CONSTRUCTION OF BALANCED INCOMPLETE BLOCK DESIGNS FROM ASSOCIATION MATRICES

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In Chapter I, balanced incomplete block designs and m-class association schemes are defined. Necessary and sufficient conditions are found in Chapter II for a BIB incidence matrix to be (i) a linear combination of association matrices or (ii) a matrix formed by placing association matrices side by side (this is referred to as a juxtaposition of association matrices). In Chapters III and IV, various methods of constructing association schemes with two and three classes are discussed; using the results from Chapter II, the schemes are then examined to determine whether BIB designs can be constructed from them.

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LIST OF SYMBOLS

Symbol	Meaning
J_v	vxv matrix of 1's.
$J_{v,b}$	vxb matrix of 1's.
I_v	vxv identity matrix.
x'	transpose of the vector x.
N'	transpose of the matrix N.
$\|N\|$	determinant of the matrix N.

CHAPTER I

INTRODUCTION

1.1. Balanced Incomplete Block (BIB) Designs [3], [13].[1/]

A balanced incomplete block (BIB) design is an arrangement of v elements (objects, varieties, treatments) in b subsets, called blocks, such that

(i) each block contains exactly k elements, all distinct,

(ii) each element appears in exactly r blocks,

(iii) each pair of distinct elements occur together in exactly λ blocks. The numbers

$$v,\ b,\ r,\ k,\ \lambda \tag{1.1.1}$$

are called the parameters of the design; all must be positive integers. The parameters k and r are referred to as the block size and the number of replications, respectively. We shall refer to a BIB design with the parameters (1.1.1) as a BIB (v, b, r, k, λ).

It is easy to see that the parameters (1.1.1) must satisfy the relations

$$bk = vr \tag{1.1.2}$$

[1/] Numbers in square brackets refer to bibliography entries.

and

$$\lambda(v-1) = r(k-1) \quad . \tag{1.1.3}$$

Suppose α and β are two distinct treatments of a BIB design. Then α and β occur together in λ blocks, and α occurs in a total of r blocks; thus $\lambda \leq r$. In case $\lambda = r$, we see from (1.1.3) that $v = k$, and the design is a randomized blocks design; then for a "proper" BIB design, i.e., one that is not a randomized blocks design, we have

$$0 < \lambda < r \quad . \tag{1.1.4}$$

Now consider the vxb matrix $N = (\,(n_{ij})\,)$ $(i = 1,2, \ldots, v;$ $j = 1,2, \ldots, b)$, where

$$n_{ij} = \begin{cases} 1 & \text{if the } \underline{ith} \text{ treatment occurs in the } \underline{jth} \text{ block,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.5}$$

We shall call N the incidence matrix for the BIB (v, b, r, k, λ). It is clear from (1.1.5) that

$$N' J_v = k\, J_{b,v} \tag{1.1.6}$$

and

$$NN' = r\, I_v + \lambda(J_v - I_v) \quad . \tag{1.1.7}$$

Conversely, if (1.1.4) holds and if N is a vxb matrix of 0's and 1's satisfying (1.1.6) and (1.1.7), then N is an incidence matrix for a BIB (v, b, r, k, λ) [20]. Thus we have the following lemma.

Lemma 1.1.1

Suppose N is a vxb matrix of 0's and 1's, and suppose λ and r are integers such that $0 < \lambda < r$. Then the necessary and sufficient condition for N to be an incidence matrix for a BIB (v, b, r, k, λ) is that

(i) $N' J_v = k J_{b,v}$ and

(ii) $NN' = r I_v + \lambda(J_v - I_v)$.

Now from (1.1.7) we see that $|NN'| = rk(r-\lambda)^{v-1}$. Then (1.1.4) implies that $|NN'| > 0$ for a proper BIB design. Hence NN', which is a vxv matrix, has rank v. We know that the rank of N can not exceed b, the number of columns of N. Then $b \geq \text{rank } N = \text{rank } NN' = v$; thus for a proper BIB design, we have the inequality

$$(1.1.8) \qquad b \geq v \quad ,$$

which was first proved by R. A. Fisher [11].

For any mxn matrix A of 0's and 1's, the matrix $J_{m,n} - A$ shall be called the complement of A. We now prove the following lemma.

Lemma 1.1.2

Suppose N is an incidence matrix for a BIB (v, b, r, k, λ), and suppose $k < v-1$. Then the complement of N is an incidence matrix for a BIB $(v^*, b^*, r^*, k^*, \lambda^*)$, where

$$v^* = v,\ b^* = b,\ r^* = b-r,\ k^* = v-k, \quad \lambda^* = b-2r+\lambda.$$

<u>Proof:</u> Denote the complement of N by N^*; i.e.,

$$N^* = J_{v,b} - N. \text{ From (1.1.5), } N' J_v = k\, J_{b,v};$$

then

$$(N^*)' J_v = (J_{v,b} - N)' J_v = J_{b,v} J_v - kJ_{b,v} = (v-k)\, J_{b,v} = k^* J_{b,v}.$$

From (1.1.7),

$$NN' = r\, I_v + \lambda(J_v - I_v);$$

then

$$\begin{aligned} N^*(N^*)' &= (J_{v,b} - N)\,(J_{v,b} - N)' \\ &= J_{v,b}\, J_{b,v} - N\, J_{b,v} - J_{v,b}\, N' + NN' \\ &= b\, J_v - r\, J_v - r\, J_v + r\, I_v + \lambda(J_v - I_v) \\ &= (b - 2r + \lambda)\, J_v + (r-\lambda)\, I_v \\ &= (b-r)\, I_v + (b - 2r + \lambda)\,(J_v - I_v) \\ &= r^*\, I_v + \lambda^*(J_v - I_v). \end{aligned}$$

From (1.1.4) we see that $\lambda < r$; adding $(b-2r)$ to both sides of the inequality, we have $\lambda^* < r^*$. Now $\lambda^* = \dfrac{r^*(k^*-1)}{v^*-1} = \dfrac{(b-r)(v-k-1)}{v-1}$.

Then if $k < v-1$, $\lambda^* > 0$, and by Lemma 1.1.1 N^* is an incidence matrix for a BIB(v^*, b^*, r^*, k^*, λ^*).

(Note: The only case not allowed in Lemma 1.1.2, when $k = v-1$, gives a trivial example of a BIB design; some references, for example [20], omit the case $k = v-1$ altogether in considering BIB designs.)

A BIB design with $v = b$ and $r = k$ is known as a symmetric BIB design. In this paper we shall be concerned largely with such designs.

Now the relations (1.1.2) and (1.1.3) are necessary, but not in general sufficient, for the existence of a BIB(v, b, r, k, λ). However, Hanani has shown [14] that (i) for $k = 3,4$, and any λ or (ii) for $k = 5$ and $\lambda = 1,4,20$, the relations (1.1.2) and (1.1.3) are both necessary and sufficient for the existence of a BIB (v, b, r, k, λ).

Fisher and Yates [12] have tabled BIB designs with $r \leq 15$. Rao [19] has studied BIB designs with $r = 11$ to 15. Sprott [25] lists designs with $r = 16$ to 20. The literature contains a large number of articles showing the impossibility of certain BIB designs: for example, [21] and [22].

Numerous methods may be employed in the construction of BIB designs. A few examples are the method of differences [1], finite geometrics [1], construction from known BIB designs [3], orthogonal arrays [9], Hadamard matrices [23], and association schemes [23]. The last method has been used to construct some symmetric BIB designs. This work will be concerned with an analysis of methods of constructing BIB designs from association matrices. Two methods will be employed. The first, linear combinations of association matrices, is quite similar to a method suggested by Shrikhande and Singh in [23];

however, the result in the present work was arrived at independently. The second method, juxtaposition of association matrices, is believed to be new.

1.2. Association Schemes [2]

An association scheme in m associate classes is a set of v elements (objects, treatments, varieties) which satisfies the following conditions:

(i) any two treatments are either 1st, 2nd, ..., or mth associates, and the relation of association is symmetrical;

(ii) each element has exactly n_i ith associates ($i = 1,2,\ldots, m$), where the number n_i is independent of the element chosen;

(iii) if α and β are ith associates, then the number of elements which are jth associates of α and kth associates of β is p^i_{jk}, and p^i_{jk} is independent of the pair of ith associates chosen ($i, j, k = 1,2, \ldots, m$).

The numbers

$$(1.2.1) \qquad v,\ n_i,\ p^i_{jk}$$

are called the parameters of the association scheme; all must be positive integers.

The following relations among the parameters (1.2.1) are easily shown:

$$(1.2.2) \qquad p^i_{jk} = p^i_{kj}\ ;$$

$$(1.2.3) \qquad \sum_{i=1}^{m} n_i = v-1 \quad ;$$

$$(1.2.4) \qquad \sum_{k=1}^{m} p_{jk}^{i} = n_j \quad \text{if} \quad i \neq j \ ,$$

$$= n_j - 1 \text{ if } i = j \ ;$$

$$(1.2.5) \qquad n_i \, p_{jk}^{i} = n_j \, p_{ik}^{j} = n_k \, p_{ij}^{k} \ .$$

It is useful to make the convention that each element is the zero-th associate of itself and of no other elements. Then we must have

$$(1.2.6) \qquad n_o = 1 \ ;$$

$$(1.2.7) \qquad p_{ij}^{o} = p_{ji}^{o} = 0 \text{ if } i \neq j \ ,$$

$$= n_j \text{ if } i = j \ ;$$

$$(1.2.8) \qquad p_{ko}^{i} = p_{ok}^{i} = 0 \quad \text{if } i \neq k \ ,$$

$$= 1 \quad \text{if } i = k \ .$$

Then (1.2.3) and (1.2.4) become

$$(1.2.9) \qquad \sum_{i=o}^{m} n_i = v \ ,$$

$$(1.2.10) \qquad \sum_{k=o}^{m} p_{jk}^{i} = n_j \quad .$$

For the case $m = 2$, it is sufficient to specify v, n_1, p_{11}^1, and p_{11}^2, and the other parameters are then determined; see, for example, [2].

Given an m-class association scheme, we call the matrices B_i $(i = 0, 1, \ldots, m)$ the association matrices of the scheme, where

(1.2.11)
$$B_i = ((b_{\alpha i}^{\beta})) = \begin{bmatrix} b_{1i}^1 & b_{1i}^2 & \cdots & b_{1i}^v \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ b_{vi}^1 & b_{vi}^2 & \cdots & b_{vi}^v \end{bmatrix} \quad \text{and}$$

(1.2.12) $b_{\alpha i}^{\beta} = 1$ if α and β are ith associates,
$= 0$ otherwise.

Clearly, we have

(1.2.13) $$B_0 = I_v$$

and

(1.2.14) $$B_0 + B_1 + \cdots + B_m = J_v \, .$$

Also, the linear form $c_0 B_0 + c_1 B_1 + \cdots + c_m B_m$ is equal to the zero matrix if and only if $c_0 = c_1 = \cdots = c_m = 0$; i.e., the B_i's are linearly independent. The association matrices satisfy the relation

(1.2.15) $$B_k B_j = B_j B_k = p_{jk}^0 B_0 + p_{jk}^1 B_1 + \cdots + p_{jk}^m B_m$$
$$(j, k = 0, 1, \ldots, m) \, .$$

The result (1.2.15), along with the linear independence of the B_i's , will be quite important in the proofs of the theorems in Chapter II .

1.3. Partially Balanced Incomplete Block (PBIB) Designs [2]

Given an m-class association scheme of v treatments, a partially balanced incomplete block (PBIB) design is an arrangement of the v treatments in b blocks such that

(i) each block contains exactly k treatments, all distinct;

(ii) each treatment occurs in exactly r blocks ;

iii) if two treatments α and β are ith associates, they occur together in exactly λ_i blocks, where λ_i is independent of the choice of the particular pair of ith associates α and β $(i = 1, 2, \ldots, m)$.

Clearly, if $\lambda_1 = \lambda_2 = \ldots = \lambda_m$, the design is a BIB design.

PBIB designs were introduced by Bose and Nair [6]; a slightly generalized definition was given by Nair and Rao [16]. The definition in the preceding paragraph is that of Nair and Rao. Bose and Shimamoto [8] introduced the concept of association schemes and based the definition of PBIB designs on these schemes. The present work will not be concerned with PBIB designs, but they are mentioned because of their direct connection with association schemes.

CHAPTER II

THE PROBLEM AND FUNDAMENTAL RESULTS

The problem considered here is that of construction of BIB designs from the association matrices B_i by the following two methods:

(i) to obtain a matrix of the form $B_{i_1} + B_{i_2} + \dots + B_{i_t}$ which will be a BIB incidence matrix, and

(ii) to obtain a matrix of the form $[\, B_{i_1} \vdots B_{i_2} \vdots \dots \vdots B_{i_s} \,]$ which will be a BIB incidence matrix.

A matrix of type (i) will be referred to as a linear combination of association matrices, and a matrix of type (ii) will be called a juxtaposition of association matrices. The two theorems which follow give the necessary and sufficient conditions for (i) and (ii).

Theorem 2.1 (Linear Combinations)

Suppose we have an m-class association scheme in v elements, with association matrices $B_o = I_v, B_1, B_2, \dots, B_m$. Suppose $i_1, i_2, \dots, i_t$ are distinct integers such that $i_j \in (0,1,2,\dots,m)$ for $j = 1,2, \dots, t \leq m$. Then the necessary and sufficient condition for $C = B_{i_1} + B_{i_2} + \dots + B_{i_t}$ to be an incidence matrix for a BIB $(v, v, r, r, \lambda^{(i_1,i_2,\dots,i_t)})$, where

$$r = n_{i_1} + n_{i_2} + \dots + n_{i_t}$$

and

$$\lambda^{(i_1, i_2, \ldots, i_t)} = \frac{r(r-1)}{v-1} ,$$

is that $\lambda^{(i_1, i_2, \ldots, i_t)}$ be a positive integer and $\sum_{j=1}^{t} p^k_{i_j i_j}$ + $2 \sum_{j<\ell}^{t} p^k_{i_j i_\ell} = \lambda^{(i_1, i_2, \ldots, i_t)}$ for $k = 1, 2, \ldots, m$.

Proof: (For convenience, we shall denote $\lambda^{(i_1, i_2, \ldots, i_t)}$ by λ during the proof.) Suppose λ is a positive integer. Then the required necessary and sufficient condition is that

$$(1) \quad C J_v = r J_v$$

and

$$(2) \quad CC' = (r-\lambda) I_v + \lambda J_v = r I_v + \lambda(J_v - I_v).$$

Now

$$C J_v = (B_{i_1} + B_{i_2} + \ldots + B_{i_t}) J_v$$

$$= n_{i_1} J_v + n_{i_2} J_v + \ldots + n_{i_t} J_v$$

$$= r J_v .$$

Then (1) is always satisfied when the B_i's are association matrices.

$$CC' = (B_{i_1} + B_{i_2} + \ldots + B_{i_t}) (B_{i_1} + \ldots + B_{i_t})'$$

$$= B_{i_1} B_{i_1} + B_{i_2} B_{i_2} + \ldots + B_{i_t} B_{i_t} + 2 \sum_{j<\ell}^{t} B_{i_j} B_{i_\ell}$$

$$= \sum_{k=o}^{m} p_{i_1 i_1}^{k} B_k + \sum_{k=o}^{m} p_{i_2 i_2}^{k} B_k + \ldots + \sum_{k=o}^{m} p_{i_t i_t}^{k} B_k$$

$$+ 2 \sum_{k=o}^{m} \sum_{j<\ell}^{t} p_{i_j i_\ell}^{k} B_k .$$

Since $p_{ii}^{o} = n_i$ and $p_{ij}^{o} = 0$ for $i \neq j$, we have

$$CC' = r I_v + \left(\sum_{j=1}^{t} p_{i_j i_j}^{1} + 2 \sum_{j<\ell}^{t} p_{i_j i_\ell}^{1} \right) B_1$$

$$+ \left(\sum_{j=1}^{t} p_{i_j i_j}^{2} + 2 \sum_{j<\ell}^{t} p_{i_j i_\ell}^{2} \right) B_2$$

$$+ \ldots + \left(\sum_{j=1}^{t} p_{i_j i_j}^{m} + 2 \sum_{j<\ell}^{t} p_{i_j i_\ell}^{m} \right) B_m .$$

Now

$$\lambda (B_1 + B_2 + \ldots + B_m) = \lambda (J_v - I_v) .$$

Then, since the B_i's are linearly independent, the necessary and sufficient condition for (2) to hold is that

$$\sum_{j=1}^{t} p_{i_j i_j}^{k} + 2 \sum_{j<\ell}^{t} p_{i_j i_\ell}^{k} = \lambda, \quad k = 1, 2, \ldots, m .$$

Theorem 2.2 (Juxtaposition)

Suppose we have an m-class association scheme in v elements, with association matrices $B_o = I_v, B_1, B_2, \ldots, B_m$. Suppose

$i_1, i_2, \ldots, i_t$ are distinct integers such that $i_j \epsilon (1,2,\ldots, m)$ for $j = 1, 2, \ldots, t \leq m$. Then the necessary and sufficient condition for $D = [\, B_{i_1} \vdots B_{i_2} \vdots \ldots \vdots B_{i_t} \,]$ to be an incidence matrix for a BIB $(v, tv, tk, k, \lambda_{i_1 i_2 \ldots i_t})$, where

$$\lambda_{i_1 i_2 \ldots i_t} = \frac{tk(k-1)}{v-1}, \text{ is that}$$

(i) $\lambda_{i_1 i_2 \ldots i_t}$ be a positive integer ,

(ii) $n_{i_1} = n_{i_2} = \ldots = n_{i_t} = k$,

(iii) $\sum_{j=1}^{t} p^1_{i_j i_j} = \sum_{j=1}^{t} p^2_{i_j i_j} = \ldots = \sum_{j=1}^{t} p^m_{i_j i_j} = \lambda_{i_1 i_2 \ldots i_t}$.

Proof: (For convenience, we shall denote $\lambda_{i_1 i_2 \ldots i_t}$ by λ.) Suppose λ is a positive integer, and suppose $n_{i_1} = n_{i_2} = \ldots = n_{i_t} = k$. (It is obvious that (i) and (ii) must be satisfied for D to be a BIB incidence matrix.) Then the required necessary and sufficient condition is that

(1) $D' J_v = k \; J_{tv,v}$

and

(2) $DD' = (tk - \lambda) I_v + \lambda J_v$

$$= tk\, I_v + \lambda(J_v - I_v).$$

Now

$$D'J_v = \begin{vmatrix} B_{i_1} \\ \cdots \\ B_{i_2} \\ \cdots \\ \vdots \\ \cdots \\ B_{i_t} \end{vmatrix} J_v = \begin{vmatrix} kJ_v \\ \cdots \\ kJ_v \\ \cdots \\ \vdots \\ \cdots \\ kJ_v \end{vmatrix} = k\, J_{tv,v} \;.$$

Then (1) is always satisfied.

$$DD' = \left[B_{i_1} \vdots B_{i_2} \vdots \cdots \vdots B_{i_t} \right] \begin{vmatrix} B_{i_1} \\ \cdots \\ B_{i_2} \\ \cdots \\ \vdots \\ \cdots \\ B_{i_t} \end{vmatrix}$$

$$= B_{i_1} B_{i_1} + B_{i_2} B_{i_2} + \cdots + B_{i_t} B_{i_t} = tk\, I_v$$

$$+ \sum_{c=1}^{m} p^c_{i_1 i_1} B_c + \sum_{c=1}^{m} p^c_{i_2 i_2} B_c + \cdots + \sum_{c=1}^{m} p^c_{i_t i_t} B_c$$

$$= tk\, I_v + \Big(\sum_{j=1}^{t} p^1_{i_j i_j}\Big) B_1 + \Big(\sum_{j=1}^{t} p^2_{i_j i_j}\Big) B_2 + \cdots + \Big(\sum_{j=1}^{t} p^m_{i_j i_j}\Big) B_m \;.$$

Now

$$\lambda(B_1 + B_2 + \cdots + B_m) = \lambda(J_v - I_v) \;.$$

Then, by the linear independence of the B_i's, the necessary and sufficient condition for (2) to hold is that

$$\sum_{j=1}^{t} p^{1}_{i_j i_j} = \sum_{j=1}^{t} p^{2}_{i_j i_j} = \cdots = \sum_{j=1}^{t} p^{m}_{i_j i_j} = \lambda .$$

If we want an m-class association scheme in v elements such that $[\, B_{i_1} \vdots\, B_{i_2} \vdots\, \cdots \vdots\, B_{i_t} \,]$ and $[\, B_{j_1} + B_{j_2} + \cdots + B_{j_s} \,]$ are both BIB incidence matrices, where $s \leq t$ and the B_j's form a subset of the B_i's, then the necessary and sufficient condition is found by combining the conditions from Theorems 2.1 and 2.2.

If we want the scheme such that $[\, B_i \vdots\, B_j \,]$ and $[B_i + B_j]$ are both BIB incidence matrices, we have

$$\lambda_{ij} = \frac{2k(k-1)}{v-1} \quad \text{and} \quad \lambda^{(ij)} = \frac{2k(2k-1)}{v-1}, \text{ where } n_i = n_j = k.$$

Note that $\lambda^{(ij)} - 2\,\lambda_{ij} = \frac{2k}{v-1}$; but $2k < v-1$ if the scheme has more than two associate classes, and in any event $2k$ is not greater than $v-1$. Then $(\lambda^{(ij)} - 2\,\lambda_{ij})$ is an integer only if $2k=n_1+n_2=v-1$. In this case there are only two associate classes and $B_1 + B_2$ is trivially an incidence matrix for a BIB, since $B_1+B_2 = J_v - I_v$.

In Theorem 2.1, let $t = 2$ and $m = 3$. Then we see that the necessary and sufficient condition for $B_i + B_j$ to be a BIB incidence matrix is that $\lambda^{(ij)}$ be an integer and

$$p^{1}_{ii} + p^{1}_{jj} + 2\,p^{1}_{ij} = p^{2}_{ii} + p^{2}_{jj} + 2\,p^{2}_{ij}$$

$$= p^{3}_{ii} + p^{3}_{jj} + 2\,p^{3}_{ij} = \lambda^{(ij)} .$$

Letting t = 1 in Theorem 2.1, we get the condition for B_i to be a BIB incidence matrix; viz., that

$$p_{ii}^{1} = p_{ii}^{2} = \ldots = p_{ii}^{m} = \lambda^{(i)},$$

where $\lambda^{(i)}$ is a positive integer. The following corollary shows that in this case it is enough to show that $p_{ii}^{1} = p_{ii}^{2} = \ldots = p_{ii}^{m}$; the common value must then be $\lambda^{(i)}$. Shrikhande and Singh [23] proved that the condition in Theorem 2.1 allows an m-class scheme to be collapsed into a two-class scheme such that $p_{11}^{1} = p_{11}^{2}$ or $p_{22}^{1} = p_{22}^{2}$. However, Theorem 2.1 and the following corollary were proved independently.

<u>Corollary 2.3</u> (to Theorem 2.1)

For an m-class association scheme in v elements with association matrices $B_o = I_v, B_1, B_2, \ldots, B_m$, the necessary and sufficient condition for B_i to be an incidence matrix for a BIB$(v,v,n_i,n_i,\lambda^{(i)})$, where $\lambda^{(i)} = \dfrac{n_i(n_i-1)}{v-1}$, is that

$$p_{ii}^{1} = p_{ii}^{2} = \ldots = p_{ii}^{m} = x, \quad \text{for some positive integer } x.$$

<u>Proof:</u> Suppose $p_{ii}^{1} = p_{ii}^{2} = \ldots = p_{ii}^{m}$ for some $i \in (1,2,\ldots,m)$; let x be the common value. Now

$$\sum_{j=1}^{m} p_{ij}^{i} = n_i - 1, \quad \text{since} \quad p_{io}^{i} = 1;$$

also,

$$n_i\, p_{ij}^{i} = n_j\, p_{ii}^{j}, \quad \text{or} \quad p_{ij}^{i} = \frac{n_j}{n_i}\, p_{ii}^{j} \quad .$$

Then
$$\sum_{j=1}^{m} p_{ij}^{i} = \sum_{j=1}^{m} \frac{n_j}{n_i} p_{ii}^{j} ,$$

and we have $n_1 p_{ii}^{1} + n_2 p_{ii}^{2} + \ldots + n_m p_{ii}^{m} = n_i(n_i - 1)$, or $x(n_1 + n_2 + \ldots + n_m) = n_i(n_i - 1)$. Since

$$n_1 + n_2 + \ldots + n_m = v\text{-}1, \quad x = \frac{n_i(n_i-1)}{v-1} = \lambda^{(i)} ,$$

and the condition in Theorem 1 is satisfied if x is a positive integer.

In Theorem 2.2, taking $t = 2$ and $m = 3$, we see that the condition for $[\, B_i \vdots B_j \,]$ to b a BIB incidence matrix is that $n_i = n_j = k$ (say), $\lambda_{ij} = \frac{2k(k-1)}{v-1}$ be a positive integer, and $p_{ii}^{1} + p_{jj}^{1} = p_{ii}^{2} + p_{jj}^{2} = p_{ii}^{3} + p_{jj}^{3} = \lambda_{ij}$.

For the case $t = 3$ and $m = 3$, we must have $n_1 = n_2 = n_3 = k$, $\lambda_{123} = \frac{3k(k-1)}{v-1}$ a positive integer, and

$$p_{11}^{1} + p_{22}^{1} + p_{33}^{1} = p_{11}^{2} + p_{22}^{2} + p_{33}^{2} = p_{11}^{3} + p_{22}^{3} + p_{33}^{3} = \lambda_{123}.$$

In this case $3k = v\text{-}1$ and $\lambda_{123} = k\text{-}1$.

If $t = 2$ and $m = 2$, the condition for $[\, B_1 \vdots B_2 \,]$ to be a BIB incidence matrix is that $n_1 = n_2$ and $p_{11}^{1} + p_{22}^{1} = p_{11}^{2} + p_{22}^{2} = \lambda_{12}$. For this case,

$$\lambda_{12} = \frac{(n_1+n_2)(n_1-1)}{n_1+n_2} = n_1\text{-}1 \text{ (equivalently, } \lambda_{12} = n_2\text{-}1).$$

In Chapters III and IV various methods of constructing association schemes of two and three associate classes are discussed. These association schemes are examined for their possible use in constructing BIB designs in the special cases considered in Theorems 2.1 and 2.2. Whenever the parameter $\lambda^{(i)}$ or $\lambda^{(ij)}$ appears it will be understood to be defined as in Theorem 2.1; a parameter of the form λ_{ij} or λ_{ijk} is defined as in Theorem 2.2.

CHAPTER III

TWO-CLASS ASSOCIATION SCHEMES

3.1. Introduction

Several methods of constructing two-class association schemes will be discussed. The parameters resulting from each method will be examined in order to determine whether

(a) B_1,

(b) B_2,

(c) $[B_1 \vdots B_2]$

can be BIB incidence matrices. The methods discussed can be found in [2], [8], and [15]. The necessary and sufficient conditions in each case are as follows, from Theorems 2.1 and 2.2:

(a) $p^1_{11} = p^2_{11}$;

(b) $p^1_{22} = p^2_{22}$;

(c) $n_1 = n_2$ and $p^1_{11} + p^1_{22} = p^2_{11} + p^2_{22} = n_1 - 1$.

We note that if $N_{v,b}$ is a BIB incidence matrix, then the pairs $\binom{0}{0}$ must occur an equal number of times for any two rows, as well as the pairs $\binom{1}{1}$. Then the complement of N, or $J_{v,b} - N$, is also a BIB incidence matrix. Thus it is unnecessary to consider the combinations $B_o + B_1$ and $B_o + B_2$, since $B_o + B_1 = J_v - B_2$ and $B_o + B_2 = J_v - B_1$.

3.2. Group Divisible (GD) Scheme [8]

3.2.1. The scheme

Suppose, for integers $\ell \geq 2$ and $n \geq 2$, there is a set of $v = \ell n$ elements. Let the elements be arranged in a rectangular array with ℓ rows and n columns. Call any two elements which appear together in a row first associates; if two elements are in different rows they are second associates. Then the rectangular array gives us a two-class association scheme with the following parameters.

$$v = \ell n$$

$$n_1 = n-1$$

$$n_2 = n(\ell-1)$$

$$(3.2.1)\quad p^1_{11} = n-2 \qquad p^2_{11} = 0$$

$$p^1_{12} = 0 \qquad p^2_{12} = n-1$$

$$p^1_{22} = n(\ell-1) \qquad p^2_{22} = n(\ell-2)$$

3.2.2. Designs from B_1

We need $p^1_{11} = p^2_{11}$; but $p^2_{11} = 0$. Then B_1 can not be a BIB incidence matrix.

3.2.3. Designs from B_2

The condition is that $p^1_{22} = p^2_{22}$; then $n(\ell-1) = n(\ell-2)$, and $n=0$. Thus B_2 can not be a BIB incidence matrix.

3.2.4. Designs from [$B_1 \vdots B_2$]

We must have $n_1 = n_2$ and $p^1_{11} + p^1_{22} = p^2_{11} + p^2_{22} = n_1 - 1$. But $n_1 = n_2$ for $n - 1 = n(\ell - 1)$, or $\ell = \frac{2n-1}{n}$. There are no integers $\ell, n \geq 2$ which satisfy the above; thus [$B_1 \vdots B_2$] can not be a BIB incidence matrix.

Then GD schemes can not be used to construct any BIB design such that B_1, B_2, or [$B_1 \vdots B_2$] is the incidence matrix.

3.3. Triangular Association Scheme [8]

3.3.1. Definition of the scheme

Suppose, for some positive integer n, there is a set of $v = \binom{n}{2} = \frac{n(n-1)}{2}$ elements. Arrange the v elements in an nxn array as follows: leave the leading diagonal positions blank, and fill the $\frac{n(n-1)}{2}$ positions above the leading diagonal with the v elements; fill the remaining $\frac{n(n-1)}{2}$ positions so as to make the array symmetric with respect to the diagonal. Define first associates as two elements which appear in the same row (equivalently, the same column) of the resulting array; if two treatments do not appear in the same row, they are second associates.

The v elements might also be considered as unordered pairs (i,j), where $i \neq j$ and $i,j = 0,1, \ldots, n-1$. Then two elements are first associates if they differ in exactly one coordinate; otherwise they are second associates.

Such an array is an association scheme, called a triangular association scheme. The parameters of the scheme are as follows.

$$v = \frac{n(n-1)}{2}$$

$$n_1 = 2n - 4$$

$$n_2 = \frac{(n-2)(n-3)}{2}$$

(3.3.1)
$$p^1_{11} = n-2 \qquad p^2_{11} = 4$$
$$p^1_{12} = n-3 \qquad p^2_{12} = 2n - 8$$
$$p^1_{22} = \frac{(n-3)(n-4)}{2} \qquad p^2_{22} = \frac{(n-4)(n-5)}{2}$$

From the parameter values, we see that $n \geq 4$. An example of a triangular scheme, with n=4, is the following.

x 1 2 3
1 x 4 5
2 4 x 6
3 5 6 x

Element	First associates	Second associates
1	2 3 4 5	6
2	1 3 4 6	5
3	1 2 5 6	4
4	1 2 5 6	3
5	1 3 4 6	2
6	2 3 4 5	1

An examination of the parameters (3.3.1) shows that in one case B_1 will be a BIB incidence matrix; B_2 and $[B_1 \vdots B_2]$ can never be BIB incidence matrices.

3.3.2. Designs from B_1.

We must have $p^1_{11} = p^2_{11}$; then n=6 and $p^1_{11} = p^2_{11} = 4$. So for n=6, B_1 is a BIB incidence matrix. The parameters and scheme for this case are the following.

$v = 15$

$n_1 = 8$

$n_2 = 6$

$p^1_{11} = 4$ $\qquad$ $p^2_{11} = 4$

$p^1_{12} = 3$ $\qquad$ $p^2_{12} = 4$

$p^1_{22} = 3$ $\qquad$ $p^2_{22} = 1$

x	1	2	3	4	5
1	x	6	7	8	9
2	6	x	10	11	12
3	7	10	x	13	14
4	8	11	13	x	15
5	9	12	14	15	x

Element	First associates								Second associates					
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1	3	4	5	6	10	11	12	7	8	9	13	14	15
3	1	2	4	5	7	10	13	14	6	8	9	11	12	15
4	1	2	3	5	8	11	13	15	6	7	9	10	12	14
5	1	2	3	4	9	12	14	15	6	7	8	10	11	13
6	1	2	7	8	9	10	11	12	3	4	5	13	14	15
7	1	3	6	8	9	10	13	14	2	4	5	11	12	15
8	1	4	6	7	9	11	13	15	2	3	5	10	12	14
9	1	5	6	7	8	12	14	15	2	3	4	10	11	13
10	2	3	6	7	11	12	13	14	1	4	5	8	9	15
11	2	4	6	8	10	12	13	15	1	3	5	7	9	14
12	2	5	6	9	10	11	14	15	1	3	4	7	8	13
13	3	4	7	8	10	11	14	15	1	2	5	6	9	12
14	3	5	7	9	10	12	13	15	1	2	4	6	8	11
15	4	5	8	9	11	12	13	14	1	2	3	6	7	10

B_1 is an incidence matrix for a BIB(15, 15, 8, 8, 4).

3.3.3. Designs from B_2

The condition is that $p^1_{22} = p^2_{22}$; then

$$\frac{(n-3)(n-4)}{2} = \frac{(n-4)(n-5)}{2} , \quad \text{or} \quad n=4.$$

But in this case the common value of p^1_{22} and p^2_{22} is 0. Then B_2 can be a BIB incidence matrix.

3.3.4. Designs from $[\, B_1 \vdots B_2 \,]$

We must have $n_1 = n_2$ and $p^1_{11} + p^1_{22} = p^2_{11} + p^2_{22} = \lambda_{12}$. Now $n_1 = n_2$ for $2(n-2) = \frac{(n-2)(n-3)}{2}$, or $n=7$. Then $n_1 = n_2 = 10$ and $v=21$; $\lambda_{12} = \frac{2(10)(9)}{20} = 9$. For $n=7$, $p^1_{11} + p^1_{22} = 5 + \frac{4(3)}{2} = 11 \neq \lambda_{12}$. Then $[\, B_1 \vdots B_2 \,]$ can not be a BIB incidence matrix.

3.4. Pseudo-Cyclic Association Scheme [8], [15]

3.4.1. Definition

Suppose there is a set of v elements; denote them by the integers 1, 2, ..., v. Suppose there is a set of integers $(d_1, d_2, \ldots, d_{n_1})$ satisfying the following conditions:

(i) the d's are distinct, and $0 < d_j < v$ $(j = 1,2,\ldots, n_1)$;

(ii) among the $n_1(n_1-1)$ differences $d_i - d_j (i \neq j;\ i,j=1,2,\ldots, n_1)$ reduced (mod v), each of the numbers $d_1, d_2, \ldots, d_{n_1}$ occurs α times and each of the numbers $e_1, e_2, \ldots, e_{n_2}$ occurs β times, where $d_1, d_2, \ldots, d_{n_1}, e_1, e_2, \ldots, e_{n_2}$ are all the integers 1, 2, ..., v-1. Clearly, $n_1 \alpha + n_2 \beta = n_1(n_1-1)$.

Given the element k (k = 1, 2, ..., v), define its first associates as the elements $k+d_1$, $k+d_2$, ..., $k+d_{n_1}$ (mod v); the remaining $(v-n_1-1)$ elements are the second associates of k. Then we have an association shceme, called a cyclic association scheme, with the following parameters.

$$
\begin{array}{lll}
 & v & \\
 & n_1 & \\
 & n_2 = v-n_1-1 & \\
(3.4.1) & p^1_{11} = \alpha & p^2_{11} = \beta \\
 & p^1_{12} = n_1-\alpha-1 & p^2_{12} = n_1-\beta \\
 & p^1_{22} = n_2-n_1+\alpha+1 & p^2_{22} = n_2-n_1+\beta-1
\end{array}
$$

We see that, given v, the set of d's completely determines such a scheme. A few examples of cyclic association schemes are given below.

Some Cyclic Association Schemes

v	n_1	n_2	Set of d's
13	6	6	2, 5, 6, 7, 8, 11
17	8	8	3, 5, 6, 7, 10, 11, 12, 14
29	14	14	1, 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28

All the known cyclic association schemes are such that $v = 4u+1$, $n_1 = n_2 = 2u$, and $\alpha = u-1$, for some positive integer u. Then the association scheme has the following parameters.

$$
\begin{array}{lll}
 & v = 4u + 1 & \\
(3.4.2) & n_1 = n_2 = 2u & \\
 & p^1_{11} = u-1 & p^2_{11} = u
\end{array}
$$

$$p^1_{12} = u \qquad p^2_{12} = u$$

$$p^1_{22} = u \qquad p^2_{22} = u-1$$

Following the nomenclature in [15], we will call any association scheme satisfying the parameters (3.4.2) pseudo-cyclic, whether or not it is obtainable by the cyclic method described in [8].

Let us examine the parameters (3.4.2) to determine whether B_1, B_2, or $[\, B_1 \vdots B_2 \,]$ can be a BIB incidence matrix.

3.4.2. Designs from B_1

We must have $p^1_{11} = p^2_{11}$; but $p^1_{11} = u-1$ and $p^2_{11} = u$. Then B_1 can not be a BIB incidence matrix.

3.4.3. Designs from B_2

We need $p^2_{11} = p^2_{22}$; but $p^2_{11} = u$ and $p^2_{22} = u-1$. Then B_2 can not be a BIB incidence matrix.

3.4.4. Designs from $[\, B_1 \vdots B_2 \,]$

The condition is that $n_1 = n_2$ and $p^1_{11} + p^1_{22} = p^2_{11} + p^2_{22} = \lambda_{12} = n_1 - 1$. Now $n_1 = n_2 = 2u$ for any pseudo-cyclic scheme, and $p^1_{11} + p^1_{22} = 2u-1 = p^2_{11} + p^2_{22}$. Thus for any pseudo-cyclic association scheme, with parameters (3.4.2), $[\, B_1 \vdots B_2 \,]$ is an incidence matrix for a BIB $(4u+1,\ 2(4u+1),\ 4u,\ 2u,\ 2u-1)$. An example is the following:

$$u = 2,\ v = 9,\ n_1 = n_2 = 4 .$$

(Note that this scheme is not obtainable by the cyclic method.)

Element	First associates	Second associates
1	2 3 4 5	6 7 8 9
2	1 3 6 7	4 5 8 9
3	1 2 8 9	4 5 6 7
4	1 5 6 8	2 3 7 9
5	1 4 7 9	2 3 6 8
6	2 4 7 8	1 3 5 9
7	2 5 6 9	1 3 4 8
8	3 4 6 9	1 2 5 7
9	3 5 7 8	1 2 4 6

$p^1_{11} = 1 \qquad p^2_{11} = 2$

$p^1_{12} = 2 \qquad p^2_{12} = 2$

$p^1_{22} = 2 \qquad p^2_{22} = 1$

$$[B_1 \vdots B_2] = \left[\begin{array}{ccccccccc|ccccccccc}
0&1&1&1&1&0&0&0&0 & 0&0&0&0&0&1&1&1&1\\
1&0&1&0&0&1&1&0&0 & 0&0&0&1&1&0&0&1&1\\
1&1&0&0&0&0&0&1&1 & 0&0&0&1&1&1&1&0&0\\
1&0&0&0&1&1&0&1&0 & 0&1&1&0&0&0&1&0&1\\
1&0&0&1&0&0&1&0&1 & 0&1&1&0&0&1&0&1&0\\
0&1&0&1&0&0&1&1&0 & 1&0&1&0&1&0&0&0&1\\
0&1&0&0&1&1&0&0&1 & 1&0&1&1&0&0&0&1&0\\
0&0&1&1&0&1&0&0&1 & 1&1&0&0&1&0&1&0&0\\
0&0&1&0&1&0&1&1&0 & 1&1&0&1&0&1&0&0&0
\end{array}\right]$$

$[B_1 \vdots B_2]$ is a incidence matrix for a BIB(9, 18, 8, 4, 3).

3.5. Singly Linked Block (SLB) Association Scheme [2], [8]

3.5.1. The scheme

Suppose N' is an incidence matrix for a BIB design with b treatments, v blocks, k replications, block size r, and $\lambda = 1$;i.e. every pair of treatments occurs together in exactly one block. Then $bk = vr$ and $b-1 = k(r-1)$; this gives us $v = \frac{k(rk-k+1)}{r}$ and $b=rk-k+1$.

It has been shown that in this case N is an incidence matrix for a PBIB design with v treatments, b blocks, r replications, k plots per block, $\lambda_1 = 1$, and $\lambda_2 = 0$. Defining first associates as two treatments which appear together in some block of the derived PBIB design, we get a two-class association scheme called a singly linked block (SLB) association scheme, with the following parameters.

$$v = \frac{k(rk-k+1)}{r}$$

$$n_1 = r(k-1)$$

$$n_2 = \frac{(k-r)(r-1)(k-1)}{r} \tag{3.5.1}$$

$$p^1_{11} = k-2+(r-1)^2 \qquad p^2_{11} = r^2$$

$$p^1_{12} = (r-1)(k-r) \qquad p^2_{12} = r(k-r-1)$$

$$p^1_{22} = \frac{(r-1)(k-r)(k-r-1)}{r} \qquad p^2_{22} = (k-r)^2 + 2(r-1)\frac{k(k-1)}{r}$$

For $r=2$ the SLB scheme is the same as the triangular scheme with $m = k+1$.

An examination of the parameters (3.5.1) shows that one series of SLB schemes is such that one of the association matrices is a BIB incidence matrix.

3.5.2. Designs from B_1

The condition is that $p^1_{11} = p^2_{11}$; then $(k-2) + (r-1)^2 = r^2$, or $k = 2r + 1$. In this case the SLB scheme has the following parameters, in terms of r.

(3.5.2)

$$v = (2r-1)(2r+1)$$
$$n_1 = 2r^2$$
$$n_2 = 2(r-1)(r+1)$$
$$p^1_{11} = r^2 \qquad p^2_{11} = r^2$$
$$p^1_{12} = (r-1)(r+1) \qquad p^2_{12} = r^2$$
$$p^1_{22} = (r-1)(r+1) \qquad p^2_{22} = r^2-3$$

Then B_1 is an incidence matrix of a BIB $(4r^2-1,\ 4r^2-1,\ 2r^2,\ 2r^2,\ r^2)$. In this case the original BIB of interest, from which the SLB scheme is obtainable, has $r(2r-1)$ treatments, $(4r^2-1)$ blocks, $(2r+1)$ replications, r plots per block, and $\lambda = 1$.

3.5.3. Designs from B_2

We must have $p^1_{22} = p^2_{22}$; hence

$$\frac{(r-1)(k-r)(k-r-1)}{r} = (k-r)^2 + 2(r-1) - \frac{k(k-1)}{r}\ , \text{ or } k = 2r-1.$$

In this case the SLB scheme has the following parameters.

(3.5.3)

$$v = \frac{(2r-1)(2r^2-3r+2)}{r}$$
$$n_1 = 2r(r-1)$$
$$n_2 = \frac{2(r-1)^3}{r}$$
$$p^1_{11} = r^2-2 \qquad p^2_{11} = r^2$$
$$p^1_{12} = (r-1)^2 \qquad p^2_{12} = r(r-2)$$
$$p^1_{22} = \frac{(r-1)^2(r-2)}{r} \qquad p^2_{22} = \frac{(r-1)^2(r-2)}{r}$$

But note that in (3.5.3), p_{22}^1 and p_{22}^2 can never be integers for $r > 2$, and for $r = 2$ the common value is 0. Then B_2 can never be a BIB incidence matrix.

3.5.4. Designs from $[B_1 \vdots B_2]$

We need $n_1 = n_2$ and $p_{11}^1 + p_{22}^1 = p_{11}^2 + p_{22}^2 = \lambda_{12}$. Now $n_1 = n_2$ for $r(k-1) = \frac{(k-r)(r-1)(k-1)}{r}$, or $k = \frac{r(2r-1)}{r-1}$. In this case $n_1 = n_2 = \frac{r(2r^2-2r+1)}{(r-1)}$ and $\lambda_{12} = \frac{2r^3-2r^2+1}{(r-1)}$. Now $p_{11}^1 + p_{22}^1 =$ $k-2 + (r-1)^2 + \frac{(r-1)(k-r)(k-r-1)}{r} = \frac{2r^3-2r^2+r+1}{r-1}$ for $k = \frac{r(2r-1)}{r-1}$; then $p_{11}^1 + p_{22}^1 = \lambda_{12}$ when $2r^3-2r^2+1 = 2r^3-2r^2 + r+1$, i.e., for $r = 0$. But there can be no such association scheme, and so $[B_1 \vdots B_2]$ can not be a BIB incidence matrix.

3.6. Latin Square ($L_g(n)$) and Pseudo-Latin Square Association Schemes [2], [8], [15]

3.6.1. Definition of the scheme

Suppose we have a set of $v = n^2$ elements, arranged in an nxn array. Letting two elements which appear in the same row or the same column be first associates and two elements which do not appear together in a row or column be second associates, we can define an $L_2(n)$ association scheme.

For $3 \leq g \leq n+1$, if a set of (g-2) mutually orthogonal nxn Latin squares exists, we can define a Latin square ($L_g(n)$) association scheme from the nxn array of the v elements in the following manner. If two elements appear in the same row or column of the array, or if they correspond to the same symbol in one of the (g-2) Latin squares, they are first associates; otherwise the two elements are second associates.

For the case g=4, n=4, we can take the Latin squares LS, and LS_2, where

LS_1 =

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

and

LS_2 =

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

If the 16 elements are arranged in the array

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix},$$

then the first associates of the element 8 are 5,6,7,4,12,16,3,9,14, 2,11,and 13; for 8 corresponds to the symbol 3 in LS_1 and to the to the symbol 2 in LS_2.

For $2 \leq g \leq n+1$, the Latin square association scheme has the following parameters.

$$v = n^2$$

$$n_1 = g(n-1)$$

$$n_2 = (n-g+1)(n-1)$$

$$(3.6.1) \qquad p_{11}^1 = (g-1)(g-2)+n-2 \qquad p_{11}^2 = g(g-1)$$

$$p_{12}^1 = (n-g+1)(g-1) \qquad p_{12}^2 = g(n-g)$$

$$p_{22}^1 = (n-g+1)(n-g) \qquad p_{22}^2 = (n-g)^2 + g-2 \quad .$$

Following the nomenclature in [15], let us call an association scheme with the parameters (3.6.1) a pseudo-Latin square association scheme, whether or not it is obtainable from a set of (g-2) mutually orthogonal Latin squares. For example, an $L_3(6)$ scheme can be obtained from any 6 x 6 Latin square; its complement, or the scheme obtained by interchanging first and second associate classes, has the parameters of an $L_4(6)$ scheme, but no pair of mutually orthogonal 6 x 6 Latin squares exists.

An examination of the parameters (3.6.1) shows that two distinct series of pseudo-Latin square association schemes are such that BIB designs can be constructed from the association matrices.

3.6.2. Designs from B_1.

The condition is that $p_{11}^1 = p_{11}^2$; then $(g-1)(g-2) + n-2 = g(g-1)$, or $n = 2g$. In this case the parameters of the association scheme are as follows.

$$
(3.6.2) \qquad
\begin{aligned}
v &= 4g^2 \\
n_1 &= g(2g-1) \\
n_2 &= (g+1)(2g-1) \\
p^1_{11} &= g(g-1) & p^2_{11} &= g(g-1) \\
p^1_{12} &= (g-1)(g+1) & p^2_{12} &= g^2 \\
p^1_{22} &= g(g+1) & p^2_{22} &= (g+2)(g-1)
\end{aligned}
$$

Then B_1 is an incidence matrix for a BIB($4g^2$, $4g^2$, $g(2g-1)$, $g(2g-1)$, $g(g-1)$). The first three members of the series $n = 2g$ (letting $g = 2,3$, and 4, respectively) exist, and they give the following BIB designs: (16, 16, 6, 6, 2), (36, 36, 15, 15, 6) and (64, 64, 28, 28, 12).

3.6.3. Designs from B_2

The condition is that $p^1_{22} = p^2_{22}$; then $(n-g+1)(n-g) = (n-g)^2 + g-2$, or $n = 2(g-1)$. In this case the parameters are the following.

$$
(3.6.3) \qquad
\begin{aligned}
v &= 4(g-1)^2 \\
n_1 &= g(2g-3) \\
n_2 &= (g-1)(2g-3) \\
p^1_{11} &= (g-2)(g+1) & p^2_{11} &= g(g-1) \\
p^1_{12} &= (g-1)^2 & p^2_{12} &= g(g-2) \\
p^1_{22} &= (g-1)(g-2) & p^2_{22} &= (g-1)(g-2)
\end{aligned}
$$

It should be noted that the scheme with the parameters (3.6.3) is the complement of the scheme with parameters (3.6.2), which was shown to be such that B_1 is a BIB incidence matrix. Then we get no new designs by considering B_2

3.6.4. Designs from $[B_1 \vdots B_2]$

We need $n_1 = n_2$ and $p_{11}^1 + p_{22}^1 = p_{11}^2 + p_{22}^2 = \lambda_{12}$. Now $n_1 = n_2$ for $g(n-1) = (n-g+1)(n-1)$, or $n = 2g-1$; then $n_1 = n_2 = 2g(g-1)$, and $\lambda_{12} = 2g^2-2g-1$. For

$$n = 2g-1,$$

$$p_{11}^1 = g^2-g-1 \quad \text{and} \quad p_{22}^1 = g(g-1);$$

then

$$p_{11}^1 + p_{22}^1 = \lambda_{12} .$$

Also, $p_{11}^2 = g(g-1)$ and $p_{22}^2 = (n-g)(n-g-1) + n-2$

$$= g^2-g-1 ;$$

then

$$p_{11}^2 + p_{22}^2 = 2g^2-2g-1 = \lambda_{12} .$$

Then for any $L_g(n)$ association scheme with $n=2g-1$, $[B_1 \vdots B_2]$ is a BIB incidence matrix. In this case the parameters of the scheme are as follows.

$$v = (2g-1)^2$$

$$n_1 = n_2 = 2g(g-1)$$

$$(3.6.4) \qquad \begin{array}{ll} p_{11}^1 = g^2-g-1 & p_{11}^2 = g(g-1) \\ p_{12}^1 = g(g-1) & p_{12}^2 = g(g-1) \\ p_{22}^1 = g(g-1) & p_{22}^2 = g^2-g-1 \end{array}$$

For such a scheme, $[B_1 \vdots B_2]$ is an incidence matrix for a BIB($(2g-1)^2$,

$2(2g-1)^2$, $4g(g-1)$, $2g(g-1)$, $2g^2-g-1)$. Letting $g = 2$, 3, and 4, respectively, we see that the corresponding schemes exist, and we obtain the following BIB designs: (9, 18, 8, 4, 3), (25, 50, 24, 12, 11), and (49, 98, 48, 24, 23).

3.7. Negative Latin Square ($NL_g(n)$) Association Scheme [15]

3.7.1. Definition

It has been found that in many cases negative values of g and n will result in non-negative integers for the $L_g(n)$ parameters (3.6.1).

The simplest case is for $g = -1$ and $n = -4$; the resulting scheme has the following parameters.

$$v = 16$$

$$n_1 = 5$$

$$n_2 = 10$$

$$p^1_{11} = 0 \qquad p^2_{11} = 2$$

$$p^1_{12} = 4 \qquad p^2_{12} = 3$$

$$p^1_{22} = 6 \qquad p^2_{22} = 6$$

Substituting $-g$ for g and $-n$ for n in (3.6.1), we get the following set of parameters.

$$\begin{aligned} v &= n^2 \\ n_1 &= g(n+1) \\ n_2 &= (n-g-1)(n+1) \\ p^1_{11} &= (g+1)(g+2)-n-2 \qquad & p^2_{11} &= g(g+1) \\ p^1_{12} &= (n-g-1)(g+1) \qquad & p^2_{12} &= g(n-g) \\ p^1_{22} &= (n-g-1)(n-g) \qquad & p^2_{22} &= (n-g)^2-(g+2) \end{aligned} \tag{3.7.1}$$

An association scheme with the parameters (3.7.1) will be called a <u>negative Latin square</u> ($NL_g(n)$) association scheme.

Let us examine the parameters (3.7.1) to determine whether B_1, B_2, and $[B_1 \vdots B_2]$ can be BIB incidence matrices.

3.7.2. <u>Designs from B_1</u>

We must have $p_{11}^1 = p_{11}^2$; then $(g+1)(g+2)-n-2= g(g+1)$, or $n=2g$. In this case the $NL_g(n)$ parameters are the following, in terms of g.

$$
\begin{aligned}
v &= 4g^2 \\
n_1 &= g(2g+1) \\
n_2 &= (g-1)(2g+1) \\
p_{11}^1 &= g(g+1) \qquad & p_{11}^2 &= g(g+1) \\
p_{12}^1 &= (g-1)(g+1) \qquad & p_{12}^2 &= g^2 \\
p_{22}^1 &= g(g-1) \qquad & p_{22}^2 &= (g-2)(g+1)
\end{aligned} \tag{3.7.2}
$$

Then B_1 is an incidence matrix for a BIB $(4g^2, 4g^2, g(2g+1), g(2g+1), g(g+1))$. For g=2, the $NL_g(2g)$ scheme exists [15], and we get the BIB (16, 16, 10, 10, 6). For n=3, the existence of the scheme is unknown [15]. Letting n=4, the scheme exists [15], and we get the BIB (64, 64, 36, 36, 20).

3.7.3. <u>Designs from B_2</u>

The condition is that $p_{22}^1 = p_{22}^2$; then $(n-g-1)(n-g) = (n-g)^2 - (g+2)$, or $n=2(g+1)$. It is easily verified that the resulting scheme is just the complement of the $NL_g(n)$ scheme with $n=2g$. Then we can not obtain any further BIB designs by considering B_2 .

3.7.4. Designs from $[B_1 \vdots B_2]$

We need $n_1 = n_2$ and $p_{11}^1 + p_{22}^1 = p_{11}^2 + p_{22}^2 = \lambda_{12}$. Now $n_1 = n_2$ for $g(n+1) = (n-g-1)(n+1)$, or $n = 2g+1$; then $n_1 = n_2 = 2g(g+1)$, and $\lambda_{12} = 2g^2+2g-1$. For $n = 2g+1$, the parameters are as follows.

$$
\begin{aligned}
&v = (2g+1)^2 \\
&n_1 = n_2 = 2g(g+1) \\
(3.7.3)\qquad &p_{11}^1 = g^2+g-1 \qquad && p_{11}^2 = g(g+1) \\
&p_{12}^1 = g(g+1) && p_{12}^2 = g(g+1) \\
&p_{22}^1 = g(g+1) && p_{22}^2 = g^2+g-1
\end{aligned}
$$

Then $p_{11}^1 + p_{22}^1 = 2g^2+2g-1 = \lambda_{12}$ and $p_{11}^2 + p_{22}^2 = 2g^2 + 2g-1 = \lambda_{12}$. Then for $n = 2g+1$, a negative Latin square association scheme is such that $[B_1 \vdots B_2]$ is an incidence matrix for a BIB$((2g+1)^2, 2(2g+1)^2, 4g(g+1), 2g(g+1), 2g^2+2g-1)$. Note that a member of this series of BIB designs is also obtainable from a scheme with $L_{g+1}(2g+1)$ parameters, if such a scheme exists.

3.8. Pseudo-Geometric Association Scheme [2]

3.8.1 Partial Geometry

A partial geometry (r,k,t) is a system of points and lines, and a relation of incidence which satisfies the following **axioms:**

(i) any two distinct points are incident with not more than one line;

(ii) each point is incident with r lines;

(iii) each line is incident with k points;

(iv) if the point P is not incident with the line ℓ, then there are exactly t lines ($t \geq 1$) which are incident with P and also incident with

some point incident with ℓ.

Clearly, we have

(3.8.1) $1 \leq t \leq k$, $1 \leq t \leq r$, where r and k are ≥ 2.

It is easily seen from an examination of the four axioms above that given a partial geometry (r,k,t), we can obtain a dual partial geometry (k,r,t) by changing points to lines and lines to points.

The number of points v and the number of lines b in a partial geometry (r,k,t) satisfy the relations

$$v = \frac{k[(r-1)(k-1)+t]}{t} \tag{3.8.2}$$

and

$$b = \frac{r[(r-1)(k-1)+t]}{t} . \tag{3.8.3}$$

For convenience, we may use the ordinary geometric language when referring to partial geometries. Thus if a point and line are incident, we say that the point lies on the line (is contained in the line) and that the line passes through the point. A line which contains two points P and Q joins P and Q. If a point P lies on two lines ℓ and m, we say that ℓ and m intersect at P.

Let us call the points of a partial geometry treatments and call the lines blocks. The relation of incidence will then be that of a treatment's being contained in a block. Call two treatments first associates if they occur together in a block; otherwise they are second associates. Thus we see that a partial geometry (r,k,t) is equivalent to a PBIB design with parameters

$$v,\ b,\ r,\ k,\ \lambda_1 = 1, \quad \lambda_2 = 0, \tag{3.8.4}$$

where v and b are given by (3.8.2) and (3.8.3). The parameters of the corresponding association scheme are the following.

$$
\begin{aligned}
& v = \frac{k[(r-1)(k-1)+t]}{t} \\
& n_1 = r(k-1) \\
& n_2 = \frac{(r-1)(k-1)(k-t)}{t} \\
& p_{11}^1 = (t-1)(r-1)+k-2 \qquad && p_{11}^2 = rt \\
& p_{12}^1 = (r-1)(k-t) && p_{12}^2 = r(k-t-1) \\
& p_{22}^1 = \frac{(r-1)(k-t)(k-t-1)}{t} && p_{22}^2 = \frac{(r-1)(k-1)(k-t)}{t} \\
& && \quad - r(k-t-1)-1
\end{aligned}
\tag{3.8.5}
$$

We will call any association scheme with the parameters (3.8.5) and for which (3.8.1) holds a pseudo-geometric association scheme, since such a scheme may exist without bring derived from a partial geometry (r,k,t). However, if a pseudo-geometric scheme is a scheme derived from a partial geometry, we will call it a geometric association scheme.

In this section we consider no methods of constructing partial geometries and the corresponding schemes. However, several of the association schemes mentioned earlier in this chapter are special cases of pseudo-geometric schemes. In particular, a partial geometry $(r,k,r-1)$ gives rise to an $L_r(k)$ scheme. Thus a pseudo-Latin square scheme is just a special case of a pseudo-geometric scheme. Also, a partial geometry (r,k,r) gives us an SLB association scheme; thus we might introduce the term pseudo-SLB scheme, corresponding to a pseudo- geometric scheme with the appropriate parameters. It has been noted previously that a triangular scheme is a special case of an SLB scheme; hence we might speak of a pseudo-triangular scheme as a special case of a pseudo-geometric scheme.

Let us now examine the parameters (3.8.5) for possible BIB designs. From the preceding paragraph, we know that we will obtain some; however, there may be BIB designs obtainable from pseudo-geometric schemes which are not pseudo-Latin square or pseudo-SLB shcemes.

3.8.2. Designs from B_1.

For B_1 to be a BIB incidence matrix, we need $p^1_{11} = p^2_{11}$; then $(t-1)(r-1)+k-2 = rt$, or $t = k-r-1$. In this case the scheme has the following parameters.

$$v = \frac{rk(k-2)}{k-r-1}$$

$$n_1 = r(k-1)$$

$$n_2 = \frac{(r-1)(r+1)(k-1)}{k-r-1}$$

(3.8.6)

$$p^1_{11} = r(k-r-1) \qquad p^2_{11} = r(k-r-1)$$

$$p^1_{12} = (r-1)(r+1) \qquad p^2_{12} = r^2$$

$$p^1_{22} = \frac{r(r-1)(r+1)}{k-r-1} \qquad p^2_{22} = \frac{r^3+r-2k+2}{k-r-1}$$

Thus a pseudo-geometric association scheme with the parameters (3.8.6) is such that B_1 is a BIB incidence matrix. From (3.8.1), we see that in this case

(3.8.7) $$r+2 \leq k \leq 2r+1 \quad .$$

Then B_1 is an incidence matrix for a BIB $\left(\frac{rk(k-2)}{k-r-1}, \frac{rk(k-2)}{k-r-1}, r(k-1), r(k-1), r(k-r-1)\right)$. Now for the case $k = 2r$, the scheme (3.8.6) is of pseudo-Latin square type, and the BIB design obtained from B_1 is given in section 3.6.2. If $k = 2r + 1$, we get a scheme with SLB parameters, and the BIB design is given in section 3.5.2.

3.8.3. Designs from B_2

We need $p^1_{22} = p^2_{22}$; then $\frac{(r-1)(k-t)(k-t-1)}{t} = \frac{(r-1)(k-1)(k-t)}{t} - r(k-t-1) - 1$, or $t = k-r+1$. In this case the pseudo-geometric scheme has the following parameters.

$$v = \frac{k[r(k-2)+2]}{k-r+1}$$

(3.8.8)

$$n_1 = r(k-1)$$

$$n_2 = \frac{(r-1)^2(k-1)}{k-r+1}$$

$$p^1_{11} = r(k+1)-r^2-2 \qquad p^2_{11} = r(k-r+1)$$

$$p^1_{12} = (r-1)^2 \qquad p^2_{12} = r(r-2)$$

$$p^1_{22} = \frac{(r-1)^2(r-2)}{k-r+1} \qquad p^2_{22} = \frac{(r-1)^2(r-2)}{k-r+1}$$

The inequalities (3.8.1) impose the further restriction

$$r \leq k \leq 2r-1 \quad . \tag{3.8.9}$$

Then a pseudo-geometric scheme with parameters (3.8.8) is such that the association matrix B_2 is an incidence matrix for a BIB

$$\left(\frac{k[r(k-2)+2]}{k-r+1}, \frac{k[r(k-2)+2]}{k-r+1}, \frac{(r-1)^2(k-1)}{k-r+1}, \frac{(r-1)^2(k-1)}{k-r+1}, \frac{(r-1)^2(r-2)}{k-r+1} \right).$$

Note that for $k = 2(r-1)$, the scheme (3.8.8) is a psuedo-Latin square scheme; for $k = 2r-1$, it has SLB parameters.

3.8.4. Designs from $[B_1 \vdots B_2]$

For $[B_1 \vdots B_2]$ to be a BIB incidence matrix, the condition is that $n_1 = n_2$ and $p^1_{11} + p^1_{22} = p^2_{11} + p^2_{22}$. Now $n_1 = n_2$ for $r(k-1) = \frac{(r-1)(k-1)(k-t)}{t}$, or $t = \frac{k(r-1)}{2r-1}$. In this case $n_1 = n_2 = r(k-1)$, and $\lambda_{12} = rk-r-1$. Now if $t = \frac{k(r-1)}{2r-1}$, $p^1_{11} + p^1_{22} = \frac{2r^2k-4r^2+1}{2r-1}$ and $p^2_{11} + p^2_{22} = \frac{2r^2k-2rk-2r+1}{2r-1}$; then for equality we must have $2r^2k-4r^2+1 = 2r^2k-2rk-2r+1$, or $k=2r-1$. Then $k=2r-1$, $t=r-1$, $p^1_{11}+p^1_{22} = p^2_{11} + p^2_{22} = 2r^2-2r-1 = \lambda_{12}$, and the association scheme is as follows.

$$v = (2r-1)^2$$

$$n_1 = 2r(r-1)$$

$$n_2 = 2r(r-1) \qquad (3.8.10)$$

$$\begin{array}{ll} p^1_{11} = r^2-r-1 & p^2_{11} = r(r-1) \\ p^1_{12} = r(r-1) & p^2_{12} = r(r-1) \\ p^1_{22} = r(r-1) & p^2_{22} = r^2-r-1 \end{array}$$

For a scheme with parameters (3.8.10), $[B_1 \vdots B_2]$ is an incidence matrix for a BIB($(2r-1)^2$, $2(2r-1)^2$, $4r(r-1)$, $2r(r-1)$, $2r^2-r-1$). Note that **this** series of designs is included in the series obtained from pseudo-cyclic schemes such that $[B_1 \vdots B_2]$ is an BIB incidence matrix. The members of the present series are the members of the series obtained from pseudo-cyclic schemes for which $u = r^2-r$ for some $r \geq 2$ (see section 3.4).

CHAPTER IV

THREE-CLASS ASSOCIATION SCHEMES

4.1. Introduction

In this chapter we shall discuss some methods of constructing three-class association schemes and determine whether linear combinations or juxtapositions of the association matrices obtained can be incidence matrices for BIB designs. The necessary and sufficient conditions for the cases considered are the following: for

$$B_i \qquad (i=1,2,3),\ p^1_{ii} = p^2_{ii} = p^3_{ii}\ ;$$

for

$$B_i + B_j \ (i \neq j;\ i,j = 1,2,3),$$

$$p^1_{ii} + p^1_{jj} + 2p^1_{ij} = p^2_{ii} + p^2_{jj} + 2p^2_{ij} = p^3_{ii} + p^3_{jj} + 2p^3_{ij}$$

$$= \lambda^{(ij)};$$

for

$$[B_i \vdots B_j],\quad n_i = n_j$$

and

$$p^1_{ii} + p^1_{jj} = p^2_{ii} + p^2_{jj} = p^3_{ii} + p^3_{jj} = \lambda_{ij};$$

for

$$[B_1 \vdots B_2 \vdots B_3],\quad n_1 = n_2 = n_3$$

and

$$p_{11}^{1} + p_{22}^{1} + p_{33}^{1} = p_{11}^{2} + p_{22}^{2} + p_{33}^{2} = p_{11}^{3} + p_{22}^{3} + p_{33}^{3} = \lambda_{123}.$$

It should be noted that it is unnecessary to consider the combinations $B_o + B_k$; for $B_o + B_k$ will be a BIB incidence matrix if and only if $B_i + B_j$ is, since $B_o + B_k = J-(B_i + B_j)$ $(i \neq j \neq k;\ i,j,k = 1,2,3)$.

4.2. Group Divisible (GD) m-Associate Scheme [17]

4.2.1. Definition of the scheme

Suppose we have the number of elements

$$v = N_1 N_2 \dots N_m, \text{ where all the } N_i\text{'s are } \geq 2.$$

We can denote an element by an ordered m-triple $(i_1, i_2, \dots, i_m)$, where $i_j \in (0,1,\dots,N_j-1)$ for $j = 1,2,\dots, m$. Let two elements which have only the first $(m-j)$ coordinates in common be jth associates $(j=1,2,\dots,m)$. Then we have an association scheme, called a group divisible m-associate scheme, with the following parameters: for $i=1,2, \dots, m$, we have

$$v = N_1 N_2 \dots N_m,$$

$$n_i = N_m N_{m-1} \dots N_{m-i+2} (N_{m-i+1} - 1),$$

(4.2.1)

$$P^i = (p_{jk}^i) = \left[\begin{array}{c|c|c} O_{(i-1)\times(i-1)} & x_{i-1} & O_{(i-1)\times(m-i)} \\ \hline x'_{i-1} & \multicolumn{2}{c}{\multirow{2}{*}{$D_{(m-i+1)\times(m-i+1)}$}} \\ \cline{1-1} O_{(m-i)\times(i-1)} & \multicolumn{2}{c}{} \end{array}\right] \quad (j,k=1,2,3),$$

where O_{sxt} is the sxt matrix of zeros, x_{i-1} is a column vector of order (i-1) with elements $n_1, n_2, \ldots, n_{i-1}$, respectively, and $D_{(m-i+1)x(m-i+1)}$ is a diagonal matrix with diagonal elements

$$[N_m N_{m-1} \cdots N_{m-i+2}(N_{m-i+1}-2)],\ n_{i+1},\ n_{i+2},\ \ldots,\ n_m, \text{ respectively.}$$

For a 3-class GD scheme, we have $N_1, N_2', N_3 \geq 2$, with the parameters given below.

$$v = N_1 N_2 N_3$$

$$n_1 = N_3-1$$

$$n_2 = N_3(N_2 -1)$$

$$n_3 = N_3N_2 (N_1 - 1)$$

(4.2.2)

$$p^1_{11} = N_3-2 \qquad p^2_{11} = 0 \qquad p^3_{11} = 0$$

$$p^1_{12} = 0 \qquad p^2_{12} = N_3-1 \qquad p^3_{12} = 0$$

$$p^1_{13} = 0 \qquad p^2_{13} = 0 \qquad p^3_{13} = N_3-1$$

$$p^1_{22} = N_3(N_2-1) \qquad p^2_{22} = N_3(N_2-2) \qquad p^3_{22} = 0$$

$$p^1_{23} = 0 \qquad p^2_{23} = 0 \qquad p^3_{23} = N_3(N_2-1)$$

$$p^1_{33} = N_3N_2(N_1-1) \qquad p^2_{33} = N_3N_2(N_1-1) \qquad p^3_{33} = N_3N_2(N_1-2)$$

An example is the following.

$$N_1 = 3,\ N_2 = N_3 = 2$$

$$v = 12$$

$$n_1 = 1$$
$$n_2 = 2$$
$$n_3 = 8$$

$p^1_{11} = 0$ $p^2_{11} = 0$ $p^3_{11} = 0$

$p^1_{12} = 0$ $p^2_{12} = 1$ $p^3_{12} = 0$

$p^1_{13} = 0$ $p^2_{13} = 0$ $p^3_{13} = 1$

$p^1_{22} = 2$ $p^2_{22} = 0$ $p^3_{22} = 0$

$p^1_{23} = 0$ $p^2_{23} = 0$ $p^3_{23} = 2$

$p^1_{33} = 8$ $p^2_{33} = 8$ $p^3_{33} = 4$

Element	3-coordinate representation	First associates	Second associates	Third associates
1	(0,0,0)	2	3,4	5,6,7,8,9,10,11,12
2	(0,0,1)	1	3,4	5,6,7,8,9,10,11,12
3	(0,1,0)	4	1,2	5,6,7,8,9,10,11,12
4	(0,1,1)	3	1,2	5,6,7,8,9,10,11,12
5	(1,0,0)	6	7,8	1,2,3,4,9,10,11,12
6	(1,0,1)	5	7,8	1,2,3,4,9,10,11,12
7	(1,1,0)	8	5,6	1,2,3,4,9,10,11,12
8	(1,1,1)	7	5,6	1,2,3,4,9,10,11,12
9	(2,0,0)	10	11,12	1,2,3,4,5, 6, 7, 8
10	(2,0,1)	9	11,12	1,2,3,4,5, 6, 7, 8
11	(2,1,0)	12	9,10	1,2,3,4,5, 6, 7, 8
12	(2,1,1)	11	9,10	1,2,3,4,5, 6, 7, 8

Let us investigate the possibility of constructing BIB design from 3-class GD schemes.

4.2.2. Linear combinations

(i) $B_1 + B_2$

If $B_1 + B_2$ is a BIB incidence matrix, then $p^1_{11} + p^1_{22} + 2p^1_{12} = p^2_{11} + p^2_{22} + 2p^2_{12} = p^3_{11} + p^3_{22} + 2p^3_{12} = \lambda^{(12)}$. But $p^3_{11} + p^3_{22} + 2p^3_{12} = 0$;

therefore, $B_1 + B_2$ can not be a BIB incidence matrix.

(ii) $\underline{B_1 + B_3}$

Now $p_{11}^1 + p_{33}^1 + 2p_{13}^1 = p_{11}^2 + p_{33}^2 + 2p_{13}^2$

for

$$N_3 - 2 + N_3 N_2(N_1-1) = N_3 N_2(N_1-1), \quad \text{or} \quad N_3 = 2;$$

$$p_{11}^3 + p_{33}^3 + 2p_{13}^3 = N_3 N_2(N_1-2) + N_3 - 1$$

$$= 2N_1 N_2 - 4N_2 + 1 \text{ for } N_3 = 2, \quad \textbf{and}$$

$$p_{11}^1 + p_{33}^1 + 2p_{13}^1 = p_{11}^2 + p_{33}^2 + 2p_{13}^2 = 2N_1 N_2 - 2N_2.$$

Then

$$p_{11}^1 + p_{33}^1 + 2p_{13}^1 = p_{11}^3 + p_{33}^3 + 2p_{13}^3 \quad \textbf{for}$$

$$2N_1N_2 - 4N_2 + 1 = 2N_1N_2 - 2N_2, \quad \text{or } N_2 = {}^1/_2.$$

Then $B_1 + B_3$ can not be a BIB incidence matrix.

(iii) $\underline{B_2 + B_3}$

Now $p_{22}^1 + p_{33}^1 + 2p_{23}^1 = p_{22}^2 + p_{33}^2 + 2p_{23}^2$

for

$$N_3(N_2-1) + N_3N_2 (N_1-1) = N_3(N_2-2) + N_3N_2(N_1-1), \quad \text{or } N_3 = 0.$$

Then $B_2 + B_3$ can not be a BIB incidence matrix.

(iv) $\underline{B_1}$

Since $p_{11}^2 = 0 = p_{11}^3$, B_1 can not be a BIB incidence matrix.

(v) $\underline{B_2}$

B_2 can not be a BIB incidence matrix, for $p_{22}^3 = 0$.

(vi) $\underline{B_3}$

Now $p_{33}^1 = p_{33}^2 = N_3N_2(N_1-1)$; but $p_{33}^3 = N_3N_2(N_1-2)$, and $p_{33}^1 = p_{33}^2 = p_{33}^3$ for $N_2N_3 = 0$. Then B_3 can not be a BIB incidence matrix.

4.2.3. Juxtapositions

(i) $\underline{[B_1 \vdots B_2]}$

We have $n_1 = n_2$ for $N_3-1 = N_3(N_2-1)$, or $N_2 = \dfrac{2\ N_3-1}{N_3}$.; there are no integers $N_2, N_3 \geq 2$ which satisfy the above relation. Then $[B_1 \vdots B_2]$ can not be a BIB incidence matrix.

(ii) $\underline{[B_1 \vdots B_3]}$

Now $n_1 = n_3$ for $N_3-1 = N_3N_2(N_1-1)$, or

$$N_3 = \frac{1}{1 + (N_2-N_2N_1)} .$$

We see that $N_2 - N_2N_1 = N_2(1-N_2) \leq -2$; for, N_1, $N_2 \geq 2$ and $1-N_1 \leq -1$. Then the above expression for N_3 is negative; therefore $[B_1 \vdots B_3]$ can not be a BIB incidence matrix.

(iii) $\underline{[B_2 \vdots B_3]}$

Now $n_2 = n_3$ for $N_3(N_2-1) = N_3N_2(N_1-1)$, or $N_1 = \dfrac{2N_2-1}{N_2}$; thus we have the same situation as for $[B_1 \vdots B_2]$, and $[B_2 \vdots B_3]$ can not be a BIB incidence matrix.

(iv) $\underline{[B_1 \vdots B_2 \vdots B_3]}$

Since no two of the n_i's can be the same, certainly all three can not be equal; then $[B_1 \vdots B_2 \vdots B_3]$ can not be a BIB incidence matrix.

4.3. Tetrahedral Association Scheme [4]

4.3.1. Definition of scheme

A three-class association scheme, called a tetrahedral scheme, can be defined in a manner analogous to the definition of the two-class triangular scheme. Suppose there exists a set of $v = \binom{n}{3}$ elements, for some positive integer n; we can denote the v elements by unordered triples (x_1, x_2, x_3), where $x_1 \neq x_2 \neq x_3$ and x_1, x_2, and x_3 range from 0 to n-1. The elements can then be considered as points in three-dimensional Euclidean space; two elements with the same coordinates, in any order, will be considered the same. For i=1,2,3, call two elements ith associates if they differ in exactly i coordinates; for example, the elements (1,2,3) and (1,4,2) are first associates. This definition of association gives us an association scheme in three classes, with the following parameters.

$$v = \frac{n(n-1)(n-2)}{6}$$

$$n_1 = 3(n-3)$$

$$n_2 = \frac{3(n-3)(n-4)}{2}$$

$$n_3 = \frac{(n-3)(n-4)(n-5)}{6} \tag{4.3.1}$$

$$\begin{array}{lll} p_{11}^1 = n-2 & p_{11}^2 = 4 & p_{11}^3 = 0 \\ p_{12}^1 = 2(n-4) & p_{12}^2 = 2(n-4) & p_{12}^3 = 9 \\ p_{13}^1 = 0 & p_{13}^2 = n-5 & p_{13}^3 = 3(n-6) \\ p_{22}^1 = (n-4)^2 & p_{22}^2 = \frac{(n-5)(n+2)}{2} & p_{22}^3 = 9(n-6) \end{array}$$

$$p_{23}^1 = \frac{(n-4)(n-5)}{2} \qquad p_{23}^2 = (n-5)(n-6) \qquad p_{23}^3 = \frac{3(n-6)(n-7)}{2}$$

$$p_{33}^1 = \frac{(n-4)(n-5)(n-6)}{6} \quad p_{33}^2 = \frac{(n-5)(n-6)(n-7)}{6} \quad p_{33}^3 = \frac{(n-6)(n-7)(n-8)}{6}$$

We see that for such a scheme to exist we must have $n \geq 6$.

Let us examine the parameters of the tetrahedral scheme and attempt to construct BIB designs.

4.3.2. Linear combinations

(i) $B_1 + B_2$

Now $p_{11}^1 + p_{22}^1 + 2p_{12}^1 = (n-2) + (n-4)^2 + 4(n-4) = n^2-3n-2$, and

$p_{11}^2 + p_{22}^2 + 2p_{12}^2 = 4 + \frac{(n-5)(n+2)}{2} + 4(n-4) = \frac{n^2+5n-34}{2}$. Then we must have $n^2-3n-2 = \frac{n^2+5n-34}{2}$, or $n^2-11n+30 = 0$; in order for tetrahedral scheme to exist, we must take $n = 6$. But $p_{11}^3 + p_{22}^3 + 2p_{12}^3 = 9(n-6)+18$ $= 18$ for $n = 6$, whereas $p_{11}^1 + p_{22}^1 + 2p_{12}^1 = 16$; then B_1+B_2 can not be a BIB incidence matrix.

(ii) $B_1 + B_3$

We have $p_{11}^1 + p_{33}^1 + 2p_{13}^1 = n-2 + \frac{(n-4)(n-5)(n-6)}{6} =$

$\frac{n^3-15n^2+80n-132}{6}$, and $p_{11}^2 + p_{33}^2 + 2p_{13}^2 = 4 + \frac{(n-5)(n-6)(n-7)}{6} + 2(n-5)$

$= \frac{n^3-18n^2+119n-246}{6}$. The condition then requires

$n^3 - 15n^2 + 80n - 132 = n^3 - 18n^2 + 119n - 246$, or $n^2 - 13n + 38 = 0$.

Since the above has no integral solution, $B_1 + B_3$ can not be a BIB incidence matrix.

(iii) $\underline{B_2 + B_3}$

Now $p^1_{22} + p^1_{33} + 2p^1_{23} = (n-4)^2 + \frac{(n-4)(n-5)(n-6)}{6} + (n-4)(n-5)$

$= \frac{n^3 - 3n^2 - 28n + 96}{6}$, and $p^2_{22} + p^2_{33} + 2p^2_{23} = \frac{(n-5)(n+2)}{2} +$

$\frac{(n-5)(n-6)(n-7)}{6} + 2(n-5)(n-6) = \frac{n^3 - 3n^2 - 34n + 120}{6}$. Then we must

have $n^3 - 3n^2 - 28n + 96 = n^3 - 3n^2 - 34n + 120$, or $n = 4$. Since a tetrahedral scheme requires $n \geq 6$, $B_2 + B_3$ can not be a BIB incidence matrix.

(iv) $\underline{B_1}$

Since $p^3_{11} = 0$, B_1 can not be a BIB incidence matrix.

(v) $\underline{B_2}$

Now $p^1_{22} = p^2_{22}$ for $(n-4)^2 = \frac{(n-5)(n+2)}{2}$, or $n^2-13n+42 = 0$. Then $n = 6$ or $n = 7$. For $n = 6$, $p^1_{22} = p^2_{22} = 4$; but $p^3_{22} = 9(n-6) = 0$. For $n = 7$, $p^1_{22} = p^2_{22} = 9$, and $p^3_{22} = 9$. In this case B_2 is an incidence matrix for a BIB(35, 35, 18, 18, 9).

(vi) $\underline{B_3}$

Now $p^1_{33} = p^2_{33}$ for $\frac{(n-4)(n-5)(n-6)}{6} = \frac{(n-5)(n-6)(n-7)}{6}$; then n must be 6 for a tetrahedral scheme to exist. But for $n = 6$, $p^1_{33} = p^2_{33} = p^3_{33} = 0$. Then B_3 can not be a BIB incidence matrix.

4.3.3. Juxtapositions

(i) $\underline{[B_1 \vdots B_2]}$

Now $n_1 = n_2$ for $3(n-3) = \frac{3(n-3)(n-4)}{2}$, or $n = 6$; then $n_1=n_2=9$

and v = 20. But for n = 6, $\lambda_{12} = \frac{2(9)(8)}{19} = \frac{144}{19}$, and λ_{12} is not an integer. Then $[B_1 \vdots B_2]$ can not be a BIB incidence matrix.

(ii) $\underline{[B_1 \vdots B_3]}$

Now $n_1 = n_3$ for $3(n-3) = \frac{(n-3)(n-4)(n-5)}{6}$, or $n^2 - 9n+2 = 0$.

Since the above has no integral solution, $[B_1 \vdots B_3]$ can not be a BIB incidence matrix.

(iii) $\underline{[B_2 \vdots B_3]}$

Now $n_2 = n_3$ for $\frac{3(n-3)(n-4)}{2} = \frac{(n-3)(n-4)(n-5)}{6}$, or n = 14; then $n_1 = n_2 = 165$ and v = 364. But for n = 14, $\lambda_{23} = \frac{330(329)}{363} = \frac{3290}{11}$, and λ_{23} is not an integer. Then $[B_2 \vdots B_3]$ can not be a BIB incidence matrix.

(iv) $\underline{[B_1 \vdots B_2 \vdots B_3]}$

From (ii) above we see that $n_1 \neq n_3$ for all integral n. Then $[B_1 \vdots B_2 \vdots B_3]$ can not be a BIB incidence matrix.

4.4. Cubic Association Scheme [18]

4.4.1. The scheme

Suppose we have a set of $v = n^3$ elements, for some integer $n \geq 2$. Consider the v elements as ordered triples (x_1, x_2, x_3), where x_1, x_2, and x_3 range from 0 to n-1. Call two treatments ith associates if they have exactly i coordinates different (i = 1, 2, 3). Equivalently, we can consider the elements as points in three-dimensional Euclidean space. Then the first associates of a point α are those points lying on the three lines through α which are perpendicular to the coordinate planes; the second associates of α are the remaining points lying in the three planes determined by the first associates of α; other points are third associates of α.

The resulting scheme is an association scheme in three classes, called a cubic scheme. The parameters are the following.

$$v = n^3$$

$$n_1 = 3(n-1)$$

$$n_2 = 3(n-1)^2$$

$$n_3 = (n-1)^3 \tag{4.4.1}$$

$p^1_{11} = n-2$	$p^2_{11} = 2$	$p^3_{11} = 0$
$p^1_{12} = 2(n-1)$	$p^2_{12} = 2(n-2)$	$p^3_{12} = 3$
$p^1_{13} = 0$	$p^2_{13} = n-1$	$p^3_{13} = 3(n-2)$
$p^1_{22} = 2(n-1)(n-2)$	$p^2_{22} = n^2-2n+2$	$p^3_{22} = 6(n-2)$
$p^1_{23} = (n-1)^2$	$p^2_{23} = 2(n-2)(n-1)$	$p^3_{23} = 3(n-2)^2$
$p^1_{33} = (n-1)^2(n-2)$	$p^2_{33} = (n-1)(n-2)^2$	$p^3_{33} = (n-2)^3$

An examination of the above parameters shows that we can obtain one BIB design from the association matrices of a cubic scheme.

4.4.2. Linear combinations

(i) $B_1 + B_2$

Now $p^1_{11} + p^1_{22} + 2p^1_{12} = (n-2) + 2(n-1)(n-2) + 4(n-1) = 2n^2-n-2$, and $p^2_{11} + p^2_{22} + 2p^2_{12} = 2 + n^2-2n+2+4n-8 = n^2+2n-4$; then we must have $2n^2-n-2 = n^2+2n-4$, or $n^2-3n+2=0$. The only $n \geq 2$ which satisfies this relation is $n = 2$; then $p^1_{11} + p^1_{22} + 2p^1_{12} = p^2_{11} + p^2_{22} + 2p^2_{12} = 4$. But in this case $p^3_{11} + p^3_{22} + 2p^1_{13} = 6n - 12 + 6 = 6$; therefore B_1+B_2

can not be a BIB incidence matrix.

(ii) $\underline{B_1 + B_3}$

Now $p^1_{11} + p^1_{33} + 2p^1_{13} = (n-2) + (n-1)^2\ (n-2) = n^3-4n^2+6n-4$, and

$p^2_{11} + p^2_{33} + 2p^2_{13} = 2 + (n-1)(n-2)^2 + 2n-2 = n^3-5n^2 + 10n-4$; then we need $n^3-4n^2+6n-4= n^3-5n^2+10n-4$, or $n^2-4n=0$. In this case $n = 4$ and $p^1_{11} + p^1_{33} + 2p^1_{13} = p^2_{11} + p^2_{33} + 2p^3_{13} = 20$. Now $p^3_{11} + p^3_{33} + 2p^3_{13}$ $=$

$(n-2)^3 + 6(n-2) = 20$ for $n = 4$; $v = 64$, $n_1 = 9$, $n_2 = 27$, $n_3 = 27$, and $\lambda^{(13)} = \frac{36.35}{63} = 20$. So $B_1 + B_3$ is an incidence matrix for a BIB (64, 64, 36, 36, 20).

(iii) $\underline{B_2 + B_3}$

We have $p^1_{22} + p^1_{33} + 2p^1_{23} = 2(n-1)(n-2) + (n-1)^2(n-2) + 2(n-1)^2$

$= n^3-5n+4$, and $p^2_{22} + p^2_{33} + 2p^2_{23} = n^2-2n+2+(n-1)(n-2)^2 + 4(n-2)(n-1)$

$= n^3-6n+6$; our condition requires that $n^3-5n+4 = n^3-6n+6$, or $n = 2$. In this case $p^1_{22} + p^1_{33} + 2p^1_{23} = p^2_{22} + p^2_{33} + 2p^2_{23} = 2$; but $p^3_{22} + p^3_{33} +$

$2p^3_{23} = 6(n-2) + (n-2)^3 + 6(n-2)^2 = 0$ for $n = 2$. Then $B_2 + B_3$ can not be a BIB incidence matrix.

(iv) $\underline{B_1}$

Now $p^3_{11} = 0$; then B_1 can not be a BIB incidence matrix.

(v) $\underline{B_2}$

We have $p^1_{22} = p^2_{22}$ for $2(n-1)(n-2) = n^2-2n+2$, or $n^2-4n+2 = 0$; this relation has no integral solution, and B_2 can not be a BIB incidence matrix.

(vi) B_3

Now $p^1_{33} = p^2_{33}$ for $(n-1)^2(n-2) = (n-1)(n-2)^2$, or $n = 2$. But in this case $p^1_{33} = p^2_{33} = p^3_{33} = 0$. Then B_3 can not be a BIB incidence matrix.

4.4.3. Juxtapositions

(i) $[B_1 \vdots B_2]$

We have $n_1 = n_2$ for $3(n-1) = 3(n-1)^2$, or $n = 2$; then $n_1=n_2=3$ and $v = 8$. But in this case $\lambda_{12} = \frac{2(3)(2)}{7} = \frac{12}{7}$, and λ_{12} is not an integer. Thus $[B_1 \vdots B_2]$ can not be a BIB incidence matrix.

(ii) $[B_1 \vdots B_3]$

Now $n_1 = n_3$ for $3(n-1) = (n-1)^3$, or $n^2-2n-2 = 0$. Since this relation has no integral solution, $[B_1 \vdots B_3]$ can not be a BIB incidence matrix.

(iii) $[B_2 \vdots B_3]$

Now $n_2 = n_3$ for $3(n-1)^2 = (n-1)^3$, or $n = 4$; in this case $n_2 = n_3 = 27$ and $v = 64$. But for $n = 4, \lambda_{23} = \frac{2(27)(26)}{63} = \frac{156}{7}$, and λ_{23} is not an integer. Thus $[B_2 \vdots B_3]$ can not be a BIB incidence matrix.

(iv) $[B_1 \vdots B_2 \vdots B_3]$

The condition requires $n_1 = n_2 = n_3$; but we saw in (ii) above that n_1 and n_3 can not be equal. Then $[B_1 \vdots B_2 \vdots B_3]$ can not be a BIB incidence matrix.

4.5. Rectangular Association Scheme [26]

4.5.1. The scheme

Suppose we have a set of $v = \ell n$ elements for some integers $\ell, n \geq 2$. Then we can arrange the v elements in a rectangular array with ℓ rows and n columns. If two elements appear in the same row, call them first associates; if they appear in the same column, call them second associates; otherwise call them third associates. Then we have a three-class association scheme, called a rectangular scheme, with the following parameters.

$$v = \ell n$$

$$n_1 = n-1$$

$$n_2 = \ell-1$$

$$n_3 = (\ell-1)(n-1) \qquad (4.5.1)$$

$$\begin{array}{lll}
p^1_{11} = n-2 & p^2_{11} = 0 & p^3_{11} = 0 \\
p^1_{12} = 0 & p^2_{12} = 0 & p^3_{12} = 1 \\
p^1_{13} = 0 & p^2_{13} = n-1 & p^3_{13} = n-2 \\
p^1_{22} = 0 & p^2_{22} = \ell-2 & p^3_{22} = 0 \\
p^1_{23} = \ell-1 & p^2_{23} = 0 & p^3_{23} = \ell-2 \\
p^1_{33} = (\ell-1)(n-2) & p^2_{33} = (\ell-2)(n-1) & p^3_{33} = (\ell-2)(n-2)
\end{array}$$

Let us examine the parameters of the rectangular scheme to determine whether linear combinations or partitions of the association matrices can give us BIB designs.

4.5.2. Linear combinations

(i) $B_1 + B_2$

Now $p_{11}^1 + p_{22}^1 + 2p_{12}^1 = n-2$, and $p_{11}^2 + p_{22}^2 + 2p_{12}^2 = \ell-2$; then we must have $\ell=n$. Now $p_{11}^3 + p_{22}^3 + 2p_{12}^3 = 2$, which tells us that the common value of ℓ and n must be 4. In this case $v = 16$, $n = 3$, $n_1 = 3$, $n_2 = 3$, $n_3 = 9$, and $\lambda_{12} = \frac{(6)(5)}{15} = 2$. Then $B_1 + B_2$ is an incidence matrix for a BIB(16, 16, 6, 6, 2).

(ii) $\underline{B_1 + B_3}$

Now $p_{11}^1 + p_{33}^1 + 2p_{13}^1 = n-2 + (\ell-1)(n-2) = \ell(n-2)$, and $p_{11}^2 + p_{33}^2 + 2p_{13}^2 = (\ell-2)(n-1) + 2(n-1) = \ell(n-1)$; the condition requires that $\ell(n-2) = \ell(n-1)$, which is an impossible relation for positive ℓ, n. Then $B_1 + B_3$ cannot be a B IB incidence matrix.

(iii) $\underline{B_2 + B_3}$

We have $p_{22}^1 + p_{33}^1 + 2p_{23}^1 = (\ell-1)(n-2) + 2(\ell-1) = n(\ell-1)$, and $p_{22}^2 + p_{33}^2 + 2p_{23}^2 = \ell-2 + (\ell-2)(n-1) = n(\ell-2)$. Then we must have $n(\ell-1) = n(\ell-2)$, which can not be satisfied for positive ℓ, n. So $B_2 + B_3$ can not be a BIB incidence matrix.

(iv) $\underline{B_1}$

Since $p_{11}^2 = p_{11}^3 = 0$, B_1 can not be a BIB incidence matrix.

(v) $\underline{B_2}$

Since $p_{22}^1 = p_{22}^3 = 0$, B_2 can not be a BIB incidence matrix.

(vi) $\underline{B_3}$

We need $p_{33}^1 = p_{33}^2 = p_{33}^3$; but $p_{33}^1 = p_{33}^3$ for $(\ell-1)(n-2) = (\ell-2)(n-2)$, or $n = 2$. In this case $p_{33}^1 = p_{33}^3 = 0$, so B_3 can not be a BIB incidence matrix.

4.5.3. Juxtapositions

(i) $[B_1 \vdots B_2]$

We need $p_{11}^1 + p_{22}^1 = p_{11}^2 + p_{22}^2 = p_{11}^3 + p_{22}^3$; but $p_{11}^3 + p_{22}^3 = 0$. Then $[B_1 \vdots B_2]$ can not be a BIB incidence matrix.

(ii) $[B_1 \vdots B_3]$

Now $n_1 = n_3$ for $n-1 = (\ell-1)(n-1)$, or $\ell=2$. The condition requires $p_{11}^1 + p_{33}^1 = p_{11}^2 + p_{33}^2 = p_{11}^3 + p_{33}^3$; but for $\ell=2$, $p_{11}^2 + p_{33}^2 = p_{11}^3 + p_{33}^3 = 0$. So $[B_1 \vdots B_3]$ can not be a BIB incidence matrix.

(iii) $[B_2 \vdots B_3]$

We have $n_2 = n_3$ for $\ell-1 = (\ell-1)(n-1)$, or $n = 2$. The condition requires $p_{22}^1 + p_{33}^1 = p_{22}^2 + p_{33}^2 = p_{22}^3 + p_{33}^3$; but for $n = 2$, $p_{22}^1 + p_{33}^1 = p_{22}^3+p_{33}^3=0$. Therefore $[B_2 \vdots B_3]$ can not be a BIB incidence matrix.

(iv) $[B_1 \vdots B_2 \vdots B_3]$

Now $n_1 = n_2 = n_3$ for $\ell=n=2$; but then $p_{11}^1 + p_{22}^1 + p_{33}^1 = p_{11}^2 + p_{22}^2 + p_{33}^2 = p_{11}^3 + p_{22}^3 + p_{33}^3 = 0$. So $[B_1 \vdots B_2 \vdots B_3]$ can not be a BIB incidence matrix.

4.6. Three-Class Association Scheme from an Orthogonal Array [24]

4.6.6. Orthogonal arrays

An orthogonal array (N, m, s, t) is an $m\times N$ rectangular array of N assembles, with m contraints, in s symbols (e.g., the elements of the array may be the integers $0, 1, \ldots, s-1$), such that in any t-rowed submatrix of the array each of the s^t possible column vectors appears exactly λ times, where $\lambda s^t = N$. λ is called the index of the array.

4.6.2. Definition of the scheme

Suppose we have an orthogonal array $(n^2, \beta_1 + \beta_2, n, 2)$, where n_1, β_1, and β_2 are positive integers such that $n \geq 2$ and $\beta_1 + \beta_2 \leq n$. We see that the index $\lambda = 1$ in this case. Consider the n^2 assemblies as treatments. Define two treatments α and β as first associates if the columns corresponding to α and β are alike in exactly one position in the first β_1 rows; let α and β be second associates if the columns corresponding to them coincide in exactly one position in the remaining β_2 rows; otherwise α and β will be third associates. Since the array has strength 2 and index 1, we see that the definition of association is unambiguous; for two columns of the array can be alike in at most one position. Then we have a three-class association scheme with the following parameters.

$$v = n^2$$

$$n_1 = \beta_1(n-1)$$

$$n_2 = \beta_2(n-1)$$

$$n_3 = \beta_3(n-1), \text{ where } \beta_3 = n+1 - (\beta_1 + \beta_2)$$

(4.6.1)

$$p^1_{11} = n-2+(\beta_1-1)(\beta_1-2) \qquad p^2_{11} = \beta_1(\beta_1-1) \qquad p^3_{11} = \beta_1(\beta_1-1)$$

$$p^1_{12} = \beta_2(\beta_1-1) \qquad p^2_{12} = \beta_1(\beta_2-1) \qquad p^3_{12} = \beta_1\,\beta_2$$

$$p^1_{13} = \beta_3(\beta_1-1) \qquad p^2_{13} = \beta_1\,\beta_3 \qquad p^3_{13} = \beta_1(\beta_3-1)$$

$$p^1_{22} = \beta_2(\beta_2-1) \qquad p^2_{22} = n-2+(\beta_2-1)(\beta_2-2) \qquad p^3_{22} = \beta_2(\beta_2-1)$$

$$p^1_{23} = \beta_2\,\beta_3 \qquad p^2_{23} = \beta_3(\beta_2-1) \qquad p^3_{23} = \beta_2(\beta_3-1)$$

$$p^1_{33} = \beta_3(\beta_3-1) \qquad p^2_{33} = \beta_3(\beta_3-1) \qquad p^3_{33} = n-2+(\beta_3-1)(\beta_3-2)$$

Now let us attempt to construct BIB designs from linear combinations and juxtapositions of the association matrices B_i.

4.6.3. Linear combinations

(i) B_1

The requirement is that $p_{11}^1 = p_{11}^2 = p_{11}^3$. Now $p_{11}^2 = p_{11}^3 = \beta_1(\beta_1 - 1)$;

$$p_{11}^1 = n-2 + (\beta_1-1)(\beta_1-2) = \beta_1(\beta_1-1) \quad \text{for } \beta_1 = {}^n/_2 .$$

Then n must be even and we must take $\beta_1 = {}^n/_2$; in this case the association scheme with parameters (4.6.1) is such that B_1 is an incidence matrix for a BIB(n^2, n^2, $\frac{n}{2}(n-1)$, $\frac{n}{2}(n-1)$, $\frac{n}{4}(n-2)$).

(ii) B_2

We need $p_{22}^1 = p_{22}^2 = p_{22}^3$. Now $p_{22}^1 = p_{22}^3 = \beta_2(\beta_2-1)$, and $p_{22}^2 = n-2+(\beta_2-1)(\beta_2-2)$. Then our condition is satisfied for $\beta_2 = {}^n/_2$, in which case B_2 will be an incidence matrix for a BIB(n^2,n^2,$\frac{n}{2}(n-1)$, $\frac{n}{2}(n-1)$, $\frac{n}{4}(n-2)$); thus we get no new designs from considering B_2.

(iii) B_3

We see that $p_{33}^1 = p_{33}^2 = \beta_3(\beta_3-1)$, and $p_{33}^3 = n-2 + (\beta_3-1)(\beta_3-2)$; thus if $\beta_3 = {}^n/_2$, we get the same series of BIB designs as obtained from B_1 and B_2.

(iv) $B_1 + B_2$

The condition in this case is that $p_{11}^1 + p_{22}^1 + 2p_{12}^1 = p_{11}^2 + p_{22}^2 + 2p_{12}^2 = p_{11}^3 + p_{22}^3 + 2p_{12}^3 = \lambda^{(12)}$, where $\lambda^{(12)} =$

$\dfrac{(n_1+n_2)(n_1+n_2-1)}{v-1}$ is a positive integer. Now $p_{11}^1 + p_{22}^1 + 2p_{12}^1 =$

$n-2+(\beta_1-1)(\beta_1-2) + \beta_2(\beta_2-1) + 2\ \beta_2(\beta_1-1) \ = \ n + (\beta_1+\beta_2)^2-3(\beta_1+\beta_2)$;

the value of $p_{11}^2 + p_{22}^2 + 2p_{12}^2$ is the same. We have $p_{11}^3 + p_{22}^3 + 2p_{12}^3 =$

$\beta_1(\beta_1-1) + \beta_2(\beta_2-1) + 2\beta_1\beta_2 = (\beta_1 + \beta_2)^2 - (\beta_1+\beta_2)$. Then we must have

$n + (\beta_1+\beta_2)^2 - 3(\beta_1+\beta_2) = (\beta_1+\beta_2)^2 - (\beta_1+\beta_2)$, or $\beta_1+\beta_2 = \frac{n}{2}$.

In this case the common value of $p_{11}^i + p_{22}^i + 2p_{12}^i$ is $\frac{n(n-2)}{4}$, which is also the value of $\lambda^{(12)}$. Thus if n is an even positive integer and the orthogonal array $(n^2, \frac{n}{2}, n, 2)$ exists, the derived association scheme with parameters given by (4.6.1), where β_1 and β_2 are any positive integers such that $\beta_1+\beta_2 = \frac{n}{2}$, is such that B_1+B_2 is an incidence matrix for a BIB$(n^2, n^2, \frac{n}{2}(n-1), \frac{n}{2}(n-1), \frac{n(n-2)}{4})$. Again, we have the same series of designs as we got from considering B_i alone. However, it is worthwhile to consider B_1+B_2 also, since it may allow us to construct designs not obtainable from the B_i alone. For example, let $n = 6$. Then to construct the BIB(36, 36, 15, 15, 6) from B_1, B_2, or B_3 would require the existence of an orthogonal array (36, 4,6,2); we see this by letting β_1, β_2, or β_3 be $\frac{n}{2}$, or 3. But the existence of such an array is equivalent to the existence of a pair of orthogonal Latin squares of side 6, which is impossible; then the orthogonal array (36,4,6,2) does not exist, and $B_i\,(i=1,2,3)$ can not be an incidence matrix for the BIB(36,36,15,15,6). However, the orthogonal array (36,3,6,2), which is equivalent to one Latin square of side 6, does exist. Letting $\beta_1 = 1$, $\beta_2 = 2$, we see that the association scheme with parameters (4.6.1) is then such that B_1+B_2 is an incidence matrix for a BIB(36,36, 15,15,6).

(v) $\underline{B_1+B_3}$

The requirement for B_1+B_3 to be a BIB incidence matrix is that

$$p_{11}^1 + p_{33}^1 + 2p_{13}^1 = p_{11}^2 + p_{33}^2 + 2p_{13}^2 = p_{11}^3 + p_{33}^3 + 2p_{13}^3 = \lambda^{(13)}.$$

A little manipulation like that done when considering B_1+B_2 will show that the requirement is satisfied when $\beta_1+\beta_3 = \frac{n}{2}$ for some even positive integer n; then we get the same BIB designs by considering B_1+B_3 as we got from B_1+B_2.

(vi) $\underline{B_2+B_3}$

As in parts (iv) and (v), B_2+B_3 will be an incidence matrix for the BIB(n^2, n^2, $\frac{n}{2}(n-1)$, $\frac{n}{2}(n-1)$, $\frac{n(n-2)}{4}$) when $\beta_2+\beta_3 = \frac{n}{2}$ and the orthogonal array (n^2, $\beta_2+\beta_3$, n, 2) exists.

4.6.4. Juxtapositions

(i) $\underline{[B_1 \vdots B_2]}$

The requirement for $[B_1 \vdots B_2]$ to be a BIB incidence matrix is that $n_1 = n_2$ and $p_{11}^1 + p_{22}^1 = p_{11}^2 + p_{22}^2 = p_{11}^3 + p_{22}^3 = \lambda_{12}$, where $\lambda_{12} = \frac{2n_1(n_1-1)}{v-1}$ is a positive integer. Now $n_1 = n_2$ for $\beta_1 = \beta_2$; let x be the common value of β_1 and β_2. Then $p_{11}^1 + p_{22}^1 = p_{11}^2 + p_{22}^2 = n+2x^2-4x$, and $p_{11}^3 + p_{22}^3 = 2x^2 - 2x$; we see that for equality we must have $n + 2x^2 - 4x = 2x^2 - 2x$, or $x = \frac{n}{2}$. In this case $p_{11}^i + p_{22}^i = \frac{n(n-2)}{2}$, and $n_1 = n_2 = \frac{n(n-1)}{2}$; then $\lambda_{12} = \frac{n(n-2)}{2}$ and our requirement is satisfied. Then if n is an even integer > 2 and the orthogonal array

$(n^2, n, n, 2)$ exists, the scheme with parameters (4.6.1) is such that $[B_1 \vdots B_2]$ is an incidence matrix for a BIB$(n^2, 2n^2, n(n-1), \frac{n(n-1)}{2}, \frac{n(n-2)}{2})$. The simplest example is for $n = 4$; then we get the BIB(16, 32, 12,6, 4).

(ii) $\underline{[B_1 \vdots B_3]}$, $\underline{[B_2 \vdots B_3]}$

For $[B_1 \vdots B_3]$, the requirement is that $n_1 = n_3$ and $p_{11}^1 + p_{33}^1 = p_{11}^2 + p_{33}^2 = p_{11}^3 + p_{22}^3 = \lambda_{13}$. By the same reasoning as for $[B_1 \vdots B_2]$, we see that if $\beta_1 = \beta_3 = \frac{n}{2}$ and $\beta_2 = 1$ and if the orthogonal array $(n^2, \frac{n+2}{2}, n, 2)$ exists, then the scheme with parameters (4.6.1) is such that $[B_1 \vdots B_3]$ is an incidence matrix for a BIB $(n^2, 2n^2, n(n-1), \frac{n(n-1)}{2}, \frac{n(n-2)}{2})$. Therefore we gain nothing extra by considering $[B_1 \vdots B_3]$. Similarly, we get the same series of designs by considering $[B_2 \vdots B_3]$.

(iii) $\underline{[B_1 \vdots B_2 \vdots B_3]}$

We need $n_1 = n_2 = n_3$; then $\beta_1 = \beta_2 = \beta_3 = \frac{n+1}{3}$, and $n_1 = n_2 = n_3 = \frac{n^2-1}{3}$. We require also that $p_{11}^1 + p_{22}^1 + p_{33}^1 = p_{11}^2 + p_{22}^2 + p_{33}^2 = p_{11}^3 + p_{22}^3 + p_{33}^3 = n_1 - 1 = \frac{n^2-4}{3}$. We see from (4.6.1) that for $\beta_1 = \beta_2 = \beta_3 = \frac{n+1}{3}$ the requirement is satisfied. Then if n is greater than 2 and $n \equiv 2 \pmod 3$ and if the orthogonal array $(n^2, \frac{2(n+1)}{3}, n, 2)$ exists, then the association scheme with parameters (4.6.1) is such that $[B_1 \vdots B_2 \vdots B_3]$ is an incidence matrix for a BIB$(n^2, 3n^2, n^2-1, \frac{n^2-1}{3}, \frac{n^2-4}{3})$. The simplest case, $n = 5$, gives us the BIB(25,75,24,8,7).

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