EXTENSIONS OF THE ANALYSIS OF RANDOMIZED COMPLETE BLOCKS USING WEIGHTED RANKINGS

by

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1. INTRODUCTION

Experimental designs with random block structure have received much attention from statisticians for many years. Such interest is still being maintained, both in theoretical aspects, as is shown by the work of Dey (1986) and of Nigam, Puri, and Gupta (1988), and in specific applications, as is seen in Fleiss (1981) or Giesbrecht (1986).

The classical analysis of these designs rests on the assumption of a linear model in which the dependent variable has a normal distribution with expectation

$$\mu_{ij} = \mu + \tau_i + \beta_j\,, \quad i = 1, \ldots, I, \;\; j = 1, \ldots, J$$

indicating that each of the J treatments has been randomly assigned to one of the J cells or plots which constitute each of the I blocks.

For the study of this design in a distribution-free framework various methodologies have been proposed: some based solely on the intrablock information (for example, Friedman, 1937; Benard and van Elteren, 1953; Mack and Skillings, 1980), and others aimed at exploiting the interblock information also. In this second category, an asymptotically distribution-free procedure is that of aligned ranks, due to Hodges and Lehmann (1962) and Salter and Fawcett (1985), while a strictly distribution-free procedure is that of weighted rankings: Quade (1972), Silva (1977), Silva and Quade (1980, 1983).

In Section 2 of this work we review these proposals briefly, dedicating Sections 3 and 4 to the study of extensions of the method of weighted rankings in two directions of interest: a) balanced incomplete blocks incorporating linear and exponential weights and b) multiple observations per cell. The use of certain alternative procedures is discussed in both situations.

2. NONPARAMETRIC METHODS FOR THE ANALYSIS OF RANDOMIZED BLOCK DESIGNS

The classical procedure for the nonparametric analysis of randomized complete block designs (RCB) with one observation per cell is that proposed by Friedman (1937), using independent rankings within each block:

$$\{\{R_{ij} \mid R_{ij} = 1, \ldots J\}, \;\; i = 1, \ldots, I\}\,.$$

From the set $\{x_{ij}\,;\;\; i = 1 \ldots I, \;\;\; j = 1 \ldots J\}$ of observed responses we can derive the value of the statistic

$$FR = \frac{12I}{J(J+1)} \sum_j \{R_{\cdot j} - \tfrac{1}{2}(J+1)\}^2\,. \tag{2.1}$$

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Clearly FR uses only the "intrablock" information, so permutational arguments justify its character as a "distribution-free" procedure and permit the determination of its exact distribution under the null hypothesis of "absence of treatment effects" or $H_0 : \sum_j \tau_j^2 = 0$; and, asymptotically, FR $\approx \chi^2(J-1)$.

For more than one observation per cell, Conover (1971), Mehra and Sarangi (1967), and Benard and van Elteren (1953), among others, have proposed variants of FR which we shall discuss in Section 4.

As a way to "recover interblock information", Hodges and Lehmann (1962) proposed the use of ranks over all IJ observations, after "aligning" them by subtracting from each one of them a constant characterizing the block, for example, the arithmetic mean (or another measure of location). The "alignment" has as its object the removal of additive block effects, requiring that the indicated constant be a symmetric function of its J arguments and satisfying $g(x+a) = g(x)+a$. In addition it is assumed that the vector of "aligned observations" within each block has a symmetric joint distribution.

If r_{ij} $(= 1, \ldots ,IJ)$ is the overall rank of the j-th aligned observation within the i-th block, then the proposed statistic ("ranking after alignment") is

$$\text{RAL} = \frac{(J\text{-}1) \sum_{j=1}^{J} (\sum_{i=1}^{I} z_{ij})^2}{\sum_{i,j} z_{ij}^2 - \sum_i (\sum_j z_{ij})^2/J} .$$

For the case of linear ranks $z_{ij} = r_{ij} - r_{i.}/J$ we have

$$\text{RAL} = \frac{(J-1)[\Sigma_j r_{.j}^2 - IJ(IJ+1)/4]}{\sum_{i,j}(r_{ij} - \bar{r}_{i.})^2} \tag{2.2}$$

and the computational formula

$$\text{RAL} = \frac{1.5J(J-1)[4 \,\Sigma_j \, r_{.j}^2 - IJ^2(IJ+1)^2]}{J^2 I(IJ+1)(2IJ+1) - 6 \,\Sigma_i \, r_i^2} . \tag{2.2.1}$$

The exact distribution of RAL, under H_0, depends on the observed "configuration" (set of ranks located in each block) and thus RAL is only conditionally distribution-free. The original proposal of Hodges and Lehmann for $J=2$, based on the use of a Wilcoxon statistic on the aligned observations was extended to $J>2$ by Mehra and Sarangi (1967) using a Kruskal-Wallis statistic.

Asymptotically, under H_0, RAL $\approx \chi^2(J-1)$. Various theoretical aspects of this statistic are discussed in detail by Sen (1968), Puri and Sen (1971, Chapter 7; 1985, Chapter 7) and Tardif (1980, 1985).

This procedure was considered again by Salter and Fawcett (1985) in their proposed "aligned rank transform" which incorporates into the ideas of "align" and then "assign ranks" the rank transform of Conover and Iman (1981), Hora and Conover (1984), and Hora and Iman (1983). It is proposed that, after aligning the observations in each block and assigning ranks to the resulting residuals, one use the variance ratio statistic of the traditional ANOVA

$$\mathrm{ART} = \frac{(1/I) \sum_j [r_{.j} - (1/J) \sum_i r_{i.}]^2/(J-1)}{[\sum_{i,j} (r_{ij} - r_{..}/IJ)^2 - \sum(r_{.j} - \sum r_{i.}/J)^2/I]/(IJ - I - J + 1)} \tag{2.3}$$

We shall consider this statistic as one of the competing methods in Section 3, keeping in mind in any case that it is not "distribution-free" and that asymptotically it is distributed as F(J-1, IJ-I-J+1).

Another procedure which recovers interblock information and is strictly distribution-free is that proposed by Quade (1972) in introducing the idea of weighted rankings. Although it is possible to trace its origin to Tukey (1953), this proposal was made operational in the cited work and in Silva (1977), Quade (1979), and Silva and Quade (1980, 1983), and was studied further by Ferretti and Yohai (1986) and by Tardif (1987).

The basic idea of this procedure is to take into account the apparent variability within each block by supposing that a greater variability will reflect a greater discrimination of the block in which it occurs with respect to the differences in the effects of the "treatments". In consequence, if D is the variability statistic which has been chosen (invariant under translation and symmetric in its J arguments), we may associate with it a "score" $s(Q_i)$ where Q_i is the rank of the value of D observed in the i-th block. Such a score for the i-th block will serve as a weighting for the score associated with the observed response for the j-th treatment in that block.

> We may "suppose the observations on different treatments are more distinct in some blocks than in the others; then it seems intuitively reasonable that the ordering of the treatments that these blocks suggest is more likely to reflect the underlying true ordering. These same blocks might more or less equivalently be described as having greater variability, although the word observed is to be emphasized because [as we shall see in what follows, a basic assumption is that] all blocks are identically distributed except for additive block effects. Thus, these blocks, which may be referred to as more credible with respect to treatment ordering, will be given greater weight in the analysis." (Quade, 1979)

Appealing to the following basic assumptions:

I) The blocks $(x_{i1}, \ldots, x_{iJ})$, $i = 1, \ldots, I$, are mutually independent.

II) There exist quantities $\beta_1, \ldots, \beta_I$ such that the vectors $(x_{i1} - \beta_i, \ldots, x_{iJ} - \beta_i)$, $i = 1, \ldots, I$, are identically distributed.

III) There are no ties within the blocks: $P\{X_{ij} = X_{ij'}) = 0$ for $j \neq j'$.

IV) There are no ties among the measures of variability: $P(D_i = D_{i'}) = 0$ for $i \neq i'$.
it can be shown that the statistic

$$\mathrm{QQ} = \frac{(J-1) \sum_j (\sum_i s_{Q_i} t_{R_{ij}})^2}{(\Sigma s_i^2)(\Sigma t_j^2)} \tag{2.4}$$

is strictly distribution-free and asymptotically $\chi^2(J-1)$. Assumptions III and IV are convenient but not indispensable. The regularity conditions that must be satisfied by both sets of scores are studied by Silva (1977), and are verified, in particular, if we use the linear ranks (1, ... ,J) and (1, ... ,I), obtaining

$$QQ = \frac{72\ \Sigma_j\ [\Sigma_i\ Q_i(R_{ij} - 1/2(J+1)]^2}{J(J+1)\ I\ (I+1)\ (2I+1)} \tag{2.4.1}$$

The exact distribution of this statistic for small designs (J = 3, I = 3, ... ,7; J = 4, I = 3, 4; J = 5, I = 3) was evaluated by Quade in his initial paper of 1972; in the same context comparative studies of this method versus parametric and nonparametric competitors have been carried out (Silva, 1977; Silva and Quade, 1980). In this work Monte Carlo simulation was used in combination with the notion of expected significance level (ESL) as defined by Dempster and Schatzoff (1965) and recommendations made by Joiner (1969).

In essence the ESL "is equivalent to the complement of the power function averaged over all values of α". Estimators of the ESL of a statistic under the null hypothesis H_0 vs an alternative hypothesis H_1, and of their corresponding standard errors, can be obtained directly starting from random samples of size n simulated under both hypotheses.

By evaluating the difference between the ESLs of the two statistics, and its standard error, it was possible to compare various variations on the weighted rankings test with the principal competitors existing at that time: Friedman, ranking after alignment, and ANOVA, for designs with J = 3, 4, 5 and I = 3, 4, 5, 6. For the random error distributions considered in this work -- normal, uniform, and double exponential -- the weighted rankings statistics performed better (had significantly smaller ESL) than Friedman for the first two distributions and especially for the second. For a larger number of blocks (I = 20) it is relevant to mention the results of Ferretti and Yohai (1986), whose study of the empirical power for normal errors, for Student's t with 3 DF, and Cauchy, was fully consistent with the aforementioned.

An extensive Monte-Carlo study for small designs (J = 3, ... ,9; I = 3, ... ,9) was carried out by Fawcett and Salter (1984), comparing the classical F-test, Friedman's test (FR), weighted rankings, ranking after alignment (RAL), and the "rank transform" (RT) (Iman and Conover, 1981). Keeping in mind that the last two are not strictly distribution-free procedures, one must emphasize that F, RA, and RT had the greatest power when the "classical" conditions are satisfied. In contrast, under an ad-hoc symmetric distribution, nonnormal, with mean 0 and variance 1, the weighted rankings statistic was clearly superior for small and moderate treatment differences.

For $n \to \infty$, comparative studies have been based on the notion of asymptotic relative efficiency (ARE). Silva (1977, 1981) and Silva and Quade (1983) estimated the ARE of the statistics FR, RAL, and QQ discussed previously, with respect to the F of classical ANOVA; they used a result of Hannan (1956) which says that if T_h and T_k have asymptotic noncentral X^2 distributions with $(J-1)$ DF and noncentrality parameters Δ_{nh} and Δ_{nk} then

$$ARE(T_h, T_k) = \lim_{n \to \infty} (\Delta_{nh}/\Delta_{nk})\ . \tag{2.5}$$

Adapting a theorem of Fieller (Finney, 1978), it was possible to associate a fiducial interval to each of these estimated AREs (J = 2, 3, 4, 5; I = 200). We shall give details of this procedure in Section 3.

"For normal errors, both weighted-rankings statistics [QQ_1 using the sample variance as the measure of "credibility" and QQ_2 using the sample range] appear to have asymptotic efficiency well above that of Friedman's statistic [FR] but somewhat less than that of the ranking-after-alignment statistic [RAL], which is itself almost fully efficient. For uniform errors, the weighted-rankings statistics appear the most efficient of those considered, followed by the ordinary variance ratio and ranking after alignment, with Friedman's in last place again. For double-exponential errors, in contrast, weighted rankings (or at least the two versions considered here) may be the least efficient of the five, and ranking after alignment the best." (Silva and Quade, 1983)

Of course, the ARE of weighted rankings for $J = 2$ is already known exactly for various error distributions, since this test reduces to the signed-ranks test of Wilcoxon. In addition, for normal errors, with the range as the measure of intrablock variability, the work of Iman, Hora, and Conover (1984), Ferretti and Yohai (1986), and Tardif (1987) has shown that the ARE of weighted rankings with respect to Friedman is greater than 1 for $2 \leq J \leq 7$, but less for $J \geq 8$.

Tardif showed that an upper bound on the ARE of the weighted rankings statistics, attainable for $J = 2$, is the ratio of the largest characteristic root (LCR) of Σ to the sum of its characteristic roots, where $\Sigma = [Cov(V_h, V_{h'})]$ with $V_h = -f_h/f$ and f the joint density function, under H_0, of $J-1$ "aligned observations" within any block.

3. EXTENSION TO BALANCED INCOMPLETE BLOCKS

3.1 The principal nonparametric competitors

A design of undeniable practical utility is that of balanced incomplete blocks (BIB), proposed by Yates (1936), but whose analysis, from a nonparametric perspective, began with Durbin (1951). He adapted Friedman's test to this situation, proposing the statistic

$$DD = \frac{12(J-1)}{IJ(K^2-1)} \sum_j [R_{.j} - \tfrac{1}{2} J(K+1)]^2 \tag{3.1}$$

where K is the number of cells per block, J the number of treatments, I the number of blocks, R the number of occurrences of each treatment, and R_{ij} = the within-block rank of the observation X_{ij}. Asymptotically, under H_0, $DD \simeq \chi^2(J-1)$.

The same is true of the version of ranking after alignment adapted to this situation (Silva, 1977):

$$RAL = \frac{(J-1)\,(\Sigma_j r^2_{.j} - \tfrac{1}{4} IKR(IK+1)^2)}{\sum_{i,j} I_{ij}(r_{ij} - \bar{r}_{i.})^2 + \frac{K(I-R)}{I-1} \sum_i (\bar{r}_{i.} - \bar{r}_{..})^2} \tag{3.2}$$

where $I_{ij} = 1$ if the j-th treatment is observed in the i-th block, and 0 otherwise.

In the same manner, the weighted rankings statistic has the following form for a BIB design (Silva, 1977, p. 76):

$$QQ = \frac{(J-1)\, \Sigma_j \, [\Sigma_i \, I_{ij} s_{Q_i} \, t_{R_{ij}}]^2}{(\sum_i s_i^2)\, (\sum_j t_j^2)} \qquad (3.3)$$

which for $t_j = j - \frac{1}{2}(K+1)$, $j = 1, \ldots, K$, and $s_{n,i} = i$, $i = 1, \ldots, I$, reduces to

$$QQ = \frac{72\,(J-1)\, \Sigma\, H_j^2}{RJ(K^2-1)\,(I+1)\,(2I+1)} \qquad (3.3.1)$$

with $H_j = \Sigma\, I_{ij} s_{Q_i} [R_{ij} - \frac{1}{2}(K+1)]$, $j = 1, \ldots, J$.

Comparisons between these statistics and the F of ANOVA in terms of ARE were carried out by simulation for three basic designs and three error distributions by Vergara and Silva (1985).

As an illustration we shall consider the following:

Example 3.1 (Fleiss, 1981) "Consider the reliability study design laid out in Table [3.1.1], where each entry is the rating [on a scale of depression] given by the indicated rater to the indicated subject. Note the following features of the design: 1) Each of the 10 subjects is rated by three raters; 2) Each of the six raters rates five subjects; and 3) Each pair of raters jointly rate two subjects. These features characterize the study as a balanced incomplete blocks design."

In reference to our previous notation, we recognize in this example $I = 10$, $J = 6$, $K = 3$, and $R = 5$ ($\lambda = 2$). In Table 3.2 we have the within-block ranks (R_{ij}), the mean ($\bar{x}_i$) and the variance (D_i) of each block , and finally the ranks (Q_i) of the sample variances.

Table 3.1.1 — Results of a study of the reliability of a scale of depression designed as BIB.

Rater	**Subject**										**Average**
	1	2	3	4	5	6	7	8	9	10	
1	10	3	7	3	20						8.6
2	14	3				20	5	14			11.2
3	10		12			14			12	18	13.2
4		1		8			8		17	19	10.6
5				5	26	20		18	12		16.2
6			9		20		14	15		13	14.2
Average	11.3	2.3	9.3	5.3	22.0	18.0	9.0	15.7	13.7	16.7	12.3

According to the analysis of variance reported by Fleiss we have F(observers) = 7.12/9.23 = 0.7714 with $P(F_{(5,\,15)} > 0.7714) = 0.5849$.

Table 3.1.2 — Calculation of the statistic DD

R_{ij}	1	2	3	4	5	6	$\bar{x}_i$	D_i	Q_i
1	1.5	3	1.5				11.3	5.33	10
2	2.5	2.5		1			2.3	1.15	1
3	1		3			2	9.3	2.52	3.5
4	1			3	2		5.3	2.52	3.5
5	1.5				3	1.5	22.0	3.46	7.5
6		2.5	1		2.5		18.0	3.46	7.5
7		1		2		3	9.0	4.58	9
8		1			3	2	15.7	2.08	2
9			1.5	3	1.5		13.7	2.89	5
10			2	3		1	16.7	3.21	6
$R_{\cdot j}$	7.5	10.0	9.0	12.0	12.0	9.5			

Accordingly, $1/2J(K+1) = 12$ and $DD = \frac{12 \cdot 5}{6 \cdot 10 \cdot 8}(4.5^2 + 2^2 + 3^2 + 2.5^2 + 0 + 0) = 4.9375$ with $P(\chi^2(5) > 4.9375) = 0.4236$.

Table 3.1.3 — Calculation of the statistic RAL

r_{ij}	1	2	3	4	5	6	r_i	$\bar{r}_{i\cdot}$
1	12	26	12				50	16.67
2	18.5	18.5		12			49	16.33
3	4.5		26			16.5	47	15.67
4	4.5			26	16.5		47	15.67
5	6.5				29	6.5	42	14.00
6		21.5	1.5		21.5		44.5	14.83
7		1.5		14		30	45.5	15.17
8		9			23.5	15	47.5	15.83
9			9	28	9		46.0	15.33
10			20	23.5		3	46.5	15.50
$r_{\cdot j}$	46	76.5	68.5	103.5	99.5	71		

We have $IKR(IK+1)^2/4 = 36037.5$, $\Sigma r^2_{\cdot j} = 1532.56$, $K(I-R)/(I-1) = 1.66$, $\sum_{i,j}(r_{ij} - \bar{r}_{i\cdot})^2 = 2222.833$, and $\sum_i (\bar{r}_{i\cdot} - \bar{r}_{\cdot\cdot})^2 = 5.05$; thus $RAL = \frac{5 \cdot 2276.5}{2231.259} = 5.10$ with $P(\chi^2(5) > 5.10) = 0.4038$.

Table 3.1.4 – Calculation of the statistic QQ

$I_{ij}b_Q(R_{ij} - \frac{1}{2}(K+1))$	1	2	3	4	5	6
1	-5.0	10.0	-5.0			
2	0.5	0.5		-1.0		
3	-3.5		3.5			0.0
4	-3.5			3.5	0.0	
5	-3.75				7.5	-3.75
6		3.75	-7.5		3.75	
7		-9.0		0.0		9.0
8		-9.0			9.0	0.0
9			-2.5	5.0	-2.5	
10			0.0	6.0		-6.0
T_j	-15.25	-3.75	-11.5	13.5	17.75	-0.75

Thus $QQ = \frac{72.5}{5 \cdot 6 \cdot 8 \cdot 11 \cdot 21} \cdot 876.75 = 5.6932$ with $P(\chi^2(5) > 5.69) = 0.3372$.

Making use of the conditional likelihood approach of Cox (1972, 1975), Downton (1976) derived nonparametric tests for various situations with asymptotically optimal results. Starting from independent within-block ranks, he associated with each one of them an "exponential score" $t_{r,n}$ defined as

$$t_{r,n} = \sum_{s=n-r+1}^{n} s^{-1} . \qquad (3.4)$$

This "exponential score" is the expected value of the r-th order statistic of n observations from a standard exponential distribution.

Example 3.2 - Some exponential scores are:

i) For $n = 3$
$t_{1,3} = 1/3 = 2/6$ $\quad t_{2,3} = 1/3 + 1/2 = 5/6$
$t_{3,3} = 1/3 + 1/2 + 1 = 11/6$ $\quad$ and $\Sigma t = 3$.

ii) For $n = 6$
$t_{1,6} = 1/6 = 10/60$ $\quad t_{2,6} = 1/6 + 1/5 = 22/60$
$t_{3,6} = 1/6 + 1/5 + 1/4 = 37/60$ $\quad t_{4,6} = 57/60$ $\quad t_{5,6} = 87/60$
$t_{6,6} = 147/60$ $\quad$ and $\Sigma t = 6$.

iii) For $n = 10$
$t_{1,10} = 1/10 = 504/5040$ $\quad t_{2,10} = 19/90 = 1064/5040$
$t_{3,10} = 1694/5040$ $\quad t_{4,10} = 2414/5040$ $\quad t_{5,10} = 3254/5040$
$t_{6,10} = 4262/5040$ $\quad t_{7,10} = 5522/5040$ $\quad t_{8,10} = 7202/5040$
$t_{9,10} = 9722/5040$ $\quad t_{10,10} = 14762/5040$ $\quad$ and $\Sigma t = 10$.

We shall discuss Downton's statistic in its general form in Section 4, considering now its form for a BIB design

$$DWN = \frac{J-1}{I(K-t_{k,k})} \sum_j U^2{}_j \tag{3.5}$$

where $U_j = \frac{IK}{J} - \sum_i t^{(i)}_{r_j,k}$ is the exponential score if the j-th treatment was applied in the i-th block, and is 0 otherwise. Downton showed that this statistic is strictly distribution free and asymptotically, under H_0, distributed as $\chi^2(J-1)$. In Table 3.1.5 we apply this statistic to Example 3.1:

Table 3.1.5 – Calculation of the statistic DWN

$t^{(i)}_{r_j,k}$	1	2	3	4	5	6
1	3.5/6	11/6	3.5/6			
2	8/6	8/6		2/6		
3	2/6		11/6			5/6
4	2/6			11/6	5/6	
5	3.5/6				11/6	3.5/6
6		8/6	2/6		8/6	
7		2/6		5/6		11/6
8		2/6			11/6	5/6
9			3.5/6	11/6	3.5/6	
10			5/6	11/6		2/6
6 Σ	19	31	25	40	38.5	26.5
6 U_j	11	-1	5	-10	-8.5	3.5

Thus $DWN = \frac{5}{10 \cdot (3\text{-}11/6)} \cdot \frac{331.25}{36} = 3.9464$ with $P(\chi^2(5) > 3.94) = .5572$.

We propose to make use of these "exponential scores" to generate two variations on the weighted rankings statistic:

(i) QQE, defining $t_{R_{ij}}$ as $t^{(i)}_{R_j,k} - \bar{t}$, and (ii) QQEE, adding to the preceding definition that $s_Q = t_{Q,n}$; in both cases we shall use D_i = the variance of the i-th block.

We shall illustrate the preceding proposal by applying it to Example 3.1, for which we shall refer to the following table.

Table 3.1.6 – Calculation of the statistics QQE and QQEE.

$6t'_j$	1	2	3	4	5	6	s_Q	$5040s'_Q$
1	-2.5	5	-2.5				10	14762
2	2	2		-4			1	504
3	-4		5			-1	3.5	2054
4	-4			5	-1		3.5	2054
5	-2.5				5	-2.5	7.5	6362
6		2	-4		2		7.5	6362
7		-4		-1		5	9	9722
8		-4			5	-1	2	1064
9			-2.5	5	-2.5		5	3254
10			-1	5		-4	6	4262
Row a	-69.75	23.00	-56.00	59.50	46.50	-3.25		
Row b	-68234	44398	-64480	36112	39665	12539		

In Row "a" of this table we have the values $6 \cdot H_j$, (see Formula 3.3.1) which are multiples of the column sums of the products of (i) "exponential scores minus their mean" t'_j for the treatments and (ii) "linear weights" for the blocks. Thus $\Sigma t^2_j = 42/36$, $\Sigma s^2_i = 10 \cdot 11 \cdot 21/6$ (not considering, for simplicity, the ties) and $\Sigma H^2_j = 14243.125/36$ so

$$QQE = \frac{5 \cdot 6 \cdot 14243.125}{42 \cdot 10 \cdot 11 \cdot 21} = 4.4042 \text{ with } P(\chi^2 > 4.4042) = 0.4928 .$$

On the other hand, in Row "b" of Table 3.6 we have the values $6 \cdot H_j \cdot 5040$, which are multiples of the column sums of the product of the "exponential scores minus their mean" t'_j for the treatments by the exponential scores s_Q for the blocks. Thus $\Sigma t^2_j = 42/36$, $\Sigma s^2_i = 17.071$ (= 86038/5040) (not considering, for simplicity, the ties) and $\Sigma H^2_j = 544.034/36$; then

$$QQEE = \frac{5 \cdot 544.034}{42 \cdot 17.071} = 3.7941 \quad \text{with } P(\chi^2 > 3.7941) = 0.5794 .$$

3.2 Comparative study of the AREs

In this subsection we shall consider the following competing statistics for the problem of testing the null hypothesis of "absence of treatment effects" in a BIB design: (1) F from the parametric analysis of variance; (2) Durbin's DD; (3) RAL from ranking after alignment; (4) QQ from weighted rankings with linear scores for treatments and for blocks; (5) Downton's DWN; (6) QQE from weighted rankings with exponential scores for the treatments and linear for the blocks; (7) QQEE from weighted rankings with exponential scores for the treatments and the blocks; (8) RT from the rank transform (Conover and Iman, 1981); and (9) ART from the "aligned rank transform".

We shall keep in mind that, of these, only DD, QQ, DWN, QQE, and QQEE are strictly free of distributional assumptions. On the other hand, $(J-1)F$, DD, RAL, QQ, DWN, QQE, QQEE, $(J-1)RT$ and $(J-1)ART$ are asymptotically distributed as $\chi^2(J-1)$, under the null hypothesis. Let us say that $T_u \sim \chi^2(J-1)$, $u = 1, \ldots, 9$, under H_0.

Under the local alternative hypotheses $H_1 : \tau_I = \tau/\sqrt{I}$ we have that $T_u \approx \chi^2(J-1, \Delta u)$ asymptotically. Proceeding as in Silva and Quade (1983), we consider, for sufficiently large I, K vectors of treatment parameters of the form $\phi_k \tau$, k = 1, ... ,K ; then the respective noncentrality parameters Δ_{uk} will be proportional to $\phi^2{}_k$ for each T_u. In other words,

$$\Delta_{uk} = \beta_u \phi^2{}_k \tag{3.6}$$

so if we take estimators Δ_{uk} for the succession of values of $\phi^2{}_k$, it will be straightforward to estimate β_u. One can reason analogously for T_v , and following Hannan (1956), the ARE(T_u,T_v) can be estimated as $\Delta_{uk}/\Delta_{vk} = \beta_u/\beta_v$. A minimum variance unbiased estimator of the noncentrality parameters of the statistics T_u with asymptotic χ^2 distribution (Silva and Quade, 1983) is:

$$\hat{\Delta}_{uk} = \frac{\sum_{i=1}^{L} T_{ui}}{L} - (J-1) = Y_{uk} \tag{3.7}$$

where L is the number of independent observations on the statistic T_u and J is the number of treatments. If L is sufficiently large so that one may assume that the distribution of the Y_u is approximately normal, then we can apply a theorem of Fieller to obtain fiducial intervals for ARE(T_u,T_1), u = 2, ... ,9, whose point estimates are $R_u = \hat{\beta}_u/\hat{\beta}_1 = b_u/b_1$ and in which the $\hat{\beta}$'s are obtained by adjusting the following multiple regression model:

$$E(Y_{uk}) = \beta_0 + \beta_1 Z_{1k} + \beta_2 Z_{2k} + \ldots + \beta_9 Z_{9k} \tag{3.8}$$

where

$$Z_{uv} = \phi^2{}_k \quad \text{if } u = v \text{ and 0 otherwise.}$$

$(1-\alpha)100\%$ fiducial limits on ARE(T_u,T_1) , u = 2, ... ,9, are (Finney, 1978)

$$R_u^{(I)}, R_u^{(S)} = \left(R_u - \frac{g w_{1u}}{w_{11}} \pm \frac{t\sqrt{CME}}{b_1} A\right) \Big/ (1-g) \tag{3.9}$$

where

w_{ij} is the ij element of the inverse of the matrix of sums of squares and crossproducts of the Zs

t is the $(1-\alpha/2)$ percentile of Student's t

$$g = \frac{t^2 CME w_{11}}{b_1},$$

MSE is the mean squared error in model (3.8)

$$A = w_{11} - 2R_u w_{1u} + R_u{}^2 w_{11} + \left(w_{uu} - \frac{w_{1u}}{w_{11}}\right)$$

Experimental development and results

The values of the statistics T_u, u = 1, ... ,9, were obtained by computational simulation of three BIB designs corresponding to plans 11*, 11.1a*, and 11.4 of Cochran and Cox (1957), each replicated sufficiently many times so that the three designs were composed of 200 blocks. The simulations took place for five error distributions (normal, uniform, Laplace or double exponential, logistic, and extreme value), all with mean 0 and variance 1. For reasons made clear in Silva and Quade (1983), K = 2 was chosen (with ${\phi_1}^2 = 1$ and ${\phi_2}^2 = 2$). In addition, two samples generated from different random numbers were used. Thus there were four independent estimates of Δ_u , u = 1, ... ,9, for adjusting the regression model (3.8).

Using a FORTRAN program given in Appendix A, L = 500 independent experiments were simulated for each alternative hypothesis (2 hypotheses), each sample (2 samples), and each error distribution (5 distributions). For Design 1 and the normal error distribution, 1000 experiments were simulated, but this resulted in excessive simulation time, so afterwards only L = 500 experiments were considered.

The simulations took place on an IBM 4381 computer of the University of Santiago de Chile (USACH) under VM/SP, Release 5.0. The random numbers were generated using the VARGEN subroutines (TUCC, 1971) and the SAS statistical package was used to adjust the regression model and to obtain point and interval estimates of the AREs.

The BIB designs used appear as follows:

DESIGN 1

1	2	3
1	2	4
1	3	4
2	3	4

J = 4
K = 3
R_o = 3
I_o = 4
# of replications = 50
R = 150
I = 200

DESIGN 2

1	2	3	4
1	2	3	5
1	2	4	5
1	3	4	5
2	3	4	5

J = 5
K = 4
R_o = 4
I_o = 5
# of replications = 40
R = 160
I = 200

DESIGN 3

1	2	5
1	2	6
1	3	4
1	3	6
1	4	5
2	3	4
2	3	5
2	4	6
3	5	6
4	5	6

J = 6, K = 3
I_o = 10, R_o = 5
# of replications = 20
R = 100, I = 200

The treatment vectors τ used under H_1 were:

DESIGN 1

$$H_1 : \phi_i \underset{\sim}{\tau} = \phi_i\,(-0.06250,\ -0.03125,\ 0.03125,\ 0.06250)$$

DESIGN 2

$$H_1 : \phi_i \underset{\sim}{\tau} = \phi_i\,(-0.06250,\ -0.03125,\ 0.0,\ 0.03125,\ 0.06250)$$

DESIGN 3

$$H_1 : \phi_i \underset{\sim}{\tau} = \phi_i(-0.6250,\ -0.03125,\ -0.015625,\ 0.015625,\ 0.03125,\ 0.06250)$$

where $\phi_i^2 = 1$ or 2 for i = 1 or 2, respectively, in the three designs.

In Tables 3.2.1, 3.2.2, and 3.2.3 are shown the estimates of $ARE(T_u/T_1)$, $u = 2, \ldots, 9$, and the exact values(*) where they are available in the corresponding literature, and in Table 3.2.4 the statistics are ordered from largest to smallest, according to the estimated AREs, for the five error distributions considered and the three designs.

(*) Exact expressions available for determining the AREs in the case of a normal error distribution are:

$$ARE(DD,F) = \frac{3K}{\pi(K+1)} \qquad \text{(van Elteren and Noether, 1979)}$$

$$ARE(RAL,DD) = \frac{K+1}{K} \qquad \text{(Puri and Sen, 1971)}$$

$$ARE(RAL,DD) = \frac{3K}{\pi(K-1)[1+(6/\pi)\sin^{-1}(\frac{1}{2}(K-1))} \qquad \text{(Puri and Sen, 1971)}$$

Observing the results summarized in Tables 3.2.1 to 3.2.4, we can conclude that for the normal error distribution, in general, the eight statistics analyzed are less efficient than the statistic F of the classical ANOVA (as was to be expected) and that among them those with greatest ARE are the statistics ART, RT, and RAL, generally in that order; among QQ, QQE, QQEE, DD and DWN (statistics strictly free of distributional assumptions) the leader in efficiency is the weighted rankings statistic of Quade (QQ). In two of the three designs analyzed the statistic DWN turns out to have the smallest estimated ARE.

The statistics RT, ART (statistics constructed starting from F) and RAL (ranking after alignment) are those which show the best estimated ARE, greater than 1 (better in efficiency than F) for the Laplace (double exponential), logistic, and extreme value error distributions.

For the uniform error distribution, the statistics QQE (weighted rankings with exponential scores for the treatments and linear scores for the blocks), QQEE (weighted rankings with exponential scores for both treatments and blocks), and QQ (weighted rankings with linear scores for both treatments and blocks) are the ones with greatest asymptotic efficiency, in the order stated (except that in Design 3 QQ is slightly exceeded by RT). The estimated AREs of the first two statistics are greater than 1. The least efficient statistics for this error distribution are DD (Durbin) and DWN (Downton).

TABLE 3.2.1: POINT AND INTERVAL ESTIMATES OF THE ARE FOR DIFFERENT STATISTICS AND ERROR DISTRIBUTIONS – DESIGN 1

ERROR DISTRIBUTION	STATISTIC (T_u)	ESTIMATE ARE (T_u, T_1)		FIDUCIAL LIMITS (95%) $R_u(I)$	$R_u(S)$
NORMAL	QQ	0.869		0.819	0.921
	DD	0.684	(0.716)	0.636	0.732
	RAL	0.934	(0.966)	0.883	0.988
	RT	0.988		0.936	1.043
	ART	0.967		0.916	1.022
	DWN	0.671		0.623	0.719
	QQE	0.791		0.743	0.842
	QQEE	0.785		0.736	0.835
UNIFORM	QQ	1.001		0.956	1.049
	DD	0.699	(0.750)	0.658	0.741
	RAL	0.877		0.834	0.921
	RT	0.972		0.927	1.018
	ART	0.901		0.858	0.946
	DWN	0.670		0.629	0.711
	QQE	1.289		1.236	1.345
	QQEE	1.228		1.177	1.282
LAPLACE	QQ	0.954		0.881	1.034
	DD	1.058	(1.125)	0.981	1.142
	RAL	1.147		1.066	1.235
	RT	1.476		1.379	1.584
	ART	1.175		1.093	1.265
	DWN	1.012		0.936	1.094
	QQE	0.730		0.661	0.802
	QQEE	0.709		0.641	0.780
LOGISTIC	QQ	0.874		0.829	0.920
	DD	0.781		0.738	0.825
	RAL	1.018		0.970	1.067
	RT	1.094		1.045	1.146
	ART	1.045		0.997	1.096
	DWN	0.750		0.707	0.794
	QQE	0.720		0.678	0.764
	QQEE	0.700		0.657	0.743
EXTREME VALUE	QQ	0.954		1.202	2.210
	DD	0.942		1.067	1.983
	RAL	1.074		1.261	2.313
	RT	1.243		1.457	2.649
	ART	1.114		1.300	2.381
	DWN	0.973		1.113	2.060
	QQE	0.732		0.932	1.764
	QQEE	0.753		0.962	1.811

NOTE: The numbers in parentheses are the exact values of these AREs.

TABLE 3.2.2: POINT AND INTERVAL ESTIMATES OF THE ARE FOR DIFFERENT STATISTICS AND ERROR DISTRIBUTIONS – DESIGN 2

ERROR DISTRIBUTION	STATISTIC (T_u)	ESTIMATE ARE (T_u, T_1)		FIDUCIAL LIMITS (95%) $R_u(I)$	FIDUCIAL LIMITS (95%) $R_u(S)$
NORMAL	QQ	0.826		0.711	0.952
	DD	0.748	(0.764)	0.635	0.869
	RAL	0.939	(0.965)	0.820	1.073
	RT	0.950		0.831	1.085
	ART	0.967		0.846	1.103
	DWN	0.726		0.614	0.846
	QQE	0.692		0.580	0.811
	QQEE	0.669		0.557	0.786
UNIFORM	QQ	1.085		1.028	1.145
	DD	0.727	(0.800)	0.678	0.778
	RAL	0.858		0.807	0.912
	RT	0.973		0.919	1.030
	ART	0.879		0.827	0.933
	DWN	0.704		0.655	0.755
	QQE	1.309		1.245	1.378
	QQEE	1.284		1.221	1.352
LAPLACE	QQ	1.229		1.125	1.348
	DD	1.426	(1.200)	1.310	1.560
	RAL	1.508		1.386	1.649
	RT	1.929		1.777	2.108
	ART	1.571		1.445	1.718
	DWN	1.318		1.208	1.443
	QQE	0.859		0.771	0.955
	QQEE	0.847		0.758	0.942
LOGISTIC	QQ	0.853		0.812	0.896
	DD	0.812		0.771	0.854
	RAL	1.012		0.967	1.058
	RT	1.087		1.041	1.135
	ART	1.043		0.998	1.090
	DWN	0.756		0.716	0.797
	QQE	0.656		0.616	0.696
	QQEE	0.633		0.594	0.673
EXTREME VALUE	QQ	0.944		1.162	3.340
	DD	0.947		1.172	3.367
	RAL	1.094		1.331	3.813
	RT	1.200		1.486	4.261
	ART	1.124		1.364	3.906
	DWN	0.980		1.249	3.581
	QQE	0.678		0.849	2.517
	QQEE	0.705		0.890	2.661

NOTE: The numbers in parentheses are the exact values of these AREs.

TABLE 3.2.3: POINT AND INTERVAL ESTIMATES OF THE ARE FOR DIFFERENT STATISTICS AND ERROR DISTRIBUTIONS – DESIGN 3

ERROR DISTRIBUTION	STATISTIC (T_u)	ESTIMATE ARE (T_u, T_1)		FIDUCIAL LIMITS (95%) $R_u(I)$	$R_u(S)$
NORMAL	QQ	0.851		0.656	1.083
	DD	0.694	(0.716)	0.501	0.906
	RAL	0.924	(0.966)	0.726	1.167
	RT	0.949		0.749	1.196
	ART	0.971		0.770	1.221
	DWN	0.694		0.501	0.906
	QQE	0.734		0.541	0.950
	QQEE	0.730		0.538	0.946
UNIFORM	QQ	0.926		0.822	1.041
	DD	0.647	(0.750)	0.549	0.748
	RAL	0.871		0.769	0.982
	RT	0.969		0.863	1.087
	ART	0.913		0.810	1.027
	DWN	0.649		0.552	0.751
	QQE	1.184		1.067	1.317
	QQEE	1.140		1.026	1.270
LAPLACE	QQ	0.841		0.736	0.955
	DD	0.927	(1.125)	0.819	1.046
	RAL	1.176		1.056	1.314
	RT	1.406		1.271	1.566
	ART	1.238		1.114	1.381
	DWN	0.929		0.822	1.049
	QQE	0.577		0.476	0.680
	QQEE	0.596		0.495	0.699
LOGISTIC	QQ	0.930		0.806	1.070
	DD	0.944		0.820	1.086
	RAL	1.079		0.947	1.232
	RT	1.219		1.078	1.387
	ART	1.121		0.987	1.279
	DWN	0.992		0.865	1.138
	QQE	0.699		0.582	0.824
	QQEE	0.708		0.590	0.833
EXTREME VALUE	QQ	0.820		0.751	0.893
	DD	0.740		0.672	0.811
	RAL	1.027		0.951	1.108
	RT	1.069		0.992	1.152
	ART	1.075		0.998	1.158
	DWN	0.735		0.667	0.806
	QQE	0.637		0.570	0.705
	QQEE	0.636		0.569	0.704

NOTE: The numbers in parentheses are the exact values of these AREs.

TABLE 3.2.4: RANKING OF THE ESTIMATED AREs FOR THE DIFFERENT STATISTICS, ERROR DISTRIBUTIONS, AND DESIGNS

DESIGN	RANKING	ERROR DISTRIBUTION				
		NORMAL	UNIFORM	LAPLACE	LOGISTIC	EXTREME VALUE
1	1	RT	QQE*	RT*	RT*	RT*
	2	ART	QQEE*	ART*	ART*	ART*
	3	RAL	QQ*	RAL*	RAL*	RAL*
	4	QQ	RT	DD*	QQ	DWN
	5	QQE	ART	DWN*	DD	QQ
	6	QQEE	RAL	QQ	DWN	DD
	7	DD	DD	QQE	QQE	QQEE
	8	DWN	DWN	QQEE	QQEE	QQE
2	1	ART	QQE*	RT*	RT*	RT*
	2	RT	QQEE*	ART*	ART*	ART*
	3	RAL	QQ*	RAL*	RAL*	RAL*
	4	QQ	RT	DD*	QQ	DWN
	5	DD	ART	DWN*	DD	DD
	6	DWN	RAL	QQ*	DWN	QQ
	7	QQE	DD	QQE	QQE	QQEE
	8	QQEE	DWN	QQEE	QQEE	QQE
3	1	ART	QQE*	RT*	RT*	ART*
	2	RT	QQEE*	ART*	ART*	RT*
	3	RAL	RT	RAL*	RAL*	RAL*
	4	QQ	QQ	DWN	DWN	QQ
	5	QQE	ART	DD	DD	DD
	6	QQEE	RAL	QQ	QQ	DWN
	7	DD	DWN	QQEE	QQEE	QQE
	8	DWN	DD	QQE	QQE	QQEE

*: ARE > 1

4. ANALYSIS OF RANDOMIZED BLOCKS WITH UNEQUAL NUMBERS OF OBSERVATIONS PER CELL

In the preceding sections we have considered randomized block designs with one observation in each cell (CB), or with either 0 or 1 under conditions of "balance" (BIB); now we shall look at the situation in which cell (i, j) has H_{ij} observations

$$X_{ij,h} = \mu_{ij} + \epsilon_{ij,h} \quad i = 1, \ldots, I \quad j = 1, \ldots, J \quad h = 1, \ldots, H_{ij} \tag{4.1}$$

Special cases of interest are:

(i) $H_{ij} = H$ = constant for all (i, j);

(ii) $H_{ij} = H_j$ = constant within the j-th column (i.e., the same number M of observations in each block); and

(iii) $H_{ij} = H_{i.} \cdot H_{.j}/H_{..}$ = proportional frequencies (i.e., M_i observations in each block).

Apart from the usual difficulty of assuming a normal distribution for $\epsilon_{ij,k}$ -- which motivates the study of "distribution-free" alternatives -- the lack of balance further complicates the analysis, leading to diverse proposals for standardization, as we shall see.

The natural extension of Friedman's test in case (i) was made by Conover (1971, p. 273). Defining "reduced ranks" within each block

$$t_{ijh} = R_{ijk} - \tfrac{1}{2}(HJ+1) \quad h = 1, \ldots, H \text{ for all } (i, j) \tag{4.2}$$

he proposed the statistic

$$FR = \frac{12}{H^2IJ(HJ+1)} \sum_j^J [R_{.j.} - \tfrac{1}{2} HI(HJ+1)]^2 \tag{4.3}$$

$$= \frac{12}{H^2IJ(HJ+1)} \sum_j^J R^2_{.j.} - 3I(HJ+1) \tag{4.3.1}$$

which is strictly distribution-free and asymptotically $\chi^2(J-1)$.

For case (ii), $M = H_{i.}$ = constant, (all the cells associated with a given treatment have the same size, although this may vary across treatments), Mehra and Sarangi (1967, p. 102) proposed a distribution-free procedure which uses the following statistic based on independent rankings

$$MS = \frac{12}{M(M+1)I} \sum_j \frac{1}{H_j} [R_{.j.} - \tfrac{1}{2} H_jI(M+1)]^2 \tag{4.4}$$

$$= \frac{12}{M(M+1)I} \sum_j \frac{1}{H_j} R^2_{.j.} - 3I(M+1) . \tag{4.4.1}$$

This statistic, under H_0, is asymptotically distributed as $\chi^2(J-1)$.

In the same article Mehra and Sarangi also proposed a generalization of the conditional test of Hodges and Lehmann (RAL) for case (ii) $H_{ij} = H_j$ for all i. In this case

$$MS' = \frac{M-1}{M(\Sigma z_i^2)} \sum \frac{1}{H_j} [r_{.j.} - \tfrac{1}{2} H_j I(IM+1)]^2 \tag{4.5}$$

where $z_i^2 = \frac{1}{M} \sum_{h,j} (r_{ijh} - \bar{r}_{i..})^2$, $M = \Sigma H_{ij}$ and $\bar{r}_{i..} = \frac{1}{M} \sum_{h,j} r_{ijh}$. Under H_0, conditional on the observed configuration, MS′ is distribution-free; asymptotically it is distributed as $\chi^2(J-1)$.

For normal errors ARE(MS′, MS) varies from 1.5 to 1.0 as J varies from 2 to ∞; whereas ARE(MS′, F) = $3/\pi$ for J = 2 and then decreases from 0.9662 to 0.9549 as J goes from 3 to ∞.

The same authors studied an extension of MS′, defining in general $z_i^2 = \sum_{h,j} (r_{ijh} - \bar{r}_{i..})^2 / M_i$, $M_i = \sum_j H_{ij}$, $\bar{r}_{i..} = \frac{1}{N_i} \sum_{h,j} r_{ijh}$, $V_j = (r_{.j.} - \sum_i \bar{r}_{i..} H_{ij})/\sqrt{H_j}$ and proposing the statistic

$$MS'' = \underline{V}'\underline{\Sigma}^{-1}\underline{V} \tag{4.6}$$

where

$$\underline{V} = [V_1 \dots V_{j-1} \quad V_{j+1} \dots V_J]' \quad \text{and} \quad \underline{\Sigma} = [\sigma_{jj'}] \tag{4.6.1}$$

with $\sigma_{jj} = \sum_j \frac{M_i - H_{ij}}{M_i - 1} H_{ij} z_i^2$ and $\sigma_{jj'} = -\sum_i H_{ij} H_{ij'} \left(\frac{z_i^2}{M_i - 1}\right)$.

Mehra and Sarangi showed that the statistic (4.6) is, under H_0 and for the given configuration of ranks, asymptotically $\chi^2(J-1)$ and, in fact, is a re-expression of the statistic of Benard and van Elteren (1960).

Benard and van Elteren, studying the most general case, worked with intrablock ranks R_{ijh} reduced to

$$t_{ijh} = R_{ijh} - \tfrac{1}{2}(M_i + 1) \tag{4.7}$$

where M_i is the number -- not necessarily constant -- of observations in the i-th block.

Under the null hypothesis of no differences among the treatments, the ranks are distributed at random within each block, and therefore, for $U_{ij} = \sum_h t_{ijh}$, one has

$$\mathrm{Var}[U_{ij}] = H_{ij}(H_{i.} - H_{ij})W_i$$

and

$$\mathrm{Cov}[U_{ij}, U_{ij'}] = -H_{ij}H_{ij'} W_i \tag{4.7.1}$$

where

$$W_i = (H_i^3 - T_i)/(12H_i(H_i + 1)) \quad \text{if there are ties}$$

$$= (H_{i.} + 1)/12 \quad \text{if there are no ties.}$$

Let

$\underline{U} = [U_{.1}, U_{.2}, \dots, U_{.J}]'$; $\underline{V} = \text{Cov}(\underline{U})$ has elements $v_{jj} = \sum_i H_{ij}(H_{i.} - H_{ij})W_i$ and $v_{jj'} = -\sum_i H_{ij}H_{ij'}W_i$ so that the blocks are mutually independent.

Let $\underline{V}_U$ be defined as the $(J+1)\times(J+1)$ "bordered matrix" $\underline{V}_U = \begin{bmatrix} \underline{V} & \underline{U} \\ \underline{U}' & 0 \end{bmatrix}$; form V^* by removing one row and one column of V; remove one row and one column (other than the last) from $\underline{V}_U$ to form $\underline{V}_U{}^*$. The statistic proposed by Benard and van Elteren is

$$\text{BVE} = \frac{|\det V_u{}^*|}{|\det V^*|} = U'\,V^-\,U \tag{4.8}$$

where $\underline{V}^-$ is a generalized inverse of $\underline{V}$. Its asymptotic distribution is $\chi^2(J-1)$, but its exact distribution would be very cumbersome. Hutchinson (1977) restudied this statistic from the computational point of view; his FORTRAN program calculates BVE for observed data and associates to it an empirical significance level estimated on the basis of 1000 random permutations of the intrablock ranks, the 1000 corresponding values of BVE and the proportion of those which exceed the observed value of BVE.

Norwood, Sampson, et al (1989) proposed a method of multiple comparisons suitable for complementing an initial analysis based on the statistic BVE.

In 1979 Prentice studied a modification of BVE, reexpressing it as

$$B = \underline{U}'\,\underline{V}^{-1}\,\underline{U} \tag{4.9}$$

where $\underline{U}$ is the $(J-1)$-element column vector

$$U_j = \sum_{I(j)} \{R_{ij} - \tfrac{1}{2}(M_i + 1)\} \tag{4.9.1}$$

in which M_i is the number of observations in the i-th block, I(j) the set of blocks in which treatment j appears and $\underline{V}$ the $(J-1)\times(J-1)$ matrix of elements

$$v_{jj} = \sum_{I(j)} (M_i^2 - 1)/12 \quad \text{and} \quad v_{jj'} = -\sum_{I(j)\cap I(j')} (M_i + 1)/12 \,.$$

Replacing the "reduced ranks" $R_{ij} - \frac{1}{2}(M_i + 1)$ by new "standardized ranks" $\{R_{ij} - 1/2(M_i + 1)\}/\{M_i + 1\}$ remedies situations where "... an object ranked first out of four objects [would] have the same reduced rank as an object ranked 49th out of 100 objects" (Prentice, 1979, p. 168).

The statistic proposed by Prentice is

$$C = \underline{Y}'\underline{W}^{-1}\underline{Y} \tag{4.9.2}$$

where $\underline{Y}$ has dimension $J-1$ and elements $Y_j = \sum_{I(j)} \{R_{ij}/(M_i + 1) - 1/2\}/12$ while $\underline{W}$ has dimension

$(J-1)\times(J-1)$ and elements $w_{jj} = \{\sum_{I(j)}(M_i-1)/(M_i+1)\}/12$ and $w_{jj'} = -\{\sum_{I(j)\cap I(j')} 1/(M_i+1)\}/12$. Complementing the work of van Elteren and Noether (1959), Prentice showed that his statistic C is asymptotically more powerful than BVE for a wide class of BIB designs.

Downton (1976) bases his argument on the assumption that the observations X_{ijh} have a distribution function of the form

$$F_{ij}(x) = 1-(1-F_i(x))\exp(\tau_j) \tag{4.10}$$

In the i-th block there are $(H_{i.})!$ permutations of the observations, and thus the s-th of these permutations has the following likelihood, conditional on the observed values (Cox, 1972)

$$\exp(L_i(\tau)) = \prod_{j=1}^{J} \exp(\tau_i)H_{ij} \Big/ \prod_{k=1}^{H_i} T_{s,k} \tag{4.11}$$

where $T_{s,k} = \sum_{j=k}^{J} H_{ij}\exp(\tau_j)$ (Cox, 1975).

The log-likelihood for the complete experiment, based on the ranks, is $L(\underline{\tau}) = \Sigma L_i(\underline{\tau})$, whose first partial derivative with respect to τ_j is, under H_0,

$$U_{ij}(\underline{0}) = H_{ij} - \sum_{k=1}^{H_i}(m_{j,k}^{(i)})/(H_i-k+1) \tag{4.12}$$

where $m_{j,k}^{(i)}$ is the number of observations on the j-th treatment which have rank not less than k in the i-th block. If in this block the observations with the j-th treatment have ranks $r_{j,1}^{(i)}, \ldots, r_{j,H_{ij}}^{(i)}$ then $U_{ij}(\underline{0}) = H_{ij} - \sum_{k=1}^{H_{ij}} t_{R_{kj},H_i}^{(i)}$ where $t_{R_{kj},H_i}^{(i)}$ is the "exponential score" corresponding to the rank $R_{kj..}$.
That is, $t_{r,n} = \sum_{s=n-r+1}^{n} 1/s$ and $\underline{U} = (U_2, \ldots, U_J)'$ where

$$U_j = \Sigma_i U_{ij}(\underline{0}) = \Sigma_i H_{ij} - \sum_i\sum_k t_{R_{kj},H_i} \tag{4.13}$$

The general statistic of Downton is

$$DWN = \underline{U}'\underline{V}^{-1}\underline{U} \tag{4.14}$$

where, under H_0, the matrix $\underline{V}$ of dimension $(J-1)\times(J-1)$ has elements
$v_{jj} = \Sigma_i H_{ij}(H_{i.}-H_{ij})(H_{i.}-1)^{-1}(1-t_{H,H}/H_{i.})$ and
$v_{jj'} = -\Sigma_i H_{ij}H_{ij'}(H_{i.}-1)^{-1}(1-t_{H..,H..}/H_{i.})$. Asymptotically, under H_0, $DWN \approx \chi^2(J-1)$.

For the situation where $H_{ij} \geq 1$ for all (i, j), Mack and Skillings (1980) propose a two-way nonparametric analysis of variance based on assigning ranks within each row (level of factor A) and constructing for each treatment a "modified sum of ranks"

$$R_{.j.} = \sum_i R_{ij.}/H_{i.} \quad \text{with } H_{i.} = \sum_j H_{ij} \tag{4.15}$$

Under H_0, $E[R_{.j.}] = \sum_i H_{ij}(H_{i.}+1)/2H_{i.}$ and

$$\begin{aligned} \sigma_{jj'} &= \sum_i H_{ij}(H_{i.}-H_{ij})(H_{i.}+1)/12H_{i.}^2 && \text{if } j=j' \\ &= -\Sigma H_{ij}H_{ij'}(H_{i.}+1)/12H_{i.}^2 && \text{if } j\neq j'. \end{aligned} \tag{4.16}$$

The statistic of Mack and Skillings is

$$MSK = \underline{R}'\underline{\Sigma}^{-}\underline{R} \tag{4.16.1}$$

where $\underline{R}$ is the J-dimensional vector with elements $R_{.j.} - E[R_{.j.}]$ and $\underline{\Sigma}^{-}$ is any generalized inverse of the matrix $\underline{\Sigma} = [\sigma_{jj'}]$ just described. Alternatively,

$$MSK = \underline{R}_1'\underline{\Sigma}_{11}^{-1}\underline{R}_1 \tag{4.16.2}$$

where $\underline{R}_1$ consists of the first $J-1$ elements of $\underline{R}$ and $\underline{\Sigma}_{11}$ is the upper left submatrix of dimension $(J-1)\times(J-1)$ of $\underline{\Sigma}$.

Asymptotically, MSK has the $\chi^2(J-1)$ distribution, and for the case of proportional frequencies $H_{ij} = H_{i.}H_{.j}/H_{..}$ it reduces to

$$MSK = \frac{12}{H_{..}(H_{..}+1)} \sum_j H_{.j}\left[\sum_i \frac{R_{ij.}}{H_{ij}} - \frac{H_{..}+1}{2}\right]^2 \tag{4.17}$$

In particular, for $H_{ij} \equiv 1$, MSK reduces to FR; these authors show that ARE(MSK,BVE) > 1.

For the case of randomized blocks with $H_{ij} = 1$ or 0, the same authors (Skillings and Mack, 1981) propose assigning the rank R_{ij} $(1, \ldots, K_i)$ to the observation X_{ij} if it is present and $R_{ij} = (K_i+1)/2$ if it is absent; to each treatment they associate an <u>adjusted</u> <u>sum</u>

$$R_j = \sum_i \frac{12}{\sqrt{(K_i+1)}}[R_{ij} - \tfrac{1}{2}(K_i+1)] \tag{4.18}$$

and the working statistic is analogous to (4.16.1) or (4.16.2) with

$$\sigma_{jj'} = -m_{jj'} \quad \text{if } j\neq j' \quad \text{and} \quad \sigma_{jj} = \sum_{t\neq j} m_{jt} \quad \text{if } j=j' \tag{4.18.1}$$

where m_{jt} indicates the number of blocks in which treatments j and t appear simultaneously.

Already in 1983, de Kroon and van der Laan had proposed another generalization of Friedman's test based on the following standardization

$$t_{ijh} = \frac{R_{ijh} - \tfrac{1}{2}(H_{i.}+1)}{\sqrt{[H_{i.}(H_{i.}+1)/12]}} \tag{4.19}$$

For the case of proportional frequencies their statistic is

$$KL = \sum_{i,j} H_{ij}(\bar{t}_{.j.})^2 = \sum_j H_{.j}(\bar{t}_{.j.})^2 = \sum_j \frac{1}{H_{.j}}(t_{.j.})^2 \tag{4.20}$$

and, if $H_{ij} = H_j$ = constant for i = 1, ... ,I

$$KL = \frac{12}{MI(M+1)} \sum_j \frac{1}{H_j} \{R_{\cdot j \cdot} - \tfrac{1}{2} H_j I(M+1)\}^2 , \quad M = \sum_j H_{ij} = \text{constant} \tag{4.21}$$

coinciding with the statistic of Benard and van Elteren.

In more general form

$$KL = \underline{R}' \Sigma^- \underline{R} \tag{4.22}$$

where $\underline{R} = [R_j]$, $R_j = \Sigma t_{ijh}$ j = 1, ... ,J and Σ^- a generalized inverse of $\Sigma = [\sigma_{jj'}]$ for $\sigma_{jj'} = H_{\cdot j} - \sum_i H_{ij}^2/H_{i\cdot}$ if $j = j'$ and $\sigma_{jj'} = -\sum_i H_{ij} H_{ij'}/H_{i\cdot}$ if $j \neq j'$.

The work of Burnett and Willan (1988) is interesting; they show that for some error distributions (logistic, extreme value, and double exponential) one or another of the above statistics provides a uniformly most powerful test. Groggel and Skillings (1986) studied a rank test for multifactorial designs and Thompson and Amman (1989) extended the "rank transform" to two-factor models with N replications per cell.

The procedures described so far attempt to optimize the intrablock information by combining it with various non-stochastic weightings. The use of the information contained in the observations themselves for the same purpose corresponds to the idea of Quade (1972, 1979) and, in another form, to that of Rothe (1983).

Rothe developed the idea of block rank statistics, defined as

$$BRS = | \sum_i w_i \underline{D}^{(i)}/\sqrt{I} |^2 \tag{4.23}$$

where w_i uses some additional information contained in the observations, but in such a manner that BRS will be nonparametric: $\underline{D}^{(i)}$ is a $J \times J$ matrix associated with the vector of ranks within the i-th block such that

$$\underline{D}^{(i)} = [D_{ik1}] \tag{4.24}$$

where $D_{ik1} = 1 - 1/p$ if $R_{i1} = k$ and $D_{ik1} = -1/p$ if $R_{i1} \neq k$. ($|\underline{A}|^2$ indicates the sum of squares of all the elements of the matrix $\underline{A}$).

Tardif (1987) considers the family of Rothe statistics while incorporating the possibility of more than one observation per cell and assuming interchangeable errors within each block instead of requiring independence, thus including the original statistic of Quade. Previously Tardif (1980, 1985) had studied this same design using ranking after alignment.

Considering a total of N observations divided into J blocks <u>of</u> <u>M</u> <u>observations</u> <u>each</u> ($H_{ij} \geq 1$, for all i, j), Tardif presents the weighted rankings statistic in the form

$$TT = \Sigma_j S_j^2/(\sigma_t^2 \Sigma_i b_i^2) \tag{4.25}$$

where $\sigma_t^2 = \Sigma_j (t_j - \bar{t})^2/(M-1)$, b_i for i = 1, ... ,I are the scores or weights for each of the blocks and $S_j = \Sigma_i b_{Q_i} R_{ijh}$ the sum of scores associated with the j-th treatment, and

$$t_{ij} = \sum_h (t_{R_{ijh}} - t_i)/\sqrt{H_{ij}} \ . \tag{4.26}$$

Seeking the broadest possible generalization of the method of weighted rankings for the analysis of randomized blocks, we propose associating with each treatment a statistic

$$T_j = \underline{b}'\underline{t}_j = \sum_{i,h} b_{Q_i} t_{ijh} I_{ij} \qquad j = 1, \dots, J \tag{4.27}$$

where

$$t_{ijh} = \frac{t_{R_{ijh}} - \frac{1}{2}(H_{i\cdot} + 1)}{\sqrt{H_{i\cdot}(H_{i\cdot} + 1)/12}} \qquad \begin{array}{l} h = 1, \dots, H_{ij} \\ j = 1, \dots, J \\ i = 1, \dots, I \end{array} \tag{4.28}$$

and I_{ij} is an indicator function for the presence or absence of the j-th treatment in the i-th block (we assume at least two non-empty cells per block).

Standard analyses show that, under H_0 , this score t_{ijh} has expectation 0 and (for $h_{i\cdot} \to \infty$) variance asymptotically equal to 1. In fact,

$$V[t_{ijh}] = \frac{H_{i\cdot} - 1}{H_i}, \quad cov[t_{ijh}, t_{ijh'}] = -\frac{1}{H_{i\cdot}} \text{ and } cov[t_{ijh}, t_{ij'h'}] = -\frac{1}{H_{i\cdot}} \tag{4.29}$$

As a result,

$$V[t_{ij\cdot}] = H_{ij}(1 - H_{ij}/H_{i\cdot}) \tag{4.30}$$

and

$$cov[t_{ij\cdot}, t_{ij'}] = \sum_h \sum_{h'} cov[t_{ijh}, t_{ij'h'}] = -H_{ij}H_{ij'}/H_{i\cdot} \ .$$

Finally, we have

$$V[T_j] = V[\Sigma_i b_{Q_i} t_{ij\cdot}] = \Sigma_i b_i^2 H_{ij}(1 - H_{ij}/H_{i\cdot}) \tag{4.31}$$

and

$$\begin{aligned} cov[T_j, T_{j'}] &= \Sigma_i {b_{Q_i}}^2 cov[t_{ij\cdot}, t_{ij'\cdot}] \\ &= -\Sigma_i b_i^2 H_{ij} H_{ij'}/H_{i\cdot} \end{aligned} \tag{4.32}$$

That is, $\underline{T}$ will approach a variance-covariance matrix $\underline{\Sigma}$ $(J \times J)$ of rank $(J-1)$ with elements of the form (4.31) and (4.32), respectively. The generalized weighted rankings statistic will be

$$QQ^* = \underline{T}\underline{\Sigma}^{-}\underline{T} = \underline{T}_1 \underline{\Sigma}_{11}^{-1} \underline{T}_1 \tag{4.33}$$

where $\underline{\Sigma}^{-}$ is a generalized inverse of the $\underline{\Sigma}$ just defined, $\underline{\Sigma}_{11}$ is (for example) the upper left submatrix of $\underline{\Sigma}$ and $\underline{T}_1$ is the subvector $[T_1 \dots T_{J-1}]$.

We note that for the case of one observation per cell, we have $H_{i\cdot} = J$, $i = 1, \dots, I$ and by (4.28), (4.30), (4.31), and (4.32) QQ^* reduces to (2.4.1); analogously, for a BIB design we have $H_{i\cdot} = K$, reducing (4.33) to (3.3.1).

By suitably repeating the tedious steps described in Silva (1977), Sections 5.1.3 and 3.2, we can show that (4.32) has an <u>asymptotic</u> $\chi^2(J-1)$ <u>distribution</u>, central under H_0 and noncentral under local alternative hypotheses.

We shall illustrate the proposal we have made for using weighted rankings by means of a simple example with artificial data, and two examples from the literature.

Example 4.1 – The following table serves to illustrate the situation which we have called case (i); the parentheses indicate the intrablock ranks.

	j = 1		j = 2				j = 3			
i = 1	27	(5)	22	(2)	26	(4)	25	(3)	21	(1)
i = 2	31	(5)	28	(2)	29	(3)	27	(1)	30	(4)
i = 3	25	(1)	30	(4)	31	(5)	26	(2)	28	(3)
i = 4	22	(2)	19	(1)	25	(3)	26	(4)	29	(5)

Clearly $H_{i.} = M = 5$ for $i = 1, \ldots, 4$, $(H_{i.} + 1)/2 = 3$, and $\sqrt{H_{i.}(H_{i.} + 1)/12} = \sqrt{5/2}$. Next we have the intrablock variances, and the scores for each block and each cell (defining $a = \sqrt{2/5}$).

t_{ijh}	j = 1	j = 2		j = 3		s_i^2	${}^{b}Q_i$
i = 1	2a	– a	a	0	– 2a	6.7	3
i = 2	2a	– a	0	– 2a	a	6.7	3
i = 3	– 2a	a	2a	– a	0	6.7	3
i = 4	– a	– 2a	0	a	2a	6.7	3
T_j	0	– 3a		3a			

We calculate T_j applying (4.27). Using (4.31) and (4.32) we obtain

$$\underline{\Sigma} = \begin{bmatrix} 24 & -12 & -12 \\ -12 & 36 & -24 \\ -12 & -24 & 36 \end{bmatrix}$$

and from (4.33)

$$QQ^* = [0 \quad -3a]\begin{bmatrix} 24 & -12 \\ -12 & 36 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -3a \end{bmatrix} = [0 \quad -3a]\,\frac{1}{720}\begin{bmatrix} 36 & 12 \\ 12 & 24 \end{bmatrix}\begin{bmatrix} 0 \\ -3a \end{bmatrix}$$

$$= \frac{216a^2}{720} = \frac{3}{10}\cdot\frac{2}{5} = 0.12$$

For this example it is easy to verify that the statistics of Mehra and Sarangi, of Benard and van Elteren, of Mack and Skillings, and of Kroon and van der Laan coincide in the value 0.15.

Example 4.2 – (Patterson and Thompson, 1971) The general case without empty cells.

X_{ijh}	Treatment 1	Treatment 2	s_i^2	$1/2(H_i+1)$	$\sqrt{12/H_i(H_i+1)}$
i = 1	3 2	2 3	0.333	2.5	0.7746
i = 2	2 3 5 6 7	8 8 9	6.286	4.5	0.4082
i = 3	3	4 4 3	1.100	3.5	0.5345

Here the scores for cell and block are, respectively

t_{ijh}	Treatment 1		Treatment 2		s_{Qi}
i = 1	0.7746	-0.7746	-0.7746	0.7746	1
i = 2	-1.4289 -0.6124 0.2041	-1.0206 -0.2041	0.8165 1.4289	0.8165	3
i = 3	-0.2182		-0.5345 -0.5345 1.3363	0.5345 -1.3362	2
T_j	-10.2547		10.2547		

Since
$$H_{i1}(1-H_{i1}/H_i) = 1, \quad 15/8 \quad \text{and} \quad 5/6 \quad \text{for} \quad i = 1,2,3,$$
$$H_{i2}(1-H_{i2}/H_i) = 1, \quad 15/8 \quad \text{and} \quad 5/6 \quad \text{for} \quad i = 1,2,3,$$
$$-(H_{i1}\cdot H_{i1})/H_i = -1, \quad -15/8 \quad \text{and} \quad -5/6 \quad \text{for} \quad i = 1,2,3,$$

respectively, we have

$$\underline{\Sigma} = \begin{bmatrix} 509/24 & -509/24 \\ -509/24 & 509/24 \end{bmatrix}$$

and $QQ^* = (-10.2547)(24/509)(10.2547) = 4.958$ with $P(\chi^2(1) > 4.958) \approx 0.026$. The classical analysis of variance gives us $F = 15.168$ with $P(F(1,14) > 15.168) = 0.028$ for the treatment effects adjusted for blocks, in an interesting concordance with our result.

Example 4.3 – (Skillings and Mack, 1981) The general case with some empty cells. The observations correspond to assembly times for a product, considering four assembly procedures and nine operators. The missing values correspond to machinery failure or to absenteeism.

	j = 1	j = 2	j = 3	j = 4
i = 1	3.2 (1)	4.1 (3)	3.8 (2)	4.2 (4)
i = 2	3.1 (1)	3.9 (3)	3.4 (2)	4.0 (4)
i = 3	4.3 (2)	3.5 (1)	4.6 (3)	4.8 (4)
i = 4	3.5 (1)	3.6 (2)	3.9 (3)	4.0 (4)
i = 5	3.6 (1)	4.2 (4)	3.7 (2)	3.9 (3)
i = 6	4.5 (2)	4.7 (3)	3.7 (1)	--
i = 7	--	4.2 (2)	3.4 (1)	--
i = 8	4.3 (1)	4.6 (3)	4.4 (2)	4.9 (4)
i = 9	3.5 (1)	--	3.7 (2)	3.9 (3)

The following intermediate calculations permit the block scores and the cell scores to be adjusted in accordance with our proposal:

s_i^2	b_i	$1/2(H_i+1)$	$\sqrt{H_i(H_1+1)/12}$	$H_{ij}(1-H_{ij}/H_i)$	b_i^2/H_i
0.2025	6	2.5	1.291	0.75	9.00
0.1800	5	2.5	1.291	0.75	6.25
0.3267	9	2.5	1.291	0.75	20.25
0.0567	2	2.5	1.291	0.75	1.00
0.0700	3.5	2.5	1.291	0.75	3.06
0.2800	7	2.0	1.000	0.67	16.33
0.3200	8	1.5	0.707	0.50	32.00
0.0700	3.5	2.5	1.291	0.75	3.06
0.0400	1	2.0	1.000	0.67	0.33

In consequence, the scores t_{ijh}, the vector $\underline{T}$ and the matrix $\underline{\Sigma}$ are, respectively,

t_{ijh}				
	−1.1619	0.3873	−0.3873	1.1619
	−1.1619	0.3873	−0.3873	1.1619
	−0.3873	−1.1619	0.3873	1.1619
	−1.1619	−0.6455	0.3873	1.1619
	−1.1619	1.1619	−0.3873	0.3873
	0	1	−1	--
	--	0.7071	−0.7071	--
	−1.1619	0.3873	−0.3873	1.1619
	−1	--	0	1

$$\underline{T} = [\Sigma\, b_i\, t_{ijh}] = [-27.7236 \quad 11.1077 \quad -15.3679 \quad 31.9839]$$

$$\Sigma = \begin{bmatrix} 161.208 & -58.958 & -59.291 & -42.958 \\ -58.958 & 192.542 & -90.958 & -42.625 \\ -59.292 & -90.958 & 193.208 & -42.958 \\ -42.958 & -42.625 & -42.958 & 128.542 \end{bmatrix}$$

Thus $QQ^* = \underline{T}_1 \underline{\Sigma}_{11}^{-1} \underline{T}' = 11.166$ with $P(\chi^2 > 11.166) = 0.0109$. Skillings and Mack report MSK = 15.49 with P-value = 0.0014, which shows at least in this situation greater power for our method. In addition, we may note that the classical analysis (GLM-SAS) gives F = 3.75 with P-value = 0.027.

5. PERMUTATION TEST WITH GENERALIZED WEIGHTED RANKINGS

Following the line of thought of Edgington (1987) and Welch (1990), not to mention the earlier experiments of Hutchinson (1977), one may recommend placing the use of the generalized weighted rankings statistic (QQ^*) in the context of randomized tests. This will free us from dependence on asymptotic distributions with slow convergence and from the search for exact distributions which are tedious to produce by means of systematic permutations except for small designs.

A short SAS program permits us to: (i) evaluate the statistic QQ^* for the observed data, (ii) generate a "null realization", that is, under the hypothesis of absence of treatment effects, of the experimental design given in our data, evaluate QQ^* in this new situation, (iii) compare QQ_c^* of the preceding step with QQ_{obs}^* of the first step, and (iv) repeat steps (ii) and (iii) a sufficiently large number C of times, recording the number R of occurrences of $QQ_{obs}^* > QQ_c^*$.

The indicated null realization QQ_c^* is obtained starting from the random permutation of the H_i ranks of the i-th block for $i = 1, \ldots, I$, and from a similar permutation of the I ranks of the measures of variability. Taking into account the random nature of the indicated permutations of ranks and the properties of the structure of each block, that is the set (H_{ij}; $j = 1, \ldots, J$) of frequencies within each block, the proportion R/C is a reasonable estimator of the empirical level of significance (P-value) corresponding to the situation under study, no matter how unbalanced the design may be.

Using $C = 5000$ for Example 3.1 ($J = 3$ treatments) the value $QQ^*_{obs} = 0.12$ was obtained, with P-value $= 0.9394$ and mean QQ^*_c 1.75, with 199 seconds of CPU time required. For Example 3.2 ($J = 2$), with $C = 5000$ $QQ^*_{obs} = 4.958$ was obtained, with P-value $= 0.0136$ and mean QQ^*_c 0.88, using 178 seconds of CPU time. Under the same conditions, for Example 3.3 ($J = 4$) $QQ^*_{obs} = 11.26$ was obtained, with P-value $= 0.0002$ and mean Q^*_c equal to 2.41, with 493 seconds of CPU time used.

These results are illustrative of better power of this randomized test based on weighted rankings relative to the procedures mentioned in the preceding section. The mean values of the "null realizations" of this statistic clearly differ from $J - 1$, the asymptotic expectation under H_0, illustrating the inadequacy of the $\chi^2(J-1)$ approximation.

The above was worked out under version 5.16 of SAS under VM/CMS on an IBM 4381 computer.

6. DISCUSSION

The use in QQ^* of scores (s_{Q_i}) for the blocks, assigned as a function of the discriminability of the block, allows us (as has been shown in our previous papers) to recover "interblock information". The incorporation of an indicator function and an adequate standardization allows us to expand the applicability of the weighted rankings statistic to unbalanced cases. Finally, utilization of the concept of random permutation provides us with the ability to use QQ^* as a nonparametric test which optimizes the use of information and which is not tied to a questionable asymptotic approximation. The use of systematic permutations gives an exact test, but nevertheless it is expensive for problems of practical interest.

The work developed in this two-year project has affirmed the validity of the method of weighted rankings as an efficient tool for the analysis of randomized block designs, in spite of numerous more recent competitors. Incorporation of exponential scores has increased the potential of the method, especially in the case of uniform errors. The program BIB200, a computational subproduct of the project, includes all the possibilities we have discussed.

In the course of the study there have emerged, as is natural, new questions to explore, such as the use of different nonlinear scores, perhaps dependent (in a suitably broad sense) on the distribution of the random component of the model. Analogously, in the extension to the most general case (Section 4) we may conjecture that the convergence to a χ^2 distribution will be slower and less "regular" for the pattern $[H_{ij}]$. The exact null distribution will be needed more in such cases, or, in practical terms, a more computationally efficient randomization test. We hope to continue development of these approaches since they may provide better solutions to problems frequent in statistical practice.

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Appendix A

Program BIB200

```
C
      DIMENSION SADJ(5),YY(800),IND(800),EM(1165)
      DIMENSION TR(5),BE(200),BM(200),RS(200,5),Y(5),RY(5)
      DIMENSION YA(5),YAL(800),RYAL(800),RAFA(200,5)
      DIMENSION SW(200),RSW(200),HN(5),DB(5),SCA(5),SRA(200)
      DIMENSION S(6),NDF(6),N(5)
        DIMENSION HNE(5),HNEE(5),RSE(200,5) ,REX(4),EX(4)
        DIMENSION RTY(800),RTS(6),RSS(6),RSADJ(5),RTSADJ(5)
        DIMENSION REY(5),RESW(200),REW(200,5),DW(5)
      DIMENSION KC(2,10) ,TRAT(5)
        DOUBLE PRECISION S,RTS,RSS
C     REAL*8 YY,GM,SADJ,EF,S,EM,FA,FR,FNR
      INTEGER*2 FLAG,DIST,DISTV,DISTR
      INTEGER T,B,R
      INTEGER*4 SEED
      DATA BE/200*0./
      DATA KC/20*0/
      CXCX =1
C     SEED =524287
C     SEEDB=54743017
C-----------------------------------------(PARAMETER CARD 1)
C     DEFINE THE DIMENSION OF THE BIB DESIGN
      READ (1,99) T,RT,B,RB,KK,RK,R,RR,NN,RNN,NS
   99 FORMAT (5(I4,F4.0),I4)
      LV=B*KK
      WRITE (3,199) T,B,KK,NN
  199 FORMAT (1X,I4,' TREATMENTS,',I4,' BLOCKS,',I4,' CELLS PER BLOCK,
     XI6,' EXPERIMENTS',/)
C     DESCRIBE THE BIB DESIGN (PARAMETER CARDS 2 AND 3)
      READ (1,249) (N(I),I=1,5),IOPT
      READ (1,239) (IND(I),I=1,LV)
  249 FORMAT (12I3)
  239 FORMAT (60I1)
      WRITE (3,259) (N(I),I=1,5),IOPT
C     WRITE (3,269) (IND(I),I=1,LV)
  259 FORMAT (1X,5I6,/)
  269 FORMAT (1X,3I2,/)
C
C     DEFINE TREATMENT EFFECTS (PARAMETER CARD 4)
      READ (1,89) (TR(J),J=1,T)
   89 FORMAT (10F9.5)
C      WRITE (3,189) (TR(J),J=1,T)
  189 FORMAT (///,' TREATMENT EFFECTS H1   ',10F9.5,/)
C     DEFINE ERROR DISTRIBUTION AND ITS PARAMETERS
C  ------------------------------------------- (PARAMETER CARD 5)
```

```
      READ (1,79)  DISTR,DIST,SEED,AXAX,BXBX
   79 FORMAT (2I2,I8,2F8.4)
      WRITE (3,179)  DIST,SEED,AXAX,BXBX
  179 FORMAT (1X,' DIST: ',I2,' SEED:',I10,' PARAM:',2F8.4,/)
C
C ---------------------- DEFINE CRITICAL VALUES (PARAMETER CARD 6)
      READ (1,69) CH5,CH1,FF5,FF1,FM5,FM1
   69 FORMAT (10F8.5)
  169 FORMAT (1X,' CRIT. VALUES :',F10.5,/)
      FM5=CH5/(RT-1.)
      FM1=CH1/(RT-1.)
      WRITE (3,169) CH5,CH1,FF5,FF1,FM5,FM1
C--------------------- DEFINE OPTIONAL OUTPUT (PARAMETER CARD 7)
      READ (1,249) LOB,LST
      WRITE (3,149) LOB,LST
  149 FORMAT (1X,'LOB :',I2,' LST:',I2,/)
C---------- DEFINE BLOCK-EFFECTS DISTRIBUTION (PARAMETER CARD 8)
      READ (1,29)  DISTV
   29 FORMAT (I2)
      IF (DISTR) 8001 ,8003,8002
C     GENERATE ERROR NORMAL (0,1)
8001  WRITE (3,222)
      GOTO 200
C   GENERO DISTRIBUCION UNIFORME (-V¯3,+V¯3)
8002  WRITE (3,333)
      GOTO 200
8003  IF (DISTV) 7001 ,7002,7003
7001  WRITE(3,701)
      GOTO 200
7002  WRITE(3,702)
      GOTO 200
7003  WRITE(3,703)
      GOTO 200
701   FORMAT(5X,'    ERROR LAFLACE ,MEDIA 0, VAR=1')
C702   FORMAT(3X,'    ERROR D. CAUCHY ,MEDIA 0, VAR=1')
702   FORMAT(5X,'    ERROR D. LOGISTICA ,MEDIA 0, VAR=1')
703   FORMAT(5X,'    ERROR D.VALOR EXTREMO (MENOR) ,MEDIA 0, VAR=1')
333   FORMAT(5X,'    ERROR UNIFORME (-V¯3,V¯3)  ')
222   FORMAT(3X,'    ERROR NORMAL (0,1)     ' )
C---- DEFINE NUMERO DE HIFOT ALTERNATIVAS A FLANTEAR(PARAMETER CARD 9)
200   READ(1,2002) L1,L2
2002  FORMAT(2I2)
      DO 2000 L=  L1,L2
C     WRITE(5,2005)  L
      WRITE(3,2005)  L
2005  FORMAT(/,5X,'K  = ', I4 )
              DO 8000 I=1 ,T
                     AL=L**0.5
                      TRAT(I) = AL*IR(I)
2000  CONTINUE
C              BLANQUEO CONTADOR DEL NUMERO DE RECHAZOS
C
             DO 7000 I=1,2
               DO 9001 J=1,NS
              KC(I,J)=0.0
```

```
 9001  CONTINUE
 9000  CONTINUE
 C
       SDD=0
       SDD2=0
       SHH=0
       SHH2=0
       SRAL=0
       SRAL2=0
       SFF=0
       SFF2=0
          SRTF=0
          SRTF2=0
          SRSF=0
          SRSF2=0
          SDWN=0
          SDWN2=0
            SHHE=0.
            SHHE2=0.
            SHHEE=0.
            SHHEE2=0.
            TKK = 0.
            DO 40 JC=1,KK
    40      TKK = TKK + 1./(KK -(JC-1))
C
C     GENERATE THE SAMPLES
      DO 60 K=1,NN
C
C     GENERATE AND PROCESS ONE SAMPLE
      H=0.
      HE=0.
      HEE=0.
      D=0.
          DOW=0.
   45 DO 50 I=1,B
      BS=0.
      SRA(I)=0.
      BB=0.
      DO 55 JA=1,T
      RS(I,JA)=0.
      DB(JA)=0.
      HN(JA)=0.
      HNE(JA)=0.
      HNEE(JA)=0.
          DW(JA)=0.
      RAFA(I,JA)=0.
          REW(I,JA)=0.
      SCA(JA)=0.
   55 CONTINUE
      DO 1 J=1,KK
      IF (DISTR ) 1001 ,1003,1002
C     GENERATE ERROR NORMAL (0,1)
1001  CALL VARGEN(DIST.SEED,FLAG,AXAX,BXBX,Y(J),D)
CC1001  CALL GAUSS(SEED,CXCX,AXAX,Y(J))
      GO TO 25
```

```
C   GENERO DISTRIBUCION UNIFORME (-V¯3,+V¯3)
1002  CALL VARGEN(DIST,SEED,FLAG,AXAX,BXBX,Y(J),D)
C1002  CALL RANDU(SEED,IXXX,Y(J))
      Y(J)= (3**0.5 )*(2*Y(J) - 1)
      GO TO 25
1003  CALL VARGEN(DIST,SEED,FLAG,AXAX,BXBX,Y(J),D)
C1003  CALL RANDU(SEED,IXXX,Y(J))
      IF (DISTV) 20,21,22
C
C GENERA DISTRIBUCION LAPLACE
C
20    IF (Y(J)-0.5) 23,23,24
   23 Y(J)= BXBX*ALOG(2.*Y(J))
      GO TO 25
   24 Y(J)=(-1.)*BXBX*ALOG(2.*(1.-Y(J)))
      GO TO 25
C
C  GENERA DISTRIBUCION CAUCHY
C
C     CALL RANDU(SEED,IXXX,Y(J))
C21    Y(J)= BXBX*TAN(3.14159*(Y(J)-0.5))
C      GO TO 25
C
C   GENERO DISTRIBUCION LOGISTICA
C
21     Y(J)=0.5513288956*ALOG(Y(J)/(1-Y(J)))
       GO TO 25
C
C   GENERO DISTRIBUCION VALOR EXTREMO (MENOR)
C
22    Y(J)=0.4500487906 + 0.7796969*ALOG(ALOG(1/Y(J)))
   25 JA=(I-1)*KK + J
      JT=IND(JA)
      YA(J)=Y(J)
      Y(J)=Y(J)+TRAT(JT)+BE(I)
      YAL(JA)=Y(J)
      YY(JA)=DBLE(Y(J))
      BB=BB+Y(J)
      BS=BS+Y(J)*Y(J)
    1 CONTINUE
      BM(I)=BB/RK
      SW(I)=BS - (BB*BB)/RK
      IF (LOB.EQ.0) GO TO 11
      WRITE (3,299) (YA(J),J=1,KK)
      WRITE (3,299) (Y(J),J=1,KK),BE(I),SW(I),BM(I)
  299 FORMAT (1X,13F10.4)
   11 CONTINUE
C     RANK THIS BLOCK AND STORE ITS RANKING
C
         CALL RANK2(Y,RY,REY,KK)
      JB1=KK*(I-1) + 1
      JB2=KK*I
      DO 3 JB=JB1,JB2
      DO 2 J=1,T
      IF (IND(JB).NE.J) GO TO 2
```

```
      JC=JB - JB1 + 1
      RS(I,J)=RY(JC) - (RK+1.)/2.
          RSE(I,J)=REY(JC)- 1.
          REW(I,J)=REY(JC)
    2 CONTINUE
    3 CONTINUE
   50 CONTINUE
CC       RANK TRANSFORM
         CALL RANK(YAL,RTY,LV)
      CALL RANK2(SW,RSW,RESW,B)
      DO 80 JA=1,T
      DO 81 I=1,B
      HN(JA)=HN(JA) + RS(I,JA)*RSW(I)
        HNE(JA)=HNE(JA) + RS(I,JA)*RESW(I)
        HNEE(JA)=HNEE(JA) + RSE(I,JA)*RESW(I)
      DB(JA)=DB(JA) + RS(I,JA)
          DW(JA)=DW(JA) + REW(I,JA)
   81 CONTINUE
      D=D + DB(JA)*DB(JA)
      H=H + HN(JA)*HN(JA)
        HE=HE + HNE(JA)*HNE(JA)
        HEE=HEE + HNEE(JA)*HNEE(JA)
          DOW=DOW + (RR-DW(JA))*(RR-DW(JA))
   80 CONTINUE
CC
       SRESW2=0.
       DO 83 I=1,B
         SRESW2 =SRESW2 +RESW(I)*RESW(I)
83       CONTINUE
C
          SEX=0.
          SREX2=0.
          DO 84 I =1,KK
           EX(I) = 1./(KK-I+1)
              SEX = SEX +EX(I)
              REX (I)= SEX
              SREX2=SREX2 +(REX(I)-1.)*(REX(I)-1.)
84            CONTINUE
      FA=(12.*(RT-1.))/(RR*RT*(RK-1.)*(RK+1.))
      DD=FA*D
      HH=(6.*FA*H)/((RB+1.)*(2.*RB+1.))
          HHE=(12*(RT-1.)*HE)/(SRESW2*RK*(RK-1.)*(RK+1.))
          HHEE=((RT-1.)*HEE)/(SRESW2*SREX2)
          WW = (RT-1.)/(RB*(RK-TKK))
        DWM=WW*DOW
      DO 90 I=1,B
      DO 90 J=1,KK
      JA=KK*(I-1) + J
   90 YAL(JA)=YAL(JA) - BM(I)
      CALL RANK(YAL,RYAL,LV)
      DO 91 I=1,B
      JB1=KK*(I-1) + 1
      JB2=KK*I
      DO 93 JB=JB1,JB2
      SRA(I)=SRA(I) + RYAL(JB)
```

```
      DO 92 J=1,T
      IF (IND(JB).NE.J) GO TO 92
      RAFA(I,J)=RYAL(JB)
      SCA(J)=SCA(J) + RYAL(JB)
   92 CONTINUE
   93 CONTINUE
   91 CONTINUE
      SN=0.
      SD1=0.
      SD2=0.
      DO 101 J=1,T
  101 SN=SN+(SCA(J) - RR*(RB*RK+1.)/2.)**2
      DO 102 I=1,B
      CC=SRA(I)/RK
      SD2=SD2 + ( CC - (RB*RK+1.)/2.)**2
      JB1=KK*(I-1) + 1
      JB2=KK*I
      DO 103 JB=JB1,JB2
  103 SD1=SD1 + (RYAL(JB)-CC)**2
  102 CONTINUE
      RAL=((RT-1.)*SN)/(SD1+RK*(RB-RR)*SD2/(RB-1.))
      CALL ABIBAN(YY,N,IND,IOPT,EM,GM,S,SADJ,NDF,EF,IER)
      FA=(NDF(4)+NDF(5))/(NDF(3)*(S(4)+S(5)))
      FR=SADJ(1)*FA
      FNR=S(3)*FA
      FF=FR
CC    FF=SNGL(FR)
CC      RANK TRANSFORM
      CALL ABIBAN(RTY,N,IND,IOPT,EM,GM,RTS,RTSADJ,NDF,EF,IER)
      RTA=(NDF(4)+NDF(5))/(NDF(3)*(RTS(4)+RTS(5)))
      RTF=RTSADJ(1)*RTA
      RTFNR=RTS(3)*RTA
C
      CALL ABIBAN(RYAL,N,IND,IOPT,EM,GM,RSS,RSADJ,NDF,EF,IER)
      RSA=(NDF(4)+NDF(5))/(NDF(3)*(RSS(4)+RSS(5)))
      RSF=RSADJ(1)*RSA
      RSFNR=RSS(3)*RSA
      IF (LOP.EQ.0) GO TO 111
      DO 7 I=1,B
      WRITE (3,299) (REW(I,J),J=1,T)
CC     WRITE (3,299) (RS(I,J),J=1,T)
    7 CONTINUE
CC      DO 94 I=1,B
CC     WRITE(3,299) (RAFA(I,J),J=1,T),SRA(I)
CC 94 CONTINUE
      WRITE (3,299) (DW(J),J=1,T)
             WRITE (3,299) DOW,WW,DWN
      WRITE (3,299) (SCA(J),J=1,T)
      WRITE (3,39) GM,SADJ(1),SADJ(2),EF
      WRITE (3,39) (S(J),J=1,6)
      WRITE (3,49) (NDF(J),J=1,6),IER
C      WRITE (3,1299) SN,SD1,SD2
C 1299 FORMAT(/,1X,3F14.4)
      WRITE (3,39) FR,FNR
  111 IF (LST.EQ.0) GO TO 112
```

```
        WRITE (3,289) K,HH,DD,RAL,FF,RTF,RSF,DWN,HHE,HHEE
  289 FORMAT (/,1X,I4,10F9.4)
   39 FORMAT (/,10D12.5)
   49 FORMAT (/,10(7X,I3))
  112 CONTINUE
C
C  ESCRIBO LAS ESTADISTICAS EN ARCHIVO EXTERNO
C
C      WRITE (5,389) HH,DD,RAL,FF,RTF,RSF,DWN,HHE,HHEE
C 389 FORMAT (1X,9F8.4)
C
      IF (HH.GE.CH5) KC(1,1)=KC(1,1) + 1
      IF (HH.GE.CH1) KC(2,1)=KC(2,1) + 1
      IF (DD.GE.CH5) KC(1,2)=KC(1,2) + 1
      IF (DD.GE.CH1) KC(2,2)=KC(2,2) + 1
      IF (RAL.GE.CH5) KC(1,3)=KC(1,3) + 1
      IF (RAL.GE.CH1) KC(2,3)=KC(2,3) +1
      IF (FF.GE.FF5) KC(1,4)=KC(1,4) + 1
      IF (FF.GE.FF1) KC(2,4)=KC(2,4) + 1
      IF (FF.GE.FM5) KC(1,5)-KC(1,5) + 1
      IF (FF.GE.FM1) KC(2,5)=KC(2,5) + 1
      IF (RTF.GE.FM5) KC(1,6)=KC(1,6) + 1
      IF (RTF.GE.FM1) KC(2,6)=KC(2,6) + 1
      IF (RSF.GE.FM5) KC(1,7)=KC(1,7) + 1
      IF (RSF.GE.FM1) KC(2,7)=KC(2,7) + 1
       IF (DWN.GE.CH5) KC(1,8)=KC(1,8) + 1
       IF (DWN.GE.CH1) KC(2,8)=KC(2,8) +1
          IF (HHE.GE.CH5) KC(1,9)=KC(1,9) + 1
          IF (HHE.GE.CH1) KC(2,9)=KC(2,9) + 1
          IF (HHEE.GE.CH5) KC(1,10)=KC(1,10) + 1
          IF (HHEE.GE.CH1) KC(2,10)=KC(2,10) + 1
      SDD=SDD+DD
      SDD2=SDD2+DD*DD
      SHH=SHH+HH
      SHH2=SHH2 + HH*HH
      SRAL=SRAL+RAL
      SRAL2=SRAL2+RAL*RAL
      SFF=SFF+FF
      SFF2=SFF2+FF*FF
C
      SRTF=SRTF+RTF
      SRTF2=SRTF2+RTF*RTF
      SRSF=SRSF+RSF
      SRSF2=SRSF2+RSF*RSF
       SDWN=SDWN+DWN
       SDWN2=SDWN2+DWN*DWN
       SHHE =SHHE +HHE
       SHHE2=SHHE2+HHE*HHE
       SHHEE =SHHEE +HHEE
       SHHEE2=SHHEE2+HHEE*HHEE
C
   60 CONTINUE
C
C     CALCULO PROMEDIO Y VARIANZA DE LAS ESTADISTICAS
      AHHH=SHH/NN
```

```
      VHH=(SHH2 -(SHH*SHH)/NN)/(NN-1)
      AMDD=SDD/NN
      VDD=(SDD2 -(SDD*SDD)/NN)/(NN-1)
      AMRAL=SRAL/NN
      VRAL=(SRAL2 -(SRAL*SRAL)/NN)/(NN-1)
      AMFF=SFF/NN
      VFF =(SFF2 -(SFF*SFF)/NN)/(NN-1)
C
      AMRTF=SRTF/NN
      VRTF =(SRTF2 -(SRTF*SRTF)/NN)/(NN-1)
      AMRSF=SRSF/NN
      VRSF =(SRSF2 -(SRSF*SRSF)/NN)/(NN-1)
      AMDWN=SDWN/NN
      VDWN =(SDWN2 -(SDWN*SDWN)/NN)/(NN-1)
C
          AMHHE  =  SHHE/NN
          VHHE   = (SHHE2 -(SHHE*SHHE)/NN)/(NN-1)
          AMHHEE =  SHHEE/NN
          VHHEE  = (SHHEE2 -(SHHEE*SHHEE)/NN)/(NN-1)
CC
      WRITE(3,189) (TRAT(I) , I=1,5)
      WRITE(3,109) AMHH,VHH,AMDD,VDD,AMRAL,VRAL,AMFF,VFF,AMRTF,VRTF,AMR
     1F,VRSF,AMDWN,VDWN,AMHHE,VHHE,AMHHEE,VHHEE
109    FORMAT(//,10X,
     1//,10X,'MEDIA HH = ',F12.6,10X,'VARIANZA HH = ',F14.6,
     2//,10X,'MEDIA DD = ',F12.6,10X,'VARIANZA DD = ',F14.6,
     3//,10X,'MEDIA RAL = ',F12.6,10X,'VARIANZA RAL = ',F14.6,
     4//,10X,'MEDIA FF = ',F12.6,10X,'VARIANZA FF = ',F14.6,
     5//,10X,'MEDIA RTF = ',F12.6,10X,'VARIANZA RTF = ',F14.6,
     6//,10X,'MEDIA RST= ',F12.6,10X,'VARIANZA RST= ',F14.6 ,
     7//,10X,'MEDIA DWN= ',F12.6,10X,'VARIANZA DWN= ',F14.6
     8//,10X,'MEDIA HHE= ',F12.6,10X,'VARIANZA HHE= ',F14.6
     9//,10X,'MEDIA HHEE= ',F12.6,10X,'VARIANZA HHEE= ',F14.6)
      WRITE (3,59)
   59 FORMAT(//,15X,'REJ. BASED ON CHI-SQ. APPROX.',//,1X,'ALPHA',5X,'
     1',5X,'DD',3X,'RAL',4X,'FF',4X,'FM',3X,'RTF',3X,'RST',3X,'DWN',
     23X,'HHE',2X,'HHEE',/)
      WRITE (3,159) (KC(1,J),J=1,NS)
  159 FORMAT (/,' 0.05',2X,10I6)
      WRITE (3,359) (KC(2,J),J=1,NS)
  359 FORMAT (/,' 0.01',2X,10I6)
2000  CONTINUE
      STOP
      END
C
      SUBROUTINE ABIBAN(Y,N,IND,IOPT,EM,GM,S,SADJ,NDF,EF,IER)
      DIMENSION Y(800),IND(800),EM(1163),NDF(6),S(6),N(5),SADJ(5)
CC    DIMENSION Y(9),IND(36),EM(6),NDF(6),S(6),N(5),SADJ(9)
CC    DIMENSION Y(1),IND(1),EM(1),NDF(1),S(1),N(1),SADJ(1)
      DOUBLE PRECISION Z,XN,SUM,Z1,SUM1,S
      DATA ZERO/0.0/,ONE/1.0/
      IER = 0
      NS = N(1)
      NK = N(2)
      NB = N(3)
```

```
      NR = N(4)
      NT = N(5)
      IF (NT .GE. 3 .AND. NT .GT. NK) GO TO 5
C     TERMINAL ERROR
      IER = 129
      GO TO 9000
    5 IF (NR .GE. 1 .AND. NB .GT. NT/NK) GO TO 10
C     TERMINAL ERROR
      IER = 130
      GO TO 9000
   10 NIND = NK*NB
      GO TO 15
C
      ENTRY ABALAT(Y,N,IND,IOPT,EM,GM,S,SADJ,NDF,EF,IER)
C
      NK = N(2)
      NT = NK*NK
      NR = NK+1
      NB = NT+NK
      NS = N(1)
      IER = 0
      NIND = NT*NR
   15 IER = 131
      IF (N(1) .LT. 1 .OR. N(2) .LT. 2) GO TO 9000
      IER = 0
      K = NIND*(NT+1)/2
      J = 0
      DO 20 I = 1,NIND
   20 J = J+IND(I)
      IF (J .EQ. K) GO TO 25
C                 TERMINAL ERROR - INPUT EXPERIMENTAL
C                 DESIGN IS NOT A BALANCED INCOMPLETE
C                 BLOCK OR A BALANCED LATTICE DESIGN
      IER = 132
      GO TO 9000
   25 NKB = NB*NK
      NN = NB+NT+NR+NKB
      NT1 = NT-1
      NK1 =NK-1
      NTK = NT-NK
      NSK = NS*NK
C                 INITIALIZE VECTOR
      DO 30 I = 1,NN
   30 EM(I) = ZERO
      K = 1
      KK = 1
      KKK = 1
      NBKS = NB*NSK
      NBL = NBKS/(NR*NS)
      N2 = NB
      N3 = NB+NT
      N4 = N3+NR
      XN = 1.D0/NS
      GN = ZERO
      LL = N3+1
```

```
      DO 45 I = 1,NB
         DO 45 J = 1,NK
            Z = 0.0D0
            DO 35 L = 1,NS
               Z = Z+Y(K)
   35       K = K+1
         NTR = IND(KK)
C              CALCULATE TREATMENT BLOCK TOTAL
      IF (NS .GT. 1) EM(N4+KK) = Z
C              CALCULATE BLOCK TOTALS
      EM(I) = EM(I)+Z
C              CALCULATE TREATMENT TOTALS
      EM(NTR+N2) = EM(NTR+N2)+Z
C              CALCULATE TOTAL FOR GROUP OF
C              REPLICATES
      IF (KKK .LE. NRL) GO TO 40
      LL = LL+1
      KKK = KKK-NRL
   40 EM(LL) = EM(LL)+Z
      KKK = KKK+1
      GM = GM+Z
   45 KK = KK+1
C              CALCULATE GRAND MEAN
      GGM = GM*NK1
      XX = NTK
      XY = NT1*NSK
      GM = GM/NBKS
C              CALCULATE MEAN FOR GROUPS OF REPLICATES
      XN = 1.D0/(NRL*NS)
      N1 = N3+1
      N4 = N3+NR
      DO 50 I = N1,N4
   50 EM(I) = EM(I)*XN
      IF (NR .EQ.1) GO TO 60
      SUM = 0.D0
      DO 55 I = N1,N4
         Z = EM(I)-GM
   55 SUM = SUM+Z*Z
      S(1) = SUM*NRL*NS
      GO TO 65
   60 S(1) = ZERO
   65 NDF(1) = NR-1
C              CALCULATE BLOCK MEANS
      SUM = 0.D0
      XN = 1.D0/NSK
      DO 70 I = 1,NB
      EM(I) = EM(I)*XN
      Z = EM(I)-GM
   70 SUM = SUM+Z*Z
      S(2) = SUM*NSK-S(1)
      NDF(2) = NB-NR
      NN = NKB
      C = FLOAT(NN)/NT
      X = NBKS*GGM
      EF = FLOAT(NT)/FLOAT(NBKS)
```

```
      Z1 = 0.D0
      Z = 0.D0
C                 CALCULATE ADJUSTED TREATMENT MEANS
      DO 80 I = 1,NT
         NN = NB+I
      SUM = EM(NN)
      SADJ(I) = SUM
      SUM1 = 0.D0
      XN = SUM*EF-GM
      Z1 = Z1+XN*XN
      L = 1
      DO 75 J = 1,NB
         DO 75 K = 1,NK
            IF (IND(L).NE.I) GO TO 75
            SUM1 = SUM1+EM(J)
   75         L = L+1
      SUM = SUM-NS*SUM1
      EM(NN) = XX*SADJ(I)+GGM-XY*SUM1
   80 Z = Z+SUM*SUM
      EF = FLOAT(NT*NK1)/FLOAT(NK*NT1)
      XX = Z/(EF*C*NS)
      NDF(3) = NT1
C                 CALCULATE TREATMENT BLOCK MEANS
      IF (NS .EQ. 1) GO TO 90
      SUM = 0.D0
      NBK = NKB+N4
      N1 = N4+1
      XN= 1.D0/NS
      K=1
      DO 85 I = N1,NBK
         EM(I) = EM(I)*XN
         DO 85 J=1,NS
            Z=Y(K)-EM(I)
            K=K+1
   85 SUM = SUM+Z*Z
      S(5) = SUM
      NDF(5) = NKB*(NS-1)
      GO TO 95
   90 NDF(5) = 0
      S(5)= ZERO
   95 SUM = 0.D0
      DO 100 I = 1,NBKS
         Z = Y(I)-GM
  100 SUM = SUM+Z*Z
      S(6) = SUM
      NDF(6) = NBKS-1
      S(4) = S(6)-S(1)-S(2)-XY-S(5)
      NDF(4) = NDF(6)-NDF(1)-NDF(2)-NDF(3)-NDF(5)
      S(3) = NS*C*Z1
      EE = S(4)/NDF(4)
      XY = S(2)+XX-S(3)
      IF (IOPT .NE. 0) GO TO 105
C                 INTRABLOCK ANALYSIS
      THETA = ONE/(NT*NK1)
      GO TO 110
```

```
C                   INTERBLOCK ANALYSIS
  105 EB = XY/NDF(2)
      THETA = ZERO
      IF (EB.GE.EE) THETA = (NDF(2)*(EB-EE))/(NT*NK1*NDF(2)*EB+EE*
     * NTK*(NDF(2)-NDF(3)))
  110 N1 = NB+1
      N2 = NB+NT
      K = 1
      Z = FLOAT(NT)/FLOAT(NBKS)
      DO 115 I = N1,N2
         FM(I) = (SADJ(K)+THETA*EM(I))*Z
         K = K+1
  115 CONTINUE
      SADJ(1) = XX
      SADJ(2) = XY
      IF (IOPT .EQ. 0) SADJ(2) = ZERO
      IF (NR.EQ.1) GO TO 120
      EF = (S(4)+XY)/(EE*(ONE+NTK*THETA)*(NKB-NR-NT1))
      IF (IOPT .EQ.0) GO TO 9005
      IF (EB .LT. EE) IER = 37
      IF (EF.LT.ONE) IER = 38
      IF (IER) 9000,9005,9000
  120 EF = ZERO
      GO TO 9005
 9000 CONTINUE
 9005 CONTINUE
CC    CALL UERTST (IER,6HABIBAN)
C     WRITE (3,9875)
C9875  FORMAT ('DATOS CALCULADOS DENTRO DE LA SUBRUTINA ABIBAN') ,
C      WRITE(3,9876) NDF(1),NDF(2),NDF(3),NDF(4),NDF(5),NDF(6),
C     1S(1),S(2),S(3),S(4),S(5)
C 9876 FORMAT(5X,'NDF(1)=',I4,2X,'NDF(2)=',I4,
C     1'NDF(3)=',I4,2X,'NDF(4)=',I4,2X,'NDF(5)=',I4,2X,'NDF(6)=',I4,
C     2/ 5X,'S(1)=', F12.4,2X,'S(2) =',F12.4,2X,'S(3)=', F12.4,/,5X,
C     3'S(4) =',F12.4,2X,'S(5)=',F12.4,/)
C      WRITE (3,9877)
C9877  FORMAT(  25X,'SALI DE LA SUBRUTINA ABIBAN')
C9005  RETURN
      RETURN
      END
C SUBROUTINE RANK
C PURPOSE  RANK A VECTOR OF VALUES
C USAGE CALL RANK(A,R,N)
C DESCRIPTION OF PARAMETERS
C      A - INPUT VECTORS OF N VALUES
C      R - OUTPUT VECTORS OF LENGTH N. SMALLEST VALUE IS RANKED 1
C            LARGEST IS RANKED N. TIES ARE ASSIGNED AVERAGE OF TIED RANKS
C      N - NUMBERS  OF VALUES
C
C  METHOD
C        VECTOR IS SEARCHED FOR SUCCESSIVELY LARGER ELEMENTS.
C        IF TIES OCCUR , THEY ARE LOCATED AND THEIR RANK VALUE COMPUTE
C        FOR EXAMPLE , IF TWO VALUES ARE TIED FOR SIX RANK,THEY ARE AS
C         ASSIGNED RANK OF 6.5 ( =( 6 + 7 )/2)
C
```

```
C ****************************************************************************
C
      SUBROUTINE RANK (A,R,N)
      DIMENSION A(1) ,R(1)
C        INITIALIZATION
C
      DO 10 I=1,N
C         FIND RANK OF DATA
C
      R(I) = 0.0
   10 CONTINUE
      DO 100 I = 1,N
C         TEST WHETHER DATA POINT IS ALREADY RANKED
C
      IF (R(I))  20 , 20 , 100
C           DATA POINT TO BE RANKED
C
   20 SMALL=0.0
      EQUAL=0.0
      X= A(I)
            DO 50 J = 1  , N
                   IF ( A(J) - X )  30 ,40 , 50
C         COUNT NUMBER OF DATA POINTS WHICH ARE SMALLER
   30 SMALL = SMALL + 1.0
      GO TO 50
C         COUNT NUMBER OF DATA POINTS WHICH ARE EQUALS

   40 EQUAL = EQUAL + 1.0
      R(J)  = -1.0
   50 CONTINUE
C         TEST FOR TIE
      IF(EQUAL-1.0)   60 ,60, 70
C         STORE RANK OF DATA  POINT WHERE NO TIE
   60 R(I)  = SMALL + 1.0
      GO TO 100
C           CALCULATE RANK OF TIED DATA POINTS
C
   70  P = SMALL + (EQUAL + 1.0 )*0.5
                   DO 90 J = 1  , N
                         IF ( R(J) + 1.0)  90 ,80 ,90
   80  R(J) = P
   90  CONTINUE
  100  CONTINUE
       RETURN
       END
C SUBROUTINE RANK2
C PURPOSE  RANK LINEAR AND EXPONENTIALLY
C  A VECTOR OF VALUES
C USAGE CALL RANK2(A,R,N)
C DESCRIPTION OF PARAMETERS
C     A - INPUT VECTORS OF N VALUES
C     R - OUTPUT VECTORS OF LENGTH N. SMALLEST VALUE IS RANKED 1/N.
C        LARGEST IS RANKED .... TIES ARE ASSIGNED AVERAGE OF TIED RANKS
C     N - NUMBERS  OF VALUES
C
```

```
C  METHOD
C        VECTOR IS SEARCHED FOR SUCCESSIVELY LARGER ELEMENTS.
C        IF TIES OCCUR , THEY ARE LOCATED AND THEIR RANK VALUE COMPUTE
C        FOR EXAMPLE , IF TWO VALUES ARE TIED FOR SIX RANK,THEY ARE AS
C         ASSIGNED RANK OF 6.5 ( =( 6 + 7 )/2)
C
C *******************************************************************
C
      SUBROUTINE RANK2 (A,R,RE,N)
      DIMENSION A(1),R(1),RE(1)
C          INITIALIZATION
        DO 10 I=1,N
C           FIND RANK OF DATA
        R(I) = 0.0
   10   CONTINUE
        DO 100 I = 1,N
C           TEST WHETHER DATA POINT IS ALREADY RANKED
        IF (R(I))  20 , 20 , 100
C            DATA POINT TO BE RANKED
   20   SMALL=0.0
        EQUAL=0.0
        X= A(I)
              DO 50 J = 1  , N
                     IF ( A(J) - X )  30 ,40 , 50
C          COUNT NUMBER OF DATA POINTS WHICH ARE SMALLER
   30   SMALL = SMALL + 1.0
        GO TO 50
C          COUNT NUMBER OF DATA POINTS WHICH ARE EQUALS

   40   EQUAL = EQUAL + 1.0
        R(J)  = -1.0
   50   CONTINUE
C          TEST FOR TIE
        IF(EQUAL-1.0)   60 ,60, 70
C          STORE RANK OF DATA  POINT WHERE NO TIE
   60   R(I)= SMALL + 1.0
        IR  = SMALL + 1.0
          RE(I) = 0.0
          DO 55  J=1,IR
            RE(I) = RE(I) + 1.0/(N - (J-1))
   55     CONTINUE
        GO TO 100
C           CALCULATE RANK OF TIED DATA POINTS
C
   70    R = SMALL + (EQUAL + 1.0 )*0.5
            IAA=SMALL
            IA = SMALL + 1
            IB = SMALL + EQUAL
             RE(I), = RE(IAA)
           DO 75 J=IA,IB
             RE(I) = RE(I) + 1.0/(N-(J-1))
   75      CONTINUE
           Q = RE(I)/EQUAL
                    DO 90 J = 1  , N
                          IF ( R(J) + 1.0)  90 ,80 ,90
```

```
   80 R(J) = P                                                          BIB07640
        RE(J) = Q                                                       BIB07650
   90 CONTINUE                                                          BIB07660
  100 CONTINUE                                                          BIB07670
      RETURN                                                            BIB07680
      END                                                               BIB07690
```