The Effect of Couple-Stress on the Pure Bending of a Prismatic Bar

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Summary

An evaluation of the applicability of the couple-stress theory to the stress analysis of graphite structures is performed by solving a pure bending problem. The differences between solutions from the couple-stress theory and from the classical theory of elasticity are compared. It is found that the differences are sufficient to account for the inconsistencies which have often been observed between the classical elasticity theory and actual behavior of graphite under bend and tensile loadings. An experimental procedure to measure the material constants in the couple-stress theory is also suggested.

The linear couple-stress theory, the origins of which go back to the turn of the last century, adds linear relations between couple-stresses and rotation gradients to the classical stress-strain law. By adopting the classical assumption that the plane cross section remains plane after deformation, the pure-bending problem is reduced to a plane couple-stress problem with traction-free boundary conditions. A general solution for an isotropic elastic prismatic bar under pure bending is then obtained using the Airy stress function and another stress function which accounts for the couple-stresses. For a cylindrical bar, it reduces to a simple series solution.

The moment-curvature and stress-curvature relations derived for a cylindrical bar from the general solution are used to examine the effect of couple-stresses. Numerical compilation of relations indicates that the couple stress parameters can be practically determined by measuring the moment-curvature ratio of various diametered specimens under bending. Although there is not sufficient data for such evaluation at present, it appears that the theory is consistent with the limited bend and tensile strength data of cylindrical specimens for H-451 graphite.

1. Introduction

It has been observed in laboratories that both the apparent strength and the apparent Young's modulus of graphite calculated from classical linear elasticity are higher for specimens under bending than for those under uniaxial tensile loads [1]. This discrepancies can be partially explained by an application of nonlinear theories [2,3]. To interpret the indications that the apparent strengths of some materials are affected by strain gradients, stress concentration problems in couple-stress theory were solved by a number of authors [4 - 7]. The consideration of stresses as functions of strains as well as rotational gradients will certainly provide some flexibility that is lacking in the classical theory of elasticity for a satisfactory explanation of the phenomenon.

A comprehensive description of the couple-stress theory of linear elasticity can be found in Mindlin and Tiersten [8]. This theory assumes linear relations between the couple-stresses and the rotation gradients in addition to the classical stress-strain relations. The proportional constants in the linear relations contain only two independent couple-stress parameters for an isotropic solid. Both parameters have the dimensions of the square of a length times the modulus of, say, shear. As suggested by Mindlin [4], the length, ℓ, has an order of magnitude about equal to the radius of a grain of a granular material. In this paper, we shall investigate whether a direct measurement of ℓ and hence a quantitative evaluation of the couple-stress effect are possible by solving the pure bending problem in the couple-stress theory.

In Section 1, a general solution of the couple-stress theory is obtained for an isotropic elastic prismatic bar under pure bending. By adopting the classical assumption that the plane cross section remains plane after deformation, the pure bending problem reduces to a two-dimensional couple-stress problem. Applying the stress functions similar to those of Mindlin for a plane stress problem [4], the solution is expressed in terms of the Airy stress function that satisfies the biharmonic equation and another stress function that satisfies the harmonic or Helmholtz equation.

The general solution is applied to the pure bending of a solid cylinder in Section 2. From the solution, a moment-curvature relation and a stress-curvature relation are derived. Numerical results and discussions are given in Section 3. The result indicates that the couple-stress effect is significant if the radius of the specimen is less than 10 times the length, ℓ. The estimated length, ℓ, for H451 graphite based on a limited number of bend strength tests is 1 mm. The maximum grain size of the material is about 1.5 mm. It appears that the couple-stress parameters for graphite can be determined by performing bend tests on various sized specimens of diameters less than 20 mm.

2. General Solution for Pure Bending of Prismatical Bar

Consider a prismatic bar bent in one of its principal planes by two equal and opposite couples. Assuming that the plane cross section remains plane after deformation and taking the origin of the coordinates at the centroid of the cross section and x_1x_3 - plane in the principal plane of bending, the displacements, $u_i^*(x_j)$, can be expressed as

$$u_1^*(x_i) = -\frac{x_3^2}{2R} - m\,x_3 + u_1(x_\alpha) \quad ,$$

$$u_2^*(x_i) = -n\,x_3 + u_2(x_\alpha) \quad ,$$

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$$u_3^*(x_i) = \frac{x_1 x_3}{R} + m\, x_1 + n\, x_2 + p \quad , \tag{1}$$

where m, n, p and R are constants. The usual indicial notation is used with the understanding that Latin and Greek subscripts have the respective ranges (1,2,3) and (1,2). The strain components, e_{ij}, and the gradients of rotations, κ_{ij}, are obtained by differentiating the kinematic relations of (1):

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \equiv u_{(\alpha,\beta)} \quad , \quad e_{33} = x_1/R \quad ,$$

$$\kappa_{\alpha 3} = \frac{1}{2}\varepsilon_{\gamma\beta}\, u_{\beta,\gamma\alpha} \quad , \quad \kappa_{32} = -\frac{1}{R} \quad ,$$

$$e_{\alpha 3} = e_{3\alpha} = \kappa_{\alpha\beta} = \kappa_{31} = \kappa_{33} = 0 \quad , \tag{2}$$

where R is the radius of curvature under bending, and $\varepsilon_{\alpha\beta}$ is the two-dimensional alternator, i.e., $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = 1$, and $\varepsilon_{21} = -1$. The constitutive relations of the couple stress theory, [4], in the present circumstances reduce to

$$e_{\alpha\beta} = \frac{1}{2\mu}\left[\tau_{(\alpha\beta)} - \nu\, \tau_{\gamma\gamma}\, \delta_{\alpha\beta}\right] - \nu \frac{x_1}{R}\delta_{\alpha\beta} \quad ,$$

$$\tau_{33} = \frac{2\mu}{1-2\nu}\left[\frac{(1-\nu)x_1}{R} + \nu\, e_{\alpha\alpha}\right] \quad ,$$

$$\sigma_{\alpha 3} = 4\mu\ell^2\left(\kappa_{\alpha 3} - \frac{\eta}{R}\delta_{\alpha 2}\right) \quad ,$$

$$\sigma_{3\alpha} = 4\mu\ell^2\left(\eta\, \kappa_{\alpha 3} - \frac{1}{R}\delta_{\alpha 2}\right) \quad ,$$

$$\tau_{3\alpha} = \tau_{\alpha 3} = \sigma_{\alpha\beta} = \sigma_{33} = 0 \quad , \tag{3}$$

where μ is the sheer modulus; ν is Poisson's ratio; ℓ and η are material constants of the couple stress theory; $\delta_{\alpha\beta}$ is the Kronecker delta; $\tau_{(\alpha\beta)}$ is the symmetric part of stress tensor $\tau_{\alpha\beta}$; σ_{ij} are the couple stresses. From (2) and (3), we obtain

$$\sigma_{\alpha 3} = 2\ell^2\left[\varepsilon_{\beta\gamma}\, \tau_{(\alpha\gamma),\beta} + \nu\, \varepsilon_{\alpha\beta}\, \tau_{\gamma\gamma,\beta} - \frac{2\mu}{R}(\nu + \eta)\, \delta_{2\alpha}\right] \quad .$$

$$\sigma_{3\alpha} = \eta\, \sigma_{\alpha 3} - (1 - \eta^2)\frac{4\mu\ell^2}{R}\delta_{2\alpha} \quad ,$$

$$\tau_{33} = \frac{E}{R}x_1 + \nu\, \tau_{\alpha\alpha} \quad , \tag{4}$$

where $E = 2\mu(1+\nu)$ is the Young's modulus.

Similar to Mindlin's [4] generalization of the Airy stress function in the classical theory of plane strain, the equilibrium and the compatibility conditions of the present problem are satisfied if one defines the stress functions ϕ and ψ as follows:

$$\tau_{\alpha\beta} = \varepsilon_{\gamma\alpha}\, \varepsilon_{\rho\beta}\, \phi_{,\gamma\rho} + \psi_{,\beta\gamma} \quad ,$$

$$\sigma_{\alpha 3} = \psi_{,\alpha} \quad . \tag{5}$$

Substitution from (5) into (4) yields

$$(\ell^2 \nabla^2 \psi - \psi)_{,\alpha} = 2(1 - \nu) \ell^2 \varepsilon_{\alpha\beta}\nabla^2 \phi_{,\beta} + \frac{4\mu\ell^2}{R} (\nu + \eta) \delta_{2\alpha} \quad . \tag{6}$$

Eq. (6) differs from the stress functions of the plane strain problem by the presence of the last term on the right hand side. Nevertheless, it requires,

$$\nabla^4 \phi = 0 \quad \text{and} \quad \ell^2 \nabla^4 \psi - \nabla^2 \psi = 0 \quad , \tag{7}$$

∇^2 is the Laplace operator. Hence ϕ is biharmonic in x_1 and x_2, and ψ can be expressed in terms of harmonic functions plus solutions to the Helmholtz equation, $\ell^2\nabla^2\psi - \psi = 0$. The general solution of (6) and (7) in cylindrical coordinates, $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \arctan (x_2/x_1)$, is given in the following:

$$\psi = \sum_{n=o}^{\infty} \left[A_n I_n\left(\frac{r}{\ell}\right) + B_n K_n\left(\frac{r}{\ell}\right) \sin n\theta + \left[C_n I_n\left(\frac{r}{\ell}\right) + D_n K_n\left(\frac{r}{\ell}\right)\right] \cos n\theta\right] ,$$

$$+ A_o \theta I_o\left(\frac{r}{\ell}\right) + B_o \theta K_o\left(\frac{r}{\ell}\right) - \frac{4\mu\ell^2(\nu + \eta)}{R} r \sin \theta$$

$$+ \sum_{n=-\infty}^{\infty} 8(1 - \nu) \ell^2 \left[E_n \sin n\theta + F_n \cos n\theta\right] r^n$$

$$\varphi = \sum_{\substack{n=-\infty \\ n\neq-1}}^{\infty} \frac{1}{n+1}\left(E_n \cos n\theta + F_n \sin n\theta\right) r^{n+2}$$

$$+ E_{-1} \left[(\ell n\, r) \cos\theta + \theta \sin\theta\right] r + F_{-1}\left[(\ell n\, r) \sin\theta - \theta \cos\theta\right] r$$

$$+ \sum_{n=-\infty}^{\infty} (G_n \cos n\theta + H_n \sin n\theta) r^n + G\, \ell n\, r + H\, \theta \quad . \tag{8}$$

where $K_n (r/\ell)$ and $I_n (r/\ell)$ are modified Bessel functions. The coefficients can be determined by boundary conditions. In the next section, we obtain the solution for a solid cylinder under pure bending. For a prismatic bar of arbitrary cross section, the problem becomes more difficult. However, a boundary-point-matching technique can be used to obtain numerical solutions.

3. Cylinder Under Pure-Bending

For a solid cylinder of radius "a" subjected to pure-bending, the coefficients of $K_n(r/\ell)$, r^{-n}, $\ell n\, r$ and θ, in Eq. (8) must vanish for a bounded and single valued solution at $r = 0$. Furthermore, the traction-free surface conditions require,

$$\tau_{rr} = \frac{1}{r} \phi_{,r} + \frac{1}{r^2} \phi_{,\theta\theta} - \left(\frac{\psi}{r}\right)_{,r\theta} = 0 \quad ,$$

$$\tau_{r\theta} = - \left(\frac{\psi}{r}\right)_{,r\theta} - \frac{1}{r} \psi_{,r} - \frac{1}{r^2} \psi_{,\theta\theta} = 0 \quad ,$$

$$\sigma_{r3} = \psi_{,r} = 0 \quad , \quad \text{on } r = a \quad . \tag{9}$$

The relations between the stresses and the stress functions are obtained from (5) through coordinate transformation, [2].

Substituting (8) into (9) we have

$$A_1 = \frac{4\mu(\nu + \eta)a^2\ell}{R}\left\{\frac{a}{\ell} I_1 \left(\frac{a}{\ell}\right) + \left[\left(\frac{a^2}{\ell^2}\right) + 8(1 - \nu)\right] I_2 \left(\frac{a}{\ell}\right)\right\}^{-1} ,$$

$$E_1 = A_1 \ I_2\left(\frac{a}{\ell}\right)/a^2\ell \quad , \tag{10}$$

and the remaining coefficients in (8) equal zero. From (4), (8) and (10), we obtain the following moment-curvature and the axial stress-curvature relations:

$$M_2 = \int_0^a \int_0^{2\pi} \left(- x_1 \ \tau_{33} + \sigma_{32}\right) r \ d\theta dr = - C_m \frac{EI}{R} \quad ,$$

$$\tau_{33} = C_t \frac{Ex_1}{R} \quad ,$$

where

$$C_t = 1 + \frac{8\mu(\nu + \eta)}{(1 + \nu)} \ I_2 \left(\frac{a}{\ell}\right)\left\{\frac{a}{\ell} I_1 \left(\frac{a}{\ell}\right) + \left[\frac{a^2}{\ell^2} + 8(1 - \nu)\right] I_2 \left(\frac{a}{\ell}\right)\right\}^{-1} \quad ,$$

$$C_m = 1 + 8(1 - \eta^2) \ \ell^2/(1 + \nu) \ a^2 + (C_t - 1) \ (1 - \eta/\nu) \quad ,$$

$$I = \pi \ a^4/4 \quad . \tag{11}$$

R is the radius of curvature of bending of the beam.

4. Numerical Results and Discussions

The axial stress coefficient, C_t, and the moment coefficient, C_m, defined in (11) reflect the effect of couple-stress on the pure bending of cylindrical bars. The values are not always equal to one as in the classical beam theory.

Numerical calculations of C_t and C_m for different Poisson's ratios and couple-stress parameters η and ℓ are illustrated in Figures 1 and 2, respectively. The variation of C_t as shown in Figure 1 is less than 17%, whereas the value of C_m varies with a^2/ℓ^2 at small a/ℓ. It has been suggested by Mindlin that the couple-stress theory starts to break down when a/ℓ decreases to an order of 1. Figure 2 indicates that C_m rapidly approaches infinity when a/ℓ approaches 1.

The apparent stress, $\tau_{33}^{(a)} = -M_2 \ x_1/I$, of the classical beam theory of linear elasticity can be related to the true stress, τ_{33}, of the couple-stress theory as the following

$$\tau_{33}^{(a)} = (C_m/C_t) \ \tau_{33} \quad . \tag{12}$$

For this reason the ratio of C_m/C_t is plotted as a function of η in Figure 3 and as a function of a/ℓ in Figure 4. The ratio is relatively insensitive to η for $-0.5 < \eta < 0.5$ for $a/\ell \geq 2.5$ as shown in Figure 3. This suggests that the apparent strengths of bend specimens can be used to estimate the value of ℓ if η falls into this range. A limited number of experiments performed on cylindrical specimens under bending, [9], indicate that the apparent bend strength of H-451 graphite with 6.35 mm diameter is about 1.7 times the apparent tensile strength. From Figure 4 we find $a/\ell = 3.2$ for $C_m/C_t = 1.7$ and $\eta = 0.0$. It yields ℓ

= 1.0 mm, which is indeed comparable to the maximum grain size, 1.5 mm, of H-451 graphite.

The couple stress parameters can be determined from moment-curvature measurements in pure bending tests on cylindrical specimens of different diameters. A curve for C_m versus a/ℓ can be established by the initial slopes (i.e., the apparent moduli) of the moment-curvature curves obtained from these measurements. If this curve is identifiable with one of those in Fig. 2, the values for ℓ and η can be obtained. The consistency of the theory can then be evaluated by using either Eq. (11) or, equivalently, any of Figures 1, 3, or 4.

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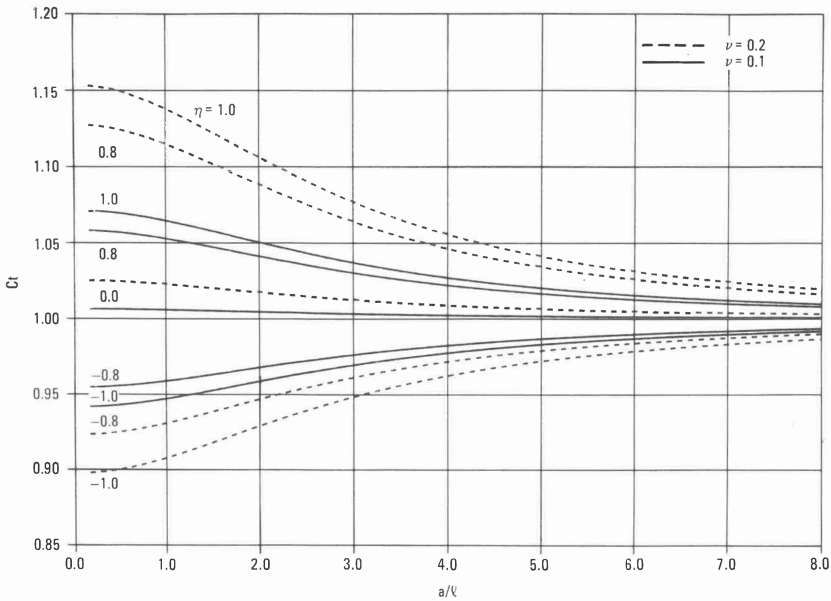

Fig. 1. The axial coefficient, C_t versus a/ℓ for different couple-stress parameters, η, and Poisson's ratios, ν.

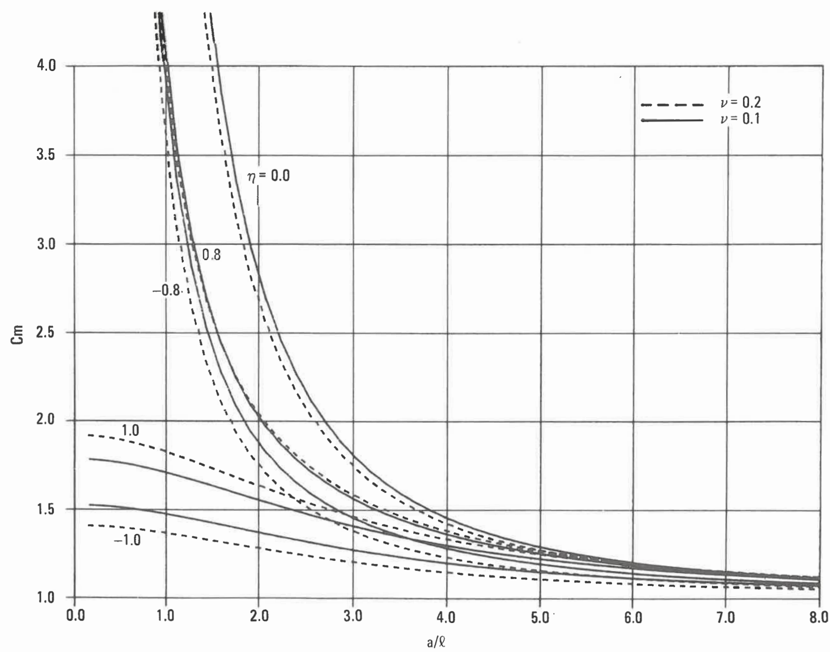

Fig. 2. The moment coefficient C_m versus a/ℓ for different couple-stress parameters, η, and Poisson's ratios, ν.

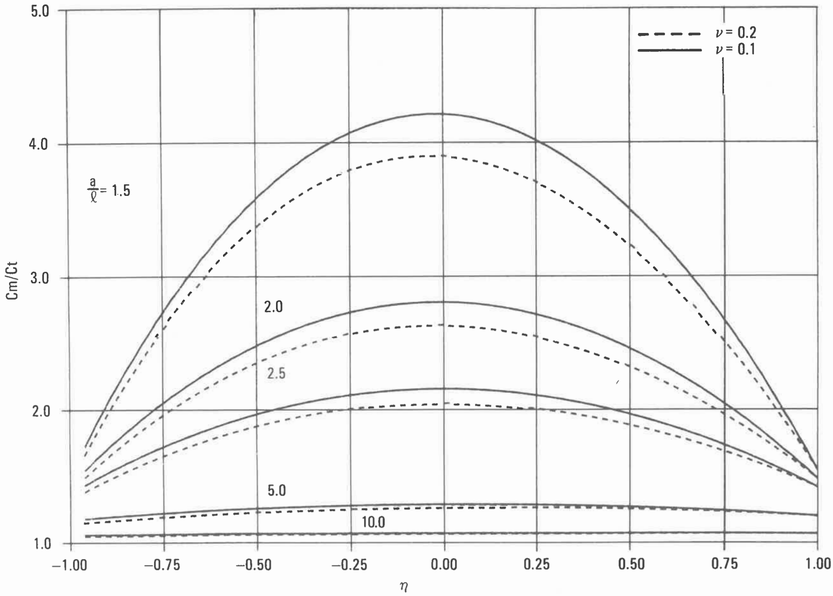

Fig. 3. The radio C_m/C_t versus η for different a/ℓ and ν.

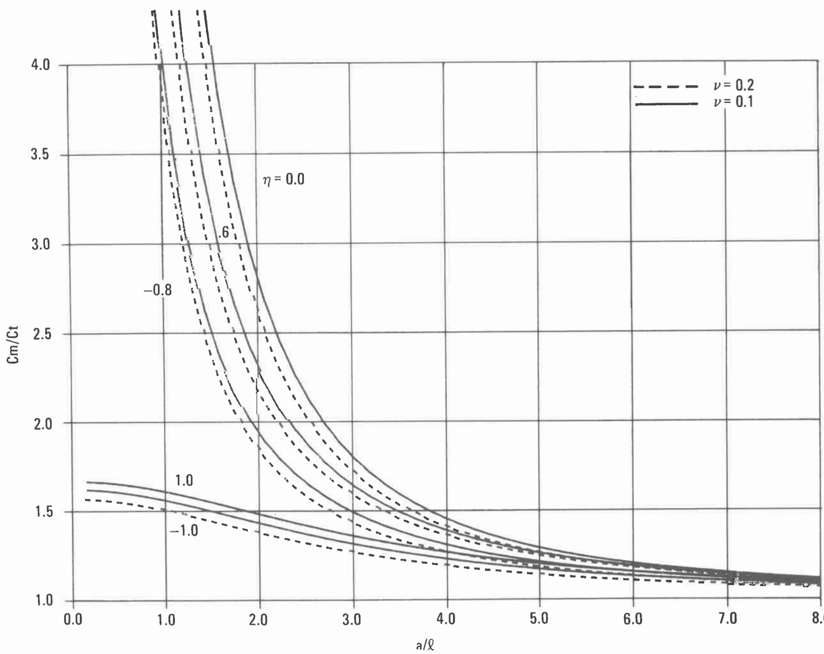

Fig. 4. The ratio C_m/C_t versus a/ℓ for different couple-stress parameters, η, and Poisson's Ratios, ν.