A Simplified Inelastic Seismic Analysis of the Maximum and Limit States of Structure

S. H. Lee, J. Zarka
Ecole Polytechnique, Palaiseau, France

ABSTRACT

A new direct method to evaluate the maximum and limit states of inelastic structures under dynamic loading, is presented. As in our previous method (Zarka et al., 1985), it is based on the full analytical solutions of a single-degree-of-freedom (SDOF) system which represents a kinematically hardening material and which is subjected to an instantaneous constant force (step loading). We propose rules to obtain the inelastic displacements, plastic strains, and residual stresses fields with only ***two or three static elastic analyses****. Full details of the method could be found in Lee (1989).*

INTRODUCTION

During the last three decades, with the development of computers, many powerful methods of dynamic inealstic analysis such as direct integration methods and modal superposition methods were developed. Nevertheless, this type of analysis is very expensive and generally very complicate to use. The methods used in the domain of the dynamic plasticity adopt the step-by-step incremental procedure, which often leads to many difficult problems such as instability of algorithms, slow convergence of equilibrium iterations, inaccuracy of results in cases of severe plasticity and at last very long and expensive calculations.

Therefore, it was necessary to develop a simplified method having the following characteristics :

1) inexpensive direct analysis instead of the step-by-step incremental analysis,
2) fast convergence, unconditional stability, practically acceptable accuracy, and easy usage.

In the spirit of above, Zarka *et al.* (1985) proposed a new approach to the dynamic analysis by expanding the simplified methods during quasi-static cyclic loadings (1979, 1989). This new method gives a good approximation of limit states in the case of very large loadings. We have presented an extension of the direct method to evaluate the maximum and limit states, to more general cases of dynamic loadings (see for more details Lee, 1989).

The new direct method follows the basic idea that the maximum response to arbitrary dynamic loadings is qualitatively same as that to instantaneous constant forces and the complete analytical solutions for a SDOF system subjected to a constant force enables us to reduce our dynamic problem to a static elastoplastic problem which can be resolved by the simplified standard method. With a particular dynamic load filtering scheme, the general dynamic loads or the ground accelerations can be transformed into an equivalent constant force which gives the same maximum response as that to the general dynamic loads under consideration.

In this special paper, we focus on the new algorithm for evaluating the maximum and limit states of the structure subjected to instantaneous constant forces.

ANALYTICAL DETERMINATION OF THE MAXIMUM AND LIMIT STATES OF A SDOF Elastoplastic system during an instantaneous constant force

Equations of Motion

A simple elastoplastic system with kinematically hardening material is shown in Fig. 1. It is composed of one mass, one elastic spring, one inelastic internal spring, and one friction block. The governing equation of motion is written as

$$m\ddot{x} + k\,(x - \alpha) = f(t) = F \tag{1}$$

with the initial conditions : $x(0) = x_0,\ \dot{x}(0) = \dot{x}_0\ ,\ \alpha(0) = \alpha_0,$

where m denotes the mass, k the stiffness of elastic spring, $\ddot{x}(t)$, $\dot{x}(t)$, $x(t)$, and $\alpha(t)$ are, respectively, the acceleration, the velocity, the displacement, and the inelastic displacement at time t, $f(t)$ is the time dependent force.

The equation of evolution is given in the symbolic form (Moreau, 1971) as follows :

$$\dot{\alpha} \in \partial\Psi_{C_0}(\sigma) \tag{2}$$

i.e.

$$\sigma = k\,(x - \alpha) - h\alpha \in C_0 = [\ -S\ ,\ S\ ] \tag{3}$$

and

$$\begin{cases} \dot{\alpha} \equiv 0 & \text{if } |\,\sigma\,| < S \\ \dot{\alpha} \geq 0 & \text{if } \sigma = S \\ \dot{\alpha} \leq 0 & \text{if } \sigma = -S \end{cases} \tag{4}$$

where C_0 is the fixed convex set (here a segment) which defines the elastic domain of the model, $\dot{\alpha}(t)$ is the velocity of inelastic displacement, S the threshold of plasticity, $\sigma(t)$ the stress on the friction block, and h the stiffness of the internal spring or the strain hardening coefficient.

Determination of the Maximum State

As shown by Newmark *et al.* (1971), the trajectories in the phase plane $(x, \dot{x}/\omega)$ are deduced from the first integral and are drawn in Fig. 2. In the phases of the elastic loadings or unloadings, they are half-circles centered in $(F/k + \alpha_0, 0)$ and with the radius :

$$r^{el} = \sqrt{(\dot{x}_0/\omega)^2 + [x_0 - (F/k + \alpha_0)]^2} \tag{5}$$

where ω is the angular frequency of the elastic system defined by $\sqrt{k/m}$.

During the plastic loadings and reloadings, the trajectories in the phase plane $(x, \dot{x}/\omega_t)$ represent also half-circles centered in $(\hat{F}/k_t, 0)$ and with the radius :

$$r = \sqrt{[\dot{x}(t_y)/\omega_t]^2 + [x(t_y) - \hat{F}/k_t]^2} \tag{6}$$

with

$$\hat{F} = F - kS/(k + h) \ \text{ if } \dot{x} \geq 0\ ;\ \hat{F} = F + kS/(k + h) \ \text{ if } \dot{x} \leq 0 \tag{7}$$

where $k_t = kh/(k + h)$ is the tangential stiffness, ω_t the angular frequency of the tangential system given by $\sqrt{k_t/m}$, and $\dot{x}(t_y)$ and $x(t_y)$ are, respectively, the velocity and displacement at the time when the system begins to be plastified.

The first radius of the elastoplastic system can be obtained as follows :

$$r_0 = \sqrt{[k/k_t]\,[(\dot{x}_0/\omega)^2 + [x_0 - (F/k + \alpha_0)]^2 - (h\alpha_0/k - (F - \epsilon S)/k)^2] + [k\alpha_0/k_t - (F - \epsilon S)/k_t]^2} \tag{8}$$

with

$$\epsilon = \dot{x}_0/|\dot{x}_0| \ \text{ if } \dot{x}_0 \neq 0\ ,\ \ \epsilon = F/|F| \ \text{ if } \dot{x}_0 = 0 \text{ and } F \neq 0\ ,\ \ \epsilon = -\sigma_0/|\sigma_0| \ \text{ if } \dot{x}_0 = F = 0 \tag{9}$$

and

$$\sigma_0 = kx_0 - (k + h)\,\alpha_0\ . \tag{10}$$

The maximum state can be determined graphically or analytically according to the initial conditions and the applied force. For simplicity, in this paper, we shall treat only the case $F\,\dot{x}_0 \geq 0$, where the maximum state is reached at the first extremum of the response and the maximum total displacement and inelastic displacements are given by :

$$x_{max} = \frac{F}{k_t} - \frac{\epsilon S}{h} + \epsilon r_0 \tag{11}$$

$$\alpha_{max} = \frac{F - \epsilon S}{h} + \epsilon\,\frac{k}{k+h}\,r_0\ . \tag{12}$$

It is worth noting that *for zero initial velocity, the maximum state is independent of the mass. The maximum displacements depend of the ratios of k/h and F/S. For all k/h, if the force is very large (F >> S), the inelastic displacement is effectively equal to the double of the static reponse, i.e. 2(F - ϵS)/h, which is also the maximum displacement of the SDOF rigid-platic system. The maximum total displacement and the maximum stress are equal to the double of the static elastic response for the elastic case* (Lee, 1989).

Determination of the Limit State

After some elastoplastic cycles, the system vibrates purely elastically, it reaches a limiting state where it shakesdown. The limit inelastic displacement can be considered as the last extreme displacement of the elastoplastic vibration. It is determined after calculating successively all extreme values of the elastoplastic response. In the phase plane, the radii of the first integrals reduce progressively each time the sign of the velocity changes. The system reaches its limiting state if the radius of the first integral of the elastoplastic system becomes inferior to a certain value, which is calculated from the criteria of plasticity, i.e.,

$$r_{n-1} \leq S/k_t \tag{13}$$

where n denotes nth extremum where the structure shakesdown.

The limiting total displacement and inelastic displacement are given in the quasi-analytical form

$$x_{lim} = \frac{F}{k_t} - (-1)^{n+1} \frac{\epsilon S}{h} + (-1)^{n+1} \epsilon r_{n-1} \tag{14}$$

$$\alpha_{lim} = \frac{F - (-1)^{n+1} \epsilon S}{h} + (-1)^{n+1} \epsilon \frac{k}{k+h} r_{n-1} \tag{15}$$

with

$$r_n = \sqrt{4(r_{n-1} - S/k_t)(S/k) + (r_{n-1} - 2S/k_t)^2}\ , \quad n \geq 1 \tag{16}$$

The limit state is reached in an equilibrium domain dependent of the material and independent of the initial conditions; but the limiting values are functions of the initial conditions. The equilibrium domain grows in function of k/h from F/h to [(F-S)/h, (F+S)/h] which is the fixed equilibrium domain of the rigid-plastic system (Lee, 1989).

MAXIMUM AND LIMIT STATES OF THE MULTI-DEGREE-OF-FREEDOM (MDOF) SYSTEM

Maximum State of the Elastic System

Consider now a n-DOF elastic system. It is known that the mode shapes matrix and the natural frequencies matrix are given by :

$$\mathbf{\Phi} = [\Phi_1\ \Phi_2 \ldots \Phi_n]\ ;\ \ \mathbf{\Omega} = \mathrm{diag}[\omega_i],\ i = 1,2, ..., n \tag{17}$$

with orthogonality properties written as :

$$\mathbf{\Phi}^T \mathbf{M}\, \mathbf{\Phi} = \mathbf{I},\ \mathbf{\Phi}^T \mathbf{K}\, \mathbf{\Phi} = \mathbf{\Omega}^2 \tag{18}$$

where **M** is the mass matrix and **K** the stiffness matrix.

Considering the orthogonality and the following transformation of the displacements $\mathbf{X} = \mathbf{\Phi Y}$, the equation of motion for the generalized modal displacement in the case of constant forces **F** is written as :

$$\ddot{\mathbf{Y}} + \mathbf{\Omega}^2\, \mathbf{Y} = \mathbf{\Phi}^T\mathbf{F}(t) = \mathbf{\Phi}^T\mathbf{F}. \tag{19}$$

This equation is composed of n individual equations and the solution of each equation can be found by the Duhamel integral. When the initial conditions are equal to zero, we have :

$$\mathbf{Y} = [y_1\ y_2 \ldots y_n]^T \tag{20}$$

$$y_i = \frac{1}{\omega_i} \int_0^t \Phi_i^T\ \mathbf{F} \sin \omega_i(t - \tau)\, d\tau,\ \ i = 1, 2, ..., n\ . \tag{21}$$

Since all the degrees-of-freedom are decoupled, it is evident that the maximum state of the structure is reached when all the degree-of-freedom reach their maximum state at an instant t. This maximum state is possible mathematically if there exists no energy dissipation in the structure and a certain numerical error is permitted. So, we have :

$$\mathbf{Y}_{max} = 2\ \mathbf{\Omega}^{-2}\mathbf{\Phi}^T\ \mathbf{F}. \tag{22}$$

The maximum displacement is found as follows :

$$\mathbf{X}_{max} = \mathbf{\Phi}\ \mathbf{Y}_{max} = 2\ [\mathbf{\Phi}\ \mathbf{\Omega}^{-2}\mathbf{\Phi}^T]\ \mathbf{F} = 2\ \mathbf{K}^{-1}\mathbf{F}. \tag{23}$$

This means that the maximum displacements of the elastic system subjected to a constant force are equal to the double of the static response to the same force applied statically and that ***at the time of the maximum state of the structure, the displacement of each degree-of-freedom is equal to the maximum value of the individual SDOF system whose characteristics are same as those of the DOF under considering****. These results are also true for the stresses.*

Hypothesis for the Maximum and Limit States of the Elastoplastic System

During the dynamic elastoplastic response, there exist temporal plastic energy dissipation and also spatial plastic energy distribution.

The result for the MDOF elastic system allows to make the hypothesis that *each degree-of-freedom of a MDOF elastoplastic system behaves as an independent SDOF elastoplastic system and that the maximum/limit state of the structure is attained when all the DOF reach their maximum/limit state.* This hypothesis enables to determine the local maximum/limit elastic stress field of a MDOF elastoplastic system from the known analytical solutions due to the temporal plastic energy dissipation for the SDOF system. So, we reach a static elastoplastic problem.

Using the above informations on the maximum and limit states, we impose as fictive loads the initial strains due to the dynamic elastic stresses on the structure and then we solve a static elastoplastic problem, from which we can find the spatial plastic energy distribution in the structure.

NUMERICAL METHODS

With our hypothesis, the analysis of the maximum and limit states is reduced to a static elastoplastic problem that we will solve with the quasi-static simplified method proposed by Zarka *et al.* (1979).

Review of the Evolution of Structures under Quasi-static Loadings

In this paper, we consider only the kinematically hardening materials.

The structure of a finite volume V with its boundary ∂V is subjected to body forces $\mathbf{X}^d(t)$ in V, surface forces $\mathbf{f}_i^d(t)$ on $\partial_{f_i}V$ of ∂V, displacements $\mathbf{u}_j^d(t)$ on $\partial_{u_j}V$, and initial strains $\epsilon_I^d(t)$ in V.

First we compute the **purely elastic response**. Assuming that the structure is made of a purely elastic material, we can obtain the displacement field $\mathbf{u}^{el}(t)$, the strain field $\epsilon^{el}(t)$ and the stress field $\sigma^{el}(t)$ using the routine ELAS :

$$\textbf{ELAS}\ (V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ \mathbf{X}^d,\ \mathbf{f}_i^d,\ \mathbf{u}_i^d,\ \epsilon_I^d,\ \mathbf{M}), \tag{24}$$

where the first three arguments represent the geometry of the structure, the subsequent four arguments are the loadings, and the last one denotes the material.

These fields are such that $\mathbf{u}^{el}(t)$ and $\epsilon^{el}(t)$ are kinematically admissible (K.A.) with $\mathbf{u}_j^d(t)$ on $\partial_{u_j}V$,

and
$$\sigma^{el}(t) = \mathbf{L}(\epsilon^{el}(t) - \epsilon_I^d(t))\ (\text{or } \epsilon^{el}(t) = \mathbf{M}\sigma^{el}(t) + \epsilon_I^d(t)) \tag{25}$$

is statically admissible (S.A.) with $\mathbf{X}^d$ in V and $\mathbf{f}_i^d(t)$ on $\partial_{f_i}V$, where $\mathbf{M}$ and $\mathbf{L}=\mathbf{M}^{-1}$ are the symmetric positive definite elastic matrices.

Then, we consider the **real response**. In the real structure, we have the actual displacement and strain fields $\mathbf{u}(t)$ and $\epsilon(t)$ K.A. with $\mathbf{u}_j^d(t)$, and the actual stress field $\sigma(t)$ S.A. with $\mathbf{X}^d(t)$ and $\mathbf{f}_i^d(t)$. We can rewrite them as follows :

$$\mathbf{u}(t) = \mathbf{u}^{el}(t) + \mathbf{u}^{ine}(t)\ ;\ \epsilon(t) = \epsilon^{el}(t) + \epsilon^{ine}(t)\ ;\ \sigma(t) = \sigma^{el}(t) + \rho(t) \tag{26}$$

with
$$\epsilon^{ine}(t) = \mathbf{M}\rho(t) + \epsilon^p(t), \tag{27}$$

where $\mathbf{u}^{ine}$ and ϵ^{ine} are, respectively, the inelastic displacement and strain fields K.A. with 0 on $\partial_{u_j}V$, ρ denotes the residual stress field S.A. with **0** in V and 0 on ∂_{f_i}, and ϵ^p is the plastic strain field.

With our tool ELAS and the associated arguments :

$$\textbf{ELAS}\ (V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ \mathbf{0}^d,\ \mathbf{0}_i^d,\ \mathbf{0}_j^d,\ \epsilon^p,\ \mathbf{M}), \tag{28}$$

we can obtain
$$\mathbf{u}^{ine} \Rightarrow \epsilon^{ine} \Rightarrow \rho = \mathbf{L}(\epsilon^{ine} - \epsilon^p). \tag{29}$$

We then introduce **the structural transformed parameters**. The yield criterion can be written as :

$$f(\sigma - \mathbf{y}) = f(\mathbf{S} - \mathbf{y}) = f(\mathbf{S}^{el} + \text{dev}\ \rho - \mathbf{y}) = f(\mathbf{S}^{el} - \mathbf{Y}) \leq 0 \tag{30}$$

with
$$\mathbf{y} = \mathbf{C}\epsilon^p \Leftrightarrow \epsilon^p = \mathbf{C}^{-1}\mathbf{y}\ ;\ \mathbf{Y} = \mathbf{C}\epsilon^p - \text{dev}\ \rho, \tag{31}$$

where $\mathbf{S}$ and $\mathbf{S}^{el}$ denote, respectively, the deviatoric part of the actual and elastic stresses, $\mathbf{C}$, the hardening modulus, and $\mathbf{y}$, $\mathbf{Y}$, the internal parameters and the structural transformed parameters, respectively.

Now the inelastic strain field can be rewritten in the $\mathbf{Y}$ space as :

$$\epsilon^{ine} = \mathbf{M}\rho + \epsilon^p = \mathbf{M}\rho + \mathbf{C}^{-1}(\mathbf{Y} + \text{dev}\ \rho) = (\mathbf{M} + \mathbf{C}^{-1}\text{dev})\rho + \mathbf{C}^{-1}\mathbf{Y} = \mathbf{M}'\rho + \mathbf{C}^{-1}\mathbf{Y} \tag{32}$$

where $\mathbf{M}' = (\mathbf{M} + \mathbf{C}^{-1}\text{dev})$ is the modified elastic coefficient matrix.

When $\mathbf{Y}$ is supposed known, the ELAS routine is used to determine the inelastic fields :

$$\textbf{ELAS}\ (V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ \mathbf{0}^d,\ \mathbf{0}_i^d,\ \mathbf{0}_j^d,\ \mathbf{C}^{-1}\mathbf{Y},\ \mathbf{M}') \tag{33}$$

which gives :

$$\mathbf{u}^{ine} \Rightarrow \epsilon^{ine} \Rightarrow \rho = \mathbf{L}'(\epsilon^{ine} - \mathbf{C}^{-1}\mathbf{Y}) \Rightarrow \mathbf{y} = \mathbf{Y} + \text{dev}\rho \Rightarrow \epsilon^p = \mathbf{C}^{-1}\ \mathbf{y},\ \text{where } \mathbf{L}' = \mathbf{M}'^{-1}. \tag{34}$$

In many cases, there will exist at the same time an elastic part V_e and a plastic part V_p in the volume V. In this case, the field ϵ^p in V_e and the field $\mathbf{Y}$ on V_p are known and so we can calculate the inelastic fields using ELAS as follows :

$$\textbf{ELAS}\ (V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ \mathbf{0}^d,\ \mathbf{0}_i^d,\ \mathbf{0}_j^d,\ \epsilon^p \text{ in } V_e,\ \mathbf{C}^{-1}\mathbf{Y} \text{ in } V_p,\ \mathbf{M} \text{ in } V_e,\ \mathbf{M}' \text{ in } V_p) \tag{35}$$

which gives the fields $\mathbf{u}^{ine}$, from which the other associated unknown fields can be calculated using Eqs. (29) in V_e and Eqs. (34) in V_p .

New Algorithm for the Dynamic Analysis

The general idea of our approach consists in :

Once the local elastic stress field is known after a static elastic analysis, we treat the SDOF elastoplastic mechanism at each point of the structure, and then we calculate, from the known analytical solutions, the local maximum or limit dynamic elastic stress field. In a second step, with the assumed transformed parameters **Y** in the plastic zone V_p and the zero plastic strains in the elastic zone V_e, the inelastic displacement field is determined using ELAS. According to the amount of the plasticity, we need some iteration procedures (usually a very few iterations). See Lee and Zarka (1989) and Lee (1989) for full details.

EXAMPLES OF APPLICATIONS

Our method was verified varying the divers parameters of loadings and materials with a cantilver beam model. In all cases, the results agree very well with the exact solutions and direct integration method (see our accompanying invited paper in Division L).

One-bay Two-story Plane Frame

The frame shown in Fig. 3 was analysed. We model the beam using the 4x4 integration points per finite element and we use the Gauss quadrature. The element tangential stiffness matrix was constructed considering the yield criteria at each point of the element. The kinematically hardening elastoplastic behavior was assumed. For the appreciation of the results, we used the Newmark direct integration method (1959) with pseudo force method and $\beta = 1/4$, $\gamma = 1/2$.

The material data used in this problem were as follows ; elastic modulus E = 210 x 10^6 kN/m²; yield stress σ_y = 250 x 10^3 kN/m²; mass density ρ = 800 kg sec²/m⁴. Three different hardening moduli h = E, 0.1E, and 0.01E were tried. The section data were as follows : b x d = 0.6 m x 0.6 m for columns, b x d = 0.3 m x 0.6 m for beams.

The lateral forces F = 3600 kN, (approximately the static initial yielding load), were applied at the nodes 3 and 5. The time step sizes and duration of the calculation for the Newmark method were, respectively, Δt = 0.01 sec.= $0.014T_1$ and 2 sec. in all the three analyses. The comparison of the results of the simplified (SM) and Newmark (CM) methods is shown in Table 1. The simplified method was averagely about 40 times faster than the Newmark method in computaion times. Fig. 4 shows the displacement histories at the node 5 in the case of h/E = 0.1 and the maximum and limiting values obtained with the simplified method.

Table 1 Unit : x 10^{-2}m

node	h/E = 1		h/E = 0.1		h/E = 0.01	
	SM	CM	SM	CM	SM	CM
3	2.7705	2.7894	2.7669	2.9419	2.7131	2.9833
5	6.0135	6.0846	6.0130	6.4115	5.9054	6.4971

CONCLUSIONS

The new direct method based on the complete analytical solutions gives very satisfactory results without the instability and inaccuracy encountered often in the step-by-step incremental procedures. The drastic saving in cost during the inelastic dynamic analysis is remarkable.

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ILLUSTRATIONS

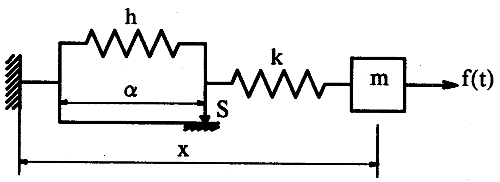

Fig. 1(a) : Single-degree-of-freedom elastoplastic model with kinematically hardening material

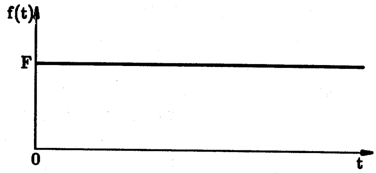

Fig. 1(b) : Instantaneous constant force

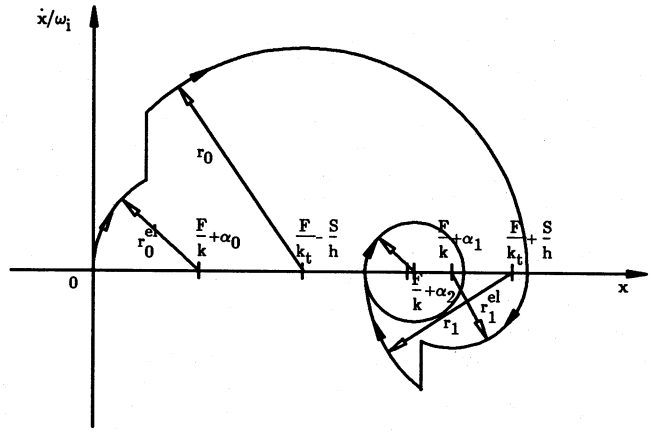

Fig. 2 : Trajectories in phase plane, $\omega_i = \omega$ for the elastic response, ω_t for the elastoplastic response

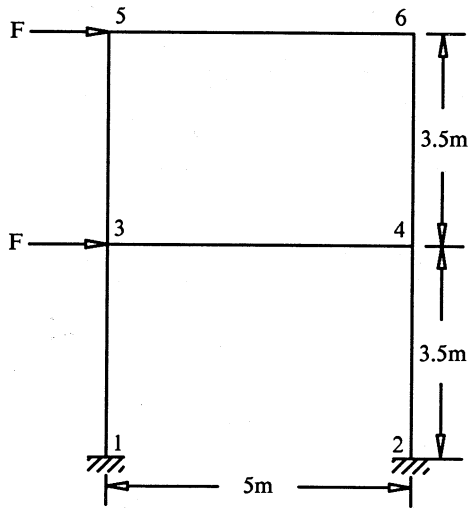

Fig. 3 : One-bay two-story plane frame

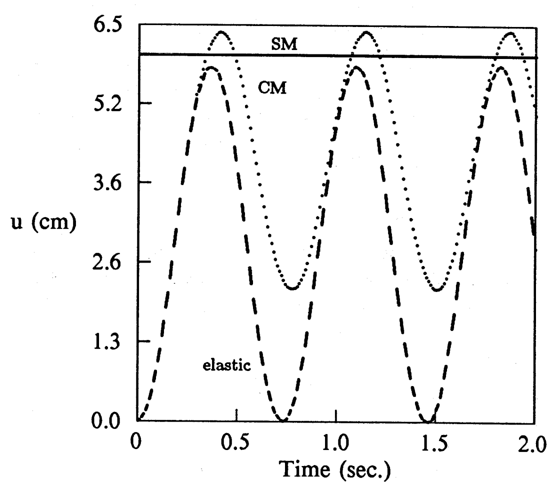

Fig. 4 : Elastic and elastoplastic lateral displacement-time histories at node 5 of the plane frame by the Newmark method and the maximum and limit displacements obtained for the simplified method (h/E = 0.1)

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