]). *For a non-zero $f \in \mathbb{R}[x]$, there exists $g \in \mathbb{R}[x]$ such that $v(gf) = p(f)$.*

Poincaré proved that, for a polynomial f , there exists a multiplier g that makes Descartes' Rule of Signs exact, in that $v(gf) = p(f)$.

Example 2.3.1. *Let $f = x^2 - 2x + 4$ and $g = x^2 + 2x + 4$. Then $gf = x^4 + 4x^2 + 16$, $v(gf) = 0$ and $v(gf) = p(f)$.*

Definition 2.3.1. For a non-zero polynomial $f \in \mathbb{R}[x]$, an exactifying multiplier for Descartes' Rule of Signs is a polynomial $g \in \mathbb{R}[x]$ such that $v(gf) = p(f)$.

Poincaré's theorem guarantees the existence of a polynomial multiplier g that serves to make Descartes' Rule of Signs exact. Martin Avendaño proved that such a g with non-negative coefficients exists, and found a witness for this g , which we will detail here.

Avendaño first proved two smaller lemmas.

Lemma 2.3.1 (Avendaño 2010 [4]). *Let $f \in \mathbb{R}[x]$ be a monic quadratic polynomial with no positive real roots. Then there exists a monic polynomial $g \in \mathbb{R}[x]$ with all non-negative coefficients such that gf has all non-negative coefficients.*

Proof. Let $f = x^2 + bx + c$.

1. Case 1: f has two non-positive real roots, $\alpha, \beta \in \mathbb{R}_{\leq 0}$. Then

$$f = (x - \alpha)(x - \beta) = x^2 + (-\alpha - \beta)x + \alpha\beta.$$

Since $\alpha, \beta \in \mathbb{R}_{\leq 0}$, $-\alpha - \beta, \alpha\beta \in \mathbb{R}_{\geq 0}$ and f itself has non-negative coefficients. Hence, $g = 1$ will suffice.

2. Case 2: f has two complex roots, α and $\bar{\alpha}$, where $\arg(\alpha) \in (0, \pi)$. For any α , $\arg(\alpha) = \frac{\pi}{2^k}$ or $\frac{\pi}{2^k} < \arg(\alpha) < \frac{\pi}{2^{k-1}}$ for some $k \in \mathbb{N}$ (so $\frac{\pi}{2^k} \leq \arg(\alpha) < \frac{\pi}{2^{k-1}}$). We induct on k .

- (a) Base Case: $k = 1$. Then $\frac{\pi}{2} \leq \arg(\alpha) < \pi$. In this case, $\alpha = -r + si$, $\bar{\alpha} = -r - si$, where $r, s \in \mathbb{R}_{\geq 0}$.

$$\begin{aligned} f &= (x - (-r + si))(x - (-r - si)) \\ &= (x + r - si)(x + r + si) \\ &= x^2 + 2rx + (r^2 + s^2) \end{aligned}$$

In this case, $2r, r^2 + s^2 \in \mathbb{R}_{\geq 0}$, so f has non-negative coefficients and $g = 1$ will suffice.

- (b) $k > 1$: Assume any monic quadratic with two complex roots $\alpha, \bar{\alpha}$ such that $\frac{\pi}{2^{k-1}} \leq \arg(\alpha) < \frac{\pi}{2^{k-2}}$ has a monic polynomial $g \in \mathbb{R}[x]$ such that gf has all non-negative coefficients. Consider the polynomial $f(x)f(-x)$:

$$\begin{aligned} f(x)f(-x) &= (x^2 + bx + c)(x^2 - bx + c) \\ &= x^4 + (2c - b^2)x^2 + c^2 \\ &= x^4 + dx^2 + e \end{aligned}$$

where $d = 2c - b^2$ and $e = c^2$. Then $f(x)f(-x) = \tilde{f}(x^2)$ where $\tilde{f} = x^2 + dx + e$. We have roots α^2 and $\bar{\alpha}^2 = \overline{\alpha^2}$ of \tilde{f} since $\tilde{f}(\alpha^2) = f(\alpha)f(-\alpha) = 0$ and $\tilde{f}(\bar{\alpha}^2) = f(\bar{\alpha})f(-\bar{\alpha}) = 0$. Now

$$\begin{aligned} \frac{\pi}{2^k} &\leq \arg(\alpha) < \frac{\pi}{2^{k-1}} \\ \iff \frac{\pi}{2^k} &\leq \frac{1}{2} \arg(\alpha^2) < \frac{\pi}{2^{k-1}} \\ \iff \frac{\pi}{2^{k-1}} &\leq \arg(\alpha^2) < \frac{\pi}{2^{k-2}} \end{aligned}$$

Then \tilde{f} satisfies the inductive hypothesis. There exists some \tilde{g} with non-negative coef-

ficients such that $\tilde{g}\tilde{f}$ has all non-negative coefficients. Consider $g = f(-x)\tilde{g}(x^2)$.

$$\begin{aligned} gf &= f(-x)\tilde{g}(x^2)f(x) \\ &= \tilde{g}(x^2)f(x)f(-x) \\ &= \tilde{g}(x^2)\tilde{f}(x^2) \end{aligned}$$

We know $\tilde{g}\tilde{f}$ has all non-negative coefficients, so all that remains is to show that g has non-negative coefficients. Since $\arg(\alpha) < \frac{\pi}{2}$, $\alpha = r + si$, $\bar{\alpha} = r - si$ for $r, s \in \mathbb{R}_{\geq 0}$. Then

$$\begin{aligned} f &= (x - (r + si))(x - (r - si)) \\ &= (x - r - si)(x - r + si) \\ &= x^2 - 2rx + (r^2 + s^2) \end{aligned}$$

Hence, $f(-x)$ has non-negative coefficients and $g = f(-x)\tilde{g}(x^2)$ must also have non-negative coefficients. □

Example 2.3.2. Let $f = x^2 - 2x + 4$. Then $b = -2$ and $c = 4$. By the proof of Lemma 2.3.1,

$$\begin{aligned} \tilde{f} &= x^2 + (2c - b^2)x + c^2 \\ &= x^2 + 4x + 16 \\ \tilde{g} &= 1 \\ g &= f(-x)\tilde{g}(x^2) \\ &= (x^2 + 2x + 4)(1) \\ gf &= (x^2 + 2x + 4)(x^2 - 2x + 4) \\ &= x^4 + 4x + 16. \end{aligned}$$

Lemma 2.3.1 leads to a simple proof for the existence of such a g for monic polynomials of arbitrary degree with no positive real roots.

Lemma 2.3.2 (Avenidaño 2010 [4]). *Let $f \in \mathbb{R}[x]$ be a monic polynomial with no positive real roots. Then there exists a monic polynomial $g \in \mathbb{R}[x]$ with all non-negative coefficients such that gf has all non-negative coefficients.*

Proof. Let

$$f = (x - r_1) \cdots (x - r_t) f_1 \cdots f_m$$

where r_1, \dots, r_t are real roots and f_1, \dots, f_m are the irreducible quadratic factors of f over \mathbb{R} . Then the f_i have non-real roots and by Lemma 2.3.1, for each f_i , there exists a g_i such that $g_i f_i$ has all non-negative coefficients.

Let $g = g_1 \cdots g_m$. Then

$$\begin{aligned} g f &= g_1 \cdots g_m (x - r_1) \cdots (x - r_t) f_1 \cdots f_m \\ &= (x - r_1) \cdots (x - r_t) g_1 f_1 \cdots g_m f_m \end{aligned}$$

and $g f$ has all non-negative coefficients. □

We are now ready to prove the existence of Avendaño's exactifying multiplier.

Theorem 2.3.2 (Avendaño 2010 [4]). *Let $f \in \mathbb{R}[x]$ be a non-zero polynomial. Then there exists $g \in \mathbb{R}[x]$ with all non-negative coefficients such that $v(gf) = p(f)$.*

Proof. We will divide the proof into conceptual steps.

1. Let $f = a(x - r_1) \cdots (x - r_t) h$ where $a \in \mathbb{R}$, $r_i \in \mathbb{R}_{>0}$ and h is monic with no positive real roots. Let $n = \deg(f)$ and $t = p(f)$. Then $l = \deg(h) = \deg(f) - t = n - t$.
2. From Lemma 2.3.2, there exists a monic q such that $v(qh) = 0$. Let $N = \deg(qh)$.
3. Let $p = a(x^{N+1} - r_1^{N+1}) \cdots (x^{N+1} - r_t^{N+1})$. Note that $p = a$ if f has no positive real roots.

$$\begin{aligned} p &= a \prod_{j=1}^t (x^{N+1} - r_j^{N+1}) \\ &= a \sum_{j=0}^t (-1)^j e_j(r_1^{N+1}, \dots, r_t^{N+1}) (x^{N+1})^{t-j} \end{aligned}$$

where $e_j(r_1^{N+1}, \dots, r_t^{N+1})$ is the elementary symmetric polynomial of degree j in $(r_1^{N+1}, \dots, r_t^{N+1})$. In this form, we see immediately that $v(p) = t$.

4. Note that the degree gap between two consecutive terms of p is $N + 1$. Note also that qh has all non-negative coefficients and $\deg(qh) = N$. Now consider pqh :

$$\begin{aligned} pqh &= \left(a \sum_{j=0}^t (-1)^j e_j(r_1^{N+1}, \dots, r_t^{N+1}) (x^{N+1})^{t-j} \right) qh \\ &= a \sum_{j=0}^t (-1)^j e_j(r_1^{N+1}, \dots, r_t^{N+1}) (x^{N+1})^{t-j} qh \end{aligned}$$

In order to count the sign changes in pqh , fix a j :

$$\begin{aligned}
& (-1)^j e_j(r_1^{N+1}, \dots, r_t^{N+1})(x^{N+1})^{t-j} qh \\
&= (-1)^j e_j(r_1^{N+1}, \dots, r_t^{N+1}) x^{(N+1)(t-j)} \sum_{i=0}^N b_i x^i \quad \text{where } qh = \sum_{i=0}^N b_i x^i \\
&= \sum_{i=0}^N (-1)^j e_j(r_1^{N+1}, \dots, r_t^{N+1}) b_i x^{i+(N+1)(t-j)}
\end{aligned}$$

This polynomial has all same sign coefficients, based on the sign of $(-1)^j e_j(r_1^{N+1}, \dots, r_t^{N+1})$, since the b_i are all non-negative.

Additionally, there are no common terms when j varies. Consider terms contributed by the $(j-1)$ -th, j -th, and $(j+1)$ -th terms of pqh :

$$\begin{aligned}
& \text{highest degree term for } j+1: (N+1)(t-j)-1 \\
& \text{lowest degree term for } j: (N+1)(t-j)+0 \\
& \text{highest degree term for } j: (N+1)(t-j)+N \\
& \text{lowest degree term for } j-1: (N+1)(t-j)+N+1
\end{aligned}$$

Hence, we have preserved the number of sign changes of p and $v(pqh) = t$.

5. Note

$$\begin{aligned}
pqh &= a(x^{N+1} - r_1^{N+1}) \dots (x^{N+1} - r_t^{N+1}) qh \\
&= \frac{x^{N+1} - r_1^{N+1}}{x - r_1} \dots \frac{x^{N+1} - r_t^{N+1}}{x - r_t} q (a(x - r_1) \dots (x - r_t) h) \\
&= (x^N + r_1 x^{N-1} + \dots + r_1^N) \dots (x^N + r_t x^{N-1} + \dots + r_t^N) qf \\
&= gf
\end{aligned}$$

where $g = (x^N + r_1 x^{N-1} + \dots + r_1^N) \dots (x^N + r_t x^{N-1} + \dots + r_t^N) q$. Hence $v(gf) = t$.

6. Since f has t positive roots, we see that gf has at least t positive real roots. By Descartes' Rule of Signs, the number of positive real roots of gf is at most $v(gf) = t$. Thus gf has exactly t positive roots, i.e. Descartes' Rule of Signs is exact for gf .

□

Example 2.3.3. Let $f = (x - 1)(x^2 - 2x + 4)$. From the proof of Theorem 2.3.2, we have $n = 3$, $t = 1$, $r_1 = 1$ and $h = x^2 - 2x + 4$. From Example 2.3.2, we have $q = x^2 + 2x + 4$ and $qh = x^4 + 4x + 16$. Then

$$\begin{aligned}
 N &= 4 \\
 p &= x^5 - 1^5 = x^5 - 1 \\
 g &= (x^4 + x^3 + x^2 + x + 1) q \\
 gf &= (x^4 + x^3 + x^2 + x + 1)(x^2 + 2x + 4)(x - 1)(x^2 - 2x + 4) \\
 &= x^9 + 4x^7 + 16x^5 - x^4 - 4x^2 - 16.
 \end{aligned}$$

Note that $v(gf) = 1 = p(f)$.

2.4 A Degree Bound from Multivariate Certificates of Positivity

Recent research in multivariate polynomials has lent itself to the derivation of a degree bound on an exactifying multiplier for Descartes' Rule of Signs. This branch of research was motivated by David Hilbert's Seventeenth Problem:

"When can a non-negative form be expressed as a quotient of sums of squares of forms?" [17]

One way to answer Hilbert's question is to find a "certificate of positivity" for a given multivariate polynomial - an expression that serves to demonstrate that the polynomial is positive over the given set. We will first discuss some basic results on certificates of positivity and then detail the work by Pólya, which led to the bound in question.

To formalize this concept, let

$$\begin{aligned}
 f &\in \mathbb{R}[x_1, \dots, x_n] \\
 G &= \{g_1, \dots, g_r\} \subset \mathbb{R}[x_1, \dots, x_n] \\
 S &= \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}.
 \end{aligned}$$

The goal is to determine if $f(x_1, \dots, x_n) > 0$ for all $x_1, \dots, x_n \in S$ and, if so, provide a certificate of positivity.

2.4.1 Background on Certificates of Positivity

Hilbert himself suggested that expressing a polynomial as a quotient of sums of squares would serve as a certificate of positivity.

Example 2.4.1. Let $f(x) = x^2 - 2x + 4$. Since

$$f(x) = x^2 - 2x + 4 = (x - 1)^2 + (\sqrt{3})^2,$$

$f(x)$ is non-negative over the reals. Note that, in this example, $G = \emptyset$ and $S = \mathbb{R}$.

Then the goal of this technique is, given a polynomial f and a set G , to find an equivalent expression for f that is a quotient of sums of squares or to conclude that none exists. In order to create an algorithm to find such an expression, we need to have an upper bound on the degree of the polynomial components of the sum of squares or the algorithm may never terminate.

Once a bound is found, we immediately have the following algorithm:

1. Search for a certificate of a low degree.
2. If successful, then claim that f is positive and return the found certificate.
3. If not, then increase the degree and repeat.
4. If the degree reaches the bound, we stop and claim that f is not positive.

Due to the importance of the challenge, there has been much effort yielding significant progress. We will list several key results - the certificate paired with the relevant bound.

Throughout the following theorems, let

$$\begin{aligned} f &= \sum_e a_e x^e \\ d &= \deg(f). \end{aligned}$$

Emil Artin proved that a non-negative polynomial over \mathbb{R}^n can, in fact, always be written as a sum of squares. The subsequent research involves polynomials that are positive over more restrictive sets G .

Table 2.1 Certificate and Bound for $G = \emptyset$

Certificate	Bound
Artin 1927 [2]	Lombardi-Perucci-Roy 2014 [20]
$\forall_{x \in S} f(x) > 0 \implies \exists_{p, q \neq 0} f = \frac{\sum_i p_i^2}{q^2}$	$\deg(p_i), \deg(q) \leq 2^{2^{2^{d^{4^n}}}}$

It is important to note that, while a bound on the degree of the certificate of positivity exists, this bound is not currently practical in terms of implementation in an algorithm.

Example 2.4.2. Let $f = x^2 - 2x + 4$ and $G = \emptyset$. Then $n = 1$ and $d = 2$. Thus the bound on the Artin certificate, from Table 2.1, is given by

$$\deg p, \deg q \leq 2^{2^{2^{d^{4^n}}}} = 2^{2^{2^4}} \approx 10^{10^{20000}}$$

However, in Example 2.4.1, we provided a certificate of degree 1,

$$f = x^2 - 2x + 4 = (x - 1)^2 + (\sqrt{3})^2.$$

Table 2.2 Certificate and Bound for G arbitrary

Certificate	Bound
Krivine 1964 [19] / Stengle 1974 [33]	Lombardi-Perucci-Roy 2014 [20]
$\forall_{x \in S} f(x) > 0 \implies \exists_{p, q \neq 0} f = \frac{1 + \sum_{T \subset G} \sum_i p_{T,i}^2 \prod_{g \in T} g}{\sum_{T \subset G} \sum_i q_{T,i}^2 \prod_{g \in T} g}$	$\deg(p_{T,i}), \deg(q_{T,i}) \leq 2^{2^{\left(2^{d^{4^n}} + r^{2^n} d^{16^n \log_2 d}\right)}}$

Table 2.3 Certificate and Bound for G such that S is compact

Certificate	Bound
Schmüdgen 1991 [29]	Schweighofer 2004 [31]
$\forall_{x \in S} f(x) > 0 \implies \exists_{p \neq 0} f = \sum_{T \subset G} \sum_i p_{T,i}^2 \prod_{g \in T} g$	$\deg(p_{T,i}) \leq c d^2 \left(1 + \left(d^2 n^d \frac{U'}{L'} \right)^c \right)$ $U' = \max_e a_e \frac{e_1! \cdots e_n!}{(e_1 + \cdots + e_n)!} r^{e_1 + \cdots + e_n}$ <p style="text-align: center;">where r is such that $S \subset (-r, r)^n$</p> $L' = \min_{x \in S} f(x)$ <p>c is some constant</p>

The final certificate requires a definition first.

Definition 2.4.1. A set S is archimedean if $\exists_{k \in \mathbb{N}} k - (x_1^2 + \cdots + x_n^2) \in \{s_0 + s_1 g_1 + \cdots + s_r g_r : s_i\}$ is a sum of squares.

Table 2.4 Certificate and Bound for G such that S is archimedean

Certificate	Bound
Putinar 1993 [26]	Nie-Schweighofer 2007 [22]
$\forall_{x \in S} f(x) > 0 \implies \exists_{p \neq 0} f = \sum_i p_{0,i}^2 + \sum_{g \in G} \sum_i p_{g,i}^2 g$	$\deg(p_{0,i}), \deg(p_{g,i}) \leq c e \left(d^2 n^d \frac{U'}{L'} \right)^c$ $U' = \max_e a_e \frac{e_1! \cdots e_n!}{(e_1 + \cdots + e_n)!} r^{e_1 + \cdots + e_n}$ <p style="text-align: center;">where r is such that $S \subset (-r, r)^n$</p> $L' = \min_{x \in S} f(x)$ <p>c is some constant</p>

In each of these cases, the bound on the degree of the certificate is impractical. To make an algorithm feasible, we must improve upon these bounds.

2.4.2 Pólya's Certificate of Positivity

George Pólya considered the case when $G = \{x_1, x_2, \dots, x_n\}$ and proved that, in this case, sums of squares are not needed. In Section 2.5, we will show how this multiplier relates to Descartes' Rule of Signs and provides a degree bound on the exactifying multiplier for a fundamental set of polynomials.

Theorem 2.4.1 (Pólya 1928 [24]). *Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be monic and $G = \{x_1, x_2, \dots, x_n\}$. Then*

$$\forall_{x \in S} f(x) > 0 \implies \exists_{\substack{p, q \neq 0 \\ \text{coeffs}(p) \geq 0}} f = \frac{p}{q}$$

where $\text{coeffs}(p)$ represents "the coefficients of p ".

Rearranging the statement provides an equivalent form that we will use:

$$\forall_{x \in S} f(x) > 0 \implies \exists_{\substack{p, q \neq 0 \\ \text{coeffs}(p) \geq 0}} qf = p.$$

Hence, Pólya proved the existence of a polynomial multiplier that serves as a certificate of positivity when $G = \{x_1, x_2, \dots, x_n\}$. In addition to proving the existence of such a multiplier, Pólya also provided the shape of a witness.

Theorem 2.4.2 (Polya 1928 [24]). *Let $f \in \mathbb{R}[x_1, \dots, x_n]$. If $f > 0$ for $x \in \Delta_n$, then there exists $k \in \mathbb{N}_0$ such that the polynomial $(x_1 + \dots + x_n)^k f$ has all coefficients of the same sign.*

2.4.3 Powers-Reznick's Degree Bound on Pólya's Certificate of Positivity

Powers and Reznick studied the witness from Theorem 2.4.2 in order to compute a bound on the degree of the multiplier.

Theorem 2.4.3 (Powers-Reznick 2001 [25]). *Let $f = \sum_{|p|=d} a_p x^p$ be such that*

$$\forall_{x \in \Delta_n} f(x) > 0$$

where

$$\Delta_n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \geq 0, x_1 + \dots + x_n = 1\}.$$

Then all the coefficients of

$$(x_1 + \dots + x_n)^{[k]} f(x)$$

are non-negative if

$$k \geq \frac{d(d-1)U}{2L} - d$$

where

$$U = \max_{|p|=d} \frac{p_1! \cdots p_n!}{d!} |a_p|$$

$$L = \min_{x \in \Delta_n} f(x).$$

Proof. Let

$$f = \sum_{|p|=d} a_p x^p \quad g = \sum_{|q|=k} b_q x^q.$$

Note

$$\begin{aligned} fg &= \sum_{|p|=d} a_p x^p \sum_{|q|=k} b_q x^q = \sum_{\substack{|p|=d \\ |q|=k}} a_p b_q x^{p+q} = \sum_{|r|=d+k} \sum_{\substack{|p|=d \\ |q|=k \\ p+q=r}} a_p b_q x^r \\ &= \sum_{|r|=d+k} \sum_{\substack{|p|=d \\ p \leq r}} a_p b_{r-p} x^r = \sum_{|r|=d+k} c_r x^r \end{aligned}$$

where

$$c_r = \sum_{\substack{|p|=d \\ p \leq r}} a_p b_{r-p}.$$

Now we consider a specific g . For some k ,

$$g = (x_1 + \dots + x_n)^k.$$

The multinomial theorem states that the coefficients of g are given by

$$b_q = \frac{k!}{\prod_{i=1}^n q_i!}.$$

Thus

$$c_r = \sum_{\substack{|p|=d \\ p \leq r}} a_p \frac{k!}{\prod_{i=1}^n (r_i - p_i)!}.$$

Note

$$\begin{aligned}
c_r &= \frac{k!}{r_1! \cdots r_n!} \sum_{\substack{|p|=d \\ p \leq r}} a_p \prod_{i=1}^n \frac{r_i!}{(r_i - p_i)!} \\
&= \frac{k!}{r_1! \cdots r_n!} \sum_{\substack{|p|=d \\ p \leq r}} a_p \prod_{i=1}^n \prod_{j=0}^{p_i-1} (r_i - j) \quad \text{since } \prod_{i=1}^n \frac{r_i!}{(r_i - p_i)!} = r_i(r_i - 1) \cdots (r_i - p_i + 1) = \prod_{j=0}^{p_i-1} (r_i - j) \\
&= \frac{k!(d+k)^d}{r_1! \cdots r_n!} \sum_{\substack{|p|=d \\ p \leq r}} a_p \frac{\prod_{i=1}^n \prod_{j=0}^{p_i-1} (r_i - j)}{(d+k)^d} \\
&= \frac{k!(d+k)^d}{r_1! \cdots r_n!} \sum_{\substack{|p|=d \\ p \leq r}} a_p \frac{\prod_{i=1}^n \prod_{j=0}^{p_i-1} (r_i - j)}{\prod_{i=1}^n (d+k)^{p_i}} \quad \text{since } |p| = d \\
&= \frac{k!(d+k)^d}{r_1! \cdots r_n!} \sum_{\substack{|p|=d \\ p \leq r}} a_p \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \\
&\quad \text{since } \frac{\prod_{i=1}^n \prod_{j=0}^{p_i-1} (r_i - j)}{\prod_{i=1}^n (d+k)^{p_i}} = \prod_{i=1}^n \left(\frac{r_i - 0}{d+k} \right) \left(\frac{r_i - 1}{d+k} \right) \cdots \left(\frac{r_i - (p_i - 1)}{d+k} \right) \\
&= \frac{k!(d+k)^d}{r_1! \cdots r_n!} u_r
\end{aligned}$$

where

$$u_r = \sum_{\substack{|p|=d \\ p \leq r}} a_p \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k}.$$

Let

$$U = \max_{|p|=d} \frac{p_1! \cdots p_n!}{d!} |a_p|$$

$$L = \min_{x \in \Delta_n} f(x).$$

Note

$$\begin{aligned}
u_r &= \sum_{\substack{|p|=d \\ p \leq r}} a_p \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \\
&= \sum_{\substack{|p|=d \\ p \leq r}} a_p \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i}{d+k} - \sum_{\substack{|p|=d \\ p \leq r}} a_p \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i}{d+k} + \sum_{\substack{|p|=d \\ p \leq r}} a_p \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \\
&= \sum_{\substack{|p|=d \\ p \leq r}} a_p \prod_{i=1}^n \left(\frac{r_i}{d+k} \right)^{p_i} - \sum_{\substack{|p|=d \\ p \leq r}} a_p \left(\prod_{i=1}^n \left(\frac{r_i}{d+k} \right)^{p_i} - \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \right) \\
&\text{by simplifying } \prod_{j=0}^{p_i-1} \frac{r_i}{d+k} \\
&= f\left(\frac{r}{d+k}\right) - \sum_{\substack{|p|=d \\ p \leq r}} a_p \left(\prod_{i=1}^n \left(\frac{r_i}{d+k} \right)^{p_i} - \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \right) \\
&\text{since } f = \sum_{|p|=d} a_p x^p \text{ and } \left(\frac{r}{d+k} \right) = \left(\frac{r_1}{d+k}, \dots, \frac{r_n}{d+k} \right) \\
&= f\left(\frac{r}{d+k}\right) - \sum_{\substack{|p|=d \\ p \leq r}} \frac{d!}{p_1! \cdots p_n!} \frac{p_1! \cdots p_n!}{d!} a_p \left(\prod_{i=1}^n \left(\frac{r_i}{d+k} \right)^{p_i} - \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \right) \\
&\geq L - U \sum_{\substack{|p|=d \\ p \leq r}} \frac{d!}{p_1! \cdots p_n!} \left(\prod_{i=1}^n \left(\frac{r_i}{d+k} \right)^{p_i} - \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \right) \\
&\text{since } \sum_{i=1}^n \frac{r_i}{d+k} = 1 \text{ and } \frac{r_i}{d+k} \geq 0 \text{ so } \frac{r}{d+k} \in \Delta_n \\
&\geq L - U \left(\sum_{\substack{|p|=d \\ p \leq r}} \frac{d!}{p_1! \cdots p_n!} \prod_{i=1}^n \left(\frac{r_i}{d+k} \right)^{p_i} - \sum_{\substack{|p|=d \\ p \leq r}} \frac{d!}{p_1! \cdots p_n!} \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \right) \\
&= L - U \left(\left(\frac{r_1}{d+k} + \cdots + \frac{r_n}{d+k} \right)^d - \sum_{\substack{|p|=d \\ p \leq r}} \frac{d!}{p_1! \cdots p_n!} \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \right)
\end{aligned}$$

by the multinomial theorem

$$\begin{aligned}
&= L-U \left(1 - \sum_{\substack{|p|=d \\ p \leq r}} \frac{d!}{p_1! \cdots p_n!} \prod_{i=1}^n \prod_{j=0}^{p_i-1} \frac{r_i - j}{d+k} \right) \text{ since } |r| = d+k \\
&= L-U \left(1 - \sum_{\substack{|p|=d \\ p \leq r}} \frac{d!}{p_1! \cdots p_n!} \prod_{i=1}^n \prod_{j=0}^{p_i-1} \left(\frac{r_i}{d+k} - jt \right) \right) \text{ where } t = \frac{1}{d+k} \\
&= L-U \left(1 - \sum_{\substack{|p|=d \\ p \leq r}} \frac{d!}{p_1! \cdots p_n!} \prod_{i=1}^n \left(\frac{r_i}{d+k} \right)_t^{p_i} \right) \text{ where } t = \frac{1}{d+k} \\
&= L-U \left(1 - \prod_{j=0}^{d-1} \left(1 - \frac{j}{k+d} \right) \right) \text{ by the iterated Chu-Vandermonde identity [25] [3].}
\end{aligned}$$

Note that if $w_j = \frac{j}{d+k}$, then $0 \leq w_j < 1$ and $0 < 1 - w_j \leq 1$ for all j . We will prove by induction that

$$\prod_{j=0}^{d-1} (1 - w_j) \geq 1 - \sum_{i=0}^{d-1} w_i.$$

The base case is clear. Assume

$$\begin{aligned}
\prod_{j=0}^{k-1} (1 - w_j) &\geq 1 - \sum_{i=0}^{k-1} w_i \quad \text{for } k < d. \\
\prod_{j=0}^k (1 - w_j) &= (1 - w_k) \prod_{j=0}^{k-1} (1 - w_j) \\
&\geq (1 - w_k) \left(1 - \sum_{i=0}^{k-1} w_i \right) \\
&= 1 + w_k \sum_{i=0}^{k-1} w_i - \sum_{i=0}^{k-1} w_i - w_k \\
&= 1 + w_k \sum_{i=0}^{k-1} w_i - \sum_{i=0}^k w_i \\
&\geq 1 - \sum_{i=0}^k w_i.
\end{aligned}$$

Hence

$$\prod_{j=0}^{d-1} (1 - w_j) \geq 1 - \sum_{i=0}^{d-1} w_j.$$

Then

$$\begin{aligned} 1 - \prod_{j=0}^{d-1} (1 - w_j) &\leq 1 - \left(1 - \sum_{i=0}^{d-1} w_j\right) \\ 1 - \prod_{j=0}^{d-1} (1 - w_j) &\leq \sum_{i=0}^{d-1} w_j \\ L - U \left(1 - \prod_{j=0}^{d-1} (1 - w_j)\right) &\geq L - U \sum_{i=0}^{d-1} w_j \\ &\geq L - U \frac{1 + \dots + d - 1}{d + k} \\ &= L - U \frac{d(d-1)}{d + k}. \end{aligned}$$

Hence $u_r \geq 0$ if

$$\begin{aligned} \frac{L}{U} &\geq \frac{d(d-1)}{2(d+k)} \\ k &\geq \frac{d(d-1)U}{2L} - d. \end{aligned}$$

□

2.5 Specializing the Powers-Reznick Degree Bound to b_{PR} , a Univariate Degree Bound

We can specialize b_{PR} from Theorem 2.4.3 by considering the two variable case and then setting $x_2 = 1$. The witness from Theorem 2.4.2 becomes $(x + 1)^k$.

Example 2.5.1. Let $f = x^2 - 2x + 4$. Note

$$\begin{aligned} (x + 1) f &= x^3 - x^2 + 2x + 4 \\ (x + 1)^2 f &= x^4 + x^2 + 6x + 4. \end{aligned}$$

Then the k from Theorem 2.4.2 is $k = 2$. Since $(x + 1)^2$ and $x^4 + x^2 + 6x + 4$ are clearly positive for $x > 0$, we must also have that $f > 0$ for $x > 0$.

Now let's specialize b_{PR} to the univariate case. This corresponds to the case for Descartes' Rule of Signs when $p(f) = 0$.

Theorem 2.5.1 (Powers-Reznick 2001 [25]). *If $f \in \mathbb{R}[x]$ has no positive real roots and*

$$k \geq \left[\binom{n}{2} \frac{\max_{0 \leq i \leq n} \{|a_i|/\binom{n}{i}\}}{\min_{x \in [0,1]} \left\{ (1-x)^n f\left(\frac{x}{1-x}\right) \right\}} \right] - n = b_{PR}(f),$$

where $n = \deg(f)$, then $(x+1)^k f(x)$ has all its coefficients of the same sign.

Proof. We specialize Theorem 2.4.3 to the case where $f \in \mathbb{R}[x_1, x_2]$. Let

$$\hat{f}(x_1, x_2) = \sum_{p_1+p_2=n} \hat{a}_{p_1 p_2} x_1^{p_1} x_2^{p_2} \text{ be such that}$$

$$\forall_{(x_1, x_2) \in \Delta_2} \hat{f}(x_1, x_2) > 0$$

where

$$\Delta_2 = \{(x_1, x_2) : x_1, x_2 \geq 0, x_1 + x_2 = 1\}.$$

Then all the coefficients of

$$(x_1 + x_2)^{[k]} \hat{f}(x_1, x_2)$$

are positive if

$$k \geq \frac{n(n-1)U}{2L} - n$$

where

$$U = \max_{p_1+p_2=n} \frac{p_1! p_2!}{n!} |\hat{a}_{p_1 p_2}|$$

$$L = \min_{(x_1, x_2) \in \Delta_2} \hat{f}(x_1, x_2).$$

1. We will rewrite parts of the above.

(a) Note

$$\begin{aligned}
& \forall_{(x_1, x_2) \in \Delta_2} \hat{f}(x_1, x_2) > 0 \\
& \iff \forall_{0 \leq x_1 \leq 1} \hat{f}(x_1, 1-x_1) > 0 \\
& \iff \forall_{0 \leq x_1 \leq 1} \sum_{p_1+p_2=n} \hat{a}_{p_1 p_2} x_1^{p_1} (1-x_1)^{p_2} > 0 \\
& \iff \forall_{0 \leq x_1 \leq 1} \sum_{i=0}^n \hat{a}_{i, n-i} x_1^i (1-x_1)^{n-i} > 0 \\
& \iff \forall_{0 \leq x_1 \leq 1} (1-x_1)^n \sum_{i=0}^n \hat{a}_{i, n-i} \left(\frac{x_1}{1-x_1} \right)^i > 0 \\
& \iff \forall_{0 \leq x_1 \leq 1} (1-x_1)^n f\left(\frac{x_1}{1-x_1} \right) > 0 \quad \text{where } f(x) = \sum_{i=0}^n a_i x^i \text{ and } a_i = \hat{a}_{i, n-i} \\
& \iff \forall_{0 \leq x} f(x) > 0.
\end{aligned}$$

(b) Note

$$\begin{aligned}
& \text{all coefficients of } (x_1 + x_2)^{[k]} \hat{f}(x_1, x_2) \text{ are positive} \\
& \iff \text{all coefficients of } (x_1 + x_2)^{[k]} \sum_{p_1+p_2=n} \hat{a}_{p_1 p_2} x_1^{p_1} x_2^{p_2} \text{ are positive} \\
& \iff \text{all coefficients of } (x_1 + x_2)^{[k]} \sum_{i=0}^n \hat{a}_{i, n-i} x_1^i x_2^{n-i} \text{ are positive} \\
& \iff \text{all coefficients of } x_2^{[k]} \left(\frac{x_1}{x_2} + 1 \right)^{[k]} x_2^n \sum_{i=0}^n \hat{a}_{i, n-i} \left(\frac{x_1}{x_2} \right)^i \text{ are positive} \\
& \iff \text{all coefficients of } x_2^{[k]} \left(\frac{x_1}{x_2} + 1 \right)^{[k]} x_2^n \sum_{i=0}^n a_i \left(\frac{x_1}{x_2} \right)^i \text{ are positive} \\
& \iff \text{all coefficients of } x_2^{[k]+n} \left(\frac{x_1}{x_2} + 1 \right)^k f\left(\frac{x_1}{x_2} \right) \text{ are positive} \\
& \iff \text{all coefficients of } \left(\frac{x_1}{x_2} + 1 \right)^{[k]} f\left(\frac{x_1}{x_2} \right) \text{ with respect to } \frac{x_1}{x_2} \text{ are positive} \\
& \iff \text{all coefficients of } (x+1)^{[k]} f(x) \text{ are positive.}
\end{aligned}$$

(c) Note

$$\begin{aligned}
 U &= \max_{p_1+p_2=n} \frac{p_1!p_2!}{n!} |\hat{a}_{p_1,p_2}| = \max_{0 \leq i \leq n} \frac{i!(n-i)!}{n!} |\hat{a}_{i,n-i}| \\
 &= \max_{0 \leq i \leq n} \frac{i!(n-i)!}{n!} |a_i| = \max_{0 \leq i \leq n} \frac{|a_i|}{\binom{n}{i}}.
 \end{aligned}$$

(d) Note

$$\begin{aligned}
 L &= \min_{(x_1,x_2) \in \Delta_2} \hat{f}(x_1, x_2) \\
 &= \min_{(x_1,x_2) \in \Delta_2} \sum_{p_1+p_2=n} \hat{a}_{p_1,p_2} x_1^{p_1} x_2^{p_2} \\
 &= \min_{0 \leq x \leq 1} \sum_{i=0}^n \hat{a}_{i,n-i} x^i (1-x)^{n-i} \\
 &= \min_{0 \leq x \leq 1} (1-x)^n \sum_{i=0}^n \hat{a}_{i,n-i} \left(\frac{x}{1-x}\right)^i \\
 &= \min_{0 \leq x \leq 1} (1-x)^n \sum_{i=0}^n a_i \left(\frac{x}{1-x}\right)^i \\
 &= \min_{0 \leq x \leq 1} (1-x)^n f\left(\frac{x}{1-x}\right).
 \end{aligned}$$

2. Combining the above two, we obtain the following.

Let $f(x) = \sum_{i=0}^n a_i x^i$ be such that

$$\forall_{0 \leq x} f(x) > 0.$$

Then all coefficients of $(x+1)^{[k]} f(x)$ are positive if

$$k \geq \frac{n(n-1)U}{2} \frac{U}{L} - n$$

where

$$\begin{aligned}
 U &= \max_{0 \leq i \leq n} \frac{|a_i|}{\binom{n}{i}} \\
 L &= \min_{0 \leq x \leq 1} (1-x)^n f\left(\frac{x}{1-x}\right).
 \end{aligned}$$

3. Finally, note

$$\begin{aligned}
 [k] &= \left\lceil \frac{n(n-1)U}{2L} - n \right\rceil \\
 &= \left\lceil \binom{n}{2} \frac{\max_{0 \leq i \leq n} \{|a_i|/\binom{n}{i}\}}{\min_{x \in [0,1]} \left\{ (1-x)^n f\left(\frac{x}{1-x}\right) \right\}} \right\rceil - n.
 \end{aligned}$$

□

Example 2.5.2. Let $f = x^2 - 2x + 4$. Then

$$\begin{aligned}
 \max \left\{ 1, \frac{|a_1|}{2}, |a_0| \right\} &= \max \left\{ 1, \frac{|-2|}{2}, |4| \right\} \\
 &= 4 \\
 \min_{x \in [0,1]} (1-x)^2 f\left(\frac{x}{1-x}\right) &= \min_{x \in [0,1]} (1-x)^2 \left[\left(\frac{x}{1-x}\right)^2 - 2\left(\frac{x}{1-x}\right) + 4 \right] \\
 &= \min_{x \in [0,1]} [x^2 - 2x(1-x) + 4(1-x)^2] \\
 &= \min_{x \in [0,1]} (7x^2 - 10x + 4) \\
 &= \frac{3}{7}.
 \end{aligned}$$

Thus,

$$b_{PR}(f) = \left\lceil \frac{4}{3/7} \right\rceil - 2 = \left\lceil \frac{28}{3} \right\rceil - 2 = 8.$$

Hence, $(x+1)^8 f$ will have all non-negative coefficients. However, we saw in Example 2.5.1 that $(x+1)^2 f$ was sufficient.

Chapter 3 will provide a lower bound than the one provided in Theorem 2.5.1 on an exactifying multiplier for Descartes' Rule of Signs for the case when $p(f) = 0$.

CHAPTER

3

A NEW DEGREE BOUND FOR POLYNOMIALS WITH NO POSITIVE REAL ROOTS (B_{HR})

In Sections 2.1 and 2.3, we discussed the existence of exactifying multipliers for Descartes' Rule of Signs and a construction of such a multiplier given by Avendaño. In Section 2.5, we discussed a witness for the multiplier given by Pólya, $g = (x + 1)^k$, for the foundational case of polynomials with no positive real roots and the bound, $b_{PR}(f)$, on k provided by Powers and Reznick. In this Chapter, we will provide a new bound on the degree of an exactifying multiplier for Descartes' Rule of Signs for polynomials with no positive real root, give an explicit witness for the certificate when $\deg(f) = 2$, and prove its optimality when $\deg(f) = 2$.

3.1 Main Results

In this section, we will state the main results relating to exactifying multipliers for Descartes' Rule of Signs when the polynomial has no positive real roots.

Theorem 3.1.1 (Bound). *Let $f \in \mathbb{R}[x]$ be monic without positive real roots. Then there exists monic polynomial $g \in \mathbb{R}[x]$ of degree at most $b_{HR}(f)$ such that $\text{coeffs}(gf) \geq 0$, where*

$$b_{HR}(f) = \sum_{i=1}^m \left(\left\lceil \frac{\pi}{\arg(\alpha_i)} \right\rceil - 2 \right)$$

and $\alpha_1, \dots, \alpha_m$ are all the non-real roots of f with positive imaginary part (multiple roots are repeated). (Note that $b_{HR}(f) = 0$ if all roots of f are real.)

In the case of quadratic polynomials with no positive real roots, we can prove the optimality of this bound and provide a witness for the certificate of that exact degree.

Theorem 3.1.2 (Optimality for Degree 2). *Let $f = x^2 + a_1x + a_0 \in \mathbb{R}[x]$ be without real roots. Then the smallest degree of non-zero $g \in \mathbb{R}[x]$ such that $\text{coeffs}(gf) \geq 0$ is*

$$b_{HR}(f) = \left\lceil \frac{\pi}{\arg(\alpha)} \right\rceil - 2$$

where α is the non-real root of f with positive imaginary part.

Remark 3.1.1. Note that $b_{HR}(f) = 0$ when f has two non-positive real roots and is, hence, clearly optimal.

Theorem 3.1.3 (Multiplier for Degree 2). *Let $f = x^2 + a_1x + a_0 \in \mathbb{R}[x]$ be without real roots. Then a witness for $\text{coeffs}(gf) \geq 0$ is given by*

$$g = \begin{vmatrix} a_2 & & & x^0 \\ a_1 & \ddots & & \vdots \\ a_0 & & \ddots & \vdots \\ & \ddots & & \ddots & \vdots \\ & & a_0 & a_1 & x^s \end{vmatrix}$$

where

$$s = b_{HR}(f) = \left\lceil \frac{\pi}{\arg(\alpha)} \right\rceil - 2$$

where again α is the non-real root of f with positive imaginary part.

Conjecture 3.1.1 (Angle-Based Optimality). *We have*

$$\forall_{\theta \in (0, \pi]^k} \exists_{r \in [0, \infty)^k} \text{opt}(f) = b_{HR}(f)$$

where

$$f_{\theta, r} = \prod_{\substack{1 \leq i \leq k \\ \theta_i = \pi}} (x + r_i) \prod_{\substack{1 \leq i \leq k \\ \theta_i \neq \pi}} (x^2 - 2r_i \cos \theta_i + r_i^2).$$

3.2 Reducing the Problem via Linear Algebra

We will provide a series of lemmas to reformat the problem. Let $f \in \mathbb{R}[x]$ be monic without losing generality. Assume that f does not have any positive real roots. The problem is to find a non-zero $g \in \mathbb{R}[x]$ such that $\text{coeffs}(gf) \geq 0$. We will reduce the problem to that of linear algebra. Let

$$\begin{aligned} f &= a_n x^n + \dots + a_0 x^0 \\ g &= b_s x^s + \dots + b_0 x^0 \end{aligned}$$

where $a_n = 1$ and $b_s = 1$. We first rewrite them using vectors. Let

$$a = \begin{bmatrix} a_0 & \dots & a_n \end{bmatrix} \quad b = \begin{bmatrix} b_0 & \dots & b_s \end{bmatrix}$$

and let

$$x_k = \begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix}.$$

Then we can write f and g compactly as

$$f = a x_n, \quad g = b x_s.$$

Let

$$A_s = \begin{bmatrix} a_0 & \dots & \dots & a_n \\ & \ddots & & \ddots \\ & & a_0 & \dots & \dots & a_n \end{bmatrix} \in \mathbb{R}^{(s+1) \times (s+n+1)}.$$

Lemma 3.2.1. $\text{coeffs}(gf) = bA_s$

Proof. Note

$$\begin{aligned}
 gf &= (bx_s)(ax_n) \\
 &= b(x_s a x_n) \\
 &= b \begin{bmatrix} x^0 \\ \vdots \\ x^s \end{bmatrix} \begin{bmatrix} a_0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x^0 \\ \vdots \\ x^n \end{bmatrix} \\
 &= b \begin{bmatrix} a_0 & \cdots & \cdots & a_n & & \\ & \ddots & & & \ddots & \\ & & a_0 & \cdots & \cdots & a_n \end{bmatrix} \begin{bmatrix} x^0 \\ \vdots \\ x^{s+n} \end{bmatrix} \\
 &= bA_s x_{s+n}.
 \end{aligned}$$

Hence $\text{coeffs}(gf) = bA_s$. □

We partition A_s into two submatrices as $A_s = [L_s | R_s]$ where

$$L_s = \begin{bmatrix} a_0 & \cdots & a_{n-1} \\ & \ddots & \vdots \\ & & a_0 \end{bmatrix} \in \mathbb{R}^{(s+1) \times n} \quad \text{and} \quad R_s = \begin{bmatrix} a_n & & & & \\ \vdots & \ddots & & & \\ \vdots & & \ddots & & \\ a_0 & & & \ddots & \\ \vdots & \ddots & & & \ddots \\ & & a_0 & \cdots & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{(s+1) \times (s+1)}.$$

Let

$$\begin{aligned}
 c &= bR_s \in \mathbb{R}^{1 \times (s+1)} \\
 T_s &= R_s^{-1}L_s \in \mathbb{R}^{(s+1) \times n}.
 \end{aligned}$$

The rows of T_s are indexed from 0 to s , while the columns are indexed from 0 to $n-1$.

Lemma 3.2.2. $\text{coeffs}(gf) = c[T_s|I]$

Proof. Note

$$\begin{aligned}
 bA_s &= b(R_s R_s^{-1})A_s \\
 &= (bR_s)(R_s^{-1}A_s) \\
 &= (bR_s)(R_s^{-1}[L_s|R_s]) \\
 &= (bR_s)[R_s^{-1}L_s|R_s^{-1}R_s] \\
 &= (bR_s)[R_s^{-1}L_s|I] \\
 &= c[T_s|I] \quad \text{where } c = bR_s \text{ and } T_s = R_s L_s.
 \end{aligned}$$

□

We will present an example here to show the various matrices associated with f and g .

Example 3.2.1. Let $f = x^2 - 2x + 4$ and $g = x^3 + 4x^2 + 14x + 20$. Then

$$\begin{aligned}
 a &= \begin{bmatrix} 4 & -2 & 1 \end{bmatrix} \\
 b &= \begin{bmatrix} 20 & 14 & 4 & 1 \end{bmatrix} \\
 A_3 &= \begin{bmatrix} 4 & -2 & 1 & 0 & 0 & 0 \\ 0 & 4 & -2 & 1 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 & 0 \\ 0 & 0 & 0 & 4 & -2 & 1 \end{bmatrix} \\
 L_3 &= \begin{bmatrix} 4 & -2 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\
 R_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 \\ 0 & 4 & -2 & 1 \end{bmatrix} \\
 c = bR_3 &= \begin{bmatrix} 20 & 14 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 \\ 0 & 4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 2 & 1 \end{bmatrix}
 \end{aligned}$$

$$T_3 = R_3^{-1} L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 \\ 0 & 4 & -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 & -2 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & 0 \\ 0 & 8 \\ -32 & 16 \end{bmatrix}.$$

Lemma 3.2.3. *We have*

$$\exists_{g \neq 0, \deg(g) \leq s} \text{coeffs}(gf) \geq 0 \iff \text{ConvexHull}(T_s) \cap \mathbb{R}_{\geq 0}^n \neq \emptyset$$

where T_s is viewed as a set of row vectors.

Proof. Note

$$\begin{aligned} & \exists_{g \neq 0, \deg(g) \leq s} \text{coeffs}(gf) \geq 0 \\ \iff & \exists_{c \neq 0} c [T_s | I] \geq 0 \quad (\text{from Lemma 3.2.2}) \\ \iff & \exists_{c \neq 0} c T_s \geq 0 \text{ and } c \geq 0 \\ \iff & \exists_{c \geq 0, c \neq 0} c T_s \geq 0 \\ \iff & \exists_{c \geq 0, c_0 + \dots + c_s = 1} c T_s \geq 0 \\ \iff & \text{ConvexHull}(T_s) \cap \mathbb{R}_{\geq 0}^n \neq \emptyset. \end{aligned}$$

□

Example 3.2.2. *Let $f = x^2 - 2x + 4$ and $g = x^3 + 4x^2 + 14x + 20$. Since*

$$\begin{aligned} gf &= (x^3 + 4x^2 + 14x + 20)(x^2 - 2x + 4) \\ &= x^5 + 2x^4 + 10x^3 + 8x^2 + 16x + 80, \end{aligned}$$

Lemma 3.2.3 states that

$$\text{ConvexHull}(T_3) \cap \mathbb{R}_{\geq 0}^2 \neq \emptyset$$

where T_3 is from Example 3.2.1. Hence, there must exist $\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \geq 0$ with $c_1 + c_2 + c_3 + c_4 = 1$ such that

$$c_1 \begin{bmatrix} 4 & -2 \end{bmatrix} + c_2 \begin{bmatrix} 8 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 8 \end{bmatrix} + c_4 \begin{bmatrix} -32 & 16 \end{bmatrix} \geq 0.$$

Then

$$\begin{bmatrix} 4c_1 + 8c_2 - 32c_4 & -2c_1 + 8c_3 + 16c_4 \end{bmatrix} \geq 0.$$

Note, for example, that $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$ will do, since

$$\begin{aligned} & 0 \begin{bmatrix} 4 & -2 \end{bmatrix} + 1 \begin{bmatrix} 8 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 8 \end{bmatrix} + 0 \begin{bmatrix} -32 & 16 \end{bmatrix} \\ & = \begin{bmatrix} 8 & 0 \end{bmatrix}. \end{aligned}$$

Let

$$f = a_n \prod_{i=1}^n (x - \alpha_i).$$

Lemma 3.2.4. *The entries of T_s are given by*

$$T_{skl} = -\frac{|N|}{|D|}$$

where

$$D = \begin{bmatrix} \alpha_1^0 & \cdots & \alpha_n^0 \\ \vdots & & \vdots \\ \alpha_1^{n-1} & \cdots & \alpha_n^{n-1} \end{bmatrix}$$

and N is obtained from D by replacing the l -th row with $\begin{bmatrix} \alpha_1^{k+n} & \cdots & \alpha_n^{k+n} \end{bmatrix}$.

Proof. Note

$$\begin{aligned} x_s f &= A_s x_{s+n} \\ &= R_s R_s^{-1} A_s x_{s+n} \\ &= R_s R_s^{-1} [L_s | R_s] x_{s+n} \\ &= R_s [R_s^{-1} L_s | R_s^{-1} R_s] x_{s+n} \\ &= R_s [T_s | I] x_{s+n} \\ &= R_s \left(T_s \begin{bmatrix} x^0 \\ \vdots \\ x^{n-1} \end{bmatrix} + \begin{bmatrix} x^n \\ \vdots \\ x^{s+n} \end{bmatrix} \right). \end{aligned}$$

By evaluating the above on each root, we have

$$\begin{bmatrix} \alpha_i^0 \\ \vdots \\ \alpha_i^s \end{bmatrix} f(\alpha_i) = R_s \left(T_s \begin{bmatrix} \alpha_i^0 \\ \vdots \\ \alpha_i^{n-1} \end{bmatrix} + \begin{bmatrix} \alpha_i^n \\ \vdots \\ \alpha_i^{s+n} \end{bmatrix} \right).$$

Since $f(\alpha_i) = 0$, we have

$$0 = R_s \left(T_s \begin{bmatrix} \alpha_i^0 \\ \vdots \\ \alpha_i^{n-1} \end{bmatrix} + \begin{bmatrix} \alpha_i^n \\ \vdots \\ \alpha_i^{s+n} \end{bmatrix} \right).$$

Since R_s is an invertible matrix, we have

$$0 = T_s \begin{bmatrix} \alpha_i^0 \\ \alpha_i^1 \\ \vdots \\ \alpha_i^{n-1} \end{bmatrix} + \begin{bmatrix} \alpha_i^n \\ \vdots \\ \alpha_i^{s+n} \end{bmatrix}.$$

Rearranging,

$$T_s \begin{bmatrix} \alpha_i^0 \\ \alpha_i^1 \\ \vdots \\ \alpha_i^{n-1} \end{bmatrix} = - \begin{bmatrix} \alpha_i^n \\ \vdots \\ \alpha_i^{s+n} \end{bmatrix}.$$

Combining the above equations for all the roots, we have

$$T_s \begin{bmatrix} \alpha_1^0 & \cdots & \alpha_n^0 \\ \vdots & & \vdots \\ \alpha_1^{n-1} & \cdots & \alpha_n^{n-1} \end{bmatrix} = - \begin{bmatrix} \alpha_1^n & \cdots & \alpha_n^n \\ \vdots & & \vdots \\ \alpha_1^{s+n} & \cdots & \alpha_n^{s+n} \end{bmatrix}.$$

By applying Cramer's rule, we have

$$T_{skl} = - \frac{|N|}{|D|}$$

where

$$D = \begin{bmatrix} \alpha_1^0 & \cdots & \alpha_n^0 \\ \vdots & & \vdots \\ \alpha_1^{n-1} & \cdots & \alpha_n^{n-1} \end{bmatrix}$$

and N is obtained from D by replacing the l -th row with $[\alpha_1^{k+n} \cdots \alpha_n^{k+n}]$. □

Remark 3.2.1. Note that T_{skl} does not depend on s . Thus we will often write it as T_{kl} .

Example 3.2.3. Let $f = x^2 - 2x + 4$. Note that the roots of f are $\alpha_1 = 1 + i\sqrt{3}$ and $\alpha_2 = 1 - i\sqrt{3}$. Note

$$\begin{aligned} T_{10} &= -\frac{\begin{vmatrix} \alpha_1^3 & \alpha_2^3 \\ \alpha_1^1 & \alpha_2^1 \end{vmatrix}}{\begin{vmatrix} \alpha_1^0 & \alpha_2^0 \\ \alpha_1^1 & \alpha_2^1 \end{vmatrix}} = -\frac{\alpha_1^3 \alpha_2^1 - \alpha_1^1 \alpha_2^3}{\alpha_2 - \alpha_1} = -\alpha_1 \alpha_2 \left(\frac{\alpha_1^2 - \alpha_2^2}{\alpha_2 - \alpha_1} \right) = \alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \\ &= (1 + i\sqrt{3})(1 - i\sqrt{3})(1 - i\sqrt{3} + 1 + i\sqrt{3}) = 8. \end{aligned}$$

Note, in particular, that this matches T_{10} from Example 3.2.1.

3.3 Proof of Bound b_{HR} (Theorem 3.1.1)

Lemma 3.3.1. Let $f \in \mathbb{R}[x]$ be such that $\deg(f) = 2$ without real roots. Let the roots be $\alpha = r e^{i\theta}$ and $\bar{\alpha} = r e^{-i\theta}$. Then we have

$$\begin{aligned} T_{k0} &= +r^{k+2} \frac{\sin(k+1)\theta}{\sin \theta} = r^2 \frac{\operatorname{Im} \alpha^{k+1}}{\operatorname{Im} \alpha} \\ T_{k1} &= -r^{k+1} \frac{\sin(k+2)\theta}{\sin \theta} = -\frac{\operatorname{Im} \alpha^{k+2}}{\operatorname{Im} \alpha}. \end{aligned}$$

Proof. From Lemma 3.2.4 we have

$$\begin{aligned} T_{k0} &= -\frac{\begin{vmatrix} \alpha^{k+2} & \bar{\alpha}^{k+2} \\ \alpha^1 & \bar{\alpha}^1 \end{vmatrix}}{\begin{vmatrix} \alpha^0 & \bar{\alpha}^0 \\ \alpha^1 & \bar{\alpha}^1 \end{vmatrix}} = -\frac{\alpha^{k+2} \bar{\alpha} - \alpha \bar{\alpha}^{k+2}}{\bar{\alpha} - \alpha} = -r^{k+2} \frac{+2i \sin(k+1)\theta}{-2i \sin \theta} \\ &= +r^{k+2} \frac{\sin(k+1)\theta}{\sin \theta} = r^2 \frac{\operatorname{Im} \alpha^{k+1}}{\operatorname{Im} \alpha} \\ T_{k1} &= -\frac{\begin{vmatrix} \alpha^0 & \bar{\alpha}^0 \\ \alpha^{k+2} & \bar{\alpha}^{k+2} \end{vmatrix}}{\begin{vmatrix} \alpha^0 & \bar{\alpha}^0 \\ \alpha^1 & \bar{\alpha}^1 \end{vmatrix}} = -\frac{\bar{\alpha}^{k+2} - \alpha^{k+2}}{\bar{\alpha} - \alpha} = -r^{k+1} \frac{-2i \sin(k+2)\theta}{-2i \sin \theta} \\ &= -r^{k+1} \frac{\sin(k+2)\theta}{\sin \theta} = -\frac{\operatorname{Im} \alpha^{k+2}}{\operatorname{Im} \alpha}. \end{aligned}$$

□

Example 3.3.1. Let $f = x^2 - 2x + 4$. Note that the roots of f are $\alpha = 1 + i\sqrt{3} = 2e^{i\frac{\pi}{3}}$ and $\bar{\alpha} = 1 - i\sqrt{3} = 2e^{-i\frac{\pi}{3}}$.

$$T_{10} = (2)^2 \frac{\operatorname{Im}(1 + i\sqrt{3})^{1+1}}{\operatorname{Im}(1 + i\sqrt{3})} = 4 \frac{\operatorname{Im}(-2 + 2i\sqrt{3})}{\operatorname{Im}(1 + i\sqrt{3})} = 4 \frac{2\sqrt{3}}{\sqrt{3}} = 8$$

Note again that this matches Example 3.2.1.

Lemma 3.3.2. Let $f \in \mathbb{R}[x]$ be such that $\deg(f) = 2$ without real roots. Let $\alpha, \bar{\alpha}$ be the roots of f .

$$s = b_{HR}(f) = \left\lceil \frac{\pi}{\arg(\alpha)} \right\rceil - 2.$$

Then

$$\exists_{g \neq 0, \deg(g) \leq s} \operatorname{coeffs}(gf) \geq 0.$$

Proof. Note

$$\begin{aligned} & \exists_{g \neq 0, \deg(g) \leq s} \operatorname{coeffs}(gf) \geq 0 \\ \iff & \operatorname{ConvexHull}(T_s) \cap \mathbb{R}_{\geq 0}^n \neq \emptyset \quad (\text{from Lemma 3.2.3}) \\ \iff & T_{s0}, T_{s1} \geq 0 \\ \iff & \sin(s+1)\theta \geq 0 \wedge \sin(s+2)\theta \leq 0 \quad (\text{from Lemma 3.3.1}) \\ \iff & 0 < (s+1)\theta \leq \pi \wedge \pi \leq (s+2)\theta < 2\pi \\ \iff & s \leq \frac{\pi}{\theta} - 1 \wedge s \geq \frac{\pi}{\theta} - 2 \\ \iff & \frac{\pi}{\theta} - 2 \leq s \leq \frac{\pi}{\theta} - 1 \\ \iff & s = \left\lceil \frac{\pi}{\theta} \right\rceil - 2. \end{aligned}$$

□

Example 3.3.2. Let $f = x^2 - 2x + 4$. Note that the roots of f are $\alpha = 2e^{i\frac{\pi}{3}}$ and $\bar{\alpha} = 2e^{-i\frac{\pi}{3}}$.

$$s = b_{HR}(f) = \left\lceil \frac{\pi}{\pi/3} \right\rceil - 2 = [3] - 2 = 1$$

According to the proof for Lemma 3.3.2, T_{10} and T_{11} must be non-negative. This can be verified in Example 3.2.1.

In addition, we know that

$$\operatorname{ConvexHull}(T_1) \cap \mathbb{R}_{\geq 0}^2 \neq \emptyset.$$

In fact, we can see that

$$0 \begin{bmatrix} 4 & -2 \end{bmatrix} + 1 \begin{bmatrix} 8 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 0 \end{bmatrix} \in \text{ConvexHull}(T_1) \cap \mathbb{R}_{\geq 0}^2.$$

Note that this is a truncation of the vector provided in Example 3.2.2. Since c_3 and c_4 were 0, they were unnecessary.

Finally, this means that a g with degree 1 exists. Let $g = x + c$. Let's find g :

$$\begin{aligned} gf &= (x + c)(x^2 - 2x + 4) \\ &= x^3 + (c - 2)x^2 + (4 - 2c)x + 4c \end{aligned}$$

Then we have $c - 2 \geq 0$, $4 - 2c \geq 0$ and $4c \geq 0$. Hence $c = 2$ and $g = x + 2$.

Proof of Theorem 3.1.1. Let

$$f = (x - r_1) \cdots (x - r_t) f_1 \cdots f_m$$

where r_1, \dots, r_t are non-positive real roots and f_1, \dots, f_m are the irreducible quadratic factors of f over \mathbb{R} . Let α_i be the non-real root of f_i with positive imaginary part. From Lemma 3.3.2, there exists non-zero $g_i \in \mathbb{R}[x]$ such that $\text{coeffs}(g_i f_i) \geq 0$ and $\deg(g_i) \leq \left\lceil \frac{\pi}{\arg(\alpha_i)} \right\rceil - 2$. Let $g = g_1 \cdots g_m$.

Then $\text{coeffs}(gf) \geq 0$ and $\deg(g) \leq \sum_{i=1}^m \left(\left\lceil \frac{\pi}{\arg(\alpha_i)} \right\rceil - 2 \right) = b_{HR}(f)$. \square

3.4 Proof of Optimality of b_{HR} for Degree 2 (Theorem 3.1.2)

Proof of Theorem 3.1.2. Let $s = b_{HR}(f)$. When $\frac{\pi}{2} \leq \theta < \pi$, it is obvious since $s = 0$ and f will have all non-negative coefficients already. Thus assume that $0 < \theta < \frac{\pi}{2}$. Let g be such that $g \neq 0$ and $\deg(g) = t < s$.

$$\begin{aligned} & \forall_{0 \leq t < s} 0 \leq t < \left\lceil \frac{\pi}{\theta} \right\rceil - 2 \\ \iff & \forall_{0 \leq t < s} 0 \leq t < \frac{\pi}{\theta} - 2 \quad (\text{since } t \in \mathbb{Z}) \\ \iff & \forall_{0 \leq t < s} \sin(t + 2)\theta > 0 \\ \iff & \forall_{0 \leq t < s} T_{t1} < 0 \\ \iff & \forall_{0 \leq t < s} \text{ConvexHull}(T_t) \cap \mathbb{R}_{\geq 0}^2 = \emptyset \\ \iff & \nexists_{\substack{g \neq 0 \\ \deg(g) < s}} \text{coeffs}(gf) \geq 0. \end{aligned}$$

Hence, there can be no $g \in \mathbb{R}[x]$ with $\deg(g) < s$ such that $\text{coeffs}(gf) \geq 0$. □

3.5 Proof of Certificate for b_{HR} for Degree 2 (Theorem 3.1.3)

Proof of Theorem 3.1.3. From Lemma 3.3.1, we have $T_{s0}, T_{s1} \geq 0$. Thus it suffices to choose g such that $c = [0 \ \cdots \ 0 \ 1]$ since

$$\text{coeffs}(gf) = c [T_s | I] = [T_{s0} \ T_{s1} \ 0 \ \cdots \ 0 \ 1] \geq 0.$$

Now let us find an explicit expression for g .

$$\begin{aligned} g &= c R_s^{-1} x_s \quad \text{since } c = b R_s \\ &= [0 \ \cdots \ 0 \ 1] R_s^{-1} x_s \\ &= (R_s^{-1} x_s)_s \\ &= \frac{\begin{vmatrix} a_2 & & & x^0 \\ a_1 & \ddots & & \vdots \\ a_0 & & \ddots & \vdots \\ & \ddots & & \vdots \\ & & a_0 & a_1 & x^s \end{vmatrix}}{\begin{vmatrix} a_2 & & & \\ a_1 & \ddots & & \\ a_0 & & \ddots & \\ & \ddots & & \\ & & a_0 & a_1 & a_2 \end{vmatrix}} && \text{from Cramer's rule} \\ &= \frac{\begin{vmatrix} a_2 & & & x^0 \\ a_1 & \ddots & & \vdots \\ a_0 & & \ddots & \vdots \\ & \ddots & & \vdots \\ & & a_0 & a_1 & x^s \end{vmatrix}}{\begin{vmatrix} a_2 & & & \\ a_1 & \ddots & & \\ a_0 & & \ddots & \\ & \ddots & & \\ & & a_0 & a_1 & a_2 \end{vmatrix}} && \text{since } a_2 = 1 \end{aligned}$$

□

Example 3.5.1. Let $f = x^2 - 2x + 4$. Then $s = b_{HR}(f) = 1$ from Example 3.3.2. By Theorem 3.1.3,

$$g = \begin{vmatrix} a_2 & x^0 \\ a_1 & x^1 \end{vmatrix} = \begin{vmatrix} 1 & x^0 \\ -2 & x^1 \end{vmatrix} = x + 2.$$

Example 3.5.2. Let $f = x^4 - 4x^3 + 10x^2 - 12x + 8$. Note

$$f = (x^2 - 2x + 2)(x^2 - 2x + 4)$$

$$f_1 = x^2 - 2x + 2$$

$$f_2 = x^2 - 2x + 4.$$

The root of f_1 with positive imaginary part is $\alpha_1 = 1 + i$ and the root of f_2 with positive imaginary part is $\alpha_2 = 1 + i\sqrt{3}$. Then

$$\begin{aligned} \arg(\alpha_1) &= \frac{\pi}{4} \\ \arg(\alpha_2) &= \frac{\pi}{3}. \end{aligned}$$

Hence,

$$s = b_{HR}(f) = \left(\left\lceil \frac{\pi}{\pi/4} \right\rceil - 2 \right) + \left(\left\lceil \frac{\pi}{\pi/3} \right\rceil - 2 \right) = 3.$$

We can find g_1 , corresponding to f_1 , and we have g_2 from Example 3.5.1:

$$g_1 = \begin{vmatrix} 1 & 0 & x^0 \\ -2 & 1 & x^1 \\ 2 & -2 & x^2 \end{vmatrix} = x^2 + 2x + 2$$

Then $g = g_1 g_2 = (x^2 + 2x + 2)(x + 2)$ and we should have $\text{coeffs}(gf) \geq 0$. Indeed,

$$\begin{aligned} gf &= (x^2 + 2x + 2)(x + 2)(x^4 - 4x^3 + 10x^2 - 12x + 8) \\ &= x^7 + 8x^4 + 4x^3 + 32. \end{aligned}$$

3.6 Progress on Proof of Angle-Based Optimality (Conjecture 3.1.1)

Definition 3.6.1. The optimal degree of f , denoted $\text{opt}(f)$, is defined as

$$\text{opt}(f) = \min_{g, \text{coeffs}(gf) \geq 0} \deg(g).$$

Lemma 3.6.1. Let $f' = (x+r)f$. Then $\exists_{r>0} \text{opt}(f') = \text{opt}(f)$.

Proof. Let $f \in \mathbb{R}[x]$. Let $f' = (x+r)f$. We need to show $\exists_{r>0} \text{opt}(f') = \text{opt}(f)$. We will divide the proof into several claims.

$$\text{C1: } \exists_{r>0} \text{opt}(f') = \text{opt}(f) \iff \exists_{r>0} \text{CH}(T_{f',0}^*, \dots, T_{f',s-1}^*) \cap \mathbb{R}_{\geq 0}^n = \emptyset$$

where

- $n = \deg(f)$
- $s = \text{opt}(f)$
- The $T_{f',i}$ are the rows of T_{s-1} for f' .
- $T_{f',i}^*$ is obtained from $T_{f',i}$ by deleting the first element.

Proof: Note

$$\begin{aligned} & \text{opt}(f') = \text{opt}(f) \\ \iff & \text{opt}(f') \geq s \quad \text{since } \text{opt}(f') \leq \text{opt}(f) + \text{opt}(x+r) = \text{opt}(f) = s \\ \iff & \neg \exists_{h \neq 0, \deg(h) < s} \text{coeffs}(hf') \geq 0 \\ \iff & \neg (\text{CH}(T_{f',0}, \dots, T_{f',s-1}) \cap \mathbb{R}_{\geq 0}^{n+1} \neq \emptyset) \\ \iff & \text{CH}(T_{f',0}, \dots, T_{f',s-1}) \cap \mathbb{R}_{\geq 0}^{n+1} = \emptyset \\ \iff & \text{CH}(T_{f',0}^*, \dots, T_{f',s-1}^*) \cap \mathbb{R}_{\geq 0}^n = \emptyset. \end{aligned}$$

$$\text{C2: } \exists_{r>0} \text{opt}(f') = \text{opt}(f) \iff \exists_{r>0} \varepsilon_{f'}(r) > 0$$

where $\varepsilon_{f'}(r)$ stands for the minimum Euclidean distance between $\text{CH}(T_{f',0}^*, \dots, T_{f',s-1}^*)$ and $\mathbb{R}_{\geq 0}^n$, that is,

$$\varepsilon_{f'}(r) := \min_{\substack{x \in \text{CH}(T_{f',0}^*, \dots, T_{f',s-1}^*) \\ y \in \mathbb{R}_{\geq 0}^n}} \|x - y\|$$

Proof: Immediate from the above claim.

C3: $\varepsilon_{f'}(r)$ is continuous at $r = 0$.

Proof: We will divide it into several steps.

(a) Note

$$\begin{aligned}
\varepsilon_{f'}(r) &= \min_{\substack{c \in \mathbb{R}_{\geq 0}^s \\ c_0 + \dots + c_{s-1} = 1 \\ y \in \mathbb{R}_{\geq 0}^n}} \left\| \sum_{i=0}^{s-1} c_i T_{f',i}^* - y \right\| \\
&= \min_{\substack{c \in \mathbb{R}_{\geq 0}^s \\ c_0 + \dots + c_{s-1} = 1 \\ y \in \mathbb{R}_{\geq 0}^n}} \left\| \left[\sum_{i=0}^{s-1} c_i T_{f',i,1}^* - y_1, \dots, \sum_{i=0}^{s-1} c_i T_{f',i,n}^* - y_n \right] \right\| \\
&= \min_{\substack{z \in \mathbb{R}_{\geq 0}^{s+n} \\ z_0 + \dots + z_{s-1} = 1}} \left\| \left[\sum_{i=0}^{s-1} z_i T_{f',i,1}^* - z_s, \dots, \sum_{i=0}^{s-1} z_i T_{f',i,n}^* - z_{s+n-1} \right] \right\| \quad \text{where } z = (c, y) \\
&= \min_{z \in C} p(z, r)
\end{aligned}$$

where $p(z, r)$ is Euclidean distance and

$$C = \{z \in \mathbb{R}_{\geq 0}^{s+n} : z_1 + \dots + z_s = 1\}.$$

(b) Note, since $p(z, r)$ is the distance between two closed sets, the minimum distance is realized by a point in each set. Hence,

$$\varepsilon_{f'}(r) = \min_{z \in C} p(z, r) = \inf_{z \in C} p(z, r).$$

(c) Note the following:

- i. By Section 3.1.5 of [7], $p(z, r)$ is a convex function since $p(z, r)$ is a norm.
- ii. By Section 3.2.5 of [7], $\varepsilon_{f'}(r) = \inf_{z \in C} p(z, r)$ is a convex function, since C is convex and $p(z, r)$ is bounded below.
- iii. By Corollary 3.5.3 in [21], since $\varepsilon_{f'}(r)$ is a convex function defined on a convex set \mathbb{R} , $\varepsilon_{f'}(r)$ is continuous on the relative interior of \mathbb{R} , $\text{ri}(\mathbb{R}) = \mathbb{R}$.
- iv. Hence, $\varepsilon_{f'}(r)$ is continuous at $r = 0$.

C4: $\varepsilon_{f'}(0) > 0$.

Proof: Let $r = 0$. Then $f' = x \cdot f$.

(a) Subclaim: $T_{f',i,j}^* = T_{f,i,j}$. Note that these are the entries of $T_{f',s-1}^*$ and $T_{f,s-1}$. We will prove the claim using the fact that $T_{s-1} = R_{s-1}^{-1} L_{s-1}$ for any f .

- i. Note $R_{f,s-1} = R_{f',s-1}$. This is clear from the definition of R_{s-1} . Hence $R_{f,s-1}^{-1} = R_{f',s-1}^{-1}$.
- ii. Note that $L_{f,i,j} = L_{f',i,j+1}$. This is clear from the definition of L_{s-1} , since $L_{f',s-1}$ is composed of $L_{f,s-1}$ with a column of zeroes added on the left.

iii. Note that for $i = 0, \dots, s-1$ and $j = 0, \dots, n-1$,

$$\begin{aligned}
T_{f',i,j}^* &= T_{f',i,j+1} \\
&= \sum_{k=0}^{s-1} R_{f',i,k}^{-1} L_{f',k,j+1} \\
&= \sum_{k=0}^{s-1} R_{f,i,k}^{-1} L_{f,k,j} \\
&= T_{f,i,j}
\end{aligned}$$

Hence $T_{f',i,j}^* = T_{f,i,j}$.

(b) Subclaim: $\varepsilon_{f'}(0) = \varepsilon_f$

where ε_f stands for the minimum Euclidean distance between $\text{CH}(T_{f,0}, \dots, T_{f,s-1})$ and $\mathbb{R}_{\geq 0}^n$, that is,

$$\varepsilon_f := \min_{\substack{x \in \text{CH}(T_{f,0}, \dots, T_{f,s-1}) \\ y \in \mathbb{R}_{\geq 0}^n}} \|x - y\|.$$

To see this, note

$$\begin{aligned}
\varepsilon_{f'}(0) &= \min_{\substack{c \in \mathbb{R}_{\geq 0}^s \\ c_0 + \dots + c_{s-1} = 1 \\ y \in \mathbb{R}_{\geq 0}^n}} \left\| \sum_{i=0}^{s-1} c_i T_{f',i}^* - y \right\| \\
&= \min_{\substack{c \in \mathbb{R}_{\geq 0}^s \\ c_0 + \dots + c_{s-1} = 1 \\ y \in \mathbb{R}_{\geq 0}^n}} \left\| \sum_{i=0}^{s-1} c_i T_{f,i} - y \right\| \\
&= \min_{\substack{x \in \text{CH}(T_{f,0}, \dots, T_{f,s-1}) \\ y \in \mathbb{R}_{\geq 0}^n}} \|x - y\| \\
&= \varepsilon_f
\end{aligned}$$

(c) Subclaim: $\varepsilon_f > 0$.

Note since $\text{opt}(f) = s$, we have $\text{CH}(T_{f,0}, \dots, T_{f,s-1}) \cap \mathbb{R}_{\geq 0}^n = \emptyset$. Thus $\varepsilon_f > 0$.

(d) From the above two subclaims, we have $\varepsilon_{f'}(0) > 0$.

From the above three claims, we immediately have

$$\exists_{r>0} \text{opt}(f') = \text{opt}(f).$$

□

We will show two examples here, one where $\text{opt}(f') \neq \text{opt}(f)$ and one where $\text{opt}(f') = \text{opt}(f)$.

Example 3.6.1. Let $f = x^2 - 2x + 4$ and $f' = (x + 2)(x^2 - 2x + 4)$. We know from Example 3.3.2 and Theorem 3.1.2 that $\text{opt}(f) = 1$. Hence, $\text{opt}(f') \leq 1$.

$$\begin{aligned} f' &= (x + 2)(x^2 - 2x + 4) \\ &= x^3 + 8 \end{aligned}$$

Hence, for this choice of $r = 2$, we have $\text{opt}(f') = 0$ and hence $\text{opt}(f') \neq \text{opt}(f)$.

Example 3.6.2. Let $f = x^2 - 2x + 4$ and $f' = (x + 1)(x^2 - 2x + 4)$. We know from Example 3.3.2 and Theorem 3.1.2 that $\text{opt}(f) = 1$. Hence, $\text{opt}(f') \leq 1$.

$$\begin{aligned} f' &= (x + 1)(x^2 - 2x + 4) \\ &= x^3 - x^2 + 2x + 4 \end{aligned}$$

Hence, for this choice of $r = 1$, we have $\text{opt}(f') \neq 0$. Then since $\text{opt}(f') \leq \text{opt}(f)$, we must have $\text{opt}(f') = \text{opt}(f)$.

Conjecture 3.6.1. Let $f' = (x^2 + 2r \cos \theta + r^2)f$ for $0 < \theta < \pi$. Then $\exists_{r>0} \text{opt}(f') = \text{opt}(f) + \left\lceil \frac{\pi}{\theta} \right\rceil - 2$.

Proposed Proof of Conjecture 3.1.1. We will use induction on k .

1. The claim holds for $k = 1$.

(a) $\theta_1 = \pi$

Since $\text{opt}(f_1) = 0$ and $b_{HR}(f_1) = 0$.

(b) $\theta_1 \neq \pi$

From Theorem 3.1.2.

2. Assume that the claim holds for $\leq k$ where $k \geq 1$.

3. The claim holds for k .

Let $\theta \in (0, \pi]^k$. We need to find $r \in [0, \infty)^k$ such that

$$\text{opt}(f_{(\theta_1, \dots, \theta_k), (r_1, \dots, r_k)}) = b_{HR}(f_{(\theta_1, \dots, \theta_k), (r_1, \dots, r_k)}).$$

By the induction hypothesis, we have

$$\text{opt}(f_{(\theta_1, \dots, \theta_{k-1}), (r_1, \dots, r_{k-1})}) = b_{HR}(f_{(\theta_1, \dots, \theta_{k-1}), (r_1, \dots, r_{k-1})})$$

for some $(r_1, \dots, r_{k-1}) \in [0, \infty)^{k-1}$.

(a) $\theta_k = \pi$

i. By Lemma 3.6.1, we have

$$\text{opt}(f_{(\theta_1, \dots, \theta_k)}, (r_1, \dots, r_{k-1}, r_k)) = \text{opt}(f_{(\theta_1, \dots, \theta_{k-1})}, (r_1, \dots, r_{k-1}))$$

for some $r_k \in [0, \infty)$.

ii. Since $\theta_k = \pi$, we have

$$b_{HR}(f_{(\theta_1, \dots, \theta_{k-1})}, (r_1, \dots, r_{k-1})) = b_{HR}(f_{(\theta_1, \dots, \theta_k)}, (r_1, \dots, r_k)).$$

iii. From a.i and a.ii, we have

$$\text{opt}(f_{(\theta_1, \dots, \theta_k)}, (r_1, \dots, r_k)) = b_{HR}(f_{(\theta_1, \dots, \theta_k)}, (r_1, \dots, r_k)).$$

(b) $\theta_k \neq \pi$

i. By Conjecture 3.6.1, we have

$$\text{opt}(f_{(\theta_1, \dots, \theta_k)}, (r_1, \dots, r_{k-1}, r_k)) = \text{opt}(f_{(\theta_1, \dots, \theta_{k-1})}, (r_1, \dots, r_{k-1}))$$

for some $r_k \in [0, \infty)$.

ii. Since $\theta_k \neq \pi$, we have

$$b_{HR}(f_{(\theta_1, \dots, \theta_{k-1})}, (r_1, \dots, r_{k-1})) + \left\lceil \frac{\pi}{\theta_k} \right\rceil - 2 = b_{HR}(f_{(\theta_1, \dots, \theta_k)}, (r_1, \dots, r_k)).$$

iii. From b.i and b.ii, we have

$$\text{opt}(f_{(\theta_1, \dots, \theta_k)}, (r_1, \dots, r_k)) = b_{HR}(f_{(\theta_1, \dots, \theta_k)}, (r_1, \dots, r_k)).$$

□

CHAPTER

4

EXTENDING THE DEGREE BOUND TO POLYNOMIALS WITH ARBITRARILY MANY POSITIVE REAL ROOTS (B_{ARB})

In Section 2.3, we discussed the work by Poincaré, which showed that there exists a multiplier g for a polynomial f that will produce a product gf such that the number of positive real roots of f is equal to the number of sign differences in consecutive non-zero coefficients of gf . In this sense, Descartes' Rule of signs can be "exact". For an example of an exactifying multiplier for a polynomial with a positive real root, see Example 2.3.3 in Chapter 2. We also discussed the work by Avendaño to construct an exactifying multiplier for arbitrary polynomials.

The multiplier provided in Chapter 3 can be used to find a witness for the exactifying multiplier for arbitrary polynomials. We include a count of the degree of this multiplier.

4.1 Main Results

In this section, we will state the main results relating to exactifying multipliers for Descartes' Rule of Signs when the polynomial has arbitrarily many positive real roots.

Theorem 4.1.1 (Bound). *Let $f \in \mathbb{R}[x]$. Then there exists a polynomial $g \in \mathbb{R}[x]$ of degree at most $b_{ARB}(f)$ such that Descartes' Rule of Signs is exact for $g f$ where*

1. $b_{ARB}(f) = (t + 1)s + t(n - t)$
2. $s = \sum_{i=1}^m \left(\left\lceil \frac{\pi}{\arg(\alpha_i)} \right\rceil - 2 \right)$
3. $n = \deg(f)$
4. $t = p(f)$, the number of positive real roots of f , counting multiplicities
5. $\alpha_1, \dots, \alpha_m$ are all the non-real roots of f with positive imaginary part (multiple roots are repeated).

Theorem 4.1.2 (Multiplier). *Let $f \in \mathbb{R}[x]$. Then a witness for the exactifying multiplier is given by*

$$g = q \cdot \prod_{i=1}^t (x^{s+l} + r_i x^{s+l-1} + \dots + r_i^{s+l})$$

where

1. r_i are the positive real roots of f
2. $t = p(f)$
3. h is the monic factor of f with $p(h) = 0$
4. $s = b_{HR}(h)$
5. $l = \deg(h)$
6. q is the exactifying multiplier for h .

4.2 Proof of Multiplier (Theorem 4.1.2)

Recall that

- $p(f)$ = the number of positive real roots of f (counting multiplicity)
 $v(f)$ = the number of sign differences between consecutive non-zero coefficients.

Example 4.2.1. Let $f = (x-1)(x^2-2x+4)$ and $g = (x^3+x^2+x+1)(x+2)$.

Note that

$$\begin{aligned} p(f) &= 1 \\ gf &= (x^3+x^2+x+1)(x+2)(x-1)(x^2-2x+4) \\ &= x^7+8x^4-x^3-8 \end{aligned}$$

We can see that $v(gf) = 1$. Hence, there exists a multiplier g that produces a product gf in which $p(f) = v(gf)$.

Proof of Theorem 4.1.2. We will divide the proof into conceptual steps.

1. Let $f = a(x-r_1)\cdots(x-r_t)h$ where $a \in \mathbb{R}$, $r_i \in \mathbb{R}_{>0}$ and h is monic with no positive real roots. Let $l = \deg(h) = \deg(f) - t = n - t$.
2. Note $s = b_{HR}(h)$.
3. From Theorem 3.1.1, there exists a monic q such that $\deg(q) = s$ and $v(qh) = 0$.
4. Let $p = a(x^{s+l+1} - r_1^{s+l+1})\cdots(x^{s+l+1} - r_t^{s+l+1})$. Note that $p = a$ if f has no positive real roots.

$$\begin{aligned} p &= a \prod_{j=1}^t (x^{s+l+1} - r_j^{s+l+1}) \\ &= a \sum_{j=0}^t (-1)^j e_j(r_1^{s+l+1}, \dots, r_t^{s+l+1}) (x^{s+l+1})^{t-j} \end{aligned}$$

where $e_j(r_1^{s+l+1}, \dots, r_t^{s+l+1})$ is the elementary symmetric polynomial of degree j in $(r_1^{s+l+1}, \dots, r_t^{s+l+1})$. In this form, we see immediately that $v(p) = t$.

5. Note that the degree gap between two consecutive terms of p is $s+l+1$. Note also that qh has all non-negative coefficients and $\deg(qh) = s+l$. Now consider pqh :

$$\begin{aligned} pqh &= \left(a \sum_{j=0}^t (-1)^j e_j(r_1^{s+l+1}, \dots, r_t^{s+l+1}) (x^{s+l+1})^{t-j} \right) qh \\ &= a \sum_{j=0}^t (-1)^j e_j(r_1^{s+l+1}, \dots, r_t^{s+l+1}) (x^{s+l+1})^{t-j} qh \end{aligned}$$

In order to count the sign changes in pqh , fix a j :

$$\begin{aligned}
& (-1)^j e_j(r_1^{s+l+1}, \dots, r_t^{s+l+1}) (x^{s+l+1})^{t-j} qh \\
&= (-1)^j e_j(r_1^{s+l+1}, \dots, r_t^{s+l+1}) x^{(s+l+1)(t-j)} \sum_{i=0}^{s+l} b_i x^i \quad \text{where } qh = \sum_{i=0}^{s+l} b_i x^i \\
&= \sum_{i=0}^{s+l} (-1)^j e_j(r_1^{s+l+1}, \dots, r_t^{s+l+1}) b_i x^{i+(s+l+1)(t-j)}
\end{aligned}$$

This polynomial has all same sign coefficients, based on the sign of $(-1)^j e_j(r_1^{s+l+1}, \dots, r_t^{s+l+1})$, since the b_i are all non-negative.

Additionally, there are no common terms when j varies. Consider terms contributed by the $(j-1)$ -th, j -th, and $(j+1)$ -th terms of pqh :

$$\begin{aligned}
& \text{highest degree term for } j+1: & (s+l+1)(t-j)-1 \\
& \text{lowest degree term for } j: & (s+l+1)(t-j)+0 \\
& \text{highest degree term for } j: & (s+l+1)(t-j)+s+l \\
& \text{lowest degree term for } j-1: & (s+l+1)(t-j)+s+l+1
\end{aligned}$$

Hence, we have preserved the number of sign changes of p and $v(pqh) = t$.

6. Note

$$\begin{aligned}
pqh &= a(x^{s+l+1} - r_1^{s+l+1}) \dots (x^{s+l+1} - r_t^{s+l+1}) qh \\
&= \frac{x^{s+l+1} - r_1^{s+l+1}}{x - r_1} \dots \frac{x^{s+l+1} - r_t^{s+l+1}}{x - r_t} q (a(x - r_1) \dots (x - r_t) h) \\
&= (x^{s+l} + r_1 x^{s+l-1} + \dots + r_1^{s+l}) \dots (x^{s+l} + r_t x^{s+l-1} + \dots + r_t^{s+l}) qf \\
&= gf
\end{aligned}$$

where $g = q \cdot \prod_{i=1}^t (x^{s+l} + r_i x^{s+l-1} + \dots + r_i^{s+l})$. Hence $v(gf) = t$.

7. Since f has t positive roots, we see that gf has at least t positive real roots. By Descartes' Rule of Signs, the number of positive real roots of gf is at most $v(gf) = t$. Thus gf has exactly t positive roots, i.e. Descartes' Rule of Signs is exact for gf .

□

Now, we will construct the g given in Example 4.2.1 using the format detailed in Theorem 4.1.2.

Example 4.2.2. Let $f = (x - 1)(x^2 - 2x + 4)$. Note $n = 3$, $h = x^2 - 2x + 4$, $t = 1$, $l = 2$, $s = b_{HR}(h) = 1$, and $r_1 = 1$.

From Example 3.5.1, we know the certificate for $h = x^2 - 2x + 4$ from Theorem 3.1.3 is $q = x + 2$. The polynomial p from Step 4 of the proof of Theorem 4.1.2 is $p = x^4 - 1$ (which has exactly one sign change).

$$\begin{aligned} pqh &= (x^4 - 1)(x + 2)(x^2 - 2x + 4) \\ &= (x - 1)(x^3 + x^2 + x + 1)(x + 2)(x^2 - 2x + 4) \\ &= x^7 + 8x^4 - x^3 - 8 \end{aligned}$$

Note that pqh has exactly one sign change, per the claim, and

$$\begin{aligned} pqh &= (x^3 + x^2 + x + 1)(x + 2)(x - 1)(x^2 - 2x + 4) \\ &= gf. \end{aligned}$$

Then $g = (x^3 + x^2 + x + 1)(x + 2)$.

4.3 Proof of Bound b_{ARB} (Theorem 4.1.1)

Proof of Theorem 4.1.1. We compute the degree of the multiplier given in Theorem 4.1.2.

$$\deg(g) = (s + l)t + s = (s + (n - t))t + s = (t + 1)s + t(n - t) = b_{ARB}(f).$$

□

Example 4.3.1. Let $f = (x - 1)(x^2 - 2x + 4)$. Note $n = 3$, $h = x^2 - 2x + 4$, $t = 1$, and $s = b_{HR}(h) = 1$.

$$b_{ARB}(f) = (t + 1)s + t(n - t) = 4$$

We will show, via a counterexample, that this degree bound on the exactifying multiplier is not optimal.

Example 4.3.2. Let $f = (x-1)(x^2-2x+4)$. The bound from Theorem 4.1.1 is 4. However, a multiplier of degree 3 exists, which will produce a product with exactly one sign change.

$$\begin{aligned}g &= x^3 + 3x^2 + 3x + 2 \\gf &= (x^3 + 3x^2 + 3x + 2)(x-1)(x^2-2x+4) \\ &= x^6 + 7x^3 - 8\end{aligned}$$

Hence, $\deg(g) = b_{ARB}(f)$ is not optimal.

CHAPTER

5

COMPARING B_{PR} TO B_{HR}

The new bound from Theorem 3.1.1 is optimal for quadratic polynomials. We are curious to know how it compares to the previous bound by Powers and Reznick given in Theorem 2.5.1.

For a polynomial f , let

$b_{PR}(f)$ be the Powers-Reznick bound for f
 $b_{HR}(f)$ be the new bound for f .

First, we will show a general comparison of the two bounds for a large set of random polynomials. In addition, we will compare b_{PR} to b_{HR} specifically for quadratic polynomials, where it is known that b_{HR} is optimal.

5.1 Comparison between b_{PR} and b_{HR} for Random Polynomials

It is easy to find examples of polynomials for which b_{HR} is smaller than b_{PR} .

Example 5.1.1. Let $f = x^4 - 4x^3 + 10x^2 - 12x + 8$. Then

$$b_{PR}(f) = 491$$

$$b_{HR}(f) = 3 \text{ from Example 3.5.2.}$$

In general, b_{HR} is a lower bound than b_{PR} . To examine this relationship in better detail, we did the following. For each degree n , we:

1. Computed 1000 polynomials of degree n by selecting random coefficients, $\{f_1, \dots, f_{1000}\}$.
2. Computed $b_{PR}(f_i)$ and $b_{HR}(f_i)$ for each polynomial in $\{f_1, \dots, f_{1000}\}$.
3. Computed the average $b_{PR}(\text{avg})$ of the set $\{b_{PR}(f_i) : 1 \leq i \leq 1000\}$ and $b_{HR}(\text{avg})$ of $\{b_{HR}(f_i) : 1 \leq i \leq 1000\}$.
4. Plotted the point $\left(n, \frac{b_{HR}(\text{avg})}{b_{PR}(\text{avg})}\right)$.

The resulting graphs show the ratio of $b_{HR}(\text{avg})$ to $b_{PR}(\text{avg})$ as n increases and the rate of decrease, which appears to be linear in n .

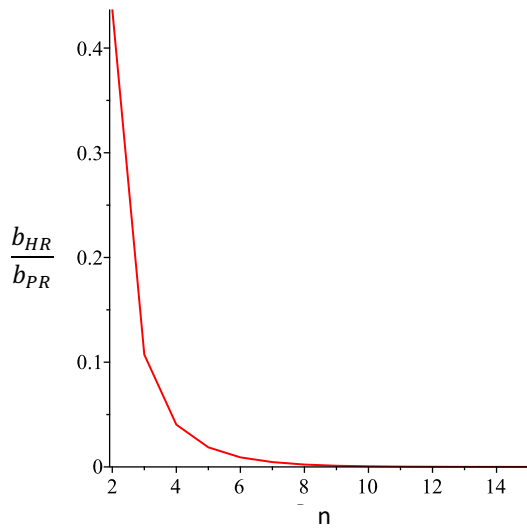


Figure 5.1 Graph of Ratio $\frac{b_{HR}}{b_{PR}}$

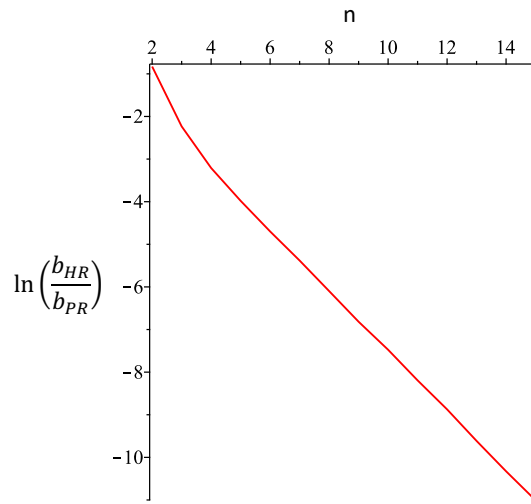


Figure 5.2 Rate of Decrease of Ratio $\frac{b_{HR}}{b_{PR}}$

Hence, in general, b_{HR} is exponentially smaller in n than b_{PR} .

5.2 Comparison between b_{PR} and b_{HR} for Quadratic Polynomials

In order to compare the bounds for quadratic polynomials, we will give an explicit formula for $b_{PR}(f)$ in terms of the coefficients of f when $f = x^2 + a_1x + a_0$.

First, let $f = x^2 + a_1x + a_0$ have no positive real roots. Then the Powers-Reznick bound from Theorem 2.5.1 is given by

$$b_{PR}(f) = \left[\frac{\max \left\{ 1, \frac{|a_1|}{2}, |a_0| \right\}}{\min_{x \in [0,1]} (1-x)^2 f \left(\frac{x}{1-x} \right)} \right] - 2.$$

The bound says that $(x+1)^{b_{PR}(f)}f$ will have all non-negative coefficients. For an example of the computation of b_{PR} in the quadratic case, reference Example 2.5.2 in Chapter 2.

We could determine b_{PR} in terms of the coefficients via a careful case analysis. There are 3 possible cases for the numerator. For the denominator, the cases may be analyzed according to the leading coefficient of $g = (1-x)^2 f \left(\frac{x}{1-x} \right)$ and the location of its vertex with respect to $[0, 1]$, e.g. a positive leading coefficient of g and a vertex $v < 0$ implies that $\min_{x \in [0,1]} g = g(0)$. Hence, there are 8 possible cases for the denominator, since g may be linear.

Using brute force, we must consider 24 total cases. However, many of these cases can be combined and some are not realizable. We present in the following theorem a nice summary via trial and error.

Theorem 5.2.1. Let $f = x^2 + a_1x + a_0 \in \mathbb{R}[x]$ have no positive real roots. Then $b_{PR}(f)$ is as follows:

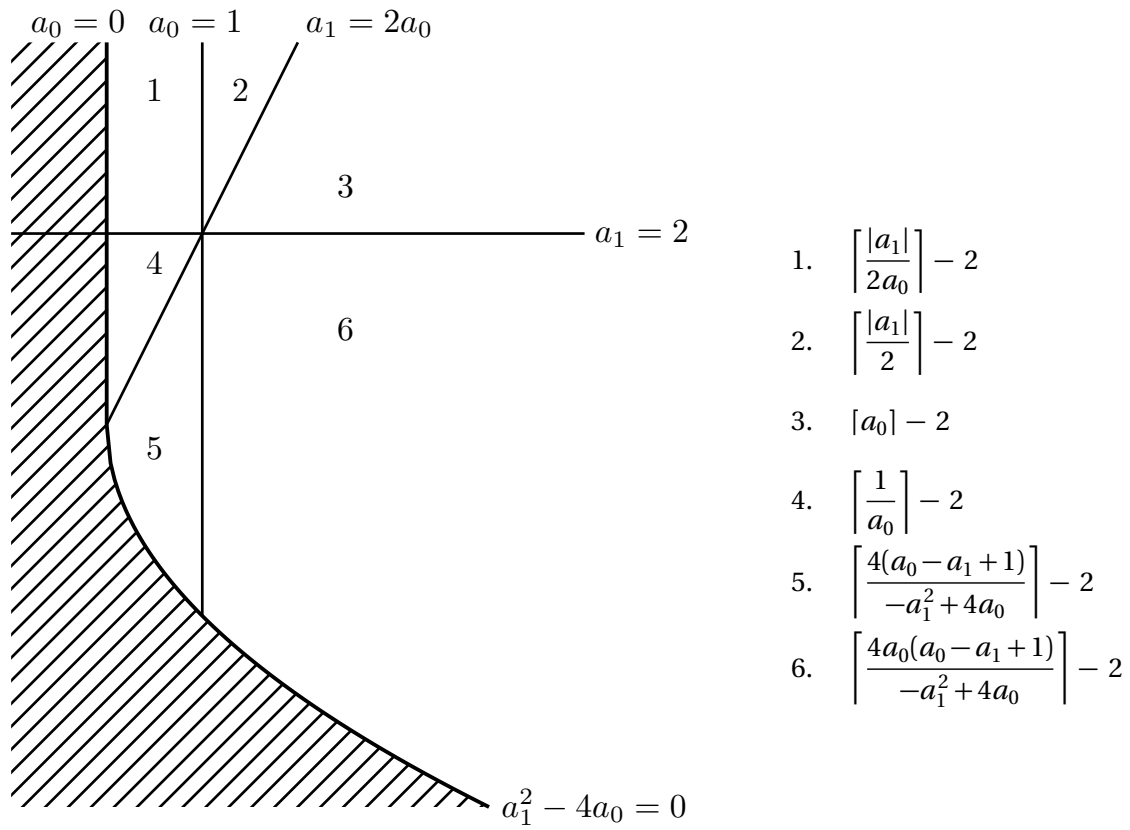


Figure 5.3 Quadratic b_{PR} by Coefficients of f

Proof. Let

$$\begin{aligned}
 f &= (x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2 \\
 b_{PR}(f) &= \left\lceil \frac{N}{D} \right\rceil - 2 \\
 N &= \max \left\{ 1, \frac{|r_1 + r_2|}{2}, |r_1 r_2| \right\} \\
 D &= \min_{x \in [0,1]} g(x) \\
 g &= (1-x)^2 f \left(\frac{x}{1-x} \right) \\
 &= (1-x)^2 \left[\left(\frac{x}{1-x} \right)^2 - (r_1 + r_2) \left(\frac{x}{1-x} \right) + r_1 r_2 \right] \\
 &= (1 + r_1 + r_2 + r_1 r_2)x^2 - (r_1 + r_2 + 2r_1 r_2)x + r_1 r_2 \\
 &= (r_1 + 1)(r_2 + 1)x^2 - (r_1 + r_2 + 2r_1 r_2)x + r_1 r_2 \\
 a_1 &= -r_1 - r_2 \\
 a_0 &= r_1 r_2 \\
 b_2 &= a_0 - a_1 + 1 \\
 b_1 &= a_1 - 2a_0 \\
 b_0 &= a_0.
 \end{aligned}$$

Note that $a_0 > 0$.

$$\begin{aligned}
 N &= \max \left\{ 1, \frac{|a_1|}{2}, a_0 \right\} \\
 D &= \min_{x \in [0,1]} g(x) \\
 g &= b_2 x^2 + b_1 x + b_0 \\
 v &= \frac{-b_1}{2b_2} = \frac{-a_1 + 2a_0}{2(a_0 - a_1 + 1)} \\
 g(v) &= \frac{-a_1^2 + 4a_0}{4(a_0 - a_1 + 1)}
 \end{aligned}$$

Note that if $a_1 < 0$ and f has two real roots, then the roots must be positive. Hence, $a_1 < 0$ implies f has two non-real roots. Then we must have $a_1^2 - 4a_0 < 0$ when $a_1 < 0$.

Region 1. $a_1 \geq 2 \wedge a_1 \geq 2a_0 \wedge a_0 < 1$

1. Note

$$\begin{aligned} & a_1 \geq 2 \\ \implies & |a_1| \geq 2a_0 \\ \iff & \frac{|a_1|}{2} \geq a_0 \\ & a_1 \geq 2 \\ \implies & |a_1| \geq 2 \\ \iff & \frac{|a_1|}{2} \geq 1. \end{aligned}$$

Thus

$$\max\left\{1, \frac{|a_1|}{2}, a_0\right\} = \frac{|a_1|}{2}.$$

2. Note

$$\begin{aligned} & a_1 \geq 2 \\ \iff & a_1 - (a_0 + 1) \geq 2 - (a_0 + 1) \\ \iff & -a_0 + a_1 - 1 \geq 1 - a_0 \\ \implies & -b_2 > 0 \text{ since } 1 - a_0 > 0 \\ \iff & b_2 < 0 \\ & a_1 \geq 2a_0 \\ \iff & 0 \geq -a_1 + 2a_0 \\ \iff & -b_1 \leq 0. \end{aligned}$$

Thus $b_2 < 0$ and $v \geq 0$, so

$$\min_{x \in [0,1]} g(x) = \min_{x \in [0,1]} \{g(0), g(1)\} = \min_{x \in [0,1]} \{a_0, 1\} = a_0.$$

3. Note

$$b_{PR}(f) = \left\lceil \frac{|a_1|}{2a_0} \right\rceil - 2.$$

Region 2. $a_1 \geq 2 \wedge a_1 \geq 2a_0 \wedge a_0 \geq 1$

1. Note

$$\begin{aligned}
 & a_1 \geq 2 \\
 \implies & |a_1| \geq 2a_0 \\
 \iff & \frac{|a_1|}{2} \geq a_0 \\
 & a_1 \geq 2 \\
 \implies & |a_1| \geq 2 \\
 \iff & \frac{|a_1|}{2} \geq 1.
 \end{aligned}$$

Thus

$$\max\left\{1, \frac{|a_1|}{2}, a_0\right\} = \frac{|a_1|}{2}.$$

2. Note

$$\begin{aligned}
 & a_1 \geq 2a_0 \\
 \iff & -a_0 \geq a_0 - a_1 \\
 \implies & -1 \geq a_0 - a_1 \text{ since } -1 \geq -a_0 \\
 \iff & 0 \geq a_0 - a_1 + 1 \\
 \iff & b_2 \leq 0 \\
 & a_1 \geq 2a_0 \\
 \iff & 0 \geq -a_1 + 2a_0 \\
 \iff & -b_1 \leq 0.
 \end{aligned}$$

3. If $b_2 < 0$, then $v \geq 0$ and

$$\min_{x \in [0,1]} g(x) = \min_{x \in [0,1]} \{g(0), g(1)\} = \min_{x \in [0,1]} \{a_0, 1\} = 1.$$

4. If $b_2 = 0$, then

$$\begin{aligned}
 & a_0 - a_1 + 1 = 0 \\
 \iff & a_0 = a_1 - 1 \\
 & b_1 = a_1 - 2a_0 \\
 \implies & b_1 = -a_1 + 2 \\
 \implies & b_1 \leq 0 \text{ since } 2 - a_1 \leq 0 \\
 & -b_1 \leq 0 \wedge b_1 \leq 0 \\
 \implies & b_1 = 0 \\
 & b_2 = 0 \wedge b_1 = 0 \\
 \implies & b_0 = 1.
 \end{aligned}$$

Thus g is constant and $g = 1$, so

$$\min_{x \in [0,1]} g(x) = 1.$$

5. Note

$$b_{PR}(f) = \left\lceil \frac{|a_1|}{2} \right\rceil - 2.$$

Region 3. $a_1 \geq 2 \wedge a_1 < 2a_0 \wedge a_0 \geq 1$

1. Note

$$\begin{aligned} & a_1 \geq 2 \\ \implies & |a_1| < 2a_0 \\ \iff & \frac{|a_1|}{2} < a_0 \\ & a_1 \geq 2 \\ \implies & |a_1| \geq 2 \\ \iff & \frac{|a_1|}{2} \geq 1. \end{aligned}$$

Thus

$$\max \left\{ 1, \frac{|a_1|}{2}, a_0 \right\} = a_0.$$

2. Note

$$\begin{aligned} & a_1 < 2a_0 \\ \iff & 0 < -a_1 + 2a_0 \\ \iff & -b_1 > 0. \end{aligned}$$

3. Note if $b_2 < 0$, then $\nu < 0$ and

$$\min_{x \in [0,1]} g(x) = g(1) = 1.$$

4. Note if $b_2 > 0$, then $\nu > 0$ and

$$\begin{aligned} & a_1 \geq 2 \\ \iff & a_1 - (2a_1 - 2a_0) \geq 2 - (2a_1 - 2a_0) \\ \iff & -a_1 + 2a_0 \geq 2a_0 - 2a_1 + 2 \\ \iff & \frac{-a_1 + 2a_0}{2(a_0 - a_1 + 1)} \geq 1. \end{aligned}$$

Thus $\nu \geq 1$ and

$$\min_{x \in [0,1]} g(x) = g(1) = 1.$$

5. Note if $b_2 = 0$, then since $b_1 < 0$,

$$\min_{x \in [0,1]} g(x) = g(1) = 1.$$

6. Thus

$$b_{PR}(f) = [a_0] - 2.$$

Region 4. $0 \leq a_1 < 2 \wedge a_1 \geq 2a_0 \wedge a_0 < 1$

1. Note

$$\begin{aligned} & 0 \leq a_1 < 2 \\ \implies & |a_1| \geq 2a_0 \\ \iff & \frac{|a_1|}{2} \geq a_0 \\ & 0 \leq a_1 < 2 \\ \implies & |a_1| < 2 \\ \iff & \frac{|a_1|}{2} < 1. \end{aligned}$$

Thus

$$\max \left\{ 1, \frac{|a_1|}{2}, a_0 \right\} = 1.$$

2. Note

$$\begin{aligned} & a_1 \geq 2a_0 \\ \iff & 0 \geq -a_1 + 2a_0 \\ \iff & -b_1 \leq 0. \end{aligned}$$

3. Note if $b_2 < 0$, then $\nu \geq 0$ and

$$\begin{aligned} & a_1 < 2 \\ \iff & a_1 - (2a_1 - 2a_0) < 2 - (2a_1 - 2a_0) \\ \iff & -a_1 + 2a_0 < 2a_0 - 2a_1 + 2 \\ \iff & \frac{-a_1 + 2a_0}{2(a_0 - a_1 + 1)} > 1. \end{aligned}$$

Thus $\nu > 1$ and

$$\min_{x \in [0,1]} g(x) = g(0) = a_0.$$

4. Note if $b_2 > 0$, then $\nu \leq 0$ and

$$\min_{x \in [0,1]} g(x) = g(0) = a_0.$$

5. Note if $b_2 = 0$, then since $b_1 \geq 0$,

$$\min_{x \in [0,1]} g(x) = g(0) = a_0.$$

6. Thus

$$b_{PR}(f) = \left\lceil \frac{1}{a_0} \right\rceil - 2.$$

Region 5. $-\sqrt{4a_0} \leq a_1 < 2 \wedge a_1 < 2a_0 \wedge a_0 < 1$

1. Note

$$\begin{aligned} &-\sqrt{4a_0} \leq a_1 < 2 \wedge a_0 < 1 \\ \implies &|a_1| \leq 2 \\ \iff &\frac{|a_1|}{2} \leq 1. \end{aligned}$$

2. Note if $|a_1| < 2a_0$, then $\frac{|a_1|}{2} < a_0$. Thus

$$\max \left\{ 1, \frac{|a_1|}{2}, a_0 \right\} = 1.$$

3. Note if $|a_1| \geq 2a_0$, then $\frac{|a_1|}{2} \geq a_0$. Thus

$$\max \left\{ 1, \frac{|a_1|}{2}, a_0 \right\} = 1.$$

4. Note

$$\begin{aligned}
& a_1 < 2a_0 \\
\iff & a_1 - (2a_1 - 2) < 2a_0 - (2a_1 - 2) \\
\iff & -a_1 + 2 < 2a_0 - 2a_1 + 2 \\
\implies & 0 < 2a_0 - 2a_1 + 2 \text{ since } 0 < 2 - a_1 \\
\iff & b_2 > 0 \\
& a_1 < 2a_0 \\
\iff & 0 < -a_1 + 2a_0 \\
\iff & -b_1 > 0.
\end{aligned}$$

Thus $b_2 > 0$, $v > 0$, and

$$\begin{aligned}
& a_1 < 2 \\
\iff & a_1 - (2a_1 - 2a_0) < 2 - (2a_1 - 2a_0) \\
\iff & -a_1 + 2a_0 < 2a_0 - 2a_1 + 2 \\
\iff & \frac{-a_1 + 2a_0}{2(a_0 - a_1 + 1)} < 1.
\end{aligned}$$

Thus $v \in (0, 1)$ and

$$\min_{x \in [0,1]} g(x) = g(v) = \frac{-a_1^2 + 4a_0}{4(a_0 - a_1 + 1)}.$$

5. Thus

$$b_{PR}(f) = \left\lceil \frac{4(a_0 - a_1 + 1)}{-a_1^2 + 4a_0} \right\rceil - 2.$$

Region 6. $-\sqrt{4a_0} < a_1 < 2 \wedge a_1 \geq -2a_0 \wedge a_0 \geq 1$

1. Note if $-2 \leq a_1 < 2$, then

$$\begin{aligned}
& |a_1| \leq 2 \\
\iff & \frac{|a_1|}{2} \leq 1 \\
& |a_1| \leq 2 \wedge 2a_0 \geq 2 \\
\implies & |a_1| \leq 2a_0 \\
\iff & \frac{|a_1|}{2} \leq a_0.
\end{aligned}$$

Thus

$$\max \left\{ 1, \frac{|a_1|}{2}, a_0 \right\} = a_0.$$

2. Note if $a_1 < -2$, then

$$\begin{aligned} & |a_1| > 2 \\ \Leftrightarrow & \frac{|a_1|}{2} > 1 \\ & -a_1 \leq 2a_0 \\ \Rightarrow & |a_1| \leq 2a_0 \\ \Leftrightarrow & \frac{|a_1|}{2} \leq a_0. \end{aligned}$$

Thus

$$\max\left\{1, \frac{|a_1|}{2}, a_0\right\} = a_0.$$

3. Note

$$\begin{aligned} & a_1 < 2 \wedge a_0 \geq 1 \\ \Rightarrow & -a_1 + 2a_0 > 0 \\ \Leftrightarrow & -b_1 > 0 \\ & -a_1 + 2a_0 > 0 \\ \Rightarrow & a_1 < 2a_0 \\ \Leftrightarrow & a_1 - (2a_1 - 2) < 2a_0 - (2a_1 - 2) \\ \Leftrightarrow & -a_1 + 2 < 2a_0 - 2a_1 + 2 \\ \Rightarrow & 0 < 2a_0 - 2a_1 + 2 \text{ since } 0 < 2 - a_1 \\ \Leftrightarrow & b_2 > 0. \end{aligned}$$

Thus $b_2 > 0$, $\nu > 0$, and

$$\begin{aligned} & a_1 < 2 \\ \Leftrightarrow & a_1 - (2a_1 - 2a_0) < 2 - (2a_1 - 2a_0) \\ \Leftrightarrow & -a_1 + 2a_0 < 2a_0 - 2a_1 + 2 \\ \Leftrightarrow & \frac{-a_1 + 2a_0}{2(a_0 - a_1 + 1)} < 1. \end{aligned}$$

Thus $\nu \in (0, 1)$ and

$$\min_{x \in [0,1]} g(x) = g(\nu) = \frac{-a_1^2 + 4a_0}{4(a_0 - a_1 + 1)}.$$

4. Thus

$$b_{PR}(f) = \left\lceil \frac{4a_0(a_0 - a_1 + 1)}{-a_1^2 + 4a_0} \right\rceil - 2.$$

□

Example 5.2.1. Let $f = x^2 - 2x + 4$. Then $a_1 = -2$ and $a_0 = 4$. Hence, this f falls in Region 6 from Theorem 5.2.1. Then

$$\begin{aligned}
 b_{PR}(f) &= \left\lceil \frac{4a_0(a_0 - a_1 + 1)}{-a_1^2 + 4a_0} \right\rceil - 2 \\
 &= \left\lceil \frac{4(4)(4 - (-2) + 1)}{-(-2)^2 + 4(4)} \right\rceil - 2 \\
 &= \left\lceil \frac{16(7)}{12} \right\rceil - 2 \\
 &= \left\lceil \frac{28}{3} \right\rceil - 2 \\
 &= 8.
 \end{aligned}$$

Note that this matches the $b_{PR}(f)$ calculated in Example 2.5.2.

To further illuminate b_{PR} , below is an additional graph of b_{PR} in terms of the coefficients a_0 and a_1 , color-coded by region from Theorem 5.2.1.

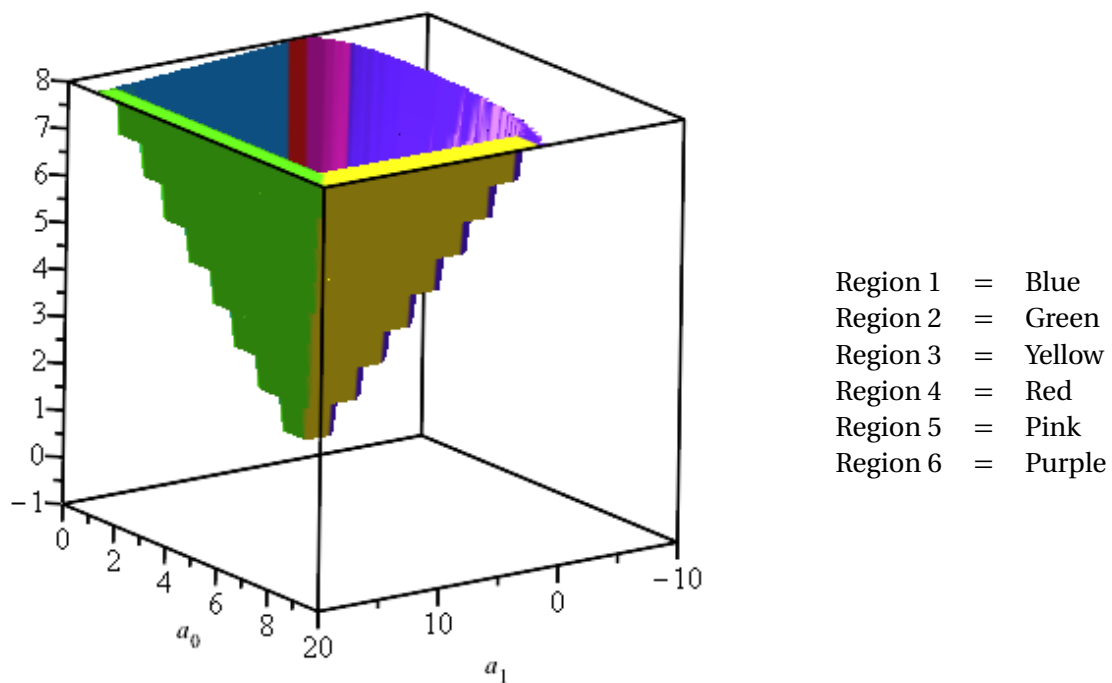


Figure 5.4 Quadratic b_{PR} by Region

Of particular note, consider any $f = x^2 + a_1x + a_0$ where $a_1, a_0 \geq 0$. This f will have all non-negative coefficients and, hence, can have no positive real roots. It is visually obvious that the multiplier $g = 1$, with degree 0, will suffice. However, $b_{PR}(f)$ is clearly positive for most of these polynomials.

Example 5.2.2. Let $f = x^2 + 2x + 10$. Then

$$b_{PR}(f) = 8.$$

However, we have $v(f) = 0$.

Finally, we will compare b_{PR} and b_{HR} in terms of a_0 and a_1 .

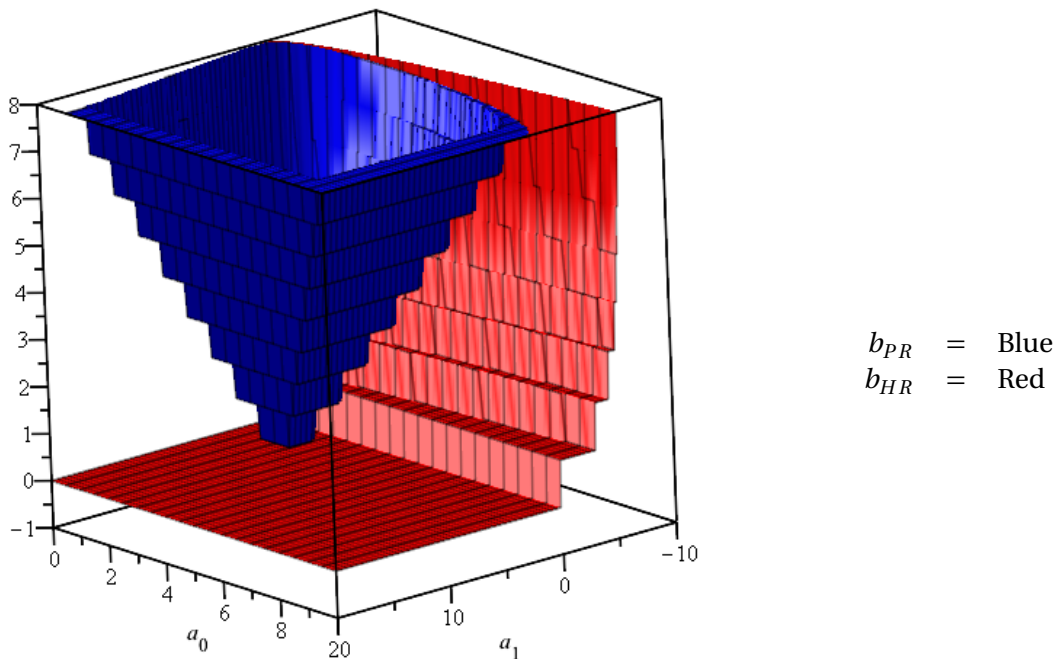


Figure 5.5 Quadratic b_{PR} Compared to Quadratic b_{HR}

In this comparison, two notable things are clear.

1. b_{HR} is less than b_{PR} .
2. b_{HR} captures those polynomials for which a degree 0 multiplier is clearly sufficient - those polynomials with $a_1, a_0 \geq 0$.

Example 5.2.3. *Let $f = x^2 + 2x + 10$. Then*

$$b_{HR}(f) = 0.$$

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APPENDIX

APPENDIX

A

MAPLE CODES

A.1 Utilities

```
restart:
with(LinearAlgebra):
with(simplex):
with(RandomTools):
with(Statistics):
with(SolveTools[Inequality]):
with(plots):
with(combinat,powerset):
kernelopts(opaquemodules=false):
Plot:-Inequality:-refine_bounding_box := (r,x,a,b,y)->a..b:
kernelopts(opaquemodules=true):

# returns a quadratic polynomial with roots of radius r and argument Pi/(t+2)
QF := proc(r,t)
  # r is a positive integer.
  local a1,a0,f;
  a1 := -2*r*round(cos(Pi/(t+2))*2^10)/2^10;
  a0 := r^2;
  f := x^2 + a1 *x + a0;
  return f;
end:
```

A.2 Bounds

```
# returns  $b_{\{HR\}}(f)$  for a given polynomial
HRB := proc(f)
  local R,s,r;
  R := [fsolve(f,complex)];
  R := select(r->Im(r)>0,R);
  s := add(ceil(Pi/argument(r))-2,r=R);
  s := max(s,0);
  return s;
end:

# returns the minimum on [0,1] of the denominator g from  $b_{\{PR\}}(f)$ 
UMin01 := proc(g)
  local gp,rs,vs;
  gp := diff(g,x);
  rs := {0,1,fsolve(gp)};
  rs := select(r->0<=r and r<=1,rs);
  vs := map(r->abs(eval(g,x=r)),rs);
  return min(vs);
end:

# returns  $b_{\{PR\}}(f)$  for a given polynomial
PRB := proc(f)
  local d,M,g,m,p,i;
  d := degree(f);
  M := max(seq(abs(coeff(f,x,i))/binomial(d,i), i=0..d));
  g := simplify((1-x)^d*eval(f,x=x/(1-x)));
  g := evalf(g);
  m := UMin01(g);
  if m = 0 then
    p := infinity;
  else
    p := evalf(binomial(d,2)*(M/m)-d);
    p := max(ceil(p),0);
  fi;
  return p;
end:
```

A.3 Derivation Matrices

```
# returns A_s for a given polynomial
Am := proc(f,s,tr:=false)
  local n,A;
  n := degree(f,x);
  A := Matrix(s+1,s+n+1,(i,j)->coeff(f,x,j-i));
end:

# returns L_s for a given polynomial
Lm := proc(f,s)
  local A,n,L;
  A := Am(f,s);
  n := ColumnDimension(A) - RowDimension(A);
  L := SubMatrix(A,[1..s+1],[1..n]);
  return L;
end:

# returns R_s for a given polynomial
Rm := proc(f,s)
  local A,n,R;
  A := Am(f,s);
  n := ColumnDimension(A) - RowDimension(A);
  R := SubMatrix(A,[1..s+1],[n+1..s+n+1]);
  return R;
end:

# returns T_s for a given polynomial
Tm := proc(f,s)
  local A,L,R,T;
  L := Lm(f,s);
  R := Rm(f,s);
  T := map(expand,R^(-1).L);
  return T;
end:
```

A.4 Optimality Testing

```
# returns a C-vector for a given polynomial
CV := proc(f,s)
  local T,Tv,C,S,c,i,j;
  T := Tm(f,s);
  C := {add(c[i-1],i=1..RowDimension(T))=1};
  C := {op(C),seq(add(c[i-1]*T[i,j],i=1..RowDimension(T))>=0,j=1..ColumnDimension(T))};
  S := maximize(c[s],C,NONNEGATIVE);
  if S = {} then return none fi;
  c := eval([seq(c[i-1],i=1..RowDimension(T))],S);
  return c;
end:
```

```
# returns opt(f) for a given polynomial
OPB := proc(f,hrb:=20)
  local s,c,R,G,g,T,i;
  for s from 0 to hrb do
    c := CV(f,s);
    if c <> none then
      R := Rm(f,s);
      T := Tm(f,s);
      G := convert(Matrix(c).R^(-1),list);
      g := add(G[i+1]*x^i,i=0..s);
      return s,evalf(c),g;
    fi;
  od;
  return "Increase hrb";
end:
```

A.5 Comparing the Bounds on Random Inputs

```
# returns bound comparisons based on random polynomials from Chapter 5
CompRCoeffsU := proc(db)
  local d,i,f, Rs,rs,r,ra, HRBs,hrbs,hrb,hrba, PRBs,prbs,prb,prba;
  HRBs := [];
  PRBs := [];
  Rs := [];
  for d from db[1] to db[2] do
    hrbs := [];
    prbs := [];
    rs := [];
    for i from 1 to 1000 do
      while true do
        f := Generate(polynom(rational(range=-1..1,denominator=2^20), x, degree = d));
        f := evalf(expand(f));
        if not (coeff(f,x,d) > 0 and nops(select(r->r>0,[fsolve(f)]))=0) then next fi;
        prb := PRB(f);
        hrb := HRB(f);
        if hrb = 0 or prb = infinity then next fi;
        r := evalf(hrb/prb);
        break;
      od:
      hrbs := [op(hrbs), hrb];
      prbs := [op(prbs), prb];
      rs := [op(rs), r];
    od:
    hrba := Mean(hrbs);
    prba := Mean(prbs);
    ra := Mean(rs);
    HRBs := [op(HRBs), [d,hrba]];
    PRBs := [op(PRBs), [d,prba]];
    Rs := [op(Rs), [d,ra] ];
  od:
  return HRBs,PRBs,Rs;
end:
```

```

SetState(state=1234567);
HRBs,PRBs,Rs := CompRCoeffsU([2,15]):
save HRBs,PRBs,Rs, "CompRCoeffsU2.m":
read "CompRCoeffsU2.m";
pointplot(HRBs,color=red, labels=[typeset(Degree),typeset(Average)],connect=true);
pointplot(PRBs,color=blue, labels=[typeset(Degree),typeset(Average)],connect=true);
pointplot(Rs,color=red, labels=[typeset(Degree),typeset(Ratio)], connect=true);
Rp := map(r->[r[1],ln(r[2])],Rs):
P1 := pointplot(Rp,color=red,
labels=[typeset(Degree),typeset(Ratio)],connect=true,thickness=1):
P1;
X := map(r->r[1],Rp):
Y := map(r->r[2],Rp):
g := Fit(a*d+b,X,Y,d);
P2 := plot(g,d=2..15,linestyle=dash,color=blue,thickness=4):
display(P1,P2);

```

A.6 Comparing the Bounds on Quadratic Inputs

```

# returns bound comparisons based on quadratic polynomials from Chapter 5
plot_bound := proc(bf,ar,vr,c)
local P,a1,PP,a0,P_plot;
P := [];
for a1 from -ar to 2*ar by 0.1 do
PP := [];
for a0 from 0 to ar by 0.1 do
PP := [op(PP), [a1,a0,`if(a1<=0 and a0<=a1^2/4,1000,bf(x^2+a1*x+a0))]];
od:
P := [op(P), PP];
od:
P_plot := surfdata(P,color=c,view=vr, labels=[a[1],a[0],b]):
return P_plot;
end:

```

```

PRB_ec := (a1,a0)-> piecewise(
  a1<=0 and a0<=a1^2/4,          [1000,          0],
  a1>=2 and a0>=1 and a1<=2*a0, [ceil(a0)-2,    20],
  a1>=2 and a0>=1 and a1> 2*a0, [ceil(abs(a1)/2)-2, 40],
  a1>=2 and a0<=1,              [ceil(abs(a1)/(2*a0))-2, 80],
  a1< 2 and a0>=1,              [ceil(4*a0*(a0-a1+1)/(-a1^2+4*a0))-2, 100],
  a1< 2 and a0<=1 and a1<=2*a0, [ceil(4 * (a0-a1+1)/(-a1^2+4*a0))-2, 120],
  a1< 2 and a0<=1 and a1> 2*a0, [ceil(1/a0)-2,      140]):

```

```

PRB_e := (a1,a0)-> max(PRB_ec(a1,a0)[1],0):

```

```

PRB_c := (a1,a0)-> PRB_ec(a1,a0)[2]:

```

```

plot_exp := proc(ar,vr)
  return plot3d(PRB_e(a1,a0), a1=-ar..2*ar, a0=0.001..ar, view=vr,
    style=patchngrid, numpoints=10000, color=PRB_c(a1,a0),
    labels=[a[1],a[0],b],lightmodel=none);
end:

```

```

ar := 10:
vr := [-ar..2*ar,0..ar,-1..8]:
PPRB := plot_bound(PRB,ar,vr,blue):
PHRB := plot_bound(HRB,ar,vr,red):
PPRB_e := plot_exp(ar,vr):
display(PPRB_e);
display([PHRB,PPRB]);

```

A.7 Witness for the Quadratic Multiplier

```
# returns two animations detailing certificates for quadratic polynomials
cert := proc(t,ver)
  local f,s,c,h,opt,Pf,Ph;
  f := QF(1,t);
  h := ver(f);
  if nops(select(z->z<0,{coeffs(evalf(expand(f*h))))})>1 then print("ERROR") fi;
  opt := style=point,symbol=solidcircle,axiscoordinates=polar;
  Pf := complexplot([fsolve(f,complex)],opt,symbolsize=20,color=red);
  Ph := complexplot([fsolve(h,complex)],opt,symbolsize=20,color=blue);
  display(Pf,Ph);
end:

anim_cert := proc(max_t,ver)
  display([seq(cert(t/10,ver),t=0..max_t*10)],insequence=true);
end:

ver1 := f->OPB(f)[3]:
anim_cert(6,ver1);

ver2 := proc(f)
  local s,h;
  s := OPB(f)[1];
  h := mul(x^2-2*cos((2*i+1)*Pi/(s+2))*x+1,i=1..iquo(s,2));
  if irem(s,2) = 1 then h := h*(x+1); fi;
  return h;
end:
anim_cert(6,ver2);
```