SOME PROPERTIES OF A BAYES TWO-STAGE TEST FOR THE MEAN[1]

by

Morris Skibinsky
Institute of Statistics
University of North Carolina

Institute of Statistics
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"Beware the Jabberwock, my son!
  The jaws that bite, the claws that catch!
Beware the Jubjub bird, and shun
  The frumious Bandersnatch!"

He took his vorpal sword in hand;
  Long time the manxome foe he sought--

. . . .

One, two! One, two! And through and through
  The vorpal blade went snicker-snack!
He left it dead, and with its head
  He went galumphing back.

. . . .

'Twas brillig, and the slithy toves
  Did gyre and gimble in the wabe;
All mimsy were the borogoves,
  And the mome raths outgrabe.

Jabberwocky
(Lewis Carroll)

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Morris Skibinsky

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INTRODUCTION

A fairly general statement of Wald's decision problem [5][1], for two stages may be expressed as follows. Let X be a random variable with frequency function, $f_\theta(x)$, where θ may be a real or vector valued parameter in some subset, Ω, of the real line or of a finite dimensional euclidian space. Suppose we are given a sequence of independent observations, $x_1, x_2, \ldots$, on X. On the basis of these observations we are to select one of a finite number of possible alternative courses of action, $A_0, A_1, \ldots, A_k$, which comprise a set $\mathcal{A}$ of possible alternatives, by using the following decision rule.

We are given numbers, q_m, $m=1,2,\ldots$; functions, $p_\nu(\underline{x}_m)$, defined for all points $\underline{x}_m$ in m-space, and for $m=1,2,\ldots$, $\nu=0,1,2,\ldots$; and functions, $\delta_{m+\nu}(A_i, \underline{x}_{m+\nu})$, defined for all $A_i \epsilon \mathcal{A}$, all points $\underline{x}_{m+\nu}$ in $(m+\nu)$-space, and for $m=1,2,\ldots,\nu=0,1,2,\ldots$; such that always

$$(0.1) \qquad q_m,\ p_\nu(\underline{x}_m),\ \delta_{m+\nu}(A_i, \underline{x}_{m+\nu}) \geq 0 \quad ,$$

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[1] Numbers in square brackets refer to bibliography.

and

$$(0.2) \quad \sum_{m=1}^{\infty} q_m = \sum_{\nu=0}^{\infty} p_\nu(\underline{x}_m) = \sum_{A_i \in \mathcal{A}} \delta_{m+\nu}(A_i, \underline{x}_{m+\nu}) = 1.$$

<u>Rule</u>

1. Take m observations, with probability q_m.
2. If the observed sample is $\underline{x}_m$, take a second sample of ν observations, with probability $p_\nu(\underline{x}_m)$.
3. If the total observed sample is $\underline{x}_{m+\nu}$, accept alternative A_i with probability $\delta_{m+\nu}(A_i, \underline{x}_{m+\nu})$.

The problem is to find sequences q, p, and δ, subject to the indicated restrictions, which are optimum in some sense. In particular, suppose we are given a loss function, $W(\Theta, A_i)$, defined for all $\Theta \epsilon \Omega$, and all $A_i \epsilon \mathcal{A}$, non-negative and bounded, representing the loss incurred by accepting alternative A_i when Θ is the true parameter; a c.d.f., $\lambda(\Theta)$, defined over Ω; and suppose the cost per observation to be a constant, c. We may then seek those sequences, q, p, δ, for which the average expected loss is minimum, i.e. a Bayes solution.

Using the above rule, the expected number of observations required, given Θ as the true parameter, is

$$(0.3) \qquad E_\Theta n = \sum_{m=1}^{\infty} q_m \left[m + \sum_{\nu=0}^{\infty} \nu E_\Theta p_\nu(\underline{x}_m) \right] ,$$

where E_Θ denotes the expectation of a function of the observations on X, when Θ is the true parameter value. The probability, given Θ, that the rule will accept A_i is

$$(0.4) \qquad \sum_{m=1}^{\infty} q_m \sum_{\nu=0}^{\infty} E_\Theta \left[p_\nu(\underline{x}_m) \delta_{m+\nu}(A_i, \underline{x}_{m+\nu}) \right] .$$

Hence, the risk or expected loss incurred by use of the rule, given Θ, is

$$(0.5) \; \sum_{m=1}^{\infty} q_m \left\{ cm + \sum_{\nu=0}^{\infty} E_\Theta \left[\{ c\nu + \sum_{A_i \in \Omega} W(\Theta, A_i) \delta_{m+\nu}(A_i, \underline{x}_{m+\nu}) \} p_\nu(\underline{x}_m) \right] \right\} ,$$

and the average risk over Ω is, after making the permissible indicated interchange of integration and summation operations,

$$(0.6) \qquad \sum_{m=1}^{\infty} q_m \left\{ cm + \sum_{\nu=0}^{\infty} \int_{\mathcal{X}_{m+\nu}} p_\nu(\underline{x}_m) \left[c\nu \int_{\Omega} f_\Theta(\underline{x}_{m+\nu}) d\lambda \right. \right.$$

$$+ \sum_{A_i \in \mathcal{A}} \delta_{m+\nu}(A_i, \underline{x}_{m+\nu}) \int_{\Omega} W(\Theta, A_i) f_\Theta(\underline{x}_{m+\nu}) d\lambda \Big] d\underline{x}_{m+\nu} \quad ,$$

where $X_{m+\nu}$ stands for the $(m+\nu)$-dimensional observation space, and where by $f_\Theta(\underline{x}_{m+\nu})$, we denote the joint frequency function of the first $m+\nu$ observation on X. The existance of a sequence, δ, which for any fixed sequences, q, p, and fixed point, $\underline{x}_{m+\nu}$, in $(m+\nu)$-space, will minimize (0.6), is immediately apparent. Let

$$(0.7) \qquad a_j(\underline{x}_{m+\nu}) = \int_{\Omega} W(\Theta, A_j) f_\Theta(\underline{x}_{m+\nu}) d\lambda,$$

and suppose $\min(a_1, a_2, \ldots)$ is unique, then such a sequence is

$$(0.8) \quad \delta_{m+\nu}(A_j, \underline{x}_{m+\nu}) = \begin{cases} 1 \ , & a_j = \min(a_1, a_2, \ldots) \\ \\ 0 \ , & \quad \neq \end{cases} \qquad j=1,2,\ldots \quad .$$

Modifications of (0.8) for which the restriction of a unique minimum may be removed are easily made.

To complete our Bayes solution, we need sequences, q and p which will minimize (0.6) when the sequence δ is as defined by (0.8) or a suitable modification. Let

$$(0.9) \ R_\nu(A_i, \underline{x}_m) = \{(x_{m+1}, \ldots, x_{m+\nu}) : \delta_{m+\nu}(A_i, \underline{x}_{m+\nu}) = 1\} \ , \quad \nu=1,2,\ldots,$$

then the average risk may be written

$$(0.10) \qquad \sum_{m=1}^{\infty} q_m \Big(cm + \int_{\mathcal{X}_m} \sum_{\nu=0}^{\infty} p_\nu(\underline{x}_m)\, \mathcal{G}_\nu(\underline{x}_m)\, d\underline{x}_m\Big) \quad ,$$

where, for m=1,2,...,

$$(0.11)\ \mathcal{G}_\nu(\underline{x}_m) = \begin{cases} \displaystyle\int_{\Omega} \Big[c\nu + \sum_{A_i \in \mathcal{A}} W(\Theta,A_i) P_\Theta(R_\nu(A_i,\underline{x}_m))\Big] f_\Theta(\underline{x}_m)\, d\lambda, \nu=1,2,\ldots \\ \displaystyle\int_{\Omega} \Big[\sum_{A_i \in \mathcal{A}} W(\Theta,A_i) \delta_m(A_i,\underline{x}_m)\Big]\ f_\Theta(\underline{x}_m)\, d\lambda \qquad ,\nu=0 \quad , \end{cases}$$

and $P_\Theta(R_\nu(A_i,\underline{x}_m))$ represents the conditional probability of accepting A_i, given that the first sample is $\underline{x}_m$ and that ν observations are taken in the second. Now

$$(0.12) \quad \int_{\Omega} \min_{A_i \in \mathcal{A}} W(\Theta,A_i) f_\Theta(\underline{x}_m)\, d\lambda \leq \mathcal{G}_\nu(\underline{x}_m) - \Big[c \int_{\Omega} f_\Theta(\underline{x}_m)\, d\lambda\Big] \cdot \nu$$

$$\leq \int_{\Omega} \max_{A_i \in \mathcal{A}} W(\Theta,A_i) f_\Theta(\underline{x}_m)\, d\lambda,$$

so that $\mathcal{G}_\nu(\underline{x}_m)$ is non-negative and has at least one absolute minimum with respect to ν, for

$$(0.13) \quad \nu \le \frac{\int_{\Omega} \max_{A_i, A_j \in \mathcal{A}} \left[W(\theta, A_i) - W(\theta, A_j) \right] f_\theta(\underline{x}_m) d\lambda}{c \int_{\Omega} f_\theta(\underline{x}_m) d\lambda} \quad .$$

It cannot have an absolute minimum w.r.t. ν, for ν greater than this number. For each $\underline{x}_m$, let $\nu(\underline{x}_m)$ be a value of ν for which $\mathcal{R}_\nu(\underline{x}_m)$ is absolutely minimum. One minimizing sequence, p, is then seen to be

$$(0.14) \quad p_\nu(\underline{x}_m) = \begin{cases} 1 \;, & \nu = \nu(\underline{x}_m) \\ 0 \;, & \ne \end{cases} \;, \qquad \begin{matrix} \nu=0,1,2,\ldots \\ m=1,2,\ldots \end{matrix} \quad .$$

If we use this sequence, the average risk may, by (0.10), be written

$$(0.15) \qquad \sum_{m=1}^{\infty} q_m \mathcal{H}_m \quad ,$$

where

$$(0.16) \quad \mathcal{H}_m = cm + \int_{\mathcal{X}_m} \mathcal{R}_{\nu(\underline{x}_m)}(\underline{x}_m)\, d\underline{x}_m \quad , \quad m=1,2,\ldots \quad .$$

Suppose there exists a positive integer m = m*, say, such that

$$(0.17) \qquad \mathcal{H}_{m^*} = \min_{m} \mathcal{H}_m \quad ,$$

then a sequence, q, for which the average risk is minimum is

$$(0.18) \qquad q_m = \begin{cases} 1 \ , & m = m^* \\ 0 \ , & \neq \end{cases} \quad , \quad m=1,2,\ldots,$$

and this in a formal sense would complete the Bayes solution.

Using this solution to our decision rule, gives us, by (0.3),

$$(0.19) \qquad E_{\Theta} n = m^* + E_{\Theta} \nu(\underline{x}_{m^*}) \quad .$$

By (0.4), the probability, given Θ, that the rule will accept A_i is

$$(0.20) \qquad E_{\Theta} \delta_{m^*+\nu(\underline{x}_{m^*})} \left(A_i, \underline{x}_{m^*+\nu(\underline{x}_{m^*})}\right) \quad .$$

The minimum average risk over Ω among all such rules is

$$(0.21) \qquad cm^* + \int_{\mathcal{X}_{m^*}} \mathcal{Q}_{\nu(\underline{x}_{m^*})}(\underline{x}_{m^*})\, d\underline{x}_{m^*} \quad .$$

Consider now that the size of the first sample, m, is given and suppose the set $\mathcal{A}$ contains only the two alternatives, A_0, A_1. The above rule may then be simplified as follows. Given functions, $p_\nu(\underline{x}_m)$, $\phi_{m+\nu}(\underline{x}_{m+\nu})$, defined for $\nu=0,1,\ldots$ and, respectively for all points $\underline{x}_m$ in m-space and all points $\underline{x}_{m+\nu}$ in $(m+\nu)$-space, such that always

$$(0.22) \qquad 0 \leq p_\nu(\underline{x}_m),\ \phi_{m+\nu}(\underline{x}_{m+\nu}) \leq 1 \quad , \qquad \sum_{\nu=0}^{\infty} p_\nu(\underline{x}_m) = 1 \quad ,$$

1. Take m observations
2. If the observed sample is $\underline{x}_m$, take a second sample of ν observations, with probability $p_\nu(\underline{x}_m)$.
3. If the total observed sample is $\underline{x}_{m+\nu}$, accept A_1 with probability $\phi_{m+\nu}(\underline{x}_{m+\nu})$. Accept A_0 with one minus this probability.

The sequence, ϕ, is obviously related to the sequence, δ, of the more general rule. Again, suppose we are given the non-negative loss functions, $W_i(\Theta) = W(\Theta,A_i)$, defined for i=0,1, and for all $\Theta \in \Omega$, representing the loss incurred by accepting A_i when Θ is the true parameter; a c.d.f., $\lambda(\theta)$, defined over Ω, and suppose the cost per observation to be a constant, c.

The average risk, over Ω, involved in the use of this rule is

$$(0.23)\quad cm + \sum_{\nu=0}^{\infty} \int_{\mathcal{X}_{m+\nu}} p_\nu(\underline{x}_m) \left\{ \int_\Omega \left[c\nu + W_0(\theta) \right] f_\theta(\underline{x}_{m+\nu})\, d\lambda \right.$$

$$\left. + \phi_{m+\nu}(\underline{x}_{m+\nu}) \int_\Omega \left[W_1(\theta) - W_0(\theta) \right] f_\theta(\underline{x}_{m+\nu}) d\lambda \right\} d\underline{x}_{m+\nu} \quad .$$

A sequence, ϕ, which minimizes this, for any fixed sequence, p, is clearly seen to be

$$(0.24)\quad \phi_{m+\nu}(\underline{x}_{m+\nu}) = \begin{cases} 1 \, , & \int_\Omega W_1(\theta) f_\theta(\underline{x}_{m+\nu}) d\lambda < \int_\Omega W_0(\theta) f_\theta(\underline{x}_{m+\nu}) d\lambda \\ 0 \, , & \qquad\qquad\qquad \geq \end{cases} \quad .$$

Let

$$(0.25)\quad R_\nu(\underline{x}_m) = \{(x_{m+1}, \ldots, x_{m+\nu}) : \phi_{m+\nu}(\underline{x}_{m+\nu}) = 1\} \quad , \quad \nu = 0, 1, \ldots ,$$

where ϕ is as defined by (0.24), and the relation of this set to (0.9) is obvious, then the average risk may be written

$$(0.26) \qquad cm + \int_{\mathcal{X}_m} \sum_{\nu=0} p_\nu(\underline{x}_m) \mathcal{Q}_\nu(\underline{x}_m)\, d\underline{x}_m$$

where

$$(0.27) \mathcal{Q}_\nu(\underline{x}_m) = \begin{cases} \int_\Omega \{c\nu + W_0(\Theta)\,[1 - P_\Theta(R_\nu(\underline{x}_m))] + W_1(\Theta) P_\Theta(R_\nu(\underline{x}_m))\} f_\Theta(\underline{x}_m) d\lambda, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nu=1,2,\ldots \\ \int_\Omega \{W_0(\Theta)\,[1 - \phi_m(\underline{x}_m)] + W_1(\Theta)\phi_m(\underline{x}_m)\} f_\Theta(\underline{x}_m) d\lambda, \quad \nu=0 \quad . \end{cases}$$

Clearly this is a special case of (0.11), so that by (0.12), it is non-negative and has at least one absolute minimum with respect to ν in the interval defined by (0.13). It cannot have an absolute minimum outside of this interval. The sequence, p, which minimizes (0.26) is just (0.14) for our given value of m, and this again in a formal sense, completes our Bayes solution.

Using this solution, the expected size of the second sample becomes

$$(0.28) \qquad E_\Theta \nu(\underline{x}_m) \quad .$$

The probability, given Θ, that the rule will accept A_1 is

$$E_\theta \phi_{m+\nu(\underline{x}_m)}(\underline{x}_{m+\nu(\underline{x}_m)}) \quad , \tag{0.29}$$

and the minimum average risk is just (0.21), with m^* replaced by our given value of m.

In this paper, we are concerned with a particularization of the general Bayes problem outlined above. In this case, Ω consists of two points on the real line, say θ_0 and θ_1, with $\theta_0 < \theta_1$. We prefer alternative A_0, when $\theta = \theta_0$, A_1, when $\theta = \theta_1$. We are given the following apriori distribution over Ω

$$\lambda(\theta_i) = g_i \ , \ i=0,1 \ , \quad g_0 + g_1 = 1 \ , \ g_0, g_1 \neq 0. \tag{0.30}$$

Our loss functions are

$$W_i(\theta_j) = \begin{cases} W_i \ , \ i \neq j \\ 0 \ , \ i = j \end{cases} \quad , \quad i,j,=0,1 \quad . \tag{0.31}$$

Let

$$\underline{x}_\nu' = (x_{m+1}, \ldots, x_{m+\nu}) \tag{0.32}$$

(0.33) $$r_m = \frac{f_{\theta_1}(\underline{x}_m)}{f_{\theta_0}(\underline{x}_m)}\ , m=1,2,\ldots, \qquad r_\nu' = \frac{f_{\theta_1}(\underline{x}_\nu')}{f_{\theta_0}(\underline{x}_\nu')}\ , \quad \nu=1,2,\ldots,$$

and note that

(0.34) $$r_{m+\nu} = r_m r_\nu' \quad .$$

Let

(0.35) $$\lambda = \frac{W}{g} \quad , \quad W = \frac{W_1}{W_0} \quad , \quad g = \frac{g_1}{g_0} \quad ,$$

then the sequence of decision functions for which the average risk is a minimum is, by (0.24)

(0.36) $$\phi_{m+\nu}(\underline{x}_{m+\nu}) = \begin{cases} 1 \ , & r_{m+\nu} > \lambda \\ & \\ 0 \ , & \quad\ \leq \end{cases} \quad , \quad \nu=0,1,\ldots \quad .$$

By (0.25)

(0.37) $$R_\nu(\underline{x}_m) = \{x_\nu' : r_\nu' > \frac{\lambda}{r_m}\} \quad ,$$

so that by (0.27)

$$(0.38)\ \mathcal{Q}_{\nu}(\underline{x}_m) = \begin{cases} (c\sum_{i=0}^{1} g_i f_{\theta_i}(\underline{x}_m))\cdot\nu + W_0 g_1 f_{\theta_1}(\underline{x}_m) P_{\theta_1}(\underline{x}_{\nu}^{'}: r_{\nu}^{'} \leq \frac{\lambda}{r_m}) \\ \qquad + W_1 g_0 f_{\theta_0}(\underline{x}_m) P_{\theta_0}(\underline{x}_{\nu}^{'}: r_{\nu}^{'} > \frac{\lambda}{r_m}) \ , \quad \nu=1,2,\ldots \\ W_0 g_1 [\underline{1}-\phi_m(\underline{x}_m)] + W_1 g_0 f_{\theta_0}(\underline{x}_m)\phi_m(\underline{x}_m) \end{cases}$$

$$= \begin{cases} W_1 g_0 f_{\theta_0}(\underline{x}_m), r_m > \lambda \\ W_0 g_1 f_{\theta_1}(\underline{x}_m), r_m \leq \lambda \end{cases} \ , \quad \nu=0 \quad .$$

As justification for pursuing a Bayes approach to this problem, it may be noted that Wald and Wolfowitz in their paper on the "Optimum Character of the Sequential Probability Ratio Test" [4], proved, for the problem of deciding between two simple alternatives, that for arbitrary apriori probabilities, g_0, g_1, and cost c, every sequential probability ratio test can be regarded as a Bayes solution w.r.t. some values $\overline{W}_0$, $\overline{W}_1$, say, of W_0, W_1, and hence that

$$(0.39)\quad \sum_{i=0}^{1} g_i [\overline{W}_i\alpha_i(S_0)+cE_i^0 n] \leq \sum_{i=0}^{1} g_i [\overline{W}_i\alpha_i(S_1)+cE_i^1 n] \quad ,$$

where S_0 is any sequential probability ratio test for deciding between two simple alternatives, S_1, any other test for the same purpose; $\alpha_i(S_j)$, $i,j=0,1$, is the probability, under S_j, of rejecting H_i when it is true; $E_i^j n$ is the expected number of observations under S_j, when H_i is true (existence assumed). From this it follows, almost immediately, that

$$\alpha_i(S_1) \leq \alpha_i(S_0) \ , \ i=0,1 \quad \Longrightarrow E_i^0 n \leq E_i^1 n \quad , \quad i=0,1 \quad .$$

It will be shown that in certain special cases, similar properties hold for the Bayes two-stage test.

CHAPTER I

GENERAL PROPERTIES OF THE SECOND SAMPLE SIZE FUNCTION IN THE NORMAL CASE

1. Nature of the Defining Equation.

In the following sections, we consider the particular case outlined in the latter part of our introduction, when

$$(1.1)\quad f_\theta(\underline{x}_N) = (\sqrt{2\pi})^{-N} e^{-\frac{1}{2}\sum_{i=1}^{N}(x_i-\theta)^2} , \quad N = 1,2,\ldots \; .$$

We let

$$(1.2)\quad s_m = \sum_{i=1}^{m} x_i \; , \; m = 1,2,\ldots \; , \quad s_\nu' = \sum_{i=m+1}^{m+\nu} x_i \; , \; \nu = 1,2,\ldots \; ,$$

so that

$$s_{m+\nu} = s_m + s_\nu' \; .$$

Let

$$d = \theta_1 - \theta_0 \quad , \quad \bar{\theta} = \frac{1}{2}(\theta_0 + \theta_1)$$

$$(1.5)\qquad t_N = s_N - \bar{\theta}N - \frac{1}{d}\log\lambda\ , \quad N = 1,2,\ldots\ ,$$

then by (0.33),

$$(1.6)\qquad r_m = \lambda e^{dt_m}\ , \quad m = 1,2,\ldots\ ,$$

and by (0.34),

$$(1.7)\qquad r'_\nu = \frac{r_{m+\nu}}{r_m} = e^{d(s'_\nu - \bar{\theta}\nu)}\ , \quad \nu = 1,2,\ldots\ .$$

Thus, by (0.36)

$$(1.8)\qquad \phi_{m+\nu}(\underline{x}_{m+\nu}) = \begin{cases} 1\ , & t_{m+\nu} > 0 \\ 0\ , & \phantom{t_{m+\nu}} \leq \end{cases}\ , \quad \nu = 0,1,\ldots\ .$$

Using (0.38), we have that the size of the second sample for which the average risk is minimum, is given by the integral value of ν which minimizes (absolutely) the following function of ν.

$$(1.9)\ G_\nu(t_m) = \frac{\mathcal{R}_\nu(\underline{x}_m)}{w_0 g_1 f_{\theta_1}(\underline{x}_m)} = A(t_m)\,\nu + \frac{1}{\sqrt{2\pi}}\int_{h^+(\nu,t_m)}^{\infty} e^{-\frac{1}{2}y^2}dy$$

$$+ \frac{e^{-dt_m}}{\sqrt{2\pi}}\int_{h^-(\nu,t_m)}^{\infty} e^{-\frac{1}{2}y^2}dy\ ,$$

where

$$(1.10) \qquad A(t_m) = c\left(\frac{1}{W_0} + \frac{1}{W_1} e^{-dt_m} \right)$$

$$(1.11) \qquad h^{\pm}(\nu,t_m) = \frac{1}{2} d\nu^{\frac{1}{2}} \pm t_m \nu^{-\frac{1}{2}} .$$

Now (1.9) can have an absolute minimum with respect to ν only for a value of ν which satisfies the inequality

$$G_\nu(t_m) \leq G_0(t_m) = \begin{cases} 1 & , \quad t_m \leq 0 \\ e^{-dt_m} & , \quad \geq \end{cases} .$$

Hence, an upper bound to any value of ν for which (1.9) is absolutely minimum is immediately seen to be

$$\frac{G_0(t_m)}{A(t_m)} = \begin{cases} \frac{1}{c}\left(\frac{1}{W_0} + \frac{1}{W_1} e^{-dt_m}\right)^{-1} & , \quad t_m \leq 0 \\ \frac{1}{c}\left(\frac{1}{W_1} + \frac{1}{W_0} e^{dt_m}\right)^{-1} & , \quad t_m \geq 0 \end{cases} .$$

We shall, for convenience, in the following, drop the m subscript from t_m, and write some of the above functions without their arguments, when this practice will cause no confusion. If we regard (1.9) as a continuous function of ν, ν any non-negative real number, we have

$$(1.12) \qquad \frac{\partial}{\partial\nu} G_\nu(t) = A(t) - \frac{d}{2\sqrt{2\pi}}\, \nu^{-\frac{1}{2}}\, e^{-\frac{1}{2}h^{+2}} \quad ,$$

$$(1.13) \qquad \frac{\partial^2}{\partial\nu^2} G_\nu(t) = \frac{d}{16\sqrt{2\pi}}\, \nu^{-\frac{5}{2}} (d^2\nu^2 + 4\nu - 4t^2)\, e^{-\frac{1}{2}h^{+2}}$$

We have, first of all, that

$$(1.14) \qquad \frac{\partial^2}{\partial\nu^2} G_\nu(t) \lesseqgtr 0 \ , \quad \nu \lessgtr \frac{2}{d^2}\left[(d^2t^2+1)^{\frac{1}{2}} - 1 \right] = \mathcal{M}(t), \text{ say} \ .$$

$$(1.15) \qquad \frac{\partial}{\partial\nu} G_\nu(t) = 0 \quad ,$$

if and only if,

$$(1.16) \qquad \log\nu = 2\log\eta(t) - \tfrac{1}{4}d^2\nu - t^2\nu^{-1} = f_t(\nu), \text{ say} \quad ,$$

where

$$(1.17)\qquad \eta(t) = \frac{dZ}{2\sqrt{2\pi}}\left[\underline{W}_0 e^{\frac{1}{2}dt} + \underline{W}_1 e^{-\frac{1}{2}dt}\right]^{-1} ,$$

$$(1.18)\qquad Z = \frac{1}{c}\min(W_0,W_1)\ ,\quad \underline{W}_i = \frac{1}{W_i}\min(W_0,W_1),\ i = 0,1\ .$$

We note here that in all the work which follows, we assume that d and Z are both positive numbers.

Now

$$(1.19)\ \lim_{\nu\to 0}\left[\frac{\partial}{\partial\nu}G_\nu(0)\right] = -\infty,\quad \lim_{\nu\to\infty}\left[\frac{\partial}{\partial\nu}G_\nu(0)\right] = A(0)>0,\ \mathcal{M}(0)=0,$$

so that when t=0, (1.15) has exactly one root in ν. This root is positive and is obviously the value of ν for which $G_\nu(0)$ is an absolute minimum. On the other hand, when $t \neq 0$,

$$(1.20)\qquad \lim_{\nu\to 0}\left[\frac{\partial}{\partial\nu}G_\nu(t)\right] = \lim_{\nu\to\infty}\left[\frac{\partial}{\partial\nu}G_\nu(t)\right] = A(t) > 0\ .$$

Thus, by (1.14), disregarding the case t = 0, (1.15) has

$$(1.21)\ \begin{matrix}2\\1\\0\end{matrix}\text{ rts. in }\nu,\text{when }\frac{\partial}{\partial\nu}G_\nu(t)\Big|_{\nu=\mathcal{M}(t)}\begin{matrix}<\\=\\>\end{matrix}0\text{ i.e. when }\log\mathcal{M}(t)\begin{matrix}<\\=\\>\end{matrix}f_t(\mathcal{M}(t)).$$

Now

$$(1.22)\qquad \log\mathcal{M}(t) = \log\frac{2}{d^2} + \log\left[(d^2t^2+1)^{\frac{1}{2}} - 1\right]$$

is an increasing function of $|t|$, with unique minimum = -oo at $t = 0$. It $\to$ oo, as $t \to \pm$ oo and has negative second derivative for all t.

$$(1.23) \qquad f_t(\mathcal{M}(t)) = 2 \log \eta(t) - (d^2t^2+1)^{\frac{1}{2}}$$

has a unique maximum at a value of t which is $\lesseqgtr 0$, according as $W_0 \lesseqgtr W_1$. It tends to -oo, as $t \to \pm$ oo and has negative second derivative for all t. It follows that there exist two numbers, call them $t^{\pm}$, both of which depend upon the parameters d, Z, W, such that

$$(1.24) \qquad t^- < 0 < t^+$$

and such that (1.15) has

$$(1.25) \qquad \begin{matrix} 2 & & t^- < t < t^+, \quad t \neq 0 \\ 1 & \text{rts. in } \nu, \text{ when} & t = t^- \text{ or } t^+ \text{ or } 0 \\ 0 & & t < t^- \text{ or } > t^+ \end{matrix} \quad .$$

In the first case above, the two roots of (1.15) lie one above, one below the inflection point $\nu = \mathcal{M}(t)$, so that by (1.14), $G_\nu(t)$ is relatively maximum at the first, relatively minimum at the second.

We have discussed the unique root of (1.15) when $t = 0$. When $t = t^{\pm}$, the unique root, $\nu = \mathcal{M}(t^{\pm})$, is an inflection point of zero slope of $G_\nu(t^{\pm})$. The slope of $G_\nu(t^{\pm})$ is thus ≥ 0, $\nu \geq 0$.

When $t < t^-$ or $> t^+$, the slope of $G_\nu(t)$ is > 0, $\nu \geq 0$. It follows that when $t \leq t^-$ or $\geq t^+$, the absolute minimum of $G_\nu(t)$ is at $\nu = 0$.

We define the function

$$(1.26) \qquad \nu^*(t) = \begin{cases} \text{larger root of } (1.15) \ , & t^- < t < t^+, \ t \neq 0 \\ \text{unique root of } (1.15) \ , & t = 0 \text{ or } t^- \text{ or } t^+ \\ 0 \ , & t < t^- \text{or} > t^+ \end{cases} .$$

If now for every t, we take $\nu(t)$ to be the value of ν for which $G_\nu(t)$ is absolutely minimum (choosing the smallest number when this value is not unique), we have clearly that

$$(1.27) \qquad \nu(t) = \begin{cases} 0 \ , & G_0(t) \leq G_{\nu^*}(t) \\ \nu^*(t) \ , & G_0(t) > G_{\nu^*}(t) \end{cases} .$$

From the above discussion, it is apparent that

$$(1.28) \qquad \{t : G_0(t) > G_{\nu^*}(t)\} \subset \{t : t^- < t < t^+\} .$$

Recall from (1.15), (1.16) the significance of the equation

$$\log \nu = f_t(\nu) .$$

Let us consider, in the following that

$$(1.29) \qquad W = \frac{W_1}{W_0} \neq 1,$$

then

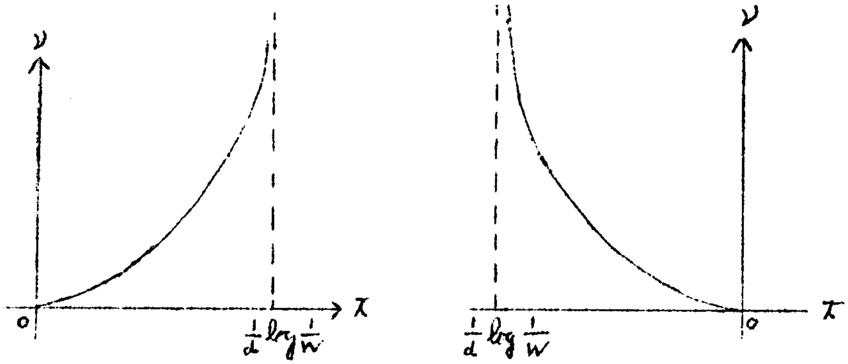
$$(1.30) \qquad \frac{\partial}{\partial t} f_t(\nu) = d\,\frac{1-We^{dt}}{1+We^{dt}} - \frac{2t}{\nu} = 0$$

if and only if

$$(1.31) \qquad \nu = \frac{2t}{d}\,\frac{1+We^{dt}}{1-We^{dt}} = \mathcal{V}_W(t), \text{ say.}$$

We consider, of course, only non-negative values of ν, so that when $W<1$, the curve (1.31) is defined only on the interval $0 \leq t < \frac{1}{d} \log \frac{1}{W}$. In this interval it is convex and has positive slope everywhere. When $W > 1$, the curve (1.31) is defined only on the interval $\frac{1}{d} \log \frac{1}{W} < t \leq 0$. In this interval, it is also convex , but now has negative slope everywhere. In both cases, we have

$$(1.32) \qquad \mathcal{V}_W(0) = 0\ , \quad \lim_{t \to \frac{1}{d}\log\frac{1}{W}} \mathcal{V}_W(t) = \infty$$



$\mathcal{V}_W(t)$, W<1 $\qquad$ $\mathcal{V}_W(t)$, W >1

Figure 1.1

The role of the function $\mathcal{V}_W(t)$ is indicated more fully, by the following analysis

$$\begin{array}{llll} & & W < 1 & W > 1 \\ & \left\{ \begin{array}{l} > 0, \text{ All } \nu \quad , \\ \substack{\leq \\ >} 0, \ \nu \substack{\leq \\ >} \mathcal{V}_W(t), \\ < 0, \text{ All } \nu \quad , \end{array} \right. & \begin{array}{l} t < 0 \\ 0 \leq t < \frac{1}{d} \log \frac{1}{W} \\ t \geq \frac{1}{d} \log \frac{1}{W} \end{array} & \begin{array}{l} t \leq \frac{1}{d} \log \frac{1}{W} \\ \frac{1}{d} \log \frac{1}{W} < t \leq 0 \\ t > 0 \end{array} \end{array} \qquad .$$

$$(1.33) \quad \frac{\partial}{\partial t} f_t(\nu)$$

Consider now the equation

$$(1.34) \qquad \log \mathcal{V}_W(t) = f_t(\mathcal{V}_W(t)) \quad .$$

When $W < 1$, both sides of (1.34) are defined only in the interval $0 \leq t < \frac{1}{d} \log \frac{1}{W}$. The L. H. S. is a continuous increasing function of t which $\rightarrow -\infty$ as $t \rightarrow 0$, and $\rightarrow +\infty$ as $t \rightarrow \frac{1}{d} \log \frac{1}{W}$. The R. H. S. is a continuous, concave, decreasing function of t, equal to a constant, when $t = 0$, and $\rightarrow -\infty$ as $t \rightarrow \frac{1}{d} \log \frac{1}{W}$. When $W > 1$, both sides of (1.34) are defined only in the interval $\frac{1}{d} \log \frac{1}{W} < t \leq 0$. The L. H. S. is a continuous decreasing function of t which $\rightarrow -\infty$ as $t \rightarrow 0$, and $\rightarrow +\infty$ as $t \rightarrow \frac{1}{d} \log \frac{1}{W}$. The R. H. S. is a continuous concave increasing function of t, equal to a constant when $t = 0$, and $\rightarrow -\infty$ as $t \rightarrow \frac{1}{d} \log \frac{1}{W}$. Thus in both cases, the solution to (1.34), call it T_W, is unique.

When $W = 1$, (1.30) holds true if and only if $t = 0$. In this case (1.33) may be written

$$(1.35) \qquad \frac{\partial}{\partial t} f_t(\nu) \underset{>}{\leq} 0, \text{ all } \nu,\ t \underset{<}{\geq} 0 \quad ,$$

and we may appropriately define

$$(1.36) \qquad T_1 = 0.$$

Lemma 1.

$$(1.37) \quad \nu^*(t) \begin{cases} \text{is an increasing function of } t\ ,\ t^- \leq t < T_W \\ \text{has a unique maximum at } t{=}T_W, \text{which is equal to } \mathcal{V}_W(T_W) \\ \text{is a decreasing function of } t\ ,\ T_W < t \leq t^+ \end{cases} .$$

Proof.

It is obvious from the definition of T_W that

$$(1.38) \qquad \nu^*(T_W) = \mathcal{V}_W(T_W)\ ,\quad W \neq 1\ ,$$

and that this relationship must also hold in the limit as $W \to 1$. Thus, to prove the lemma, we need only show that

(1) $\nu^*(t') < \nu^*(t)$, all t, t' such that $t^- \leq t' < t \leq T_W$

(2) $\nu^*(t') < \nu^*(t)$, all t, t' such that $T_W \leq t < t' \leq t^+$.

I. We first suppose that $W < 1$

1a) Let t be any number such that $t^- \leq t < T_W$, then by (1.33)

$$f_t(\nu) \lesseqgtr f_{T_W}(\nu) \;, \quad \nu \gtreqless \nu_0 \;, \tag{1.39}$$

where ν_0 is a number such that

$$\mathcal{V}_W(T_W) > \nu_0 > \begin{cases} \mathcal{V}_W(t), & 0 \leq t < T_W \\ 0 \;, & t^- \leq t < 0 \;. \end{cases} \tag{1.40}$$

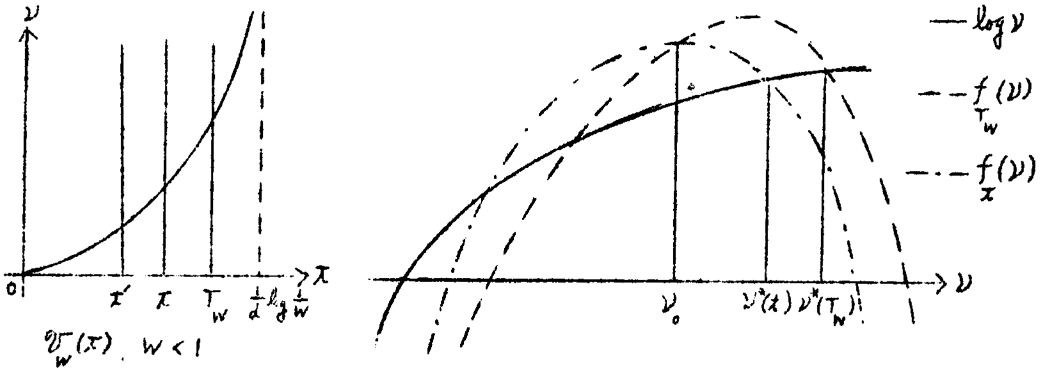


Figure 1.2

It follows, using (1.38), that

$$\nu^*(T_W) > \nu^*(t) > \nu_0 \quad . \tag{1.41}$$

1b) Now let t be any number such that $0 < t < T_W$, t' any number such that $t^- \leq t' < t$, then by (1.33)

$$f_{t'}(\nu) \lesseqgtr f_t(\nu) \quad , \quad \nu \gtreqless \nu_1 \;, \tag{1.42}$$

where ν_1 is a number such that

(1.43) $$\mathcal{V}_W(t) > \nu_1 > \begin{cases} \mathcal{V}_W(t'), & 0 \leq t' < t \\ 0 & , t^- \leq t' < 0 \end{cases} .$$

By the second inequality in (1.40) and the second inequality in (1.41), we have further, that

(1.44) $$\nu^*(t) > \mathcal{V}_W(t) .$$

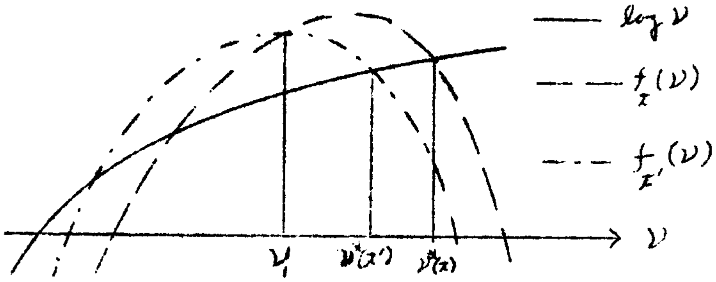


Figure 1.3

Thus, it follows that

$$\nu^*(t) > \nu^*(t') .$$

1c) Finally, let t, t' be any two numbers such that $t^- \leq t' < t \leq 0$, then by (1.33)

(1.46) $$f_{t'}(\nu) < f_t(\nu) , \text{ all } \nu > 0 ,$$

from which it immediately follows that

(1.47) $$\nu^*(t) > \nu^*(t') .$$

Q.E.D. (1), W < 1.

2a) Let t be any number such that $T_W < t \leq t^+$, then by (1.33)

$$f_t(\nu) \lessgtr f_{T_W}(\nu) \quad , \quad \nu \lessgtr \nu_2 \quad , \tag{1.48}$$

where ν_2 is a number such that

$$\mathcal{V}_W(T_W) < \nu_2 < \begin{cases} \mathcal{V}_W(t) \ , & T_W < t < \frac{1}{d} \log \frac{1}{W} \\ \infty \ , & \frac{1}{d} \log \frac{1}{W} \leq t \leq t^+ \end{cases} \quad . \tag{1.49}$$

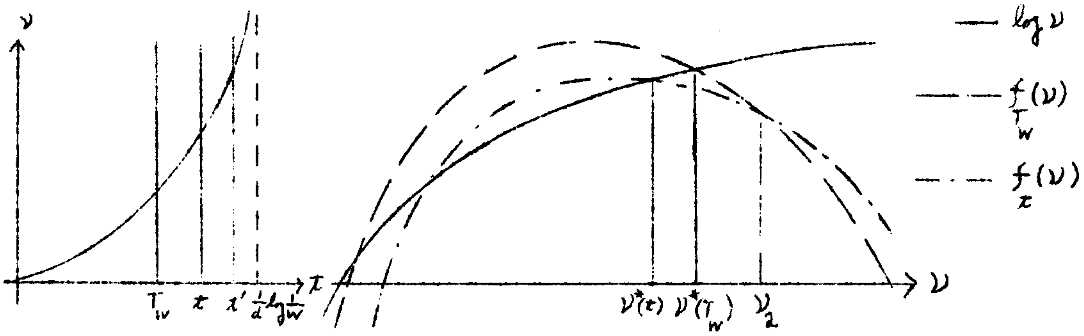


Figure 1.4

It follows, using (1.38), that

$$\nu^*(T_W) > \nu^*(t) \quad . \tag{1.50}$$

2b) Now let t be any number such that $T_W < t < \frac{1}{d} \log \frac{1}{W}$, t' any number such that $t < t' \leq t^+$, then by (1.33)

$$f_{t'}(\nu) \lesseqgtr f_t(\nu) \ , \quad \nu \lesseqgtr \nu_3 \tag{1.51}$$

where ν_3 is a number such that

$$(1.52) \qquad \mathcal{V}_W(t) < \nu_3 < \begin{cases} \mathcal{V}(t') \ , & t < t' < \frac{1}{d} \log \frac{1}{W} \\ \infty \ , & \frac{1}{d} \log \frac{1}{W} \leq t' \leq t^+ \end{cases} .$$

By (1.50), (1.49),

$$(1.53) \qquad \nu^*(t) < \mathcal{V}_W(t) \quad .$$

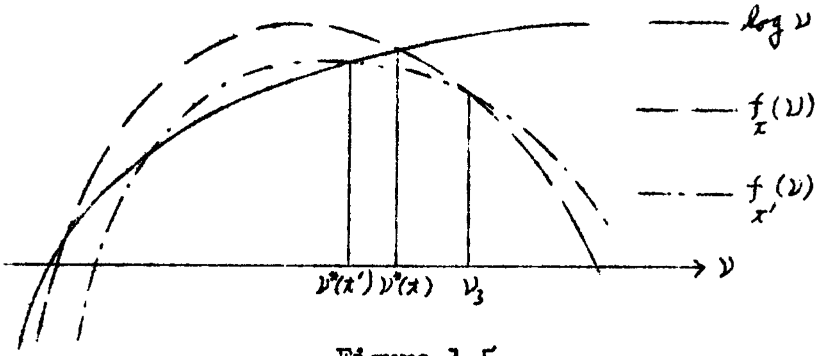


Figure 1.5

Thus

$$(1.54) \qquad \nu^*(t) > \nu^*(t') \quad .$$

2c) Finally, let t, t' be any numbers such that $\frac{1}{d} \log \frac{1}{W} \leq t < t' \leq t^+$, then by (1.33)

$$f_{t'}(\nu) < f_t(\nu) \ , \ \text{all } \nu > 0 \quad ,$$

from which it follows immediately that

$$\nu^*(t) > \nu^*(t')$$

Q.E.D. (2), W < 1.

II. Suppose now that $W = 1$. We have by (1.35) that if t, t' are any two numbers such that $t^- \leq t' < t \leq 0$, then

$$f_{t'}(\nu) < f_t(\nu), \text{ all } \nu > 0 \quad ,$$

so that

$$\nu^*(t) > \nu^*(t') \quad .$$

If t, t' are any two numbers such that $0 \leq t < t' \leq t^+$, we have the same result. Q.E.D. (1), (2), W=1.

III. The proof for $W > 1$ proceeds in a strictly analogous manner to that given for $W < 1$.

Q.E.D.

Lemma 2.

$\nu^*(t)$ is a continuous function of t, $t^- \leq t \leq t^+$.

Proof.

The lemma will be proved if for any given $\varepsilon > 0$ and any arbitrary but fixed t in the indicated intervals, we can prove the following four statements to be true

1) $t^- \leq t < T_W$: $\exists^{2}\ \delta > 0,\ \cdot\ni\cdot^{2}\ 0 < t' - t < \delta \Rightarrow 0 < \nu^*(t') - \nu^*(t) < \varepsilon$

2) $t^- < t \leq T_W$: $\exists\ \delta > 0,\ \cdot\ni\cdot\ 0 < t - t' < \delta \Rightarrow 0 < \nu^*(t) - \nu^*(t') < \varepsilon$

3) $T_W \leq t < t^+$: $\exists\ \delta > 0,\ \cdot\ni\cdot\ 0 < t' - t < \delta \Rightarrow 0 < \nu^*(t') - \nu^*(t) < \varepsilon$

4) $T_W < t \leq t^+$: $\exists\ \delta > 0,\ \cdot\ni\cdot\ 0 < t - t' < \delta \Rightarrow 0 < \nu^*(t) - \nu^*(t') < \varepsilon$.

We shall prove, below, only statements 1 and 2. Statements 3 and 4 may be proved in strictly analogous fashion.

Proof of Statement 1.

We are given that t is an arbitrary but fixed number in the interval, $t^- \leq t < T_W$. Let

$$\xi_1 = \log\left(1 + \frac{\varepsilon}{\nu^*(T_W)}\right) \tag{1.55}$$

Now

$$\frac{\partial}{\partial \nu} f_t(\nu) < \frac{\partial}{\partial \nu} \log \nu\ , \qquad \nu > \nu^*(t)\ , \quad t^- \leq t \leq t^+ \quad ,$$

from which it follows that

$$f_t(\nu^*(t) \cdot e^{\xi_1}) < \log \nu^*(t) + \xi_1 . \tag{1.56}$$

Since $f_t(\nu)$ is a continuous function of t, all t, all $\nu > 0$,

---

[2] We use the notation, $\exists$, to mean "there exists"; $\cdot\ni\cdot$, to mean "such that".

we can find a positive number, δ, which is $\leq T_W - t$ and such that

(1.57) $0<t'-t<\delta \Rightarrow f_{t'}(\nu^*(t)e^{\xi_1}) < \log \nu^*(t) + \xi_1$.

But by lemma 1, this implies that

$$(1.58) \qquad \nu^*(t) < \nu^*(t') < \nu^*(t)e^{\xi_1} ,$$

which in turn implies that

$$(1.59) \quad 0 < \nu^*(t') - \nu^*(t) < \nu^*(t)\cdot(e^{\xi_1}-1) < \nu^*(T_W)\cdot(e^{\xi_1}-1) = \varepsilon .$$

Q.E.D.

Proof of Statement 2.

We are given that t is an arbitrary fixed number in the interval $t^- < t \leq T_W$. Let

$$(1.60) \qquad \xi_2 = \log \left(1 - \frac{\varepsilon}{\nu^*(T_W)}\right)^{-1} .$$

If

$$(1.61) \qquad f_t(\nu^*(t)e^{-\xi_2}) \leq \log \nu^*(t) - \xi_2 ,$$

then

$$(1.62) \qquad \nu^*(t)e^{-\xi_2} < \mathcal{M}(t)$$

Now

(1.63) $$f_t(\mathcal{M}(t)) > \log \mathcal{M}(t) \quad .$$

Hence, by the continuity of $f_t(\nu)$, we can find a positive number, δ, such that

(1.64) $$0 < t - t' < \delta \Rightarrow f_{t'}(\mathcal{M}(t)) > \log \mathcal{M}(t) \quad .$$

But by Lemma 1, this implies that

(1.65) $$\mathcal{M}(t) < \nu^*(t') < \nu^*(t) \quad .$$

Thus, by (1.62)

(1.66) $$0 < \nu^*(t) - \nu^*(t') < \nu^*(t)\cdot(1-e^{-\xi_2}) \leq \nu^*(T_W)\cdot(1-e^{-\xi_2}) = \varepsilon \; .$$

If, on the other hand,

(1.67) $$f_t(\nu^*(t)e^{-\xi_2}) > \log \nu^*(t) - \xi_2 \quad ,$$

we can, by the continuity of $f_t(\nu)$, find a positive number, δ', such that

(1.68) $$0 < t - t' < \delta' \Rightarrow f_{t'}(\nu^*(t)\cdot e^{-\xi_2}) > \log \nu^*(t) - \xi_2 \quad .$$

But by Lemma 1, this implies that

(1.69) $$\nu^*(t)\, e^{-\xi_2} < \nu^*(t') < \nu^*(t) \quad ,$$

and this leads to the conclusion (1.66).

Q.E.D.

Lemma 3.

$$(1.70) \qquad G_{\nu*(t)}(t) \begin{matrix} < \\ = \\ > \end{matrix} G_0(t) \quad \begin{matrix} , \; t^{\overset{-}{*}} < t < t^{\overset{+}{*}} \\ , \; t = t^{\overset{-}{*}} \text{ or } t^{\overset{+}{*}} \\ , \; t < t^{\overset{-}{*}} \text{ or } t^{\overset{+}{*}} \end{matrix} \quad ,$$

where $t^{\overset{-}{*}}$, $t^{\overset{+}{*}}$ are two numbers which are dependent upon the parameters d, Z, W, and such that

$$(1.71) \qquad t^- < t^{\overset{-}{*}} < 0 < t^{\overset{+}{*}} < t^+ \quad .$$

Proof.

By reference to (1.9), we have that

$$(1.72) \qquad G_0(t) = \begin{cases} 1 & , \; t \leq 0 \\ e^{-dt} & , \; t \geq 0 \end{cases} \quad ,$$

and that

$$(1.73) \quad G_{\nu*(t)}(t) = \frac{1}{Z} (\underline{W}_0 + \underline{W}_1 e^{-dt}) \nu*(t) + \frac{1}{\sqrt{2\pi}} \int_{h^+(\nu*(t),t)}^{\infty} e^{-\frac{1}{2}y^2} dy$$

$$+ \frac{e^{-dt}}{\sqrt{2\pi}} \int_{h^-(\nu*(t),t)}^{\infty} e^{-\frac{1}{2}y^2} dy$$

By Lemma 2, this is a continuous function of t, $t^- \leq t \leq t^+$. It is easy to verify that

$$(1.74) \qquad \frac{\partial}{\partial t}\left[G_{\nu*(t)}(t)\right] < 0 \quad , \quad t^{-} \leq t \leq t^{+} \quad .$$

Now previous discussion, see (1.19) - (1.25), has shown that

$$(1.75) \qquad G_{\nu*(t^{-})}(t^{-}) > G_{0}\,(t^{-}) \; , \qquad G_{\nu*(0)}(0) < G_{0}(0).$$

Hence, by (1.72), (1.74), there exists in the open interval, $t^{-}<t<0$, a unique value of t, call it $t^{\overset{-}{*}}$, which is dependent upon the parameters d, Z, W of (1.72) and (1.73) and such that for all t in the interval $t^{-}\leq t\leq 0$,

$$(1.76) \qquad G_{\;*(t)}(t) \gtrless G_{0}(t) \quad , \quad t \lesseqgtr t^{\overset{-}{*}} \quad ,$$

To complete the proof, we have by (1.72) that

$$(1.77) \qquad e^{dt}\, G_{0}(t) = \begin{cases} 1 & , \quad t \geq 0 \quad . \\ e^{dt} & , \quad t \leq 0 \quad . \end{cases}$$

It is easy to verify that

$$(1.78) \qquad \frac{\partial}{\partial t}\left[\, e^{dt}\, G_{\nu*(t)}(t)\right] > 0 \quad , \quad t^{-} \leq t \leq t^{+} \quad .$$

By the discussion referred to above,

$$(1.79) \quad e^{dt^{+}} G_{\nu*(t^{+})}(t^{+}) > e^{dt^{+}} G_{0}(t^{+}) \quad , \; e^{d.0} G_{\nu*(0)}(0) < e^{d.0} G_{0}(0).$$

Hence, by (1.77), (1.78), there exists in the open interval, $0<t<t^{+}$, a unique value of t, call it $t^{\overset{+}{*}}$, which is dependent

upon the parameters, d, Z, W of (1.72) and (1.73), and such that for all t in the interval $0 \leq t \leq t^{+}$,

$$(1.80) \qquad G_{\nu^*(t)}(t) \ \substack{\geq \\ <} \ G_0(t) \quad , \quad t \ \substack{\geq \\ <} \ t^{\overset{+}{*}} \quad .$$

This completes the proof.

Note that the proof of the above lemma demonstrates the existence of and uniquely defines $t^{\overset{+}{\overline{*}}}$, respectively, as the positive and negative roots of

$$(1.81) \qquad G_{\nu^*(t)}(t) = G_0(t) \qquad ,$$

in the interval $t^{-} \leq t \leq t^{+}$.

Lemma 3 and (1.27) now give us

Theorem 1.

$$(1.82) \qquad \nu(t) = \begin{cases} \nu^*(t) & , \quad t^{\overline{*}} < t < t^{\overset{+}{*}} \\ \\ 0 & , \quad t \leq t^{\overline{*}} \text{ or } \geq t^{\overset{+}{*}} \end{cases} \qquad ,$$

where $t^{\overline{*}}$, $t^{\overset{+}{*}}$ are, respectively, the unique negative and positive roots in t of (1.81) which are dependent upon the parameters d, Z, W, and such that

$$(1.83) \qquad t^{-} < t^{\overline{*}} < 0 < t^{\overset{+}{*}} < t^{+}$$

From Lemmas 1 and 2 and Theorem 1, we get

Theorem 2.

$\nu(t)$ is a continuous function of t, $t^{\overline{*}} < t < t^{\overset{+}{*}}$.

If $t^{\overline{*}} \leq T_W \leq t^{\overset{+}{*}}$,

$$\nu(t) \begin{cases} \text{is an increasing function of } t, & t^{\overline{*}} < t < T_W \\ \text{has a unique maximum at } t = T_W, & \\ \text{is a decreasing function of } t, & T_W < t < t^{\overset{+}{*}} \quad . \end{cases}$$

If $T_W < t^{\overline{*}}$

$\nu(t)$ is a decreasing function of t, $t^{\overline{*}} < t < t^{\overset{+}{*}}$.

If $T_W > t^{\overset{+}{*}}$,

$\nu(t)$ is an increasing function of t, $t^{\overline{*}} < t < t^{\overset{+}{*}}$.

The following interesting peculiarity of the second sample size function, $\nu(t)$, is easily deducible from the above results.

Theorem 3.

The second sample size function, $\nu(t)$, has discontinuities at the points $t^{\overset{+}{*}}$, $t^{\overline{*}}$.

$$\begin{aligned} \nu(t^{\overset{+}{*}} - 0) - \nu(t^{\overset{+}{*}} + 0) &> \nu\ (t^{+}), && T_W \leq t^{\overset{+}{*}} \\ &> \nu^*(t^{-}), && T_W > t^{\overset{+}{*}} \quad , \end{aligned} \tag{1.84}$$

$$\begin{aligned} \nu(t^{\overline{*}} + 0) - \nu(t^{\overline{*}} - 0) &> \nu^*(t^{-}), && T_W \geq t^{\overline{*}} \\ &> \nu^*(t^{+}), && T_W < t^{\overline{*}} \quad . \end{aligned} \tag{1.85}$$

Recall that

$$\mathcal{M}(t^{\pm}) = \nu^*(t^{\pm}) \quad , \tag{1.86}$$

then Theorem 3 implies

$$(1.87) \qquad \nu(t) > \text{Min}\,[\mathfrak{m}(t^{+}), \mathfrak{m}(t^{-})], \qquad t^{\overset{-}{*}} < t < t^{\overset{+}{*}} \quad .$$

By Theorem 1, we may now modify the statement of our decision rule as follows. First, compute the numbers, $t^{\overset{\pm}{*}}$ (see (1.81)).

1. Take m observations.
2. If the observed sample is $\underline{x}_m$, compute t_m(1.5) .
   a) If $t_m \leq t^{\overset{-}{*}}$, accept A_0.
   b) If $t_m \geq t^{\overset{+}{*}}$, accept A_1.
   c) If $t^{\overset{-}{*}} < t_m < t^{\overset{+}{*}}$, take $\nu(t_m)$ additional observations.
3. If (2c) occurs and the observed total sample is $\underline{x}_{m+\nu}$, compute $t_{m+\nu}$.
   a) If $t_{m+\nu} \leq 0$, accept A_0.
   b) If $t_{m+\nu} > 0$, accept A_1 .

Note that in general $\nu(t_m)$ will not be integral, in which case we shall approximate the test by taking the nearest integral value.

2. An Important Identity.

In this section, two further lemmas are proved and an important identity established. These results then lead, in the following two sections, to the derivation of an asymptotic expansion for the second sample size function.

We define the function

$$(2.1) \qquad M_\nu(\mu) = \left(\frac{1}{4} d^2 + \frac{1}{\nu} \right) \mu - \frac{1}{4} d^2, \qquad 0 \leq \mu \leq 1, \quad \nu > 0.$$

Note that for every fixed $\nu > 0$, (2.1) is a linear function of μ, which intersects the lines $\mu = 0$ and $\mu = 1$ at $-\frac{1}{4} d^2$ and $\frac{1}{\nu}$, respectively. If for any arbitrary but fixed value of μ in the half open interval, $0 < \mu \leq 1$, we set

$$(2.2) \qquad \frac{\partial}{\partial \nu} f_t(\nu) = M_\nu(\mu) \quad ,$$

and solve for ν, we get

$$(2.3) \qquad \nu = \frac{2}{d^2}\left[\left(\frac{d^2 t^2}{\mu} + 1\right)^{\frac{1}{2}} - 1\right] = \mathcal{M}_\mu(t), \text{ say.}$$

Consider the non-negative ν axis in the t, ν plane,

$$(2.4) \qquad t = 0, \ \nu \geq 0 \quad ,$$

to correspond to the case, $\mu = 0$. It then follows that (2.3), for $0 < \mu \leq 1$, plus (2.4), for $\mu = 0$, represent a family with parameter, μ, $0 \leq \mu \leq 1$, the individual curves of which (ignoring points at infinity) are loci of points in the upper half t, ν plane which satisfy (2.2). In particular, note that $\mathcal{M}(t)$, defined by (1.14), is the particular curve of this family for which $\mu = 1$.

For reasons which will presently become apparent, we now consider the solutions in t of the equation

$$f_t(\mathscr{M}_\mu(t)) = \log \mathscr{M}_\mu(t). \tag{2.5}$$

First, suppose that μ is an arbitrary but fixed number in the half open interval $0 < \mu \leq 1$.

$$\log \mathscr{M}_\mu(t) = \log \frac{2}{d^2} + \log \left[\left(\frac{d^2t^2}{\mu} + 1 \right)^{\frac{1}{2}} - 1 \right] \tag{2.6}$$

is an increasing function of $|t|$, with unique minimum = -oo at $t = 0$. It ——> oo as t ——> $\pm$ oo and is concave for all $t \neq 0$.

$$f_t(\mathscr{M}_\mu(t)) = 2 \log \frac{dZ}{2\sqrt{2\pi}} - 2 \log \left[W_1 e^{-\frac{1}{2}dt} + W_0 e^{\frac{1}{2}dt} \right] \tag{2.7}$$

$$- \tfrac{1}{2} \left[(\mu+1) \left(\frac{d^2t^2}{\mu} + 1 \right)^{\frac{1}{2}} + \mu - 1 \right]$$

is increasing to a unique maximum between $t = 0$ and $t = \frac{1}{d} \log \frac{1}{W}$, and then decreasing. It tends to -oo as t ——> $\pm$ oo and is concave for all t.

It follows that there exist two and only two values of t, call them t_μ^- , t_μ^+ , which satisfy (2.5), which must in general depend upon the parameters μ, d, Z, W, and such that

$$t_\mu^- < 0 < t_\mu^+ \ , \ 0 < \mu \leq 1. \tag{2.8}$$

When $\mu = 0$, we consider equation (1.16) over (2.4). In this case, obviously, $t = 0$, and we define

$$(2.9) \qquad t_0^{\pm} = 0.$$

Note that the numbers, $t^{\pm}$, defined in the discussion above (1.24), (1.25) are the particular cases $t_1^{\pm}$ of the above solutions to (2.5).

Lemma 4.

Let μ, d, W be arbitrary but fixed numbers, $0 < \mu \leq 1$, d, W > 0, then $t_\mu^-(Z)$ is a continuous decreasing function of Z, $t_\mu^+(Z)$, a continuous increasing function of Z. Furthermore,

$$(2.10) \qquad \lim_{Z \longrightarrow 0} t_\mu^{\pm}(Z) = 0 \quad , \quad \lim_{Z \longrightarrow oo} t_\mu^{\pm}(Z) = \pm\, oo.$$

Proof.

If we denote the function (2.7) by $F_Z(t)$ to indicate its dependence upon Z, we have for any positive Δ, Z,

$$(2.11) \qquad F_{Z+\Delta}(t) - F_Z(t) = 2 \log \left(1 + \frac{\Delta}{Z}\right) > 0 \quad ,$$

which quantity is independent of t. Further

$$(2.12) \qquad \lim_{Z \longrightarrow 0} F_Z(t) = -oo, \quad \lim_{Z \longrightarrow oo} F_Z(t) = oo, \quad \text{all } t.$$

Hence, by the above description of the function, $\log \mathcal{M}_\mu(t)$, which itself is independent of Z, we have that t_μ^- (Z) is a decreasing, t_μ^+ (Z), an increasing function of Z, and that the limiting relations (2.10) hold.

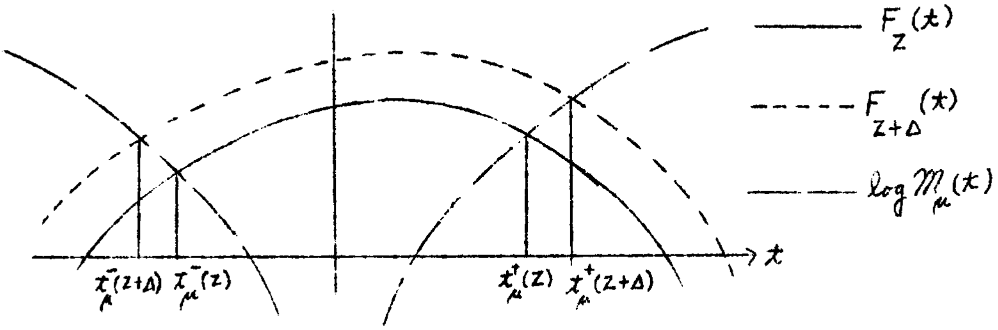


Figure 2.1, W < 1

We shall here only prove continuity for t_μ^+ (Z). Continuity for t_μ^- (Z) may be shown in strictly analogous fashion. For any ϵ, Z > 0, let

$$(2.13) \qquad \Delta_{\epsilon,Z} = \log \mathcal{M}_\mu (t_\mu^+ (Z) + \epsilon) - F_Z(t_\mu^+(Z) + \epsilon) \quad .$$

Then, since $F_Z(t)$ is a continuous function of Z, all Z > 0, all t, we can find a $\delta = \delta_{\epsilon,Z} > 0$, such that

$$(2.14) \quad 0 < Z' - Z < \delta_{\epsilon,Z} \Longrightarrow F_{Z'} (t_\mu^+(Z) + \epsilon) - F_Z(t_\mu^+(Z) + \epsilon) < \Delta_{\epsilon,Z}.$$

But this implies that

$$(2.15) \qquad t_\mu^+ (Z') - t_\mu^+(Z) < \epsilon$$

Q.E.D.

Lemma 5.

Let d, Z, W be fixed positive numbers, then t_μ^- is a continuous decreasing function of μ, t_μ^+ is a continuous increasing function of μ, $0 \leq \mu \leq 1$.

Proof.

By (2.5), the definition of $t_\mu^\pm$ (see the discussion above (2.8)), and the definition of $\nu^*(t)$ (see (1.26), (1.16)), we have the following identities in μ,

$$(2.16) \qquad \mathcal{M}_\mu(t_\mu^\pm) \equiv \nu^*(t_\mu^\pm), \quad 0 < \mu \leq 1 .$$

Thus, besides being respectively the unique positive and negative solutions in t to (2.5), $t_\mu^\pm$ may equally as well be considered, respectively, as the unique positive and negative solutions in t to

$$(2.17) \qquad \mathcal{M}_\mu(t) = \nu^*(t), \quad 0 < \mu \leq 1.$$

Now, $\mathcal{M}_\mu(t)$ is, for every fixed μ, $0 < \mu \leq 1$, a continuous, increasing, convex function of $|t|$, with

$$(2.18) \qquad \mathcal{M}_\mu(0) \equiv 0 \quad , \ 0 < \mu \leq 1.$$

For every fixed $t \neq 0$, $\mathcal{M}_\mu(t)$ is a continuous decreasing function of μ. $\nu^*(t)$, on the other hand, as described by Lemmas 1 and 2, is positive for all t in the interval $t_1^- \leq t \leq t_1^+$, continuous, increasing to a unique maximum and then decreasing in this interval. Also it is obviously independent of μ.

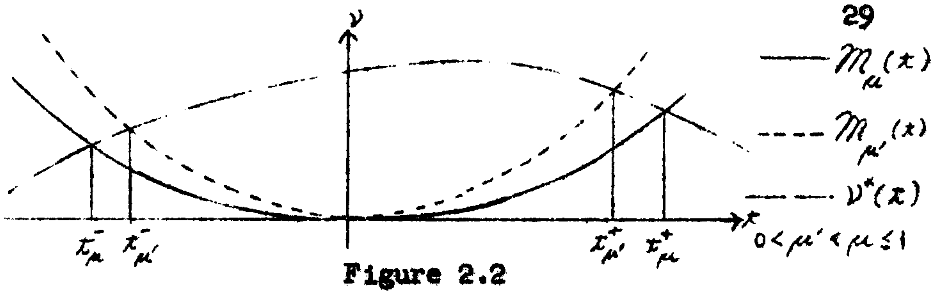


Figure 2.2

Hence t_μ^- is a decreasing function of μ, t_μ^+ is an increasing function of μ. By (2.8), (2.9) this is true for all μ, including the left endpoint, in the closed interval $0 \leq \mu \leq 1$.

We shall prove continuity only for t_μ^+. An essentially similar argument holds for t_μ^-.

Let μ be any arbitrary but fixed number in the half open interval, $0 < \mu \leq 1$. Let ϵ be any given positive number. Two all inclusive but mutually exclusive possibilities may occur. First, suppose that

$$(2.19) \qquad 0 < t_\mu^+ \leq \epsilon \quad ,$$

then since t_μ^+ is an increasing function of μ, we have that

$$(2.20) \quad 0 < \mu - \mu' < \mu \Longrightarrow 0 < t_{\mu'}^+ < t_\mu^+ \leq \epsilon \Longrightarrow 0 < t_\mu^+ - t_{\mu'}^+ < \epsilon.$$

Now, suppose that

$$(2.21) \qquad t_\mu^+ > \epsilon,$$

then clearly, we have that

$$(2.22) \qquad \nu^*(t_\mu^+ - \epsilon) > \mathcal{M}_\mu(t_\mu^+ - \epsilon) \quad .$$

Hence, since $\mathcal{M}_\mu(t)$ is a continuous decreasing function of μ

for all $t \neq 0$, we can find a $\delta = \delta_\epsilon > 0$, such that

$$(2.23) \quad 0 < \mu - \mu' < \delta_\epsilon \Longrightarrow \mathcal{M}_\mu(t_\mu^+ - \epsilon) < \mathcal{M}_{\mu'}(t_\mu^+ - \epsilon) < \nu^*(t_\mu^+ - \epsilon).$$

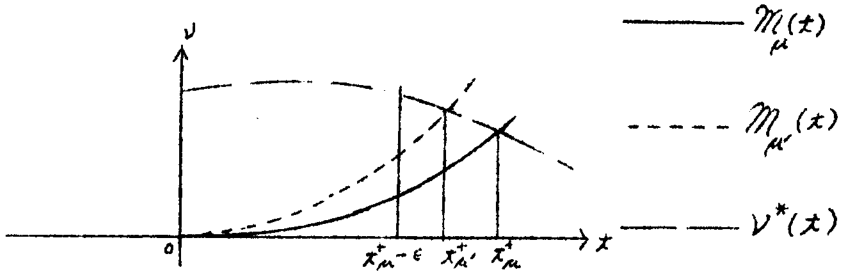


Figure 2.3

But this implies that

$$(2.24) \quad t_\mu^+ - \epsilon < t_{\mu'}^+ < t_\mu^+ \quad .$$

Hence,

$$(2.25) \quad 0 < t_\mu^+ - t_{\mu'}^+ < \epsilon \; .$$

This proves the continuity of t_μ^+ for $0 < \mu \leq 1$. We now prove t_μ^+ to be continuous on the right at $\mu = 0$. We have for any $t \neq 0$

$$(2.26) \quad \lim_{\mu \longrightarrow 0} \mathcal{M}_\mu(t) = \infty.$$

Hence, we can certainly find a $\delta = \delta_\epsilon > 0$ such that

$$(2.27) \quad 0 < \mu' - 0 < \delta_\epsilon \Longrightarrow \mathcal{M}_{\mu'}(\epsilon) > \nu^*(\epsilon)$$

But by (2.9) this implies that

$$(2.28) \quad 0 < t_{\mu'}^+ - t_0^+ = t_{\mu'}^+ - 0 < \epsilon.$$

Q.E.D.

By Lemma 5 and (2.16) it now follows that

$$(2.29) \qquad \lim_{\mu \longrightarrow 0} \mathcal{M}_{\mu}(t_{\mu}^{\pm}) = \nu^*(0) \quad ,$$

and this in turn implies that $t_0^{\pm} = 0$ are the limiting solutions to (2.17) as $\mu \longrightarrow 0$.

By (2.16), (2.29), for arbitrary but fixed μ, $0 \leq \mu \leq 1$, the point

$$(2.30) \qquad (t, \nu) = (t_{\mu}^{\pm} , \nu^*(t_{\mu}^{\pm}))$$

lies on a curve of the family (2.3), (2.4) and hence satisfies the relationship (2.2). This gives us the following identities in μ.

$$(2.31) \quad \frac{\partial}{\partial \nu} f_{t_{\mu}^{\pm}}(\nu) \Big|_{\nu=\nu^*(t_{\mu}^{\pm})} \equiv M_{\nu^*(t_{\mu}^{\pm})}(\mu) \quad , \quad 0 \leq \mu \leq 1,$$

which may be written

$$(2.32) \quad \mu \equiv (t_{\mu}^{\pm})^2 \left[\nu^*(t_{\mu}^{\pm}) \left(\frac{1}{4} d^2 \nu^*(t_{\mu}^{\pm}) + 1\right) \right]^{-1} = \mu_{t_{\mu}^{\pm}} \quad , \quad \text{say.}$$

Thus

$$(2.33) \quad \begin{aligned} t_{\mu}^{+} = t' &\Longrightarrow \mu = \mu_{t'} \quad , \quad 0 \leq t' \leq t_1^{+} \\ t_{\mu}^{-} = t' &\Longrightarrow \mu = \mu_{t'} \quad , \quad t_1^{-} \leq t' \leq 0 \quad . \end{aligned}$$

On the other hand,

$$(2.34) \quad \mu_t = \mu' \Longrightarrow M_{\nu^*(t)}(\mu') = \frac{\partial}{\partial \nu} f_t(\nu)\Big|_{\nu=\nu^*(t)} \Longrightarrow \nu^*(t) = \mathcal{M}_{\mu'}(t).$$

But, by the discussion above (2.17), this implies that

$$(2.35) \qquad \mu_t = \mu' \Longrightarrow t = t^{\pm}_{\mu'} \,, \quad 0 < \mu' \leq 1.$$

By (2.29), this holds, as well, in the limit as $\mu' \longrightarrow 0$.

By Lemma 2,

$$(2.36) \qquad \mu_t = t^2 \left[\nu^*(t) \cdot \left(\frac{1}{4} d^2 v^*(t) + 1\right) \right]^{-1}$$

is a continuous function of t, $t_1^- \leq t \leq t_1^+$. By (2.32), (2.33), (2.35), and by Lemma 5

$$(2.37) \qquad \mu_t \begin{cases} \text{is a decreasing function of } t, & t_1^- \leq t < 0 \\ \text{has a unique minimum} = 0 \text{ at } \mu = 0 & \\ \text{is an increasing function of } t, & 0 < t \leq t_1^+ \end{cases} .$$

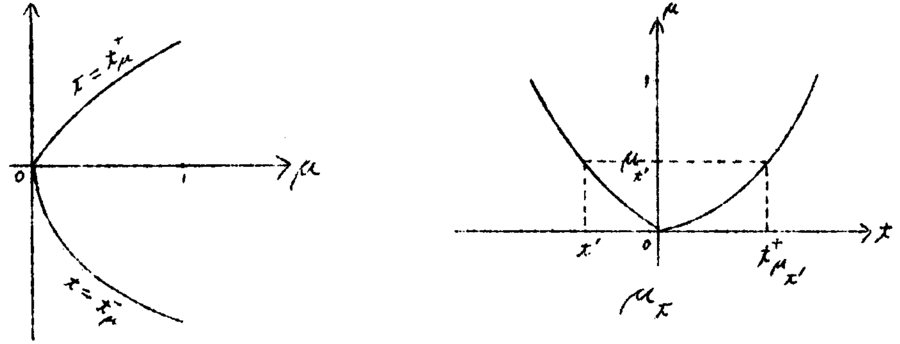


Figure 2.4

We thus arrive, finally, by (2.16), (2.29) at the important identity

$$(2.38) \qquad \nu^*(t) \equiv \mathcal{M}_{\mu_t}(t) \ , \quad t_1^- \leq t \leq t_1^+ \, , \quad t \neq 0 \ ,$$

$$(2.39) \qquad \nu^*(0) = \lim_{t \longrightarrow 0} \mathcal{M}_{\mu_t} (t) \ .$$

CHAPTER II

ASYMPTOTIC PROPERTIES OF BAYES TWO-STAGE TEST

3. Asymptotic Expression for the Second Sample Size Function.

Equation (2.5) may be written in the form

$$(3.1) \qquad 2 \log Z = d|t|(a(\mu) + \epsilon_{\mu}(t)),$$

where μ is any fixed number in the half open interval $0 < \mu \leq 1$,

$$(3.2) \qquad a(\mu) = \frac{(\mu^{\frac{1}{2}} + 1)^2}{2\mu^{\frac{1}{2}}}$$

$$(3.3) \quad \epsilon_{\mu}(t) = \frac{1}{d|t|}\left[\log\left(\frac{16\pi}{d^4} \cdot \mathcal{W}(t) \cdot \left[\left(\frac{d^2t^2}{\mu}+1\right)^{\frac{1}{2}}-1\right]\right) - \frac{1}{2}(1-\mu)\right.$$

$$\left. + \frac{\mu+1}{2\mu^{\frac{1}{2}}}\left[\left(1 + \frac{\mu}{d^2t^2}\right)^{\frac{1}{2}} - 1\right]\right]$$

$$(3.4) \qquad \mathcal{W}(t) = \begin{cases} (\underline{W}_0 + \underline{W}_1 e^{-dt})^2, & t \geq 0 \\ (\underline{W}_1 + \underline{W}_0 e^{dt})^2, & t \leq 0 \quad . \end{cases}$$

If we now substitute into (3.1) the solutions $t = t_{\mu}^{\pm}(Z)$, we get the following result,

$$(3.5) \qquad t^{\pm}_{\mu}(Z) = \pm \frac{4\mu^{\frac{1}{2}}\log Z}{d(\mu^{\frac{1}{2}}+1)^2}\,(1-\epsilon^{\pm}_{2\mu}(Z)) \quad ,$$

where

$$(3.6) \qquad \epsilon^{\pm}_{2\mu}(Z) = \frac{\epsilon^{\pm}_{1\mu}(Z)}{1+\epsilon^{\pm}_{1\mu}(Z)} \quad ,$$

$$(3.7) \quad \epsilon^{\pm}_{1\mu}(Z) = \frac{1}{a(\mu)}\,\epsilon_{\mu}(t^{\pm}_{\mu}(Z) =$$

$$\frac{2\mu^{\frac{1}{2}}}{d(\mu^{\frac{1}{2}}+1)^2\,|t^{\pm}_{\mu}(Z)|}\left[\log\left(\frac{16\pi}{d^4}\cdot\mathscr{W}(t^{\pm}_{\mu}(Z))\cdot\left[\left(\frac{d^2(t^{\pm}_{\mu}(Z))^2}{\mu}+1\right)-1\right]\right)^{\frac{1}{2}}-\frac{1}{2}(1-\mu)\right]$$

$$+\frac{\mu+1}{(\mu^{\frac{1}{2}}+1)^2}\left[\left(1+\frac{\mu}{d^2(t^{\pm}_{\mu}(Z))^2}\right)^{\frac{1}{2}}-1\right] .$$

Note that $\epsilon^{\pm}_{1\mu}(Z)$ and hence $\epsilon^{\pm}_{2\mu}(Z)$ are positive decreasing functions respectively of $|t^{\pm}_{\mu}(Z)|$ for sufficiently large values of $|t^{\pm}_{\mu}(Z)|$. Hence, by Lemma 4, they are positive decreasing functions of Z for sufficiently large Z. Also

(3.8) $$\lim_{Z \to \infty} \epsilon^{\pm}_{2\mu}(Z) = \lim_{|t^{\pm}_{\mu}(Z)| \to \infty} \epsilon^{\pm}_{2\mu}(Z) = 0.$$

Now we define

(3.9) $$\epsilon^{\pm}_{10}(Z) = \lim_{\mu \to 0} \frac{1}{a(\mu)} \epsilon_{\mu}(t^{\pm}_{\mu}(Z)) =$$

$$2\left[(d^2\nu^*(0)+2)^2-4\right]^{-\frac{1}{2}}\left[2\log\left(\frac{8\pi}{d^2}(\underline{W}_0+\underline{W}_1)^2\nu^*(0)\right)-1\right]$$

$$+\left[1-4(d^2\nu^*(0)+2)^{-2}\right]^{-\frac{1}{2}} - 1,$$

(3.10) $$\epsilon^{\pm}_{20}(Z) = \frac{\epsilon^{\pm}_{10}(Z)}{1+\epsilon^{\pm}_{10}(Z)} \quad .$$

Clearly, then, the relationships (3.5) will hold for $\mu=0$. By (1.26), (1.16), $\nu^*(0)$ is the unique solution in ν to the equation

(3.11) $$\log \nu = 2 \log \left[\frac{dZ}{2\sqrt{2\pi}}(\underline{W}_0+\underline{W}_1)^{-1}\right] - \frac{1}{4} d^2\nu \ .$$

This may be written

$$2 \log Z = \frac{1}{4} d^2 \nu + \log \nu + 2 \log \frac{2\sqrt{2\pi}(\underline{W}_0 + \underline{W}_1)}{d} \quad .$$

Substituting into this equation its unique solution in ν, we have

$$(3.12) \quad 2 \log Z = \frac{1}{4} d^2 \nu^*(0) + \log \nu^*(0) + 2 \log \frac{2\sqrt{2\pi}(\underline{W}_0 + \underline{W}_1)}{d} \quad .$$

Now, as Z increases, the R.H.S. of (3.12) must, to maintain the equality, also increase. But the R.H.S. is an increasing function of $\nu^*(0)$. Hence $\nu^*(0)$ is an increasing function of Z. Further, we have that

$$(3.13) \qquad \lim_{Z \to \infty} \nu^*(0) = \infty \quad .$$

We can now see that $\epsilon^{\pm}_{10}(Z)$ and hence $\epsilon^{\pm}_{20}(Z)$ are positive decreasing functions of Z for sufficiently large Z and that

$$\lim_{Z \to \infty} \epsilon^{\pm}_{20}(Z) = \lim_{\nu^*(0) \to \infty} e^{\pm}_{20}(Z) = 0 \ .$$

Hence, finally, for all μ in the closed interval $0 \leq \mu \leq 1$, $\epsilon^{\pm}_{2\mu}(Z)$ are positive decreasing functions of Z, for sufficiently large Z, and

$$\lim_{Z \longrightarrow \infty} \epsilon^{\pm}_{2\mu}(Z) = 0 \quad , \quad 0 \leq \mu \leq 1 \quad . \tag{3.14}$$

By (2.32) - (2.35), we have the identities

$$t^{+}_{\mu_t}(Z) \equiv t, \quad 0 \leq t \leq t^{+}_{1} \tag{3.15}$$

$$t^{-}_{\mu_t}(Z) \equiv t, \quad t^{-}_{1} \leq t \leq 0 \quad . \tag{3.16}$$

Thus, by (3.5), we have

$$\frac{4\mu_t^{\frac{1}{2}} \log Z}{d(\mu_t^{\frac{1}{2}}+1)^2} (1 - \rho_t(Z)) \equiv |t|, \; t^{-}_{1} \leq t \leq t^{+}_{1}, \tag{3.17}$$

where

$$\rho_t(Z) = \begin{cases} \epsilon^{+}_{2\mu_t}(Z), & 0 \leq t \leq t^{+}_{1} \\ \epsilon^{-}_{2\mu_t}(Z), & t^{-}_{1} \leq t \leq 0 \quad . \end{cases} \tag{3.18}$$

By some simple manipulation of (3.17), we get, for $t_1^- \leq t \leq t_1^+$, $t \neq 0$,

$$(3.19) \quad \frac{1}{2}\mu_t - \left(\frac{2(1 - \rho_t(Z))\log Z}{d|t|} - 1\right)\mu_t^{\frac{1}{2}} + \frac{1}{2} = 0 \quad .$$

The case, t=0, is excluded to avoid dividing by zero. The roots of this quadratic in $\mu_t^{\frac{1}{2}}$ are

$$(3.20) \quad \frac{2(1-\rho_t(Z)\log Z}{d|t|} - 1 \pm \left[\left(\frac{2(1-\rho_t(Z)\log Z}{d|t|} - 1\right)^2 - 1\right]^{\frac{1}{2}} \quad .$$

To choose between these roots, we observe that $\mu_t^{\frac{1}{2}}$ is real and that $0 \leq \mu_t^{\frac{1}{2}} \leq 1$. Hence the correct root must be so restricted. First, if either root is to be real, we must have

$$(3.21) \qquad \frac{2(1-\rho_t(Z)\log Z}{d|t|} - 1 \geq 1 \quad \text{or} \quad \leq -1 \quad .$$

If the second of these two possibilities were true, both roots would be negative. It follows that the first inequality of (3.21) must hold. However, in this case, the root (3.20) with the positive second term would always be ≥ 1 and hence could not satisfy our requirements. There remains only the root with negative second term, which by the first inequality (3.21) is readily seen to satisfy them. Thus,

$$(3.22)\quad \mu_t^{\frac{1}{2}} = \frac{2(1-\rho_t(Z)\log Z}{d|t|} - 1 - \left[\left(\frac{2(1-\rho_t(Z))\log Z}{d|t|} - 1\right)^2 - 1\right]^{\frac{1}{2}},\ t_1^- \leq t \leq t_1^+,\ t \neq 0.$$

By (2.36), $\mu_t^{\frac{1}{2}}$ is continuous and equal to zero at $t=0$. The limit of the R.H.S. above as $t \longrightarrow 0$ is 0. Hence the above relationship holds in the limit as $t \longrightarrow 0$.

Now let

$$(3.23)\qquad \xi(t) = \frac{d|t|}{\log Z}\ ,\quad t_1^- \leq t \leq t_1^+ \quad .$$

Using (3.5), we have

$$(3.24)\qquad \begin{aligned} 0 &= \xi(0) \leq \xi(t) \leq \xi(t_1^+(Z)) = 1-\epsilon_{21}^+(Z),\ 0 \leq t \leq t_1^+\ , \\ 0 &= \xi(0) \leq \xi(t) \leq \xi(t_1^-(Z)) = 1-\epsilon_{21}^-(Z),\ t_1^- \leq t \leq 0\quad . \end{aligned}$$

Both upper bounds are by (3.14), < 1 for sufficiently large Z. Substituting $\xi(t)$ into (3.22), we have

$$(3.25)\quad \mu_t^{\frac{1}{2}} = \frac{2}{\xi(t)}(1-\rho_t(Z)) - 1 - \left[\left(\frac{2}{\xi(t)}(1-\rho_t(Z)) - 1\right)^2 - 1\right]^{\frac{1}{2}},\ t_1^- \leq t \leq t_1^+,\ t \neq 0,$$

and this relationship holds in the limit as t → 0. We now rearrange the R.H.S. of (3.25) for greater convenience.

$$(3.26)\quad \mu_t^{\frac{1}{2}}=\left(\frac{2}{\xi(t)}-1\right)(1-\rho_{1t}(Z)) - \left[\frac{4}{\xi^2(t)}(1-\xi(t))(1-\rho_{1t}(Z))^2-\rho_{1t}(Z)(2-\rho_{1t}(Z))\right]^{\frac{1}{2}},$$

where, again this holds for $t\neq 0$ in the closed interval $t_1^- \leq t \leq t_1^+$ and in the limit as $t \longrightarrow 0$, and

$$(3.27)\qquad \rho_{1t}(Z) = \frac{\rho_t(Z)}{1-\frac{1}{2}\xi(t)}\quad ,\quad t_1 \leq t \leq t_1^+$$

Again, rearranging the R.H.S., we have

$$(3.28)\quad \mu_t^{\frac{1}{2}}=\frac{2}{\xi(t)}\left[1-\frac{1}{2}\xi(t)-(1-\xi(t))^{\frac{1}{2}}\right](1-\rho_{1t}(Z))+\rho_{2t}(Z),\ t_1^- \leq t \leq t_1^+ ,$$

where

$$(3.29) \qquad \rho_{2t}(Z) = \frac{2}{\xi(t)}\,(1-\xi(t))^{\frac{1}{2}}(1-\rho_{1t}(Z)) - \left[\frac{4}{\xi^2(t)}(1-\xi(t))(1-\rho_{1t}(Z))^2 - \rho_{1t}(Z)(2-\rho_{1t}(Z))\right]^{\frac{1}{2}},$$

$$t_1^- \leq t \leq t_1^+,\ t \neq 0,$$

and we define

$$(3.30) \qquad \rho_{20}(Z) = \lim_{t \longrightarrow 0} \rho_{2t}(Z) = 0.$$

Finally, we can write

$$(3.31) \qquad \mu_t^{\frac{1}{2}} = \frac{1-(1-\xi(t))^{\frac{1}{2}} + 2\rho_{3t}(Z)}{1+(1-\xi(t))^{\frac{1}{2}}} \quad , \quad t_1^- \leq t \leq t_1^+ \quad ,$$

where

$$(3.22) \quad \rho_{3t}(Z)=\frac{1}{2}\left[1+(1-\xi(t))^{\frac{1}{2}}\right]\rho_{2t}(Z)-\frac{1}{2}\left[1-(1-\xi(t))^{\frac{1}{2}}\right]\rho_{1t}(Z), t_1^- \leq t \leq t_1^+.$$

By (2.38), (2.3),

$$(3.33)\quad v^*(t) \equiv \mathcal{M}_{\mu_t}(t) = \frac{2}{d^2}\left[\left(\frac{d^2t^2}{\mu_t} + 1\right)^{\frac{1}{2}} - 1\right]$$

$$\equiv \frac{2|t|}{d\mu_t^{\frac{1}{2}}} - \frac{2}{d^2}\left\{1 - \left[\left(\frac{d^2t^2}{\mu_t} + 1\right)^{\frac{1}{2}} - \frac{d|t|}{\mu_t^{\frac{1}{2}}}\right]\right\}, t_1^- \leq t \leq t_1^+,\ t \neq 0,$$

and by (2.39), this relationship holds in the limit as t ⟶ 0.

By (3.17),

$$(3.34)\qquad \frac{|t|}{\mu_t^{\frac{1}{2}}} = \frac{4(1-\rho_t(Z))\log Z}{d(\mu_t^{\frac{1}{2}} + 1)^2}\ ,\quad t_1^- \leq t \leq t_1^+\ ,\quad t \neq 0.$$

By (3.31)

$$(3.35)\qquad \mu_t^{\frac{1}{2}} + 1 = \frac{2(1 + \rho_{3t}(Z))}{1 + (1 - \xi(t))^{\frac{1}{2}}}\quad ,\qquad t_1^- \leq t \leq t_1^+$$

Thus we get

$$(3.36)\quad \frac{|t|}{\mu_t^{\frac{1}{2}}} = \frac{1}{d}\left[1 + (1-\xi(t)^{\frac{1}{2}}\right]^2 (1 - \rho_{4t}(z))\log Z,\ t_1^- \leq t \leq t_1^+,\ t \neq 0,$$

where

$$(3.37)\qquad \rho_{4t}(Z) = \frac{\rho_t(Z) + \rho_{3t}(Z)\,(2 + \rho_{3t}(Z))}{(1+\rho_{3t}(Z))^2}\ ,\quad t_1^- \leq t \leq t_1^+ .$$

Substituting (3.36) into (3.33), we have

$$(3.38)\ \nu^*(t) = \frac{2}{d^2}\left[1+(1-\xi(t))^{\frac{1}{2}}\right]^2 \cdot (1-\rho_{6t}(Z))\log Z,\ t_1^- \leq t \leq t_1^+ ,$$

where

$$(3.39)\ \rho_{6t}(Z) = \rho_{4t}(Z) + \frac{1-\rho_{5t}(Z)}{\left[1+(1-\xi(t))^{\frac{1}{2}}\right]^2 \log Z},\quad t_1^- \leq t \leq t_1^+ ,$$

$$(3.40)\qquad \rho_{5t}(Z) = \left[\left[1 + (1-\xi(t))^{\frac{1}{2}}\right]^4 (1-\rho_{4t}(Z))^2 \log^2 Z + 1\right]^{\frac{1}{2}}$$

$$-\left[1+(1-\xi(t))^{\frac{1}{2}}\right]^2 (1-\rho_{4t}(Z))\log Z,\ t_1^- \leq t \leq t_1^+ .$$

Using (3.14) together with the conclusion above it, we have by (3.18) that

$$(3.41)\qquad \rho_t(Z) > 0,\ \text{suff. large } Z,\ \lim_{Z\to\infty} \rho_t(Z) = 0,\ t_1^- \leq t \leq t_1^+ ,$$

and hence it can be shown in each case for $t_1^- \leq t \leq t_1^+$,

$$(3.42)\quad \rho_{it}(Z) \geq 0, \text{ suff. large } Z, \ \lim_{Z \to \infty} \rho_{it}(Z) = 0, \quad i = 1, \ldots, 6.$$

Let

$$(3.43)\qquad u(t) = 1 + (1 - \xi(t))^{\frac{1}{2}} \;,$$

then by (3.38)

$$(3.44)\qquad \nu^*(t) = \frac{2}{d^2} u^2(t)\,(1 - \rho_{6t}(Z)) \log Z \;, \quad t_1^- \leq t \leq t_1^+ \;.$$

Substituting this expression into (1.11), we get

(3.45)

$$h^{\pm}(\nu^*(t),t) = \frac{u^2(t)(1 - \rho_{6t}(Z)) \pm \frac{dt}{\log Z}}{u(t)(1 - \rho_{6t}(Z))^{\frac{1}{2}}} \cdot \left(\tfrac{1}{2} \log Z\right)^{\frac{1}{2}}, \quad t_1^- \leq t \leq t_1^+ \;.$$

Rearrangement of the R. H. S. gives us

$$(3.46)\quad h^{\pm}(\nu^*(t),t) = \frac{J(\pm t) - \frac{1}{2} u(t) \cdot \rho_{6t}(Z)}{(1 - \rho_{6t}(Z))^{\frac{1}{2}}} \cdot (2 \log Z)^{\frac{1}{2}}, \quad t_1^- \leq t \leq t_1^+.$$

where

$$(3.47)\qquad J(t) = \begin{cases} (1 - \xi(t))^{\frac{1}{2}} \;, & t \leq 0 \\ 1 \;, & t \geq 0 \;. \end{cases}$$

We shall use the following well known result. e.g., see [2] .

$$(3.48)\quad \int_{y}^{\infty} e^{-\frac{1}{2}x^{2}}dx = e^{-\frac{1}{2}y^{2}}y^{-1}\sum_{j=0}^{N} c_{j}y^{-2j} + (-1)^{N+1}R_{N},\ N = 0,1,2,\ldots,$$

where y is a positive number,

$$c_{j} = \frac{(-1)^{j}(2j)!}{2^{j}\ j!}\ ,\quad j = 0,1,\ldots\ ,$$

(3.49)

$$R_{N} = \frac{(2N+2)!}{2^{N+1}(N+1)!}\int_{y}^{\infty} X^{-2(N+1)}e^{-\frac{1}{2}X^{2}}dX,\quad N = 0,1,\ldots\ ,$$

and where

$$(3.50)\qquad 0 < R_{N} \leq e^{-\frac{1}{2}y^{2}}\,|c_{N+1}|\cdot y^{-(2N+3)}$$

Lemma 6.

Let d, W be arbitrary but fixed positive numbers. Let

$$(3.51)\qquad \mathcal{T}_{\delta}(Z) = \frac{1}{d}(1 - \delta)\log Z\ \ ,$$

then for any fixed positive $\delta < 1$, no matter how small, we can by taking Z sufficiently large, obtain for all t in the closed interval

$$(3.52) \qquad |t| \leq \mathcal{T}_\delta(Z),$$

the inequality

$$(3.53) \qquad G_{\nu*(t)}(t) < G_0(t) \quad .$$

Proof.

By (1.72), the lemma will be proved, if we can show that

$$(3.54) \qquad \lim_{Z \longrightarrow \infty} G_{\nu*(t)}(t) = 0 \quad , \quad -\mathcal{T}_\delta \leq t \leq 0 \quad ,$$

$$(3.55) \qquad \lim_{Z \longrightarrow 0} e^{dt} G_{\nu*(t)}(t) = 0 \; , \quad 0 \leq t \leq \mathcal{T}_\delta \quad .$$

By (1.74), (1.78), respectively, (3.54) and (3.55) will be true, if

$$(3.56) \qquad \lim_{Z \longrightarrow \infty} G_{\nu^*(-\mathcal{T}_\delta)}(-\mathcal{T}_\delta) = 0$$

$$(3.57) \qquad \lim_{Z \longrightarrow \infty} e^{d\mathcal{T}_\delta} G_{\nu^*(\mathcal{T}_\delta)}(\mathcal{T}_\delta) = 0 \quad .$$

We first note that for any fixed positive $\delta < 1$, no matter how small, we have by (3.5), (3.8), for $\mu = 1$, that

(3.58) $$t_1^-(Z) < -\Upsilon_\delta(Z) < 0 < \Upsilon_\delta(Z) < t_1^+(Z),$$

for suff. large Z.

Hence, by (3.42)

(3.59) $$\lim_{Z \longrightarrow \infty} \rho_{6,\pm\Upsilon_\delta}(Z) = 0 \ .$$

We shall prove, here, only (3.56). (3.57) can be proved in strictly analogous fashion.

By (1.73)

(3.60) $$G_{\nu^*(-\Upsilon_\delta)}(-\Upsilon_\delta) = \frac{1}{Z}(\underline{W}_0 + \underline{W}_1 e^{d\Upsilon_\delta})\,\nu^*(-\Upsilon_\delta) + \frac{1}{\sqrt{2\pi}}\int_{h^+(\nu^*(-\Upsilon_\delta),-\Upsilon_\delta)}^{\infty} e^{-\frac{1}{2}x^2}dx + \frac{e^{d\Upsilon_\delta}}{\sqrt{2\pi}}\int_{h^-(\nu^*(-\Upsilon_\delta),-\Upsilon_\delta)}^{\infty} e^{-\frac{1}{2}x^2}dx \ .$$

By (3.46)

(3.61) $$h^-(\nu^*(-\Upsilon_\delta),-\Upsilon_\delta) = \frac{1-\frac{1}{2}(1+\delta^{\frac{1}{2}})\rho_{6,-\Upsilon_\delta}}{(1-\rho_{6,-\Upsilon_\delta})^{\frac{1}{2}}}(2\log Z)^{\frac{1}{2}}$$

$$= \left[2(1-\hat{\rho}_\delta(Z)\log Z\right]^{\frac{1}{2}} \ ,$$

where

$$(3.62)\quad \hat{\rho}_{\delta}(Z) = \left[\delta^{\frac{1}{2}} - \frac{1}{4}(1+\delta^{\frac{1}{2}})^{2}\,\rho_{6,-\mathcal{T}_{\delta}}\right] \frac{\rho_{6,-\mathcal{T}_{\delta}}}{1-\rho_{6,-\mathcal{T}_{\delta}}}$$

and by (3.59)

$$(3.63)\qquad \lim_{Z \longrightarrow \infty} \hat{\rho}_{\delta}(Z) = 0.$$

By (3.48) - (3.50), taking N = 0, y = (3.61), we have that

$$(3.64)\quad \frac{1}{\sqrt{2\pi}} \int_{h^{-}(\nu^{*}(-\mathcal{T}_{\delta}),-\mathcal{T}_{\delta})}^{\infty} e^{-\frac{1}{2}x^{2}}\,dx \;<\; \frac{1}{2}\left[\pi\,(1-\hat{\rho}_{\delta}(Z))\log Z\right]^{-\frac{1}{2}} Z^{-1+\hat{\rho}_{\delta}(Z)}$$

On the other hand,

$$(3.65)\qquad e^{d\,\mathcal{T}_{\delta}} = Z^{1-\delta} .$$

Hence

$$(3.66)\quad \frac{e^{d\,\mathcal{T}_{\delta}}}{\sqrt{2\pi}} \int_{h^{-}(\nu^{*}(-\mathcal{T}_{\delta}),-\mathcal{T}_{\delta})}^{\infty} e^{-\frac{1}{2}x^{2}}\,dx \;<\; \frac{1}{2}\left[\pi(1-\hat{\rho}_{\delta}(Z))\log Z\right]^{-\frac{1}{2}} Z^{\hat{\rho}_{\delta}(Z)-\delta} .$$

But, by (3.63), the R.H.S. and hence the L.H.S. ⟶ 0 as $Z \longrightarrow \infty$.

By (3.46)

$$(3.67)\quad h^{+}(v^{*}(-\mathcal{J}_{\delta}),-\mathcal{J}_{\delta}) = \frac{\delta^{\frac{1}{2}} - (1+\delta^{\frac{1}{2}})\rho_{6,-\mathcal{J}_{\delta}}}{(1-\rho_{6,-\mathcal{J}_{\delta}})^{\frac{1}{2}}}\,(2\log Z)^{\frac{1}{2}}$$

$$= \left[2\,(\delta - \hat{\rho}_{\delta}(Z))\log Z\right]^{\frac{1}{2}} \quad .$$

Clearly, this ——> oo, as Z ——> oo and hence the second term R.H.S. of (3.60) ——> 0, as Z ——> oo.

Finally, by (3.44), (3.65), the first term R.H.S. of (3.60) may be written

$$(3.68)\qquad \frac{2}{d^{2}}\,(1+\delta^{\frac{1}{2}})^{2}\,(1-\rho_{6,-\mathcal{J}_{\delta}})\cdot\left[\frac{\underline{W}_{0}\log Z}{Z} + \frac{\underline{W}_{1}\log Z}{Z^{\delta}}\right] ,$$

and this also ——> 0, as Z ——> oo .

Q.E.D.

As a direct consequence of Lemmas 3 and 6, we now have the following.

Theorem 4.

Let d, W be arbitrary but fixed positive numbers. Let $t^{\ddagger}$, $t^{\overline{*}}$ be the two numbers defined by Lemma 3. Then for any fixed positive $\delta < 1$, no matter how small, we have, by taking Z sufficiently large that

$$(3.69)\qquad \mathcal{J}_{\delta}(Z) < t^{\ddagger}(Z) < t_{1}^{+}(Z) \quad ,$$

$$(3.70)\qquad t_{1}^{-}(Z) < t^{\overline{*}}(Z) < -\mathcal{J}_{\delta}(Z) \quad .$$

4. Expansion of the Second Sample Size Function.

In this section, we expand the function, $\nu^*(t)$ (3.38), in terms of the parameter Z, to terms of order $o(\frac{1}{\log Z})$; our result holding for all t in the closed interval, $t \leq \mathcal{J}_\delta(Z)$.

In the following pages, we shall for convenience, and where no confusion is likely, make use of the abbreviated notation indicated below.

$$(4.1) \qquad \rho = \rho_t(Z) \ , \qquad \rho_i = \rho_{it}(Z) \ , \ i = 1,2,\ldots \ .$$

$$(4.2) \qquad \xi = \xi(t) = \frac{d\,|t|}{\log Z}$$

$$(4.3) \qquad \zeta = (1 - \xi)^{\frac{1}{2}} \ , \quad u = 1 + \zeta \ , \quad w = 1 - \zeta \quad .$$

We shall, for the same reason, where no confusion is likely, drop the arguments from functions elsewhere introduced. For reference to the original definitions of the infinitesimals (4.1), i = 1,...,6, see (3.18), (3.27) - (3.40), in the previous section.

By (3.18), (3.6),

$$\rho = \varepsilon_{1\mu_t}(Z) \cdot (1 - \rho) , \quad t_1^- \le t \le t_1^+ \tag{4.4}$$

where

$$\varepsilon_{1\mu_t}(Z) = \begin{cases} \varepsilon^+_{1\mu_t}(Z) , & 0 \le t \le t_1^+ \\ \varepsilon^-_{1\mu_t}(Z) , & t_1^- \le t \le 0 \end{cases} . \tag{4.5}$$

By (3.7), (3.15), (3.16), (3.34), we have, for $t_1^- \le t \le t_1^+$,

$$\begin{aligned} \varepsilon_{1\mu_t}(Z) = \frac{1}{2(1-\rho) \log Z} \Bigg\{ \log \left(\frac{16\pi}{d^4} \cdot \mathcal{W}(t) \cdot \left[\left(\frac{16(1-\rho)^2 \log^2 Z}{(\mu_t^{\frac{1}{2}}+1)^4} + 1 \right)^{\frac{1}{2}} - 1 \right] \right) \\ - \tfrac{1}{2}(1-\mu_t) \Bigg\} + \frac{\mu_t + 1}{(\mu_t^{\frac{1}{2}}+1)^2} \left[\left(1 + \frac{(\mu_t^{\frac{1}{2}}+1)^4}{16(1-\rho)^2 \log^2 Z} \right)^{\frac{1}{2}} - 1 \right] , \end{aligned} \tag{4.6}$$

so that by (4.4),

$$\rho = \frac{1}{\log Z} \left[\tfrac{1}{2} \log \log Z + \beta + \rho_7 \right] , \quad t_1^- \le t \le t_1^+ , \tag{4.7}$$

where

$$\beta = \beta_t(Z) = \log \left(\frac{8\sqrt{\pi}}{d^2} \; \frac{(\mathcal{W}(t))^{\frac{1}{2}}}{u_t^{\frac{1}{2}} + 1} \right) - \tfrac{1}{4}(1 - \mu_t) . \tag{4.8}$$

(4.9) $$\rho_{7} = \rho_{71} + \rho_{72} \quad ,$$

(4.10) $$\rho_{71} = \tfrac{1}{2} \log \left[\left((1-\rho)^2 + \frac{b^2}{\log^2 Z} \right)^{\frac{1}{2}} - \frac{b}{\log Z} \right] \quad ,$$

(4.11) $$\rho_{72} = \frac{1}{4b} (\mu_t + 1)(1-\rho) \left[\left(\log^2 Z + \frac{b^2}{(1-\rho)^2} \right)^{\frac{1}{2}} - \log Z \right] \quad ,$$

(4.12) $$b = b_t(Z) = \frac{1}{4} (\mu_t^{\frac{1}{2}} + 1)^2 \quad .$$

We note, first of all, that for all $Z > 0$, $t_1^- \leq t \leq t_1^+$,

(4.13) $$\log \left(\frac{4\sqrt{\pi}}{d^2} \cdot \frac{\min(W_0, W_1)}{\max(W_0, W_1)} \right) - \frac{1}{4} < \beta_t(Z) < \log \left(\frac{8\sqrt{\pi}}{d^2} \cdot \left[1 + \frac{\min(W_0, W_1)}{\max(W_0, W_1)} \right] \right),$$

and that

(4.14) $$\lim_{Z \longrightarrow \infty} \rho_{7t}(Z) = 0 \quad , \quad t_1^- \leq t \leq t_1^+ \quad .$$

Hence

(4.15) $$\rho^k = \left[\rho_t(Z) \right]^k = o\left(\frac{1}{\log^j Z}\right) \ , \quad k \geq j+1 \ , \quad j = 1, 2, \ldots \ .$$

and this holds for all t in the closed interval $t_1^- \leq t \leq t_1^+$.

The results which follow depend upon (4.15) and are obtained by straightforward algebraic methods. They are presented without detailed derivation. (4.10) and (4.11) can be put in the following form

$$(4.16) \qquad \rho_{71} = -\tfrac{1}{2}\left(\rho + \frac{b}{\log Z}\right) + o\left(\frac{1}{\log Z}\right) \quad , \quad t_1^- \leq t \leq t_1^+ \quad ,$$

$$(4.17) \qquad \rho_{72} = \frac{(\mu_t+1)b}{8 \log Z} + o\left(\frac{1}{\log Z}\right) \quad , \quad t_1^- \leq t \leq t_1^+ \quad .$$

Adding these together and using (3.31), we get

$$(4.18) \quad \rho_7 = -\tfrac{1}{2}\left[\rho + \frac{u^2+2\zeta}{2u^4 \log Z}\right] + o\left(\frac{1}{\log Z}\right) \quad , \quad t_1^- \leq t \leq t_1^+ \quad .$$

(4.8) can be put in similar form.

$$(4.19) \quad \beta = \log v - \frac{\zeta}{u^2} - \frac{\xi+2w}{2u^2}\,\rho + o\left(\frac{1}{\log Z}\right) \quad , \quad |t| \leq \mathfrak{J}_\delta \quad ,$$

where

$$(4.20) \qquad v = v_t(Z) = \frac{4\sqrt{\pi}}{d^2} \cdot u \cdot (\mathcal{W}(t))^{\frac{1}{2}} \quad .$$

Note that here we find the remainder to be of the indicated order only for t in the closed interval, t $\le \mathcal{J}_\delta$, where $\mathcal{J}_\delta$ is defined by (3.51) and again δ may be any positive fixed number < 1, no matter how small. Substituting (4.18), (4.19) into (4.7), we find after one iteration that for $|t| \le \mathcal{J}_\delta$,

$$(4.21) \quad \rho = \frac{1}{\log Z}\left(\log V - \frac{\zeta}{u^2}\right) - \frac{1}{\log^2 Z}\left(\frac{2}{u^2}\log V + \frac{w^2-2\zeta}{4u^4}\right) + o\left(\frac{1}{\log^2 Z}\right) ,$$

where

$$(4.22) \qquad V = V_t(Z) = v\sqrt{\log Z} \quad .$$

After, progressively expressing ρ_1 to ρ_5 in terms of ρ, we arrive at the following result for ρ_6, valid for $|t| \le \mathcal{J}_\delta$.

$$(4.23) \quad \rho_6 = \frac{1}{u^2 \log Z} + \frac{1}{\zeta}\rho + \frac{w^2}{4\zeta^3}\rho^2 - \frac{1}{2u^4\log^2 Z} + o\left(\frac{1}{\log^2 Z}\right).$$

Substituting into this our derived expression (4.21) for ρ, we get, again for $|t| \le \mathcal{J}_\delta$,

$$(4.24) \quad \rho_6 = \frac{\log V}{\zeta \log Z} + \frac{\log V}{2\zeta^2 \log^2 Z} \left(\frac{w^2}{2\zeta} \log V - 1 \right) + o\left(\frac{1}{\log^2 Z} \right) .$$

Finally, we have from (3.44) that

$$(4.25) \qquad \nu^*(t) = \frac{2}{d^2} \; u^2 \left[\log Z - \rho_6 \log Z \right], \; t_1^- \leq t \leq t_1^+ ,$$

so that the desired expression for this function in the interval $|t| \leq \mathcal{T}_\delta$ is

$$(4.26) \quad \nu^*(t) = \frac{2}{d^2} u^2 \left[\log Z - \frac{1}{\zeta} \log V - \frac{\log V}{2\zeta^2 \log Z} \left(\frac{w^2}{2\zeta} \log V - 1 \right) \right] + o\left(\frac{1}{\log Z} \right) .$$

5. Expected Value of the Second Sample Size Function.

By our original assumption (1.1) and by (1.5) the frequency function of $t = t_m$ is

$$p_\theta(t) = \frac{1}{\sqrt{2\pi m}}\, e^{-\frac{1}{2m}(t - q_\theta)^2} , \tag{5.1}$$

where

$$q_\theta = \begin{cases} \frac{1}{2}md - \frac{1}{d}\log\lambda , & \theta = \theta_1 \\ -\frac{1}{2}md - \frac{1}{d}\log\lambda , & \theta = \theta_0 \end{cases} . \tag{5.2}$$

We will need the following

Lemma 7.

$$\int_{\mathcal{J}_\delta}^{\infty} p_\theta(t)dt < \left[\sqrt{2\pi}\,\beta_-\, Z^{\frac{1}{2}\beta_-^2 \log Z} \log Z\right]^{-1} ,$$

$$\int_{-\infty}^{-\mathcal{J}_\delta} p_\theta(t)dt < \left[\sqrt{2\pi}\,\beta_+\, Z^{\frac{1}{2}\beta_+^2 \log Z} \log Z\right]^{-1} ,$$

where

$$\beta_{\pm} = \frac{1}{d\sqrt{m}}\left(1 - \delta \pm \frac{dq_\theta}{\log Z}\right) . \tag{5.4}$$

Proof.

(5.5)

$$\int_{\mathcal{J}_\delta}^{oo} p_\Theta(t)dt = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{m}}(\mathcal{J}_\delta - q_\Theta)}^{oo} e^{-\frac{1}{2}x^2} dx \ , \qquad \int_{-oo}^{-\mathcal{J}_\delta} p_\Theta(t)dt = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{m}}(\mathcal{J}_\delta + q_\Theta)}^{oo} e^{-\frac{1}{2}x^2} dx.$$

By (3.48) - (3.50), we have, taking N=0, and recalling the definition (3.51) of $\mathcal{J}_\delta$,

(5.6)

$$\frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{m}}(\mathcal{J}_\delta \pm q_\Theta)}^{oo} e^{-\frac{1}{2}x^2} dx < \frac{\sqrt{m}}{\sqrt{2\pi}(\mathcal{J}_\delta \pm q_\Theta)} e^{-\frac{1}{2m}(\mathcal{J}_\delta \pm q_\Theta)^2} =$$

$$\frac{d\sqrt{m}}{\sqrt{2\pi}\left(1-\delta \pm \frac{dq_\Theta}{\log Z}\right)\log Z} e^{-\frac{1}{2d^2 m}\left(1-\delta \pm \frac{dq_\Theta}{\log Z}\right)^2 \log^2 Z} =$$

$$\left[\sqrt{2\pi}\ \beta_\pm\ Z^{\frac{1}{2}\beta_\pm^2 \log Z} \log Z\right]^{-1}$$

Q.E.D.

As a result of the above lemma, we have, certainly that

$$(5.7) \quad \int_{-oo}^{-\mathcal{J}_\delta} p_\Theta(t)dt \quad , \quad \int_{\mathcal{J}_\delta}^{oo} p_\Theta(t)dt = o\left(\frac{1}{Z^j \log Z}\right) \quad ,$$

all finite positive j.

We define the notation, E_Θ^δ, by the identity

$$(5.8) \qquad E_\theta^{\delta}\,\mathcal{K}(t) \equiv \int_{-\Upsilon_\delta}^{\Upsilon_\delta} \mathcal{K}(t)\, p_\theta(t)\, dt \qquad .$$

By Theorem 1,

$$(5.9) \qquad E_\theta\, \nu(t) = E_\theta^{\delta}\, \nu^*(t) + \psi_1 \qquad ,$$

where

$$(5.10) \qquad \psi_1 = \int_{\Upsilon_\delta}^{t^*} \nu^*(t)\, p_\theta(t)\, dt + \int_{t^*}^{-\Upsilon_\delta} \nu^*(t)\, p_\theta(t)\, dt \qquad .$$

By Theorem 4, Lemma 1, we have, for sufficiently large Z,

$$(5.11) \qquad 0 < \psi_1 < \max \left[\nu^*(-\Upsilon_\delta), \nu^*(\Upsilon_\delta)\right] \cdot \left(\int_{\Upsilon_\delta}^{\infty} p_\theta(t)dt + \int_{-\infty}^{-\Upsilon_\delta} p_\theta(t)dt \right) .$$

By (3.38)

$$(5.12) \qquad \nu^*(\pm \Upsilon_\delta) = \frac{2}{d^2}\,(1 + \delta^{\frac{1}{2}})^2 (1 - \rho_{6,\pm\Upsilon_\delta}) \log Z \qquad ,$$

so that, by (3.59), (5.7),

$$(5.13) \qquad \psi_1 = o\left(\frac{1}{Z}\right) .$$

Hence,

$$(5.14) \qquad E_\theta\, \nu(t) = E_\theta^{\ \delta} \nu^*(t) + o(\tfrac{1}{Z}) \ .$$

For the remainder of this section, all relationships in t should be understood to hold for all t in the closed interval $|t| \leq \mathcal{T}_\delta$. By (4.26),

$$(5.15) \quad \tfrac{1}{2} d^2 E_\theta^{\ \delta}\ \nu^*(t) = \log Z \cdot E_\theta^{\ \delta} u^2 - E_\theta^{\ \delta}\left(\frac{u^2}{\zeta} \log V\right)$$

$$+ \frac{1}{2 \log Z}\ E_\theta^{\ \delta}\left(\frac{u^2}{\zeta^2} \log V\right) - \frac{1}{4 \log Z}\ E_\theta^{\ \delta}\left(\frac{\xi^2}{\zeta^3} \log^2 V\right)$$

$$+ o\left(\frac{1}{\log Z}\right) \quad .$$

First, we note that

$$(5.16) \quad \xi^{j+1} = \left(\frac{d}{\log Z}\right)^{j+1} \cdot |t|^{j+1} = o\left(\frac{1}{\mathrm{lcg}^j\ Z}\right) \cdot |t|^{j+1} \ ,$$

$$j = 1,\ 2,\ \ldots,$$

We shall now consider, one at a time, the terms on the R.H.S. of (5.15). Using (5.16) we have that

$$(5.17) \qquad \zeta = (1 - \xi)^{\frac{1}{2}} = 1 - \tfrac{1}{2}\xi - \tfrac{1}{8}\xi^2 - o\left(\frac{1}{\log^2\ Z}\right) \cdot |t|^3 \ ,$$

so that

$$(5.18) \quad u^2 = 2 - \xi + 2\zeta = 4 - 2\xi - \frac{1}{4}\ \xi^2 - o\left(\frac{1}{\log^2 Z}\right) \cdot |t|^3 \ ,$$

Hence,

$$(5.19) \qquad E_\theta^\delta\, u^2 = 4\, E_\theta^\delta(1) - 2\, E_\theta^\delta\, \xi - \frac{1}{4}\, E_\theta^\delta\, \xi^2 + o\left(\frac{1}{\log^2 Z}\right)$$

With the help of Lemma 7, we have, after some integration, that

$$(5.20) \qquad E_\theta^\delta(1) = 1 + o\left(\frac{1}{Z \log Z}\right)$$

$$(5.21) \qquad E_\theta^\delta\, \xi = \frac{\ell_{1\theta}}{\log Z} + o\left(\frac{1}{Z \log Z}\right) \quad ,$$

where

$$(5.22) \qquad \ell_{1\theta} = \left(\frac{2m}{\pi}\right)^{\frac{1}{2}} d\, e^{-\frac{1}{2m}\, q_\theta^{\,2}} + \frac{dq_\theta}{\sqrt{2\pi}} \int_{-q_\theta/\sqrt{m}}^{q_\theta/\sqrt{m}} e^{-\frac{1}{2}x^2} dx \quad .$$

$$(5.23) \qquad E_\theta^\delta\, \xi^2 = \frac{d^2}{\log^2 Z}\,(m + q_\theta^{\,2}) + o\left(\frac{1}{Z \log Z}\right) \quad .$$

Thus,

$$(5.24) \quad \log Z \cdot E_\theta^\delta\, u^2 = 4 \log Z - 2\, \ell_{1\theta} - \frac{d^2}{4 \log Z}\,(m + q_\theta^{\,2}) + o\left(\frac{1}{Z}\right).$$

We may now consider the second term R.H.S. of (5.15). It is easily verified that

$$(5.25) \qquad \frac{u^2}{\zeta} = 4 + \frac{\xi^2}{\zeta u^2} \quad .$$

Thus, by (4.22), (4.20),

$$(5.26) \quad \frac{u^2}{\zeta} \log V = 4 \log V + \frac{\log v + \frac{1}{2}\log \log Z}{\zeta\, u^2 \log^2 Z}\, d^2 t^2$$

$$= 4 \log V + o\left(\frac{1}{\log Z}\right) \cdot t^2 \quad .$$

By (4.22), (4.20),

$$(5.27) \quad \log V = \tfrac{1}{2} \log \log Z + \log \frac{4\sqrt{\pi}}{d^2} + \log (\mathcal{W}(t))^{\frac{1}{2}} + \log u.$$

Now, $u = 1 + \zeta$ may be written

$$(5.28) \quad u = 2 - \tfrac{1}{2}\xi - \frac{1}{2u^2}\, \xi^2 = 2(1 - \tfrac{1}{4}\, \xi)\left(1 - \frac{\xi^2}{(4-\xi)u^2}\right) \quad .$$

Hence, using (5.16), it is easy to show that

$$(5.29) \quad \log u = \log 2 - \tfrac{1}{4}\, \xi + o\left(\frac{1}{\log Z}\right) \cdot t^2 \quad .$$

Thus, by (5.26)

$$(5.30) \quad \frac{u^2}{\zeta} \log V = 2 \log \log Z + 4 \log \frac{8\sqrt{\pi}}{d^2} + 4 \log (\mathcal{W}(t))^{\frac{1}{2}}$$

$$- \xi + o\left(\frac{1}{\log Z}\right) \cdot t^2 \quad ,$$

so that we have

$$(5.31) \quad E_\theta^{\delta}\left(\frac{u^2}{\zeta} \log V\right) = 2 \log \log Z + 4 \log \frac{8\sqrt{\pi}}{d^2} + 4\ell_{2\theta} - \frac{\ell_{1\theta}}{\log Z}$$

$$+ o\left(\frac{1}{\log Z}\right) \quad ,$$

where

$$(5.32) \qquad \ell_{2\theta} = E_{\theta} \log (\mathcal{W}(t))^{\frac{1}{2}} \quad .$$

Now consider the third term R.H.S. (5.15). We have

$$(5.33) \qquad \frac{u^2}{\zeta^2} = 4 + \frac{2u^2 - \xi}{\zeta^2 u^2} \xi \qquad ,$$

so that

$$(5.34) \quad \frac{1}{2 \log Z} \cdot \frac{u^2}{\zeta^2} \log V = \frac{2 \log V}{\log Z}$$

$$+ \frac{(2u^2-\xi)(\log v + \frac{1}{2}\log \log Z)}{2\zeta^2 u^2 \log^2 Z} d|t| = \frac{2 \log V}{\log Z} + o\left(\frac{1}{\log Z}\right) \cdot |t|$$

$$= \frac{\log \log Z}{\log Z} + \frac{2}{\log Z} \left(\log \frac{8\sqrt{\pi}}{d^2} + \log (\mathcal{W}(t))^{\frac{1}{2}}\right)$$

$$+ o\left(\frac{1}{\log Z}\right) \cdot |t| + o\left(\frac{1}{\log Z}\right) \cdot t^2 \quad .$$

Hence,

$$(5.35) \quad \frac{1}{2 \log Z} E_{\theta}^{\delta}\left(\frac{u^2}{\zeta^2} \log V\right) = \frac{\log \log Z}{\log Z} + \frac{2}{\log Z}\left(\log \frac{8\sqrt{\pi}}{d^2} + \ell_{2\theta}\right)$$

$$+ o\left(\frac{1}{\log Z}\right) \quad .$$

By (5.16), it is clear that the fourth term R.H.S. (5.15) is $o\left(\frac{1}{\log Z}\right)$.

Collecting terms (5.24), (5.31), (5.35), we have, consulting (5.14), (5.15),

$$(5.36)\quad E_\theta \nu(t) = \frac{8}{d^2} \log \frac{Z}{\sqrt{\log Z}} - \frac{4}{d^2} \ell_{4\theta} + \frac{2 \log \log Z}{d^2 \log Z} + \frac{\ell_{5\theta}}{2d^2 \log Z} + o\left(\frac{1}{\log Z}\right) ,$$

where

$$(5.37)\qquad \ell_{5\theta} = 4\ell_{4\theta} - d^2(m + q_\theta{}^2) ,$$

$$(5.38)\qquad \ell_{4\theta} = 2 \log \frac{8\sqrt{\pi}}{d^2} + \ell_{3\theta} ,$$

$$(5.39)\qquad \ell_{3\theta} = \ell_{1\theta} + 2\ell_{2\theta} .$$

$\ell_{1\theta}$ and $\ell_{2\theta}$ are defined by (5.22) and (5.32), respectively.

With respect to our apriori distribution (0.15), the unconditional expected value of the second sample size function is

$$(5.40)\quad E\nu(t) = \frac{8}{d^2} \log \frac{Z}{\sqrt{\log Z}} - \frac{4}{d^2} \ell_4 + \frac{2 \log \log Z}{d^2 \log Z} + \frac{\ell_5}{2d^2 \log Z} + o\left(\frac{1}{\log Z}\right) ,$$

where we take

$$(5.41) \qquad \ell_j = g_0 \ell_{j\theta_0} + g_1 \ell_{j\theta_1} \quad , \qquad j = 1, 2, \ldots \quad .$$

Note that

$$(5.42) \qquad \ell_4 = 2 \log \frac{8\sqrt{\pi}}{d^2} + \ell_3$$

and, by (5.2),

$$(5.43) \quad \ell_5 = 4\ell_4 - md^2 - \frac{1}{4} m^2 d^4 - \log^2 \lambda + (g_1 - g_0) md^2 \log \lambda \quad .$$

The difference between the expected values of the second sample size function at θ_1 and at θ_0 is

$$(5.44) \quad E_{\theta_1} \nu(t) - E_{\theta_0} \nu(t) = \frac{4}{d^2} (\ell_{3\theta_0} - \ell_{3\theta_1})$$

$$+ \frac{2(\ell_{3\theta_1} - \ell_{3\theta_0}) + md^2 \log \lambda}{d^2 \log Z} + o\left(\frac{1}{\log Z}\right) .$$

This difference is, of course, equal to zero, when $\frac{W_1}{W_0} = \frac{g_1}{g_0} = 1.$

When the ratio, $W = \frac{W_1}{W_0}$ is close to one, it may be convenient to use bounds for $\ell_{2\theta}$. By (5.32), (3.4), we have that

$$(5.45) \quad \log\left(\frac{\min(W_0, W_1)}{\max(W_0, W_1)}\right) < \ell_{2\theta} < \log 2 \quad , \qquad \theta = \theta_0, \theta_1 \quad .$$

6. Error Probabilities.

The probability that our Bayes decision rule accepts alternative A_1 when the true parameter value is Θ is given by

$$(6.1) \qquad E_\Theta \phi_{m+\nu(t_m)} \left(t_{m+\nu(t_m)} \right) = Q_\Theta, \text{ say.}$$

This may be written in the form

$$(6.2) \quad Q_\Theta = P_\Theta(t^{\bar{*}} < t < t^{\overset{+}{*}}) \, E_\Theta \left[\phi_{m+\nu(t)} \left(t_{m+\nu(t)} \right) \middle| \, t^{\bar{*}} < t < t^{\overset{+}{*}} \right]$$

$$+ \, P_\Theta(t \leq t^{\bar{*}} \text{ or } \geq t^{\overset{+}{*}}) \, E_\Theta \left[\phi_{m+\nu(t)} \left(t_{m+\nu(t)} \right) \middle| \, t \leq t^{\bar{*}} \text{ or } \geq t^{\overset{+}{*}} \right],$$

where, as before, we write t for t_m.

By Theorem 1, (1.82), the second term in (6.2) may be written

$$(6.3) \quad P_\Theta(t \leq t^{\bar{*}} \text{ or } \geq t^{\overset{+}{*}}) \, E_\Theta \left[\phi_m(t) \, \middle| \, t \leq t^{\bar{*}} \text{ or } t^{\overset{+}{*}} \right] =$$

$$\int_{-\infty}^{t^{\bar{*}}} p_\Theta(t) \, E_\Theta \left[\phi_m(t) \, \middle| \, t \right] dt + \int_{t^{\overset{+}{*}}}^{\infty} p_\Theta(t) \, E_\Theta \left[\phi_m(t) \, \middle| \, t \right] dt.$$

By (1.8), this gives us

$$(6.4) \quad \int_{-\infty}^{t^{\bar{*}}} p_\Theta(t) P_\Theta(t > 0 | t) dt + \int_{t^{\overset{+}{*}}}^{\infty} p_\Theta(t) P_\Theta(t > 0 | t) dt = \int_{t^{\overset{+}{*}}}^{\infty} p_\Theta(t) \, dt \quad .$$

The first term R.H.S. (6.2) may be written

$$\int_{t^{\overline{*}}}^{t^{\overset{+}{*}}} p_\theta(t)\ E_\theta\left[\phi_{m+\nu(t)}\left(t_{m+\nu(t)}\right) \,\middle|\, t\right] dt$$

$$= \int_{t^{\overline{*}}}^{t^{\overset{+}{*}}} p_\theta(t)\ P_\theta(t_{m+\nu(t)} > 0 \,|\, t)\ dt$$

(6.5)

$$= \int_{t^{\overline{*}}}^{t^{\overset{+}{*}}} p_\theta(t) P_\theta(\hat{s}_{\nu(t)} > \bar{\theta}\nu(t) - t \,|\, t)\ dt \quad ,$$

where $\bar{\theta}$ is defined by (1.4) and $\hat{s}_\nu$ is normally distributed with mean $\theta\nu(t)$ and variance $\nu(t)$. Note that $\hat{s}_\nu$ is a generalization of s'_ν (1.2), for which ν is not restricted to integral values. It follows that

$$(6.6)\quad P_\theta\left(\hat{s}_{\nu(t)} > \bar{\theta}\nu(t) - t \,|\, t\right) = P_\theta\left(\frac{\hat{s}_{\nu(t)} - \theta\nu(t)}{\sqrt{\nu(t)}} > h_\theta(t) \,\middle|\, t\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{h_\theta(t)}^{\infty} e^{-\frac{1}{2}x^2} dx \quad ,$$

where

$$(6.7)\quad h_\Theta(t) = \frac{(\bar{\Theta}-\Theta)\nu(t)-t}{\sqrt{\nu(t)}} = \begin{cases} -\frac{1}{2}d\sqrt{\nu(t)} - \frac{t}{\sqrt{\nu(t)}} = -h^{+}(\nu(t),\ t), & \Theta=\Theta_1 \\ \frac{1}{2}d\sqrt{\nu(t)} - \frac{t}{\sqrt{\nu(t)}} = \ h^{-}(\nu(t),\ t), & \Theta=\Theta_0 \end{cases}$$

Thus, the first term R.H.S. (6.2) may be put in the form

$$(6.8)\qquad \frac{1}{\sqrt{2\pi}} \int_{t^{\overset{-}{*}}}^{t^{\overset{+}{*}}} p_\Theta(t) \int_{h_\Theta(t)}^{\infty} e^{-\frac{1}{2}x^2} dx\, dt \qquad .$$

Hence,

$$(6.9)\qquad Q_\Theta = \int_{t^{\overset{+}{*}}}^{\infty} p_\Theta(t)dt + \frac{1}{\sqrt{2\pi}} \int_{t^{\overset{-}{*}}}^{t^{\overset{+}{*}}} p_\Theta(t) \int_{h_\Theta(t)}^{\infty} e^{-\frac{1}{2}x^2} dx\, dt.$$

Now, by (3.46) and Theorem 1,

$$(6.10)\quad h^{\pm}(\nu(t),t) = \left[2(J^2(\pm t) - \rho_8)\log Z\right]^{\frac{1}{2}}, \qquad t^{\overset{-}{*}} < t < t^{\overset{+}{*}} \quad ,$$

where

$$(6.11)\qquad \rho_8 = \zeta\rho_6 - \frac{1}{4}w^2\rho_6^2 - \frac{w^2}{4(1-\rho_6)}\rho_6^3 \qquad .$$

By (4.23), we have

$$(6.12) \qquad \rho_8 = \frac{\log V}{\log Z} - \frac{\log V}{2\,\zeta\,\log^2 Z} + o\left(\frac{1}{\log^2 Z}\right), \qquad |t| \le \mathcal{T}_\delta .$$

By (6.9), Q_θ may be written in the following form

$$(6.13) \qquad Q_\theta = \frac{1}{\sqrt{2\pi}} \int_{-\mathcal{T}_\delta}^{\mathcal{T}_\delta} p_\theta(t) \int_{h_\theta(t)}^{\infty} e^{-\frac{1}{2}x^2}\, dx + I_\theta(Z) \quad ,$$

where

$$(6.14) \qquad I_\theta(Z) = \int_{t^{\overset{+}{*}}}^{\infty} p_\theta(t)\, dt + \frac{1}{\sqrt{2\pi}} \int_{t^{\overset{-}{*}}}^{-\mathcal{T}_\delta} p_\theta(t) \int_{h_\theta(t)}^{\infty} e^{-\frac{1}{2}x^2}\, dx + \frac{1}{\sqrt{2\pi}} \int_{\mathcal{T}_\delta}^{t^{\overset{+}{*}}} p_\theta(t) \int_{h_\theta(t)}^{\infty} e^{-\frac{1}{2}x^2}\, dx$$

By Theorem 4,

$$(6.15) \qquad 0 < I_\theta(Z) < 2 \int_{\mathcal{T}_\delta}^{\infty} p_\theta(t)dt + \int_{-\infty}^{-\mathcal{T}_\delta} p_\theta(t)\, dt \quad .$$

By Lemma 7, the R.H.S. of the above inequality is of order less than $\frac{1}{Z \log Z}$.

Hence

$$(6.16)\qquad I_\theta(Z) = o\left(\frac{1}{Z \log Z}\right) .$$

By (6.13), (6.7), (6.10), (3.47)

$$(6.17)\quad Q_{\theta_0} = \frac{1}{\sqrt{2\pi}} \int_{-\mathfrak{J}_\delta}^{0} p_{\theta_0}(t) \int_{\sqrt{2(1-\rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2}\, dx\, dt$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{0}^{\mathfrak{J}_\delta} p_{\theta_0}(t) \int_{\sqrt{2(\zeta^2-\rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2}\, dxdt + o\left(\frac{1}{Z \log Z}\right) .$$

For the remainder of this section, all relationships in t should be understood to hold for all t in the closed interval $|t| \leq \mathfrak{J}_\delta$.

By (3.48) - (3.50), we have, taking N = 1,

$$(6.18)\quad \int_{\sqrt{2(1-\rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2} dx = e^{-(1-\rho_8)\log Z} \cdot \frac{1-[2(1-\rho_8)\log Z]^{-1}}{(2(1-\rho_8)\log Z)^{\frac{1}{2}}} + R_1 ,$$

$$(6.19) \qquad 0 < R_1 < 3 \left[2(1-\rho_8) \log Z \right]^{-\frac{5}{2}} e^{-(1-\rho_8)\log Z} .$$

By (6.12)

$$(6.20) \qquad -(1-\rho_8)\log Z = -\log Z + \log V - \frac{\log V}{2\zeta\log Z} + o\left(\frac{1}{\log Z}\right) ,$$

so that

$$(6.21) \qquad e^{-(1-\rho_8)\log Z} = \frac{V}{Z}\, e^{-\frac{\log V}{2\zeta\log Z} + o\left(\frac{1}{\log Z}\right)}$$

$$= \frac{v\sqrt{\log Z}}{Z}\left(1 - \frac{\log V}{2\zeta\log Z}\right) + o\left(\frac{1}{Z\log Z}\right) .$$

Thus by (6.19)

$$(6.22) \qquad R_1 = o\left(\frac{1}{Z\log Z}\right) .$$

Again, using (6.12), we have that

$$(6.23) \qquad \begin{aligned} (1-\rho_8)^{-\frac{1}{2}} &= 1 + \frac{\log V}{2\log Z} + o\left(\frac{1}{\log Z}\right) , \\ \left[2(1-\rho_8)\log Z\right]^{-1} &= \frac{1}{2\log Z} + o\left(\frac{1}{\log Z}\right) \end{aligned}$$

Hence

$$(6.24)\quad \int_{\sqrt{2(1-\rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \frac{v}{\sqrt{2}\ Z}\left(1 - \frac{\log V}{2\zeta\log Z}\right)\left(1 + \frac{\log V}{2\log Z}\right)\left(1 - \frac{1}{2\log Z}\right) + o\left(\frac{1}{Z\log Z}\right)$$

If we multiply out and substitute for v its definition (4.20), the R.H.S. above may be written

$$(6.25)\quad \frac{2\sqrt{2\pi}}{d^2 Z}\,(\mathcal{W}(t))^{\frac{1}{2}}\left[\left(1 - \frac{1}{2\log Z}\right)\cdot u - \frac{\xi\log V}{2\,\zeta\,\log Z}\right] + o\left(\frac{1}{Z\log Z}\right) .$$

Using (5.16), we find that

$$(6.26)\quad u = 2 - \tfrac{1}{2}\xi + o\left(\frac{1}{\log Z}\right)\cdot t^2 \quad , \quad \frac{\xi\log V}{2\,\zeta\,\log Z} = o\left(\frac{1}{\log Z}\right)\cdot |t|$$

Thus we have

$$(6.27)\quad \frac{1}{\sqrt{2\pi}}\int_{\sqrt{2(1-\rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \frac{2}{d^2 Z}\,(\mathcal{W}(t))^{\frac{1}{2}}\left(2 - \tfrac{1}{2}\xi - \frac{1}{\log Z}\right) + o\left(\frac{1}{Z\log Z}\right)\cdot(1 + |t| + t^2) .$$

Now consider the inner integral of the second term R.H.S. (6.14). By (3.48) - (3.50), taking N = 1,

$$(6.28)\quad \int_{\sqrt{2(\zeta^2-\rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2}\,dx = e^{-(\zeta^2-\rho_8)\log Z} \cdot \frac{1 - [2(\zeta^2-\rho_8)\log Z]^{-1}}{(2(\zeta^2-\rho_8)\log Z)^{\frac{1}{2}}} + R_1' ,$$

$$(6.29)\qquad 0 < R_1' < 3\,[2(\zeta^2-\rho_8)\log Z]^{-\frac{5}{2}}\; e^{-(\zeta^2-\rho_8)\log Z}$$

By (6.12)

$$(6.30)\quad -(\zeta^2-\rho_8)\log Z = -\log Z + d|t| + \log V - \frac{\log V}{2\,\zeta\log Z} + o\left(\frac{1}{\log Z}\right),$$

so that

$$(6.31)\quad e^{-(\zeta^2-\rho_8)\log Z} = \frac{V}{Z}\, e^{d|t|}\, e^{-\frac{\log V}{2\zeta\log Z} + o\left(\frac{1}{\log Z}\right)}$$

$$= \frac{v\sqrt{\log Z}}{Z}\, e^{d|t|}\left(1 - \frac{\log V}{2\,\zeta\log Z}\right)$$

$$+ o\left(\frac{1}{Z\log Z}\right)\cdot e^{d|t|} \quad .$$

Thus by (6.29)

(6.32) $$R_1' = o\left(\frac{1}{Z \log Z}\right) \cdot e^{d|t|} \quad .$$

By (6.12)

(6.33) $$(\zeta^2 - \rho_8)^{-\frac{1}{2}} = \frac{1}{\zeta}\left(1 + \frac{\log V}{2\zeta^2 \log Z}\right) + o\left(\frac{1}{\log Z}\right) ,$$

$$\left[2(\zeta^2 - \rho_8) \log Z\right]^{-1} = \frac{1}{2\zeta^2 \log Z} + o\left(\frac{1}{\log Z}\right) .$$

Hence

(6.34)

$$\int_{\sqrt{2(\zeta^2-\rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2} dx = \frac{v}{\sqrt{2}\ \zeta Z} e^{d|t|}\left(1 - \frac{\log V}{2\zeta \log Z}\right)\left(1 + \frac{\log V}{2\zeta^2 \log Z}\right)\left(1 - \frac{1}{2\zeta^2 \log Z}\right)$$

$$+ o\left(\frac{1}{Z \log Z}\right) \cdot e^{d|t|} .$$

If we multiply out and substitute for v its definition (4.20), the R.H.S. above may be written

(6.35) $$\frac{2\sqrt{2\pi}}{d^2 Z} e^{d|t|} \cdot (\mathcal{W}(t))^{\frac{1}{2}} \left[\frac{u}{\zeta}\left(1 - \frac{1}{2\zeta^2 \log Z}\right) + \frac{\xi \log V}{2\zeta^3 \log Z}\right]$$

$$+ o\left(\frac{1}{Z \log Z}\right) \cdot e^{d|t|}$$

Using (5.16), we find that

$$\frac{u}{\zeta} = 2 + \tfrac{1}{2}\xi + o\left(\frac{1}{\log Z}\right) \cdot t^2 \quad ,$$

(6.36)

$$\frac{1}{2\zeta^2 \log Z} = \frac{1}{2 \log Z} + o\left(\frac{1}{\log Z}\right) \cdot |t| \quad ,$$

$$(6.37) \qquad \frac{\xi \log V}{2\zeta^3 \log Z'} = o\left(\frac{1}{\log Z}\right) \cdot |t| \quad .$$

Thus we have

$$(6.38) \quad \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2(\zeta^2 - \rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \frac{2}{d^2 Z} (\mathcal{W}(t))^{\frac{1}{2}} e^{d|t|} \left(2 + \tfrac{1}{2}\xi - \frac{1}{\log Z}\right)$$

$$+ o\left(\frac{1}{Z \log Z}\right) . (1 + |t| + t^2) \cdot e^{d|t|} .$$

It is easy to verify by (5.1), (5.2), that

$$(6.39) \qquad e^{dt}\, p_{\theta_0}(t) = \frac{1}{\lambda}\, p_{\theta_1}(t)$$

Hence, recalling the definition of $\mathcal{W}(t)$ (3.4), we have by (6.17), (6.27), (6.38)

$$(6.40)\quad Q_{\Theta_0} = \frac{2}{d^2 Z}\left(2 - \frac{1}{\log Z}\right)\left(\underline{W}_1 \int_{-\mathcal{T}_\delta}^{\mathcal{T}_\delta} p_{\Theta_0}(t)dt + \frac{\underline{W}_0}{\lambda}\int_{-\mathcal{T}_\delta}^{\mathcal{T}_\delta} p_{\Theta_1}(t)\,dt\right)$$

$$+ \frac{1}{d\,Z\,\log Z}\left(\underline{W}_1 \int_{-\mathcal{T}_\delta}^{\mathcal{T}_\delta} t\,p_{\Theta_0}(t)dt + \frac{\underline{W}_0}{\lambda}\int_{-\mathcal{T}_\delta}^{\mathcal{T}_\delta} t\,p_{\Theta_1}(t)\,dt\right)$$

$$+ o\left(\frac{1}{Z\,\log Z}\right) \qquad .$$

Now by Lemma 7

$$\int_{-\mathcal{T}_\delta}^{\mathcal{T}_\delta} p_\Theta(t)dt = 1 + o\left(\frac{1}{Z\,\log Z}\right) \quad ,$$

$$(6.41)$$

$$\int_{-\mathcal{T}_\delta}^{\mathcal{T}_\delta} t\,p_\Theta(t)dt = q_\Theta + o\left(\frac{1}{Z}\right) \quad , \qquad \Theta = \Theta_0,\ \Theta_1 \qquad .$$

Hence

(6.42) $$Q_{\theta_0} = \frac{2}{d^2 Z}\left(2 - \frac{1}{\log Z}\right)\left(\underline{W}_1 + \frac{\underline{W}_0}{\lambda}\right) + \frac{1}{dZ\log Z}\left(\underline{W}_1 q_{\theta_0} + \frac{\underline{W}_0}{\lambda} q_{\theta_1}\right)$$

$$+ o\left(\frac{1}{Z \log Z}\right) \quad .$$

By (0.30), (0.35), (1.18), (5.2)

(6.43) $$\underline{W}_1 + \frac{\underline{W}_0}{\lambda} = \frac{\underline{W}_1}{g_0}$$

(6.44) $$\underline{W}_1 q_{\theta_0} + \frac{\underline{W}_0}{\lambda} q_{\theta_1} = \frac{\underline{W}_1}{g_0} (g_0 q_{\theta_0} + g_1 q_{\theta_1})$$

$$= \frac{\underline{W}_1}{g_0} \left[\frac{1}{2} \text{ md } (g_1 - g_0) - \frac{1}{d} \log \lambda\right] \quad ,$$

so that

(6.45) $$Q_{\theta_0} = \frac{\underline{W}_1}{dg_0 Z}\left[\frac{2}{d}\left(2 - \frac{1}{\log Z}\right) + \frac{1}{\log Z}(g_0 q_{\theta_0} + g_1 q_{\theta_0})\right]$$

$$+ o\left(\frac{1}{Z \log Z}\right) \quad .$$

By (6.44), we may write this

(6.46) $$Q_{\theta_0} = \frac{\underline{W}_1}{2d^2 g_0 Z}\left[8 + \frac{1}{\log Z}(md^2(g_1 - g_0) - 4 - 2 \log \lambda)\right]$$

$$+ o\left(\frac{1}{Z \log Z}\right) \quad .$$

By (6.13), (6.16)

(6.47)

$$Q_{\theta_1} = \int_{-\mathcal{T}_\delta}^{\mathcal{T}_\delta} p_{\theta_1}(t)\left(1 - \frac{1}{\sqrt{2\pi}} \int_{-h_{\theta_1}}^{\infty} e^{-\frac{1}{2}x^2} dx\right) dt + o\left(\frac{1}{Z \log Z}\right).$$

By (6.41), (6.7), (6.10), this may be written

$$(6.48)\quad Q_{\theta_1} = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\mathcal{T}_\delta}^{0} p_{\theta_1}(t) \int_{\sqrt{2(\zeta^2-\rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2} dx\, dt$$

$$- \frac{1}{\sqrt{2\pi}} \int_{0}^{\mathcal{T}_\delta} p_{\theta_1}(t) \int_{\sqrt{2(1-\rho_8)\log Z}}^{\infty} e^{-\frac{1}{2}x^2} dxdt + o\left(\frac{1}{Z \log Z}\right).$$

By (6.27), (6.38) and Lemma 7, this becomes

$$(6.49)\quad Q_{\theta_1} = 1 - \frac{2}{d^2Z}\left(2 - \frac{1}{\log Z}\right)(\underline{W}_1\lambda + \underline{W}_0) + \frac{1}{dZ\log Z}\left(\underline{W}_1\lambda q_{\theta_0} + \underline{W}_0 q_{\theta_1}\right)$$

$$+ o\left(\frac{1}{Z \log Z}\right).$$

Finally, by (6.43), (6.44)

$$(6.50)\quad Q_{\theta_1} = 1 - \frac{\lambda \underline{W}_1}{d g_0 Z}\left[\frac{2}{d}\left(2 - \frac{1}{\log Z}\right) - \frac{1}{\log Z}\left(g_0 q_{\theta_0} + g_1 q_{\theta_1}\right)\right] + o\left(\frac{1}{Z \log Z}\right)$$

$$(6.51)\quad = 1 - \frac{\lambda \underline{W}_1}{2d^2 g_0 Z}\left[8 - \frac{1}{\log Z}\left(md^2(g_1 - g_0) + 4 - 2 \log \lambda\right)\right] + o\left(\frac{1}{Z \log Z}\right) .$$

By (6.46), (6.51), we have for large Z, taking only the leading terms of the expansions,

$$(6.52)\quad Q_{\theta_0} \cong \frac{4\underline{W}_1}{d^2 g_0 Z} \quad , \quad 1 - Q_{\theta_1} \cong \frac{4\underline{W}_1}{d^2 g_0 Z}\lambda \quad .$$

Now regard d, g to be arbitrary, fixed, positive, and choose W and Z to be, respectively, the following functions of d, g:

$$(6.53)\quad W'(d,g) = \frac{\alpha_1}{\alpha_0}\, g \quad , \quad Z'(d,g) = \begin{cases} \dfrac{4}{d^2 g_0 \alpha_0} \quad , & g \le \dfrac{\alpha_0}{\alpha_1} \\ \dfrac{4}{d^2 g_1 \alpha_1} \quad , & \ge \end{cases} ,$$

where α_0, α_1 are small given positive numbers. If α_0, α_1 are sufficiently small, and W, Z chosen as above, then we have, approximately

$$(6.54) \qquad Q_{\Theta_0} \underset{\cdot}{\overset{\sim}{=}} \alpha_0 \quad , \quad 1 - Q_{\Theta_1} \underset{\cdot}{\overset{\sim}{=}} \alpha_1 \quad .$$

Let W_0', W_1' be any values of w_0, W_1, and c' any value of c, such that

$$(6.55) \qquad \frac{W_1'}{W_0'} = W' \quad , \quad c' = \frac{Z'}{\min\, (W_0', W_1')} \quad .$$

Since our two-stage test is a Bayes solution, (consider now the test with parameters, d, g, W', Z'), its average risk is minimum, i.e.,

$$(6.56) \quad \sum_{i=0}^{1} g_i \left[c'(m + E_{\Theta_i} \nu(t)) + W_i' \alpha_i \right] \leq \sum_{i=0}^{1} g_i \left[c' E_i' n + W_i' \alpha_i' \right] ,$$

where $E_i'n$ represents the expected number of observations, at Θ_i, of any other two-stage test with first sample of size m, and α_i', its probability of rejecting Θ_i, when true, $i = 0, 1$. It follows that if

$$(6.57) \qquad \alpha_i' \leq \alpha_i \quad , \quad i = 0, 1$$

then

$$(6.58) \qquad m + \sum_{i=0}^{1} g_i E_{\Theta_i} \nu(t) \leq \sum_{i=0}^{1} g_i E_i' n \quad .$$

Now, by (5.36), for small values of the error probabilities, α_i, i.e., for large values of Z'

$$(6.59) \qquad E_{\theta_i} \nu(t) \doteq \frac{8}{d^2} \log \frac{Z'}{\sqrt{\log Z'}} \quad , \quad i = 0, 1 \quad .$$

Hence, since g_0, g_1 are arbitrary, we have asymptotically for small α_i ,

$$(6.60) \qquad m + E_{\theta_i} \nu(t) \leq E_i' n \quad , \quad i = 0, 1 \quad .$$

Thus, asymptotically, for small values of the error probabilities, the Wald property mentioned in our introduction, is seen to hold for Bayes two-stage tests.

7. Comparison with one-stage Bayes Solution.

In this section, we shall compare one and two-stage Bayes solutions to our problem, for large values of Z, in terms of expected sample size; requiring of the one-stage solution, that its error probabilities be the same as those for the two-stage procedure.

If we take a single sample of size $m + \tilde{\nu}$, the Bayes solution is given by the decision function

$$(7.1) \qquad \phi_{m+\tilde{\nu}}(t_{m+\tilde{\nu}}) = \begin{cases} 1 , & t_{m+\tilde{\nu}} > 0 \\ & \\ 0 & \qquad\quad \leq \end{cases} .$$

The probability that we shall accept alternative A_1, given that the true parameter value is θ, is

$$(7.2) \quad \tilde{Q}_\theta = P_\theta(t_{m+\tilde{\nu}} > 0) = \frac{1}{\sqrt{2\pi(m+\tilde{\nu})}} \int_0^{\infty} e^{\frac{(x-\tilde{q}_\theta)^2}{2(m+\tilde{\nu})}} \, dx,$$

where

$$(7.3)\qquad \tilde{q}_\theta = E_\theta t_m + \tilde{\nu} = \begin{cases} \frac{1}{2}\tilde{n}d - \frac{1}{d}\log\lambda, & \theta = \theta_1 \\ -\frac{1}{2}\tilde{n}d - \frac{1}{d}\log\lambda, & \theta = \theta_0 \end{cases},$$

$$(7.4)\qquad \tilde{n} = m + \tilde{\nu}$$

This may be written

$$(7.5)\qquad Q_{\theta_i} = \frac{1}{\sqrt{2\pi}} \int_{y_i}^{\infty} e^{-\frac{1}{2}x^2}\, dx, \quad i=0,1,$$

where

$$(7.6)\qquad y_i = \tfrac{1}{2}(-1)^i d\tilde{n}^{\frac{1}{2}} + \frac{1}{d}\log\lambda\cdot\tilde{n}^{-\frac{1}{2}}, \qquad i=0,1.$$

To determine what value of $\underset{\cdot}{\tilde{n}}$ is required so that the error probabilities (7.5) associated with this one-stage procedure will be equal to the error probabilities Q_{θ_0}, θ_{θ_1} of our two-stage Bayes solution, we set

$$(7.7) \qquad \frac{1}{\sqrt{2\pi}} \int_{y_0}^{\infty} e^{-\frac{1}{2}x^2} \, dx = Q_{\theta_0} \quad .$$

This gives us

$$(7.8) \qquad y_0 = \frac{1}{2} d\underset{\cdot}{\tilde{n}}^{\frac{1}{2}} + \frac{1}{d} \log \lambda \cdot \underset{\cdot}{\tilde{n}}^{-\frac{1}{2}} = \chi(Q_{\theta_0}) = \chi_0 \text{ , say ,}$$

where we define the function $\chi(\alpha)$, by the identity

$$(7.9) \qquad \frac{1}{\sqrt{2\pi}} \int_{\chi(\alpha)}^{\infty} e^{-\frac{1}{2}x^2} \, dx \equiv \alpha \text{ , } \quad 0 \leq \alpha \leq 1$$

Solving for (7.8) for $\tilde{n}$, we get

$$(7.10) \qquad \tilde{n} = \frac{2}{d^2}\left[X_0^2 - \log \lambda + X_0(X_0^2 - 2 \log \lambda)^{\frac{1}{2}}\right] .$$

If we expand the radical on the right hand side above, we get

$$(7.11) \qquad \tilde{n} = \frac{4}{d^2}(X_0^2 - \log \lambda) - O(X_0^{-2}) .$$

Some investigation shows that for positive x

$$(7.12) \qquad X^2(\tfrac{1}{x}) = 2 \log x - \log \log x - \log 4\pi + O(\frac{\log \log x}{\log x}) .$$

By (6.46),

$$(7.13) \qquad Q_{\theta_0} = \frac{4\underline{W}_1}{d^2 g_0 Z}\left[1 + o(\frac{1}{\log Z})\right] ,$$

so that setting

$$\frac{1}{x} = Q_{\theta_0} \quad , \tag{7.14}$$

we get

$$2 \log x = 2 \log Z + 2 \log \frac{d^2 g_0}{4 \underline{W}_1} + o(\frac{1}{\log Z}) \tag{7.15}$$

$$\log \log x = \log \log Z + O(\frac{1}{\log Z}) \quad . \tag{7.16}$$

Substituting into (7.12), we get

$$x_0^2 = 2 \log \frac{Z}{\sqrt{\log Z}} + 2 \log \frac{d^2 g_0}{8 \underline{W}_1 \sqrt{\pi}} + O(\frac{\log \log Z}{\log Z}) \quad . \tag{7.17}$$

Substituting this, in turn, into (7.11), we get, recalling the definition (0.35) of λ,

$$\tilde{n} = \frac{8}{d^2} \log \frac{Z}{\sqrt{\log Z}} + \frac{4}{d^2} \log \left[\frac{g_0 g_1 d^4}{64\pi} \frac{\max(W_0, W_1)}{\min(W_0, W_1)} \right] + O(\frac{\log \log Z}{\log Z}). \tag{7.18}$$

If now, on the other hand, we set

$$(7.19) \qquad \frac{1}{\sqrt{2\pi}} \int_{y_1}^{\infty} e^{-\frac{1}{2}x^2} dx = Q_{\theta_1} \quad ,$$

we get

$$(7.20) \qquad -y_1 = \frac{1}{2} d\tilde{n}^{\frac{1}{2}} - \frac{1}{d} \log \lambda \cdot \tilde{n}^{-\frac{1}{2}} = X(1 - Q_{\theta_1}) \quad .$$

Solving for $\tilde{n}$ and proceding as above, we arrive at exactly the same result, (7.18).

A comparison of the leading terms on the right hand sides of (7.18) and (5.36) now clearly shows that the average number of observations required by the two-stage procedure is asymptotically equal, for large Z, to the number of observations required by the above one-stage plan, i.e.

$$(7.21) \qquad \lim_{Z \to \infty} \frac{m + E_\theta \nu(t)}{\tilde{n}} = 1 \quad .$$

Now the one-stage plan is a degenerate two-stage procedure of the type (with first sample of given size m) we have been considering, which has its second sample of constant size, $\tilde{v}$, for all possible observations in the first sample. Our two-stage Bayes solution, since it minimizes the average risk with respect to the class considered, will require not more observations on the average, than any such one-stage plan with the same error probabilities. The result (7.21) indicates, however, that the larger the value of Z, the slighter will be any improvement over the one-stage set up that results.

A similar comparison with the Sequential Probability ratio test, using Wald's approximation to the expected value shows that the ratio of expected values, two-stage over sequential probability ratio, approaches 4, in the limit as $Z \longrightarrow \infty$.

8. A Trivial Asymptotic Solution.

We have seen, by the results of section 7, that the two-stage Bayes solution to our problem is, for large values of the parameter, Z, little better, in terms of the average number of observations required, than a one-stage Bayes solution with the same power. We now examine the possibility of an asymptotic solution to our problem when $d = \theta_1 - \theta_0$ becomes small, other parameters remaining fixed. That this leads to a trivial result, is indicated by the following

Lemma 7

Let Z, W be arbitrary, fixed, positive numbers, then

$$\lim_{d \longrightarrow 0} \nu(t) \equiv 0 \quad , \quad \text{all } t. \tag{8.1}$$

Proof

By (1.22)

$$\lim_{d \longrightarrow 0} \log \mathcal{M}(t) = \log t^2 \quad , \tag{8.2}$$

which is an increasing function of $|t|$ with unique minimum $= -\infty$ at $|t| = 0$. By (1.23), (1.17),

$$\lim_{d \longrightarrow 0} f_t(\mathcal{M}(t)) = -\infty \quad , \quad \text{all } t \quad . \tag{8.3}$$

It now follows from the definition of $t^{\pm}$ (see discussion above (1.24)), that

$$(8.4) \qquad \lim_{d \longrightarrow 0} t^{\pm} = 0 \; .$$

Thus, by theorem 1,

$$\lim_{d \longrightarrow 0} \nu(t) \equiv 0 \quad , \quad \text{all } t. \qquad \textbf{Q.E.D.}$$

By some simple additional calculation, we then have

$$(8.5) \quad \lim_{d \longrightarrow 0} Q_\theta = \lim_{d \longrightarrow 0} E_\theta \; \phi_m(t) = \begin{cases} 0 & , \quad \lambda \quad > \quad 1 \\ \frac{1}{2} & \qquad\quad\;\; = \\ 1 & \qquad\quad\;\; < \end{cases} \quad .$$

CHAPTER III

NON-ASYMPTOTIC CONSIDERATION OF BAYES TWO-STAGE TEST

9. Further Properties of Second Sample Size Function.

Throughout this section we shall regard the parameters m and d to be arbitrary, positive, fixed, m, an integer, and consider the second sample size and its related functions in terms of their dependency upon the loss ratio, $W = W_1/W_0$ and the ratio, $Z = \min(W_0,W_1)/c$.

By theorem 1, we have for arbitrary fixed W, Z > 0, the inequalities

$$(9.1) \qquad t^{-} < t^{\overset{-}{*}} < 0 < t^{\overset{+}{*}} < t^{+} \quad ,$$

where by the proof of lemma 3, $t^{\overset{\pm}{*}} = t^{\overset{\pm}{*}}(W,Z)$ are the respective unique solutions in t to the equations

$$(9.2) \qquad e^{dt}G_{\nu*(t)}(t) = 1 \quad , \qquad G_{\nu*(t)}(t) = 1 \quad ,$$

and by (1.21) - (1.24), $t^{\pm} = t^{\pm}(W,Z)$ are determined uniquely as the positive and negative solutions in t to the equation

$$(9.3) \qquad f_t(\mathcal{M}(t)) = \log \mathcal{M}(t) \quad .$$

Let

$$y = dt \quad ,$$

then by (1.23)

$$(9.5)\ f_t(\mathcal{M}(t)) = 2 \log \frac{dZ}{2\sqrt{2\pi}} - |\ y\ | -(y^2+1)^{\frac{1}{2}} - K(y,W) = f(y,W,Z), \text{say},$$

where for all y, all $W \geq 0$

$$(9.6)\ K(y,W) = \begin{cases} 2 \log \left[\underline{W}_0 + \underline{W}_1 e^{-y}\right] = \begin{cases} 2 \log(W+e^{-y}), \ 0 \leq W \leq 1 \\ \\ 2 \log(1+\frac{1}{W}e^{-y}), W \geq 1 \end{cases} , y \geq 0 \\ \\ 2 \log \left[\underline{W}_1 + \underline{W}_0 e^{y}\ \right] = \begin{cases} 2 \log(1+We^{y}) \ , 0 \leq W \leq 1 \\ \\ 2 \log(\frac{1}{W}+e^{y}) \quad , W \geq 1 \end{cases} , y \leq 0. \end{cases}$$

Note that for arbitrary fixed y, K(y,W) is a continuous function of W. For $y \geq 0$, it increases from $-2y$ at $W = 0$, to a unique maximum $= 2 \log(1+e^{-y})$, at $W = 1$. It then decreases, approaching zero in the limit as $W \longrightarrow \infty$. The picture for $y \leq 0$ is immediately seen from the relationship

$$(9.7) \qquad K(-y,W) = K(y,\tfrac{1}{W}) \quad .$$

This clearly holds for all y, all $W > 0$, and in the limit as $W \to 0, \infty$. Hence, over the same range

$$(9.8) \qquad K(\,|y|\,,0) \leq K(y,W) \leq K(y,1) \quad .$$

It follows, by (9.5), that for arbitrary fixed Z and over the same range of values of y and W,

$$(9.9) \qquad f(-y,W,Z) = f(y,\tfrac{1}{W},Z) \quad ,$$

and

$$(9.10) \qquad f(y,1,Z) \leq f(y,W,Z) \leq f(\,|y|\,,0,Z) \quad .$$

It may be inferred from this, and from the description of K(y,W) that for fixed y,Z, f(y,W,Z) has a unique minimum at W = 1.

By (1.22), the R.H.S. of (9.3),

$$\text{(9.11)} \qquad \log \mathcal{M}(t) = \log\left(\frac{2}{d^2}\left[(y^2+1)^{\frac{1}{2}} - 1\right]\right) = g(y), \text{ say}$$

and this function is clearly independent of W.

Now

$$\text{(9.12)} \qquad y = dt^{\pm}(W,Z)$$

are the unique positive and negative values of y at which each member of the family of curves f(y,W,Z) intersects the curve g(y). Hence by (9.9)

$$\text{(9.13)} \qquad t^{\pm}(W,Z) = -t^{\mp}(\tfrac{1}{W}, Z) , \qquad W,Z > 0 .$$

Unique intersections at finite positive and negative values of y occur also in the limit as W ——> 0, oo. Now both upper and lower bounding curves of (9.10) are symmetric about y = 0. Their intersections with g(y) are thus respectively at

$$\text{(9.14)} \qquad y = \pm\, dt^{+}(0,Z) \quad \text{and} \quad y = \pm\, dt^{+}(1,Z) .$$

By the nature of $g(y)$ already described in terms of the variable t (see (1.22)), it then follows that for $Z > 0$, $W \geq 0$, and in the limit as $W \longrightarrow \infty$

$$t^+(1,Z) \leq \pm t^{\pm}(W,Z) \leq t^+(0,Z) \quad . \tag{9.15}$$

For arbitrary fixed $Z > 0$, $t^+(W,Z)$ has a unique minimum and $t^-(W,Z)$ has a unique maximum, at $W = 1$. Both functions are monotone for $W < 1$, and monotone (in the opposite sense) for $W > 1$, and both can be shown to be continuous in W. By (9.1), we then see that $t^{\pm *}(W,Z)$ are certainly bounded functions of W.

The function $\nu*(t) = \nu*(t,W,Z)$ is defined by (1.26) to be the larger root in ν of the equation

$$\log \nu = f_t(\nu) \;, \qquad t^- < t < t^+ \;, \tag{9.16}$$

and the unique root in ν of this equation when $t = t^{\pm}$. Let

$$C = \frac{d^2 Z}{4\sqrt{2\pi}} \;, \qquad z = \frac{dZ}{2\sqrt{2\pi}} \;, \tag{9.17}$$

then by (1.16), (1.17), (9.6), we may write this equation

$$(9.18) \quad C^2\left(\frac{\nu}{z^2}\right) + \log \nu + \frac{t^2}{\nu} = 2 \log z - 2C \frac{|t|}{z} - K\left(2C \frac{t}{z}, W\right) \quad .$$

Let

$$(9.19) \qquad \gamma = \frac{t}{z} \quad , \quad x = \frac{\nu}{z^2}$$

then the equation becomes

$$(9.20) \qquad \frac{1}{x}(Cx + |\gamma|)^2 = -\log x - K(2C\gamma, W) \quad .$$

If we apply the transformations (9.17), (9.19) to both sides of equation (9.3) and then solve for γ, the two solutions in γ which are obtained will be

$$(9.21) \qquad \gamma^{\pm}(W,C) = \frac{1}{z} t^{\pm}(W,Z) \quad ,$$

where $\gamma^{\pm}(W,C)$ are functions of W and C only. We then have that

$$(9.22) \qquad \nu^*(t,W,Z) = z^2 x(\gamma,W,C) ,$$

where $x(\gamma,W,C)$ is the larger root in x of (9.20) whenever

$$(9.23) \qquad \gamma^-(W,C) < \gamma < \gamma^+(W,C),$$

and the unique root in x for ν at the endpoints of this interval. In keeping with the definition (1.26), we define $x(\gamma,W,C)$ to be identically zero for all γ outside, and not bordering, the interval (9.23).

For arbitrary fixed $C > 0$, let S(C) be the region in the γ,W plane, $W \geq 0$, defined by (9.23) and including the bounding curves $\gamma = \gamma^{\pm}(W,C)$. It can be shown by some simple arguments which involve the equation (9.20), that $x(\gamma,W,C)$ forms a continuous bounded surface over S(C). For fixed γ, the surface is monotone in W, decreasing when $W < 1$, increasing when $W > 1$. For fixed W, its behavior as γ value may be inferred, by (9.22), from Lemmas 1 and 2.

Now for arbitrary $Z > 0$, let S'(Z) be the region in the t,W plane, $W \geq 0$, bounded by the t axis and the curves, $t = t^{\pm}(W,Z)$.

Reversing the transformations (9.17), (9.19), we have, by the above, that $\nu^*(t,W,Z)$ is continuous and bounded over $S'(Z)$. For fixed t, it is monotone in W, decreasing when $W < 1$, increasing when $W > 1$. As a function of t, for fixed W, it is described by lemmas 1 and 2.

In addition, we note that by (9.20), (9.7)

$$x(-\gamma,W,C) = x(\gamma, \frac{1}{W}, C) \tag{9.24}$$

so that by (9.22)

$$\nu^*(-t,W,Z) = \nu^*(t, \frac{1}{W}, Z). \tag{9.25}$$

The symmetry of ν^* in t when $W = 1$ is just a special case of (9.25).

On the other hand, using the results of section 2, in particular, the identity (2.38), (2.39), it is easy to show that for arbitrary fixed t,W in the region $S'(Z)$, $\nu^*(t,W,Z)$ is a continuous increasing function of Z, all $Z > 0$. Further,

$$\lim_{Z \longrightarrow 0,\infty} \nu^*(t,W,Z) = 0 , \infty \quad . \tag{9.26}$$

In addition by lemma 4, for arbitrary fixed W, $t^+(W,Z)$ increases, $t^-(W,Z)$ decreases, continuously with Z. By (2.10), $S'(Z)$ reduces to the positive W axis in the limit as Z ——> 0, while as Z ——> oo, it becomes the entire upper half t,W plane. To relate the second sample size function to the result (9.22), we apply the transformations (9.17), (9.19) to the L.H.S. of both equations (9.2). The unique solutions in γ obtained will be respectively

$$(9.27) \qquad \gamma^{\overset{+}{-}*}(W,C) = \frac{1}{z}\, t^{\overset{+}{-}*}(W,Z) \quad ,$$

where $\gamma^{\overset{+}{-}*}(W,C)$ are functions of W and C only. By theorem 1, the second sample size function may then be written

$$(9.28)\ \nu(t,W,Z) = \begin{cases} z^2 x(\gamma,W,C) & , \quad \gamma^{-*}(W,C) < \gamma < \gamma^{+*}(W,C) \\ 0 & , \quad \text{otherwise} \end{cases} .$$

Finally, it can be shown that the points of discontinuity of the second sample size function, $t^{\overset{+}{-}*}(W,Z)$ are continuous in W and Z. First,

$$(9.29)\ G_\nu(t) = \frac{1}{Z}(W_0 + W_1 e^{-dt})\cdot\nu + \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{2}d\nu^{\frac{1}{2}}+t\nu^{-\frac{1}{2}}}^{oo} e^{-\frac{1}{2}y^2}\, dy + \frac{e^{-dt}}{\sqrt{2\pi}} \int_{\frac{1}{2}d\nu^{\frac{1}{2}}-t\nu^{-\frac{1}{2}}}^{oo} e^{-\frac{1}{2}y^2}\, dy$$

is clearly continuous in ν, t, W, and Z; ν, $Z > 0$, $W \geq 0$, all t. $\nu*(t, W, Z)$ is positive and continuous in t, W, Z everywhere in

$$(9.30) \qquad \left\{ (t, W, Z): t^{-}(W, Z) \leq t \leq t^{+}(W, Z),\ Z > 0,\ W \geq 0 \right\} .$$

Hence $G_{\nu*(t)}(t)$ is continuous in t, W, Z over this range.

As was indicated in the proof of lemma 3, $G_{\nu*(t)}(t)$ is monotone decreasing in t for any fixed W, Z in (9.30). We may regard this function, for fixed Z as a continuous surface over the t, W plane. The trace of this surface in the plane of height 1 above the t, W plane will because of the continuity of the surface, its monotonicity in t for all $W \geq 0$, and because of (9.1), describe a curve which determines a single valued continuous function of W. This curve will, by (9.2), of course, be $t^{\overset{-}{*}}(W, Z)$. Exactly the same argument, with Z and W interchanged, shows the continuity and single valuedness of $t^{\overset{-}{*}}(W, Z)$ in Z. Similarly, since $e^{dt}G_{\nu*(t)}(t)$ is monotone increasing in t for all fixed W, Z in (9.30), we can infer the continuity of $t^{\overset{+}{*}}(W, Z)$ in W and Z.

10. Some Exploratory Computations in the Symmetric Case.

To obtain a more exact idea of the behavior of our two-stage Bayes solution for intermediate values of the parameters, we investigate, by means of computation, the symmetric case of equal losses and equal apriori probabilities. i.e. in this section, we shall consider that

$$(10.1) \qquad W = \frac{W_1}{W_0} = 1 \, , \qquad g = \frac{g_1}{g_0} = 1 \qquad .$$

Since we are taking $W = 1$, we cease, for convenience, to indicate it as an argument in functions dependent on it. We shall, however, where necessary for understanding, indicate functional dependence upon other parameters.

When $W = 1$, the solutions to equations (9.2), (9.3) are respectively symmetrical about $t = 0$. i.e.

$$(10.2) \qquad t^+ = - t^- = t_1 \text{ , say , } \quad t^{\overset{+}{*}} = - t^{\overset{-}{*}} = t^* \text{ , say } \; ,$$

and both these numbers depend only upon Z and d. By (9.21), (9.27)

$$(10.3) \quad \frac{1}{z} t_1 = \gamma^+ = - \gamma^- = \gamma_1 \text{, say, } \quad \frac{1}{z} t^* = \gamma^{\overset{+}{*}} = - \gamma^{\overset{-}{*}} = \gamma^* \text{, say .}$$

γ_1 and γ^* depend only on the parameter C defined by (9.17). By (9.1), we have, indicating the functional dependence on C,

$$(10.4) \qquad 0 < \gamma^*(C) < \gamma_1(C) \quad .$$

From section 9, we have that by using the transformation

$$(10.5) \qquad y = dt = 2C\gamma \quad ,$$

equation (9.3) may be written

$$(10.6) \; |y| + (y^2+1) + \log\left[(y^2+1)^{\frac{1}{2}} - 1\right] + 2\log(1+e^{-|y|}) = \log 2C^2$$

The L. H. S. of (10.6) is a continuous, increasing, concave function of $|y|$ which → -oo, oo, as y → o, oo, and which is independent of C. Hence the solution of (10.6) in $|y|$, namely

$$(10.7) \qquad |y| = y_1(C), \text{ say, } = 2C\gamma_1(C) \quad ,$$

approaches zero as C → 0. Substituting this solution into the L. H. S. of (10.6) and solving the resulting identity in C for $\gamma_1(C)$, we get

$$\gamma_1(C) = \frac{1}{2}\, e^{-\frac{1}{2} + \varepsilon_1(C)} + \varepsilon_2(C)$$

where $\varepsilon_1(C)$, $\varepsilon_2(C) \to 0$, as $C \to 0$. Thus

$$(10.8) \qquad \lim_{C \to 0} \gamma_1(C) = \frac{1}{2} e^{-\frac{1}{2}} \cong .3033 \quad .$$

Rewriting (10.6) in the form

(10.9)

$$2C|\gamma| + (4C^2\gamma^2+1)^{\frac{1}{2}} + \log\left[(4C^2\gamma^2+1)^{\frac{1}{2}} - 1\right] + 2\log(1+e^{-2C|\gamma|}) - \log 2C^2 = 0 ,$$

we find that the L. H. S. is, for fixed C, a continuous, increasing, concave function of $|\gamma|$ which $\to -\infty$, ∞, as $|\gamma| \to 0$, ∞, while for fixed γ, it is a continuous increasing function of C. Hence the solution, $|\gamma| = \gamma_1(C)$ is a decreasing function of C. It follows, by (10.8), that $\frac{1}{2} e^{-\frac{1}{2}}$ is a least upper bound for $\gamma_1(C)$, and hence, by (10.4), an upper bound for $\gamma^*(C)$.

By (9.28), the second sample size function $\nu(t)$, may be written

$$(10.10) \qquad \nu(t,Z,d) = \begin{cases} z^2 x(\gamma,C) , & |\gamma| < \gamma^*(C) \\ 0 , & \geq \end{cases} \quad .$$

$x(\gamma,C)$ is the larger root in x of the equation

$$(10.11) \qquad \frac{1}{x}(Cx + |\gamma|)^2 + 2\log(1 + e^{-2C|\gamma|}) = -\log x$$

for all γ such that $|\gamma| < \gamma_1(C)$, the unique root when $|\gamma| = \gamma_1(C)$, and identically zero for $|\gamma| > \gamma_1(C)$. Clearly, this function depends on γ and C, only and is symmetric about $\gamma = 0$.

Applying the transformations (9.17), (9.19) to the L. H. S.'s of equations (9.2), we have, for W = 1,

$$e^{dt} G_{\nu*(t)}(t) = G_{\nu*(-t)}(-t) =$$

$$(10.12) \quad \frac{C}{\sqrt{2\pi}}(1+e^{2C\gamma})\, x(C,\gamma) + \frac{1}{\sqrt{2\pi}} \int_{h^-}^{oo} e^{-\frac{1}{2}y^2} dy + \frac{e^{2C\gamma}}{\sqrt{2\pi}} \int_{h^+}^{oo} e^{-\frac{1}{2}y^2} dy ,$$

where

$$(10.13) \qquad h^{\pm} = C\sqrt{x(\gamma,C)} \pm \frac{\gamma}{\sqrt{x(\gamma,C)}} .$$

Setting (10.12) equal to 1, we obtain $\gamma^*(C)$ as the unique solution in γ.

Our decision rule for deciding between θ_0 and θ_1 may now be written: Take m observations. If the observed sample is $X_1, \ldots, X_m$, form the function

$$\gamma = \frac{t}{z} = \frac{1}{z} \left[\sum_{i=1}^{m} X_i - \frac{1}{2} m (\theta_0 + \theta_1) \right] .$$

If $\gamma \leq -\gamma^*(C)$, accept θ_0. If $\gamma \geq \gamma^*(C)$, accept θ_1. If, however, $|\ \gamma\ | < \gamma^*(C)$, take $\nu = z^2 x(\gamma, C)$ additional observations. If, now, the total observed sample is $X_1, \ldots, X_{m+\nu}$, from the function

$$t_{m+\nu} = \sum_{i=1}^{m+\nu} x_i - \frac{1}{2}(m+\nu)(\theta_0 + \theta_1)$$

If this is negative, accept θ_0, positive, accept θ_1. We need not consider the case $t_{m+\nu} = 0$, since this has probability zero.

Table I

C	$\gamma^*(C)$
.01	.2536
.10	.2535
1.00	.2373
3.00	.1833
5.00	.1472
10.00	.1007
20.00	.0644
50.00	.0334
100.00	.0196

In table I, we give, for some selected values of C, the values of the function $\gamma^*(C)$ rounded at the 4th decimal place. These numbers were obtained by setting (10.12) equal to 1 and using this equation in conjunction with (10.11). Standard tables, which are listed in the bibliography, were used for this and the remaining tables in this section. From these results, it would appear that $\gamma^*(C)$ is a decreasing function of C with L.U.B. less than .26 which is well below the upper bound given by (10.8).

Table II: $x(\gamma,C)$

γ/C	.01	1	3	5	10	50	100
0	.249994	.203888	.100859	.0582611	.0236015	.00194291	.000602768
.015	.249769	.203663	.100634	.0580360	.0233768	.00173058	.000414810
.030	.249092	.202987	.0999578	.0573602	.0227054	.00115491	
.045	.247961	.201855	.0988263	.0562310	.0215930		
.060	.246367	.200262	.0972332	.0546430	.0200387		
.075	.244304	.198197	.0951678	.0525847	.0180149		
.090	.241757	.195649	.0926143	.0500341	.0154112		
.105	.238711	.192600	.0895485	.0469495			
.120	.235144	.189027	.0859346	.0432499			
.135	.231031	.184902	.0817166	.0387638			
.150	.226338	.180186	.0768041				
.165	.221021	.174829	.0710398				

Table II (continued)

γ/C	.01	1	3	5	10	50	100
.180	.215025	.168762	.0641129				
.195	.208277	.161888					
.210	.200673	.154065					
.225	.192071	.145070					
.240	.182252						
γ*(C)	.172008	.136558	.0623741	.0342506	.0129849	.000950055	.000285660

In table II, we have tabulated, for some of the values of C in table I, and for values of γ, from $\gamma = 0$, at intervals of length .015 up to $\gamma = \gamma^*(C)$, the function, $x(\gamma,C)$. Entries were computed from (10.11) to six significant figures, the last figure being rounded.

The expected value of the second sample size function and the probabilities of wrong decisions may be easily found in terms of integrals which involve $x(\gamma,C)$. In this symmetric case, we have, of course,

(10.14) $E_{\theta_0}\nu(t) = E_{\theta_1}\nu(t) = E\nu(t)$, say; $Q_{\theta_0} = 1-Q_{\theta_1} = Q$, say.

Also by (10.1)

$$(10.15) \qquad \lambda = \frac{W}{g} = 1 \quad .$$

By (5.1), (5.2), we have, since $\nu(t)$ is symmetric about $t = 0$,

$$(10.16) \qquad E\nu(t) = \frac{1}{\sqrt{2\pi m}} \int_{-t^*}^{t^*} \nu(t)\, e^{-\frac{1}{2m}(t + \frac{1}{2}md)^2}\, dt \quad .$$

Applying the transformations (9.17), (9.19), and the additional transformation

$$(10.17) \qquad \Gamma = \frac{C}{z} \sqrt{m} = \frac{1}{2}\, d \sqrt{m}$$

we get after some simple manipulation,

$$(10.18) \qquad d^2E\nu(t) = \frac{8C^3}{\Gamma\sqrt{2\pi}}\, e^{-\frac{1}{2}\Gamma^2} \int_0^{\gamma^*(C)} x(\gamma, C)\cosh(c\gamma)\, e^{-\frac{1}{2}(\frac{C\gamma}{\Gamma})^2}\, d\gamma \quad .$$

On the other hand, by (6.9)

$$(10.19)\quad Q=\frac{1}{\sqrt{2\pi m}}\int_{t^*}^{\infty} e^{-\frac{1}{2m}(t+\frac{1}{2}md)^2}dt+\frac{1}{2\pi\sqrt{m}}\int_{-t^*}^{t^*} e^{-\frac{1}{2m}(t+\frac{1}{2}md)^2}\int_{h_{\theta_0}(t)}^{\infty} e^{-\frac{1}{2}y^2}dydt,$$

where $h_{\theta_0}(t)$ is defined by (6.7). Applying the same transformations as above, we arrive at the following result

(10.20)

$$Q=\frac{1}{\sqrt{2\pi}}\int_{\Gamma+\frac{C\gamma^*(C)}{\Gamma}}^{\infty} e^{-\frac{1}{2}y^2}dy+\frac{C}{2\pi\Gamma}e^{-\frac{1}{2}\Gamma^2}\int_{0}^{\gamma^*(C)} e^{-\frac{1}{2}(\frac{C\gamma}{\Gamma})^2}\cdot\left(e^{C\gamma}\int_{h^+}^{\infty} e^{-\frac{1}{2}y^2}dy+e^{-C\gamma}\int_{h^-}^{\infty} e^{-\frac{1}{2}y^2}dy\right)d\gamma.$$

Clearly, (10.18) and (10.20) are functions of C and Γ, only.

We now compare the two-stage test of θ_0 v.s. θ_1 which is defined by the second sample size function (10.10) and the decision function (1.8), with the analogous one-stage Bayes solution and with the sequential probability ratio test, in terms of expected sample size, requiring of these tests that they have error probabilities equal to Q.

By (10.15), (7.5), the probability that the one-stage solution accepts θ_1 when θ_0 is true, or vice versa, is given by

$$(10.21) \qquad 1 - \tilde{Q}_{\theta_1} = \tilde{Q}_{\theta_0} = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{2}d\tilde{n}^{\frac{1}{2}}}^{\infty} e^{-\frac{1}{2}y^2} \, dy ,$$

where $\tilde{n}$ is the one-stage sample size. Setting this equal to Q gives us

$$(10.22) \qquad \frac{1}{2} d\tilde{n}^{\frac{1}{2}} = \chi(Q) = \chi_0, \text{ say,}$$

where $\chi(\alpha)$, $0 \leq \alpha \leq 1$, is defined by

$$(10.23) \qquad \frac{1}{\sqrt{2\pi}} \int_{\chi(\alpha)}^{\infty} e^{-\frac{1}{2}y^2} \, dy \equiv \alpha \quad .$$

Thus

$$d^2\tilde{n} = 4X_0^2 \quad . \tag{10.24}$$

Now consider the sequential probability ratio test of θ_0 v.s. θ_1, which has symmetric error probabilities, Q. Wald [3] gives the following approximation (a lower bound) to the expected number of observations required by this test.

$$\tilde{E} = \frac{2}{d^2}(1 - 2Q) \log \frac{1-Q}{Q} \quad . \tag{10.25}$$

Multiplying both sides by d^2, we have

$$d^2\tilde{E} = 2(1 - 2Q) \log \frac{1-Q}{Q} \; . \tag{10.26}$$

Since, as we have shown, Q is a function of C and Γ, only, it follows that (10.24) and 10.26) are also functions of C and Γ, only.

Now our two-stage procedure is a Bayes solution. Hence the average risk

$$c \sum_{i=0}^{1} g_i(m+E_{\theta_i}\nu(t)) + g_0W_1Q_{\theta_0} + g_1W_0 \left[1-Q_{\theta_1}\right] \tag{10.27}$$

is a minimum among all two-stage tests with first sample of size m. When

(10.28) $$W_0 = W_1 = \overline{W}, \text{ say; } g_0 = g_1 = \frac{1}{2} \quad ,$$

i.e. when (10.1) is satisfied, the average risk is

(10.29) $$c \left[m + E\nu(t) \right] + \overline{W} Q \quad ,$$

and this is minimum among all symmetric two-stage tests with first sample of size m.

By continuity results of the previous section, Q, (10.19), is seen to be a continuous function of the parameter, Z, other parameters being fixed. By the asymptotic results of section 6, we have that

(10.30) $$\lim_{Z \longrightarrow oo} Q = 0 \quad .$$

Now

(10.31) $$C = \frac{d^2 Z}{4 \sqrt{2\pi}} \quad .$$

Hence, Q is a continuous function of C, and

$$\lim_{C \longrightarrow \infty} Q = 0 \quad . \tag{10.32}$$

On the other hand, by (10.20),

$$\lim_{C \longrightarrow 0} Q = \frac{1}{\sqrt{2\pi}} \int_{\Gamma}^{\infty} e^{-\frac{1}{2} y^2} dy \quad . \tag{10.33}$$

It follows that for fixed Γ, given any α in the interval

$$0 \leq \alpha \leq \frac{1}{\sqrt{2\pi}} \int_{\Gamma}^{\infty} e^{-\frac{1}{2} y^2} dy \ , \tag{10.34}$$

we can always find a value of C such that

$$Q(C,\Gamma) = \alpha \quad . \tag{10.35}$$

Since $\Gamma > 0$, this implies that, given any α in the interval $0 \leq \alpha < \frac{1}{2}$, we can always find a pair, (C,Γ), such that (10.35) holds. Let (C^*,Γ^*) be a particular pair such that (10.35) holds.

Recall the definition of Z (1.18). In the symmetric case, (10.28), this is

$$Z = \frac{\overline{W}}{c} \quad . \tag{10.36}$$

Now let W^*, c^* be any two positive numbers such that

$$\frac{W^*}{c^*} = Z^* = \frac{4C^* \sqrt{2\pi}}{d^2} \quad . \tag{10.37}$$

Also, let m^*, d^* be any two positive numbers, the first, integral, such that

$$\frac{1}{2} d^* \sqrt{m^*} = \Gamma^* \quad . \tag{10.38}$$

Consider the two-stage Bayes procedure with starred parameters and let $E^*\nu(t)$ be its expected second sample size. The error probability is, of course, α. Let S be any other two stage test with first sample of size m^* for deciding between θ_0 and $\theta_0 + d^*$, $E^S n$, its expected total sample size at these two points, and $\alpha(S)$, the probability of wrong decisions in its use, and suppose that

$$\alpha(S) \leq \alpha \quad . \tag{10.39}$$

Since we must have

$$c^{*}\left[m^{*} + E^{*}\nu(t)\right] + W^{*}\alpha \leq c^{*}E^{S}n + W^{*}\alpha(S) \quad , \tag{10.40}$$

it follows that

$$m^{*} + E^{*}\nu(t) \leq E^{S}n \quad . \tag{10.41}$$

Thus, the Bayes solution in the symmetric case provides a lower bound for the average number of observations required by double sampling schemes with fixed first sample size.

Table III

C	Γ	Q	$d^2(m+E\nu)$	$d^2\tilde{n}$	$d^2\tilde{E}$	R_1	R_s	R
.01	.05	.4800	$.010^{3}37$	.010055	.006399	.99489	1.563	.014
	.25	.4013	$.250^{4}71$	.250055	.157995	.99978	1.582	$.0^{3}6$
	.4	.3446	$.640^{4}42$	.640056	.399752	.99991	1.601	$.0^{4}2$
	.5	.3085	$1.0^{6}32$	$1.0^{4}55$	.618047	.99995	1.618	$.0^{4}1$
	2.5	.0062	$25.0^{8}32$	$25.0^{3}75$	10.0248	.99997	2.494	$.0^{5}5$

Table III (continued)

C	Γ	Q	$d^2(m+E\nu)$	$d^2\tilde{n}$	$d^2\tilde{E}$	R_1	R_s	R
	4.	$.0^{4}317$	$64.0^{10}15$	$64.0^{4}83$	20.7188	.999999	3.089	$.0^{6}2$
	5.	$.0^{6}287$	$100.0^{12}14$	$100.0^{4}80$	30.1290	.999999	3.319	$.0^{6}1$
1.	.05	.3258	.815	.816	.507	.999	1.609	.002
	.25	.3249	.7271	.8251	.5125	.881	1.419	.314
	.4	.2961	.9466	1.1472	.7060	.825	1.305	.455
	.5	.2709	1.2389	1.4885	.9071	.832	1.366	.429
	1.	.1455	4.0842	4.4613	2.5110	.915	1.627	.193
	1.5	.0621	9.0302	9.4561	4.7562	.955	1.899	.091
	2.	.0213	16.0097	16.4543	7.3322	.973	2.183	.049
	2.5	.0058	25.0025	25.4535	10.1598	.982	2.461	.030
	4.	$.0^{4}298$	$64.0^{4}11$	64.4529	20.8385	.993	3.071	.010
	5.	$.0^{6}270$	$100.0^{6}10$	100.4505	30.247	.996	3.306	.006
3	.25	.1722	3.5420	3.5763	2.0590	.990	1.720	.023
	.4	.1742	3.2609	3.5172	2.0280	.927	1.608	.172
	.5	.1695	3.1822	3.6563	2.1006	.870	1.515	.305
	1.	.1061	4.8508	6.2281	3.3592	.779	1.444	.480
	1.5	.0477	9.3113	11.1271	5.4181	.837	1.719	.318
	2.	.0167	16.0982	18.0900	7.8753	.890	2.044	.195

Table III (continued)

C	Γ	Q	$d^2(m+E\nu)$	$d^2\tilde{n}$	$d^2\tilde{E}$	R_1	R_s	R
	2.5	.0046	25.0256	27.0730	10.6373	.924	2.353	.125
	4.	$.0^4242$	64.0^312	66.0531	21.2609	.969	3.010	.046
	5.	$.0^6220$	100.0^511	102.05	31.06	.980	3.220	.029
5	.25	.1141	5.79	5.81	3.16	.997	1.830	.008
	.4	.1167	5.4977	5.6807	3.1036	.968	1.771	.071
	.5	.1169	5.2362	5.6725	3.0997	.923	1.689	.170
	1.	.0819	5.8002	7.7577	4.0430	.748	1.435	.527
	1.5	.0385	9.6716	12.5078	5.9399	.773	1.628	.432
	2.	.0138	16.2134	19.4130	8.3032	.835	1.953	.288
	2.5	.0039	25.0558	28.3675	11.0144	.883	2.275	.191
	4.	$.0^4204$	64.0^327	67.3174	21.5943	.951	2.964	.073
	5.	$.0^6187$	100.0^524	103.30	30.99	.968	3.227	.046
10	.4	.063	9.28	9.34	4.74	.993	1.959	.014
	.5	.0641	8.957	9.261	4.676	.967	1.915	.066
	1.	.0525	7.892	10.507	5.177	.751	1.525	.491
	1.5	.0268	10.510	14.897	6.798	.706	1.546	.542
	2.	.0100	16.487	21.653	9.008	.761	1.830	.409
	2.5	.0029	25.128	30.500	11.638	.824	2.159	.287
	4.	$.0^4155$	64.0^362	69.402	22.144	.922	2.890	.114
	5.	$.0^6143$	100.0^555	105.37	31.52	.949	3.173	.073

In table III, we have tabulated the function, (10.20), (10.18) plus $d^2m = 4\Gamma^2$, (10.24), and (10.26). Computations were made for five values of C and selected values of Γ. Gregory's formula of numerical integration [6] was employed to evaluate the integrals in (10.18) and (10.20). All entries are rounded at the last decimal place given.

Wald and Wolfowitz [4] (see introduction), have shown that the sequential probability ratio test minimizes, simultaneously at θ_0 and θ_1 , the expected number of observations among all sequential tests with the same or smaller error probabilities. Both one and two-stage tests may be regarded as degenerate sequential tests. Also, the one-stage test may be regarded as a degenerate two-stage test. We thus have that

$$\tilde{E} < m + E\nu(t) < \tilde{n} \tag{10.42}$$

and this inequality is clearly evidenced by the results in table III.

To compare the two-stage procedure with its one-stage and sequential probability ratio test analogues, we have computed the following two expectation ratios

$$R_1 = \frac{m + E\nu(t)}{\tilde{n}} , \qquad R_s = \frac{m + E\nu(t)}{\tilde{E}} \tag{10.43}$$

If we consider the "distance" (using expected number of observations as criterion) between the one-stage and sequential probability ratio tests to be 1, we can use the measure

$$(10.44) \qquad R = \frac{\tilde{n} - [m + E\nu(t)]}{\tilde{n} - \tilde{E}}$$

to indicate the fraction of this "distance" which lies between the one and two-stage tests. These measures of comparison have been tabulated in table III.

In decisions between two simple alternatives θ_0, θ_1, the problem usually specifies the distance, d, between θ_0 and θ_1, as well as the probabilities, α, β, which indicate the degree of allowable error. Table III, though inadequate for more exhaustive investigation, provides some insight into the behavior of our two-stage test, under these restrictions, in the symmetric case, $\alpha = \beta$. Examination of table III indicates that for fixed C, the expected total sample size has a minimum w.r.t. Γ, and is monotone on either side of the minimum. This minimizing value of Γ, call it $\Gamma(C)$, appears to increase with increasing C, while the corresponding value of Q decreases. If this can be taken as representing the actual case, then the "best" parameter point, (C,Γ), is of the form $(C,\Gamma(C))$, where C satisfies the equation

$$Q\,(C,\Gamma(C)) = \alpha \quad ,$$

and α is the desired error probability. If we make allowance for the inadequacy of the computations, the following table indicates the situation, approximately.

TABLE IV

C	Approximate $\Gamma(C)$	Q
.01	.05	.48
1.	.25	.32
3.	.5	.17
5.	.5	.11
10.	1.	.05

For a fixed value of d, selection of Γ in this way would insure that we took the best value of the first sample size, m.

Practical use of these tables of course requires further extension and greater accuracy in table IV, which in turn requires extension of table III to appropriate values of C and Γ. However, we may illustrate the above by an example. Suppose we desire to test θ_0 against $\theta_0 + .1$ with a probability of error approximately equal to .05. Table IV indicates that $C = 10$ and $\Gamma = 1$ (the approximate value of Γ for which $m + E\nu(t)$ is minimum at $C = 10$) have associated with them the desired error probability.

Since $d = .1$, we would take a first sample of size,

$$m = \frac{4(1)^2}{.01} = 400 \quad .$$

Consulting table III, we see that the test will require an average total of approximately 789 observations. In terms of this expected number of observations, the two-stage test would lie about one-half of the way between the analogous one-stage and sequential probability ratio tests, which would require approximately 1051, and on the average 518 observations, respectively.

A comparison of computed and asymptotic formula values of $x(\gamma,C)$ indicates a fairly close correspondence between the two for the larger values of C. e.g., at $C = 100$, the computed values of x (table II) at $\gamma = 0, .015, .0196$ are, respectively, $.0^3 60$, $.0^3 41$, $.0^3 29$, while the corresponding values obtained by using (4.26) are $.0^3 52$, $.0^3 33$, $.0^3 21$. Comparison was also made at $C = 50$ and $C = 10$, where the correspondence was found not quite so good. e.g., at $C = 10$, $\gamma = 0$, table II gives .02360, (4.26) gives .01693. Computations of $d^2 E\nu$ and Q were made only up to $C = 10$. At this value, computed and asymptotic formula values of $d^2 E\nu$ are reasonably close for $\Gamma \leq .5$. e.g., for $\Gamma = .4, .5$, table III gives $d^2 E\nu = 9.3, 9.0$, respectively, while formula (5.36) gives 8.8, 8.4. For larger values of Γ, larger values of C are needed for reasonable comparison. The leading terms of the asymptotic formula for Q (6.46) are independent of Γ and give for C = 10; Q = .036. This is not too bad an approximation for $\Gamma \leq 1.5$ (see table III), but for larger values of Γ, C = 10 is again too small for a reasonable comparison.

The results obtained here are, with an exception indicated below, not directly comparable to other two-stage procedures discussed in the literature. Owen [1] has proposed a double sample test, which at least for the example cited above, seems almost as good, in terms of expected number of observations, as the Bayes test. For $\alpha = .05$, a first sample one-half the size of the analogous one-stage sample, is taken. Depending on its outcome, following a certain intuitively proposed rule, one or the other alternatives may be accepted, or an additional sample, equal in size to the analogous one-stage sample, taken, and then the decision between alternatives made. Computations in the paper, indicate that in the symmetric case, the ratio of expected total sample size to the analogous one-stage sample size will be .757 for this procedure. This is but slightly higher than the value, .751, which may be obtained for this ratio from the above example.

As indicated by (10.41), the Bayes solution in the symmetric case provides a lower bound for the average number of observations required by double sampling schemes with fixed first sampling size and in this sense it is best for deciding between two simple alternatives. The laborious computations necessary to exhibit these tests, their average sample sizes, and error probabilities, with sufficient accuracy for practical application, is a serious drawback to their use.

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