A Numerical Model for the Determination of Periodic Solutions of Pipes Subjected to Non-Conservative Loads

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INTRODUCTION

Nonlinear vibrations of pipes are of great interest in Nuclear Engineering Design. Non-conservative loads arise, for example, in the pipe-whip phenomenon and in others problems related to flow-induced vibrations. Concerning this last topic, the determination of periodic solutions can be an important aspect of the problem if the main objective is the characterization of limit cycle solutions. In (Galeão,1980) and in (Almeida and Galeão,1987) limit cycle approximations, corresponding respectively to the problem of pipes conveying fluid and to the problem of vortex shedding due to a fluid flowing around a cylinder, were numerically obtained using a step-by-step integration procedure. In (Nayfeh and Mook,1979) many others different methods are presented for the approximation of periodic motions.

In this work a variational approach based on Hamilton's Principle combined with a Fourier series expansion is derived in order to determine the frequency and the amplitudes of periodic solutions. This model generalizes a previous one presented in (Veloso and Loula,1985).

FORMULATION

According to the Principle of Virtual Work the dynamic response of nonlinear elastic systems can be characterized by the displacement field $\mathbf{u}(\mathbf{x},t)$ which satisfies

$$\int_\Omega \rho\ddot{\mathbf{u}}\cdot\hat{\mathbf{u}}d\Omega + \int_\Omega \sigma(\mathbf{u})\cdot\hat{\varepsilon}(\hat{\mathbf{u}})d\Omega = \widehat{W}, \quad \forall\hat{\mathbf{u}}\in V, \tag{1}$$

where: $\mathbf{x}\in\Omega$, $t\in[t_1,t_2]$, and V is the space of admissible variations. In the above equation $\ddot{\mathbf{u}}(\mathbf{x},t)$ is the acceleration field, $\widehat{W}$ is the virtual work of applied external loads, and σ the stress tensor related to the strain tensor ε by the constitutive law

$$\sigma = \mathbf{D}\varepsilon; \qquad \varepsilon = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}^T\nabla\mathbf{u}), \tag{2}$$

If we are interested in periodic solutions, a very convenient approach can be achieved using Hamilton's Principle instead of the P.V.W. In this case the desired solution is the displacement field $\mathbf{u}(\mathbf{x},t)$ which satisfies

$$-\int_0^{2\pi}\int_\Omega \rho\dot{\mathbf{u}}\cdot\hat{\dot{\mathbf{u}}}\,d\Omega\,d\theta + \int_0^{2\pi}\int_\Omega \sigma(\mathbf{u})\cdot\hat{\varepsilon}(\hat{\mathbf{u}})d\Omega\,d\theta = \int_0^{2\pi}\widehat{W}\,d\theta, \quad \forall\hat{u}\in V\times[0,2\pi/\omega]. \tag{3}$$

Equation (3) was obtained by integration of eq. (1) over one period, using the fact that

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t+2\pi/\omega); \qquad \theta = \omega t, \tag{4}$$

where ω is the frequency of the periodic solution.
Using a Fourier expansion of the form

$$\mathbf{u}(\mathbf{x},\theta) = \mathbf{U}_0(\mathbf{x}) + \sum_{j=1}^{N} [\mathbf{U}_j(\mathbf{x}) cos\ j\theta + \mathbf{U}_{N+j}(\mathbf{x}) sin\ j\theta], \tag{5}$$

equation (3) becomes:

$$\int_0^{2\pi}\int_\Omega \sigma\left(\mathbf{U}_0 + \sum_{j=1}^{N}\mathbf{U}_j f_j\right)\cdot\hat{\varepsilon} d\Omega d\theta - \pi\omega^2\int_\Omega \rho\sum_{j=1}^{2N}\alpha_j^2 \mathbf{U}_j\cdot\hat{\mathbf{U}}_j\ d\Omega = \int_0^{2\pi}\widehat{W}\ d\theta \tag{6}$$

with

$$\alpha_j = \begin{cases} 1, & \text{when } 1 \le j \le N \\ j-N, & \text{when } N+1 \le j \le 2N \end{cases}; \quad f_j(\theta) = \begin{cases} cos\ j\theta & \text{for } 1 \le j \le N \\ sin\ j\theta & \text{for } N+1 \le j \le 2N. \end{cases} \tag{7}$$

In order to characterize the periodic solution, i.e., the set $\{w, \mathbf{U}_j : j = 0,\ldots,2N\}$, it is necessary to introduce an adittional relation connecting this solution with a parameter associated with the energy level of the system. (This can be done, for example, choosing a given value for the kinetic energy). To solve this nonlinear problem an incremental procedure will be used. Let $\{^n\omega, ^n\mathbf{U}_j : j = 0,\ldots,2N\}$, corresponding to a energy level ^nK, be known. Then, using the incremental relations:

$$^{n+1}\mathbf{U}_j = {}^n\mathbf{U}_j + \Delta\mathbf{U}_j; \qquad j = 0,\ldots,2N$$

$$^{n+1}\omega = {}^n\omega + \Delta\omega; \qquad ^{n+1}\varepsilon = {}^n\varepsilon + \Delta\varepsilon; \qquad ^{n+1}\sigma = {}^n\sigma + \Delta\sigma, \tag{8a-d}$$

and disregarding quadratic terms in the increments which appears in equation (6), we arrive at:

$$\int_0^{2\pi}\int_\Omega \mathbf{D}\Delta\varepsilon_l\cdot\Delta\hat{\varepsilon}_{l,0}\ d\Omega\ d\theta + \int_0^{2\pi}\int_\Omega {}^n\sigma\cdot\hat{\varepsilon}_{nl,0}\ d\Omega d\theta +$$

$$\int_0^{2\pi}\int_\Omega {}^n\sigma\cdot\Delta\hat{\varepsilon}_{l,0}\ d\Omega\ d\theta = \int_0^{2\pi} {}^{n+1}\widehat{W}_0 d\theta$$

$$-\pi\ {}^n\omega^2\alpha_j^2\int_\Omega \rho\Delta\mathbf{U}_j\cdot\Delta\hat{\mathbf{U}}_j d\Omega + \int_0^{2\pi}\int_\Omega \mathbf{D}\Delta\varepsilon_l\cdot\Delta\hat{\varepsilon}_{l,j} d\Omega d\theta$$

$$+\int_0^{2\pi}\int_\Omega {}^n\sigma\cdot\Delta\hat{\varepsilon}_{nl,j} d\Omega d\theta - 2\pi\ {}^n\omega\alpha_j^2\int_\Omega \rho\ {}^n\mathbf{U}_j\cdot\Delta\hat{\mathbf{U}}_j d\Omega\ \ \Delta w =$$

$$= \int_0^{2\pi} {}^{n+1}\widehat{W}_j d\theta - \pi\ {}^n\omega^2\alpha_j^2\int_\Omega \rho\ {}^n\mathbf{U}_j\cdot\Delta\hat{\mathbf{U}}_j d\Omega + \int_0^{2\pi}\int_\Omega {}^n\sigma\cdot\Delta\hat{\varepsilon}_l, j d\Omega d\theta \quad j = 1,\ldots,2N$$

where:

$$\Delta\varepsilon_l = \sum_{i=0}^{2N}\{\frac{1}{2}[\nabla(\Delta\mathbf{U}_i) + \nabla(\Delta\mathbf{U}_i)^T]f_i + \sum_{k=0}^{2N}\frac{1}{2}[(\nabla\ {}^n\mathbf{U}_k)^T\nabla(\Delta\mathbf{U}_i) + \nabla(\Delta\mathbf{U}_i)^T\nabla\ {}^n\mathbf{U}_k]f_i f_k\}$$

$$\Delta\varepsilon_{nl} = \sum_{i=0}^{2N}\sum_{k=0}^{2N}\{\frac{1}{2}\nabla(\Delta \mathrm{U}_i)^T\nabla(\Delta \mathrm{U}_k)\}f_i f_k. \tag{9a - d}$$

This set of $(2N+1)$ equations when coupled with the incremental relation

$$^{n+1}K = {}^{n}K + \Delta K, \tag{10}$$

completely defines the new solution $\{ {}^{n+1}\omega;\ {}^{n+1}U_j : j = 0, \ldots, 2N\}$.

NONLINEAR VIBRATION OF BEAMS

The general model previously derived will be applied to the problem of nonlinear vibration of piping systems. To this end we are going to assume that the motion of the pipe occurs in a plane, and use a beam model to represent its behavior. We also assume that shear deformations can be neglected. In this situation the only nonzero component of the strain tensor is given by:

$$\epsilon(x,z,t) = u' + 0.5u'^2 + 0.5w'^2 - z\phi[(1+u')cos\phi + w' sin\phi + 0.5z^2\phi'^2] \tag{11}$$

where $()' = \partial()/\partial x$, while $u(x,0,t)$ and $w(x,0,t)$ represent the component of the displacement of the axis of the pipe. The angle ϕ gives the rotation of the cross sectional area. According to the above assumptions

$$tan\ \phi = w'/(1+u'). \tag{12}$$

From these definitions we obtain

$$\Delta\epsilon_l = (1+\ {}^{n}u')\Delta u' +^{n} w'\Delta w' - z\{\ {}^{n}\phi' cos\ {}^{n}\phi\Delta u' +^{n} \phi' sin\ {}^{n}\phi\Delta w' +$$

$$+\ {}^{n}\phi'[\ {}^{n}w' cos\ {}^{n}\phi - (1+\ {}^{n}u')sin\ {}^{n}\phi]\Delta\phi + [(1+\ {}^{n}u')cos\ {}^{n}\phi + w' sin\ {}^{n}\phi]\Delta\phi'\} + z^2\ {}^{n}\phi'\Delta\phi$$

$$\Delta\epsilon_{nl} = \frac{1}{2}\Delta u'^2 + \frac{1}{2}\Delta w'^2 - z\{\ {}^{n}\phi' cos\ {}^{n}\phi\Delta w'\Delta\phi -\ {}^{n}\phi' sin\ {}^{n}\phi\Delta u'\Delta\phi\} +$$

$$+[{}^{n}w' cos\ {}^{n}\phi - (1+\ {}^{n}u')sin\ {}^{n}\phi]\Delta\phi\Delta\phi' + cos\ {}^{n}\phi\Delta\phi'\Delta u' + sin\ {}^{n}\phi\Delta w'\Delta\phi'\}z^2\Delta\phi'^2/2.$$

with:

$$\Delta\phi \approx (\partial\ {}^{n}\phi/\partial u')\Delta u' + (\partial\ {}^{n}\phi/\partial w')\Delta w'; \quad cos\Delta\phi \approx 1; \quad sin\Delta\phi \approx \Delta\phi. \tag{13a - c}$$

Now, if we assume that the material is isotropic and follows Hooke's law, the constitutive equation is expressed by

$$\sigma = E\epsilon, \tag{14}$$

where E is the Young's modulus.

As long as equation (5) has been used, we obtain the desired variational form in the amplitudes U_j and W_j by substituting the above expressions in eq.(9), (eliminating ϕ according to eq.(12)), and performing the integrations in the the cross sectional area.

APPROXIMATE SOLUTIONS

Following the approach described in the preceding section it is quite clear that second derivatives U_j'' and W_j'' will appear in this variational form. For this reason the finite element that will be used in this work was constructed utilizing Hermite's cubic interpolation for both axial and transversal displacements, leading to a beam element with four degrees of freedom per node. The discretized version of equation (9) gives a system of algebraic equations which can be written as:

$$\sum_{k=0}^{2N}[\,{}^{n}K_{0k}]\{\Delta D_k\} = \{\,{}^{n+1}F_0\} + \{\,{}^{n}R_0\}$$

$$-\pi\,{}^{n}\omega^2\alpha_j^2[M]\{\Delta D_j\} + \sum_{k=0}^{2N}[\,{}^{n}K_{jk}]\{\Delta D_k\} - 2\pi\,{}^{n}\omega\alpha_j^2\{\,{}^{n}P_j\}\Delta\omega =$$

$$\{\,{}^{n+1}F_j\} + \pi\,{}^{n}\omega^2\alpha_j^2\{\,{}^{n}P_j\} + \{\,{}^{n}R_j\}; \quad j = 1,\dots,2N \qquad (15a-b)$$

where : $[M]$ is the conventional mass matrix, and $[\,{}^{n}K_{jk}]$ $(j = 0,\dots,2N)$, which come from the discretization of the second and third terms in (9.b), can be seen as tangent stiffness matrices . The vector $\{\,{}^{n+1}F_j\}$ is obtained from the discretization of the term associated with the virtual work of applied external loads, and $\{\,{}^{n}R_j\}$ and $\{\,{}^{n}P_j\}$ come from the remaining terms that depend on quantities known at the preceding step. Obviously $\{\Delta D_j\}$ $(j = 1,\dots,2N)$ is the vector of nodal amplitude increments for harmonic j.

NUMERICAL RESULTS

The model developed above was applied to the determination of periodic solutions of a cantilever beam of lenght L, subjected to non-conservative follower forces acting at the free end. The data used were: $L = 1000mm$; $\rho = 7.85 \times 10^{-9} N/mm^3$; $E = 2.1 \times 10^5 N/mm^2$; $diameter = 20mm$; $thickness = 1.5mm$.

Two distinct situations were considered: (i) for the first one the force remains normal to the pipe axis and has a variable magnitude $F = F_0 w(L), (F_0 = 7)$, where $w(L)$ is the transversal displacement of the free end. (ii) for the second, a tangential follower force of constant intensity $F_c/4$ was used (F_c denotes the critical flutter load).

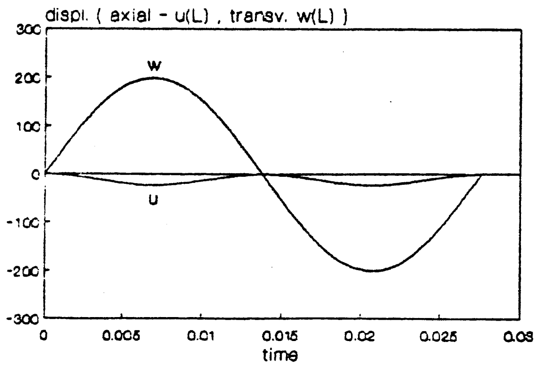
In both cases eq.(5) reduces to:

$$u(x,t) = \sum_{j=1}^{N} U_j(cos2j\omega t - 1); \qquad w(x,t) = \sum_{j=1}^{N} W_j sin(2j-1)\omega t, \qquad (16)$$

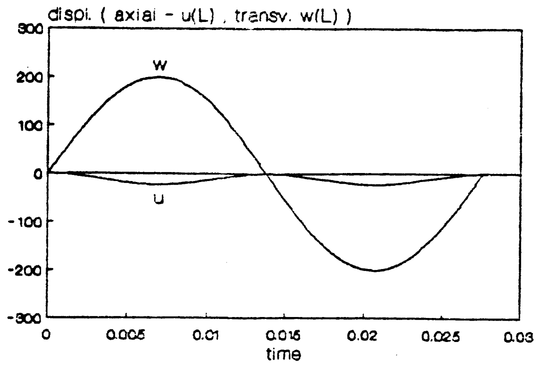
and the desired periodic solution can be parameterized by $w(L)$ choosing a given value for W_1, (i.e. fixing a certain level of energy for the system).

In figures 1 and 2 it is shown the oscillatory motion of the free end of the pipe, just over one period. Three different values of W_1 was taken in order to analyse the influence of the nonlinear terms. Continuous line curves refer to the numerical results obtained with the method proposed here, using one term (N=1) and two terms (N=2) in eq.(16), to approximate the solution time-dependence. Dashed line curves represent time-history responses obtained utilizing the total lagrangian P.V.W. plus Newmark's algorithm with a time-step $\Delta t = 2\pi/100\omega$, where ω is the frequency determined from the periodic approximations (the obtained values are listed in the figures). For both methods the same discretized model, consisting of five equal beam elements, was used.

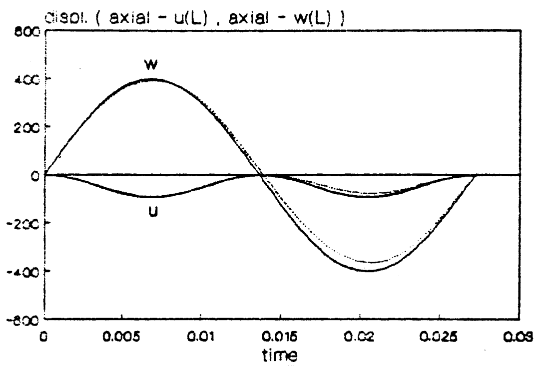
From these figures we see that for N=2 the two solutions "coincide", but for N=1 this only happens for small displacements. This shows that the one-term approximation compares fairly well with the predicted periodic solution, (dashed line), when small to moderate displacement amplitudes occur. For large amplitudes, at least a two-term approximation should be used.



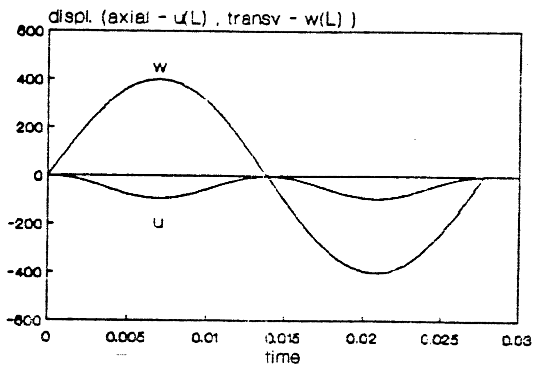
N = 1 ; $W_1 = 200mm$; $\omega = 227.67Hz$



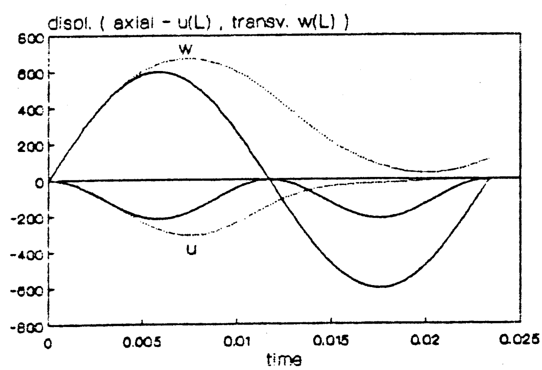
N = 2 ; $W_1 = 200mm$; $\omega = 227.59Hz$



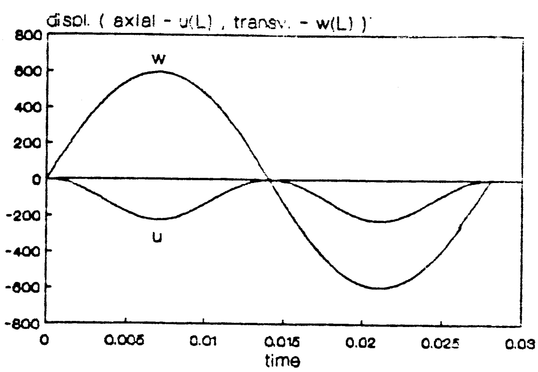
N = 1 ; $W_1 = 400mm$; $\omega = 230.69Hz$



N = 2 ; $W_1 = 400mm$; $\omega = 226.34Hz$

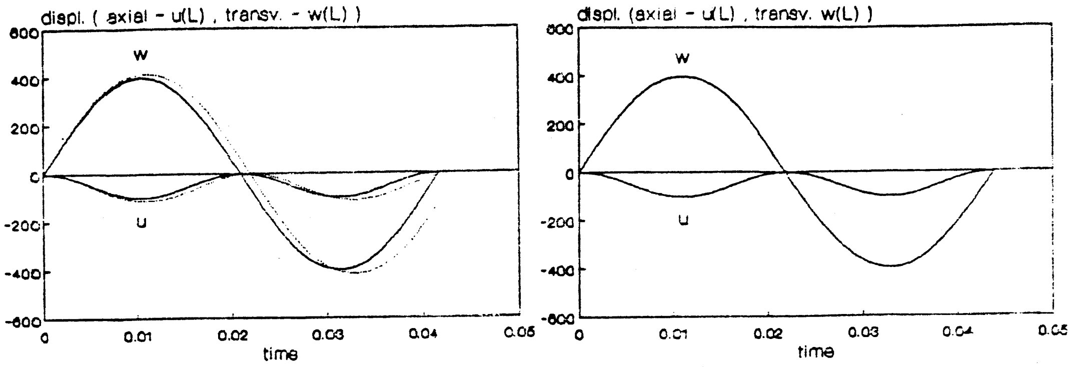


N = 1 ; $W_1 = 600mm$; $\omega = 268.61Hz$

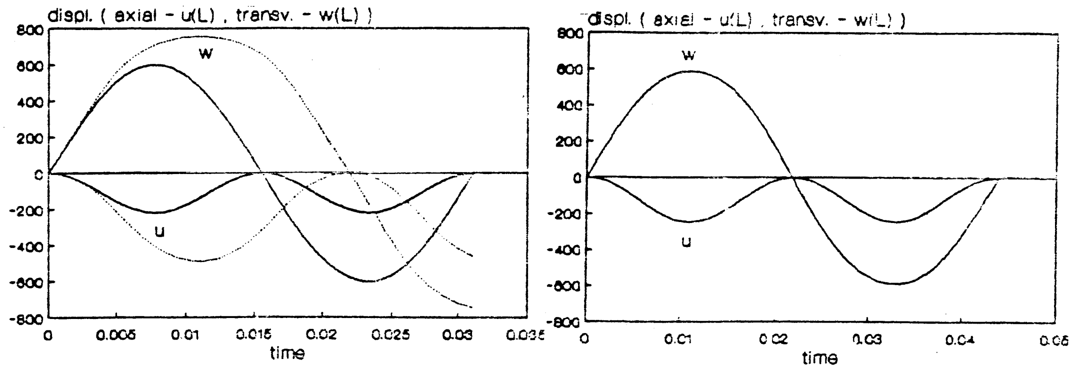


N = 2 ; $W_1 = 600mm$; $\omega = 223.34Hz$

Fig.1. Cantilever beam subjected to a normal follower force. (ω = calculated frequency)



N = 1 ; $W_1 = 400mm$; $\omega = 150.64Hz$ N = 2 ; $W_1 = 400mm$; $\omega = 143.61Hz$



N = 1 ; $W_1 = 600mm$; $\omega = 202.25Hz$ N = 2 ; $W_1 = 600mm$; $\omega = 143.59Hz$

Fig.2. Cantilever beam subjected to a tangential follower force. (ω = calculated frequency)

CONCLUSIONS

The P.V.W. combined with a step-by-step integration scheme could be used for the determination of general transient responses while the numerical model presented in this paper deals only with periodic solutions. However, from the computing time point of view, this last method is normally more advantageous than the first one. This is particularly true when the periodic solution is sufficiently regular to be well approximated by a few terms of the Fourier series expansion.

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