ON A CLASS OF NONPARAMETRIC TESTS FOR BIVARIATE INTERCHANGEABILITY*

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SUMMARY

A class of nonparametric tests is developed here for testing the hypothesis of interchangeability of the two variates in a sample from a bivariate population with an unknown distribution function. The classes of alternative hypotheses relate to possible differences in location and/or Scale parameters of the two variates, and in this light, the performance characteristics of the proposed tests are studied and compared with those of the standard parametric tests for the same problems (based on the assumption of bivariate normal distribution).

1. INTRODUCTION

In a random sample of size n drawn from a bivariate population having a continuous cumulative distribution function (cdf) of unspecified form, a class of nonparametric rank order tests is proposed and studied for testing the null hypothesis of interchangeability of the two variates against admissible families of alternative hypotheses which relate to shifts in locations or difference in dispersions of the two variates. In this context, the concept of rank permutation tests is developed, and with the aid of this, a class of genuinely distribution free rank order tests is constructed. Further, the well known theorem of Chernoff and Savage (1958) [also see Govindarajulu etal (1965)] on the asymptotic normality of a celebrated class of nonparametric test-statistics (in the case of two independent samples) is extended here to the bivariate (or paired sample) case. This takes care of the study of the asymptotic properties of the proposed class of tests.

Certain variational cases are also considered here. First, the n sample observations are drawn not from a common population but from distributions which may possibly differ from each other by shifts in location only. As we shall see later on that this situation often arises in connection with design of experiments involving only a pair of treatments which are replicated in two or more blocks. Secondly, the two variates may be actually interchangeable only when each of them is normalized by suitable location and scale parameters which are not known. In such a case, one may be interested in testing the identity of locations without assuming the scales to be the same, or in testing the equality of scales without assuming the equality of locations. These problems may be regarded as the nonparametric generalizations of the problems of paired t-test and Pitman-Morgan test for the equality of variances, respectively. Finally, in many psychometric problems, tests for interchangeability of two or more correlated variates arise in connection with testing the parallelity of two or more tests (in a battery of tests administered on a common batch of subjects). Wilks (1946) has proposed and studied the likelihood ratio test criteria L_{MVC}, L_M and L_{VC} for the hypotheses H_{MVC}, H_M and H_{VC} respectively, where H stands for the hypothesis of equality, M for the means, V for the variances and C for the covariances. Nonparametric generalizations of these likelihood ratio tests are considered here only for the bivariate case, while the more general case of $p(\geq 2)$ variates will be considered in a separate communication.

2. STATISTICAL FORMULATION OF THE PROBLEMS

Let $\underset{\sim}{X}_\alpha = (X_{1\alpha}, X_{2\alpha})$ be a bivariate random variable drawn from a population having a continuous cdf $F_\alpha(x_1, x_2)$, and let $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ be n independent random variables distributed according to the cdf's $F_1, \ldots, F_n$ respectively, where these $F_1, \ldots, F_n$ need not be identical. Let now Ω be the set of all continuous bivariate cdf's, and it is assumed that

$$F_\alpha \in \Omega \quad \text{for all } \alpha = 1, \ldots, n. \tag{2.1}$$

Let us denote the two dimensional Eucledian space by R^2, and let ω be a subset of Ω for which

$$F_\alpha(x_1, x_2) = F_\alpha(x_2, x_1) \text{ for all } (x_1, x_2) \in R^2 \tag{2.2}$$

If (2.2) holds, we say that $X_{1\alpha}$ and $X_{2\alpha}$ are interchangeable or $\underset{\sim}{X}_\alpha$ is an interchangeable vector. We are interested in testing the null hypothesis

$$H_0: \; F_\alpha \in \omega \text{ for all } \alpha = 1, \ldots, n, \tag{2.3}$$

against various types of admissible alternatives. Now, in testing the null hypothesis (2.3), we may have either of the following two situations. First, the conventional case where $\underset{\sim}{X}_\alpha$, $\alpha = 1, \ldots, n$ are n independent and identically distributed bivariate random variables (i. i. d. b. r. v.) distributed according to a common cdf $F \in \Omega$, and we want to test the null hypothesis

$$H_0: \; F \in \omega \; . \tag{2.4}$$

Secondly, $\{X_\alpha\}$'s are independent but may not be identically distributed. Often, in such a case, we have

$$\underset{\sim}{X}_\alpha = Z_\alpha \underset{\sim}{I}_2 + \underset{\sim}{Y}_\alpha , \quad \alpha = 1, \ldots, n \tag{2.5}$$

where $\underset{\sim}{I}_2$ is a 2-vector with unit elemants, $\{Z_\alpha, \; \alpha = 1, \ldots, n\}$ are n real quantities which may or may not be stochastic in nature, and $\underset{\sim}{Y}_\alpha = (Y_{1\alpha}, Y_{2\alpha})$ are such that under the null hypothesis to be tested, they are interchangeable, for each $\alpha = 1, \ldots, n$. Situation (2.5) arises in design of experiments involving only a pair of treatments which are replicated in two or more replicates. The replicate effects under additive model correspond to Z_α, $\alpha = 1, \ldots, n$, while the treatment cum error components are $(Y_{1\alpha}, Y_{2\alpha})$, $\alpha = 1, \ldots, n$. Under the assumption of no treatment effect, $\underset{\sim}{Y}_\alpha$ will be an interchangeable vector, and hence to test the null hypothesis of no treatment effect, we may desire to test for the interchangeability of $\underset{\sim}{Y}_\alpha$ without presuming the knowledge of Z_α, $\alpha = 1, \ldots, n$.

In testing the null hypothesis (2.3), the class of alternatives we are usually interested, relates to possible differences in locations or dispersions of the two variates. To pose these, we let $F_\alpha = F$ for all $\alpha = 1, \ldots, n$, where

$$F(x_1, x_2) = F_0([x_1 - \mu_1]/\delta_1, \; [x_2 - \mu_2]/\delta_2); \tag{2.6}$$

the cdf $F_o(u, v)$ being assumed to be invariant under interchange of its arguments, for all $u, v \in R^2$, and $\underset{\sim}{\mu} = (\mu_1, \mu_2)$ and $\underset{\sim}{\delta} = (\delta_1, \delta_2)$ being respectively the location and the scale vector with real elements. In the classical location problems, we are interested in testing H_o in (2.4) under the assumption $\delta_1 = \delta_2$ i. e., we want to test for the identity of locations μ_1 and μ_2 assuming the equality of δ_1 and δ_2. Similarly, in the classical scale problem, we want to test for the equality of δ_1 and δ_2 in (2.6) assuming the equality of μ_1 and μ_2. In the studentized location problem, we shall be interested in testing the null hypothesis μ_1 and μ_2 without assuming the identity of δ_1 and δ_2. Similarly, in the general scale problem, we are interested in testing the equality of δ_1 and δ_2 without assuming the equality of μ_1 and μ_2 or their difference to be known.

Finally, referred to (2.6), we may be also interested in testing the compound hypothesis

$$H_o: \mu_1 = \mu_2, \ \delta_1 = \delta_2 \tag{2.7}$$

against the set of alternatives that at least one of the two equalities does not hold. This will be the nonparametric analogue of Wilk's (1946) H_{MVC} in the bivariate case.

Suitable families of rank order tests are offered for each of the above hypotheses and the performance characteristics of these tests are studied and compared with these of the corresponding standard parametric tests based on the assumption of bivariate normal distribution.

3. RANK ORDER TESTS FOR I. I. D. B. R. V. SET

Let $\underset{\sim}{X}_\alpha = (X_{1\alpha}, X_{2\alpha})$, $\alpha = 1, \ldots, n$ be n i. i. d. b. r. v. 's distributed according to a common continuous cdf $F(x_1, x_2)$. We pool these n paired observations into a combined sample of size $N(= 2n)$, and denote these 2n observations by

$$\underset{\sim}{Z}_N = (Z_{N,1}, \ldots, Z_{N,N}) \tag{3.1}$$

where we adopt the convention that

$$Z_{N,\,2\alpha-1} = X_{1\alpha},\ Z_{N,2\alpha} = X_{2\alpha},\ \alpha = 1, \ldots, n. \tag{3.2}$$

Let us arrange the N observations in (3.1) in order of magnitude and denote them by

$$Z_{N(1)} < \ldots < Z_{N(N)}; \tag{3.3}$$

by virtue of the assumed continuity of $F(x_1, x_2)$, the possibility of ties in (3.3) may be ignored, in probability. Now, we consider a sequence of rank functions (which are real valued functions of N,) denoted as

$$E_N = (E_{N,1}, \ldots, E_{N,\,N}), \tag{3.4}$$

$E_{N,i}$ being an explicit function of i and N, for i = 1, ..., N, and defined for each N = 2, 4, ... Let us also define

$$C_{N,\,i} = \begin{cases} 1 \text{ if } Z_{N(i)} \text{ is an } X_{1\alpha}, \alpha = 1, \ldots, n; \\ 0, \text{ otherwise; for } i = 1, \ldots, N. \end{cases} \tag{3.5}$$

Let then

$$\underset{\sim}{C}_N = (C_{N,1}, \ldots, C_{N,N}), \tag{3.6}$$

so that

$$\underset{\sim}{C}_N \cdot \underset{\sim}{C}_N = \sum_{i=1}^{N} C_{N,i}^2 = \sum_{i=1}^{N} C_{N,i} = n. \tag{3.7}$$

Then, we consider a statistic

$$T_n = \underset{\sim}{C}_N \cdot \underset{\sim}{E}_N \,/\, \underset{\sim}{C}_N \cdot \underset{\sim}{C}_N = \frac{1}{n} \sum_{i=1}^{N} E_{N,i}\, C_{N,i}\,. \tag{3.8}$$

Our proposed test is then based on the rank order statistic T_n, and by considering various possible $\underset{\sim}{E}_N$ (with emphasis on different classes of alternative hypotheses) we will get a class of tests which will be studied here in detail.

Now, referred to (3.3), the rank of $X_{i\alpha}$ is denoted by $R_{i\alpha}$ for i = 1, 2, $\alpha = 1, \ldots, n$. Thus, corresponding to the vector $\underset{\sim}{X}_\alpha = (X_{1\alpha}, X_{2\alpha})$ we have a rank-pair

$$R_\alpha = (R_{1\alpha}, R_{2\alpha}),\ \alpha = 1, \ldots, n. \tag{3.9}$$

Then $(R_1, \ldots, R_n)$ will be a permutation of the numbers $(1, \ldots, N)$. We denote by R'_α the column vector corresponding to R_α in (3.9), and define a 2 x n matrix

$$\underset{\sim}{R}_N = (R'_1, \ldots, R'_n) \tag{3.10}$$

which we term the collection rank matrix. The matrix $\underset{\sim}{R}_N$ has 2n (=N) elements which are all distinct and which form a permutation of the numbers (1, ..., N). The matrix thus consists of n random rank-pairs which constitute its columns. Thus, $\underset{\sim}{R}_N$ is itself a stochastic matrix. Two such collection matrices $\underset{\sim}{R}_N$ and $\underset{\sim}{R}^*_N$ (Say,) are said to equivalent if it is possible to arrive at $\underset{\sim}{R}_N$ from $\underset{\sim}{R}_N{}^*$ by a finite number of inversions of the columns of the later. This equivalence has some statistical significance too. This means that if instead of taking the observation vector as $(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$, we take it as $(\underset{\sim}{X}_{i_1}, \ldots, \underset{\sim}{X}_{i_n})$ where $(i_1, \ldots, i_n)$ is any permutation of (1, ..., n), then the two collection rank matrices corresponding to the two observation vectors will be equivalent, (as it should be). In this manner, we achieve the uniqueness of the collection rank matrix for any given $(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$. The total number of possible realizations of $\underset{\sim}{R}_N$ (on non-equivalent sets only) is evidently equal to (2n)!/n! , and the set of all such possible realizations of $\underset{\sim}{R}_N$ is denoted by $\underset{\sim}{\ }_N$. Thus, any observed $\underset{\sim}{\ }_N$ lies in $\underset{\sim}{\ }_N$. Now, $\underset{\sim}{\ }_N$ is a 2 x n stochastic matrix with column vectors $R'_1, \ldots, R'_m$. If we permute the two rank elements within each of the n colums, we will get a set of 2^n possible rank matrices which can be obtained from $\underset{\sim}{R}_N$. This set is denoted by $S(\underset{\sim}{R}_N)_2$ so that $\underset{\sim}{R}_N \epsilon\ S(\underset{\sim}{R}_N)$. On the other-hand, $S(\underset{\sim}{\ }_N)$ is a subset of $\underset{\sim}{\ }_N$, there being $[(2n)!/\ 2^n \cdot n!]$ such subsets. Thus, we get

$$\underset{\sim}{}_{N} \in S(R_N) \subset \underset{\sim}{}_{N} \tag{3.11}$$

The set $S(\underset{\sim}{R}_N)$ will be termed the permutation-set of $\underset{\sim}{R}_N$.

The probability distribution of $\underset{\sim}{}_N$ over $\underset{\sim}{}_N$ (defined on an additive class of subsets A_N of $\underset{\sim}{}_N$) will evidently depend on the parent cdf F even under the null hypothesis (2.4). However, if the null hypothesis H_o in (2.4) holds, given $\underset{\sim}{X}_\alpha$, the first variate may assume either of the two values $X_{1\alpha}$, $X_{2\alpha}$ with equal probability (i.e., $\frac{1}{2}$) for all $\alpha = 1, \ldots, n$. Hence, given $\underset{\sim}{R}_N$, the variable $X_{1\alpha}$ can have either of the two ranks $R_{1\alpha}$, $R_{2\alpha}$ with equal probability. Thus, given $\underset{\sim}{R}_N$, the conditional distribution (under H_o) of the ranks of $(X_{1\alpha}, \alpha = 1, \ldots, n)$ over 2^n possible realizations (obtained by intra-pair permutations) would be uniform. This implies that, if corresponding to the given rank matrix $\underset{\sim}{R}_N$ we consider the permutation set $S(\underset{\sim}{R}_N)$, then given $S(\underset{\sim}{R}_N)$ there will be 2^n possible realizations of $\underset{\sim}{R}_N$ which are conditionally equally likely, viz.,

$$P\{\underset{\sim}{R}_N \mid S(\underset{\sim}{R}_N), H_o\} = 2^{-n}, \tag{3.12}$$

for any $S(\underset{\sim}{R}_N)$. Thus, if we consider the statistic T_n in (3.8), it follows from the above discussion that given the set $S(\underset{\sim}{R}_N)$ there will be 2^n equally likely (conditionally) realization of $\underset{\sim}{R}_N$ which in accordance with (3.8) will lead to 2^n realizations of T_n. This set of values of T_n is denoted by $T_n[S(\underset{\sim}{R}_N)]$. Thus, from (3.12), we get that conditionally on $T_n[S(\underset{\sim}{R}_N)]$, the permutation distribution of T_n over the 2^n elements of this set (not necessarily all distinct) would be uniform when H_o in (2.4) holds. Let us denote this permutational probability measure by $\mathcal{P}_n$, and consider a test function $\phi(\underset{\sim}{Z}_N)$ which with the aid of $\mathcal{P}_n$ associates to each $\underset{\sim}{Z}_N$ a probability of rejecting H_o in (2.4), so that $0 \leq \phi \leq 1$. Since $\mathcal{P}_n$ is a completely known probability measure, we can always select $\phi(\underset{\sim}{Z}_N)$ such that

$$E\{\phi(\underset{\sim}{Z}_N) \mid \mathcal{P}_n\} = \varepsilon : 0 < \varepsilon < 1, \tag{3.13}$$

$\mathcal{E}$ being the preassigned level of significance of our test. (3.13) implies that $E\{\phi(\underset{\sim}{Z}_N) \mid H_o\} = \epsilon$. Hence, it follows from a well-known result of Lehmann and Stein (1949) that $\phi(\underset{\sim}{Z}_N)$ has the property of $S(\epsilon)$-Structure of tests and hence is a strictly distribution free similar test of Size $\mathcal{E}$.

In actual practice, if n is not large, then corresponding to our given $\underset{\sim}{Z}_N$ we have a rank-matrix $\underset{\sim}{R}_N$ whose permutation set $S(\underset{\sim}{R}_N)$ will then consist of 2^n elements $\{\underset{\sim}{R}_N\}$, and corresponding to these elements we have the set of 2^n values of T_n which is denoted by $T_n[S(\underset{\sim}{R}_N)]$. Thus, conditioned on the given $T_n[S(\underset{\sim}{R}_N)]$, we can find out the permutation distribution function of T_n (over the 2^n elements in this set), and the same may be utilized to construct the exact test $\phi(\underset{\sim}{Z}_N)$. However, if n is large, the labour involved in this computation increases considerably, and as in the case of majority of other permutation tests (for various other problems in inference,) we shall develop first the asymptotic permutation distribution theory of T_n and use the same for the construction of large sample permutation tests.

4. ASYMPTOTIC PERMUTATION DISTRIBUTION OF T_n

For this study we shall impose certain regularity conditions on $\underset{\sim}{E}_N$ in (3.4) as well as on the parent cdf F. Extending the idea of Chernoff and Savage (1958) under the relaxed conditions of Govindavajulu et al (1965), we define

$$E_{N\alpha} = J_N\left(\frac{\alpha}{N+1}\right), \quad \alpha = 1, \ldots, N, \tag{4.1}$$

where the function J_N need be defined only at $i/(N+1)$, for $i = 1, \ldots, N$ and its domain of definition may be extended to (0, 1) by the Chernoff-Savage convention. Let us then define

$$F_{N[i]}(x) = \frac{1}{n}[\text{Number of } X_{i\alpha} \leq x], \; i = 1, 2; \tag{4.2}$$

$$H_N(x) = \frac{1}{2}[F_{N[1]}(x) + F_{N[2]}(x)]; \tag{4.3}$$

$$F_N(x_1, x_2) = \frac{1}{n}[\text{Number of } (X_{1\alpha}, X_{2\alpha}) \leq (x_1, x_2)] \tag{4.4}$$

Again, let $F_{[i]}(x)$ be the marginal cdf of $X_{i\alpha}$, $i = 1, 2$, and let

$$H(x) = \frac{1}{2}[F_{[1]}(x) + F_{[2]}(x)] \tag{4.5}$$

For the study of the asymptotic permutation distribution of T_n, we shall assume that

(c.1) $J(H) = \lim_{N = \infty} J_N(H)$ exists for $0 < H < 1$ and is not a constant.

(c.2) $$\frac{1}{N} \sum_{\alpha=1}^{N} \left[J_N\left(\frac{\alpha}{N+1}\right) - J\left(\frac{\alpha}{N+1}\right) \right] = o(N^{-\frac{1}{2}}), \tag{4.6}$$

and $$\int_{-\infty}^{\infty} \left[J_N\left(\frac{N}{N+1} H_N(x)\right) - J\left(\frac{N}{N+1} H_N(x)\right) \right] dF_{N[1]}(x) = o_p(N^{-\frac{1}{2}}). \tag{4.7}$$

(c.3) $J(H)$ is absolutely continuous in H: $0 < H < 1$, and

$$|J^{(i)}(H)| = \left| \frac{d^i}{dH^i} J(H) \right| \leq K[H(1-H)]^{-i-\frac{1}{2}+\delta}, \tag{4.8}$$

for $i = 0, 1$ and some $\delta > 0$, where K is a constant. In addition to these three conditions, we require the following two conditions only for the asymptotic convergence and non-nullity of the variance of the permutation distribution of $n^{\frac{1}{2}}T_n$.

(c.4) $$\frac{1}{N} \sum_{\alpha=1}^{N} \left[J_N^2\left(\frac{\alpha}{N+1}\right) - J^2\left(\frac{\alpha}{N+1}\right) \right] = o(1); \tag{4.9}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[J_N\left(\frac{N}{N+1} H_N(x)\right) J_N\left(\frac{N}{N+1} H_N(y)\right) - J\left(\frac{N}{N+1} H_N(x)\right) J\left(\frac{N}{N+1} H_N(y)\right) \right] dF_N(x, y) = o_p(1) \tag{4.10}$$

Also, we define

$$\nu_{ii}(F) = \int_{-\infty}^{\infty} J^2(H(x))\, dF_{[i]}(x) \text{ for } i = 1, 2; \tag{4.11}$$

$$\nu_{12}(F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(H(x))\, J(H(y))\, dF(x, y), \tag{4.12}$$

and let

$$\underset{\sim}{\nu}(F) = \begin{pmatrix} \nu_{11}(F), & \nu_{12}(F) \\ \nu_{12}(F) & \nu_{22}(F) \end{pmatrix} \tag{4.13}$$

(c.5) Let Ω_o be the set of all continuous bivariate cdf's, for which $\underset{\sim}{\nu}(F)$ in (4.13) is positive definite. Then we assume that

$$F \in \Omega_o \subset \Omega. \tag{4.14}$$

For any finite N, let us define

$$\bar{E}_N = \frac{1}{N}\sum_{\alpha=1}^{N} E_{N\alpha,\alpha} = \int_{-\infty}^{\infty} J_N\left(\frac{N}{N+1} H_N(x)\right) dH_N(x). \tag{4.15}$$

Using conditions (c.2) and (c.3) and defining $\mu = \int_0^1 J(H)\, dH$, we readily get that

$$\bar{E}_N - \mu = o(N^{-\frac{1}{2}}) \tag{4.16}$$

Let us also define

$$\nu^2 = \int_0^1 J^2(H)\, dH, \tag{4.17}$$

which by (c.1) and (c.3) will be a finite non-null quantity. Finally, we define

$$\sigma_N^2 = \frac{1}{N}\sum_{\alpha=1}^{n} [E_{N,R_{1\alpha}} - E_{N,R_{2\alpha}}]^2 \tag{4.18}$$

Using then (3.8) and (3.12), we get readily that

$$E(T_n \mid \mathcal{P}_n) = \bar{E}_N = \mu + o(N^{-\frac{1}{2}}), \tag{4.19}$$

$$V(T_n \mid \mathcal{P}_n) = \frac{1}{N}\sigma_N^2. \tag{4.20}$$

Now, from (4.5), (4.11) and (4.17) we get that

$$0 < \nu_{11}(F) + \nu_{22}(F) = \int_{-\infty}^{\infty} J^2(H(x))\, d\,[F_{[1]}(x) + F_{[2]}(x)]$$

$$= 2\int_0^1 J^2(H)\, d\,H = 2\nu^2 < \infty. \tag{4.21}$$

THEOREM 4.1 If $F \in \Omega_o$ and the conditions (c.1), (c.2), (c.3) and (c.4), (c.5) hold, then

$$\sigma_N^2 \xrightarrow{P} \frac{1}{2}[\nu_{11}(F) + \nu_{22}(F) - 2\nu_{12}(F)] = \nu^2 - \nu_{12}(F) > 0.$$

PROOF From (4.18) we get that

$$\sigma_N^2 = \frac{1}{N}\sum_{\alpha=1}^{N} E_{N,\alpha}^2 - \frac{2}{N}\sum_{\alpha=1}^{N} E_{N,R_{1\alpha}} E_{N,R_{2\alpha}} \tag{4.22}$$

Also, from (4.1), (4.3), (4.9) and condition (c.3) it is easily seen that

$$\frac{1}{N}\sum_{\alpha=1}^{N} E_{N,\alpha}^2 = \int_{\infty}^{\infty} J_N^2\left(\frac{N}{N+1} H_N(x)\right) dH_N(n) = \int_{\infty}^{\infty} J^2\left(\frac{N}{N+1} H_N(x)\right) dH_N(x) + o(1)$$

$$= \int_0^1 J^2(H)\, dH + o(1) = \nu^2 + o(1). \tag{4.23}$$

So we require to show that the second term on the right hand side of (4.22) converges to $\nu_{12}(F)$. Now, using (4.1), (4.3) and (4.12), the same can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_N\left(\frac{N}{N+1} H_N(x)\right) J_N\left(\frac{N}{N+1} H_N(y)\right) dF_N(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J\left(\frac{N}{N+1} H_N(x)\right) J\left(\frac{N}{N+1} H_N(y)\right) dF_N(x, y) + o_p(1) \tag{4.24}$$

Also, using (4.4) and (4.5), we have

$$\sup_x N^{\frac{1}{2}} \left| \frac{N}{N+1} H_N(x) - H(x) \right| = \sup_x N^{\frac{1}{2}} \left| \sum_{i=1}^{2} \left\{ \frac{N}{N+1} F_{N[i]}(x) - F_{[i]}(x) \right\} / 2 \right|$$

$$\leq 2^{-\frac{1}{2}} \left[\sum_{i=1}^{2} \left\{ \sup_x n^{\frac{1}{2}} \left| \frac{2n}{2n+1} F_{N[i]}(x) - F_{[i]}(x) \right| \right\} \right] \tag{4.25}$$

Thus, using the wellknown results on the univariate Kolmogorov-Smirnov statistic we get from (4.25) that

$$\sup_x N^{\frac{1}{2}} \left| \frac{N}{N+1} H_N(x) - H(x) \right| \text{ is bounded in probability.} \tag{4.26}$$

Again by simple algebraic manipulations we get that

$$0 \leq F_{N[i]}(x) \leq 2 H_N(x), \quad 0 \leq [1 - F_{N[i]}(x)] \leq 2 [1 - H_N(x)] \tag{4.27}$$

for i = 1, 2. Proceeding then precisely on the same line as in the proof of Theorem 5.2 of Puri and Sen (1966), we arrive at the stochastic convergence of (4.24) to $\nu_{12}(F)$. Hence, we get that

$$\sigma_N^2 \xrightarrow{P} \nu^2 - \nu_{12}(F) = \frac{1}{2}[\nu_{11}(F) + \nu_{22}(F) - 2\nu_{12}(F)] \tag{4.28}$$

(by (4.21),). It remains only to show that if $F \in \Omega_o$, then $\nu^2 - \nu_{12}(F) > 0$. From (4.28), we get that

$$\nu^2 - \nu_{12}(F) = \frac{1}{2}[I_2 \underset{\sim}{\nu}(F) I_2'], \tag{4.29}$$

where $I_2 = (1,1)$ and $\underset{\sim}{\nu}(F)$ is defined in (4.13). Since for $F \in \Omega_o$, $\nu(F)$ is positive definite and (4.29) is a quadratic form in $\underset{\sim}{\nu}(F)$, with a non-null vector I_2, it follows readily that it will be also a positive quantity.

Hence, the theorem.

<u>THEOREM 4.2</u> If $F \in \Omega_o$ and the conditions (c.1) through (c.5) hold, then under the permutational probability measure $\mathcal{P}_n$, the statistic $N^{\frac{1}{2}}(T_n - \bar{E}_N)/\sigma_N$ has asymptotically, in probability, a normal distribution with zero mean and unit variance.

<u>PROOF</u> From (3.4), (3.5), (3.8) and (4.15) we readily get that

$$T_n - \bar{E}_N = \frac{1}{n} \sum_{\alpha} \frac{1}{2}[E_{N, R_{1\alpha}} - E_{N, R_{2\alpha}}]. \tag{4.30}$$

Now, the permutational probability measure $\mathcal{P}_n$ defines a set of 2^n (conditionally) equally probable values of $T_n - \bar{E}_N$ which may be expressed as $\frac{1}{n}\sum_{\alpha=1}^{n} d_{N,\alpha}$, where $d_{N,\alpha}$, $\alpha = 1, \ldots, n$ are independent and

$$P\{d_{N,\alpha} = \frac{1}{2}[E_{N, R_{1\alpha}} - E_{N, R_{2\alpha}}] \mid \mathcal{P}_n\} = \frac{1}{2},$$
$$P\{d_{N,\alpha} = -\frac{1}{2}[E_{N, R_{1\alpha}} - E_{N, R_{2\alpha}}] \mid \mathcal{P}_n\} = \frac{1}{2}. \tag{4.31}$$

Let us now define another set of n independent random variables $\{W_{N,\alpha}, \alpha = 1, \ldots, n\}$ by

$$P\{W_{N,\alpha} = \pm \frac{1}{2} [J(\frac{R_{1\alpha}}{N+1}) - J(\frac{R_{2\alpha}}{N+1})] \mid \mathcal{P}_n\} = \frac{1}{2}, \alpha = 1, \ldots, n. \quad (4.32)$$

Then, it follows from condition (c.2), (4.30) and (4.32) that under the permutational probability measure $\mathcal{P}_n$

$$n^{\frac{1}{2}}(T_n - \bar{E}_N) \overset{P}{\sim} n^{-\frac{1}{2}} \sum_{\alpha=1}^{n} W_{N,\alpha}. \quad (4.33)$$

Hence, it is sufficient to show that $n^{-\frac{1}{2}}\Sigma_{\alpha=1}^{n} W_{N,\alpha}$ has (under the permutational probability measure $\mathcal{P}_n$) asymptotically a normal distribution. To prove this, we will use the Berry-Essen theorem [ef. Loeve (1960, P 288)], which may be stated as follows:

Let $\{Z_i\}$ be a sequence of independent random variables with means $\{\mu_i\}$, variances $\{\sigma_i^2\}$ and absolute third moments $\{\beta_i\}$, and let

$$S_n^2 = \sum_{\alpha=1}^{n} \sigma_\alpha^2, \quad \rho_n = \sum_{\alpha=1}^{n} \beta_\alpha. \quad (4.34)$$

Also, let $G_n(x)$ be the cdf of $\Sigma_{\alpha=1}^{n}(Z_\alpha - \mu_\alpha)/S_n$ and $\Phi(x)$ be the standardized normal cdf. Then there exists a finite constant $c(< \infty)$, such that for all x.

$$|G_n(x) - \Phi(x)| < C \rho_n / S_n^3. \quad (4.35)$$

Thus, if we take $W_{N,\alpha}$ for Z_α, $\alpha = 1, \ldots, n$, then from (4.32), we get that $\mu_\alpha = o$ for all $\alpha = 1, \ldots, n$. Also, precisely on the same line as in theorem 4.1, we get that

$$\frac{1}{n} S_n^2 = \frac{1}{n} \sum_{\alpha=1}^{n} E(W_{N,\alpha}^2 \; \mathcal{P}_n) = \frac{1}{2n} \sum_{\alpha=1}^{n} [J(\frac{R_{1\alpha}}{N+1}) - J(\frac{R_{2\alpha}}{N+1})]^2$$

$$= \sigma_N^2 + o_p(1) \overset{P}{\to} \nu^2 - \nu_{12}(F) > 0. \quad (4.36)$$

Again, using condition (c.3) and (4.32), we get that

$$\frac{1}{n}\rho_n = \frac{1}{n}\sum_{\alpha=1}^{n} E(|W_{N,\alpha}|^3|\mathcal{P}_n) \leq \underset{\alpha}{\text{Max}} |W_{N,\alpha}| \frac{1}{n}\sum_{\alpha=1}^{n} E(W_{N,\alpha}^2|\mathcal{P}_n)$$

$$\leq K N^{\frac{1}{2}-\delta} \cdot (\frac{1}{n} s_n^2). \tag{4.37}$$

From, (4.36) and (4.37), we readily get that

$$\rho_n / s_n^3 = o_p(N^{-\delta}), \tag{4.38}$$

and hence from (4.35), we get that under the permutational probability measure $\mathcal{P}_n$, $n^{-\frac{1}{2}} \Sigma_{\alpha=1}^{n} W_{N,\alpha}/\sigma_N$ has asymptotically, in probability a normal distribution with zero mean and unit variance. The rest of the proof follows from (4.33).

Hence, the theorem.

(It may be noted that the probability measure $\mathcal{P}_n$ being a conditional measure (depending on Z_N), the above theorem holds, in probability, i.e., for all most all Z_N.)

Before we proceed to formulate the permutation test function $\phi(Z_N)$ for both small and large samples, we will study the asymptotic distribution of $n^{\frac{1}{2}}(T_n - \mu)$ for arbitary parent cdf F, as the same will be required subsequently to study the asymptotic properties of our proposed test.

5. ASYMPTOTIC NORMALITY OF T_n FOR ARBITRARY CDF

It follows from (3.8), (4.1) (4.2) and (4.3) that

$$T_n = \int_{-\infty}^{\infty} J_N[\frac{N}{N+1} H_N (x)] dF_{N[1]}(x), \tag{5.1}$$

which has a very close analogy with Chernoff-Savage type of Statistics, (with the only difference that here there are n paired observations, while in the other case there are N independent observations). Let us then define

$$\mu_n(F) = \int_{-\infty}^{\infty} J(H(x)) dF_{[1]}(x), \tag{5.2}$$

$$\sigma_n^2 (F) = \frac{1}{2} \left\{ \int\int_{-\infty < x < y < \infty} F_{[2]} (x)[1 - F_{[2]}(y)] \, J'(H(x))J'(H(y)) \, dF_{[1]}(x) \, dF_{[1]} (y) \right.$$

$$+ \int\int_{-\infty < x < y < \infty} F_{[1]}(x)[1 - F_{[1]}(y)] \, J'(H(x))J'(H(y)) \, dF_{[2]}(x) dF_{[2]} (y)$$

$$\left. - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(x,y) - F_{[1]}(x) \, F_{[2]}(y)] \, J'(H(x)) \, J'(H(y)) dF_{[1]} (x) \, dF_{[2]}(y) \right\}$$

<u>THEOREM 5.1</u> If the conditions (c.1), (c.2) and (c.3) of section 4 hold then for any continuous F for which $\sigma_n^2(F) > 0$,

$$\lim_{n=\infty} P\{n^{\frac{1}{2}}(T_n - \mu_n(F)) | \sigma_n(F) \leq t\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2} \, dx.$$

<u>PROOF</u> If we write

(i) $F_{N[1]} (x) = F_{[1]} (x) + [F_{N[1]} (x) - F_{[1]} (x)]$,

(ii) $\int_{-\infty}^{\infty} J_N \left[\frac{N}{N+1} H_N (x)\right] dF_{N[1]} (x) = \int_{-\infty}^{\infty} J(\frac{N}{N+1} H_N (x)) \, dF_{N[1]} (x) + o_p (N^{-\frac{1}{2}})$,

(by (c.1),)

(iii) $J(\frac{N}{N+1} H_N (x)) = J(H(x)) + (H_N (x) - H(x)) \, J'(H(x)) - \frac{1}{N+1} H_N (x) \, J'(H(x))$

$$+ \left\{ J(\frac{N}{N+1} H_N (x)) - J(H (x)) - (\frac{N}{N+1} H_N (x) - H(x)) \, J'(H(x)) \right\},$$

then proceeding precisely on the same line as in the modified proof of Chernoff-Savage theorem by Govindarajulu, LeCam and Raghavachari (1965), we get after a few simple steps that

$$T_n = \mu_n (F) + B_{1,N} + B_{2,N} + \sum_{i=1}^{4} C_{i,N}, \tag{5.4}$$

where

$$B_{1,N} = \frac{1}{2}\int [F_{N[2]}(x) - F_{[2]}(x)]\, J'(H)\, dF_{[1]}(x), \tag{5.5}$$

$$B_{2,N} = \frac{1}{2}\int [F_{N[1]}(x) - F_{[1]}(x)]\, J'(H)\, dF_{[2]}(x), \tag{5.6}$$

$$C_{1,N} = -\frac{1}{N+1}\int H_N(x)\, J'(H(x))\, dF_{N[1]}(x) \tag{5.7}$$

$$C_{2,N} = \int [H_N(x) - H(x)]\, J'(H(x))\, d[F_{N[1]}(x) - F_{[1]}(x)], \tag{5.8}$$

$$C_{3,N} = \int [J(\tfrac{N}{N+1} H_N(x)) - J(H(x)) - (\tfrac{N}{N+1} H_N(x) - H(x))\, J'(H(x))]\, dF_{N[1]}(x) \tag{5.9}$$

and

$$C_{4,N} = \int [J_N(\tfrac{N}{N+1} H_N(x)) - J(\tfrac{N}{N+1} H_N(x))]\, dF_{N[1]}(x). \tag{5.10}$$

$$= o_p(N^{-\frac{1}{2}}), \text{ by (c.1)}.$$

As in the proof of Chernoff-Savage theorem we shall show later on that $C_{i,N}$, i=1, 2, 3 are all $o_p(N^{-\frac{1}{2}})$ and may be termed the higher order terms. Thus, from (5.4), we get that

$$n^{\frac{1}{2}}(T_n - \mu_n(F)) = n^{\frac{1}{2}}[B_{1,N} + B_{2,N}] + n^{\frac{1}{2}}[\sum_{i=1}^{4} C_{i,N}]. \tag{5.11}$$

With this, we consider first the asymptotic distribution of

$$n^{\frac{1}{2}}[B_{1,N} + B_{2,N}] = n^{-\frac{1}{2}}\left\{\sum_{\alpha=1}^{n} [B_1(X_{2\alpha}) - B_2(X_{1\alpha})]\right\}, \tag{5.12}$$

where

$$B_1(Y) = \frac{1}{2}\int_{-\infty}^{\infty} [F_{1[2]} - F_{[2]}]\, J'(H)\, dF_{[1]}(x), \tag{5.13}$$

$$B_2(X) = \frac{1}{2}\int_{-\infty}^{\infty} [F_{1[1]} - F_{[1]}]\, J'(H)\, dF_{[2]}(y), \tag{5.14}$$

$F_{1[1]}$ and $F_{1[2]}$ being the empirical cdf (marginal) of $(X_{1\alpha}, X_{2\alpha})$ based on a random sample of size unity. Thus, it follows from (5.12) that $B_{1,N} + B_{2,N}$ is the average of n independent and identically distributed random variables $\{[B_1(x_{2\alpha}) - B_2(x_{1\alpha})], \alpha = 1, \dots, n\}$. Hence, for the asymptotic normality of $n^{\frac{1}{2}}[B_{1,N} + B_{2,N}]$ in (5.12), we may use the classical central limit theorem, provided the variance of $[B_1(x_{2\alpha}) - B_2(x_{1\alpha})]$ is positive and finite. Now, using (5.13), we get that

$$4\,\mathrm{Var}\{B_1(y)\} = E\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_{1[2]}(x) - F_{1[2]}(x)]\,[F_{1[2]}(y) - F_{[2]}(y)] \cdot J'(H(x))J'(H(x))\, dF_{[1]}(x)\, dF_{[1]}(y)\}$$

$$= 2\int\int_{-\infty < x < y < \infty} F_{[2]}(x)\,[1-F_{[2]}(y)]\, J'(H(x))\, J'(H(y))\, dF_{[1]}(x)\, dF_{[1]}(y). \quad (5.15)$$

Similarly,

$$4\,\mathrm{Var}\{B_2(x)\} = 2\int\int_{-\infty < x < y < \infty} F_{[1]}(x)[1 - F_{[1]}(y)]\, J'(H(x))\, J'(H(y))\, dF_{[2]}(x)\, dF_{[2]}(y); \quad (5.16)$$

$$4\,\mathrm{Cov}\{B_1(y), B_2(x)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F(x,y) - F_{[1]}(x)\, F_{[2]}(y)]\, J'(H(x))\, J'(H(y))\, dF_{[1]}(x)\, dF_{[2]}(y). \quad (5.17)$$

From (5.15), (5.16) and (5.17), we get that the variance of $[B_1(x_{2\alpha}) - B_2(X_{1\alpha})]$ is equal to $\sigma_n^2(F)$, given in (5.3). Now, in the statement of the theorem, it is assumed that $\sigma_n(F)$ isnon-zero. Now, from (5.3), (5.15), (5.16) and (5.17) we get that

$$\sigma_n^2(F) \leq 2\,[\mathrm{Var}\,\{B_1(Y)\} + \mathrm{Var}\,\{B_2(X)\}]. \quad (5.18)$$

Thus it is sufficient to show that $\mathrm{Var}\{B_1(X_{2\alpha})\} + \mathrm{Var}\{B_2(X_{1\alpha})\}$ is finite. Now, by (4.5), we have

$$\text{(i)}\quad F_{[i]}(1 - F_{[i]}) \leq 4\, H(1 - H) \text{ for } i = 1, 2; \quad (5.19)$$

$$\text{(ii)}\quad dH(x) \geq \tfrac{1}{2}\, dF_{[i]}(x),\ i = 1, 2. \quad (5.20)$$

Hence, from (5.15), (5.16), (5.19) and (5.20) we get after a few simple steps

$$\mathrm{Var}\{B_1\ (X_{2\alpha})\} + \mathrm{Var}\{B_2(X_{1\alpha})\}$$

$$\leq 16 \int\int_{-\infty < x < y < \alpha} H(x)\ [1-H(y)]\ J'(H(x))\ J'(H(y))\ dH\ (x)\ dH\ (y)$$

$$= 8[\int_0^1 J^2(u)\ du - \{\int_0^1 J(u)\ du\}^2] \leq 8 \int_0^1 J^2(u)\ du$$

$$\leq 8\ K^2 \int_0^1 [u(1-u)]^{1-2\delta}\ du < \infty, \text{ by condition (c.3).} \qquad (5.21)$$

Thus, to complete the proof of the theorem, it remains only to show that $C_{i,N}$, $i = 1, 2, 3$ are all $o_p(n^{-\frac{1}{2}})$. To do this, we first note that all the elementary results of Chernoff and Savage (1958, p. 186; results 7.A.1 to 7.A.10) also hold in our case with the further simplification that $\lambda_N - \lambda_o = \frac{1}{2}$.

For brerity, these results are therefore not reproduced again. Now, as in the treatment of C_{13N} of Cherenoff and Savage (1958, p 988), we have readily

$$|C_{1,N}| \leq \frac{1}{N+1} \cdot \frac{1}{n} \sum_{\alpha=1}^{n} |J'(H(X_{1\alpha}))| \leq \frac{1}{N+1} \cdot \frac{K}{n} \sum_{\alpha=1}^{n} [H(X_{1\alpha})(1-H(X_{1\alpha}))]^{-\frac{3}{2}+\delta}$$

$$\leq \frac{K}{N+1} \cdot \frac{1}{n} \sum_{\alpha=1}^{n} [F_{[1]}\ (X_{1\alpha})\ (1 - F_{[1]}\ (X_{1\alpha}))]^{-\frac{3}{2}+\delta} = o_p(n^{-\frac{1}{2}}). \qquad (5.22)$$

Thus, we are to show that $C_{2,N}$ and $C_{3,N}$ are also $o_p(n^{-\frac{1}{2}})$.

Let now (a_N, b_N) be the interval $S_{N\epsilon}$, where

$$S_{N,\epsilon} = \{\ x:\ H(1 - H) > \eta_\epsilon /n\}. \qquad (5.23)$$

Then by the result 7.A.9 of Chernoff and Savage (1958, p 986) (with a simple extension to the bivariate case,) there exists an $\eta_\epsilon > 0$ independent of $F(x_1, x_2)$. Such that

$$P\{X_{i\alpha} \in S_{N,\epsilon}\ ,\ i = 1, 2;\ \alpha = 1, \ldots, n\} \geq 1 - \epsilon. \qquad (5.23)$$

Thus, with probability greater than $1-\epsilon$, there are no observations in the complementary region $\bar{S}_{N,\epsilon}$. Further, if $H_1 = H(a_N)$ and $H_2 = H(b_N)$, then $H_1 = 1 - H_2 < K/N$, K being a constant, which may depend on ϵ and η_ϵ. Further, it follows from lemma

4.2.2 of Govindarajulu et al (1965) that for every $\gamma > 0$, there exists a $c(\gamma) > 0$, such that with probability greater than $1 - \frac{1}{2}\gamma$, we have

$$|F_{N[i]} - F| \leq c(\gamma)\, n^{-\frac{1}{2}}[F_{[i]}\,(1 - F_{[i]})]^{(1-\delta)/2}, \tag{5.24}$$

for $i = 1, 2$. Thus, from (5.24), we get that with probability greater than $1-\gamma$,

$$|H_N(x) - H(x)| = \frac{1}{2}\left|\sum_{i=1}^{2}[F_{N[i]}(x) - F_{[i]}(x)]\right| \leq \frac{1}{2}\sum_{i=1}^{2}|F_{N[i]}(x) - F_{[i]}(x)|$$

$$\leq \frac{c(\gamma)}{2}\, n^{-\frac{1}{2}}\sum_{i=1}^{2}\{F_{[i]}(x)\,[1 - F_{[i]}(x)]\}^{(1-\delta)/2}$$

$$\leq c(\gamma)\, 2^{(1-\delta)/2}\, n^{-\frac{1}{2}}[H(x)\,[1 - H(x)]]^{(1-\delta)/2}$$

$$= c^*(\gamma)\, N^{-\frac{1}{2}}\{H(x)\,[1 - H(x)]\}^{\frac{1-\delta}{2}}, \quad 0 < c^*(\gamma) < \infty. \tag{5.25}$$

Thus, expressing the integral on the right hand side of (5.8) as the sum of two integrals over the domains of integration $S_{N,\epsilon}$ and $\bar{S}_{N,\epsilon}$ respectively, and noting that by condition (c.3) and (5.23) (with probability $> 1 - \epsilon$,)

$$\left|\int_{\bar{S}_{N,\epsilon}} [H_N(x) - H(x)]\, J'(H(x)\, d\,[F_{N[1]}(x) - dF_{[1]}(x)]\right|$$

$$\leq 2\int_{0}^{K/N}[H(x)]^{-\frac{1}{2}+\delta}[1-H(x)]^{-\frac{3}{2}+\delta}\, dH(x) + 2\int_{1-K/N}^{1}[H(x)]^{-\frac{3}{2}+\delta}\,[1-H(x)]^{-\frac{1}{2}+\delta}d[1-H(x$$

$$= o(n^{-\frac{1}{2}}); \tag{5.26}$$

we get that

$$|C_{2,N}| = \left|\int_{S_{N,\epsilon}}[H_N(x) - H(x)]\, J'(H(x))\, d[F_{N[1]}(x) - F_{[1]}(x)]\right| + o_p(n^{-\frac{1}{2}}). \tag{5.27}$$

Using then (5.25), we get that with probability greater than $1-\gamma$,

$$N^{\frac{1}{2}} \, |H_N(x) - H(x)| \, J'(H(x)) \leq c^*(\gamma) \, [H(x)\{1-H(x)\}]^{-1+\delta/2} \tag{5.28}$$

Since, the right hand side of (5.28) is integrable with respect to $H(x)$, $dF_{N[1]}(x) \leq dH_N(x)$, $H_N(x) \xrightarrow{a.s.} H(x)$, $F_{N[1]} \xrightarrow{a.s.} F_{[1]}(x)$, and $dF_{[1]}(x) \leq 2\, dH(x)$, it can be shown after a few simple steps (with the aid of(5.27) and (5.28),) that

$$N^{\frac{1}{2}} \, |C_{2,N}| \xrightarrow{p} 0 \text{ i.e., } |C_{2,N}| = o_p(N^{-\frac{1}{2}}). \tag{5.29}$$

The proof of $|C_{3,N}|$being $o_p(N^{-\frac{1}{2}})$ follows precisely on the same line as in the proof of a similar $C_{3,N}$ (for the case of two independent samples) considered by Govindarajulu et al (1965), and hence for the intended brevity of our discussion, the details are omitted.

Hence, the theorem.

<u>THEOREM 5.2</u> If the conditions of theorem 5.1 hold and in addition

$$F_{[1]}(x) = \Psi(x), \quad F_{[2]}(x) = \Psi((x - \theta_N)/\beta_N) \tag{5.30}$$

where Ψ has a continuous density $\psi(x)$ and $\theta_N \to 0$, $\beta_N \to 1$ as $N \to \infty$, then $\sigma_N^2(F)$, defined in (5.3), is positive for all $F \in \Omega_0$ (defined in (4.14),) and $n^{\frac{1}{2}}(T_n - \mu_n(F))/\sigma_n(F)$ has asymptotically a normal distribution with zero mean and unit variance.

<u>PROOF</u> In order to prove the theorem, it appears to be sufficient to show that under (5.30), $\sigma_n^2(F)$, defined in (5.3), is positive.

Then proceeding precisely on the same line as in the proof of corollary 4.12 of Govindarajulu et al (1965), we get that under (5.30) the sum of the first two integrals on the right hand side of (5.3) converges (as $n \to \infty$) to $\frac{1}{2}\{ \int_0^1 J^2(u)\,du - [\int_0^1 J(u)\,du]^2\}$. Again, by a similar technique it can be shown that the last integral on the right hand side of (5.3) converges (as $n \to \infty$) to

$$\frac{1}{2}\left\{\int_0^1 J^2(u)\,du - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F_{[1]}(x))\, J(F_{[2]}(y))\, dF(x, y)\right\} \tag{5.31}$$

Again from (4.11), (4.12) and (4.17) we get that (5.31) is nothing but $\frac{1}{2}(\nu^2 - \nu_{12}(F)) = \frac{1}{4} \{\nu_{11}(F) + \nu_{22}(F) - 2\nu_{12}(F)\} > 0$ for all $F \in \Omega_o$, by condition (4.14) and (c.5).

Hence, the theorem.

Incidentally, from theorem 4.1 and theorem 5.2, we arrive at the following result that under (5.30)

$$\sigma_N^2 \xrightarrow{P} 2\, \sigma_N^2(F) \to \nu^2 - \nu_{12}(F) > 0, \qquad 5.32)$$

for all $F \in \Omega_o$. This result will be very useful in the next section.

6. ASYMPTOTIC EFFICIENCY OF RANK ORDER TESTS

We shall now consider some specific rank order test for location or scale and study their effeciency aspects. It follows from our results in the preceeding three sections that the exact permutation test based on T_n in (3.8) reduces to a very simple form for large values of n. Also, it follows from theorem 5.2 that if we have any consistent estimate $\hat{\sigma}_n(F)$ of $\sigma_n^2(F)$ in (5.3) (under H_o), then a large sample (unconditional) test may be based on the studentized statistic $n^{\frac{1}{2}}(T_n - \mu)/ \hat{\sigma}_n(F)$, where μ is defined in (4.16). By virtue of theorems 4.2 and 5.2 and of (5.32), we may then readily conclude that the large sample permutation test and the large sample unconditional test will be stochastically equivalent for the entire class of hypotheses in (5.30). Hence to study the asymptotic power efficiency of the test based on T_n, it will be sufficient to consider only the large sample unconditional tests.

6.1 Location problem For the matched sample location problem we assume that $X_{1\alpha}$ and $X_{2\alpha} - \theta$ are statistically interchangeable for some real θ, and the null hypothesis states that $\theta = 0$. We shall consider several rank order tests based on statistics of the type (3.8), for which we assume further that $E_{N,\alpha}$ is a monotonic function of : $1 \leq \quad \leq N$. As in the case of univariate location tests, the above condition case ensure the consistency of the tests for any non-null θ. We shall consider specifically the following $\underset{\sim}{E}_N$.

(i) Median Test Here we have

$$E_{N,\alpha} = \begin{cases} 1 \text{ if } \alpha \leq [N/2], \\ 0 \text{ if } \alpha > [N/2], \end{cases} \tag{6.1}$$

([s] being the largest integer contain in s). In the univariate case, the corresponding test is propesed by Mood (1950) and is known as the median test. For the case of two independent bivariate samples, the generalization of the test is due to Chatterjee and Sen (1964).

(ii) Rank-sum test Here, we let

$$E_{N,\alpha} = \alpha, \text{ for } \alpha = 1, \ldots, N \tag{6.2}$$

In the univariate case, the corresponding test is due to Wilcoxon (1945) and Mann and Whitney (1947). For the case of two independent bivariate samples, generalization of the same has been made by Chatterjee and Sen (1964).

(iii) Ψ-Score test Let Ψ be some assumed cdf, and let in a random sample of size N drawn from a population having the cdf Ψ, $a_{N,\alpha}$ be the expected value of the α-th order Statistic, for $\alpha = 1, \ldots, N$. Then, we take

$$E_{N,\alpha} = a_{N,\alpha}, \text{ for } \alpha = 1, \ldots, N. \tag{6.3}$$

Sometimes, we also take

$$E_{N,\alpha} = \Psi^{-1}(\alpha/ (N+1)), \ 1 \leq \alpha \leq N. \tag{6.4}$$

In particular, if Ψ is a standardized normal cdf, the scores $\{a_{N,\alpha}\}$ are tested the normal scores and the test as the normal,score test. In the univariate case, tests of this type are due to Fisher and Yates, Terry, Hoeffding among many others, and a nice account of the same is given by Chernoff and Savage (1958). Generalization to the case of multivariate multi sample situations is due to Puri and Sen (1966). For the study of the asymptotic relative efficiencies (A.R.E.) of these tests as well as the parametrically optimum paired t-test, we have to select some sequence of alternative hypotheses (say, $\{H_n^{(1)}\}$) for which the power of the tests lie in the open interval (0,1). For this, we define

$$H_n^{(1)}: \ F_{[2]}(x) = F_{[1]}(x + n^{-\frac{1}{2}}\theta), \tag{6.5}$$

where θ is real and finite and the cdf $F_{[1]}$(having a continuous density $f_{[1]}(x)$,) is assumed to satisfy the conditions of lemma 7.2 of Puri (1964). Let us then define

$$A^2(F,J) = \int_0^1 J^2(u)\,du - \left\{\int_0^1 J(u)\,du\right\}^2; \tag{6.6}$$

$$\rho(F, J) = \frac{\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F_{[1]}(x))J(F_{[2]}(y))\,dF(x, y) - \{\int_0^1 J(u)\,du\}^2\right]}{A^2(F,J)}; \tag{6.7}$$

$$B(F, J) = \int_{-\infty}^{\infty} J'(F_{[1]}(x))\, f_{[1]}(x)\, dF_{[1]}(x). \tag{6.8}$$

It is then easily seen that for the sequence of alternatives $\{H_n^{(1)}\}$ in (6.5), the A.R.E. of the rank order test based on T_n is propostional to

$$[B(F, J)/A(F,J)\ \{1-\rho(F,J)\}^{\frac{1}{2}}]^2. \tag{6.9}$$

Again, if we consider the paired t-test (under the assumption of homoseedasticity, consequent on (6.5),), it can be shown that the A. R. E. of this test is propostional to

$$1/\sigma^2(1-\rho), \tag{6.10}$$

where σ^2 is the common variance and ρ is the correlation coefficient of $x_{1\alpha}$ and $x_{2\alpha}$. Thus, if we define $\xi = (\xi_1, \xi_2)$ as the (population) median-vector of $(x_{1\alpha}, x_{2\alpha})$, and write

$$\pi_{12} = P\{X_{1\alpha} \leq \xi_1, X_{2\alpha} \leq \xi_2\} , \tag{6.11}$$

then it is easily seen that the A. R. E. of the median test is

$$[f_{[1]}(\xi_1)]^2/(1-2\pi_{12}) = E_M \text{ (Say)}. \tag{6.12}$$

Similarly, if ρ_g stand for the grade correlation of $x_{1\alpha}$ and $x_{2\alpha}$ [cf. Hoeffding (1948, p. 318)], then the A. R. E. of the rank sum test will be equal to

$$12[\int_{-\infty}^{\infty} f_{[1]}^2 (x)\, dx]^2/(1- \rho_g) = E_R \text{ (say)}. \tag{6.13}$$

Finally, if we substitute $\Psi^{-1}(F)$ for $J(F)$ in (6.7) and denote the corresponding measure by ρ_ψ, the Ψ-score correlation of $x_{1\alpha}$ and $x_{2\alpha}$, the A. R. E. of the Ψ-score test would be

$$[\int_{-\infty}^{\infty} \{f_{[1]}(x)/ \psi[\psi^{-1}(F_{[1]}(x))\} dF_{[1]}(x)]^2/\sigma_\psi^2(1-\rho_\psi) = E_\psi(\text{say}), \tag{6.14}$$

where σ_ψ^2 stands for the variance of the cdf Ψ. If $\rho_e F_{[1]} \equiv \Psi$, the above reduces to $1/\sigma_\Psi^2(1-\rho_\Psi)$ and if further ψ is normal, it is easily seen that $\rho_\Psi = \rho$, and hence, the A.R.E. of the Ψ-score test will be equal to (6.10), the same as of the t-test. Thus for normal alternatives, the normal score test and the paired t-test are asymptotically power equivalent (for the sequence of alternatives $\{H_n^{(1)}\}$ in (6.5)). Further, it follows from (6.9) that the A.R.E. of any proposed test is the product of two factors; the first factor depending only on the marginal cdf $F_{[1]}$ (as $A(F, T)$ is independent of F,) while the second factor being dependent on $\rho(F, J)$, which depends on both F and J. This makes the usual univariate bounds for A.R.E. inapplicable in this case. However, if we can find bounds for the variations of $(1-\rho(F, J))/(1-\rho)$, the same can be used to study the bounds for the A.R.E. Unfortunately, this depends on the unknown F in a very involved way and no universal bounds may be attached. For the specific case of normal cdf's, it is well-known that

$$4(\pi_{12} - \tfrac{1}{4}) = \frac{2}{\pi} \mathrm{Sin}^{-1} \rho \quad \text{and} \quad \rho_g = \frac{6}{\pi} \mathrm{Sin}^{-1} \rho/2, \tag{6.15}$$

and hence, we get that the Pitman-efficiencies of the median test and the rank-sum test with respect to the paired t-test are respectively

$$(1-\rho)/\cos^{-1} \rho \quad \text{and} \quad \frac{1-\rho}{\cos^{-1}(\frac{1}{2}\rho - 1)} \tag{6.16}$$

Now, (6.16) agrees with the expressions obtained by Bickel (1965, pp 170-171) for his median and rank-sum tests for the symemtry of a bivariate cdf (around the origin). Consequently, the efficiency study made by him can be as it is applied in our case to get similar bounds for the quantities in (6.16).

Finally, in the leterature, the signed-rank test by Wilcox on (1949) and the normal score test by Govindarajulu (1960) are based on the values of $X_{1\alpha} - X_{2\alpha}$, $\alpha = 1, \ldots, n$. If we consider the asymptotic efficiency of the type of rank order test considered here with respect to the similar test according to the other method, the same will be obtained

$$(1-\rho)/[1-\rho(F, J)], \tag{6.17}$$

and hence, nothing can be said, in general, about this. However, for normal score test, (6.17) will be equal to unity, while for normal alternatives, the bounds for (6.17) for median or rank-sum test may be easily obtained by using Bickel's (1964, pp 170-171) results.

6.2 Scale problem To the best of knowledge of the author, there is no nonparametric test for the equality of Scales parameters of a bivariate distribution. Here, we assume that $X_{1\alpha}$ and $X_{2\alpha}$ have a common location (unknown but real), and $(X_{1\alpha} - \mu)$ and $\theta(X_{2\alpha} - \mu)$ are interchangeable for some real and finite (≥ 0) θ; the null hypothesis being $\theta = 1$.

In the parametric case, the test by Morgan (1939) is based upon the significance of the observed correlation between $u_\alpha = X_{1\alpha} - X_{2\alpha}$ and $V_\alpha = X_{1\alpha} + X_{2\alpha}$. This test is also related functionally to the likelihood ratio test for the same problem, and hence, possesses some optimum properties too.

The proposed rank order tests for this problem are all based on statistics of the type T_n in (3.8). As under the null hypothesis $\theta = 1$, we may take without any loss of generality that μ is the population median of either variate, and by change of origin, we can always take $\mu = 0$. We then assume that the weight-function J(F) corresponding to T_n in (3.8) satisfies the condition

$$J(F(\theta x)) - J(F(x)) \text{ is } \uparrow \text{ in } \theta \text{ for } x \geq 0, \quad \text{is } \uparrow \text{ in } \theta \text{ for } x < 0, \text{ or vice versa.} \tag{6.18}$$

It is easily seen that under (6.18), the rank order tests based on T_n in (3.8) will be consistent against any $\theta \neq 1$. The univariate scale tests satisfy (6.18) in almost all the cases. We shall consider here specifically the following rank order statistics.

(i) Rank mean-square statistic We let

$$E_{N,\alpha} = (\alpha - \frac{N+1}{2})^2 \quad \text{for } 1 \leq \alpha \leq N. \tag{6.19}$$

In the case of two independent univariate samples, the corresponding T_n is proposed by Mood (1954) for the scale problem. Multivariate extension of this is due to Puriand Sen (1966).

(ii) Symmetric rank-statistic 1. Here, we let

$$E_{N,\alpha} = \alpha(N + 1 - \alpha) \quad \text{for } \alpha = 1, \ldots, N. \tag{6.20}$$

Univariate tests of this type are due to Sen (1963) and Chatterjee (1966).

(iii) Ψ-Score statistics With the notations in (6.3) and (6.4), we define

$$E_{N,\alpha} = a^2_{N,\alpha} \text{ or } [\Psi^{-1}(\frac{\alpha}{N+1})]^2, \quad \alpha = 1, \ldots, N. \tag{6.21}$$

As for the study of the asymptotic efficiency of these tests, we proceed similarly as in the location problem and write

$$H_n^{(2)}:\ F_{[2]}(x) = F_{[1]}([1 + n^{-\frac{1}{2}}\theta]x); \quad (\text{as } \mu = 0); \tag{6.22}$$

where θ is real and finite and $F_{[1]}$ satisfies the conditions of lemma 7.2 of Puri. (1964).

We also define $A(F, J)$ and $\phi(F, J)$ as in (6.6) and (6.7), and let

$$C(F, J) = \int_{-\infty}^{\infty} x\, J'(F_{[1]}(x))\, f_{[1]}(x)\, dF_{[1]}(x). \tag{6.23}$$

It is then easily seen that for the sequence of alternatives $\{H_n^{(2)}\}$ in (6.22), the A. R. E. of the rank order test based on T_n will be proportional to

$$[C(F, J)/A(F, J)]^2/[1-\rho(F, J)], \tag{6.24}$$

while the A. R. E. of the Morgan (1939) test for the same sequence of alternatives will be proportional to

$$1/(1-\rho^2), \tag{6.25}$$

where ρ is the correlation coefficient of $(X_{1\alpha}, X_{2\alpha})$.

Now, if we work with the normal scores i.e., (6.21) when ψ is assumed to be normal and if $F_{[1]} \equiv \Psi$, it is easy to show (on using (6.7),) that $\rho(F, J)$ in this case will be nothing but ρ^2. Further, if follows from (6.23) that in this case, $C(F, J)/A(F, J)$ will be equal to unity. Hence, from (6.24) and (6.25), we will get that for normal alternatives (scale), the normal score test and the Morgan test will be asymptotically power equivalent.

The asymptotic efficiencies of the other tests may be studied precisely on the same line as in the location problem. These will naturally depend on the correlation ρ(between $X_{1\alpha}$, $X_{2\alpha}$) and, in general, no proper bounds may be attached to these.

7. TESTS FOR INTERCHANGEABILITY IN BLOCKED EXPERIMENTS

Here we shall work with the model (2.5), and will test for the interchangeability of $(Y_{1\alpha}, Y_{2\alpha})$ eliminating the effect of Z_α, for $\alpha = 1, \ldots, n$. The procedure will be a very simple modification of what we have done before. This may be termed [after Hodges and Lehmann (1962)] ranking after alignment.

We define $\bar{X}_\alpha = \frac{1}{2}(X_{1\alpha} + X_{2\alpha})$ for $\alpha = 1, \ldots, n$ and let

$$W_\alpha = X_{1\alpha} - \bar{X}_\alpha = \frac{1}{2}(X_{1\alpha} - X_{2\alpha})$$

$$= \frac{1}{2}(Y_{1\alpha} - Y_{2\alpha}) \text{ for } \alpha = 1, \ldots, n; \tag{7.1}$$

(by (2.5)),

Thus $\{W_\alpha, \alpha = 1, \ldots, n\}$ are n i. i. d. r. v. which are free from the effects of $(Z_\alpha, \alpha = 1, \ldots, n)$ and if really $Y_{1\alpha}$, $Y_{2\alpha}$ are interchangeable, then W_α will have a distribution symmetric about the origin. Thus, conditioned on the given set $\{|W_\alpha|, \alpha = 1, \ldots, n\}$. There will be 2^n possible realized values of $\{W_\alpha, \alpha = 1, \ldots, n\}$ where each W_α can assume two values $\pm |W_\alpha|$, independently of each other and with probability $\frac{1}{2}$ if H_0 is true. We now rank the values of $|W_\alpha|$ and let R_α be the rank of $|W_\alpha|$ among the n values. Then $\underset{\sim}{R} = (R_1, \ldots, R_n)$ will be a permutation of $(1, \ldots, n)$. Thus, if we define $\underset{\sim}{E}_n = (E_{n,1}, \ldots, E_{n,n})$ (where $E_{n,\alpha}$ is a function of $\alpha/(n+1)$), then we may define a rank order statistic T_n by

$$T_n = \frac{1}{n} \sum_{\alpha=1}^{n} E_{n,\alpha} d_\alpha, \tag{7.2}$$

where d_α is 1 or 0 according as W_α is positive or not. Once, we do this, we reduce the problem to the univariate problem of symmetry, and all the available tests for that may be used here. Incidentally, here, the permutation distribution of T_n will be identical with the null distribution of it, and as in our theorem 4.2, the asymptotic normality of $n^{\frac{1}{2}}(T_n - \bar{E}_n)$ will follow readily. For this, the details are omitted, and we conclude this section by a note that the tests based on T_n in (7.2) will be valid only for the location problem; while as a result of the reduction of the bivariate problem to a univariate problem (of symmetry), we are not in a position to test for the equality of the scale parameters in this manner.

8. STUDENTIZED LOCATION AND GENERAL SCALE PROBLEMS

First, referred to (2.6), we want to test for the equality of the location μ_1 and μ_2 without assuming the scales δ_1 and δ_2 to be equal or to bear a known ratio to each other. This will be then a proper nonparametric generalization of the problem of paired t-test, where actually the two variances may be arbitarily different from each other. The paired t-test is based upon the assumption of normality of the bivariate cdf F in (2.6), while our proposed tests will be valid for a wider class of cdf's.

We define a bivariate cdf F(x, y) to be diagonally symmetric around a location $\underset{\sim}{\mu} = (\mu_1, \mu_2)$ if for all (x. y) in the two dimensional real space (R^2),

$$F(\mu_1 - x, \mu_2 - y) = 1 - F(\mu_1 + x - 0, \infty) - F(\infty, \mu_2 + y - \theta)$$

$$+ F(\mu_1 + x - 0, \mu_2 + y - 0), \qquad (8.1)$$

and the class of all diagonally symmetric edf's will be denoted by Ω_s. Then, we have the following lemma, whose proof is simple and left to the reader,

<u>LEMMA 8.1</u> If $F(x, y) \epsilon\ \Omega_s$, then X - Y will have a distribution symmetric about $\delta = \mu_1 - \mu_2$.

Consequent on this, we note that if F in (2.6) is diagonally symmetric and if $\mu_1 = \mu_2$, then the cdf of $X_{1\alpha} - X_{2\alpha}$ will be symmetric about 0, no matter whatever be the values of δ_1 and δ_2. Thus, here also, if we assume that $F \epsilon\ \Omega_s$ and we work with the values of $W_\alpha = X_{1\alpha} - X_{2\alpha}$, $\alpha = 1, \ldots, n$, then we may proceed precisely on the same line as in section 7 and get a class of rank order tests which are also used to test the symmetry in the univariate case. The efficiency study of such tests have already been made in connection with the one sample location problem (univariate) by Govindarajulu (1960), and hence will not be repeated here.

Let us now consider the general scale problem. Here, referred to (2.6), we want to test for the equality of δ_1 and δ_2 without assuming the locations to be identical. As in the case of two independent samples, one way of approach would be consider suitable tests based not on the original observations but on the ones centered at the respective estimated location parameters [cf. Sukhatme (1958) and Raghavachari (1965)]. Thus, we let $\hat{\mu}_1$ and $\hat{\mu}_2$ to be some consistent estimates of μ_1 and μ_2 respectively, and regarding these we shall assume that

$$n^{\frac{1}{2}} \mid \hat{\mu}_i - \mu_i \mid , \quad i = 1, 2 \text{ are bounded in probability.} \tag{8.2}$$

Then, we define the centered observations as

$$(\hat{X}_{1\alpha}, \hat{X}_{2\alpha}) = (X_{1\alpha} - \hat{\mu}_1, X_{2\alpha} - \hat{\mu}_2) , \; \alpha = 1, \ldots, n. \tag{8.3}$$

Our proposed tests will be then based upon these centered observations.

Thus, we may now proceed with these centered observations as in section 3 and use a rank order statistic as in (3.8). This statistic will be termed a modified rank order statistic and will be denoted by $\hat{T}_n$. For small samples, the distribution of $\hat{T}_n$ will be quite involved and will fail to be distribution-free. Thus, as in the case of unpaired samples, we will confine ourselves only to the large sample case. We shall study the conditions under which

$$n^{\frac{1}{2}}(T_n - \hat{T}_n) \xrightarrow{P} 0. \tag{8.4}$$

If (8.4) holds, then asymptotically $\hat{T}_n$ will have the same properties as that of T_n, and theorems 5.1 and 5.2 will also apply to it. Consequently, the two tests based on T_n and $\hat{T}_n$ will be asymptotically power-equivalent for scale alternatives of the type (6.22).

<u>THEOREM 8.2</u> If $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2)$ satisfies (8.3), and if in addition

(i) $F_{[i]}$ (having a continuous density $f_{[i]}$) is symmetric about μ_i for $i = 1, 2$;

(ii) $J(u)$ is symmetric about $u = \frac{1}{2}$;

(iii) $J'(F_{[i]}(x-t))f_{[i]}(x-t) \leq K\, T(x)$

for all $|t| \leq c$, (c and K being constants), where $T(x)$ is quadratically integrable with respect to $F_{[i]}$, $i = 1, 2$; then the modified rank order Statistic based on the centered observations and the original rank order statistic based on the true deviations from the respective locations will be asymptotically equivalent, in probability.

PROOF By virtue of these assumptions it is easy to show that

$$\int_{-\infty}^{\infty} J'(H(x))f_{[i]}(x)\, dF_{[j]}(x) = 0 \text{ for } i, j = 1, 2. \tag{8.5}$$

Proceeding then precisely on the same line as in the proof of theorem 3.1 of Raghavachari (1965), it can be shown that all the regularity conditions for his lemma 3.3 are satisfied, and the rest of the proof can then be completed by our theorem 5.1 The details are omitted.

It is also easy to verify that the rank order statistics corresponding to (6.19), (6.20) and (6.21) all satisfy the above conditions for a wide class of diagonally symmetric (bivariate) distributions.

Hence, in large samples, we may use the rank order statistic $\hat{T}_n$ for the general scale problem, and the asymptotic power of these will be the same as in the case of tests based on T_n in (3.8).

9. TEST FOR COMPOUND SYMMETRY

Here, referred to (2.6), we want to test Wilk's (1946) hypothesis of compound symmetry, which reduces to

$$H_0: \ \mu_1 = \mu_2 \ \text{ and } \delta_1 = \delta_2, \tag{9.1}$$

against the set of alternatives that at least one of them is not true. Here, we can work with a pair of rank order statistics, say $T_n^{(1)}$ and $T_n^{(2)}$, where the first one will test the equality of locations and the second one the equality of scales. It follows from our results in section 3 that given $\underset{\sim}{R}_N$ in (3.10), there will be a set $S(R_N) \subset \underset{\sim}{\mathfrak{R}}_N$ of 2^n (conditionally) equally likely realizations of $\underset{\sim}{\mathfrak{R}}_N$, and (3.12) holds for this, when H_0 in (9.1) holds. Thus, if we define $\underset{\sim}{T}_n = (T_n^{(1)}, T_n^{(2)})$, there will be a set $\underset{\sim}{T}_n\,(S(\underset{\sim}{R}_N))$ of 2^n (conditionally) equally likely values of $\underset{\sim}{T}_n$, and hence, as in section 3, we can construct a similar size ϵ test for the hypothesis (9.1)

However, in the majority of the cases, we shall prefer to use a single valued statistic which will be a function of $\underset{\sim}{T}_n$. If we proceed as in section 4 or 5, we may deduce (under similar conditions as in there,) the joint asymptotic normality of $\underset{\sim}{T}_n$ both under the permutational as well as unconditional probability measures. Hence, it seems quite appropriate to use the quadratic form in $\underset{\sim}{T}_n$ with the discriminant as the inverse of the covariance matrix of the same.

Now, corresponding to the rank order statistic $T_n^{(i)}$, we define the sequence $\underset{\sim}{E}_N^{(i)}$ (as in (3.4)) by

$$E_N^{(i)} = (E_{N,1}^{(i)}, \dots, E_{N,N}^{(i)}), \quad i = 1, 2; \tag{9.2}$$

where

$$E_{N,\alpha}^{(i)} = J_{N(i)}\left(\frac{\alpha}{N+1}\right), \; 1 \le \alpha \le N, \; i = 1, 2. \tag{9.3}$$

The function $J_{N(i)}$ will be assumed to satisfy the regularity conditions of section 4. In addition to this, we say that $\underset{\sim}{E}_N^{(1)}$ and $\underset{\sim}{E}_N^{(2)}$ are orthogonal to each other, if

$$\sum_{\alpha=1}^{N} (E_{N,\alpha}^{(1)} - \bar{E}_N^{(1)})(E_{N,\alpha}^{(2)} - \bar{E}_N^{(2)}) = 0, \tag{9.4}$$

where $\bar{E}_N^{(i)}$, $i = 1, 2$ are defined as in (4.15). The orthogonality of $\underset{\sim}{E}_N^{(1)}$ and $\underset{\sim}{E}_N^{(2)}$ is not essential for our purpose, but we shall see later on that it simplifies the statistic appreciably and it holds in the majority of the cases. In fact, the location and scale tests considered in section 6 all satisfy this condition (under suitable choice of pairs of $T_n^{(1)}$, $T_n^{(2)}$).

If (9.4) holds, and we define the permutation variance of $T_n^{(i)}$ by $\sigma_{N,i}^2/N$, $i = 1, 2$ (as in (4.20)), then it is easy to show that under the permutational probability measure $\mathcal{P}_n$,

$$S_n = N \sum_{i=1}^{2} (T_n^{(i)} - \bar{E}_N^{(i)})^2 / \sigma_{N,i}^2 \tag{9.5}$$

has asymptotically, in probability, a χ^2 distribution with 2 d.f., and as a suitable test statistic, we may propose S_n. If (9.4) does not hold, we have to define

$$N \operatorname{Cov}(T_n^{(i)}, T_n^{(j)} \mid \mathcal{P}_n) = \sigma_{N,ij} \; ; \; i, j = 1, 2;$$

$$\Sigma_N = ((\sigma_{N,ij})), \quad \Sigma_N^{-1} = (\Sigma_N)^{-1} = ((\sigma_N^{ij})), \tag{9.6}$$

and

$$S_n = N \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_N^{ij} (T_n^{(i)} - \bar{E}_N^{(i)}) (T_n^{(j)} - \bar{E}_N^{(j)}). \tag{9.7}$$

It can be shown by a straight forward vector extension of theorem 4.2 that S_N will have asymptotically, in probability, (under $\mathcal{P}_n$), a chi square distribution with 2 d.f. Hence, in any case, we may use S_n in (9.5) or (9.7) as a suitable test-statistic, and carry out the permutation test based on it both for small and large samples.

Further, we define

$$\mu_n^{(i)}(F) = \int_{-\infty}^{\infty} J_{(i)}(H(x)) \, dF_{[1]}(x), \tag{9.8}$$

$\sigma_{n,ij}(F) = \sigma_n^2(F)$ in (5.3) with $J = J_{(i)}$ if $i=j$;

$$= \frac{1}{4} \Big\{ \iint_{-\infty < x < y < \infty} F_{[2]}(x)[1-F_{[2]}(y)] \Big\{ J'_{(1)}(H(x)) \, J'_{(2)}(H(y)) +$$

$$J'_{(1)}(H(y)) \, J'_{(2)}(H(x)) \Big\} dF_{[1]}(x) \, dF_{[2]}(y) \tag{9.9}$$

$$+ \iint_{-\infty < x < y < \infty} F_{[1]}(x)[1-F_{[1]}(y)] \Big\{ J'_{(1)}(H(x)) \, J'_{(2)}(H(y)) +$$

$$J'_{(2)}(H(x)) \, J'_{(1)}(H(y)) \Big\} dF_{[2]}(x \, dF_{[2]}(y)$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(x,y) - F_{[1]}(x) F_{[2]}(y)] \, J'_{(1)}(H(x)) \, J'_{(2)}(H(y)) \, dF_{[1]}(x) \, dF_{[2]}(y)$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(x,y) - F_{[1]}(x) F_{[2]}(y)] \, J'_{(1)}(H(y)) \, J'_{(2)}(H(x)) \, dF_{[1]}(x) \, dF_{[2]}(y) \Big\},$$

$i \neq j = 1, 2$

Then, by a more or less straight forward vector generalization of theorem 5.1, we will arrive at the (unconditional) joint asymptotic normality of

$$\sqrt{n}[(T_n^{(1)} - \mu_n^{(1)}(F)), (T_n^{(2)} - \mu_n^{(2)}(F))] \tag{9.10}$$

with a null mean vector and the dispersion matrix $\underset{\sim}{\Sigma}(F) = ((\sigma_{n,\ ij}(F)))$. Theorem 5.2 also extends directly to this case, and it can be shown easily that

$$\underset{\sim}{\Sigma}_N \overset{P}{\sim} \underset{\sim}{\Sigma}(F) \text{ under (5.30).} \tag{9.11}$$

Thus, in this case also, if we have any consistent estionator $\hat{\underset{\sim}{\Sigma}}(F)$ of $\underset{\sim}{\Sigma}(F)$, and we propose a large sample unconditional test based on the quadratic form in $\underset{\sim}{T}_n - \bar{\underset{\sim}{E}}_N$ with discriminant as the inverse of $\hat{\underset{\sim}{\Sigma}}(F)$, then the permutation test based on S_n in (9.7) will be stochastically equivalent to this test for sequences of alternatives in (5.30).

Let us now consider two specific rank-order tests for compound symmetry.

(I) <u>Rank order test, I.</u> Here, we let

$$E_{N,\alpha}^{(1)} = \alpha, \qquad 1 \leq \alpha \leq N. \tag{9.12}$$

$$E_{N,\alpha}^{(2)} = (\alpha - (N+1)/2)^2 \quad \text{for } 1 \leq \alpha \leq N. \tag{9.13}$$

Thus, $T_n^{(1)}$ will be the rank-sum test (as in (6.2),) for location, and $T_n^{(2)}$ will be the rank mean square test for Scale (as in (6.19)). It is easily seen that (9.4) holds for (9.12) and (9.13). So our statistic S_n reduces to

$$N\frac{(T_n^{(1)} - \frac{N+1}{2})^2}{\sigma_{N,1}^2} + N\frac{(T_n^{(2)} - \frac{N^2-1}{12})^2}{\sigma_{N,2}^2}, \tag{9.14}$$

where $\sigma_{N,i}^2$ is the permutation variance of $\sqrt{N}\,T_n^{(i)}$, i = 1, 2, which may be evaluated as in (4.18).

(II) <u>Ψ-score test</u> Here, we define $a_{N,i}$ as in (6.3), and let

$$E_{N,\alpha}^{(1)} = a_{N,\alpha}, \quad E_{N,\alpha}^{(2)} = (a_{N,\alpha} - \bar{A}_N)^2, \ \alpha = 1, \ldots, N, \tag{9.15}$$

where $\bar{a}_N = \frac{1}{N} \Sigma_{\alpha=1}^{N} a_{N,\alpha}$. Alternatively, we may take,

$$E_{N,\alpha}^{(1)} = \Psi^{-1}\left(\frac{\alpha}{N+1}\right), \quad E_{N,\alpha}^{(2)} = \left[\Psi^{-1}\left(\frac{\alpha}{N+1}\right) - \Psi^{-1}\left(\frac{1}{2}\right)\right]^2,$$

$$\alpha = 1, \ldots, N. \tag{9.16}$$

Again, if Ψ is a symmetric cdf, it is easily seen that (9.4) holds. (In particular, if Ψ is normal, then also (9.4) holds.) Hence, for symmetric Ψ, the statistic S_n in (9.7) reduces to a form similar to (9.14), where instead of $(N+1)/2$ and $(N^2-1)/12$, we will have to use the respective means of $E_{N,\alpha}^{(1)}$ and $E_{N,\alpha}^{(2)}$ in (9.15) or (9.16). For normal Ψ, the above statistic will be termed the normal score statistic and the test as the normal score test for compound symmetry.

As for the study of the asymptotic power properties of these tests, we may use the extension of Theorem 5.2 considered earlier, and with that for the sequence of altermatives:

$$H_n^{(3)}: \; F_{[2]}(x) = F_{[1]}([x + n^{-\frac{1}{2}} \theta_1][1 + n^{-\frac{1}{2}} \theta_2]) \tag{9.17}$$

(with θ_1, θ_2 real and finite and $F_{[1]}$ satisfying the conditions of lemma 7.2 of Puri (1964),) it can be easily shown that for normal alternatives, the normal score test and Wilks' L_{MV} test are asymptotically power-equivalent. The efficiency of the normal score test against L_{MV} test for non-normal alternatives will depend on the two unknown correlations $\rho(F, J_{(1)})$ and $\rho(F, J_{(2)})$ in (6.7), (where $J_{(1)} = \Psi^{-1}(F_{[1]}(x)$ and $J_2 = [\Psi^{-1}(F_{[1]}(x))]^2$,) and hence nothing can be said, in general, about the bounds for the same. Finally, the efficiency of the other rank test will also depend on two such unknown correlations, and hence, no proper bounds can be attached to the same. However, it can be shown that there are many situations where the proposed rank order tests are more efficiant (asymptotically) than the Wilks' L_{MV} test. As an example, we may consider the case, where

$$F(x, y) = F_{[1]}(x) F_{[2]}(y), \tag{9.18}$$

where both $F_{[1]}$ and $F_{[2]}$ are cauchy cdf's with appropriate cdf's. In this extreme case, the efficiency of the normal score test with respect to L_{MV} test will be indefinitely large; in fact L_{MV} test will be inconsistent in this case, while the other one will be reasonably efficient.

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