

A NOTE ON CONVERGENCE TO MIXTURES  
OF NORMAL DISTRIBUTIONS\*

BIOMATHEMATICS TRAINING PROGRAM

by

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ABSTRACT: Double arrays of random variables obtained by normalizing a sequence that is asymptotically close to a martingale difference sequence are considered, and conditions ensuring that the row sums converge in distribution to a mixture of normal distributions are found. The main condition is that the sums of squares in each row converge in probability to a random variable.

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1. INTRODUCTION.

Let  $\{X_{n,i} : 1 \leq i \leq k_n, n \geq 1\}$  be an array of random variables on a probability space  $(\Omega, \mathcal{B}, P)$  and let  $\mathcal{B}_{n,i}$  be the sub-sigmaalgebra that is generated by  $X_{n,1}, \dots, X_{n,i}$  ( $\mathcal{B}_{n,0} = \{\phi, \Omega\}$ ). If  $E(X_{n,i} | \mathcal{B}_{n,i-1}) = 0$ ,  $2 \leq i \leq k_n, n \geq 1$  then  $\{X_{n,i}\}$  is a martingale difference array (m.d.a.) and if furthermore it is obtained by normalizing a single sequence of martingale differences then

$$(1) \quad \mathcal{B}_{n,i} \subseteq \mathcal{B}_{n+1,i} \text{ for } n \geq 1, 1 \leq i \leq \min(k_n, k_{n+1}).$$

It is well known that if  $\sum_{i=1}^{k_n} X_{n,i}^2$  converges in probability to a constant ( $=\sigma$ , say) and certain other conditions are satisfied, then the distribution of  $\sum_{i=1}^{k_n} X_{n,i}$  converges to a normal distribution with mean zero and variance  $\sigma$  (see e.g. McLeish (1974)). In this note it is shown that if  $\sum_{i=1}^{k_n} X_{n,i}^2$  converges in probability to some random variable  $\xi$ , then under similar conditions the distribution of  $\sum_{i=1}^{k_n} X_{n,i}$  converges to a mixture of normal distributions. Eagleson (1975) observed that if  $\xi$  is measurable  $\mathcal{B}_1 = \bigcap_{n=1}^{\infty} \mathcal{B}_{n,1}$  then  $\{X_{n,i}\}$  is a martingale difference array (m.d.a.) also after conditioning on  $\xi$  and thus  $\sum_{i=1}^{k_n} X_{n,i} \xrightarrow{d} \phi(\cdot/\sqrt{\xi})^*$  under  $P(\cdot | \xi)$  (at least along subsequences satisfying  $\sum_{i=1}^{k_n} X_{n,i}^2 \xrightarrow{\text{a.s.}} \xi$ ). By taking expectations it follows that  $\sum_{i=1}^{k_n} X_{n,i} \xrightarrow{d} \int \phi(\cdot/\sqrt{x}) dF(x)$ , where  $F$  is the distribution of  $\xi$ . Hence the main result of this note is that the restriction that  $\xi$  is measurable  $\mathcal{B}_1$  is shown to be superfluous, but also our other conditions are somewhat weaker

\*  $\phi$  is the standard normal distribution function.

than those of Eagleson. (In fact Eagleson's results are phrased in terms of conditional variances instead of sums of squares, but the translation is straight-forward.) Finally, it is perhaps worth mention that the condition that  $\sum_{i=1}^{k_n} X_{n,i}^2$  converges in probability cannot be weakened very much: Dvoretzky (1972) has given an example which shows that it is not enough that  $\sum_{i=1}^{k_n} X_{n,i}^2$  converges in distribution.

## 2. CONVERGENCE TO MIXTURES OF NORMAL DISTRIBUTIONS.

We are mainly concerned with sums of martingale differences, but as is argued in Rootzén (1975) the interesting case is when the truncated variables asymptotically are martingale differences. Thus we initially don't assume that  $\{X_{n,i}\}$  is a m.d.a., but only that e.g.

$$(2) \quad \sum_{i=1}^{k_n} E\{X_{n,i} I(|X_{n,i}| \leq 1) | \mathcal{B}_{n,i-1}\} \xrightarrow{P} 0 \text{ as } n \rightarrow \infty.$$

Furthermore define

$$\xi_{n,i} = X_{n,i} I(|X_{n,i}| \leq 1) - E(X_{n,i} I(|X_{n,i}| \leq 1) | \mathcal{B}_{n,i-1}).$$

Then  $\{\xi_{n,i}\}$  is a m.d.a. and  $|\xi_{n,i}| \leq 2$  a.s.

THEOREM 1. Assume (1) and (2) are satisfied. If furthermore

$\max_{1 \leq i \leq k_n} |X_{n,i}| \xrightarrow{P} 0$  and there is a random variable  $\xi$  with distribution  $F$  such that

$$(3) \quad \sum_{i=1}^{k_n} \xi_{n,i}^2 \xrightarrow{P} \xi \text{ as } n \rightarrow \infty,$$

then  $\sum_{i=1}^{k_n} X_{n,i} \xrightarrow{d} \int \phi(\cdot/\sqrt{x}) dF(x)$  as  $n \rightarrow \infty$ .

PROOF. By definition  $\sum_{i=1}^{k_n} X_{n,i} - \sum_{i=1}^{k_n} \xi_{n,i} = \sum_{i=1}^{k_n} E\{X_{n,i} I(|X_{n,i}| \leq 1) \mid \mathcal{B}_{n,i-1}\} + \sum_{i=1}^{k_n} X_{n,i} I(|X_{n,i}| > 1)$  and thus from (2) and  $\max_{1 \leq i \leq k_n} |X_{n,i}| \xrightarrow{P} 0$  we obtain  $\sum_{i=1}^{k_n} X_{n,i} - \sum_{i=1}^{k_n} \xi_{n,i} \xrightarrow{P} 0$  as  $n \rightarrow \infty$ . Hence it is enough to prove

$$(4) \quad \sum_{i=1}^{k_n} \xi_{n,i} \xrightarrow{d} \int \phi(\cdot/\sqrt{x}) dF(x) \text{ as } n \rightarrow \infty.$$

Write  $X'_{n,i} = X_{n,i} I(|X_{n,i}| \leq 1)$ . Then  $1 \geq \max_{1 \leq i \leq k_n} |X'_{n,i}| \xrightarrow{P} 0$  and thus also

$E\{(\max_{1 \leq i \leq k_n} |X'_{n,i}|)^2\} \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\max_{1 \leq i \leq k_n} |E(X'_{n,i} \mid \mathcal{B}_{n,i-1})| \leq$

$\leq \max_{1 \leq i \leq k_n} E(\max_{1 \leq i \leq k_n} |X'_{n,i}| \mid \mathcal{B}_{n,i-1})$  and since  $E(\max_{1 \leq i \leq k_n} |X'_{n,i}| \mid \mathcal{B}_{n,i-1})$ ,

$i = 1, \dots, k_n$  is a martingale it follows from Kolmogorov's inequality for

martingales that  $\max_{1 \leq i \leq k_n} |E(X'_{n,i} \mid \mathcal{B}_{n,i-1})| \xrightarrow{P} 0$  and thus also  $\max_{1 \leq i \leq k_n} \xi_{n,i} \xrightarrow{P} 0$  as  $n \rightarrow \infty$ .

To facilitate the remainder of the proof we will without loss of generality assume that each row in the m.d.a.  $\{\xi_{n,i}\}$  is infinite and that for  $i > k_n$ ,

$\xi_{n,i} = \pm \frac{1}{n}$  with probability  $\frac{1}{2}$  each and independently of  $\mathcal{B}_{n,i-1}$ . Hence we

have  $\max_{1 \leq i} |\xi_{n,i}| \xrightarrow{P} 0$  as  $n \rightarrow \infty$  and  $\sum_{i=1}^k \xi_{n,i}^2 \xrightarrow{a.s.} \infty$  as  $k \rightarrow \infty$ , for each  $n$ .

Let  $S_n(k) = \sum_{i=1}^k \xi_{n,i}$  and introduce "the natural time-scale"

$\tau_n(t) = \inf\{k: \sum_{i=1}^k \xi_{n,i}^2 > t\}$  of the summation process based on  $\{\xi_{n,i}\}$ . Then

$$(5) \quad \sum_{i=1}^{k_n} \xi_{n,i} = S_n \circ \tau_n \left( \sum_{i=1}^{k_n} \xi_{n,i}^2 \right) - \xi_{n,k_n+1},$$

and by construction  $|\xi_{n,k_n+1}| = \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ .

The next step is to prove

$$(6) \quad d_n = |S_n \circ \tau_n \left( \sum_{i=1}^{k_n} \xi_{n,i}^2 \right) - S_n \circ \tau_n(\xi)| \xrightarrow{P} 0 \text{ as } n \rightarrow \infty.$$

For  $\epsilon > 0$  choose  $K$  such that  $P(\{\xi > K\}) < \epsilon$  and observe that for any  $\delta > 0$

$$P(\{d_n > \epsilon\}) \leq P(\{d_n > \epsilon\} \cap \{|\sum_{i=1}^{k_n} \xi_{n,i}^2 - \xi| \leq \delta\} \cap \{\xi \leq K\}) \\ + P(\{|\sum_{i=1}^{k_n} \xi_{n,i}^2 - \xi| > \delta\}) + \epsilon.$$

Here

$$P(\{d_n > \epsilon\} \cap \{|\sum_{i=1}^{k_n} \xi_{n,i}^2 - \xi| \leq \delta\} \cap \{\xi \leq K\}) \leq P(\{\sup_{\substack{|s-t| \leq \delta \\ 0 \leq s, t \leq K+\delta}} |S_n \circ \tau_n(t) - S_n \circ \tau_n(s)| > \epsilon\}) = P_n, \quad \text{say.}$$

Since  $\max_{1 \leq i \leq \tau_n(K+1)} |\xi_{n,i}| \leq \max_{1 \leq i} |\xi_{n,i}| \xrightarrow{P} 0$  as  $n \rightarrow \infty$  it follows from Theorem 1 of Rootzén (1975) that  $\{S_n \circ \tau_n(t); t \in [0, K+1]\}_{n=1}^{\infty}$ , considered as random variables in  $D(0, K+1)$  endowed with the Skorokhod topology, converge in distribution to a Brownian motion on  $[0, K+1]$ . Hence, by Theorem 15.2 of Billingsley (1968) we can make  $\limsup P_n < \epsilon$  by choosing  $\delta$  small enough. Furthermore, by (3),  $P(\{|\sum_{i=1}^{k_n} \xi_{n,i}^2 - \xi| > \delta\}) \rightarrow 0$  as  $n \rightarrow \infty$ , so

$$\limsup_{n \rightarrow \infty} P(\{d_n > \epsilon\}) \leq 2\epsilon,$$

which since  $\epsilon$  is arbitrary proves (6).

We now need the following lemma, the proof of which is given on p. 5.

LEMMA 2. If for some integer  $k_\epsilon$  the random variable  $\xi_\epsilon \geq 0$  is measurable  $\mathcal{B}_{n, k_\epsilon}$  for all large  $n$  then

$$(7) \quad S_n \circ \tau_n(\xi_\epsilon) \xrightarrow{d} \int \Phi(\cdot/\sqrt{x}) dF_\epsilon(x) \quad \text{as } n \rightarrow \infty,$$

where  $F_\epsilon$  is the distribution of  $\xi_\epsilon$ .

Since we have assumed (1) it is for  $\epsilon > 0$  possible to find an integer  $k_\epsilon$  and a random variable  $\xi_\epsilon \geq 0$ , which is measurable with respect to  $\mathcal{B}_{n, k_\epsilon}$

for all large  $n$ , such that  $P(|\xi - \xi_\epsilon| > \epsilon) \leq \epsilon$  (see e.g. Doob (1953), p. 607). From Lemma 2 we obtain that (7) holds and since  $F_\epsilon \xrightarrow{d} F$  as  $\epsilon \rightarrow 0$  also

$$(8) \quad \int \phi(\cdot/\sqrt{x}) dF_\epsilon(x) \xrightarrow{d} \int \phi(\cdot/\sqrt{x}) dF(x) \text{ as } \epsilon \rightarrow 0.$$

Moreover, by arguments precisely analogous to the proof of (6) above, we have for  $\delta > 0$  that

$$(9) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} P(\{|S_{n \circ \tau_n}(\xi_\epsilon) - S_{n \circ \tau_n}(\xi)| > \delta\}) = 0.$$

By Theorem 4.2 of Billingsley (1968) it follows from (7), (8), and (9) that

$$S_{n \circ \tau_n}(\xi) \xrightarrow{d} \int \phi(\cdot/\sqrt{x}) dF(x) \text{ as } n \rightarrow \infty,$$

which by (5) and (6) proves (4) and thus the theorem.  $\square$

PROOF OF LEMMA 2. The lemma can be reduced to the corollary on p. 561 of Eagleson (1975), but as it doesn't involve more work to give a direct proof we will do that instead. Put  $\xi'_{n,i} = \xi_{n,i}$  if  $i > k_\epsilon$  and  $\xi'_{n,i} = 0$  otherwise, let  $S'_n(k) = \sum_{i=1}^k \xi'_{n,i}$  and let  $P_\epsilon(\cdot) = P(\cdot | \xi_\epsilon)$  be a regular conditional probability given  $\xi_\epsilon$ . Since  $\xi_\epsilon \in B_{n,k_\epsilon}$  for all large  $n$ , the rows of  $\{\xi'_{n,i}\}$  are martingale differences for such  $n$  also under  $P_\epsilon$  (a.s.), cf. ref [4]. Moreover, for  $t \geq 0$

$$(10) \quad \left| \sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 - t \right| \leq \sum_{i=1}^{k_\epsilon} \xi_{n,i}^2 + \xi_{n,\tau_n(t)}^2 \leq (k_\epsilon + 1) (\max_{1 \leq i} |\xi_{n,i}|)^2.$$

As  $\max_{1 \leq i} |\xi_{n,i}| \xrightarrow{P} 0$  we can for every subsequence of  $\{1, 2, \dots\}$  find a further (infinite) subsequence  $\{n'\}$  such that  $\max_{1 \leq i} |\xi_{n',i}| \xrightarrow{\text{a.s.}} 0$ . Thus also under  $P_\epsilon$  we have a.s. that  $\max_{1 \leq i} |\xi'_{n',i}| \xrightarrow{\text{a.s.}} 0$  and  $\sum_{i=1}^{\tau_n(t)} \xi_{n',i}^2 \xrightarrow{\text{a.s.}} t$ . By

Lemma 3 of [5] this proves  $S'_{n', \tau_{n'}}(t) \xrightarrow{d} \Phi(\cdot/\sqrt{t})$  under  $P_\epsilon$  (a.s.). Since  $\xi_\epsilon$  is a constant under  $P_\epsilon$  we have in particular that  $S'_{n', \tau_{n'}}(\xi_\epsilon) \xrightarrow{d} \Phi(\cdot/\sqrt{\xi_\epsilon})$  ( $P_\epsilon$ ), a.s., and thus by taking expectations that

$$(11) \quad S'_{n', \tau_{n'}}(\xi_\epsilon) \xrightarrow{d} \int \Phi(\cdot/\sqrt{x}) dF_\epsilon(x) \quad \text{as } n' \rightarrow \infty.$$

Since a subsequence  $\{n'\}$  satisfying (11) can be extracted from any infinite sequence of integers it follows that  $S'_{n, \tau_n}(\xi_\epsilon) \xrightarrow{d} \int \Phi(\cdot/\sqrt{x}) dF(x)$  as  $n \rightarrow \infty$ , which since  $|S_{n, \tau_n}(\xi_\epsilon) - S'_{n, \tau_n}(\xi_\epsilon)| \leq k_\epsilon \max_{1 \leq i} |\xi_{n,i}| \xrightarrow{P} 0$  proves (7).  $\square$

COROLLARY 3. Assume  $\{X_{n,i}\}$  is a m.d.a. and that it satisfies (1). If  $\max_{1 \leq i \leq k_n} |X_{n,i}| \xrightarrow{P} 0$ , if  $\sum_{i=1}^{k_n} E(X_{n,i}^2 I(|X_{n,i}| > 1) | \mathcal{B}_{n,i-1}) \xrightarrow{P} 0$ , and if furthermore  $\sum_{i=1}^{k_n} X_{n,i}^2 \xrightarrow{P} \xi$  for some random variable  $\xi$  with distribution  $F$ , then  $\sum_{i=1}^{k_n} X_{n,i} \xrightarrow{d} \int \Phi(\cdot/\sqrt{x}) dF(x)$  as  $n \rightarrow \infty$ .

PROOF. We only have to show that (2) and (3) hold. Recall  $X'_{n,i} = X_{n,i} I(|X_{n,i}| \leq 1)$  and put  $X''_{n,i} = X_{n,i} - X'_{n,i}$ . Since  $E(X'_{n,i} | \mathcal{B}_{n,i-1})^2 \leq |E(X'_{n,i} | \mathcal{B}_{n,i-1})| = |E(X''_{n,i} | \mathcal{B}_{n,i-1})| \leq E(X''_{n,i}^2 | \mathcal{B}_{n,i-1})$  it follows immediately that (2) holds. Moreover, by definition

$$\begin{aligned} \sum_{i=1}^{k_n} \xi_{n,i}^2 &= \sum_{i=1}^{k_n} X_{n,i}^2 + \sum_{i=1}^{k_n} \{X''_{n,i}^2 + E(X''_{n,i} | \mathcal{B}_{n,i})^2\} \\ &\quad - 2 \sum_{i=1}^{k_n} X_{n,i} \{X''_{n,i} + E(X''_{n,i} | \mathcal{B}_{n,i})\} \\ &= \sum_{i=1}^{k_n} X_{n,i}^2 + r_n - 2 r'_n, \text{ say.} \end{aligned}$$

Here  $r_n \xrightarrow{P} 0$  as  $n \rightarrow \infty$  and using Cauchy's inequality we obtain

$$|r'_n| \leq \{2r_n \sum_{i=1}^{k_n} X_{n,i}^2\}^{1/2} \xrightarrow{P} 0 \text{ as } n \rightarrow \infty, \text{ so (3) follows} \quad \square$$



Finally, it should perhaps be mentioned that the results of this note hold also when  $\{k_n\}$  are stopping times.

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