A Necessary Condition for the Existence of Regular and Symmetrical PBIB Designs of T_m Type

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§1. Introduction and summary. A necessary condition for the existence of regular and symmetrical PBIB designs in terms of the Hasse-Minkowski p-invariant has been obtained, for triangular type by J. Ogawa [5], for T_3 type by K. Kusumoto [4], both basing upon the work of L.C.A. Corsten [1] concerning the proper space related to PBIB designs of triangular type.

In this article, the author introduces an association of T_m type as an extension of the type of association stated above, and determines the proper spaces related to PBIB designs of this type, along the line of Corsten's work. Non-existence criteria of PBIB designs of T_m type are also given with some examples. Hence, the present work is a generalization of those by J. Ogawa [5] and K. Kusumoto [4].

In the subsequent section, definition of the association of T_m type is given and the corresponding association algebra is discussed. In section 3 we discuss the proper spaces related to PBIB designs of T_m type. Section 4 is devoted to the derivation of a set of necessary conditions for the existence of PBIB designs of T_m type, and in the final section, some examples of non-existent PBIB designs of T_m type are given.

§2. Association of T_m type and the corresponding association algebra. An association of T_m type is defined as follows: Let n and m be any given positive integers such that $1 \leq m$ and $2m \leq n$, and let $v = \binom{n}{m}$. Let us take $\binom{n}{m}$ different subsets, $\{r_1, \ldots, r_m\}$'s, of $\{1, \ldots, n\}$, and we associate to each of these subsets one of the treatments, $\varphi_1, \ldots, \varphi_v$, in any but one-to-one way. Two treatments, φ_i and φ_j, which correspond to $\{r_1, \ldots, r_m\}$ and $\{r'_1, \ldots, r'_m\}$ respectively, are said to be the u-th associated if and only if $\{r_1, . , r_m\}$ and $\{r'_1, \ldots, r'_m\}$ contain exactly m-u integers in common, $u = 0, 1, \ldots, m$. Hence the number of the u-th associates of each treatment is given by

$$(2.1) \qquad n_u = \binom{m}{u}\binom{n-m}{u}, \quad (u = 0, 1, \ldots, m).$$

Parameters characterizing the association are

$$(2.2) \qquad p^t_{su} = \sum_{a=o}^{m=t} \binom{m-t}{a}\binom{t}{m-s-a}\binom{t}{m-u-a}\binom{n-m-t}{s+u-m+a}, \quad u,s,t = 0,1,\ldots,m.$$

The association matrices A_u $(u = 0,1,\ldots,m)$ generate a linear commutative algebra $\mathfrak{A}$ over the field of all rational numbers and it is called the association algebra. It can be shown [6] the regular representation of $\mathfrak{A}$ is generated by the mappings

$$(3.2) \qquad (\mathfrak{A}): \ A_u \longrightarrow \rho_u = \| p^t_{su} \| .$$

Transforming ρ_u's by a non-singular and rational matrix

$$(2.4) \qquad C = \left\| \frac{z_{st}}{n_t} \right\| , \ (s,t = 0, 1, \ldots, m)$$

with

$$(2.5) \qquad z_{st} = \sum_{a=o}^{t} (-1)^{t-a}\binom{m-a}{m-t}\binom{m-s}{a}\binom{n-m-s+a}{a} , \ (\text{cf. } (3.7)),$$

we get

$$(2.6) \qquad C \rho_u C^{-1} = \begin{Vmatrix} z_{ou} & & & 0 \\ & z_{1u} & & \\ & & \ddots & \\ 0 & & & z_{mu} \end{Vmatrix} , \quad u = 0,1,\ldots,m$$

and consequently we obtain the following (m+1) mutually orthogonal idempotent matrices belonging to   :

$$(2.7) \qquad A^{\#}_u = \frac{\alpha_u}{v} \left(\sum_{a=o}^{m} \frac{z_{ua}}{n_a} A_a \right) \qquad u = 0,1,\ldots,m$$

with respective rank

$$(2.8) \qquad \alpha_u = \text{tr. } A^{\#}_u = \binom{n}{u} - \binom{n}{u-1} = \frac{n+1-2u}{n+1-u} \binom{n}{u}, \quad u = 0,1,\ldots,m.$$

Let N be the incidence matrix of the design, then it is well-known that

$$(2.9) \qquad NN' = \sum_{u=o}^{m} \lambda_u A_u = \sum_{u=o}^{m} \rho_u A^{\#}_u ,$$

where

$$(2.10) \qquad \rho_u = \sum_{a=o}^{m} z_{ua} \lambda_a \qquad u = 0,1,\ldots,m .$$

The latent vector of NN' corresponding to the characteristic root $\rho_o = rk$ is evidently $\underline{J}'_v = (1,1,\ldots,1)$. From (2.9), it can be seen that the latent vectors of NN' corresponding to ρ_u are α_u column vectors of $A^{\#}_u$, which are linearly independent. They are all rational vectors.

<u>§3. Some properties of proper space related to PBIBD of T_m type.</u> According to L.C.A. Corsten [1], we conceive P = NN' as the matrix of the linear transformation P on a vector space $\mathcal{L}$ consisting of vectors $\underline{x}' = (x_1, x_2, \ldots, x_v)$ into itself, where the coordinate x_t corresponds to the t-th treatment. From

(2.9) the t-th coordinate y_t in $\underline{y} = P\underline{x}$ is equal to $\sum_{a=o}^{m} \lambda_a S_a$, where S_a designate the sum of the coordinates of $\underline{x}$ corresponding to the a-th associates of the treatment φ_t. $\underline{j}_v' = (1,1,\ldots,1)$ is the latent vector of P with the proper value $\sum_{a=o}^{m} \lambda_a n_a = rk$. We consider the (v-1) dimensional subspace $\mathfrak{L}^{\#}$ of $\mathfrak{L}$ orthogonal to $\underline{j}_v$. Then, for every vector $\underline{x}$ in $\mathfrak{L}^{\#}$, we have the following relation

$$(3.1) \qquad x_t + S_1 + \ldots + S_m = 0.$$

Let us construct a set of $\binom{n}{u}$ vectors of dimension v, $\left\{ \underline{c}_{i_1,\ldots,i_u} \,\middle|\, \{i_1,\ldots,i_u\} \subset \{1,\ldots,n\} \right\}$, which are contained by the vector space $\mathfrak{L}$ in the following way: let the t-th component x_t be 1 or 0, according as $\{i_1,\ldots,i_u\} \subset \{r_1,\ldots,r_m\}$ or $\{i_1,\ldots,i_u\} \not\subset \{r_1,\ldots,r_m\}$, where $\{r_1,\ldots,r_m\}$ is a set of integers corresponding to the t-th treatment φ_t, t=1,...,v. Let the $\binom{n}{u}$ dimensional subspace of $\mathfrak{L}$ spanned by those $\binom{n}{u}$ linearly independent vectors $\underline{c}_{i_1,\ldots,i_u}$'s be $\mathfrak{L}^{(u)}$, u=1,...,m, then the space $\mathfrak{L}^{(u)}$ contains the space $\mathfrak{L}^{(u-1)}$, $\mathfrak{L}^{(0)}$ being the one dimensional space spanned by $\underline{j}_v$. Hence, there exists α_u dimensional subspace $\mathfrak{L}^{\#(u)}$ of the space $\mathfrak{L}^{(u)}$ orthogonal to $\mathfrak{L}^{(u-1)}$, u=1,...,m. For any vector $\underline{x}^{(u)} = \sum_{\{i_1,\ldots,i_u\} \subset \{1,\ldots,n\}} \gamma_{i_1,\ldots,i_u} \underline{c}_{i_1,\ldots,i_u}$ in $\mathfrak{L}^{\#(u)}$, the t-th coordinate $x_t^{(u)}$ is equal to $\sum_{\{i_1,\ldots,i_u\} \subset \{r_1,\ldots,r_m\}} \gamma_{i_1,\ldots,i_u}$. It is noted that, if we replace the scalar $\gamma_{i_1,\ldots,i_u}$ by the vector $\underline{c}_{i_1,\ldots,i_u}$ in the expression $x_t^{(u)} = \sum_{\{i_1,\ldots,i_u\} \subset \{r_1,\ldots,r_m\}} \gamma_{i_1,\ldots,i_u}$, then we have a vector in $\mathfrak{L}^{(u)}$ such as $\underline{c}_t^{(u)} = \sum_{\{i_1,\ldots,i_u\} \subset \{r_1,\ldots,r_m\}} \underline{c}_{i_1,\ldots,i_u}$.

Now, since $\mathfrak{L}^{\#(k)}$ is orthogonal to $\mathfrak{L}^{(0)}$, $1 \leq k \leq m$, it holds that

$$\underline{j}_v'\underline{x}^{(k)} = \binom{n-k}{m-k} \sum_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}} \gamma_{i_1,\ldots,i_k} = 0$$

which implies that

$$(3.2) \qquad \sum_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}} \gamma_{i_1,\ldots,i_k} = 0, \; k = 1,\ldots,m.$$

For any h and k, $(h = 1,\dots,m-1,\ k = 1,\dots,m)$ the inner product of $\underline{c}_t^{(h)}$ and $\underline{x}^{(k)}$ becomes

$$(3.3)\qquad \underline{c}_t^{(h)\prime}\,\underline{x}^{(k)} = \binom{m}{h}x_t^{(k)} + \binom{m-1}{h}s_1^{(k)} + \dots + \binom{m-(m-h)}{h}s_{m-h}^{(k)}$$

$$= \binom{m-k}{h-k}\binom{n-h-k}{m-h}x_t^{(k)}$$

with the convention that $\binom{m-k}{h-k} = 0$ if $h < k$.

In the case when $h \geq k$, since, for $\underline{c}_{i_1,\dots,i_h}$ of the expression $\underline{c}_t^{(h)} = \sum_{\{i_1,\dots,i_h\} \subset \{r_1,\dots,r_m\}} \underline{c}_{i_1,\dots,i_h}$

$$\underline{c}'_{i_1,\dots,i_h}\underline{x}^{(k)} = \binom{n-h}{m-h}\Sigma\,\gamma_{j_1,\dots,j_k} + \binom{n-h-1}{m-h-1}\Sigma\gamma_{j_1,\dots,j_{k-1},j'_1}$$

$$+ \dots + \binom{n-h-k}{m-h-k}\Sigma\,\gamma_{j'_1,\dots,j'_k}$$

where, the summation sign of $\Sigma\gamma_{j_1,\dots,j_{k-u},\, j'_1,\dots,j'_u}$ designates the sum of $\gamma_{j_1,\dots,j_{k-u},j'_1,\dots,j'_u}$ for all $\{j_1,\dots,j_{k-u},j'_1,\dots,j'_u\}$ such that $\{j_1,\dots,j_{k-u}\} \subset \{i_1,\dots,i_h\}$ and $\{j'_1,\dots,j'_u\} \subset \{1,\dots,n\} \cap \overline{\{i_1,\dots,i_h\}}$, it follows from (3.2)

$$\underline{c}'_{i_1,\dots,i_h}\underline{x}^{(k)} = \left[\binom{n-h}{m-h}-\binom{n-h-k}{m-h-k}\right]\Sigma\gamma_{j_1,\dots,j_k}$$

$$+ \left[\binom{n-h-1}{m-h-1}-\binom{n-h-k}{m-h-k}\right]\Sigma\gamma_{j_1,\dots,j_{k-1},j'_1}$$

$$\cdots\cdots$$

$$+ \left[\binom{n-h-(k-1)}{m-h-(k-1)}-\binom{n-h-k}{m-h-k}\right]\Sigma\gamma_{j_1,j'_1,\dots,j'_{k-1}}\ .$$

By the relation

$$\binom{k}{u}\Sigma\gamma_{j_1,\dots,j_k} + \binom{k-1}{u-1}\Sigma\gamma_{j_1,\dots,j_{k-1},j'_1}$$

$$+ \dots + \binom{k-u}{u-u}\Sigma\gamma_{j_1,\dots,j_{k-u},j'_1,\dots,j'_u}$$

$$= \Sigma'\gamma_{j_1,\dots,j_{k-u},j'_1,\dots,j'_u}$$

where, Σ' designates the summation for all $\{j_1,\dots,j_{k-u},j'_1,\dots,j'_u\}$ such that $\{j_1,\dots,j_{k-u}\} \subset \{i_1,\dots,i_n\}$ and $\{j'_1,\dots,j'_u\} \cap \{1,\dots,n\}\ \overline{\{j_1,\dots,j_{k-u}\}}$, the above inner product can be rewritten as

$$\underline{c}'_{i_1,\dots,i_n}\underline{x}^{(k)} = \binom{n-h-k}{n-m-1}\Sigma'\gamma_{j_1,j'_1,\dots,j'_{k-1}}$$

$$+ \binom{n-h-k}{n-m-2}\Sigma'\gamma_{j_1,j_2,j'_1,\dots,j'_{k-2}}$$

$$\cdots\cdots$$

$$+ \binom{n-h-k}{n-m-k}\Sigma'\gamma_{j_1,\dots,j_k},$$

where we have the equality

$$\sum_{a=o}^{u}\binom{k-(k-u)}{u-a}\binom{n-h-k}{n-m-a} = \binom{n-h-(k-u)}{n-m}, \quad (u = 1,\dots,k).$$

On the other hand, in the case when $h < k$, since

$$0 = \underline{c}'_{i_1,\dots,i_n}\underline{x}^{(k)} = \binom{n-k-h}{n-m-1}\Sigma'\gamma_{j_1,j'_1,\dots,j'_{k-1}}$$

$$+ \binom{n-k-h}{n-m-2}\Sigma'\gamma_{j_1,j_2,j'_1,\dots,j'_{k-2}}$$

$$\cdots\cdots$$

$$+ \binom{n-k-h}{n-m-h}\Sigma'\gamma_{j_1,\dots,j_h,j'_1,\dots,j'_{k-h}},$$

it follows that

$$\Sigma''\gamma_{j_1,\dots,j_u,j'_1,\dots,j'_{k-u}} = 0, \ u = 1,\dots,k-1,$$

where, $\Sigma''\gamma_{j_1,\dots,j_u,j'_1,\dots,j'_{k-u}}$ denotes the summation for all $\{j'_1,\dots,j'_{k-u}\}$ such that $\{j'_1,\dots,j'_{k-u}\} \subset \{1,\dots,n\} \cap \overline{\{j_1,\dots,j_u\}}$ and $\{j_1,\dots,j_u\}$ being fixed.

Therefore we have

$$(3.4) \qquad \underline{c}'_{i_1,\dots,i_h}\underline{x}^{(k)} = \binom{n-h-k}{n-m-k} \{j_1,\dots,j_k\} \overset{\Sigma}{\subset} \{i_1,\dots,i_h\}\gamma_{j_1,\dots,j_k}$$

and hance

$$(3.5) \qquad \underline{c}_t^{(h)'}\ \underline{x}^{(k)} = \binom{m-k}{h-k}\binom{n-h-k}{n-m-k}x_t^{(k)} .$$

Therefore, by (3.3) and (3.5), we have the relation

$$(3.6) \qquad \binom{m}{h}x_t^{(k)} + \binom{m-1}{h}S_1^{(k)} + \dots + \binom{m-(m-h)}{h}S_{m-h}^{(k)} = \binom{m-k}{h-k}\binom{n-h-k}{n-m-k}x_t^{(k)}$$

$$h = 0,\ 1,\ \dots,\ m-1,$$

from which, we get the following equalities

$$S_u^{(k)} = z_{ku}x_t^{(k)},\ k,\ u = 1,\ \dots,\ m.$$

Now, by the argument in the preceding section, it is seen that the coordinate y_t of $\mathbf{P}\underline{\mathbf{x}}$, $\underline{x}$ being a vector belonging to $\mathcal{S}^{\#(u)}$, is equal to $(\sum_{a=o}^{m} \lambda_a z_{ua})x_t$. Therefore $\mathcal{S}^{\#(u)}$ is a proper space of NN' with proper value ρ_u, $(u = 1,\dots,m)$.

Now, let us consider a matrix $C^{(m-u)}$ of order v, whose column vectors being $\underline{c}_t^{(m-u)}$, $t=1,\dots,v$. Then, it can be seen that

$$C^{(m-u)} = \| \underline{c}_1^{(m-u)},\ \underline{c}_2^{(m-u)},\ \dots,\ \underline{c}_v^{(m-u)} \|$$

$$= \binom{m}{m-u} A_o + \binom{m-1}{m-u} A_1 + \dots + \binom{m-u}{m-u} A_u$$

where, A_i's are association matrices. Hence we get

$$A_u = \sum_{a=o}^{u} (-1)^{u-a} \binom{m-a}{m-u} C^{(m-a)} .$$

From (3.5), it follows that

$$C^{(m-a)} A_k^{\#} = \binom{m-k}{a} \binom{n-m-k+a}{a} A_k^{\#} ,$$

and hence,

$$\begin{aligned} A_u A_k^{\#} &= z_k \quad {}^{\#}_{k} \\ &= \sum_{a=o}^{u} (-1)^{u-a} \binom{m-a}{m-u} C^{(m-a)} A_k^{\#} \\ &= \sum_{a=o}^{u} (-1)^{u-a} \binom{m-a}{m-u} \binom{n-m-k+a}{a} A_k^{\#} \end{aligned}$$

from which we obtain

(3.7) $$z_{ku} = \sum_{a=o}^{u} (-1)^{u-a} \binom{m-a}{m-u} \binom{m-k}{a} \binom{n-m-k+a}{a} .$$

The Gramian P_i of the basic vectors of $\mathcal{S}^{(i)}$, the join of proper spaces $\mathcal{S}^{\#(1)}, \ldots, \mathcal{S}^{\#(i)}$, and $\mathcal{S}^{(0)}$ $(i = 1,\ldots,m)$ are, now, easily obtained as follows:

In order to calculate P_i, we simply need the inner products of the vectors $\underline{c}_{j_1,\ldots,j_i}$'s, $\{j_1,\ldots,j_i\} \subset \{1,\ldots,n\}$. Indeed, if we consider the matrix $N^{(i)'}$ whose column vectors are $\underline{c}_{j_1,\ldots,j_i}$'s, then P_i is given by

(3.8) $$P_i = N^{(i)} N^{(i)'}.$$

It should be noted that the matrix $N^{(i)}$ is the incidence matrix of the PBIB design of T_i type with parameters

$$v^{(i)} = \binom{n}{i},\ b^{(i)} = \binom{n}{m},\ k^{(i)} = \binom{m}{i},\ \lambda_a^{(i)} = \binom{n-i-a}{m-i-a},\ (a=0,1,\ldots,i),$$

and then, it is well-known that

$$P_i = \sum_{a=0}^{i} \lambda_a^{(i)} A_a^{(i)} = \sum_{a=0}^{i} \rho_a^{(i)} A_a^{(i)\#} . \tag{3.9}$$

On the other hand, by (3.4) we get

$$N^{(i)\prime} N^{(i)} A_k^{\#} = \binom{m-k}{i-k}\binom{n-i-k}{m-i} A_k^{\#} , \tag{3.10}$$

from which it follows that

$$\rho_k^{(i)} = \binom{m-k}{i-k}\binom{n-i-k}{m-i} . \tag{3.11}$$

It is also seen from (2.8) or (3.10) that

$$\alpha_k^{(i)} = \alpha_k = \frac{n+1-2k}{n+1-k}\binom{n}{k}. \tag{3.12}$$

Therefore, the determinant of P_i is given by

$$|P_i| = \prod_{k=0}^{i} \rho_k^{(i)\alpha_k} = \prod_{k=0}^{i} [\binom{m-k}{i-k}\binom{n-i-k}{m-i}^{\alpha_k} \quad i=1,\dots,m. \tag{3.13}$$

Now, let α_u linearly independent column vectors of $A_u^{\#}$ be

$$\underline{a}_1^{(u)}, \underline{a}_2^{(u)}, \dots, \underline{a}_{\alpha_u}^{(u)} , \quad u=1,\dots,m \tag{3.14}$$

and put

$$S = \| \underline{j}_v, \underline{a}_1^{(1)}, \dots, \underline{a}_{\alpha_1}^{(1)}, \dots, \underline{a}_1^{(m)}, \dots, \underline{a}_{\alpha_m}^{(m)} \|$$

then

$$S'S = \begin{Vmatrix} v & & & & 0 \\ & Q^{(1)} & & & \\ & & Q^{(2)} & & \\ & & & \ddots & \\ 0 & & & & Q^{(m)} \end{Vmatrix} \tag{3.15}$$

where

$$Q^{(u)} = \|\underline{a}_1^{(u)},\dots,\underline{a}_u^{(u)}\|'\,\|\underline{a}_1^{(u)},\dots,\underline{a}_u^{(u)}\| .$$

Moreover, it is clear that

$$(3.16) \qquad S'NN'S = \left\| \begin{matrix} \rho_o v & & & 0 \\ & \rho_1 Q^{(1)} & & \\ & & \ddots & \\ 0 & & & \rho_m Q^{(m)} \end{matrix} \right\| .$$

Since

$$\left\| \begin{matrix} v & & & 0 \\ & Q^{(1)} & & \\ & & \ddots & \\ 0 & & & Q^{(i)} \end{matrix} \right\| \sim P_i, \qquad i = 1,\dots,m.$$

It is shown easily that

$$|Q^{(i)}| \sim |P_i|\,|P_{i-1}| \quad , \; i=1,\dots,m, \text{ with } |P_o| = v.$$

Hence, by using (3.13) we get

$$(3.17) \qquad |Q^{(i)}| \sim \prod_{j=o}^{i-1} [(i-j)(n-i+1-j)]^{\alpha_j} [\binom{n-2i}{m-i}]^{\alpha_i}, \; i=1,\dots,m$$

where, as before,

$$\alpha_u = \frac{n+1-2u}{n+1-u}\binom{n}{u} \qquad u = 1,\dots,m.$$

§4. Non-existence criteria of regular and symmetrical PBIB design of T_m type.

In the present section we give a set of necessary conditions for the existence of regular and symmetrical PBIBD's of T_m type, i.e.

$$v = b \text{ and hence } r = k$$

and (m+1) characteristic roots of NN' are given by

$$(4.1) \qquad \rho_o = r^2,\; \rho_u = \sum_{a=o}^{m} z_{ua}\,\lambda_a > 0,\; u = 1,\dots,m,$$

where, as before,

$$(4.2) \qquad z_{us} = \sum_{a=o}^{s} (-1)^{s-a} \binom{m-a}{m-s} \binom{m-u}{a} \binom{n-m-u+a}{a} .$$

From (3.15), (3.16) and (4.1), it follows that

$$(4.3) \qquad \begin{Vmatrix} Q^{(1)} & & 0 \\ & \ddots & \\ 0 & & Q^{(m)} \end{Vmatrix} \sim \begin{Vmatrix} \rho_1 Q^{(1)} & & 0 \\ & \ddots & \\ 0 & & \rho_m Q^{(m)} \end{Vmatrix} .$$

Therefore, by the Hasse theorem [5], it follows that

$$(4.4) \qquad \prod_{u=1}^{m} \rho_u^{\alpha_u} \sim 1$$

and

$$(4.5) \qquad \prod_{u=1}^{m} [(-1, \rho_u)_p^{\frac{\alpha_u(\alpha_u+1)}{2}} (\rho_u, |Q^{(u)}|)_p)] \prod_{1 \leq u < h \leq m} (\rho_u, \rho_h)_p^{\alpha_u \alpha_n} = 1,$$

for all primes p, where, as before

$$(4.6) \qquad |Q^{(u)}| \sim \prod_{j=o}^{u-1} [(u-j)(n-u+1-j)]^{\alpha_j} [\binom{n-2u}{m-u}^{\alpha_u}, \quad u=1,\ldots,m.$$

with

$$\alpha_j = \frac{n+1-2j}{n+1-j} \binom{n}{j}.$$

These are necessary conditions for the existence of regular and symmetrical PBIBD's of T_m type. In cases when m=2 and m=3, these coincide with conditions obtained by J. Ogawa [5] and K. Kusumoto [4] respectively.

§5. Examples of non-existent regular and symmetrical PBIBD's of T_m type.

The following designs can be seen to be non-existent by the criteria (4.4) and (4.5).

$m=3$, $n=7$, $v=b=35$, $r=k=8$, $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=2$, $\rho_1=1$, $\rho_2=6$, $\rho_3=9$.

$m=3$, $n=7$, $v=b=35$, $r=k=12$, $\lambda_1=4$, $\lambda_2=4$, $\lambda_3=3$, $\rho_1=11$, $\rho_2=6$, $\rho_3=9$

$m=3$, $n=19$, $v=b=969$, $r=k=57$, $\lambda_1=9$, $\lambda_2=3$, $\lambda_3=3$, $\rho_1=228$, $\rho_2=126$, $\rho_3=36$.

$m=4$, $n=8$, $v=b=70$, $r=k=13$, $\lambda_1=4$, $\lambda_2=2$, $\lambda_3=1$, $\lambda_4=4$, $\rho_1=33$, $\rho_2=3$, $\rho_3=3$, $\rho_4=9$.

$m=4$, $n=8$, $v=b=70$, $r=k=16$, $\lambda_1=6$, $\lambda_2=3$, $\lambda_3=2$, $\lambda_4=4$, $\rho_1=44$, $\rho_2=18$, $\rho_3=4$, $\rho_4=6$.

$m=4$, $n=9$, $v=b=70$, $r=k=17$, $\lambda_1=5$, $\lambda_2=4$, $\lambda_3=3$, $\lambda_4=0$, $\rho_1=33$, $\rho_2=9$, $\rho_3=13$, $\rho_4=9$.

$m=4$, $n=9$, $v=b=126$, $r=k=10$, $\lambda_1=1$, $\lambda_2=1$, $\lambda_3=0$, $\lambda_4=2$, $\rho_1=19$, $\rho_2=12$, $\rho_3=2$, $\rho_4=14$.

$m=4$, $n=9$, $v=b=126$, $r=k=15$, $\lambda_1=4$, $\lambda_2=1$, $\lambda_3=1$, $\lambda_4=6$, $\rho_1=27$, $\rho_2=41$, $\rho_3=1$, $\rho_4=7$.

$m=4$, $n=9$, $v=b=126$, $r=k=15$, $\lambda_1=2$, $\lambda_2=2$, $\lambda_3=1$, $\lambda_4=2$, $\rho_1=27$, $\rho_2=13$, $\rho_3=8$, $\rho_4=17$.

$m=4$, $n=9$, $v=b=126$, $r=k=16$, $\lambda_1=3$, $\lambda_2=2$, $\lambda_3=1$, $\lambda_4=4$, $\rho_1=31$, $\rho_2=24$, $\rho_3=4$, $\rho_4=16$.

$m=4$, $n=10$, $v=b=210$, $r=k=20$, $\lambda_1=5$, $\lambda_2=2$, $\lambda_3=1$, $\lambda_4=0$, $\rho_1=100$, $\rho_2=28$, $\rho_3=16$, $\rho_4=8$.

$m=4$, $n=10$, $v=b=210$, $r=k=23$, $\lambda_1=4$, $\lambda_2=3$, $\lambda_3=1$, $\lambda_4=4$, $\rho_1=64$, $\rho_2=40$, $\rho_3=1$, $\rho_4=25$,

$m=4$, $n=10$, $v=b=210$, $r=k=30$, $\lambda_1=5$, $\lambda_2=5$, $\lambda_3=3$, $\lambda_4=4$, $\rho_1=75$, $\rho_2=27$, $\rho_3=12$, $\rho\rho_4=32$.

$m=4$, $n=10$, $v=b=210$, $r=k=34$, $\lambda_1=13$, $\lambda_2=5$, $\lambda_3=3$, $\lambda_4=8$, $\rho_1=151$, $\rho_2=103$, $\rho_3=4$, $\rho_4=8$.

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