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GROUP-INVARIANT SOLUTIONS OF NONLINEAR ELASTODYNAMIC PROBLEMS OF PLATES AND SHELLS

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Introduction.

Plates and shells are basic structural components in nuclear reactors and their equipment. The prediction of the dynamic response of these components to fast transient loadings (e.g., loadings caused by earthquakes, missile impacts, etc.) is a quite important problem in the general context of the design, reliability and safety of nuclear power stations. Due to the extreme loading conditions a more adequate treatment of the foregoing problem should rest on a suitable nonlinear shell model, which would allow large deflections of the structures regarded to be taken into account. Such a model is provided in the nonlinear Donnell-Mushtari-Vlasov (DMV) theory [1]. The governing system of equations of the DMV theory consists of two coupled nonlinear fourth order partial differential equations in three independent and two dependent variables. It is clear, as the case stands, that the obtaining solutions to this system directly, by using any of the general analytical or numerical techniques, would involve considerable difficulties.

In the present paper, the invariance of the governing equations of DMV theory for plates and cylindrical shells relative to local Lie groups of local point transformations will be employed to get some advantages in connection with the aforementioned problem. The foundations of Lie transformation group methods and its basic notions such as symmetry group, generator (basic operator), invariant solution, etc., can be found in Olver [2] and Ovsiannikov [3] and will not be referred to explicitly in the text below.

First, the symmetry of a functional, corresponding to the governing equations of DMV theory for plates and cylindrical shells will be studied. Next, the densities in the corresponding conservation laws will be determined on the basis of Noether theorem. Finally, we will study a class of invariant solutions of the governing equations. As is well known, group-invariant solutions are often intermediate asymptotics for a wider class of solutions of the corresponding equations. When such solutions are considered, the number of the independent variables can be reduced. For the class of invariant solutions studied here, the system of governing equations converts into a system of ordinary differential equations.

Fundamental equations.

The general field equations of the DMV theory are

$$\begin{aligned} &D\Delta\Delta w - d^{\alpha\mu}d^{\beta\nu}w_{;\alpha\beta}\Phi_{;\mu\nu} - d^{\alpha\mu}d^{\beta\nu}b_{\alpha\beta}\Phi_{;\mu\nu} + \rho\ddot{w} = p, \\ &(1/2Eh)\Delta\Delta\Phi + d^{\alpha\mu}d^{\beta\nu}w_{;\alpha\beta}w_{;\mu\nu} + 2d^{\alpha\mu}d^{\beta\nu}b_{\alpha\beta}w_{;\mu\nu} = 0, \end{aligned} \tag{1}$$

where w is used for the transversal displacement of the middle-surface; Φ – the stress function; D – the bending rigidity; E – the Young modulus; h – the thickness of the shell (plate); ρ – the mass per unit area of the middle-surface; $b_{\alpha\beta}$ – the curvature tensor; $d^{\alpha\beta}$ – the alternating tensor; p – the external load intensity. A semicolon is used for covariant differentiation with respect to the first fundamental form of the middle-surface; a superposed dot – for partial derivative with respect to the time t; and Δ – the Laplacian. Repeated sub- and superscripts imply summation.

The equations (1) describe the dynamic behaviour of shells (plates) when large deflections in transversal direction of the middle-surface are taken into account. Only plates ($b_{\alpha\beta} = 0$) and cylindrical shells (in Euclidean coordinates $b_{11} = b_{12} = b_{21} = 0$, $b_{22} = 1/R$, R – radius of curvature) are discussed here. In the first case, (1) are known as von Kármán equations [1].

Symmetries of the equations.

The invariance properties of the von Kármán equations have been first studied by K. Ames & W. Ames [4]. They have determined several symmetry groups of these equations. Lately, Schwarz [5] has obtained the full symmetry group of the von Kármán equations. As for the cylindrical shell equations, their invariance properties have been studied by Dzhupanov, Ivanov, Vassilev [6]. Combining the results presented in this papers, one can summarize that: the generators of the full symmetry group of the governing equations (1) are

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_4 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + \frac{x^1 x^2}{R}\frac{\partial}{\partial w},$$
$$X_5 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + 2t\frac{\partial}{\partial t} - \frac{(x^2)^2}{R}\frac{\partial}{\partial w}, \quad X_6 = \frac{\partial}{\partial w}, \quad X_7 = x^1\frac{\partial}{\partial w}, \tag{2}$$
$$X_8 = x^2\frac{\partial}{\partial w}, \quad X_9 = t\frac{\partial}{\partial w}, \quad X_{10} = tx^1\frac{\partial}{\partial w}, \quad X_{11} = tx^2\frac{\partial}{\partial w};$$

and there exist three additional symmetry groups corresponding to the operators

$$X_{12} = x^1 \mathrm{f}(t)\frac{\partial}{\partial w}, \quad X_{13} = x^2 \mathrm{g}(t)\frac{\partial}{\partial w}, \quad X_{14} = \mathrm{h}(t)\frac{\partial}{\partial w}, \tag{3}$$

where f(t), g(t) and h(t) are arbitrary functions. In the formulae (2) and (3), (x^1,x^2) are Euclidean coordinates on the middle-surface.

An invariant solution of (1) corresponds to any subgroup of the full symmetry group. Here, a special attention is paid to the invariant solutions, corresponding to the generators X_4, X_5. These solutions can be written in the form

$$w(x^1,x^2,t) = u(s) - (1/2R)(x^2)^2, \quad \Phi(x^1,x^2,t) = \varphi(s), \tag{4}$$

where

$$s = (1/2)\ln\left(\sqrt{\rho/D}\, r^2/t\right), \quad r^2 = (x^1)^2 + (x^2)^2. \tag{5}$$

The same class of invariant solutions of the von Kármán equations are considered by K.Ames & W.Ames [4]. However they have chosen the independent variable s in another form.

Variational symmetries.

Consider a functional

$$l(w,\Phi,\Sigma) = \int_\Sigma \Lambda(t,x^1,x^2,w,\Phi)d\sigma, \tag{6}$$

where Σ, $d\sigma$ – finite and infinitesimal domains of the middle-surface, Λ – Lagrangian of the mechanical system. Consider Λ in the form

$$\Lambda = \Pi - \mathrm{T}, \tag{7}$$

where

$$\mathrm{T} = (\rho/2)\dot{w}^2 ,$$
$$\Pi = (D/2)\left[(\Delta w)^2 - 2(1-\nu)d^{\alpha\mu}d^{\beta\nu}w_{;\alpha\beta}w_{;\mu\nu}\right] + (1/2)d^{\alpha\mu}d^{\beta\nu}\Phi_{;\alpha\beta}w_{;\mu}w_{;\nu} \tag{8}$$
$$- (1/2Eh)\left[(\Delta\Phi)^2 - 2(1+\nu)d^{\alpha\mu}d^{\beta\nu}\Phi_{;\alpha\beta}\Phi_{;\mu\nu}\right] + b_{22}w_{;1}\Phi_{;1} .$$

In the case, when only static problems are under consideration, and $b_{22} = 0$, (7) leads to the Lagrangian, suggested in Washizu [7] for large deflections of plates. It is easy to obtain that the Euler-Lagrange equations [3] associated with the functional (6) with Lagrangian (7), (8) coincide with (1). Further, the invariant properties of the functional (6) – (8) will be studied.

The symmetry group of a functional is a subgroup of the full symmetry group of the corresponding Euler-Lagrange equations [3]. Hence, the basic operators of the variational symmetry group of (6) with Lagrangian (7), (8) are among (2) and (3). This property helps us in obtaining the symmetry group of (6) – (8). We have simply to test if every operator from (2) and (3) satisfies the invariance criteria. We will skip the detail transformations and give only the final results:

(i) The cylindrical shell functional admits the following generators
$$X_1 , X_2 , X_3 , X_6 , X_{13} , X_{14} ; \tag{9}$$
(ii) The plate functional admits the following generators
$$X_1 , X_2 , X_3 , X_4 , X_5 , X_6 , X_{12} , X_{13} , X_{14} . \tag{10}$$

Comparison between (9) and (10) shows that the rotation group (X_4) and scaling group (X_5) are admitted by the plate functional only. The aim of this paper is to obtain invariant solutions of (1) corresponding to these two symmetry groups. That is why, only plate functional ($b_{\alpha\beta} = 0$) is studied furthermore.

Conservation laws.

According to the Noether theorem, any basic operator of the variational symmetry group (10) generates an linearly independent conservation law admitted by the solutions of the von Kármán equations. The conservation laws are usually written in the form
$$\dot{\Psi} = P^1_{;1} + P^2_{;2} , \tag{11}$$
The relation (11) implies that the rate of a function (momentum, energy, etc.) with density per unit area Ψ is equal to the flux (P^1,P^2) through the boundary of this area.

The densities, corresponding to the basic operators (10) are determined following [2,3]. They are
$$\begin{aligned}
&X_1 : \quad \Psi = -\rho\dot{w}w_{;1} ;\\
&X_2 : \quad \Psi = -\rho\dot{w}w_{;2} ;\\
&X_3 : \quad \Psi = \Pi + \mathrm{T} ;\\
&X_4 : \quad \Psi = \rho\dot{w}\left(x^1w_{;2} - x^2w_{;1}\right) ;\\
&X_5 : \quad \Psi = 2t(\Pi + \mathrm{T}) - \rho\dot{w}\left(x^1w_{;1} + x^2w_{;2}\right) ;\\
&X_6 : \quad \Psi = -\rho\dot{w} ;\\
&X_{12} , X_{13} , X_{14} : \quad \Psi = 0 .
\end{aligned} \tag{12}$$

Each of these densities satisfies particular equation in the form (11). The conservation laws for three of the densities (12) are well known. First, the density, corresponding to the operator X_6 is the transversal impulse per unit area of the middle-surface and its conservation law is the first equation of (1). Simultaneously, the conservation law, generated by X_{14} coincides with the second equation of (1). Finally, the symmetry group, which operator is X_3 describes time translation. The corresponding density is the energy per unit area of the middle-surface. This group generates the energy conservation law.

Numerical results.

Now, consider the solutions of the von Kármán equations in the form (4), i.e., invariant solutions corresponding to the operators X_4, X_5. In case of plates, the formulae (4), (5) reduce to

$$w(x^1,x^2,t)=u(s),\quad \Phi(x^1,x^2,t)=\varphi(s),\quad s=(1/2)\ln\left(\sqrt{\rho/D}\,r^2/t\right). \tag{13}$$

Substitution of (13) into (1) leads to a system of nonlinear ordinary differential equations for the unknown functions $u(s)$ and $\varphi(s)$

$$\begin{aligned} &D(u''-4u'+4u)''+(\rho e^{4s}/4-\varphi')u''+(\rho e^{4s}/2-\varphi''+2\varphi')u'=0,\\ &(\varphi''-4\varphi'+4\varphi)''+2Eh(u''-u')u'=0. \end{aligned} \tag{14}$$

Two particular analytic solutions of this system are obtained by K.Ames & W.Ames [4] on the basis of the following assumptions: u = const – for the first one and $u = \varphi$ – for the second one.

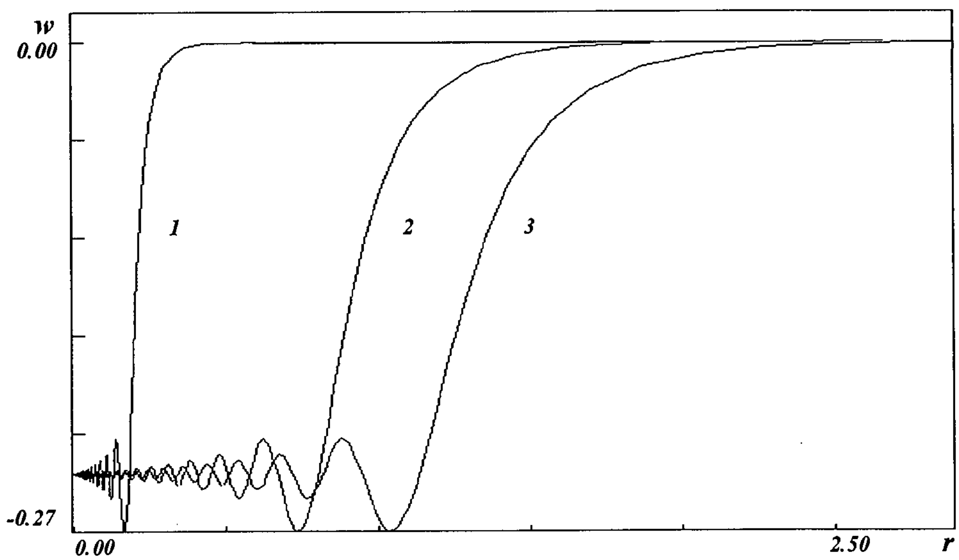

Fig.1. Transversal displacement w (in meters) of the middle-plane as a function of radius r (in meters). Curve (*1*) represents the wave shape at time t = 0.05s, (*2*) – at t = 1.00 s, (*3*) – at t = 2.00s.

Here, the system (14) is treated numerically applying a fourth-order Runge-Kutta method. An infinite plate is considered. The solution (13) describes axially symmetric propagation of bending waves with diminished velocity. In the present work, this velocity is supposed to be well below the velocities of the elastic waves in the material of the plate. The bending wave should meet this reasonable restriction even for instants $t \sim 10^{-4}$s. Thus, from (13) can be concluded that the upper boundary of the independent variable s is about 1. Therefore, a numerical solution of (14) is obtained in the interval (-20,0). The corresponding deflection is shown in Fig.1 for three instants. For $s \geq 0$ the solution $u(s)$ is sewed together with the solution $u = 0$. The initial conditions for the equations (14) are given at $s = 0$. At this point, a jump discontinuity occurs because of the sewing. That is why, to ensure energy conservation, the second derivative of u is taken smooth everywhere.

Several features of the solution in Fig.1 should be mentioned here. First, the initially flat middle-surface goes to a new position of equilibrium behind the wave. Second, oscillations with diminishing amplitudes occur behind the wave front. The existence of these oscillations is due to

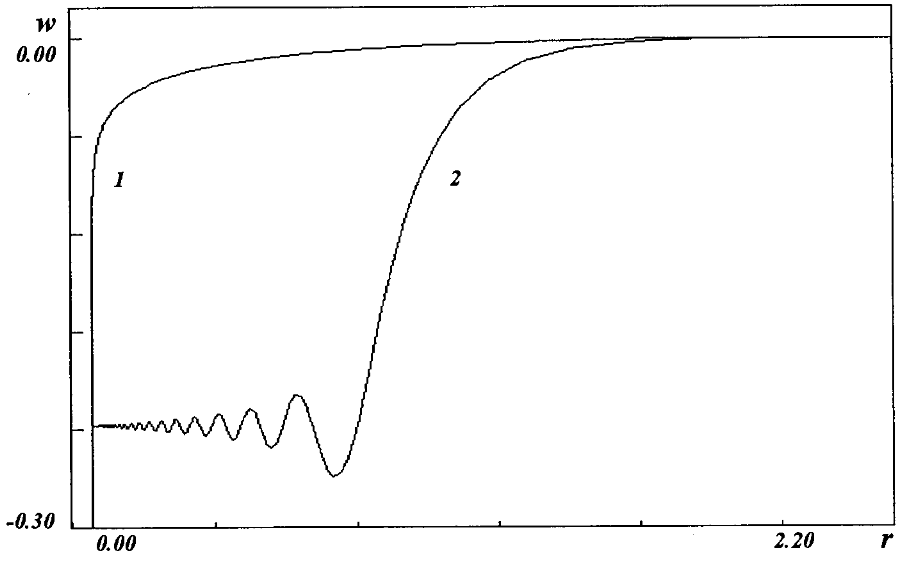

Fig.2. Comparison between invariant solutions of biharmonic equation (*1*) and von Kármán equations (*2*). This figure represents the transversal displacement w (in meters) as a function of radius r (in meters).

the nonlinearity, which can be deduced from the comparison between the invariant solutions of biharmonic equation and von Kármán equations given on the Fig.2. It is easy to find that the rotation (X_4) group and scaling (X_5) group (and thus the invariant solutions (13)) are admitted by the biharmonic equation also. The two deflection curves given on Fig.2 are obtained for the same initial conditions of the corresponding ordinary differential equations. The numerical investigations show that the absolute value of the deflection obtained from the von Kármán equations is bounded above. On the other hand, the deflection obtained from the biharmonic equation is found to be unbounded when r tends to 0.

Conclusions.

Several conclusions can be drawn on the basis of the above investigations.

First, the full variational symmetry group of the functional (6) – (8) is obtained. It is shown, that the rotation group (X_4) and scaling group (X_5) are admitted by the plate functional only.

Second, it is shown, that nine linearly independent conservation laws in the form (11) exist for the solutions of the von Kármán equations. The density of each conservation law is obtained. These conservation laws are first integrals of the equations (1) when $b_{\alpha\beta} = 0$.

Third, a new invariant solution of the von Kármán equations is obtained. It belongs to the class of scale-invariant solutions (13), already found by K.Ames & W.Ames [2]. A numerical procedure is applied to determine the deflection and stress function as functions of the independent variable s in the interval (-20,0). For s > 0, this solution is sewed together with the solution $u = 0$. The sewing is twice smooth, i.e., u and its first and second derivatives are zero at the point $s = 0$. Hence, it is easy to deduce [2,3], that the flux in (11), written for the operator X_3, also vanishes at $s = 0$. That is why, the energy conservation for the sewed solution takes place, and at any time, only finite domain of the plate is perturbed.

Finally, the values $t = 0$ and $r = 0$ are singular for the transformation (13). That is why, it is not possible to obtain numerical solution at these points. The evolution of the displacement in Fig.1 shows that this invariant solution may be considered as initiated by singular initial conditions, namely w is not zero at the point $r = 0$ only. Hence, from Fig.2 can be deduced that while the considered invariant solution of the biharmonic equation has a singularity at $r = 0$ at any time, the associated solution of the von Kármán equations is regular when the initial displacement is singular in the above sense.

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