ABSTRACT

MCALISTER, ALLISON RENEE. Frattini Properties of Leibniz Algebras. (Under the direction of Dr. Ernest Stitzinger.)

Leibniz algebras are similar in many ways to Lie algebras. When studying these algebras, a common theme is to determine which properties of Lie algebras also apply to the Leibniz algebra framework. In particular, we can consider the Frattini subalgebra, an analogue of the Frattini subgroup introduced in 1885, which has been studied in detail in Lie algebras. In this work, we obtain properties of Leibniz algebras relating to the Frattini subalgebra and the intersections of other types of subalgebras.

Frattini Properties of Leibniz Algebras

by
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DEDICATION

To all of the ones who have loved me best and made me who I am.

I am a better person for having known you.

You have my deepest gratitude and the best I have to offer.

BIOGRAPHY

Allison McAlister was born and raised in Asheboro, North Carolina. She discovered her love of mathematics as a participant in the North Carolina Governor's School program prior to her senior year of high school. After graduating from high school, Allison studied mathematics and math education at North Carolina State University, receiving a Bachelor of Science in each in May 2010. In the summer of 2010, Allison began working with a research group at North Carolina State University studying properties of Leibniz algebras under the direction of Dr. Ernest Stitzinger. She began working toward her Ph. D. in the fall of 2010 under the advisement of Dr. Stitzinger.

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Chapter 1

Introduction

Leibniz algebras were first studied by Jean-Louis Loday in 1993 [11]. Using the definition of Barnes [3], all Leibniz algebras in this work are left Leibniz algebras – algebras in which all left multiplications are derivations. Many authors instead consider right Leibniz algebras, as in [11] and [1].

Using notation analogous to Barnes in [3], left multiplication by an element, a, of the algebra will be denoted L_a. Similarly, right multiplication by a will be denoted R_a.

Now, we consider the Leibniz identity on Leibniz algebra A:

$$L_a(bc) = a(bc) = (ab)c + b(ac) \text{ for all } a, b, c \in A.$$

A summary of commonly used properties of Leibniz algebras, detailed in [3], follows. Any product of n copies is 0 unless the product is left normed. Thus, $a^n = a(a(\cdots(aa)\cdots))$. $A^1 = A$, and inductively define $A^n = AA^{n-1}$.

For a Leibniz algebra, A, consider the two-sided ideal

$$\text{Leib } (A) = \langle x^2 \mid x \in A \rangle.$$

$A/\text{Leib}(A)$ is always a Lie algebra, and hence, we see that A is a Lie algebra if and only if $\text{Leib}(A) = 0$.

In any Leibniz algebra that is not Lie, $\text{Leib}(A)$ is always a nontrivial ideal. Hence, a Leibniz algebra, A, is considered *simple* if the only proper ideals of A are $\{0\}$ and $\text{Leib}(A)$ [9]. Note that this is slightly different from the analogous Lie algebra definition.

As in Lie algebras, we consider subalgebras, ideals, normalizers, centralizers, and the

center. Many of these structures can be considered under left multiplication alone, right multiplication alone, or both multiplications. When discussing an ideal, normalizer, centralizer, center, or similar structure, it is assumed to be two-sided unless otherwise specified.

Thus, define B as an ideal of A if $AB \subseteq B$ and $BA \subseteq B$. If only the former holds, B is a left ideal of A since B is preserved by left multiplication by all elements of A. If only the latter holds, B is a right ideal of A.

Let W be a subalgebra of A. Then define the left normalizer of W in A to be

$$N_A^l(W) = \{x \in A \mid xw \in W \text{ for all } w \in W\},$$

the right normalizer of W as

$$N_A^r(W) = \{x \in A \mid wx \in W \text{ for all } w \in W\},$$

and the normalizer of W to be the intersection of the two:

$$N_A(W) = \{x \in A \mid wx \in W \text{ and } xw \in W \text{ for all } w \in W\}.$$

Define the left centralizer of W as

$$Z_A^l(W) = \{x \in A \mid xW = 0\},$$

the right centralizer of A as

$$Z_A^r(W) = \{x \in A \mid Wx = 0\},$$

and the centralizer of A as

$$Z_A(W) = \{x \in A \mid xW = 0 = Wx\}.$$

Similarly, the left center of A is defined as

$$Z^l(A) = \{x \in A \mid xa = 0 \text{ for all } a \in A\},$$

the right center is defined as

$$Z^r(A) = \{x \in A \mid ax = 0 \text{ for all } a \in A\},$$

and the center of A is defined to be

$$Z(A) = \{x \in A \mid ax = 0 = xa \text{ for all } a \in A\}.$$

The derived series, lower central series, and upper central series of a Leibniz algebra, A, are defined as usual in Lie algebras. Related definitions follow in the same vein.

A Leibniz algebra A is nilpotent of class c if $A^{c+1} = 0$, but $A^c \neq 0$. As usual, A is nilpotent if and only if the upper central series terminates at A. The largest nilpotent ideal of A, denoted $\text{Nil}(A)$, is called the *nilradical* of A. $\text{Nil}(A)$ contains all nilpotent ideals of A.

A Leibniz algebra A is solvable if $A^{(k)} = \{0\}$ for some k. The largest solvable ideal of A, called the *solvable radical* of A, is denoted $\text{Rad}(A)$. $\text{Rad}(A)$ contains all solvable ideals of A. In a slight difference from the Lie algebra definition, a Leibniz algebra is *semisimple* if $\text{Rad}(A) = \text{Leib}(A)$.

A result used often in computations is detailed as Lemma 1.9 in [3]: Let B be a minimal ideal in Leibniz algebra A. Then either $BA = 0$ or $ba = -ab$ for all $b \in B$ and all $a \in A$.

In Chapter 2, we begin with a discussion of Leibniz algebras generated by one element, called cyclic Leibniz algebras. Cyclic Leibniz algebras provide counterexamples for many extensions of results from Lie algebras to Leibniz algebras; hence, they are explored in their own right in this work. Understanding of these cyclic algebras enables construction of low dimensional examples of Leibniz algebras that are not Lie. A number of explicit computations have been detailed for cyclic Leibniz algebras over fields of characteristic 0. Some of these computations involve the Frattini subalgebra and ideal. The *Frattini subalgebra*, $F(A)$, is the intersection of all maximal subalgebras of A. As this is not generally an ideal in A, we also consider the *Frattini ideal*, $\phi(A)$, the largest ideal of A contained in $F(A)$. Next we transition to a discussion of results pertaining to the Frattini subalgebra and ideal of general Leibniz algebras in Chapter 3, including a section devoted specifically to Leibniz algebras whose Frattini ideal is equal to 0.

Elementary Leibniz algebras, algebras in which the Frattini ideal of all subalgebras is equal to 0, are discussed and classified in Chapter 4. A concept closely related to an elementary Leibniz algebra is an E-Leibniz algebra. An algebra is E if $\phi(M) \subseteq \phi(A)$ for all subalgebras M of A. A classification of E-Leibniz algebras over a field of characteristic 0 is given in Chapter 5. Also of interest are Leibniz algebras that are not elementary themselves, but all proper subalgebras are elementary. These are called minimal non-elementary Leibniz algebras. Classifications of minimal non-elementary Leibniz algebras over certain fields are given in Chapter 6.

The Frattini subalgebra and ideal share many properties with other subalgebras and ideals. We explore ideals and subalgebras that share these properties in Chapters 7 and 8. In Chapter 7, the Jacobson radical, first introduced in ring theory, is discussed. The Jacobson radical in the Leibniz algebra case is the intersection of all maximal ideals, discussed by Marshall in [12] for Lie algebras. I then consider ideals with properties in common with the Frattini ideal in Chapter 8. The Frattini subalgebra is supplemented by no proper subalgebra. Another property of the Frattini subalgebra, shown in Leibniz algebras by Barnes in [3], is that for any subalgebra W contained in $F(A)$, if A/W is nilpotent, then A is nilpotent. These ideas have been generalized in the Lie algebra case – ideals that have similar properties are called generalized Frattini ideals. I explore generalized Frattini ideals in the Leibniz algebra case.

David Towers introduced other subalgebras similar to the Frattini subalgebra in [20]. These subalgebras, $\sigma(L)$, sometimes denoted $R(L)$, and $T(L)$, are explored in [20] for various types of algebras, L. I discuss results relating to $R(A)$, the intersection of all maximal subalgebras that are ideals, and $T(A)$, the intersection of all maximal subalgebras that are not ideals, for a Leibniz algebra, A, in Chapter 8.

Chapter 2

Cyclic Leibniz Algebras

A common theme in research is determining which results from studies of Lie algebras carry over to the Leibniz algebra framework. Many results carry over successfully, while others do not. Leibniz algebras generated by one element provide counterexamples to the extension of several results from Lie algebras to Leibniz algebras. We can also construct examples of low dimension Leibniz algebras that are not Lie algebras. It would seem to be of interest to find properties of these particular algebras. In this section we study them in their own right.

Let A be a cyclic Leibniz algebra generated by a, where $a^2 \neq 0$, and let L_a denote left multiplication on A by a. Let

$$\{a, a^2, \ldots, a^n\}$$

be a basis for A, with

$$aa^j = a^{j+1} \text{for } 1 \leq j \leq n-1, \text{ and}$$

$$aa^n = \alpha_1 a + \cdots + \alpha_n a^n.$$

Using the Leibniz identity on a, a^n, and a, we find that $\alpha_1 = 0$. Consider $a(a^n a)$. On one hand, $a(a^n a) = 0$ since left multiplication by any power of a higher than 1 is identically 0, as in [3]. On the other hand, using the Leibniz identity,

$$a(a^n a) = (aa^n)a + a^n a^2 = (aa^n)a$$

since $a^n a^2 = 0$. Thus, equating these two, we see that

$$0 = (aa^n)a = (\alpha_1 a + \alpha_2 a^2 + \ldots + \alpha_n a^n)a = \alpha_1 a^2 + 0.$$

a^2 is not 0 by definition. Hence, $0 = \alpha_1$.

Thus, A^2 has basis $\{a^2, \ldots, a^n\}$. Let T be the matrix for L_a acting on A with respect to the basis, $\{a, a^2, \ldots, a^n\}$. T is the companion matrix for L_a, and its characteristic polynomial (also its minimal polynomial) is

$$p(x) = p_1(x)^{n_1} \ldots p_s(x)^{n_s},$$

where the p_j are the distinct irreducible factors of $p(x)$.

Consider the following example. This example will flow through the entire piece as a low-dimensional example of a Leibniz algebra that is not Lie.

Example 1. Let A be the Leibniz algebra over the complex numbers generated by a, with $aa^3 = a^2$.

All multiplications are enumerated below.

1. $aa = a^2$.

2. $aa^2 = a^3$.

3. $aa^3 = a^2$.

4. Left multiplication by a^k is 0 for $k > 1$, as in all Leibniz algebras.

Then the matrix representing left multiplication by a is

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the minimal polynomial for the action of L_a on A is

$$p(x) = x(x+1)(x-1).$$

Theorem 2. *Let A be a cyclic Leibniz algebra generated by a, with characteristic and minimal polynomial for L_a acting on A given by $p(x) = p_1(x)^{n_1} \cdots p_s(x)^{n_s}$. Then $\phi(A) = \{b \in A \mid q(L_a)(b) = 0\}$, where $q(x) = p_1(x)^{n_1-1} \cdots p_s(x)^{n_s-1}$.*

Proof. Let $A = W_1 \oplus \cdots \oplus W_s$ be the associated primary decomposition of A with respect to L_a. Then $W_j = \{b \in A \mid p_j(L_a)^{n_j}(b) = 0\}$. Here, $p(x)$ is also the minimal polynomial for L_a on A, and therefore, each W_j is of the form $0 \subset U_{j,1} \subset \cdots \subset U_{j,n_j} = W_j$, where all of the following hold.

1. $U_{j,i} = \{b \in A \mid p_j(L_a)^i(b) = 0\}$.

2. Each $U_{j,i+1}/U_{j,i}$ is irreducible under the induced action of L_a.

3. $\dim(U_{j,i}) = i \deg(p_j(x))$.

Since x is a factor of $p(x)$, we let $p_1(x) = x$. For $j \geq 2$, $W_j \subset A^2$, and for $i \neq n_1$, $U_{1,i} \subset A^2$. A^2 is abelian, and left multiplication by $b \in A^2$ is such that $L_b = 0$ on A. Hence,

$W_j W_k = 0$ for $1 \leq j, k \leq s$,

$W_j W_1 = 0$ for $2 \leq j$,

$W_1 W_j \subset W_j$ for $1 \leq j \leq s$,

and $U_{1,n_1-1}, W_j = 0$ for $2 \leq j$.

Hence, each $U_{j,i}$ except $U_{1,n_1} = W_1$ is an ideal in A. W_1 is generally not a right ideal.

Let $M_j = W_1 \oplus \cdots \oplus U_{j,n_j-1} \oplus \cdots \oplus W_s$. Since $\dim(A/A^2) = 1$, $A^2 = U_{1,n_1-1} \oplus W_2 \oplus \cdots \oplus W_s$, and M_1 is a maximal subalgebra of A.

We show that M_j, $j \geq 2$, is a maximal subalgebra of A. Since $a = b + c$, where $b \in W_1$ and $c \in A^2$, $L_a = L_b$. It follows that any subalgebra that contains W_1 is L_a-invariant. If M is a subalgebra of A that contains M_j properly, then $M \cap U_{j,n_j}$ contains U_{j,n_j-1} properly. Since $U_{j,n_j}/U_{j,n_j-1}$ is irreducible in $A/U_{j,n_j-1}$, $M \cap U_{j,n_j} = U_{j,n_j}$ and $M = A$. Thus, each M_j is maximal in A, and $\phi(A) \subset \bigcap_j M_j$.

Let M be a maximal subalgebra of A. If $M = A^2$, then $M = \{b \mid g(L_a)(b) = 0\}$, where $g(x) = p(x)/x$, so $M = M_1$. Suppose that $M \neq A^2$. Then $A = M + A^2$. Hence, $a = m + c$, where $m \in M$ and $c \in A^2$, and $L_a = L_m$. Hence, M is invariant under L_a. Thus, the minimum polynomial $g(x)$ for L_a on M divides $p(x)$. If there is a polynomial $h(x)$ properly between $g(x)$ and $p(x)$, then the space, H, annihilated by $h(L_a)$ is properly between M and A. H is also invariant under left multiplications by a and by any element in A^2. Hence, H is invariant under left multiplications by any element in A, and it is then a subalgebra of A. Since the minimum polynomial and characteristic polynomial for L_a on A are equal, the same is true of invariant subspaces of A. Hence, $\dim(H) = \deg(h(x))$, and H is properly between M and A, a contradiction. Thus, M is the space annihilated by $g(x) = p(x)/p_j(x)$

for some j, and $M = M_j$. Therefore, $\phi(A) = \bigcap\limits_{j=1}^{s} M_j = \{b \in A \mid q(L_a)(b) = 0\}$ where $q(x) = p_1(x)^{n_1-1} \cdots p_s(x)^{n_s-1}$. $\qquad\square$

As a special case, we obtain the following corollaries.

Corollary 3. $\phi(A) = 0$ *if and only if $p(x)$ is the product of distinct prime factors.*

Example 4. Let A be as in our running example, Example 1. Since $p(x)$ is the product of distinct linear factors, $\phi(A) = 0$.

Corollary 5. *The maximal subalgebras of A are precisely the null spaces of $r_j(L_a)$, where $r_j(x) = p(x)/p_j(x)$ for $j = 1, \ldots, s$.*

Example 6. Let A be as in our running example, Example 1. To calculate the maximal subalgebras of A, we must find the nullspaces of $r_j(L_a)$ where $r_j(x) = p(x)/p_j(x)$.

$$p_1(x) = x.$$

Then
$$r_1(x) = (x+1)(x-1) = x^2 - 1,$$

and hence,

$$r_1(L_a) = (L_a)^2 - I.$$

Any element in A is of the form $\alpha a + \beta a^2 + \gamma a^3$.
Thus, in order for

$$\left((L_a)^2 - I\right)(\alpha a + \beta a^2 + \gamma a^3) = \alpha a^3 + \beta a^2 + \gamma a^3 - \alpha a - \beta a^2 - \gamma a^3$$

$$= \alpha a^3 - \alpha a$$

to equal 0, $\alpha = 0$. Hence, the nullspace is

$$M_1 = \text{span}\ \{a^2, a^3\}.$$

$$p_2(x) = x + 1.$$

Then

$$r_2(x) = x(x - 1) = x^2 - x,$$

and hence,

$$r_2(L_a) = (L_a)^2 - L_a.$$

In order for

$$\left(L_a^2 - L_a\right)\left(\alpha a + \beta a^2 + \gamma a^3\right)$$

$$= \alpha a^3 + \beta a^2 + \gamma a^3 - \alpha a^2 - \beta a^3 - \gamma a^2$$

$$= (\alpha - \beta + \gamma)a^3 - (\alpha - \beta + \gamma)a^2$$

to equal zero, $\alpha = \beta - \gamma$.
Thus,

$$M_2 = \text{span}\{a^2 + a^3, a - a^3\}.$$

$$p_3(x) = x - 1.$$

Then

$$r_3(x) = x(x + 1) = x^2 + x,$$

and hence,

$$r_3(L_a) = (L_a)^2 + L_a.$$

In order for

$$\left(L_a^2 + L_a\right)\left(\alpha a + \beta a^2 + \gamma a^3\right)$$

$$= \alpha a^3 + \beta a^2 + \gamma a^3 + \alpha a^2 + \beta a^3 + \gamma a^2$$

$$= (\alpha + \beta + \gamma)a^3 + (\alpha + \beta + \gamma)a^2$$

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to equal 0, $\alpha = -\beta - \gamma$. Thus,

$$M_3 = \text{span } \{a - a^2, a^2 - a^3\}.$$

Now, we consider the Fitting decomposition. Let A_0 and A_1 be the Fitting null and one components of L_a acting on A. Since L_a is a derivation of A, A_0 is a subalgebra of A. L_a acts nilpotently on A_0, and $L_b = 0$ when $b \in A^2$. Therefore, for each $c \in A$, L_c is nilpotent on A_0, and A_0 is nilpotent by Engel's theorem. Let $a = b + c$, where $b \in A_0$ and $c \in A_1$. $L_c = 0$ because $A_1 \subset A^2$. Then $L_a = L_b$, and then $bA_1 = aA_1 = A_1$. For any non-zero $x \in A_1$, bx is non-zero in A_1, and x is not in the normalizer of A_0. Hence, $A_1 \cap N_A(A_0) = 0$, and $A_0 = N_A(A_0)$. Thus, A_0 is a Cartan subalgebra of A.

Conversely, let C be a Cartan subalgebra of A, and let $c \in C$. Then $c = d + e$, where $d \in A_0$ and $e \in A_1$. Since $A_1 \subset A^2$, we see that $A_1 A_1 = A_1 A_0 = 0$, $A_0 A_1 = A_1$, and $A_0 A_0 \subset A_0$. Therefore, A_1 is an abelian subalgebra of A.

Now $0 = L_c^n(c) = L_d^n(c) = L_d^n(d + e) = L_d^n(e) = L_c^n(e)$, where $d \in A_0$, which is nilpotent, and $eA = 0$. Since L_c is non-singular on A_1, $e = 0$ and $c = d$. Hence, $C \subset A_0$. Since C is a Cartan subalgebra and A_0 is nilpotent, $C = A_0$, and A_0 is the unique Cartan subalgebra of A. We record this result as the following theorem.

Theorem 7. *A has a unique Cartan subalgebra. It is the Fitting null component of L_a acting on A.*

Example 8. Let A be as in our running example, Example 1.
Then the unique Cartan subalgebra of A is

$$E_A(a) = \text{span}\{a - a^3\}.$$

Using these same ideas, we obtain a number of corollaries.

Recall the definition of a minimal ideal H of Leibniz algebra A. Non-trivial ideal H is a minimal ideal of A if H contains no other proper non-trivial ideals of A.

Corollary 9. *The minimal ideals of A are precisely $I_j = \{b \in A \mid p_j(L_a)(b) = 0\}$ for $j > 1$ and, if $n_1 > 1$, $I_1 = \{b \in A \mid p_1(L_a)(b) = 0\}$.*

Example 10. To calculate the minimal ideals in our running example, A as in 1, we calculate I_2 and I_3, where I_j is the nullspace of $p_j(L_a)$. $n_1 = 1$, so I_1 is not one of these minimal ideals.

$$p_2(x) = x + 1 \Rightarrow p_2(L_a) = L_a + I.$$

In order for

$$(L_a + I)\left(\alpha a + \beta a^2 + \gamma a^3\right)$$

$$= \alpha a^2 + \beta a^3 + \gamma a^2 + \alpha a + \beta a^2 + \gamma a^3$$

$$= \alpha a + (\alpha + \beta + \gamma)a^2 + (\beta + \gamma)a^3$$

to equal 0, $\alpha = 0$, and $\beta = -\gamma$. Hence,

$$I_2 = \text{span}\left\{a^2 - a^3\right\}.$$

$$p_3(x) = x - 1 \Rightarrow p_3(L_a) = L_a - I.$$

In order for

$$(L_a - I)\left(\alpha a + \beta a^2 + \gamma a^3\right)$$

$$= \alpha a^2 + \beta a^3 + \gamma a^2 - \alpha a - \beta a^2 - \gamma a^3$$

$$= -\alpha a + (\alpha - \beta + \gamma)a^2 + (\beta - \gamma)a^3$$

to equal 0, $\alpha = 0$, and $\beta = \gamma$. Hence,

$$I_3 = \text{span}\left\{a^2 + a^3\right\}.$$

Recall the definitions of $\text{Soc}(A)$ and $\text{Asoc}(A)$. The *socle* of A, $\text{Soc}(A)$ is the union of all minimal ideals of A and is the direct sum of some of these minimal ideals. The *abelian socle* of A, $\text{Asoc}(A)$ is the union of all abelian minimal ideals of A and is the direct sum of some of them.

Corollary 11. $Asoc(A) = \{b \in A \mid u(L_a)(b) = 0\}$, *where* $u(x) = p_2(x)\dots p_s(x)$ *if* $n_1 = 1$ *and* $u(x) = p_1(x)\dots p_s(x)$ *otherwise.*

Example 12. Continuing with A as in our running example, 1, we compute the abelian socle of A as follows.

$\text{Asoc}(A) = \{b \in A \mid u(L_a)b = 0\}$, where $u(x)$ is as defined above. Since $n_1 = 1$ in our example,

$$u(x) = (x+1)(x-1) = x^2 - 1.$$

Then

$$u(L_a) = L_a^2 - I.$$

We have already computed the nullspace of $u(L_a)$ in 6.
Hence,

$$\text{Asoc}(A) = \text{span } \{a^2, a^3\}.$$

Corollary 13. *The unique maximal ideal of A is $M_1 = \{b \in A \mid t(L_a)(b) = 0\}$, where $t(x) = p(x)/p_1(x)$.*

Example 14. Consider A as in our running example, 1. To compute the unique maximal ideal of A, we compute M_1 as in 6.

Thus, the unique maximal ideal of A is

$$M_1 = \text{span } \{a^2, a^3\}.$$

Chapter 3

The Frattini Subalgebra and Ideal

Recall that the Frattini subalgebra, $F(A)$, is the intersection of all maximal subalgebras in Leibniz algebra A. As this is not always an ideal, we consider $\phi(A)$, the Frattini ideal, which is the largest ideal contained in $F(A)$. An example from Lie algebras in which the Frattini subalgebra is not an ideal is shown:

Example 15. Let $L = \langle x, y, z \rangle$ over the field of 2 elements with multiplication $xy = z, yx = -z, yz = x, zy = -x, zx = y, xz = -y$. $F(L) = \langle x + y + z \rangle$. Consider $x(x + y + z) = xx + xy + xz = 0 + z - y$, which is not in $F(L)$. Thus, $F(L)$ is not an ideal in L. In this case, $\phi(L) = 0$.

3.1 Several Ideals of a Leibniz Algebra

The sum of nilpotent ideals is nilpotent by Corollary 3 of [8]. Therefore, define the nilradical of A to be the largest nilpotent ideal of A and denote it by $\text{Nil}(A)$.

Lemma 16. *If B is a minimal ideal in a nilpotent Leibniz algebra A, then*

1. *$B \cap Z(A) \neq 0$.*

2. *$B \subseteq Z(A)$.*

Proof. Since B is minimal, either $BA = 0$ or $ba = -ab$ for all $a \in A$ and $b \in B$, as shown in [3]. Hence, it is sufficient to work on the left of B. Let $A_{j+1}B = A(A_jB)$. If $AB = 0$, both results hold.

Suppose that $AB \neq 0$. Then there is an integer k such that $A_{k+1}B = 0$ and $A_k B \neq 0$. Hence $0 \neq A_k B \subseteq B \cap Z(A)$, so 1 holds. Since $Z(A)$ is an ideal and B is minimal, 2 holds. $\square$

Example 17. Let

$$A = \langle a, a^2, a^3, a^4 \rangle$$

be the nilpotent cyclic Leibniz algebra generated by a with $aa^4 = 0$. Then by the computations in Corollary 9, $I_1 = \text{span} \{a^4\}$ is a minimal ideal of A. $Z(A)$ is also equal to span $\{a^4\}$.

Clearly, $I_1 \cap Z(A) \neq 0$, and $I_1 \subseteq Z(A)$.

Proposition 18. *Let A be a Leibniz algebra, and let B be a minimal ideal of A. Then Nil $(A) \subseteq Z_A(B)$.*

Proof. Let $N =$Nil (A). $N \cap B$ is 0 or B since N and B are both ideals of A, with B minimal in A. If it is 0, then $NB = 0 = BN$ and $N \subseteq Z_A(B)$. If $N \cap B = B$, then B is contained in N, hence $B \cap Z(N) \neq 0$ by Lemma 16. Since B is minimal, this implies $B \subseteq Z(N)$. Hence, $NB = 0 = BN$, so $N \subseteq Z_A(B)$. $\square$

Recall the socle of A, Soc (A), and the abelian socle of A, Asoc (A). These concepts have counterparts in both Lie algebras and group theory. The following several results also have analogues in Lie algebras and groups.

Proposition 19. *Let A be a a Leibniz algebra. Then Asoc $(A) \subseteq$Nil $(A) \subseteq Z_A(Soc \ (A))$.*

Proof. The first inclusion is clear. By definition, Soc $(A) = \Sigma B_i$ where the B_i are minimal ideals of A. By the last proposition, Nil $(A) \subseteq Z_A(B_i)$ for each i. Then, Nil $(A) \subseteq$

$$\bigcap_i Z_A(B_i) = Z_A \left(\sum_i B_i \right) = Z_A(\text{Soc } (A)).$$ $\square$

3.2 Leibniz Algebras with $\phi(A) = 0$

Let A be a Leibniz algebra with $\phi(A) = 0$. If W is an abelian ideal of A, then there exists a subalgebra V of A such that A is the semidirect sum of U and V by Lemma 7.2 of [17].

Theorem 20. *Let A be a Leibniz algebra with $\phi(A) = 0$. Then Asoc $(A) =$ Nil $(A) = Z_A($Soc $(A))$.*

Proof. By Proposition 19, it is enough to show that $Z_A(\text{Soc}(A)) \subseteq \text{Asoc}(A)$. Let $C = Z_A(\text{Soc}(A))$ and $D = \text{Asoc}(A)$. Since D is abelian, A is the semidirect sum of D and a subalgebra K. Then $D + (K \cap C) = (D + K) \cap C = A \cap C = C$. Let $E = K \cap C$. Since C is an ideal, E is invariant under multiplication by K and is annihilated by D. Therefore, E is an ideal in A. Let B be a minimal ideal of A contained in E. Then $B^2 \subseteq C \cdot \text{Soc}(A) = 0$. Hence, $B \subset D$, which implies that $B \subseteq D \cap K = 0$. Thus $E = 0$, so $C \subseteq D$. $\square$

Example 21. Let A be the cyclic Leibniz algebra in our running example, Example 1.

$$\text{Asoc } (A) = \text{span } \{a^2, a^3\} = \text{Nil } (A) = Z_A(\text{Soc } (A)).$$

Corollary 22. *Let A be a Leibniz algebra with $\phi(A) = 0$. Then Nil $(A)/\phi(A) \cong$ Asoc $(A/\phi(A))$.*

Proof. This follows from the last result and Theorem 5.5 of [3]. $\square$

Theorem 23. *Let A be a Leibniz algebra with $\phi(A) = 0$. Then $A =$ Asoc $(A) \dotplus V$ where V is a Lie algebra which is isomorphic to a subalgebra of the derivation algebra of Soc (A).*

Proof. Asoc (A) is complemented in A by a subalgebra B by Lemma 7.2 of [17]. For $x \in A$, let L_x and R_x be left and right multiplication by x on Soc (A), and let $L : A \to \text{Der}(\text{Soc } (A))$ be defined as $L(x) = L_x$. L is a homomorphism, $\text{Im}(L)$ is a Lie algebra of derivations of Soc (A), and Ker $(L) = Z_A^l(\text{Soc } (A))$. Soc (A) is the direct sum of minimal ideals of A. Let W be one of these minimal ideals. When acting on W, either $R_x = 0$ for all $x \in A$ or $R_x = -L_x$ for all $x \in A$ by Lemma 1.9 of [3]. Hence, $Z_A^l(\text{Soc } (A)) = Z_A(\text{Soc } (A)) =$ Asoc (A), and Im $(L) \cong A/\text{Asoc } (A) \cong V$. $\square$

Corollary 24. *Let A be a semisimple Leibniz algebra. Then A is a Lie algebra.*

Proof. $\phi(A)$ is nilpotent by Theorem 5.5 of [3]; hence, $\phi(A) = 0$. Furthermore, $Z_A(\text{Soc } (A)) =$ Asoc $(A) = 0$. Hence, K in Theorem 23 is equal to A. $\square$

From this corollary, we immediately obtain the following extension of the classical Lie algebra result.

Corollary 25. *Let A be a semisimple Leibniz algebra over a field of characteristic zero. Then A is the direct sum of ideals that are simple Lie algebras.*

Corollary 26. *Let A be a Leibniz algebra over a field of characteristic zero with $\phi(A) = 0$. Then $A = Asoc\,(A) \dotplus V$, where $V = S \oplus Z(V)$ and S is a semisimple Lie algebra.*

Proof. By Theorem 23, $A = Asoc\,(A) \dotplus V$ where V is a Lie subalgebra of A that is isomorphic to a subalgebra D of the derivation algebra of Soc (A) and D consists of left multiplication by elements of A. Each minimal ideal is a minimal left ideal since right multiplication by $x \in A$ restricted to the minimal ideal is either 0 for all $x \in A$ or is the negative of left multiplication by x for all $x \in A$ by Lemma 1.9 of [3]. Hence, D acts completely reducibly on Soc (A). Therefore, D has the desired form by Theorem 11, chapter 2 of [10]. $\square$

Theorem 27. *Let A be a Leibniz algebra over a field of characteristic 0. Then $F(A)$ is an ideal in A.*

Proof. If B is an ideal in A and B is contained in $F(A)$, then $F(A/B) = F(A)/B$ by Proposition 4.3 of [17]. Hence, it is enough to show the result when $B = \phi(A) = 0$. In this case, we need only show that $F(A) = 0$. Then A can be written as in Theorem 23 and we use the notation presented there. Since V is a Lie algebra, $F(V) = \phi(V)$. Furthermore, V is the direct sum of ideals, each of which is simple or one dimensional. Clearly $\phi(V) = 0$. Hence, $F(A) \subseteq Asoc\,(A)$. Asoc (A) is the direct sum of minimal ideals B_i of A, each of which is abelian. Each B_i is complemented by the maximal subalgebra C_i, which is the direct sum of V and the remaining B_j. Then $\bigcap_i C_i \subseteq V$, and $F(A) \subseteq V \cap Asoc\,(A) = 0$. Therefore $F(A) = 0$. $\square$

The Frattini subalgebra of a solvable Lie algebra is an ideal, as shown by Barnes and Gastineau-Hills in [5]. This result does not carry over to Leibniz algebras, as shown in the following example.

Example 28. Let F be a field of characteristic p where p is prime and V be a vector space with basis $e_1, ..., e_p$. Define:
$x(e_j) = e_{j+1}$ with subscripts mod p
$y(e_j) = (j+1)e_{j-1}$ with subscripts mod p
$z(e_j) = e_j$.
Then $[y, x] = z$, and the other commutators between x, y, and z are 0.

Let H be the three-dimensional Lie algebra with basis $\{x, y, z\}$. Let

$$L = V \dotplus H \text{ with } [h, v] = h(v) \text{ and } VH = 0,$$

and let A be the Leibniz algebra $A = V \dotplus H$. Then

$$Z_A(V) = Z_A^l(V) = V.$$

Since $VA = 0$, $V \subseteq Z^l(A)$, the left center of A. If $xA = 0$, then $xV = 0$. Hence, $Vx = 0$, and $x \in Z_A(V)$. Hence, $x \in V$, and $V = Z^l(A)$ since V is a minimal ideal in A. By Lemma 5.12 of [3], V is complemented by a unique subalgebra in A, which is H. H is a maximal subalgebra of A and every other maximal subalgebra of A contains V. These subalgebras are precisely those of the form $V \dotplus M$ where M is maximal in H. Let Ω be the set of all maximal subalgebras in H. Then

$$F(A) = H \cap [\bigcap_{M \in \Omega} (V \dotplus M)] = H \cap [V \dotplus \bigcap_{M \in \Omega} M] = \bigcap_{M \in \Omega} M = F(H) = Fz,$$

which is not an ideal in A.

_____

Chapter 4

Elementary Leibniz Algebras

A Leibniz algebra A is called *elementary* if the Frattini ideal of every subalgebra of A is 0. The analogous concept has been studied in both group theory and Lie algebras. We first show that the direct sum of two elementary algebras is also elementary.

Lemma 29. *If $A = B \oplus C$ where B and C are both elementary, then A is elementary.*

Proof. Let S be a subalgebra of $B \oplus C$. We show that $\phi(S) = 0$. By the Second Isomorphism Theorem, $S / (B \cap S) \cong (B + S) / B$. Consider the projection map from $B + S$ onto $(B + S) \cap C$. Let $x \in B + S$. Then $x = b + c$, where $b \in B$, $c \in C$ are unique. Suppose that $x \mapsto c$. Note that $c = x - b \in C \cap (B + S)$. The kernel of this map is B. Hence, $S / (B \cap S) \cong (B + S) \cap C$. Now, $(B + S) \cap C$ is contained in C. Since C is elementary, $\phi((B + S) \cap C) = 0$. This implies that $\phi(S) \subseteq B \cap S$ and likewise, $\phi(S) \subseteq C \cap S$. Hence, $\phi(S) \subseteq B \cap C = 0$. $\qquad\square$

4.1 A^2 Nilpotent

Proposition 30. *Let A be a Leibniz algebra such that A^2 is nilpotent. Then the following are equivalent:*

1. *$\phi(A) = 0$.*

2. *Nil (A) = Soc (A), and Nil (A) is complemented by a subalgebra C.*

3. *A^2 is abelian, is a semisimple A-module, and is complemented by a subalgebra D.*

Furthermore, the complements to A^2 are precisely the Cartan subalgebras of A.

Proof. Assume that 1 holds. $\phi(A) = 0$. Then Nil (A) = Asoc (A) = Z_A(Soc (A))). Since A is solvable, Asoc (A) = Soc (A). Hence, Nil (A) is abelian and is complemented by a subalgebra. Thus, 2 holds.

Assume that 2 holds. Since A is solvable, Nil (A) =Soc (A) = Asoc (A) and, since A^2 is nilpotent, $A^2 \subseteq$ Nil (A). Hence, A^2 is abelian. Since A acts completely reducibly on Asoc (A), A acts completely reducibly on A^2, and A^2 is complemented in Asoc (A) by an ideal B. Now $B + C$ complements A^2, and 3 holds.

Assume that 3 holds. Since $\phi(A) \subseteq A^2$ always holds, there exists an A-invariant subspace B which complements $\phi(A)$ in A^2. Then $B + D$ complements $\phi(A)$ in A. Hence, $\phi(A) = 0$.

Under the conditions of the proposition, Nil $(A) = A^2 \oplus Z(A)$ and both A^2 and $Z(A)$ are the direct sums of minimal ideals of A. If B is one of these minimal ideals, then either B is central or $AB + BA = B$.

If H is a Cartan subalgebra of A, then $A = H + A^2$. $H \cap A^2$ is an ideal in A since A^2 is an abelian ideal. Hence, $H \cap A^2$ is the direct sum of minimal ideals of A. If B is one of them, then B is a minimal ideal of H which is nilpotent. Hence, $B \subseteq Z(A) \cap A^2 = 0$. Thus, $H \cap A^2 = 0$ and H is a complement to A^2 in A.

Conversely, let E be a complement to A^2 in A. Let $x \in N_A(E) \cap A^2$ where $N_A(E)$ is the normalizer of E in A. Then $xE, Ex \subseteq E \cap A^2 = 0$. Therefore, $x \in Z(A) \cap A^2 = 0$. Hence, $N_A(E) = E$ and E is a Cartan subalgebra of A. $\qquad\square$

It is not always the case that if $\phi(A) = 0$ that $\phi(M) = 0$ when M is a subalgebra of A, even if A is solvable, as the following example shows.

Example 31. Continue with the example from 28. Let H be the three-dimensional Lie algebra with basis $\{x, y, z\}$. Let

$$L = V \dotplus H, \text{ with } [h, v] = h(v) \text{ and } [v, h] = -h(v).$$

In other words, L is a solvable Lie algebra. Let

$$K = (I + R_{e_2})H.$$

R_{e_2} is a derivation since L is a Lie algebra; hence, $I + R_{e_2}$ is an automorphism, and K is a subalgebra with basis $x + 3e_1, y + e_3, z + e_2$. It is easily checked that H and K are maximal subalgebras with $H \cap K = 0$. Hence, $\phi(L) = 0$, while $\phi(H) \neq 0$.

When discussing Cartan subalgebras of Leibniz algebras, we have the following lemma. The proof is the same as the Lie algebra proof of Lemma 4 in [2].

Lemma 32. *If W is an ideal in a Leibniz algebra A, U is a subalgebra of A with $W \subseteq U$, U/W is a Cartan subalgebra of A/W, and H is a Cartan subalgebra of U, then H is a Cartan subalgebra of A.*

Theorem 33. *Let A be a Leibniz algebra with $\phi(A) = 0$ and A^2 nilpotent. Then A is elementary.*

Proof. Let M be a subalgebra of A. Suppose that $A^2 \subseteq M$. Since A^2 is nilpotent, $\phi(A^2)$ is equal to the derived algebra of A^2. Then $\phi(A^2) \subseteq \phi(A) = 0$ because the derived algebra of A^2 is an ideal in A and by Lemma 4.1 of [17]. Thus, A^2 is abelian and $A = A^2 \dotplus H$, where H is a Cartan subalgebra of A by Proposition 30. Then $M = M \cap A = M \cap (A^2 \dotplus H) = A^2 \dotplus (M \cap H)$. A acts completely reducibly on A^2 and since A^2 is abelian, H acts completely reducibly on A^2 as well. Since H is abelian, $H \cap M$ acts completely reducibly on A^2 and then also on M^2. Thus, M acts completely reducibly on M^2. Furthermore, $M = A^2 \dotplus (M \cap H) = B \oplus M^2 \dotplus (M \cap H)$ for some ideal B in M. Now $B \oplus (M \cap H)$ is a complementary subalgebra to M^2 in M. Thus, part 3 of Proposition 30 holds, and $\phi(M) = 0$.

Suppose that M does not contain A^2. Then $M + A^2$ falls in the preceding case. Hence, we may assume that $M + A^2 = A$. Since A^2 is abelian, $A^2 \cap M$ is an ideal in A. Then $M/(A^2 \cap M)$ complements $A^2/(A^2 \cap M)$ in $A/(A^2 \cap M)$, and $M/(A^2 \cap M)$ acts completely reducibly on $A^2/(A^2 \cap M)$. Hence, $M/(A^2 \cap M)$ is a Cartan subalgebra of $A/(A^2 \cap M)$ by the last proposition. Let H be a Cartan subalgebra of M. By Lemma 32, H is a Cartan subalgebra of A. Therefore, H is a complement of A^2 in A. Furthermore, $M = M \cap (H + A^2) = H + (A^2 \cap M)$ since $H \subseteq M$. Thus, H is a complement to $A^2 \cap M$ in M. $A = A^2 + M$ acts completely reducibly on A^2, and hence, M also acts completely reducibly on A since A^2 is abelian. Therefore, M acts completely reducibly on M^2 and $A^2 \cap M = M^2 \oplus (A^2 \cap Z(M))$. Since $Z(M) \subseteq H$, it follows that $Z(M) \cap A^2 = 0$. Hence, H is a complement to M^2 in M. Now M satisfies part 3 of Proposition 30, and $\phi(M) = 0$. $\qquad\qquad\square$

Example 34. Let A be as in our running example, Example 1.

Recall from Example 4 that $\phi(A) = 0$.

We know that

$$A^2 = \text{span } \{a^2, a^3\}$$

from 1.

Thus, A^2 is nilpotent, and A is elementary.

As in Lie algebras, there is a converse to Theorem 33. The Lie algebra result is shown in [19]. We show the extension to Leibniz algebras in Theorem 38. The following is a direct extension of a Lie algebra result, and the proof is the same as in the Lie algebra case as shown in Proposition 2 and Theorem 2 in [15].

Theorem 35. *Let A be a Leibniz algebra such that A^2 is nilpotent. For any subalgebra M of A, $\phi(M) \subseteq \phi(A)$.*

The following result is Lemma 7.1 of [17].

Lemma 36. *Let A be an algebra and B be an ideal in A. If U is a subalgebra of A which is minimal with respect to the property $A = B + U$, then $B \cap U \subseteq \phi(U)$.*

The next result is shown for Lie algebras as Lemma 2.3 of [18], but the proof is valid for Leibniz algebras also.

Lemma 37. *If A is an elementary Leibniz algebra and B is an ideal in A, then there exists a subalgebra C such that $A = B + C$ and $B \cap C = 0$.*

Theorem 38. *Let A be a solvable, elementary Leibniz algebra over a perfect field, K. Then A^2 is nilpotent.*

Proof. Let A be a minimal counterexample. Since $\phi(A) = 0$, $A = B + C$, where $B = \text{Nil}(A) = \text{Asoc}(A)$ and C is a Lie subalgebra. If C is nilpotent, then it is abelian, $A^2 \subseteq B$ and we are done.

Suppose that C is not nilpotent. Let M_1 be a maximal subalgebra of C containing C^2 and $M = B + M_1$. Then $A^2 \subseteq M$. Then M^2 is an ideal in A and is nilpotent by induction. Hence, $M^2 \subseteq \text{Nil}(A) = B$. Therefore, $M_1{}^2 \subseteq B \cap C = 0$, so M_1 is abelian. Hence, $A = B + (M_1 + D)$, where D is a one dimensional subalgebra with basis x.

We claim that B is the unique minimal ideal of A. Suppose that $B = B_1 + \cdots + B_t$, where each summand is a minimal ideal in A and $t > 1$. Let $S_j = B_j + C$ for each j. $S_j{}^2$ is nilpotent by induction. Since B_j is a minimal ideal in A, $S_j{}^2 = B_j + C^2$ or C^2. Hence $S_1{}^2 \cup \cdots \cup S_t{}^2$ is a Lie set whose span is A^2 and left multiplication by each s in the Lie set is nilpotent on A^2. Hence, A^2 is nilpotent by the theorem in [8], which is a contradiction. Thus, B is the unique minimal ideal in A.

We claim that $M_1 = \text{Nil}(C) = \text{Asoc}(C)$ is the unique minimal ideal in C. Suppose that $M_1 = B_1 + \cdots + B_t$, where each summand is a minimal ideal of C and $t > 1$. Let $C_j = B + B_j + D$. By induction, $C_j{}^2$ is nilpotent and $C_j{}^2 = B_jB + DB + DB_j$, since B is an abelian minimal ideal in A and B_j is an abelian minimal ideal in C. L_c is nilpotent on $C_j{}^2$ for each $c \in C_j{}^2$. Therefore, L_c is nilpotent on $C_j{}^2 + B$ since $L_c(C_j{}^2 + B) \subseteq C_j{}^2$. Since $B_iB_j = 0$, L_c is nilpotent on $A^2 = C_1{}^2 + \cdots C_t{}^2$ for all $c \in C_j{}^2$. Since $C_1{}^2 \cup \cdots \cup C_t{}^2$ is a Lie set, A^2 is

nilpotent by [8], a contradiction. Hence, $t = 1$ and $A = B + C = B + (M_1 + D)$ where B is a minimal ideal in A, M_1 is a minimal ideal in C, and D is a one dimensional subalgebra with basis x.

Let E be the algebraic closure of K. We will show that $E \otimes A^2$ is nilpotent, and hence, that A^2 is nilpotent. Since Nil $(C) =$ Asoc $(C) = M_1$, L_x acts completely reducibly on M_1. For each $m \in M_1$, L_m acts completely reducibly on B since $B \subseteq$ Asoc $(B + Km)$. Passing to $E \otimes A$, since K is perfect, the extension of these left multiplications are diagonalizable on $E \otimes M_1$ and $E \otimes B$ respectively. In the second case, the L_m commute, so the L_m are simultaneously diagonalizable on $E \otimes B$. We will show that $(E \otimes A)^2$ is nilpotent under these conditions, and hence, assume that K is algebraically closed. Hence, there is a basis of eigenvectors, $m_1, \ldots m_t$ in M_1 for L_x. Therefore, $xm_j = c_j m_j$ for some $c_j \in K$. If $c_j = 0$ for any j, then 0 is an eigenvalue of L_x on M_1. Since M_1 is irreducible over K, M_1 is one dimensional and C is two dimensional abelian, a contradiction to M_1 being the unique minimal ideal in C. Thus, $c_j \neq 0$ for all j. Also, B is the direct sum of root spaces $B_{\alpha_i} = \{b \in B | mb = \alpha_i(m)b \text{ for all } m \in M_1\}$.

Let $b_i \in B_{\alpha_i}$. Then $xb_i = b_1' + \cdots + b_n'$, where $b_j' \in B_{\alpha_j}$. Then

$$x(m_j b_i) = (xm_j)b_i + m_j(xb_i),$$

$$x\alpha_i(m_j)b_i = c_j m_j b_i + m_j \left(\sum_k b_k' \right),$$

$$\alpha_i(m_j) \left(\sum_k b_k' \right) = c_j \alpha_i(m_j)b_i + \sum_k \alpha_k(m_j)b_k',$$

and $c_j \alpha_i(m_j)b_i = \sum_k \left(\alpha_i(m_j) - \alpha_k(m_j) \right) b_k'.$

Since $c_j \neq 0$ for all j, it follows that $m_j b_i = \alpha_i(m_j)b_i = 0$ for all i and j. Therefore, $M_1 B = 0$. Since B is a minimal ideal in A, $BM_1 = 0$, and $M_1 \subseteq Z_A(\text{Asoc }(A)) =$ Asoc (A), a contradiction. Hence, no minimal counterexample exists and the result holds. $\qquad \square$

4.2 Classification of Elementary Leibniz Algebras

We first consider solvable Leibniz algebras. A construction of solvable elementary Lie algebras is given in [19]. Let A be a vector space and let B be an abelian Lie subalgebra of $gl(A)$ which acts completely reducibly on A. Forming the semidirect sum of A and B, one obtains an elementary, solvable, almost algebraic Lie algebra L. We can use this construction to obtain elementary, solvable, Leibniz algebras. Since $A = \text{Asoc}(L)$, A is the direct sum of minimal ideals of L. Let $A = A_1 + A_2$, where each summand is an ideal in L. Define the Leibniz algebra L^* to be the vector space $A + B$ with the same multiplication as in L except that $A_2 B = 0$. This algebra has $\phi(L^*) \subseteq (L^*)^2 \subseteq A$ and $\phi(L^*) \subseteq B$. Hence, $\phi(L^*) = 0$, and therefore, L^* is elementary. Following [19], such algebras will be called of type I.

Theorem 39. *Let A be a solvable Leibniz algebra over a perfect field K. The following are equivalent:*

1. *A is elementary.*

2. *$\phi(A) = 0$ and A^2 is nilpotent.*

3. *$\phi(A) = 0$ and A is metabelian.*

4. *Asoc (A) is complemented by an abelian subalgebra of A.*

5. *$A = B + E$, where B is abelian and E is elementary of type I.*

Proof. 1 implies 2: Theorem 38.

    2 implies 3: By Theorem 20 and Lemma 7.2 of [17], $A = \text{Nil}(A) + B$. Since A^2 is nilpotent, $A^2 \subseteq \text{Nil}(A)$. Hence, B is abelian.

    3 implies 4: Using Theorem 20 and Lemma 7.2 of [17], $A = \text{Nil}(A) + B$, and $A/\text{Nil}(A)$ is abelian.

    4 implies 5: Let Asoc $(A) = Z(A) + K$, where K is an ideal of A. Let $E = K + B$, where B is as in 4. Then $Z_A(K) \cap B \subseteq Z(E) \subseteq Z(A) \cap E = 0$. Hence, B is isomorphic to a subalgebra of $gl(K)$. Since K is contained in Asoc (A), K is completely reducible as a B-module. Hence, E is of type I.

    5 implies 1: As in Lie algebras, the direct sum of elementary Leibniz algebras is elementary by Lemma 29; hence, A is elementary.   □

    We now remove the condition of solvability and obtain a classification of all elementary Leibniz algebras over algebraically closed fields of characteristic 0.

Theorem 40. *Let A be a Leibniz algebra over an algebraically closed field K of characteristic 0. Then A is elementary if and only if either*

1. *A is the direct sum of copies of $sl_2(K)$, or*

2. *A has basis e_i, f_j, for $i = 1, \ldots m$ and $j = 1, \ldots n$, with $e_i f_j = \lambda_{ij} f_j$, either $e_i f_j = -f_j e_i$ or $f_j e_i = 0$, and all other products between basis elements equal to 0 or*

3. *A is the direct sum of an algebra from 1 and an algebra from 2.*

Proof. If A is semisimple, then A is a Lie algebra and it is of type 1, which follows from [18].

In general $A = B + (Z \oplus S)$, where $B = \text{Asoc}(A)$, Z is abelian, and S is as in 1. Suppose that S is not 0. We claim that $BS = SB = 0$. Since S is as in part 1, let T be one of the simple summands in S. Consider the subalgebra $C = B + T$ and let D be a minimal ideal of C contained in B. If $\dim D = p \geq 2$, then there exists $t \in T = sl_2(K)$ and $a_1, a_2, \ldots, a_p$, with $t a_i = a_{i+1}$ for $i = 1, \ldots p - 1$ and $t a_p = 0$. Either $Dt = 0$ or $td = -dt$ for all $d \in D$. In either case $\phi(E) = E^2 \neq 0$, where E is the subalgebra generated by t and the a_i. This contradiction forces $p = 1$ and $TD = 0 = DT$. Hence, $SB = 0 = BS$. Thus, Z acts completely reducibly on B. Again, since $DS = 0$ or $ds = -sd$ for all $d \in D$ and $s \in S$, Z acts completely reducibly on the left of B. Hence, L_z is semisimple for each $z \in Z$. Since Z is abelian, the left multiplications of $z \in Z$ are simultaneously diagonalizable on B, and hence, there are bases for Z and B such that the left multiplications hold as in 2 and then the right multiplications hold using Lemma 1.9 of [3].

The converse is clear. $\qquad\square$

We next obtain the result of the last theorem for Leibniz algebras over fields of characteristic $p > 3$.

First, we need the following lemma.

Lemma 41. *Let K be a perfect field. Let A be an elementary Leibniz algebra over K. Then $A = \text{Asoc}(A) + (B + S)$, where B is abelian, S is semisimple, and $BS + SB \subseteq B$.*

Proof. $A = \text{Asoc}(A) + C$ by Lemma 37. Let $B = \text{Rad}(C)$. By the same Lemma, $C = B + S$, where S is semisimple. Clearly $\text{Rad}(A) = \text{Asoc}(A) + B$. Since $\text{Rad}(A)$ is solvable and elementary, $(\text{Rad}(A))^2$ is nilpotent by Theorem 38, and hence, $B^2 \subseteq \text{Nil}(A) \cap B = \text{Asoc}(A) \cap B = 0$. $\qquad\square$

Theorem 42. *Let A be a Leibniz algebra over an algebraically closed field K of characteristic $p > 3$. Then A is elementary if and only if either*

1. *A is the direct sum of copies of $sl_2(K)$, or*

2. *A has basis e_i, f_j, for $i = 1, \ldots m$ and $j = 1, \ldots n$, with $e_i f_j = \lambda_{ij} f_j$, either $e_i f_j = -f_j e_i$ or $f_j e_i = 0$, and all other products between basis elements equal to 0, or*

   3. A is the direct sum of an algebra from 1 and an algebra from 2.

Proof. If A is solvable, then $A = \text{Asoc}(A) + C$, where C is abelian by Lemma 41. Let B be a minimal ideal of A. Then $B \subseteq \text{Nil}(B + Kc) = \text{Asoc}(B + Kc)$, and L_c acts completely reducibly on B for each $c \in C$. Hence, L_c acts diagonally on B. Since C is abelian, the L_c are simultaneously diagonalizable on B and A is as in 2, as follows from Lemma 1.9 of [3].

   If A is semisimple, then A is Lie and is as in 1 by Theorem 3.2 of [17].

   If A is neither solvable nor semisimple, then $A = B + (C + S)$, where $B = \text{Asoc}(A) \neq 0$, C is abelian, S is semisimple, and $CS + SC \subseteq C$ by the above lemma. Again S is as in 1. Let $D_i = B + S_i$, where S_i is a simple summand of S. Then $B = \text{Nil}(D_i) = \text{Asoc}(D_i)$, and B is a completely reducible D_i module. If V is an irreducible submodule of B, then we claim $\dim V = 1$. Since V is an irreducible left D_i Lie module, there exists an $e \in D_i$ such that V is a cyclic Ke module on which e acts nilpotently. Now $M = V + Ke$ is a nilpotent subalgebra since either $Ve = 0$ or $ve = -ev$ for all $v \in V$. Hence, M is abelian since it is elementary. Therefore, $eV = 0$ and $\dim V = 1$. Hence, $BS = SB = 0$.

   If $C = 0$, we are done. If not, then $C + S$ is elementary, and $\text{Nil}(C + S) = \text{Asoc}(C + S) = C$. Then as in the last paragraph, $CS = SC = 0$. Then $A = (B + C) + S$, where $B + C$ is as in 2, and so A is as in 3. □

Chapter 5

E-Leibniz Algebras

An algebra A is an E-algebra if $\phi(B) \subseteq \phi(A)$ for all subalgebras B of A. Throughout this section and the next, we will assume that the algebras are over fields of characteristic 0. Then the Frattini subalgebra is an ideal by Theorem 2.10 of [6].

5.1 Conditions for a Leibniz Algebra to be E

By Proposition 2 of [15], a Lie algebra L is an E-Lie algebra if and only if $L/\phi(L)$ is elementary. The proof carries over to Leibniz algebras, and we record it as:

Theorem 43. *Let A be a Leibniz algebra over a field of characteristic 0. Then A is an E-Leibniz algebra if and only if $A/\phi(A)$ is elementary.*

As in Lie algebras, a corollary restating Theorem 35 follows.

Corollary 44. *If A^2 is nilpotent, then A is an E-Leibniz algebra.*

Proof. By Theorem 3.5 of [6], $\phi(H) \subseteq \phi(A)$ for all subalgebras H of A when A^2 is nilpotent. Thus, $A/\phi(A)$ is elementary, and A is an E-Leibniz algebra. $\square$

5.2   Classification of E-Leibniz Algebras

We now provide a classification of E-Leibniz algebras over algebraically closed fields of characteristic zero.

Theorem 45. *Let A be a Leibniz algebra over K, an algebraically closed field of characteristic zero. Then A is an E-algebra if and only if*

1. *A is solvable, or*

2. *$A \cong sl_2(K) \oplus \ldots \oplus sl_2(K)$, or*

3. *$A = R \dotplus S$, where $S \cong sl_2(K) \oplus \ldots \oplus sl_2(K)$, R is a solvable ideal, and $RS + SR$ is contained in $\phi(L)$.*

Proof. Suppose A is solvable. Then A^2 is nilpotent, and from Corollary 44, A is an E-algebra.

Let $A = sl_2(K) \oplus \ldots \oplus sl_2(K)$. Then A is Lie, and A is elementary by Theorem 3.2 of [18]. Hence, A is an E-algebra.

Let $A = R \dotplus S$ where R is solvable, S is as in 2, and $RS + SR \subseteq \phi(A)$. S is elementary as above. $\phi(A)$ is a nilpotent ideal of A; hence, $\phi(A) \subseteq R$. In addition, $A/\phi(A) \cong R/\phi(A) \dotplus S$. Since $RS + SR \subseteq \phi(A)$, $A/\phi(A)$ is, in fact, the direct sum of $R/\phi(A)$ and S. Thus, by Theorem 4.8 of [17], $0 = \phi(A/\phi(A)) = \phi(S) \oplus \phi(R/\phi(A))$. $R/\phi(A)$ is elementary since $R/\phi(A)$ is an E-algebra and $\phi(R/\phi(A)) = 0$. Therefore, $A/\phi(A)$ is elementary by Lemma 29, and A is an E-algebra by Theorem 43.

Conversely, suppose A is an E-algebra. Then $A/\phi(A)$ is elementary from Theorem 43. $A/\phi(A)$ is the direct sum of its radical, R, and a semisimple ideal $S = sl_2(K) \oplus \ldots \oplus sl_2(K)$ from Theorem 4.1 of [6], either of which may be zero. If $S = 0$, then A is solvable as in 1. If $R = 0$, then A is the direct sum of copies of $sl_2(K)$ as in 2. If neither R nor S is 0, then $RS + SR \subseteq \phi(A)$, and since $A/\phi(A)$ is a direct sum, A is as in 3. $\square$

The following corollary addresses the special case in which A is a perfect Leibniz algebra.

Corollary 46. *Let A be a perfect Leibniz algebra ($A^2 = A$) over K, an algebraically closed field of characteristic zero. Then A is an E-algebra if and only if $A = sl_2(K) \oplus \ldots \oplus sl_2(K)$.*

Proof. Let $A = R \dotplus S$ be the Levi decomposition for Leibniz algebra A as in [4], where R is a solvable ideal of A and S is a semisimple subalgebra. $A^2 = R^2 + RS + SR + S^2$. Note that $R^2 + RS + SR \subseteq R$ since R is an ideal of A. In fact, we see that $R^2 + RS + SR = R$ since $A = A^2$. Since A is an E-algebra, $RS + SR \subseteq \phi(A)$. We claim that if $R^2 + \phi(A) = R$, then $R = 0$. If $R^2 + \phi(A) = R$, then $R^2 + \phi(A) + S = R + S = A$. This implies that $R^2 + S = A$

since no subalgebra can supplement $\phi(A)$ except A itself. Thus, $R^2 = R$, and hence, $R = 0$ since R is solvable. Therefore, $A = S$, and hence, A is Lie. Thus, by Corollary 4.5 of [18], $A = sl_2(K) \oplus \ldots \oplus sl_2(K)$. $\qquad\square$

Chapter 6

Minimal Nonelementary Leibniz Algebras

6.1 Definitions

A Leibniz algebra A is called *minimal nonelementary* if A itself is not elementary but every proper subalgebra H of A is. In [13], conditions for a Lie algebra, L, with L^2 nilpotent, to be minimal non-elementary are discussed in detail. In parts 1(b) and 3 of the following theorem, the conditions in [13] are generalized to the Leibniz case. We also obtain additional cases.

6.2 Classification of Minimal Non-Elementary Leibniz Algebras

Theorem 47. *Let A be a finite dimensional Leibniz algebra over an algebraically closed field. Suppose that A^2 is nilpotent. A is minimal non-elementary if and only if:*

1. *A is three dimensional non-nilpotent with basis x,y,z and non-zero multiplication as:*

    (a) *$xz = z$, $xy = y + z$, $zx = 0$ and $yx = 0$, with Leib(A) = $\langle y, z \rangle$, or*

    (b) *$xz = z$, $xy = y + z$, $zx = -z$, $yx = -y - z$, with Leib(A) = 0, or*

2. *A is Heisenberg, with Leib(A) = 0, or*

3. *A is generated by a, where $a^2 \neq 0$, A is nilpotent, and $\dim A \geq 2$, with Leib(A) = A^2, or*

4. *A is the four dimensional non-nilpotent algebra generated by a, b, x, and y with multiplication:*

$ax = x$, $ay = y$, $bx = x$, $by = y$, $xa = -x$, $ya = 0$, $xb = y - x$, and $yb = 0$, *with Leib(A)*
$= \langle y \rangle$.

Proof. Let A be minimal non-elementary. Since A^2 is nilpotent by supposition, A is solvable.
Suppose that A is not nilpotent. Then $\phi(A) \neq A^2$, and there exists a maximal subalgebra,
M, of A such that $A = A^2 + M$. Let B be an algebra of minimum dimension such that
$A = A^2 + B$. By Lemma 7.1 of [17], $A^2 \cap B \subseteq \phi(B) = 0$ since B is a subalgebra of A. A^2 is
nilpotent and elementary, and hence, it is abelian. Clearly, B is also abelian.

Suppose that $\dim B > 1$. For any $a \in B$, let $H(a) = A^2 + (a)$. Then $\phi(H(a)) = 0$ since all
proper subalgebras of A are elementary. Then $A^2 \subseteq \text{Nil } (H(a)) = \text{Asoc } (H(a))$. A^2 is abelian
since it is elementary and nilpotent, and it is completely reducible under the action of a.
On each minimal ideal, either $R_a = -L_a$ or $R_a = 0$ as in [3]. Hence, the minimal ideals are
one-dimensional eigenspaces for L_a and R_a acting on A^2, and L_a and R_a are simultaneously
diagonalizable on A^2. This holds for all a in B, and since the left multiplications, L_a, commute,
they are simultaneously diagonalizable.

Consider B acting on $W = A^2$ by left multiplication, and decompose A^2 as the direct sum
of weight modules, $W_\alpha = \{x \in W \mid ax = \alpha_a x \text{ for all } a \in B\}$. Each of these weight modules is
invariant under R_b, $b \in B$. If there is more than one weight module, then for each weight
α, let $B_\alpha = W_\alpha + B$. Since $\phi(B_\alpha) = 0$, $W_\alpha \subseteq \text{Asoc } (B_\alpha) = \text{Nil } (B_\alpha)$, and W_α is completely
reducible as a B-bimodule. Hence $W \subseteq \text{Asoc } (A)$. Since B complements W, Asoc (A) is
complemented and $\phi(A) = 0$, a contradiction.

Hence, there is one weight module, and each left multiplication is a scalar. Pick $a \in B$,
and suppose that $ax = \alpha x$ on W. Let $W_0 = \{x \in W \mid xa = 0\}$ and $W_1 = \{x \in W \mid xa = -\alpha x\}$.
Since W is the direct sum of one-dimensional a-invariant submodules, $W = W_0 + W_1$. If,
for each $a \in B$, $W = W_0$ or $W = W_1$, then each right multiplication is a scalar on A^2, and
$A^2 \subseteq \text{Asoc } (A)$. Again, Asoc (A) is complemented in A and $\phi(A) = 0$, a contradiction.

Thus, assume there is an $a \in B$ such that neither W_0 nor W_1 is 0. W_0 is a submodule. If
W_1 is a submodule, then by induction both components are completely reducible under B,
and $W \subseteq \text{Asoc } (A)$. Thus, Asoc (A) is complemented, and $\phi(A) = 0$, a contradiction.

Hence, assume there is an x in W_1 and a b in B such that $xb \notin W_1$. Let $N = (x, xb)$,
$bx = \beta x$, and set $y = \beta x + xb$. The following multiplications hold.

$$ax = \alpha x, \qquad ay = \alpha y, \qquad\qquad\qquad\qquad bx = \beta x, \qquad by = \beta y,$$
$$xa = -\alpha x, \quad ya = (\beta x + xb)a = \beta xa - b(xa) = 0, \quad xb = y - \beta x, \quad yb = (\beta x - b(xb) = 0.$$

Let $C = (a, b)$. Then N is a C-bimodule, and (y) is a submodule which is not comple-
mented in N, for suppose that $\sigma x + \tau y$ is C-invariant, where $\sigma \neq 0$. Then $(\sigma x + \tau y)a = -\sigma \alpha x$,
and hence, $\tau = 0$. Thus, $\sigma xb = \sigma y - \sigma \beta x$, and $\sigma = 0$ since xb is not in W_1. Therefore, no

complement exists. Hence, Asoc $(N + C)$ is not complemented in $N + C$, and $\phi(N + C) \neq 0$. Thus, $B = C$, $N = W = A^2$, dim $B = 2$, dim $A^2 = 2$, and the multiplication for B acting on A^2 is the one given in this paragraph.

Let $B = (a, b)$ and $A^2 = (x, y)$. From the forgoing,
$$ax = \alpha x, \quad ay = \alpha y, \quad bx = \beta x, \quad by = \beta y,$$
$$xa = -\alpha x, \quad ya = 0, \quad yb = 0, \quad xb = y - \beta x.$$

$N = (y)$ is a one-dimensional submodule of A, and A^2 is not completely reducible under the action of A. A simple change of basis allows us to take $\alpha = \beta = 1$, which yields the algebra in case 4.

Suppose that dim $B = 1$. Hence, $A = A^2 + B$, and A^2 is abelian. We now show that there exists a chain of ideals

$$0 \subseteq A_1 \subseteq \ldots \subseteq A_{n-1} \subseteq = A^2 \subseteq A_n = A,$$

where dim $A_i = i$. If $P \subseteq Q \subseteq A^2$ are ideals of A with Q/P irreducible under the action of A, then dim $Q/P = 1$ since the action of x on Q/P determines the action of A on Q/P, and either $R_x = -L_x$ or $R_x = 0$. Then any eigenvector of L_x in Q/P must span Q/P, and dim $Q/P = 1$. Hence, there exists a flag from 0 to L as claimed.

Now $M = A_{n-2} + B$ is a maximal subalgebra of A. Hence, $\phi(M) = 0$ by assumption. Then, $A_{n-2} \subseteq$ Asoc $(M) =$ Nil (M). Since A^2 is abelian and B is one dimensional, Asoc $(M) \subseteq$ Asoc (A). Thus, $A_{n-2} \subseteq$ Asoc (A). If $A_{n-2} \neq$ Asoc (A), then Asoc $(A) = A_{n-1} =$ Nil (A), and A splits over Nil $(A) =$ Asoc (A). Hence, $\phi(A) = 0$ from Theorem 3.1 of [6], and A is elementary. Otherwise, Asoc $(A) = A_{n-2}$, and Asoc (A) has co-dimension two in A. If Asoc (A) is not contained in $\phi(A)$, then A splits over Asoc (A) by Theorem 7.1 of [17]. Again $\phi(A) = 0$, and A is elementary. Thus, assume that Asoc $(A) = \phi(A)$. Since all minimal ideals are one-dimensional, they are eigenspaces for L_x, where B=$\langle x \rangle$. Note that $R_x = 0$ or $R_x = -L_x$ on each of these minimal ideals. Since $A^2/$Asoc (A) is one-dimensional, there exists a scalar α with $(L_x - \alpha I)^2 = 0$ and $L_x - \alpha I \neq 0$. Hence, there exist $y, z \in A^2$ with the following possible multiplications.

Case 1: Let

$$xz = cz, \quad xy = cy + z,$$
$$zx = 0, \quad yx = -dcy + ez,$$

where $d = 0$ or 1, and c is a non-zero scalar since A is not nilpotent. Using the Leibniz identity, $x(yx) = (xy)x + y(xx)$. Computations yield $-cd + ce = ce$. Hence, $d = 0$. From $y(xx) = (yx)x + x(yx)$, we see that $-cde - cd + ce = 0$. Hence, $e = 0$, and $yx = 0$. A simple change of basis allows us to take $c = 1$.

Case 2: Let

$$xz = cz, \qquad xy = cy + z,$$
$$zx = -cz, \quad yx = -dcy + ez,$$

where $d = 0$ or 1, and $c \neq 0$. Since $x(yx) = (xy)x + y(xx)$ by the Leibniz identity, we see that $-dc = -c$. Hence, $d = 1$. Similarly, $y(xx) = (yx)x + x(yx)$, and hence, $cde = -cd$. Hence, $e = -1$, and $yx = -cy - z$. Again, a change of basis allows us to choose $c = 1$.

Hence, $H = \langle x, y, z \rangle$ is such that $\phi(H) = \langle z \rangle$ and $A = H$, which is the algebra in case 1.

Now let A be nilpotent with all proper subalgebras elementary. If there exists an $a \in A$ with $a^2 \neq 0$, then the subalgebra B with basis $\{a, a^2, \ldots, a^n\}$ and $aa^n = 0$. In this case, $\phi(B) = B^2$. Since $a^2 \in \phi(B)$, $B = A$, which is the algebra in case 3.

If no such a exists, then A is Lie and hence, is Heisenberg by Theorem 4.7 of [11], yielding case 2.

Conversely, in cases 1, 2, and 4, all proper subalgebras are clearly elementary.

Suppose that A is as in case 3. Then $A = \langle a, a^2, \ldots, a^n \rangle$ with $aa^n = 0$. Since A is nilpotent, $A^2 = \phi(A)$. Now $b = \alpha_1 a + \alpha_2 a^2 + \ldots + \alpha_n a^n$ is in $\phi(A)$ if and only if $\alpha_1 = 0$. If the subalgebra B contains an element that is not in $\phi(A)$, then $B + \phi(A) = A$ since $A/\phi(A)$ is one-dimensional. Hence, $B = A$. Therefore, all proper subalgebras of A are contained in A^2, which is abelian, and hence, are elementary. Thus, A is minimal non-elementary. $\square$

We turn to the Leibniz algebra version of a Lie algebra result of Towers from [18].

Theorem 48. *Let A be a Leibniz algebra over K, an algebraically closed field of characteristic 0. A is minimal non-elementary if and only if:*

1. *A is three-dimensional non-nilpotent with basis $\{x, y, z\}$ and non-zero multiplication as:*

   (a) *$xz = z$, $xy = y + z$, $zx = 0$, and $yx = 0$, or*

   (b) *$xz = z$, $xy = y + z$, $zx = -z$, and $yx = -y - z$, or*

2. *A is Heisenberg, or*

3. *A is generated by a, where $a^2 \neq 0$, A is nilpotent, and $\dim A \geq 2$, or*

4. *A is the four-dimensional non-nilpotent algebra generated by a, b, x and y with multiplication as: $ax = x$, $ay = y$, $bx = x$, $by = y$, $xa = -x$, $ya = 0$, $xb = y - x$, and $yb = 0$.*

Proof. Suppose that A is minimal non-elementary. A is not elementary, so $\phi(A) \neq 0$. Thus, $A/\phi(A)$ can be viewed as a subalgebra of A, and hence, is elementary by assumption. Then A is an E-algebra by Corollary 44.

Suppose that A is not solvable. Then there exists a positive integer k such that $A^{(k)} = A^{(k+1)}$. If $k = 1$, then A is a perfect Leibniz algebra, and A is elementary by Corollary 46, a contradiction. If $k \geq 2$, then A^k is perfect, and $A^{(k)} \cong sl_2(K) \oplus \ldots \oplus sl_2(K)$. If R is the solvable radical of A, then $A = R \oplus A^{(k)}$. Since both summands are elementary, A is elementary by Lemma 29, a contradiction. Hence, A must be solvable. The result now follows from Theorem 47. $\qquad\square$

Chapter 7

The Jacobson Radical

7.1 Definitions

Jacobson introduced his radical in ring theory, and we consider an analogous concept in an algebra. For any algebra, L, the *Jacobson radical*, denoted $J(L)$, is defined to be the intersection of all maximal ideals of L. This concept was considered in [12] when L is a Lie algebra.

Barnes showed in [3] that if a Leibniz algebra, A, is nilpotent, that all maximal subalgebras are ideals. Then $J(A) = \phi(A)$.

Lemma 49. *For a Leibniz algebra, A, $J(A) \subseteq A^2$.*

Proof. Suppose x is not in A^2. Then there exists a complementary subspace, M, of x in A that contains A^2. Also, since $A^2 \subseteq M$, M is a maximal ideal of A, and x is not in M. Thus, if $x \in M$ for every maximal subalgebra M, then $x \in A^2$. Consequently, we see that $J(A) \subseteq A^2$. $\qquad\square$

7.2 Rad(A)

If L is a linear Lie algebra, let R denote the solvable radical of L, Rad(L), and let Rad(L^*) be the radical of the associative envelope, L^* of L. Then, by Corollary 2 on p. 45 of [10], $L \cap$ Rad(L^*) is precisely the set of all nilpotent elements of R, and $[R, L] \subseteq$ Rad (L^*).

Theorem 50. *Let A be a Leibniz algebra, and let $R = Rad\ (A)$ be the solvable radical of A. Then $AR + RA \subseteq N = Nil\ (A)$.*

Proof. Let $\mathscr{L}(A) = \{L_x \mid x \in N\}$. The map $\pi : L \to \mathscr{L}(A)$ is a homomorphism, and $\mathscr{L}(A)$ is a Lie algebra under commutation. Also, π maps R to $R(\mathscr{L}(A))$, the radical of $\mathscr{L}(A)$. By

the result in section 7.1, $[\mathscr{L}(A), R(\mathscr{L}(A))] \subseteq R(\mathscr{L}(A)^*)$. Hence, there exists an n such that $[L_{s_n}, L_{t_n}] \ldots [L_{s_2}, L_{t_2}][L_{s_1}, L_{t_1}] = 0$, where either $s_i \in A$ and $t_i \in R$, or $s_i \in R$ and $t_i \in A$. Then $L_{s_n t_n} \ldots L_{s_1 t_1} = 0$. Hence, $s_n t_n (\ldots (s_1 t_1(x))\ldots) = 0$ for all $x \in A$. Putting $x = s_0 t_0$, we have $s_n t_n (\ldots (s_1 t_1 (s_0 t_0))\ldots) = 0$. Therefore, $(AR + RA)^{n+1} = 0$. Since $AR + RA$ is an ideal in A, $AR + RA \subseteq N$.   □

Proposition 51. *If A is a Leibniz algebra over a field of characteristic 0, then $A^2 \cap R = AR + RA \subseteq N$.*

Proof. $A = R + S$ as in the Levi decomposition for Leibniz algebras in [4]. $R = \mathrm{Rad}\,(A)$, and S is a semisimple Lie subalgebra of A. Then $A^2 = S^2 + AR + RA$. Hence, $A^2 \cap R = S^2 \cap R + (AR + RA) \cap R = AR + RA$ since $S \cap R = 0$, and $AR + RA \subseteq R$.   □

Lemma 52. *If Leibniz algebra A is solvable, then $J(A) = A^2$.*

Proof. If M is a maximal ideal of A, then A/M is abelian. In fact, since A is solvable, A/M is of dimension one. Hence, $A^2 \subseteq M$ for all M, and $A^2 \subseteq J(A)$. Since $J(A)$ is always contained in A^2, we have that $A^2 = J(A)$.   □

Proposition 53. *Let A be a Leibniz algebra over a field of characteristic 0. Let $R = \mathrm{Rad}\,(A)$. Then $J(A) = AR + RA$.*

Proof. Let S be a Levi factor of A as in [4]. Then S is Lie, and $S = S_1 \oplus \ldots \oplus S_t$ where each S_i is simple. Let M_i be the sum of R and all of the S_j except S_i. Then M_i is a maximal ideal of A. Then $J(A) \subseteq \bigcap_i M_i = R$, and $J(A) \subseteq R \cap A^2$.

If M is a maximal ideal of A, then A/M is either abelian or simple. In the first case, $A^2 \subseteq M$, and in the second case, $R \subseteq M$. Clearly, the intersection of all maximal ideals M of the first type contains A^2. Similarly, the intersection of all maximal ideals M of the second type contains R. Therefore, $R \cap A^2 \subseteq J(A)$, and the result follows.   □

Corollary 54. *Let A be a Leibniz algebra over a field of characteristic 0. Then $J(A)$ is nilpotent.*

Proof. $J(A) = AR + RA \subseteq Nil(A)$ by Propositions 51 and 53.   □

Corollary 55. *$\phi(A) \subseteq J(A)$ when A is over a field of characteristic 0.*

Proof. $A = R + S$, as in the Levi decomposition in [4], and S is Lie. Hence, $\phi(S) = 0$. Thus, $\phi(A) \subseteq R$. Since $\phi(A) \subseteq A^2$, it follows that $\phi(A) \subseteq R \cap A^2 = AR + RA = J(A)$, using Propositions 51 and 53.   □

Proposition 56. *Let B be a nilpotent ideal in a Leibniz algebra, A. Then $J(B) \subseteq J(A)$.*

Proof. Since B is nilpotent, $J(B) = \phi(B) = B^2$, which is an ideal in A. Suppose that $x \in J(B)$, $x \notin J(A)$. Then let M be a maximal ideal of A such that x is not in M. Then $A = J(B) + M$, and $B = J(B) + (M \cap B)$. $M \cap B$ is a proper ideal of B that supplements $J(B)$, a contradiction, since no proper ideal of B supplements $J(B)$ in B. □

Chapter 8

Extending Frattini Properties

8.1 Generalized Frattini Property

In group theory, the intersection of all maximal subgroups of a group G is denoted $\mathrm{Frat}(G)$ and called the Frattini subgroup of G. From this context, Beidleman and Seo defined a generalized Frattini subgroup of a group G in [7]. H, a proper subgroup of G, is a generalized Frattini subgroup if whenever $G = HN_G(P)$ for any Sylow p-subgroup P of any normal subgroup K of G, then $G = N_G(P)$. Any generalized Frattini subgroup also satisfies the following property: If G is a finite group with A, B normal subgroups in G such that $B \subset \mathrm{Frat}(G)$ and A/B is nilpotent, then A is nilpotent.

As we do not have Sylow p-subgroups in Lie and Leibniz algebras, we utilize the Cartan subalgebra. A Cartan subalgebra C of a Leibniz algebra A is a subalgebra $C \subset A$ such that C is nilpotent and $N_A(C) = C$. This concept has been explored in the Lie algebra case in [14]. We generalize further to the Leibniz algebra case. Using the definition in [14], we first define the concept of a *generalized Frattini ideal* in a Leibniz algebra A as follows. A proper ideal H in A is a generalized Frattini ideal in A if for each ideal K in A and each Cartan subalgebra, C, of K, whenever $A = H + N_A(C)$, then $A = N_A(C)$.

Theorem 57. *Let H be a generalized Frattini ideal in Leibniz algebra A. Then the following are also true.*

1. *H is nilpotent.*

2. *Any ideal of A that is contained in H is also a generalized Frattini ideal in A.*

3. *$H + \phi(A)$ is a generalized Frattini ideal in A.*

4. *$H + Z(A)$ is a generalized Frattini ideal in A whenever $H + Z(A)$ is a proper subalgebra of A.*

Proof.   1. Let C be a Cartan subalgebra of H. Then $A = H + N_A(C)$ by Theorem 6.6 of Barnes in [3]. H is generalized Frattini in A, and hence, $A = N_A(C)$, which in turn implies that $H = N_H(C)$. Since C is a Cartan subalgebra of H, $N_H(C) = C$. Thus, $H = C$, and H is nilpotent.

2. Let N be an ideal of A such that $N \subseteq H$. Let K be an ideal of A, and let C be a Cartan subalgebra of K such that $A = N + N_A(C)$. Then $A = H + N_A(C)$, and hence, $A = N_A(C)$ since H is a generalized Frattini ideal in A. Thus, by definition, N is also generalized Frattini in A.

3. Since H and $\phi(L)$ are both ideals in A, $H + \phi(A)$ is also an ideal in A. Suppose K is an ideal in A with a Cartan subalgebra C of K such that $A = H + \phi(A) + N_A(C)$. Now suppose that M is a maximal subalgebra with $H + N_A(C) \subseteq M$. $\phi(A)$ is certainly contained in M since $\phi(A) \subseteq F(A) \subseteq M$. Then $H + \phi(A) + N_A(C) = M$, which is a contradiction to the earlier supposition that $H + \phi(A) + N_A(C) = A$. Therefore, no such maximal subalgebra M exists. Hence, $A = H + N_A(C)$, and in turn, $A = N_A(C)$ since H is generalized Frattini in A. Thus, by definition, $H + \phi(A)$ is generalized Frattini in A.

4. Since H and $Z(A)$ are both ideals in A, $H + Z(A)$ is also an ideal in A. Suppose K is an ideal in A, and let C be a Cartan subalgebra of K such that $A = H + Z(A) + N_A(C)$. $Z(A)$ is contained in every normalizer of a subset of A, so $Z(A)$ is certainly contained in $N_A(C)$. Thus, $A = H + N_A(C)$, and since H is generalized Frattini in A, $A = N_A(C)$. By definition, $H + Z(A)$ is also generalized Frattini in A.

$\square$

Corollary 58. *In Leibniz algebra A, $\phi(A)$ and $Z(A)$ are generalized Frattini ideals in A.*

Proof. Take $H = 0$ in (3) and (4) above. $\square$

Theorem 59. *Let H be generalized Frattini in Leibniz algebra A. If K is an ideal in A and K/H is nilpotent, then K is nilpotent.*

Proof. Let K be an ideal in A that contains H such that K/H is nilpotent. Let C be a Cartan subalgebra of K. Then $(C + H)/H$ is a Cartan subalgebra in K/H. Since K/H is nilpotent, K/H is contained in the normalizer of $(C + H)/H$ in K/H, which is $(C + H)/H$. Since $(C + H)/H$ is also contained in K/H, the two are equal. $(C + H)/H$ is Cartan, and hence is nilpotent. $\qquad\square$

Corollary 60. *A Leibniz algebra A is nilpotent if and only if A^2 is a generalized Frattini ideal in A.*

Proof. Suppose A^2 is generalized Frattini in A. A/A^2 is always nilpotent. Then by Theorem 59, A is nilpotent.

Suppose A is nilpotent. Then for any ideal H in A that contains A^2, whenever H/A^2 is nilpotent, then H is nilpotent. Hence, A^2 is generalized Frattini in A by definition. $\qquad\square$

Theorem 61. *Let H be a generalized Frattini ideal in Leibniz algebra A. If K is an ideal in A such that K^ω is contained in H, then $K^\omega = 0$.*

Proof. Suppose K is an ideal of L such that $K^\omega \subseteq H$. Then there exists a homomorphism $\Pi : K/K^\omega \to K/H$ which sends $k + K^\omega$ to $k + H$. Since K/K^ω is nilpotent, its homomorphic image, K/H is nilpotent. Thus, by Theorem 59, K is nilpotent, which is true if and only if $K^\omega = 0$. $\qquad\square$

Corollary 62. *A proper ideal K in Leibniz algebra A is nilpotent if and only if K^2 is a generalized Frattini ideal in A.*

Proof. Suppose K^2 is a generalized Frattini ideal in A. K/K^2 is always nilpotent; hence, by Theorem 59, K is nilpotent.

Suppose K is nilpotent. Then $K^2 = \phi(K) \subseteq \phi(A)$ [17]. Let J be an ideal in A such that J/K^2 is nilpotent. There exists a homomorphism $\pi : J/K^2 \to (J + \phi(A))/\phi(A)$. Thus, $(J + \phi(A))/\phi(A)$ is nilpotent, which in turn implies that J is nilpotent. Hence, K^2 is generalized Frattini in A by definition. $\qquad\square$

We now introduce an equivalent definition for an ideal being generalized Frattini in Leibniz algebra A via the following theorem.

Theorem 63. *Let H be an ideal in Leibniz algebra A. H is a generalized Frattini ideal in A if and only if for each ideal J in A that contains H, whenever J/H is nilpotent, then J is nilpotent.*

Proof. Let H be a generalized Frattini ideal in A. If J is an ideal in A with $H \subseteq J$ and J/H nilpotent, then J is nilpotent by Theorem 59.

Conversely, suppose that for each ideal J in A, we have the property that if J/H is nilpotent, then J is also nilpotent. Let K be an ideal in A, and let C be a Cartan subalgebra

of K such that $A = N_A(C) + H$. Then $(C + H)/H$ is Cartan in $(K + H)/H$. $(C + H)/H$ is an ideal in $(K + H)/H$. Now $N_{K/H}((C + H)/H) = (K + H)/H$, and $N_{K/H}((C + H)/H))$ is also $(C + H)/H$. Hence, $(C + H)/H = (K + H)/H$, and hence, $(K + H)/H$ is nilpotent. By Theorem 59, $K + H$ is nilpotent. Thus, K is nilpotent, and $C = K$. Hence, $N_A(C) = N_A(K) = A$ since K is an ideal in A. Therefore, H is generalized Frattini in A by definition. □

Corollary 64. *All proper ideals in a nilpotent Leibniz algebra are generalized Frattini.*

Example 65. Let L be the Leibniz (also Lie) algebra $\langle x, y, z \rangle$ with multiplications $zx = x$, $xz = -x$, $zy = y$, $yz = -y$, and all others 0. L is nilpotent. Then $H = \langle x \rangle$ and $K = \langle y \rangle$ are ideals that are generalized Frattini in L.

Note that the sum of two ideals with the generalized Frattini property in a Leibniz algebra, A, may not be generalized Frattini.

Example 66. In L defined as in the previous example, H and K are both generalized Frattini ideals in L. However, $H + K$, while an ideal, is not generalized Frattini in L.

Theorem 67. *Let H be a generalized Frattini ideal in Leibniz algebra A, and let K be an ideal in A that contains H. Then K/H is generalized Frattini in A/H if and only if K is generalized Frattini in A.*

Proof. Suppose K is a generalized Frattini ideal in A. Let J/H be an ideal in A/H such that $(J/H)/(K/H)$ is nilpotent. Then J/K is also nilpotent. Since K is generalized Frattini in A, J is nilpotent by Theorem 59, which implies that J/H is nilpotent. Thus, K/H is generalized Frattini in A/H by Theorem 63.

Conversely, suppose that K/H is generalized Frattini in A/H. Let J be an ideal in A containing K so that J/K is nilpotent. Then $(J/H)/(K/H)$ is nilpotent as well. Thus, J/H is nilpotent from Theorem 59. Since H is generalized Frattini in L and J/H is nilpotent, J is nilpotent by Theorem 59. Therefore, K is generalized Frattini in A by Theorem 63. □

Theorem 68. *If Nil (A) is a generalized Frattini ideal in A, then every solvable ideal of A is nilpotent.*

Proof. Suppose Nil (A) is generalized Frattini in A. Let H be a solvable ideal of A, and let k be the smallest integer such that $H^{(k+1)} = 0$. $H^{(k)}$ is abelian, and hence nilpotent. Thus, $H^{(k)} \subseteq$ Nil (A). Since Nil (A) is generalized Frattini in A, $H^{(k)}$ is generalized Frattini in A by 57. $H^{(k-1)}/H^{(k)}$ is abelian, and hence nilpotent. Thus, $H^{(k-1)}$ is nilpotent by Theorem 59, and hence, $H^{(k-1)} \subseteq$ Nil (A). Continuing in this manner, we can see that $H = H^{(1)} \subseteq$ Nil (A). Since H is an ideal in A and Nil (A) is generalized Frattini in A, H is also generalized Frattini in A by Theorem 57. $\square$

Corollary 69. *If Nil (A) is generalized Frattini in A, then A is not solvable.*

Proof. If A is solvable, then A is nilpotent by Theorem 68. Hence, $A =$ Nil (A). Then Nil (A) is not a proper ideal in A, and hence, is not generalized Frattini in A. $\square$

Example 70. Let L be the Leibniz (also Lie) algebra $\langle x, y, z \rangle$ with multiplications $zx = x$, $xz = -x$, $zy = y$, $yz = -y$, and all other multiplications 0. L is solvable, and Nil $(L) = \langle x, y \rangle$ is not generalized Frattini in L.

Theorem 71. *If H is a generalized Frattini ideal in A, then Nil $(A/H) =$ Nil $(A)/H$.*

Proof. Since H is generalized Frattini in A, H is nilpotent and $H \subseteq$ Nil (A). Then Nil $(A)/H$ is nilpotent, and hence, is contained in Nil (A/H).

Now suppose that B is an ideal in A such that $B/H =$ Nil (A/H). Then, by Theorem 59, B is nilpotent. Thus, B is contained in Nil (A), and hence, $B/H \subseteq$ Nil $(A)/H$. $\square$

Note that if H is an ideal that does not have the generalized Frattini property in A, it is possible for Nil (A/H) and Nil $(A)/H$ to be different.

Example 72. Let L be the three dimensional Lie algebra $\langle x, y, z \rangle$ with multiplication defined as

$$zx = x, xz = -x, zy = y, yz = -y \text{ and all others } 0.$$

Let H be $\langle x, y \rangle$, which is an ideal that is not generalized Frattini in L. Then Nil $(L/H) = z + H = L$, but Nil $(L)/H = 0$.

Theorem 73. *Let A be a non-nilpotent Leibniz algebra. Then Nil (A) is a generalized Frattini ideal in A if and only if Nil $(A) =$ Rad (A).*

Proof. Let N be an ideal in A containing Nil (A) such that $N/$Nil (A) is nilpotent. Then $N/$Nil (A) is solvable; hence, N is solvable. Then Nil $(A) \subseteq N \subseteq$ Rad $(A) =$ Nil (A), and $N =$ Nil (A). Thus, N is nilpotent. Therefore, Nil (A) is generalized Frattini in A by Theorem 63.

Conversely, suppose that Nil (A) is generalized Frattini in A. Then Rad (A) is nilpotent by Theorem 68; hence, Rad $(A) \subseteq$ Nil (A). Since Rad (A) is the maximal solvable subalgebra of A and Nil $(A) \neq A$, Rad $(A) =$ Nil (A). $\qquad \square$

Example 74. Let $L = gl(2, \mathbb{F})$. $Z(L) = \{\alpha I \mid \alpha \in \mathbb{F}\} =$ Nil $(L) =$ Rad (L). Hence, $Z(L)$ is generalized Frattini in L.

Consider $E_A(c) = \{x \in A \mid L_c^n(x) = 0 \text{ for some } n\}$. Although c is not contained in $E_A(c)$, by Theorem 5.1 of [3], we know there exists a $c' \in E_A(c)$ so that $E_A(c) = E_A(c')$. Thus, we delete the primes and suppose that $c \in E_A(c)$.

Theorem 75. *Let H be an ideal in Leibniz algebra A. Then H is a generalized Frattini ideal in A if and only if for each ideal K of A and each Cartan subalgebra C of K, whenever $A = H + E_A(c)$, it follows that $A = E_A(c)$, where $c \in C$ is the element such that $E_A(c)$ is minimal in the set of Engel subalgebras of A.*

Proof. Let H be a generalized Frattini ideal in A. Let K be an ideal in A, and let C be a Cartan subalgebra of K such that $A = H + E_A(c)$ for c the minimal Engel element in C. $(C + H)/H$ is a Cartan subalgebra in $(K + H)/H$ by Theorem 6.3 of [3]. C acts nilpotently on A/H, and hence on $(K + H)/H$. Thus, $(C + H)/H$ also acts nilpotently on $(K + H)/H$. Then $(C + H)/H = (K + H)/H$, so $(K + H)/H$ is nilpotent. Since H is generalized Frattini in A, $K + H$ is nilpotent by Theorem 63. Hence, K is nilpotent, and thus, $K = C$. Thus, $A = E_A(c)$.

Conversely, suppose H satisfies the conditions of the theorem. Let K be an ideal of A with $H \subseteq K$ such that K/H is nilpotent. Let C be a Cartan subalgebra of K, and let $c \in C$ be the element so that $E_A(c)$ is minimal in the set of Engel algebras of A. Then $A = K + E_A(c) = H + C + E_A(c)$. Now $C \subseteq E_A(c)$ since $C = N_K(C)$ is nilpotent. Thus, $A = H + E_A(c)$. Hence, $A = E_A(c)$ by supposition, and $K = K_K(c) \subseteq C$, so $K = C$. Therefore, K is nilpotent. Thus, H is generalized Frattini in A by Theorem 63. $\qquad \square$

Corollary 76. *The hypercenter $Z^*(A)$ is a generalized Frattini ideal in Leibniz algebra A.*

Proof. $Z^*(A) \subseteq E_A(c)$ for any $c \in C$, a nilpotent subalgebra of A. $\qquad \square$

8.2 $R(A)$ **and** $\tau(A)$

David Towers has explored two ideals that are closely related to the Frattini ideal in his 1980 work, [20]. For any nonassociative algebra L, Towers uses the symbol $\sigma(L)$ to denote the structure that we call $R(L)$, in the notation of [14]. $R(L)$ is defined to be the intersection of all maximal subalgebras of L which are also ideals of L. If no such maximal subalgebra exists, we define $R(L)$ to be equal to L. Clearly, $R(L)$ is an ideal of L. $T(L)$ is defined to be the intersection of all maximal subalgebras of L which are not ideals in L. If no such maximal subalgebras exist, we say $T(L) = L$. $T(L)$ is clearly a subalgebra, but is generally not an ideal. Let $\tau(L)$ be the largest ideal of L contained in $T(L)$.

The following lemmas are from this paper of Towers, [20]. The first two lemmas are true for any nonassociative algebra L, and in particular, for any Leibniz algebra.

Lemma 77. $F(L) = R(L) \cap T(L)$, *and* $\phi(L) = R(L) \cap \tau(L)$.

Lemma 78. $Z^*(L) \cap L^2 \subseteq \phi(L)$.

The next two lemmas from Towers [20] require that L be power solvable. An algebra A is called power solvable if all subalgebras of A that are generated by a single element are solvable. We first show that a Leibniz algebra of finite dimension is power solvable.

Lemma 79. *A Leibniz algebra A of finite dimension is power solvable.*

Proof. All finite dimensional Leibniz algebras generated by a single element, a, are cyclic of the form $A = \langle a, a^2, \ldots, a^n \rangle$, with $aa^j = a^{j+1}$ for $j < n$ and $aa^n = \alpha_1 a + \alpha_2 a^2 + \ldots \alpha_n a^n$, some linear combination of its basis elements, as in Chapter 2. As before, A^2 has basis $\{a^2, \ldots, a^n\}$. A^2 is abelian, and hence, is solvable. Then A is solvable. $\square$

Since Leibniz algebras are power solvable, the following two lemmas from Towers [20] hold for any Leibniz algebra A.

Lemma 80. $A^2 \subseteq R(A)$.

Lemma 81. *If* $\phi(A) = 0$, *then* $\tau(A) = Z(A) = Z^*(A)$.

We now return to ideals with the generalized Frattini property.

Theorem 82. *If* $\phi(A) = 0$, *then* $\tau(A)$ *is a generalized Frattini ideal in Leibniz algebra A.*

Proof. Consider $\tau(A)/\phi(A)$, which is equal to $Z(A)/\phi(A)$ by Lemma 81, and hence, is equal to $Z(A/\phi(A))$. $\phi(A)$ is generalized Frattini in A by Theorem 58. $Z(A/\phi(A))$ is also generalized Frattini in $A/\phi(A)$ by the same theorem. Hence, by Theorem 67, $\tau(A)$ is generalized Frattini in A. $\square$

Theorem 83. *Let A be a non-nilpotent Leibniz algebra with $\phi(A) = 0$. Then any ideal H that is generalized Frattini in A and is maximal with respect to the generalized Frattini property contains $\tau(A)$.*

Proof. Let H be as described. Then $H + \phi(A) = H$. $H + \phi(A)$ is a generalized Frattini ideal in A by Theorem 57. H is certainly contained in $H + \phi(A)$, and since H is maximal in A with respect to the generalized Frattini property, $H + \phi(A) = H$. Then $\phi(A) \subseteq H$.

Also, $(H + \tau(A)) / \phi(A) = H/\phi(A) + \tau(A)/\phi(A) = (H/\phi(A) + Z(A/\phi(A)))$, which has the generalized Frattini property in $A/\phi(A)$ by Theorem 57. Hence, $H + Z(A) = H + \tau(A)$. Now, $H + Z(A)$ is a generalized Frattini ideal in A by Theorem 57, and hence, $H + \tau(A)$ is also generalized Frattini in A. Since $H \subseteq H + \tau(A)$ and H is maximal with respect to the generalized Frattini property in A, $H = H + \tau(A)$, and thus, $\tau(A)$ is contained in H. $\qquad\square$

Theorem 84. *Let A be a non-nilpotent Leibniz algebra with $\phi(A) = 0$. Then any ideal H that is maximal with respect to the generalized Frattini property in A contains $Z^*(A)$.*

Proof. $(Z^*(A) + \phi(A)) / \phi(A)$ is contained in $Z^*(A/\phi(A))$, which is equal to $\tau(A)/\phi(A)$. Hence, $Z^*(A) \subseteq \tau(A)$, which is contained in H by Theorem 83. $\qquad\square$

8.3   The $\mathcal{M}(A)$ Property

In this section, we consider ideals that satisfy the property $\mathcal{M}(A)$. Given a finite-dimensional Leibniz algebra A, a proper ideal K of A is said to satisfy the property $\mathcal{M}(A)$ if and only if the following hold.

- $\phi(A/K) = 0$,

- A/K contains a unique minimal ideal, and

- $\dim(A/K) > 1$.

Example 85. Let A be as in our running example, Example 1.

Consider $K = (a^2 + a^3)$, an ideal in A.

First, observe the structure of A/K. In this quotient space, $a^3 = -a^2$.

$$A/K = \text{span} \left\{ a + K, a^2 + K, a^3 + K \right\}$$

$$= \text{span} \left\{ a + K, a^2 + K, -a^2 + K \right\}$$

$$= \text{span} \left\{ a + K, a^2 + K \right\}.$$

For brevity, we suppress the K and view A/K as having basis $\{a, a^2\}$ with multiplication defined as

$$aa^2 = a^3 = -a^2.$$

Now we verify that K is in $\mathcal{M}(A)$. Clearly, $\dim(A/K) = 2 > 1$.

Consider left multiplication by a in A/K. The matrix corresponding with this is

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

The characteristic polynomial is

$$p(x) = x^2 + x = x(x+1).$$

Since $p(x)$ is a product of distinct linear factors, $\phi(A/K) = 0$.

Next, we find the minimal ideals of A/K.

By Corollary 9, we know that since $n_1 = 1$, there is one minimal ideal. It is obtained by finding the nullspace of $p_2(L_a)$, which is span $\{a^2\}$.

Thus, A/K has a unique minimal ideal, and hence, K is in $\mathcal{M}$.

Note that $\phi(A) = 0 \subset K$.

Lemma 86. *Let K be a proper ideal of Leibniz algebra A such that K satisfies the property $\mathcal{M}(A)$. Then K contains $\phi(A)$ and A/K is non-nilpotent. In particular, A is non-nilpotent.*

Proof. $\phi(A/K) = 0$. Hence, $\phi(A)/K = 0$, which in turn implies that $\phi(A) \subseteq K$.

Let B/K be the unique minimal ideal of A/K. Suppose A/K is nilpotent. Then B/K is abelian by [16]. $A/K = \text{Nil}\,(A/K) = B/K$. Hence, A/K is abelian, which is true if and only if $\dim(A/K) = 1$. This is a contradiction to the requirement that A/K must have dimension greater than 1. Thus, A/K must be non-nilpotent. $\qquad\square$

Example 87. Let A and K be as in Example 85. Observe that A/K is non-nilpotent.

Recall that
$$A/K = \text{span}\,\{a, a^2\} \text{ with } aa^2 = -a^2.$$

Then
$$(A/K)^2 = \text{span}\,\{a^2\}.$$

$$(A/K)^3 = \text{span}\,\{a \cdot a^2\} = \text{span}\,\{a^2\} = (A/K)^2.$$

Thus, A/K is non-nilpotent.

We can also observe that A is non-nilpotent.

Theorem 88. *Let A be a solvable Leibniz algebra, and let K be an ideal of A which satisfies the property $\mathcal{M}(A)$. Then K is generalized Frattini in A if and only if K is a proper subalgebra of $Nil(A)$.*

Proof. Let B/K denote the unique minimal ideal of A/K. A is solvable, and thus, Nil $(A/K) = B/K$.

Suppose that K has the generalized Frattini property in A. By Theorem 71, $B/K = $ Nil $(A/K) = $ Nil$(A)/K$. Hence, Nil $(A) = B$, and K is a proper subalgebra of Nil (A).

Conversely, let K be a proper subalgebra of Nil (A). Then Nil $(A)/K = B/K = $ Nil (A/K). Let H be an ideal in A such that K is properly contained in H and H/K is nilpotent. Then $H/K \subseteq$ Nil $(A/K) = $ Nil $(A)/K$. Consequently, $H \subseteq$ Nil (A), and hence, H is nilpotent. Therefore, by Theorem 63, K has the generalized Frattini property in A. $\square$

Theorem 89. *Let A be a solvable Leibniz algebra, and let K be an ideal of A that has the property $\mathcal{M}(A)$. Let B/K be the unique minimal ideal of A/K. Then K has the generalized Frattini property in A if and only if $B = $ Nil (A).*

Proof. Let A be a solvable Leibniz algebra, and let K be an ideal of A that has the property $\mathcal{M}(A)$. Suppose that K also has the generalized Frattini property in A. Then Nil $(A/K) = $ Nil $(A)/K$ by Theorem 71. Since A is solvable, $B/K = $ Nil $(A/K) = $ Nil $(A)/K$. Hence, $B = $ Nil (A).

Conversely, suppose that $B = $ Nil (A). Since A is solvable, Nil $(A/K) = B/K$. Then we can say that Nil $(A) = $ Nil $(A/K) = B/K = $ Nil $(A)/K$. Then K is a proper subalgebra of Nil (A). Thus, K is a generalized Frattini ideal in A by Theorem 63. $\square$

Example 90. Let A be as in our running example, Example 1, and let K be as in 85. A is generated by one element, and hence, is solvable by Lemma 79.

Call $B = (a^2)$. Then $B/K = (a^2) + K$ is the unique minimal ideal in A/K. K is a generalized Frattini ideal in A. B is nilpotent, and hence, is contained in Nil (A).

Corollary 91. *Let A be a solvable Leibniz algebra, and let K have the property $\mathcal{M}(A)$. If K is a generalized Frattini ideal in A, then K is maximal with respect to the generalized Frattini property in A.*

Proof. Let H be a generalized Frattini ideal in A such that $K \subseteq H$. By Theorem 57, H is nilpotent, and hence, H is contained in Nil (A). Let B/K be the unique minimal ideal in A/K. By Theorem 89, $B = $ Nil (A). Hence, either $H = K$ or $H = $ Nil (A) since $K \subseteq H \subseteq$ Nil (A).

Suppose $H = \text{Nil}(A)$. Then by Theorem 68, every solvable ideal of A is nilpotent. In particular, A, being solvable, must be nilpotent. However, this contradicts Corollary 69, which says that if $\text{Nil}(A)$ is a generalized Frattini ideal in A that A is not solvable. Hence, $H = K$, which says that K is maximal with respect to the generalized Frattini property in A. $\square$

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