ABSTRACT

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Leibniz algebras are a non-commutative generalization of Lie algebras that satisfy all the defining properties of Lie algebras except for the antisymmetry of its product. In 1965, Bloh first considered Leibniz algebras, though he called them D-algebras. They were popularized by Loday, who adopted the name Leibniz algebras, in the 1990s. All Lie algebras are Leibniz algebras. A natural question to ask is which results from Lie algebras hold, sometimes with slight modifications, in Leibniz algebras? Many important results in Lie algebras have been shown to have Leibniz analogs such as Lie's Theorem, Engel's Theorem, and Cartan's criterion. In this dissertation, we are going to examine some results of subinvariance and ascendancy in Lie algebras and see if they generalize to Leibniz algebras.

A subalgebra B of Leibniz Algebra A is said to be subinvariant if there exists a chain of ideals $B = B_0 \lhd B_1 \lhd \ldots \lhd B_n = A$. The infinite dimensional analog of subinvariance is ω-ascendancy. A subalgebra B of Leibniz Algebra A is said to be ω-ascendant if there exists $B = B(0) \lhd B(1) \lhd \cdots \lhd B(\omega) = A$ where $B(\omega) = \bigcup_{k=0}^{\infty} B(k)$. In this dissertation, we give the Leibniz analog of several results in Lie algebras on subinvariance and ascendancy. For example, if a subalgebra B of Leibniz algebra A is subinvariant (respectively ω-ascendant) in two subalgebras H and K of A, we show that for certain classes of Leibniz Algebras, B is subinvariant (respectively ω-ascendant) in the subalgebra generated by H and K.

Subinvariance and Ascendancy in Leibniz Algebras

by
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BIOGRAPHY

Emma Kathryn Groves was born in Virginia and moved states several times growing up before her family settled in Brenham, TX. She graduated from Brenham High School in 2013. She then attended Southwestern University where she earned a Bachelor of Arts in Mathematics and Computer Science in 2017. Beginning in 2017, she attended North Carolina State University and earned a Master of Science in Mathematics in 2019 before completing her Doctor of Philosophy in Mathematics with a minor in Operations Research in 2023.

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TABLE OF CONTENTS

CHAPTER

1

INTRODUCTION

1.1 Introduction

Marius Sophus Lie first introduced the concept of Lie algebras to study infinitesimal transformations in the 1870s [13]. Leibniz algebras are a non-commutative generalization of Lie algebras that satisfy all the defining properties of Lie algebras except for the antisymmetry of its product. In 1965, Bloh first considered Leibniz algebras, though he called them D-algebras [3]. However, they were popularized by Loday, who adopted the name Leibniz algebras, in the 1990s [10]. Loday discovered, while studying the properties of the homology of Lie algebras, that the antisymmetry of the product was not needed to prove the derived property defined on chains. This led him to introduce the notion of right Leibniz algebras, where the right multiplication operator is a derivation. Similarly, in a left Leibniz algebra the left multiplication operator is a derivation. In this dissertation, a Leibniz algebra refers to a left Leibniz algebra.

All Lie algebras are Leibniz algebras. A natural question to ask is which results from Lie algebras hold, sometimes with slight modifications, in Leibniz algebras? Many important results in Lie algebras have been shown to have Leibniz analogs such as Lie's Theorem, Engel's Theorem, and Cartan's criterion [6]. In this dissertation, we are going to examine some results on subinvariance and ascendancy in Lie algebras and see if they generalize to Leibniz algebras. Wielandt first introduced the concept of subinvariance for groups [18]. Schenkman later popularized the idea of subinvariance in Lie algebras [14]. Chao and Stitzinger looked at subinvariance in finite dimensional, solvable Lie algebras [5]. Later, Stitzinger extended this work to infinite dimensional, locally solvable, ideally finite Lie algebras [16]. Kawamoto explored the relationship between repeated multiplication by a subalgebra and subinvariance [9] which was generalized by Tôgô and called weakly ascendant subalgebras [17]. Aldosray in [1, 2] and Maruo in [11] examined if a subalgebra B is subinvariant in two subalgebras H, K, when is B subinvariant in the subalgebra generated by H and K.

1.2 Preliminaries

In this section we will introduce the basic definitions, structures, and properties of Leibniz algebras.

Definition 1.2.1. A **left Leibniz algebra** A is a vector space over a field with a bilinear product

$$[\,,\,] : A \times A \to A$$

that satisfies the Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

for all $a, b, c \in A$. A **right Leibniz algebra** is a vector space over a field with a bilinear product that satisfies the Leibniz identity

$$[[a,b],c] = [a,[b,c]] + [[a,c],b]$$

for all $a, b, c \in A$.

Note that all Lie algebras are a Leibniz algebras. However, a Leibniz algebra is not always both a left and right Leibniz algebra as this example shows.

Example 1.2.2. Consider A, a 2-dimensional algebra with multiplications

$$[x,x] = 0,\ [x,y] = 0,\ [y,x] = x,\ [y,y] = x$$

First let us show that A is a left Leibniz algebra.

$$
\begin{aligned}
{[x,[x,x]]} &= 0 = [[x,x],x]+[x,[x,x]] \\
{[y,[x,x]]} &= 0 = [x,x]+[x,x] = [[y,x],x]+[x,[y,x]] \\
{[x,[y,x]]} = [x,x] &= 0 = [[x,y],x]+[y,[x,x]] \\
{[x,[x,y]]} &= 0 = [[x,x],y]+[x,[x,y]] \\
{[x,[y,y]]} = [x,x] &= 0 = [[x,y],y]+[y,[x,y]] \\
{[y,[x,y]]} &= 0 = [x,y]+[x,y] = [[y,x],y]+[x,[y,y]] \\
{[y,[y,x]]} = [y,x] &= x = [x,x]+[y,x] = [[y,y],x]+[y,[y,x]] \\
{[y,[y,y]]} = [y,x] &= x = [x,y]+[y,x] = [[y,y],y]+[y,[y,y]]
\end{aligned}
$$

Thus, all combinations of x and y satisfy the left Leibniz identity. Therefore, A is a left Leibniz algebra. Now consider

$$[[y,y],y] = [x,y] = 0 \neq x = [y,x]+[x,y] = [y,[y,y]]+[[y,y],y].$$

Since $[[y,y],y] = [x,y] \neq [y,[y,y]]+[[y,y],y]$ A does not satisfy the right Leibniz identity and is therefore not a right Leibniz algebra.

For a Leibniz algebra A with element a we define two multiplication operators: the left multiplication operator $L_a : A \to A$ by $L_a(b) = [a,b]$ and the right multiplication operator $R_a : A \to B$ by $R_a(b) = [b,a]$ for all $b \in A$. Due to the Leibniz identity, L_a is a derivation in a left Leibniz algebra and R_a is a derivation in a right Leibniz algebra. Throughout this work, a Leibniz algebra always refers to a left Leibniz algebra.

Definition 1.2.3. A subspace B of Leibniz algebra A is a **subalgebra** if $[B,B] \subseteq B$.

Let C be a subset of a Leibniz algebra A. Then $\langle C \rangle$ denotes the subalgebra generated by C. In other words, $\langle C \rangle$ is the smallest subalgebra of A containing C. If $B = \langle C \rangle$ has a finite set of generators, then we say that B is finitely generated.

Definition 1.2.4. A subalgebra I of Leibniz algebra A is a **left ideal** if $[A,I] \subseteq I$ and **right ideal** if $[I,A] \subseteq A$. If I is both a left and right ideal, then I is an **ideal** of A. We denote I an ideal of A as $I \lhd A$.

Like in Lie algebras, the sum and intersection of two ideals is an ideal. Unlike Lie algebras, the bracket of two ideals is not necessarily an ideal, as shown in the example below.

Example 1.2.5. Let $A = \text{span}\{x, y, v, u, w\}$ have non-zero multiplications

$$[y,v] = u,\ [v,y] = w,\ [x,y] = y = -[y,x],\ [x,u] = u,\ [x,w] = w,\ [u,x] = w,\ [w,x] = -w$$

Let $I = \text{span}\{y, u, w\}$ and $J = \text{span}\{v, u, w\}$. Both I and J are ideals of A. $[I, J] = \text{span}\{u\}$ is not and ideal of A.

Definition 1.2.6. The **left center** of A is denoted by $Z^l(A) = \{x \in A \mid [x,a] = 0 \text{ for all } a \in A\}$. The **right center** of A is denoted by $Z^r(A) = \{x \in A \mid [a,x] = 0 \text{ for all } a \in A\}$. The **center** of A is $Z(A) = Z^l(A) \cap Z^r(A)$.

For Leibniz algebra A, define $\text{Leib}(A) = \text{span}\{[a,a] \mid a \in A\}$. $\text{Leib}(A)$ is the minimal ideal of A such that $A/\text{Leib}(A)$ is a Lie algebra. Define $A^{(1)} = [A, A]$ and $A^{(i+1)} = [A^{(i)}, A^{(i)}]$ for all $i \in \mathbb{N}$. The series of ideals

$$A \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \dots$$

is called the derived series of A.

Definition 1.2.7. A Leibniz algebra A is **solvable** if $A^{(m)} = 0$ for some $m \in \mathbb{N}$.

Like in Lie algebras, the sum and intersection of two solvable ideals of a Leibniz algebra is also solvable. Therefore, all Leibniz algebras A contain a unique maximal solvable ideal denoted $\text{rad}(A)$.

Define $A^1 = [A, A]$ and $A^{i+1} = [A, A^i]$ for all $i \in \mathbb{N}$. The series of ideals

$$A \supseteq A^1 \supseteq A^2 \supseteq \dots$$

is called the lower central series of A.

Definition 1.2.8. A Leibniz algebra A is **nilpotent** if $A^m = 0$ for some $m \in \mathbb{N}$.

Like in Lie algebras, the sum and intersection of two nilpotent ideals of a Leibniz algebra is also nilpotent. Therefore, all Leibniz algebras A contain a unique maximal nilpotent ideal called the nilradical of A and is denoted $\text{nilrad}(A)$.

Definition 1.2.9. Let A be a Leibniz algebra and B be a subalgebra of A. B is **subinvariant** in A, written $B \lhd\lhd A$, if there exists a chain of ideals such that $B = B(1) \lhd B(2) \lhd \dots \lhd B(n) = A$.

Example 1.2.10. Let A be a 5-dimensional Leibniz algebra with basis $\{x_1, x_2, x_3, x_4, x_5\}$. The nonzero products are given by:

$$[x_1, x_1] = x_5, [x_1, x_2] = x_3 = -[x_2, x_1], [x_1, x_3] = x_4 = -[x_3, x_1],$$

$$[x_1, x_4] = x_5 = -[x_4, x_1]$$

We can construct the chain of ideals

$$\mathrm{span}\{x_1, x_5\} \lhd \mathrm{span}\{x_1, x_4, x_5\} \lhd \mathrm{span}\{x_1, x_3, x_4, x_5\} \lhd A.$$

Thus $\mathrm{span}\{x_1, x_5\}$ is subinvariant in A, but clearly not an ideal of A.

Now let us give an example of a subalgebra that is not subinvariant.

Example 1.2.11. Let A be a 4-dimensional Leibniz algebra with basis $\{a, b, x, y\}$. The nonzero products are

$$[a, x] = x, \; [b, y] = y.$$

$B = \mathrm{span}\{a, b\}$ is a subalgebra of A. Note that B is not an ideal of A since $[a, x] = x$. For a contradiction, let C be a three-dimensional subalgebra of A such that $B \lhd C$. Then $C = \mathrm{span}\{a, b, \alpha a + \beta b + \gamma x + \delta y\}$. $[a, \alpha a + \beta b + \gamma x + \delta y] = \gamma x$ and since $B \lhd C$, then $\gamma = 0$. Similarly, $[b, \alpha a + \beta b + \gamma x + \delta y] = \delta y$ and since $B \lhd C$, then $\delta = 0$. Thus $C = \mathrm{span}\{a, b, \alpha a + \beta b\}$ and is only two-dimensional, which contradicts the assumption that C is three-dimensional. Therefore B is not an ideal of a three-dimensional subalgebra of A or A. Thus we cannot construct a chain of ideals from B to A and B is not subinvariant in A.

The analogous structure to subinvariance in infinite dimensional Leibniz algebras is ascendancy.

Definition 1.2.12. Let A be a Leibniz algebra and B be a subalgebra of A. A is ω**-ascendant** in A if there exists a chain $B = B(0) \lhd B(1) \lhd \cdots \lhd B(\omega) = A$ where $B(\omega) = \bigcup_{k=0}^{\infty} B(k)$.

CHAPTER

2

SUBINVARIANCE

2.1 Finite Dimensional Solvable Leibniz Algebras

We will start by examining finite dimensional solvable Leibniz algebras. Let T be a linear transformation acting on a finite dimensional vector space V. Let

$$V_i^0 = \{v | T^i(v) = 0\}$$
$$V_i^1 = \{v | \exists\, w \in V \text{ such that } T^i(w) = v\}$$

Then

$$V_1^0 \subset V_2^0 \subset \cdots \subset V_r^0 = V_{r+1}^0 \cdots$$
$$V_1^1 \supset V_2^1 \supset \cdots \supset V_s^0 = V_{s+1}^0 \cdots$$

where $r = s$ and $V = V_r^0 \oplus V_r^1$. V_r^0 and V_r^1 are called the Fitting null component and Fitting one component, respectively.

Now let V be an algebra. If T is a derivation of V then V_r^0 is a subalgebra of V, T acts nilpotently on V_r^0, and T is nonsingular on V_r^1. For Leibniz algebra A and all $x \in A$, L_x is a derivation. Let $A_0(x)$ be the Fitting null component of L_x acting on A and $A_1(x)$ be the Fitting one component of L_x acting on A.

Lemma 2.1.1. *Let A be a finite dimensional solvable Leibniz algebra and D be a nilpotent subalgebra of A such that each element $x \in D$ has L_x nilpotent on A. Then D is subinvariant in A.*

Proof. Let B be a minimal ideal of A. We claim B is abelian since A is solvable.

Assume for a contradiction that B is not abelian. Then there exists $x \neq 0$ such that $x = [a, b]$, $x, a, b \in B$. Let $y \in A$. Then

$$[y, x] = [y, [a, b]] = [\underbrace{[y, a]}_{\in B}, b] + [a, \underbrace{[y, b]}_{\in B}] \in [B, B]$$
$$[x, y] = [[a, b], y]] = [a, \underbrace{[b, y]}_{\in B}] - [b, \underbrace{[a, y]}_{\in B}] \in [B, B]$$

Thus $[B, B] \lhd A$ such that $[B, B] \subsetneq B$. This contradicts the minimality of B. Thus, B is abelian.

In $D + B$, each element $x \in D$ has L_x nilpotent on $D + B$, since L_x is nilpotent on A for all $x \in D$. Using the Jacobson's Refinement to Engel's Theorem [4], $D + B$ is nilpotent. Since a Leibniz algebra is nilpotent if and only if all subalgebras are subinvariant, $D + B$ nilpotent implies D is subinvariant in $D + B$. Now $(D + B)/B$ satisfies the conditions in A/B by induction. Therefore, $D + B$ is subinvariant in A. $D \lhd\lhd D + B$ and $D + B \lhd\lhd A$ implies $D \lhd\lhd A$. □

Theorem 2.1.2. *Let A be a finite dimensional solvable Leibniz algebra and B be a subalgebra of A that is not subinvariant in A but is subinvariant in all proper subalgebras of A that contain B. Let M be a maximal subalgebra of A with $B \subseteq M$. Then there exists $x \in B$ such that $M = B + A_0(x)$.*

Proof. Note that $B + A_0(x)$ is only a vector space sum.

First, let us show that $M \subseteq B + A_0(x)$ for all $x \in B$:

We claim since $B \lhd\lhd M$, $M_1(x) \subseteq B$. Let $x \in B$. $M_1(x) = \{(L_x)^r(w) | w \in M\}$. Let $m \in M_1(x)$. Then $m = (L_x)^r(w)$ for some $w \in M$. There exists a chain of subalgebras of M such that $B = B_s \lhd \cdots \lhd B_1 = M$.

$$L_x(w) = [x, w] \in B_2 \text{ since } x \in B \subset B_2, B_2 \lhd B_1 = M \text{ and } w \in M$$
$$(L_x)^2(w) = [x, [x, w]] \in B_3 \text{ since } x \in B \subset B_3, B_3 \lhd B_2 \text{ and } [x, w] \in B_2$$

$$\vdots$$

$$(L_x)^r(w) \in B_{r+1}$$

$r+1 \geq s$ otherwise $M_1^{r+1} \neq M_1^r$, which is a contradiction. Thus, $m \in B_{r+1} \subseteq B_s = B$ which implies that $M_1(x) \subseteq B$. So, $M = M_0(x) + M_1(x) \subseteq M_0(x) + B \subseteq A_0(x) + B$.

Now we need to show $B + A_0(x) \subseteq M$ for some $b \in B$. Since $B \subseteq M$, we need to show there exists $x \in B$ such that $A_0(x) \subseteq M$.

Case 1: Suppose there exists a minimal ideal C of A such that $C \subseteq M$. Then $(B+C)/C$ satisfies the properties of the theorem in A/C. By induction, there exists $\overline{x} \in (B+C)/C$ such that $(A/C)_0(\overline{x}) \subseteq M/C$. Let $y \in A$ and $L_x^n(y) = 0$ where $\overline{x} = x + C$. Then $L_{x+C}^n(y+C) = L_x^n(y) + C$ which implies that $y + C \in (A/C)_0(\overline{x}) \subseteq M/C$. Hence $y \in M$ and $A_0(x) \subseteq M$.

Case 2: Assume no minimal ideal of A is contained in M. Then A is a solvable primitive algebra (i.e A contains a unique ideal D which is its own centralizer) with self centralizing minimal ideal D. $A = D + M$ and $D \cap M = 0$. (i.e $A = D \oplus M$ is a semidirect sum).

If B is nilpotent, then since $B \lhd\lhd M$, $A_0(x) \supseteq M$ for all $x \in B$. If $A_0(x) = A$ for all $x \in B$, then $B \lhd\lhd A$ by Lemma 2.1.1, which is a contradiction. Hence, $A_0(x) \neq A$ for some $x \in B$ and the subalgebra $A_0(x)$ contains the maximal subalgebra M, so $A_0(x) = M$ [12].

Suppose B is not nilpotent. Since $B \lhd\lhd M$, $B^w \neq 0$ and $B^w \lhd M$. Hence, B^w contains minimal ideal C of M. Since M is solvable, C is abelian. Since $C \lhd M$, $A_0(x) \supseteq M$ for all $x \in C$. Hence, either $A_0(x) = M$ for some $x \in C$ and we are done or $A_0(x) = A$ for all $x \in C$. In that case, let $E = D + C$. Then E is nilpotent by the Jacobson refinement to Engel's theorem for Leibniz algebras and $E \lhd A$ since $E/D \simeq C \lhd M \simeq A/D$. Since $C_A(D) = D$, $C_E(D) = D$. Therefore, $Z(E) \subseteq D$ ($Z(E)$ is the center of E). But $Z(E) \lhd A$ since it is characteristic in E. By the minimality of D either $Z(E) = 0$, contradicting nilpotency of E, or $Z(E) = D$ which yields $E = C_E(Z(E)) = C_E(D) = D$ and $B = 0$, which is a contradiction. Hence, $A_0(x) = M$ for some $x \in B$ and $M = A_0(x) + B$. $\square$

Theorem 2.1.3. *Let A be a finite dimensional solvable Leibniz algebra and B be a subalgebra of A. The following are equivalent:*

1. $B \lhd\lhd A$

2. *For all* $b \in B, A = B + A_0(b)$.

Proof. Assume 1. Then $A_1(b) \subseteq B$ for all $b \in B$ and $A = A_0(b) + A_1(b) \subseteq A_0(b) + B$ for all $b \in b$.

Assume 2. Let (B, A) be a minimal counterexample. Then B is subinvariant in a maximal subalgebra M of A that contains B. By Theorem 2.1.2, there exists $x \in B$ such that $M = B + A_0(x)$. This contradicts the hypothesis. □

Theorem 2.1.4. *Let A be a finite dimensional solvable Leibniz algebra and B be a subalgebra of A. The following are equivalent:*

1. $B \lhd\lhd A$
2. *For each* $x \in A$, $B \lhd\lhd \langle B, x\rangle$
3. *For each* $x \in A$, $B \lhd\lhd \langle B, R_x(B)\rangle$

Proof. Clearly $1 \implies 2 \implies 3$ since $\langle B, x\rangle \subseteq A$ and $\langle B, R_x(B)\rangle \subseteq \langle B, x\rangle$. We will show that 3 implies the equivalent condition to subinvariant in Theorem 2.1.3, for all $b \in B$ $A = B + A_0(b)$.

For each $b \in B$ and $n = \dim(A)$, $L_b^n(x) \subseteq B$ for all $x \in A$ since B is subinvariant in $\langle B, R_x(B)\rangle$. Hence $A_1(b) \subseteq B$ and $B + A_0(b) = A$ for each $b \in B$. Therefore, by Theorem 2.1.3, B is subinvariant in A □

Recall that B is a right ideal of A if $[B, A] \subset B$. Denote B a right ideal of A as $B \lhd_r A$.

Definition 2.1.5. We say that B is right subinvariant in A if there exists a chain of right ideals

$$B = B_0 \lhd_r B_1 \lhd_r \ldots \lhd_r B_m = A.$$

Theorem 2.1.6. *Let A be a finite dimensional solvable Leibniz and B be a subalgebra of A. Then B is right subinvariant in A if and only if B is subinvariant in A.*

Proof. B subinvariant in A clearly implies that B is right subinvariant in A since a chain of ideals is also a chain of right ideals.

Now let us assume that B is right subinvariant in A. Then there exists a chain of right ideals

$$B = B_0 \lhd_r B_1 \lhd_r \ldots \lhd_r B_m = A.$$

Let $b \in B$ then $L_b^m(x) \subseteq B$ for all $x \in A$ due to the chain of right ideals. Thus $A_1(x) \subseteq B$ implies $A = B + A_0(b)$ since $A = A_1(b) \oplus A_0(b)$. By Theorem 2.1.3 $B \lhd\lhd A$.

□

2.2 Curly Bracket

Let A be a Leibniz algebra and let $x, y \in A$ and B, C be subsets of A. Define the following:

$$\{x,{}_1 y\} = [x, y] - [y, x];\ \{x,{}_{i+1} y\} = [\{x,{}_i y\}, y] - [y, \{x,{}_i y\}]$$
$$\{C,{}_1 B\} = [C, B] + [-B, C];\ \{C,{}_{i+1} B\} = [\{C,{}_i B\}, B] + [-B, \{C,{}_i B\}]$$

Note that if B is a subalgebra of A then $B = -B$ so $\{C,{}_1 B\} = [C, B] + [B, C]$ and $\{C,{}_{i+1} B\} = [\{C,{}_i B\}, B] + [B, \{C,{}_i B\}]$.

Definition 2.2.1. Let H, K be subalgebras of A. We say that (H, K) is a N_k-pair if $\{K,{}_k H\} \subseteq H$ for some $k \in \mathbb{N}$. (H, K) is a N_∞-pair if for each $x \in K$ there exists $k \in \mathbb{N}$ such that $\{x,{}_k H\} \subseteq H$.

Let $N_k(H) = \{a \in A | \{a,{}_k H\} \subseteq H\}$ for $k \in \mathbb{N}$ and $N_\infty(H) = \bigcup_{k \in \mathbb{N}} N_k(H)$. Note that $N_k(H) \subseteq N_{k+1}(H)$ since for all $a \in N_k(H)$

$$\{a,{}_{k+1} H\} = [H, \{a,{}_k H\}] + [\{a,{}_k H\}, H] \subseteq [H, H] + [H, H] = [H, H] \subseteq H$$

Let $H^\omega = \bigcap_{n \in \mathbb{N}} H^n$.

Lemma 2.2.2. *Let A be a Leibniz algebra over a field, C be a subset of A and H be a subalgebra of A. Then*

1. $[H^l, C] \subseteq \{C,{}_l H\}$ *for* $l \in \mathbb{N}$.
2. $[C, H^l] \subseteq \{C,{}_l H\}$ *for* $l \in \mathbb{N}$.

Proof. We will show 1 using induction. For the base case of $l = 1$ we have

$$[H, C] \subseteq [H, C] + [C, H] = \{C,{}_1 H\}$$

Now assume $[H^l, C] \subseteq \{C, {}_l H\}$. Then

$$\begin{aligned}
[H^{l+1}, C] &= [[H, H^l], C] \\
&\subseteq [H, [H^l, C]] + [H^l, [H, C]] \\
&\subseteq [H, \{C, {}_l H\}] + [H^l, \{C, {}_1 H\}] \\
&\subseteq \{C, {}_{l+1} H\} + \{\{C, {}_1 H\}, {}_l H\} \\
&= \{C, {}_{l+1} H\} + \{C, {}_{l+1} H\} \\
&= \{C, {}_{l+1} H\}
\end{aligned}$$

Therefore $[H^l, C] \subseteq \{C, {}_l H\}$.

Next, we will show 2 using induction. For the base case of $l = 1$ we have

$$[H, C] \subseteq [H, C] + [C, H] = \{C, {}_1 H\}$$

Now assume $[C, H^l] \subseteq \{C, {}_l H\}$. Then

$$\begin{aligned}
[C, H^{l+1}] &= [C, [H, H^l]] \\
&\subseteq [[C, H], H^l] + [H, [C, H^l]] \\
&\subseteq [\{C, {}_1 H\}, H^l] + [H, \{C, {}_l H\}] \\
&\subseteq \{C, {}_{l+1} H\} + \{\{C, {}_l H\}, {}_1 H\} \\
&= \{C, {}_{l+1} H\} + \{C, {}_{l+1} H\} \\
&= \{C, {}_{l+1} H\}
\end{aligned}$$

Therefore $[C, H^l] \subseteq \{C, {}_l H\}$.

□

Lemma 2.2.3. *Let A be a Leibniz algebra such that H is a subalgebra of A and $x, y \in A$. Then $\{[x, y], {}_m H\} \subseteq \sum_{i+j=m} [\{x, {}_i H\}, \{y, {}_j H\}] + [\{y, {}_j H\}, \{x, {}_i H\}]$.*

Proof. Let us prove this lemma by induction. First, let us consider the base case of $m = 1$:

$$\begin{aligned}
\{[x, y], {}_1 H\} &= [H, [x, y]] + [[x, y], H] \\
&\subseteq [[H, x], y] + [x, [H, y]] + [x, [y, H]] + [y, [x, H]] \\
&\subseteq [x, \{y, {}_1 H\}] + [\{y, {}_1 H\}, x] + [\{x, {}_1 H\}, y] + [y, \{x, {}_1 H\}] \\
&= \sum_{i+j=1} [\{x, {}_i H\}, \{y, {}_j H\}] + [\{y, {}_j H\}, \{x, {}_i H\}]
\end{aligned}$$

Now let us assume $\{[x, y], {}_m H\} \subseteq \sum_{i+j=m} [\{x, {}_i H\}, \{y, {}_j H\}] + [\{y, {}_j H\}, \{x, {}_i H\}]$ and show

$\{[x,y], {}_{m+1}H\} \subseteq \sum_{i+j=m+1}[\{x, {}_iH\},\{y, {}_jH\}]+[\{y, {}_jH\},\{x, {}_iH\}]$:

$$\begin{aligned}
\{[x,y], {}_{m+1}H\} &= [H,\{[x,y], {}_mH\}]+[\{[x,y], {}_mH\},H] \\
&\subseteq \sum_{i+j=m} [H,[\{x, {}_iH\},\{y, {}_jH\}]]+[H,[\{y, {}_jH\},\{x, {}_iH\}]] \\
&\qquad +[[\{x, {}_iH\},\{y, {}_jH\}],H]+[[\{y, {}_jH\},\{x, {}_iH\}],H] \\
&\subseteq \sum_{i+j=m} [[H,\{x, {}_iH\}],\{y, {}_jH\}]+[\{x, {}_iH\},[H,\{y, {}_jH\}]] \\
&\qquad +[[H,\{y, {}_jH\}],\{x, {}_iH\}]+[\{y, {}_jH\},[H,\{x, {}_iH\}]] \\
&\qquad +[\{x, {}_iH\},[\{y, {}_jH\},H]]+[\{y, {}_jH\},[\{x, {}_iH\},H]] \\
&\qquad +[\{y, {}_jH\},[\{x, {}_iH\},H]]+[\{x, {}_iH\},[\{y, {}_jH\},H]] \\
&\subseteq \sum_{i+j=m} [\{x, {}_{i+1}H\},\{y, {}_jH\}]+[\{x, {}_iH\},\{y, {}_{j+1}H\}] \\
&\qquad +[\{y, {}_{j+1}H\},\{x, {}_iH\}]+[\{y, {}_jH\},\{x, {}_{i+1}H\}] \\
&= \sum_{i+j=m+1} [\{x, {}_iH\},\{y, {}_jH\}]+[\{y, {}_jH\},\{x, {}_iH\}]
\end{aligned}$$

We have shown that $\{[x,y], {}_mH\} \subseteq \sum_{i+j=m}[\{x, {}_iH\},\{y, {}_jH\}]+[\{y, {}_jH\},\{x, {}_iH\}]$.

□

Lemma 2.2.4. *Let A be a Leibniz algebra over a field and H be a subalgebra of A. Then $[H^i,H^j] \subset H^{i+j}$ for $i,j \in \mathbb{N}$.*

Proof. Let us do an induction on i. For the base case of $i=1$ we have that

$$[H,H^j] = H^{j+1} \subseteq H^{j+1}$$

for all $j \in \mathbb{N}$. Now let use assume $[H^i,H^j] \subset H^{i+j}$ for all $j \in \mathbb{N}$ and show $[H^{i+1},H^j] \subset H^{i+j+1}$

$$\begin{aligned}
[H^{i+1},H^j] &= [[H,H^i],H^j] \\
&\subseteq [H,[H^i,H^j]]+[H^i,[H,H^j]] \\
&\subseteq [H,H^{i+j}]+[H^i,H^{j+1}] \\
&\subseteq H^{i+j+1}+H^{i+j+1} \\
&= H^{i+j+1}
\end{aligned}$$

Therefore $[H^i,H^j] \subset H^{i+j}$ for all $i,j \in \mathbb{N}$.

□

Lemma 2.2.5. *Let A be a Leibniz algebra over a field and H be a subalgebra of A. Then*

1. *$N_\infty(H)$ is a subalgebra of A and $(H, N_\infty(H))$ is a N_∞ pair.*

2. *H^ω is an ideal of $N_\infty(H)$.*

Proof. First let us show 1. Let $x, y \in N_\infty(H)$. There exists some $k \in \mathbb{N}$ such that $x, y \in N_k(H)$. Let $m = 2k-1$. By Lemma 2.2.3 $\{[x,y],\, _mH\} \subseteq \sum_{i+j=m}[\{x,\, _iH\},\{y,\, _jH\}]+[\{y,\, _jH\},\{x,\, _iH\}]$.

We claim that for any $i, n \in \mathbb{N}$ with $i > n$ that $\{x,\, _iH\} = \{\{x,\, _{i-n}H\},\, _nH\}$.

$$\begin{aligned}
\{x,\, _iH\} &= L_H(\{x,\, _{i-1}H\}) + R_H(\{x,\, _{i-1}H\}) \\
&= L_H L_H(\{x,\, _{i-2}H\}) + L_H R_H(\{x,\, _{i-2}H\}) + R_H L_H(\{x,\, _{i-2}H\}) + R_H R_H(\{x,\, _{i-2}H\}) \\
&\vdots \\
&= \sum_{\alpha_l+\beta_l=1;\alpha_l,\beta_l\in 0,1} L_H^{\alpha_1} R_H^{\beta_1} \dots L_H^{\alpha_n} R_H^{\beta_n}(\{x,\, _{i-n}H\}) \\
&= \{\{x,\, _{i-n}H\},\, _nH\}.
\end{aligned}$$

Thus $\{x,\, _iH\} = \{\{x,\, _{i-n}H\},\, _nH\}$.

Let $i, j \in \mathbb{N}$ such that $i + j = m = 2k-1$. Then either $i \geq k$ or $j \geq k$. Let $i \geq k$ and consider $[\{x,\, _iH\},\{y,\, _jH\}]$. $\{x,\, _iH\} = \{\{x,\, _kH\},\, _{i-k}H\} \subseteq \{H,\, _{i-k}H\}$ since $x \in N_k(H)$. Thus $[\{x,\, _iH\},\{y,\, _jH\}] \subseteq [\{H,\, _{i-k}H\},\{y,\, _jH\}]$. Now let us show by induction that $\{H,\, _lH\} \subseteq H^{l+1}$ for $l \in \mathbb{N}$.

For the base case of $l = 1$ we have that:

$$\{H,\, _1H\} = [H,H] + [H,H] = H^2$$

Thus $\{H,\, _1H\} \subseteq H^2$. Now assume true for l and we will show for $l+1$.

$$\begin{aligned}
\{H,\, _{l+1}H\} &= [\{H,\, _lH\},H] + [H,\{H,\, _lH\}] \\
&\subseteq [H^{l+1},H] + [H,H^{l+1}] \\
&\subseteq H^{l+2} + H^{l+2} && \text{By Lemma 2.2.4} \\
&= H^{l+2}
\end{aligned}$$

Thus $\{H, {}_l H\} \subseteq H^{l+1}$. Therefore,

$$\begin{aligned}
[\{x, {}_i H\}, \{y, {}_j H\}] &\subseteq [H^{i-k+1}, \{y, {}_j H\}] \\
&\subseteq \{\{y, {}_j H\}, {}_{i-k+1} H\} && \text{By Lemma 2.2.2} \\
&= \{y, {}_{j+i-k+1} H\} \\
&= \{y, {}_{2k-1-k+1} H\} \\
&= \{y, {}_k H\} \\
&\subseteq H
\end{aligned}$$

Now let us consider $[\{y, {}_j H\}, \{x, {}_i H\}]$:

$$\begin{aligned}
[\{y, {}_j H\}, \{x, {}_i H\}] &= [\{y, {}_j H\}, \{\{x, {}_k H\}, {}_{i-k} H\}] \\
&\subseteq [\{y, {}_j H\}, \{H, {}_{i-k} H\}] \\
&\subseteq [\{y, {}_j H\}, H^{i-k+1}] \\
&\subseteq \{\{y, {}_j H\}, {}_{i-k+1} H\} \\
&= \{y, {}_{j+i-k+1} H\} \\
&= \{y, {}_{2k-1-k+1} H\} \\
&= \{y, {}_k H\} \\
&\subseteq H
\end{aligned}$$

Thus $[\{x, {}_i H\}, \{y, {}_j H\}] + [\{y, {}_j H\}, \{x, {}_i H\}] \subseteq H + H = H$. Similarly, if $j \geq k$ then $[\{x, {}_i H\}, \{y, {}_j H\}] + [\{y, {}_j H\}, \{x, {}_i H\}] \subseteq H$. Therefore,

$$\{[x, y], {}_m H\} = \sum_{i+j=m} [\{x, {}_i H\}, \{y, {}_j H\}] + [\{y, {}_j H\}, \{x, {}_i H\}] \subseteq H$$

which implies that $[x, y] \in N_m(H) \subseteq N_\infty(H)$. So $N_\infty(H)$ is a subalgebra of A.

Now let us show 2. Let $x \in N_\infty(H)$. Then $x \in N_k(H)$ for some $k \in \mathbb{N}$. Then for any $m > 0$.

$$\begin{aligned}
[x, H^\omega] &\subseteq [x, H^{k+m-1}] \\
&\subseteq \{x,\, _{k+m-1}H\} \\
&= \{\{x,\, _kH\},\, _{m-1}H\} \\
&\subseteq \{H,\, _{m-1}H\} \\
&\subseteq H^m \\
&\subseteq H^\omega
\end{aligned}$$

$$\begin{aligned}
[H^\omega, x] &\subseteq [H^{k+m-1}, x] \\
&\subseteq \{x,\, _{k+m-1}H\} \\
&\subseteq H^\omega
\end{aligned}$$

Thus $H^\omega \lhd N_\infty(H)$. $\square$

Theorem 2.2.6. *Let A be a Leibniz Algebra over a field. Let H be a subalgebra of A and B be an abelian ideal of A. If (H, B) is an N_n-pair for some $n \in \mathbb{N}$ then $H \lhd^n B + H$.*

Proof. Let $B_i = B \cap N_i(H)$ for $i \in \mathbb{N}$. We will show that $B_i + H \lhd B_{i+1} + H$ for all i. Let $g, h \in H$, $a \in B_i$ and $b \in B_{i+1}$. We claim that $[a+g, b+h] \in B_{i+1} + H$.

$$[a+g, b+h] = \underbrace{[a+b]}_{=0} + [a,h] + [g,b] + \underbrace{[g,h]}_{\in H}$$

Now let us show that $[a,h],[g,b] \in B_{i+1} = B \cap N_{i+1}(H)$. Since $B \lhd A$, $[a,h],[g,b] \in B$. Thus what is left to show is that $[a,h],[g,b] \in N_{i+1}(H)$.

$$
\begin{aligned}
\{[a,h],{}_{i+1}H\} &\subseteq \sum_{k+j=i+1} [\{a,{}_k H\},\{h,{}_j H\}]+[\{h,{}_j H\},\{a,{}_k H\}] \\
&\subseteq \sum_{k+j=i+1} [\{a,{}_k H\},\{H,{}_j H\}]+[\{H,{}_j H\},\{a,{}_k H\}] \\
&\subseteq \sum_{k+j=i+1} [\{a,{}_k H\},H^{j+1}]+[H^{j+1},\{a,{}_k H\}] \\
&\subseteq \sum_{k+j=i+1} \{\{a,{}_k H\},{}_{j+1}H\} \\
&= \{a,{}_{i+2}H\} \\
&\subseteq H
\end{aligned}
$$

because $a \in N_i(H) \subseteq N_{i+2}(H)$. Thus $[a,h] \in N_{i+1}(H)$.

$$
\begin{aligned}
\{[g,b],{}_{i+1}H\} &\subseteq \sum_{k+j=i+1} [\{g,{}_k H\},\{b,{}_j H\}]+[\{b,{}_j H\},\{g,{}_k H\}] \\
&\subseteq \sum_{k+j=i+1} [\{H,{}_k H\}],\{b,{}_j H\}]+[\{b,{}_j H\},\{H,{}_k H\}] \\
&\subseteq \sum_{k+j=i+1} [H^{k+1},\{b,{}_j H\}]+[\{b,{}_j H\},H^{k+1}] \\
&\subseteq \sum_{k+j=i+1} \{\{b,{}_j H\},{}_{k+1}H\} \\
&= \{b,{}_{i+2}H\} \\
&\subseteq H
\end{aligned}
$$

because $b \in N_{i+1}(H) \subseteq N_{i+2}(H)$.Thus $[g,b] \in N_{i+1}(H)$. Therefore $[a+g,b+h] \in B_{i+1}+H$. Similarly $[b+h,a+g]=[b,a]+[b,g]+[h,a]+[h,g] \in B_{i+1}+H$. Therefore $B_i+H \lhd B_{i+1}+H$. Since (H,B) is an N_n-pair then $B \subseteq N_n(H)$. Thus $B_n = B$. So $H = B_0 + H \lhd^n B_n + H = B + H$. □

In the next chapter, we will extend this result to ascendancy and N_∞ pairs.

2.3 Weak Subinvariance

Definition 2.3.1. A subalgebra H of Leibniz algebra A is said to be n-step weakly subinvariant if there exists an ascending chain of subspaces $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n$ such that

1. $M_0 = H$ and $M_n = A$

2. $[M_{i+1}, H], [H, M_{i+1}] \subseteq M_i$ for all $i < n$

When this is true, we write $H \leq^n A$.

Lemma 2.3.2. *Let A be a Leibniz Algebra over a field.*

a) If H is an n-step weakly subinvariant subalgebra of A and K is a subalgebra of A then $H \cap K$ is n-step weakly subinvariant in K.

b) If H is an n-step weakly subinvariant subalgebra of A and K is an ideal of A then $H + K$ is n-step weakly subinvariant in A.

c) Let f be a homomorphism of A onto $\overline{A}$, H be a subalgebra of A and $\overline{H}$ be a subalgebra of $\overline{A}$. If $H \leq^n A$ then $f(H) \leq^n \overline{A}$. If $\overline{H} \leq^n \overline{A}$ then $f^{-1}(\overline{H}) \leq^n A$.

Proof. Let $M_0 \subseteq \cdots \subseteq M_n$ be the weakly ascending chain of H in A.

a) Consider the chain $(M_0 \cap K) \subseteq \cdots \subseteq (M_n \cap K)$. $(M_0 \cap K) = H \cap K$. $(M_n \cap K) = A \cap K = A$. Let $x \in M_{i+1} \cap K$ and $y \in H \cap K$. Then $[x, y] \in M_i$ since $x \in M_{i+1}$ and $y \in H$ and $[x, y] \in K$ since $x, y \in K$. Thus $[x, y] \in M_i \cap K$ which implies that $[M_{i+1} \cap K, H \cap K] \subseteq M_i \cap K$. Similarly $[H \cap K, M_{i+1} \cap K] \subseteq M_i \cap K$. Thus $(M_0 \cap K) \subseteq \cdots \subseteq (M_n \cap K)$ is a weakly ascending chain and $H \cap K$ is n-step weakly ascendant in A.

b) Consider the chain $(M_0 + K) \subseteq \cdots \subseteq (M_n + K)$. $(M_0 + K) = H + K$. $(M_n + K) = A + K = A$. Let $m \in M_{i+1}, h \in H, k \in K$ and $j \in K$. Then

$$[m + k, h + j] = \underbrace{[m, h]}_{\in M_i} + \underbrace{[m, j]}_{\in K} + \underbrace{[k, h]}_{\in K} + \underbrace{[k, j]}_{\in K} \in M_i + K$$

which implies that $[M_{i+1} + K, H + K] \subseteq M_i + K$. Similarly $[H + K, M_{i+1} + K] \subseteq M_i + K$. Thus $(M_0 + K) \subseteq \cdots \subseteq (M_n + K)$ is a weakly ascending chain and $H + K$ is n-step weakly ascendant in A.

c) $f(M_0) \subseteq \cdots \subseteq f(M_n)$ is a weakly ascending chain for $f(H)$ in $\overline{A}$. Let $\overline{M_0} \subseteq \cdots \subseteq \overline{M_n}$ be a weakly ascending chain for $\overline{H}$ in $\overline{A}$. Then $f^{-1}(M_0) \subseteq \cdots \subseteq f^{-1}(M_n)$ is a weakly ascending chain for $f^{-1}(\overline{H})$ in A.

□

Lemma 2.3.3. *Let H be a subalgebra of Leibniz algebra A. For $n \in \mathbb{N}$, H is n-step weakly subinvariant if and only if (H, A) is a N_n-pair.*

Proof. Assume that H is n-step weakly subinvariant in A. Then we have a weakly ascending chain $M_0 \subseteq \cdots \subseteq M_n$. Then

$$\{A,{}_1H\} = [A,H]+[H,A] = [M_n,H]+[H,M_n] \subseteq M_{n-1}$$
$$\{A,{}_2H\} = [\{A,{}_1H\},H]+[H,\{A,{}_1H\}] \subseteq [M_{n-1},H]+[H,M_{n-1}] \subseteq M_{n-2}$$
$$\vdots$$
$$\{A,{}_nH\} = [\{A,{}_{n-1}H\},H]+[H,\{A,{}_{n-1}H\}] \subseteq [M_1,H]+[H,M_1] \subseteq M_0 = H$$

Thus $\{A,{}_nH\} \subseteq H$ which means (H,A) is and N_n-pair.

Now assume that (H,A) is a N_n-pair. Then $\{A,{}_nH\} \subseteq H$. We will construct a weakly ascending chain of subspaces. Put $M_i = \{A,{}_{n-i}H\}+H$ for $0 \le i \le n$. Then

1. $M_0 = \{A,{}_nH\} = H$
2. $M_n = \{A,{}_0H\}+H = A$
3. $[M_{j+1},H] = [\{A,{}_{n-(j+1)}H\},H] \subseteq \{A,{}_{n-j}H\} = M_j$
4. $[H,M_{j+1}] = [H,\{A,{}_{n-(j+1)}H\}] \subseteq \{A,{}_{n-j}H\} = M_j$.

Thus $(M_i)_{i\le n}$ is a weakly ascending chain for H in A so H is n-step weakly subinvariant in A. □

In the next chapter, we define weak ascendancy and extend this result to N_∞-pairs.

Lemma 2.3.4. *Let A be a Leibniz algebra over a field such that $A = H+K$ with H a subalgebra of A, and K and abelian ideal of A. Then H is n-step weakly subinvariant in A if and only if H is n-step subinvariant in A.*

Proof. If $H \lhd^n A$ then $H \le^n A$.

Assume that $H \le^n A$ and let $M_0 \subseteq \cdots \subseteq M_n$ be the weakly ascending chain for H in A. Then for any $i \le n$ $[M_i,H],[H,M_i] \subseteq M_i$ and $M_i = M_i \cap (H+K) = H+(M_i \cap K)$. Then

$$\begin{aligned}[M_i,M_{i+1}] &= [H+(M_i\cap K),H+(M_i\cap K)]\\ &\subseteq H^2+[H,M_{i+1}\cap K]+[M_i\cap K]+K^2\\ &\subseteq H+(M_i\cap K)\\ &= M_i\end{aligned}$$

Similarly $[M_{i+1}, M_i] \subseteq M_i$. Hence, M_i is a subalgebra of A and $M_i \lhd M_{i+1}$ for all $i < n$. Thus, H is n-step subinvariant in A. $\square$

Given a Leibniz algebra A with subalgebra H, $\mathrm{Core}_A(H)$ is the largest ideal of A contained in H. Let $\mathfrak{U}^m$ be the class of solvable Leibniz algebras with derived length less than or equal to m.

Theorem 2.3.5. *Let A be a Leibniz algebra over a field $\mathbb{F}$ and H be a subalgebra of A. If $A/\mathrm{Core}_A(H) \in \mathfrak{U}^m$ and $(H, A^{(1)})$ is an N_n-pair for some $m, n > 0$ then $H \lhd^{n(m-1)+1} A$.*

Proof. For $i > 1$ let $\overline{A} = A/A^{(i)}$ and $\overline{H} = (H + A^{(i)})/A^{(i)}$. Then $\overline{A}^{(i-1)}$ is an abelian ideal of $\overline{A}$. By assumption $(H, A^{(1)})$ is a N_n-pair which implies that $(\overline{H}, \overline{A}^{(1)})$ is a N_n-pair. Since $\overline{A}^{(k)} \subseteq \overline{A}^{(k-1)}$ for all $k \in \mathbb{N}$, then $(\overline{H}, \overline{A}^{(i-1)})$ is a N_n-pair. Thus, by Theorem 2.2.6, $\overline{H} \lhd^n \overline{A} + \overline{H}$. Therefore

$$A^{(i)} + H \lhd^n A^{(i-1)} + H.$$

$A/\mathrm{Core}_A(H) \in \mathfrak{U}^m$ implies that $A^{(m)} \subseteq \mathrm{Core}_A(H) \subseteq H$. Thus

$$H = A^{(m)} + H \lhd^{n(m-1)} A^{(1)} + H.$$

Now let us show that $A^{(1)} + H \lhd A$.

$$[A^{(1)} + H, A] = [A^{(1)}, A] + [H, A] \subseteq [A, A] + [A, A] = A^{(1)} \subseteq A^{(1)} + H$$
$$[H, A^{(1)} + H] = [A, A^{(1)}] + [A, H] \subseteq [A, A] + [A, A] = A^{(1)} \subseteq A^{(1)} + H$$

Thus $A^{(1)} + H \lhd A$. So $H \lhd^{n(m-1)+1} A$.

$\square$

Corollary 2.3.5.1. *Let A be a Leibniz algebra over a field $\mathbb{F}$ and H be a subalgebra of A. If A is solvable and (H, A) is an N_n-pair for some $n \in \mathbb{N}$ then $H \lhd\lhd A$.*

Proof. Since A is solvable, then $A/\mathrm{Core}_A(H)$ is also solvable. (H, A) an N_n-pair implies $(H, A^{(1)})$ is an N_n-pair since $A^{(1)} \subseteq A$. Thus, by Theorem 2.3.5, $H \lhd\lhd A$. $\square$

Lemma 2.3.6. *Let A be a solvable Leibniz algebra over a field, and let H be a subalgebra of A. Then H is weakly subinvariant in A if and only if H is subinvariant in A*

Proof. Assume H is subinvariant in A and has chain of ideals $H = H_0 \lhd H_1 \lhd \ldots \lhd H_n = A$. Then $\{A, {}_nH\} \subseteq H$ since $H \subseteq H_i$ for all i. Thus (H, A) is a N_n-pair which implies by Lemma 2.3.2 that H is weakly subinvariant in A.

Now assume that H is weakly subinvariant in A. Then there exists $n \in \mathbb{N}$ such that $\{A, {}_nH\} \subseteq H$. Then by Corollary 2.3.5.1 H is subinvariant in A. $\square$

Lemma 2.3.7. *Let A be a Leibniz algebra over a field. Let $X, H_i\,(i = 1,\dots,n)$ be subalgebras of A such that X is a subalgebra of H_i for all i and $\langle H_1,\dots,H_n\rangle = H_1 + \dots H_n$. If X is weakly subinvariant in H_i for all i then X is weakly subinvariant in $H_1 + \dots + H_n$.*

Proof. Let X be m_i-step weakly subinvariant in H_i for all $i = 1,\dots,n$ and let $m = \max_{i=1,\dots,n}(m_i)$. Then $\{H_i, {}_mX\} \subseteq X$ for all i. Thus $\{H_1 + \dots + H_n, {}_mX\} = \{H_1, {}_mX\} + \dots + \{H_n, {}_mX\} \subseteq X$. Thus, X is weakly subinvariant in $H_1 + \dots + H_n$. □

Corollary 2.3.7.1. *Let A be a solvable Leibniz algebra over a field. Let $X, H_i\,(i = 1,\dots,n)$ be subalgebras of A such that X is a subalgebra of H_i for all i and $\langle H_1,\dots,H_n\rangle = H_1 + \dots H_n$. If X is subinvariant in H_i for all i then X is subinvariant in $H_1 + \dots + H_n$.*

Proof. Assume X is subinvariant in H_i for all i. Then by Lemma 2.3.6 X is weakly subinvariant in H_i for all i. By Lemma 2.3.7 X is weakly subinvariant in $H_1 + \dots + H_n$. Since A is solvable so is $H_1 + \dots + H_n$. Thus, by Lemma 2.3.6 X is subinvariant in $H_1 + \dots + H_n$. □

Theorem 2.3.8. *Let A be a Leibniz algebra over a field. Let $X, H_\sigma\,(\sigma \in I)$ be subalgebras of A such that X is a subalgebra of H_σ for all σ and X is of finite codimension in $\langle H_\sigma | \sigma \in I\rangle$. If X is weakly subinvariant in H_σ for all σ then X is weakly subinvariant in $\langle H_\sigma | \sigma \in I\rangle$*

Proof. Let $H = \langle H_\sigma | \sigma \in I\rangle$ and $N_\infty(X) = \{a \in A | \{a, {}_nX\} \subseteq X$ for some $n \in \mathbb{N}\}$. By Lemma 2.2.5 $N_\infty(X)$ is a subalgebra of A. Since X is weakly subinvariant in H_σ then $\{H_\sigma, {}_nX\} \subseteq X$ for some $n \in \mathbb{N}$. Thus H_σ is a subalgebra of $N_\infty(X)$ for all $x \in I$ which implies that H is a subalgebra of $N_\infty(X)$.

Since X is of finite codimension in H by assumption, there exists $a_1,\dots,a_m \in H$ such that $H = X + \operatorname{span}\{a_1,\dots,a_m\}$. For each $i \in 1,\dots,m$ there exists $n_i \sin \mathbb{N}$ such that $\{a_i, {}_{n_i}X\} \subseteq X$. Let $n = \max_{i=1,\dots,m}(n_i)$. Then $\{H, {}_nX\} \subseteq X$. Thus X is weakly subinvariant in H. □

Corollary 2.3.8.1. *Let A be a finite dimensional Leibniz algebra over a field. Let $X, H_i\,(i = 1,\dots,n)$ be subalgebras of A such that X is a subalgebra of H_i for all i. If X is weakly subinvariant in H_i for all i then X is weakly subinvariant in $\langle H_i,\dots,H_n\rangle$.*

Corollary 2.3.8.2. *Let A be a finite dimensional Leibniz algebra over a field. Let $H_i\,(i = 1,\dots,n)$ be subalgebras of A such that $A = \langle H_1,\dots,H_n\rangle$. If H_i is weakly subinvariant in $\langle H_i, H_j\rangle$ for all $i, j = 1\dots,n$ then H_i is weakly subinvariant in A for all $i = 1,\dots,n$.*

Using Lemma 2.3.6 and Theorem 2.3.8 we get the following:

Corollary 2.3.8.3. *Let A be a solvable Leibniz algebra over a field. Let $X, H_\sigma\ (\sigma \in I)$ be subalgebras of A such that X is a subalgebra of H_σ for all σ and X is of finite codimension in $\langle H_\sigma | \sigma \in I\rangle$. If X is subinvariant in H_σ for all σ then X is subinvariant in $\langle H_\sigma | \sigma \in I\rangle$.*

Lemma 2.3.9. *Let A be a Leibniz algebra over a field. Let $A = B + X$ such that X is a subalgebra of A and B is a solvable ideal of A with $B^{(c)} = 0$. If X is m-step weakly subinvariant in A then X is cm-step subinvariant in A.*

Proof. We will prove this using induction on c. When $c = 1$ then $B^{(1)} = 0$ which implies that B is abelian.

$$\{B,\, _mX\} \subseteq \{A,\, _mX\} \subseteq X$$

since X is m step weakly subinvariant in A. Thus, by Theorem 2.2.6, $X \lhd^m B + X = A$.

For the inductive step, assume that $X \leq^m A$ implies that $X \lhd^{(c-1)m} A$. Let $N = B^{(c-1)}$. Then N is an abelian ideal of A. Let $\overline{A} = A/N$, $\overline{B} = B/N$ and $\overline{X} = X/N$. Then we have

$$\overline{A} = \overline{B} + \overline{X} \qquad \overline{B} \lhd \overline{A} \qquad \overline{B}^{(c)} = 0 \qquad \overline{X} \leq^m A.$$

By the inductive hypothesis, $\overline{X} \lhd^{m(c-1)} \overline{A}$ so $X + N \lhd^{m(c-1)} A$. Now N is an abelian ideal of $X + N$ and $X \leq^m X + N$. By Lemma 2.3.4 $X \lhd^m X + N$. Thus $X \lhd^m X + N \lhd^{m(c-1)} A$ which implies $X \lhd^{mc} A$. □

Lemma 2.3.10. *Let A be a Leibniz algebra over a field. Let X, H, K be subalgebras of A and B be an ideal of A such that $A = H + K$, $A = B + X$, and $B^{(c)} = 0$. If $X \leq^m H$ and $X \leq^m K$ then $X \lhd^{cm} A$.*

Proof. By Lemma 2.3.7 $X \leq^m H + K = A$. Then by Lemma 2.3.9 $X \lhd^{cm} A$. □

Corollary 2.3.10.1. *Let A be a Leibniz algebra over a field. Let X, H, K be subalgebras of A and B be an ideal of A such that $A = H + K$, $A = B + X$, and $B^{(c)} = 0$. If $X \lhd^m H$ and $X \lhd^m K$ then $X \lhd^{cm} A$.*

2.4 Join Subinvariance

Theorem 2.4.1. *Let A be a finite dimensional Leibniz algebra over a field with characteristic zero. Let B, H, K be subalgebras of A such that $A = H + K$, $B \subseteq H$, and $B \subseteq K$. Then $B \lhd\lhd A$ if and only if $B \lhd\lhd H$ and $B \lhd\lhd K$.*

Proof. If $B \lhd\lhd A$ then clearly $B \lhd\lhd H$ and $B \lhd\lhd K$. Now let us show the converse.

Assume A is a counter example of minimal dimension and assume that the choice of B, H, K have been made in such a way that H has maximal dimension. We have that $B \lhd\lhd H$ and $B \lhd\lhd K$ but $B \not\lhd\lhd A$. By Theorem 2.3 in [12] $B^\omega \lhd H$ and $B^\omega \lhd K$ so $B^\omega \lhd A$.

If $B^\omega \neq 0$ then by the assumption that A is a minimal counter example $B/B^\omega \lhd\lhd A/B^\omega$. Then there exists a chain of ideals

$$B/B^\omega = B(1)/B^\omega \lhd \ldots \lhd B(n)/B^\omega = A/B^\omega.$$

This gives the chain of ideals

$$B = B(1) \lhd \ldots \lhd B(n) = A$$

which implies that $B \lhd\lhd A$. This is a contradiction. Therefore $B^\omega = 0$ and B is nilpotent.

Let $b \in B$. Since a Leibniz algebra is nilpotent if and only if all its subalgebras are subinvariant then $\langle b \rangle \lhd\lhd B$. If the theorem were true for the case where $\dim(B) = 1$ it would follow that $\langle b \rangle \lhd\lhd A$ for all $b \in B$ and by Theorem 3.9 in [12] $B \subseteq \text{nilrad}(A)$. Therefore B would be subinvariant. It follows that we may assume the $B = \langle b \rangle$ for some $b \in A$.

Let $S = \text{rad}(A)$. $\text{Leib}(A) \subseteq \text{rad}(A)$. If $S = A$ then $B \lhd\lhd A$ by Theorem 2.1.4 which is again a contradiction. Thus $S \neq A$.

$S + B$ is solvable because S is solvable and B is nilpotent. Therefore it follows that $S + B \neq A$. If $S \neq 0$ then $(S + B)/S \lhd\lhd A/S$ by minimality. But $\dim(S + B)/S \leq 1$ and A/S is semisimple. Thus B is a subalgebra of S. Since L_b is nilpotent on H and K, L_b is nilpotent on S. By Lemma 2.1.1, $B \lhd\lhd S \lhd A$, which is a contradiction.

Thus $S = 0$ and $\text{Leib}(A) = 0$ which implies that A is a Lie algebra. Then $B \lhd\lhd A$ by the Theorem 1 in [2], which is a contradiction.

Thus we have shown $B \lhd\lhd A$. $\square$

Corollary 2.4.1.1. *Let A be a finite dimensional Leibniz algebra over a field of characteristic 0. Let B and $H_i (i = 1, \ldots n)$ be subalgebras of A such that $A = H_1 + \cdots + H_n$, $B \subseteq H_i$ for all*

$i = 1, \dots n$, and $\langle H_i, H_j \rangle = H_i + H_j$ for all $i, j = 1, \dots n$. Then $B \lhd\lhd A$ if and only if $B \lhd\lhd H_i$ for all $i = 1, \dots n$

Proof. If $B \lhd\lhd A$ then $B \lhd\lhd H_i$ for all $i = 1, \dots n$.

Now assume that $B \lhd\lhd H_i$ for all $i = 1, \dots n$. We will show by induction that $B \lhd\lhd H_1 + \dots H_m$ for all $m = 1, \dots n$. For the base case of $m = 1$, $B \lhd\lhd H_1$ by assumption. For the inductive step, assume that $B \lhd\lhd H_1 + \dots H_m$ for $m < n$. Since $B \lhd\lhd H_{m+1}$ by assumption, then by Theorem 2.4.1 $B \lhd\lhd H_1 + \dots + H_m + H_{m+1}$. Thus $B \lhd\lhd H_1 + \dots H_m$ for all $m = 1, \dots n$ which implies that $B \lhd\lhd H_1 + \dots + H_n = A$. □

Corollary 2.4.1.2. *Let A be a finite dimensional Leibniz algebra over a field of characteristic 0. Let H_i $i = 1, \dots n$ be subalgebras of A such that $A = \langle H_1, \dots H_n \rangle$ and $H_i \lhd\lhd \langle H_i, H_j \rangle = H_i + H_j$ for all $i, j = 1, \dots n$. Then $H_i \lhd\lhd A$ for each $i = 1, \dots n$.*

Proof. First, we will show that $H_1 + \dots H_n = A$. Clearly $H_1 + \dots H_n \subseteq A$. Now let $g \in H_i$ and $h \in H_j$. Then

$$[g, h], [h.g] \in \langle H_i, H_j \rangle = H_i + H_j \subseteq H_1 + \dots + H_n.$$

Thus $A \subseteq H_1 + \dots H_n$ which implies $A = H_1 + \dots H_n/$

Now, without loss of generality, consider H_1. Let $K_j = \langle H_1, H_j \rangle$ for $j = 2, \dots, n$. Then $A = K_2 + \dots + K_n$, $H_1 \subseteq K_j$ for $j = 2, \dots, n$ and $H_1 \lhd\lhd K_j$ $j = 2, \dots, n$. Additionally, $\langle K_i, K_j \rangle = \langle H_1 + H_i, H_1 + H_j \rangle = H_1 + H_i + H_j = K_i + K_j$ for $i, j = 2, \dots, n$. Then by Corollary 2.4.1.1 $H_1 \lhd\lhd A$. This implies that $H_i \lhd\lhd A$ for each $i = 1, \dots n$ □

Definition 2.4.2. A Leibniz algebra A is **solvable-by-finite** if there exists a solvable ideal S of A such that A/S is finite dimensional.

Definition 2.4.3. A Leibniz algebra A is **residually nilpotent** if there exists $(I_\alpha)_{\alpha \in \mathscr{A}}$ ideals of A such that A/I_α is nilpotent for all $\alpha \in \mathscr{A}$ and $\bigcap_{\alpha \in \mathscr{A}} I_\alpha = 0$. The residual R of A is the smallest ideal of A such that A/R is nilpotent. R is uniquely determined.

Lemma 2.4.4. *If Leibniz algebra A is solvable-by-finite and residually nilpotent, then A is solvable.*

Proof. Pick S to be a maximal solvable ideal of A. Let R/S be the nilpotent residual of A/S and let $I_\alpha \lhd A$ such that A/I_α is nilpotent and $\bigcap I_\alpha = 0$. Therefore $A/(I_\alpha + S)$ is nilpotent and

R is a subalgebra of $(I_\alpha + S)$. Hence $R^{(m)}$ is a subalgebra of I_α and $R^{(m)} = 0$ for some $m \in \mathbb{N}$. Thus $R \subseteq S$ and $S \subseteq R$ by definition of R. So $R = S$.

So $(A/S)^\omega = 0$ but A/S is finite dimensional by definition, so $(A/S)^n = 0$ for some $n \in \mathbb{N}$. Thus A/S is nilpotent. Since S is solvable, $S^{(j)} = 0$ for some $j \in \mathbb{N}$.

$$A^{(n+j)} \subseteq (A^n)^{(j)} = S^{(j)} = 0$$

Therefore A is solvable.

$\square$

Theorem 2.4.5. *Let A be a solvable-by-finite Leibniz algebra over a field $\mathbb{F}$ of characteristic zero and let B, H, K be subalgebras of A such that $A = H + K$, $B \lhd\lhd H$, and $B \lhd\lhd K$. Then $B \lhd\lhd A$.*

Proof. By Theorem 2.3 in [12] $B \lhd\lhd H$ and $B \lhd\lhd K$ implies that $B^\omega \lhd H$ and $B^\omega \lhd K$. Thus $B^\omega \lhd A$. B/B^ω is solvable-by-finite since A is solvable-by-finite. B/B^ω is also residually nilpotent. Thus by Lemma 2.4.4 B^ω is solvable. So without loss of generality, we may assume that B is solvable.

Let S be a solvable ideal of A such that A/S is finite dimensional. Then $(B+S)/S \lhd\lhd A/S$ by Theorem 2.4.1 which implies $B + S \lhd\lhd A$.

Next let us show $B \lhd\lhd B + S$. To do this we claim that $\{B + S, {}_m B\} \subseteq B$ for some $m \in \mathbb{N}$.

Since $B \lhd\lhd H$ and $B \lhd\lhd K$ we have $B = H_0 \lhd \ldots \lhd H_m = H$ and $B = K_0 \lhd \ldots \lhd K_n = K$ for some $m, n \in \mathbb{N}$. Let $x \in B + S$. Then $x = h + k$ for some $h \in H$ and $k \in K$ since $A = H + K$. Therefore $\{x, {}_i B\} = \{h, {}_i B\} + \{k, {}_i B\}$ for all $i \in \mathbb{N}$.

$$\{h, {}_1 B\} = [B, h] + [h, B] \subseteq H_{m-1} \text{ since } h \in H_m = H \text{ and } B \subseteq H_{m-1} \lhd H_m$$
$$\{h, {}_2 B\} = [B, \{h, {}_1 B\}] + [\{h, {}_1 B\}, B] \subseteq H_{m-2} \text{ since } \{h, {}_1 B\} \subseteq H_{m-1} \text{ and } B \subseteq H_{m-2} \lhd H_{m-1}$$
$$\vdots$$
$$\{h, {}_m B\} = [B, \{h, {}_{m-1} B\}] + [\{h, {}_{m-1} B\}, B] \subseteq H_0 = B \text{ since } \{h, {}_{m-1} B\} \subseteq H_1 \text{ and } H_0 \lhd H_1$$

Similarly $\{k, {}_n B\} \subseteq K_0 = B$. Without loss of generality, assume that $n \leq m$. Then

$$\{k, {}_m B\} = \{\{k, {}_n B\}, {}_{m-n} B\} \subseteq \{B, {}_{m-n} B\} \subseteq B.$$

Therefore $\{x,{}_m B\} = \{h,{}_m B\} + \{k,{}_m B\} \subseteq B$ which implies that $\{B+S,{}_m B\} \subseteq B$ for some $m \in \mathbb{N}$.

Since $B+S$ is solvable and $\{B+S,{}_m B\} \subseteq B$ by Theorem 2.3.5 $B \lhd\lhd B+S \lhd\lhd A$. Thus $B \lhd\lhd A$. $\square$

Corollary 2.4.5.1. *Let A be a solvable-by-finite Leibniz algebra over a field $\mathbb{F}$ with characteristic zero and H_1, H_2, H_3 be subalgebras of A such that $A = \langle H_1, H_2, H_3\rangle$. If $H_i \lhd\lhd \langle H_i, H_j\rangle = H_i + H_j$ for all $i, j = 1,2,3$ then H_1, H_2, H_3 are subinvariant in A.*

Proof. Assume that A is a Leibniz algebra over a field $\mathbb{F}$ with characteristic zero and H_1, H_2, H_3 are subalgebras of A such that $A = \langle H_1, H_2, H_3\rangle$. Also assume that $H_i \lhd\lhd \langle H_i, H_j\rangle = H_i + H_j$ for all $i, j = 1,2,3$. Let us consider H_1.

$H_1 \lhd\lhd \langle H_1, H_2\rangle = H_1 + H_2$ and $H_1 \lhd\lhd \langle H_1, H_3\rangle = H_1 + H_3$. Then by Theorem 2.4.5

$$H_1 \lhd\lhd \langle H_1, H_2\rangle + \langle H_1, H_3\rangle = H_1 + H_2 + H_3$$

.

We claim that $H_1+H_2+H_3 = A$. Clearly $H_1+H_2+H_3 \subseteq A$. Now let us show $A \subseteq H_1+H_2+H_3$. By assumption $A = \langle H_1, H_2, H_3\rangle$. Let $h \in H_1, g \in H_2$, and $l \in H_3$. Then by the assumption that $H_i \lhd\lhd \langle H_i, H_j\rangle = H_i + H_j$ for all $i, j = 1,2,3$

$$[h,g],[g,h] \in H_1 + H_2 \subseteq H_1 + H_2 + H_3$$

$$[h,l],[l,h] \in H_1 + H_3 \subseteq H_1 + H_2 + H_3$$

$$[l,g],[g,l] \in H_2 + H_3 \subseteq H_1 + H_2 + H_3.$$

Thus $A \subseteq H_1 + H_2 + H_3$ which implies $A = H_1 + H_2 + H_3$. Therefore $H_1 \lhd\lhd A$. By the same reasoning $H_2 \lhd\lhd A$ and $H_3 \lhd\lhd A$

$\square$

Let $\mathcal{N}$ and $\mathcal{X}$ be classes of Leibniz algebras over a field. Then $\mathcal{N}$-by-$\mathcal{X}$ is the class of Leibniz algebras A that have an ideal $I \in \mathcal{N}$ such that $A/I \in \mathcal{X}$.

Definition 2.4.6. A Leibniz algebra A is **join-subinvariant** if for all subalgebras H, K of A such that $H + K = A$ with X subinvariant in H and X subinvariant in K then X is subinvariant in A.

By Theorem 2.4.1, finite dimensional Leibniz algebras over a field with characteristic 0 are join-subinvariant.

Theorem 2.4.7. *Let $\mathscr{X}$ be a class of Leibniz algebras that are join-subinvariant. Then the class of solvable-by-$\mathscr{X}$ is join-subinvariant.*

Proof. Let $A = H + K$ be solvable-by-$\mathscr{X}$ and X be a subalgebra of A such that $X \lhd^m H$ and $X \lhd^m K$. Then there exists N an ideal of A and $c \in \mathbb{N}$ such that $N^{(c)} = 0$ and $A/N \in \mathscr{X}$. Since

$A/N = (H+N)/N + (K+N)/N \qquad (X+N)/N \lhd^m (H+N)/N \qquad (X+N)/N \lhd^m (K+N)/N$

and A/N is join-subinvariant there exists $f \in \mathbb{N}$ such that $(X+N)/N \lhd^f A/N$. Thus $X + N \lhd^f A$.

Since $X \lhd^m H$, $X \lhd^m K$ and $A = H + K$ then by Lemma 2.3.7 $X \leq^m A$. This implies that $X \leq^m X + N$. $N \lhd X + N$ and $N^{(c)}$ thus by Lemma 2.3.9 $X \lhd^{cm} X + N$.

Therefore $X \lhd^{cm} X + N \lhd^f A$ which implies that $X \lhd^{f+cm} A$. □

An example to the theorem above was given in Theorem 2.4.5 where we showed that solvable-by-finite Leibniz algebras are join-subinvariant.

CHAPTER

3

ASCENDANCY

3.1 Locally Solvable, Ideally Finite Leibniz Algebras

In this section, we extend the results from section 2.1 to infinite dimensional Leibniz algebras generated by finite-dimensional ideals. Stewart examined Lie algebras of this type in [15]. We will specifically consider locally solvable, ideally finite Leibniz algebras.

Definition 3.1.1. A Leibniz algebra A is said to be **locally solvable** if every finite subset of A is contained in a solvable subalgebra of A

Definition 3.1.2. A Leibniz algebra is said to be **ideally finite** if every element of the algebra is contained in a finite dimensional ideal.

Recall that B is ω-ascendant in A if there exists a chain $B = B(0) \lhd B(1) \lhd \cdots \lhd B(\omega) = A$ where $B(\omega) = \bigcup_{k=0}^{\infty} B(k)$. For each element a in Leibniz algebra A, let $A_0(a) = \{x \in A | L_a^k(x) = 0 \text{ for some } k = 1, 2, \ldots\}$ and $A_1(a) = \bigcap_{k=1}^{\infty} \text{range}(L_a^k)$.

Theorem 3.1.3. *Let A be a locally solvable, ideally finite Leibniz algebra and let B be a subalgebra of A. Then the following are equivalent.*

1. *B is ω-ascendant in A*

2. *$B + A_0(a) = A$ for all $a \in B$*

3. *$A_1(a) \subseteq B$ for all $a \in B$*

Proof. Assume 1. Then we have $B = B_0 \lhd B_1 \lhd \ldots \lhd B_\omega = A$. For $b \in B$, $L_b^k(B_k) \subseteq B$ and $L_b^j(B_k) \subseteq B$ for $j \geq k$. $A_1(b) = \bigcap_{j=1}^{\infty} \operatorname{range}(L_b^j) \subseteq B$. Hence 3 holds.

Next, let us show that 3 implies 2. Assume $A_1(a) \subseteq B$ for all $a \in B$. Let $a \in B$, $x \in A$, and let C be a finite dimension ideal which contains x. Then $C = C_0(a) + C_1(a)$. Thus

$$x \in B + C = B + C_0(a) + C_1(a) = B + C_0(a) \subseteq B + A_0(a)$$

since $C_1(a) \subseteq A_1(a) \subseteq B$. Thus $A = B + A_0(a)$ for all $a \in B$.

Last, we show that 2 implies 1. Assume that $B + A_0(a) = A$ for all $a \in B$. Since A is ideally finite, A is the union of finite dimensional ideals $\{H(\lambda)\}$. Hence each $H(\lambda)$ contains a chain $0 = H(\lambda, 0) \subset \cdots \subset H(\lambda, n(\lambda)) = H(\lambda)$ where each $H(\lambda, i)$ is an ideal in A and $H(\lambda, i)/H(\lambda, i-1)$ is an irreducible A-module. Since A is locally solvable and $H(\lambda, 1)$, $H(\beta, 1)$ are minimal $[H(\lambda, 1), H(\beta, 1)] = 0$. Consider $H(\lambda, j)$ and $H(\beta, k)$. Let $T = H(\lambda, j-1) + H(\beta, k-1)$. T is an ideal in A. $(H(\lambda, j) + T)/T$ and $(H(\beta, k) + T)/T$ are either minimal ideals or 0 in A/T. Hence $[H(\lambda, j) + T, H(\beta, k) + T] \subseteq T$ and

$$[H(\lambda, j), H(\beta, k)] \subseteq H(\lambda, j-1) + H(\beta, k-1) \tag{3.1}$$

We construct a chain of subspaces $\{\Omega(i)\}$ such that $\{B + \Omega(i)\}$ is of the desired type. Let $\Omega(0) = 0$ and $\Omega(1)$ is constructed as follows: If $H(\lambda) \nsubseteq B$ then let $r(\lambda)$ be the smallest positive integer such that $H(\lambda, r(\lambda)) \nsubseteq B$. Let $W(\lambda, 1) = \{z \in H(\lambda, r(\lambda)) \mid [B, z] \subseteq B\}$. Now $(H(\lambda, r(\lambda)) + B)/B$ is finite dimensional and since $A_0(a) + B = A$ for all $a \in B$, each $a \in B$ induces a nilpotent operator on this space. Hence $W(\lambda, 1) \neq 0$. If $H(\lambda) \subseteq B$ then let $r(\lambda)$ be the smallest positive integer such that $H(\lambda, r(\lambda)) = H(\lambda)$ and define $W(\lambda, 1)$ as above. In this case $W(\lambda, 1)$ may be 0. Since $W(\lambda, 1) \subseteq H(\lambda, r(\lambda))$ and $H(\lambda, r(\lambda) - 1) \subseteq B$, $[W(\lambda, 1), W(\beta, 1)] \subseteq [H(\lambda, r(\lambda)), H(\beta, r(\beta))] \subseteq B$ by (3.1). Also $[B, W(\lambda, 1)] \subseteq B$. Let $\Omega(1)$ be the subalgebra of A generated by all $W(\lambda, 1)$. Then $[B, B + \Omega(1)] \subseteq B$.

Now suppose that $\Omega(k)$ has been constructed and we construct $\Omega(k+1)$. If $H(\lambda) \nsubseteq B+\Omega(K)$, let $r(\lambda)$ be the smallest integer such that $H(\lambda, r(\lambda)) \nsubseteq B+\Omega(k)$. Then $(H(\lambda, r(\lambda))+B+\Omega(k)))/(B+\Omega(k))$ is finite dimensional and each element of B induces a nilpotent linear transformation on the space. Hence $W(\lambda, k+1)=\{z \in H(\lambda, r(\lambda)) \mid [B, z] \subseteq B+\Omega(k)\} \neq 0$. If $H(\lambda) \subseteq B+\Omega(k)$, let $r(\lambda)$ be the smallest positive integer such that $H(\lambda, r(\lambda))=H(\lambda)$ and define $W(\lambda, k+1)$ as before. Let $\Omega(k+1)$ be the algebra generated by all $W(\lambda, k+1)$. Then $[B, \Omega(k+1)] \subseteq B+\Omega(k)$. Also, $W(\lambda, k+1) \subseteq H(\lambda, r(\lambda))$ and $H(\lambda, r(\lambda)-1) \subseteq B+\Omega(k)$. Hence $[W(\lambda, k+1), W(\beta, k+1)] \subseteq [H(\lambda, r(\lambda), H(\beta, r(\beta))] \subseteq B+\Omega(k)$ by (3.1). Therefore, $[\Omega(k+1), \Omega(k+1)] \subseteq \Omega(k)+B$ and $[\Omega(k+1)+B, \Omega(k+1)+B] \subseteq \Omega(k)+B$.

Since $W(\lambda, 1) \subset W(\lambda, 2) \subset \ldots$ until $H(\lambda)$ is reached, $H(\lambda) \subseteq \Omega(\dim H(\lambda))$ and $A \subseteq \bigcup_{i=1}^{\infty} \Omega(i)$. Hence $\{B+\Omega(i)\}$ shows that B is ω-ascendant in A.

□

So far we have considered the case where we can construct a chain of two-sided ideals. Now let us consider a chain of right ideals. Recall that a subalgebra B of Leibniz algebra A is a right ideal if $[B, A] \subseteq B$. Denote B a right ideal of A as $B \lhd_r A$.

Definition 3.1.4. B is **right ω-ascendant** in A if there exists a chain $B=B(0) \lhd_r B(1) \lhd_r \ldots \lhd_r B(\omega)=A$ where $B(\omega)=\bigcup_{k=0}^{\infty} B(k)$.

Theorem 3.1.5. *Let A be a locally solvable, ideally finite Leibniz algebra and let B be a subalgebra of A. Then B is right ω-ascendant in A if and only if B is ω-ascendant in A.*

Proof. B ω-ascendant in A clearly implies that B is right ω-ascendant in A since a chain of ideals is also a chain of right ideals.

Now let us assume that B is right ω-ascendant in A. Then there exists a chain of right ideals

$$B=B(0) \lhd_r B(1) \lhd_r \ldots \lhd_r B(\omega)=A.$$

Let $a \in B$. Then $A_1(a)=\bigcap_{k=1}^{\infty} \operatorname{range}(L_a^k) \subseteq B$ due to the chain of ideals. Thus by Theorem 3.1.3 B is ω-ascendant in A.

□

Let $A \supseteq A^2 \supseteq \ldots$ be the lower central series of A and $A^{\omega}=\bigcap_{k=1}^{\infty}$.

Lemma 3.1.6. *Let A be a Leibniz algebra and B be ω-ascendant in A. Then $B^{\omega} \lhd A$.*

Proof. Let $B = B(0) \lhd B(1) \lhd \cdots \lhd B(\omega) = A$ where $B(\omega) = \bigcup B(i)$. Let $x \in A$. Then $x \in B(n)$ for some n and B is subinvariant in $B(n)$. Since $B^\omega \lhd B(n)$, $[x, B^\omega] \subseteq B^\omega$ and $[B^\omega, x] \subseteq B^\omega$. Thus $B^\omega \lhd A$.

□

Lemma 3.1.7. *Let A be a locally solvable, ideally finite Leibniz algebra and let B, H, K be subalgebras of A such that $B \subseteq H \cap K$. If B is ω-ascendant in H and K, then B is ω-ascendant in $\langle H, K\rangle$.*

Proof. We may assume $A = \langle H, K\rangle$. Let $B \supseteq B^2 \supseteq \dots$ be the lower central series of B with $B^\omega = \bigcap B^k$. Since B is ω-ascendant in H and K, by Lemma 3.1.6 $B^\omega \lhd H$ and $B^\omega \lhd K$ which implies that $B^\omega \lhd A$. We may assume $B^\omega = 0$ (otherwise, quotient out by B^ω). Let $a \in B$. Then $H_1(a) \bigcap_{k=1}^\infty \text{range}(L_a^k) \subseteq B$ by Theorem 3.1.3. Let $x \in H$. Then there exists k such that $L_a^k(x) \in B$ and then, since $B^\omega = 0$, $x \in A_0(a)$. Hence $H \subseteq A_0(a)$ and similarly $K \subseteq A_0(a)$. Thus $A_0(a) = \langle H, K\rangle = A$ for each $a \in B$. Therefore by Theorem 3.1.3, B is ω-ascendant in A.

□

Theorem 3.1.8. *Let A be a locally solvable, ideally finite Leibniz algebra and B be a subalgebra of A. Then there exists a unique maximal subalgebra in which B is ω-ascendant in A.*

Proof. Let $\mathscr{S}$ be the collection of subalgebras in which B is ω-ascendant. Clearly $\mathscr{S} \neq 0$. Order $\mathscr{S}$ by inclusion and let $T(1) \subseteq T(2) \subseteq \dots$ be an increasing chain in $\mathscr{S}$. Let $R = \bigcup T(i)$. If $x \in R$ then $x \in T(n)$ for some n. Let $a \in B$. Then there exists k such that $L_a^k(x) \in B$ since B is ω-ascendant in $T(n)$. Hence, $R_1(a) \subseteq B$ for each $a \in B$ and B is ω-ascendant in R. Hence $\mathscr{S}$ has a maximal element S by Zorn's Lemma. Let $U \in \mathscr{S}$. Then $U + S \in \mathscr{S}$ by Lemma 3.1.7. Hence $U \subseteq S$ and S contains all elements of $\mathscr{S}$.

□

Let $\alpha(B, A)$ denote the maximal subalgebra of A in which B is ω-ascendant. Note that $\alpha(B, A)$ is the maximal subalgebra C of A such that $C_1(a) \subseteq B$ for all $a \in B$.

Lemma 3.1.9. *Let A be a locally solvable, ideally finite Leibniz algebra and B be a subalgebra of A such that B is ω-ascendant in $\langle B, L_x(B)\rangle$ for all $x \in A$, B is not ω-ascendant in A but B is ω-ascendant in a maximal subalgebra M of A. Then there exists $x \in B$ such that $B + A_0(x) = M$.*

Proof. Since M is maximal in A and A is ideally finite, M is of finite codimension in A. Let K be a maximal ideal of A contained in M. Then K has finite codimension. Let N be a finite dimensional ideal of A which supplements M. Let $C_A(N) = \{x \in A | [x, y] = 0 \text{ and } [y, x] = 0 \text{ for all } y \in N\}$ be the centralizer of N in A. We claim that $C_A(n) \cap M \lhd A$:

Let $x \in C_A(N) \cap M$ and consider $y \in A$. Either $y \in N$ or $y \in M$. If $y \in N$ then $[x, y] = 0 = [y, x]$ which implies that $[x, y], [y, x] \in C_A(N) \cap M$. If $y \in M$, $[x, y], [y, x] \in M$ since M is a subalgebra. All that is left to show is that $[x, y], [y, x] \in C_A(N)$. Let $z \in N$. Then

$$
\begin{aligned}
[z,[x,y]] &= [\underbrace{[z,x]}_{0}, y] + \underbrace{[x, \underbrace{[z,y]}_{\in N}]}_{0} = 0 \\
[[x,y],z] &= [x,[y,z]] - [y,[x,z]] = 0 \\
[z,[y,x]] &= [[z,y],x] + [y,[z,x]] = 0 \\
[[y,x],z] &= [y,[x,z]] - [x,[y,z]] = 0
\end{aligned}
$$

Thus $[x, y], [y, x] \in C_A(N) \cap M$ which implies that $C_A(n) \cap M \lhd A$.

Let $J = C_A(N) \cap M$. Since $C_A(N)$ has finite codimension in A, so does J. Since B is ω-ascendant in M, $M \subseteq B + A_0(x)$ for all $x \in B$, hence we only deed to find $x \in B$ such that $A_0(x) \subseteq M$. $(B + K)/K = \overline{B}$ is subinvariant in $M/K = \overline{M}$ and $\overline{M}$ is self-normalizing in $A/K = \overline{A}$. Now $\overline{A}$ is a primitive algebra and by the same proof Theorem 2.1.2, there exists $x \in B$ such that $\overline{M} = \overline{B} + \overline{A}_0(\overline{x})$. Then $A_0(x) \subseteq M$ since if $y \in A$ and $L_x^n(y) = 0$ then $L_{x+K}^n(y + K) = L_x^n(y) + K = K$ and $\overline{y} \in \overline{M}$ hence $y \in M$ and $A_0(x) \subseteq M$.

□

Theorem 3.1.10. *Let A be a locally solvable, ideally finite Leibniz algebra and B be a subalgebra of A. Then the following are equivalent:*

1. *B is ω-ascendant in A.*

2. *B is ω-ascendant in every subalgebra C of A such that $B \subseteq C$ and $\dim(C/B)$ is finite.*

3. *B is ω-ascendant in $\langle B, x\rangle$ for all $x \in A$.*

4. B *is* ω*-ascendant in* $\langle B, R_x(B)\rangle$ *for all* $x \in A$.

Proof. Clearly $1 \implies 2 \implies 3 \implies 4$. What is left for us to show is 4 implies 1.
Suppose that B is ω-ascendant in $\langle B, R_x(a)\rangle$ for every $x \in A$ but B is not ω-ascendant in A. Then $\alpha(B,A) \neq A$. Let $x \in A$, $x \notin \alpha(B,A)$ and let N be a finite dimensional ideal of A containing x. Then $\alpha(B,A)$ has finite codimension in $\alpha(B,A)+N$. Hence there exists a minimal subalgebra of $\alpha(B,A)+N$ which contains $\alpha(B,A)$ properly. Thus we may assume that there exists $B \subseteq A$ such that B is ω-ascendant in $\langle B, R_x(B)\rangle$ for all $x \in A$, B is not ω-ascendant in A but B is ω-ascendant in a maximal subalgebra M of A. Then by Lemma 3.1.9, there exists $x \in B$ such that $B + A_0(x) = M$.

We show that if $y \notin M$ then $\langle B, R_y(B)\rangle = A$. Hence B is ω-ascendant, a contradiction. Let $x \in B$ such that $M = B + A_0(x)$. Suppose $\langle B, R_y(B)\rangle \neq A$. Then $\langle B, R_y(B)\rangle \subseteq \alpha(B,A) = M$ by Theorem 3.1.8 and $[x, y] \in M$. Let C be a finite dimensional ideal such that $y \in C$. Then $y \in C_0(x) + C_1(x) \subseteq A_0(x) + A_1(x)$. Let $y = s + t$ where $s \in A_0(x)$ and $t \in A_1(x)$. Then $t \notin M$ since $s \in B + A_0(x) = M$. Then $[x, t] = [x, y] - [x, s] \in M$. However, L_x is non-singular on the finite dimensional space $A_1(x)$ and $M \cap A_1(x)$ is L_x invariant. Hence if $x \in A_1(x)$ and $[z, x] \in M \cap A_1(x)$, then $z \in M \cap A_1(x)$. In particular this holds for $z = t$, a contradiction. Hence $\langle B, R_y(B)\rangle = A$ and $\alpha(B,A) = A$. Thus we have shown B is ω-ascendant in A and we are done with the proof.

□

Theorem 3.1.11. *Let A be a locally solvable, ideally finite Leibniz algebra and* B *be a subalgebra of A. Then the following are equivalent:*

1. B *is* ω*—ascendant in A.*

2. *For all* x *in A,* $x \in \langle B, R_x(B)\rangle$ *implies* $x \in B$.

Proof. Assume 2 and assume for a contradiction that B is not ω-ascendant in A. Then $M = \alpha(B,A) \neq A$. If $z \notin M$ then let N be a finite dimensional ideal containing z. $(N+M)/M$ is finite dimensional. Hence there exists a minimal subalgebra K between M and $M+N$. Thus B is ω-ascendant in the maximal subalgebra M of K but not in K. By Lemma 3.1.9 there exists $x \in B$ such that $M = B + K_0(x)$. Suppose that $y \in K_1(x)$ and $y \notin M$. If $\langle B, R_y(B)\rangle = K$ then $y \in \langle B, R_y(B)\rangle$ which implies that $y \in B \subset M$ by the assumption, a contradiction. If $\langle B, R_y(B)\rangle \neq K$ then $\langle B, R_y(B)\rangle \subseteq M$ since M is the maximal subalgebra of K by the construction of K. So $R_y(B) \subseteq M$. $y = s + t$ for some $s \in K_0(x)$ and $t \in K_1(x)$. $t \notin M$ since $y \notin M$ and $s \in K_0(x) \subseteq M$. Consider $[x, t] = [x, y] - [x, s]$. $[x, y] \in M$ since $x \in B$ and $[x, y] \in R_y(B) \subseteq M$. $[x, s] \in M$ since $s \in K_0(x)$ implies that $[x, s] \in K_0(x)$. Thus $[x, t] \in M$. L_x is nonsingular on the finite dimensional space $K_1(x)$ and $M \cap K_1(x)$ is L_x invariant. Since $[x, t] \in M \cap K_1(x)$ so is t. Hence $t \in M$, a contradiction. Therefore 2 implies 1.

Assume 1 and that x is in $\langle B, R_x(B)\rangle$. Let $B = B(1) \triangleleft B(2) \triangleleft \ldots \triangleleft B(\omega) = A$ be the ω-ascending chain from B to A. $x \in B(n)$ for some n. $B \subset B(n-1) \triangleleft B(n)$ and $x \in B(n)$ implies that $R_x(B) \subseteq B(n-1)$. This is true for all n. Thus $x \in B(1) = B$.

□

3.2 Weak Ascendancy

The following lemma is an extension of Theorem 2.2.6 to ascendancy.

Lemma 3.2.1. *Let A be a Leibniz algebra over a field. Let H be a subalgebra of A and B be an abelian ideal of A. If (H, B) is a N_∞-pair for some $n \in \mathbb{N}$ then $H \triangleleft^\omega B + H$.*

Proof. Since (H, B) is a N_∞-pair then $B \subseteq N_\infty(H)$. So

$$B = B \cap \Big(\bigcup_{i\in\mathbb{N}} N_i(H)\Big) = \bigcup_{i\in\mathbb{N}} B_i.$$

It follows that

$$\bigcup_{i\in\mathbb{N}} (B_i + H) = \bigcup_{i\in\mathbb{N}} (B_i) + H = B + H.$$

Thus $H = B_0 + H \triangleleft^\omega B + H$.

□

Definition 3.2.2. A subalgebra H of Leibniz algebra A is said to be ω**-step weakly ascendant** if there exists an ascending chain of subspaces $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\omega$ such that

1. $M_\omega = \bigcup_{i=1}^\infty M_i$
2. $M_0 = H$ and $M_\omega = A$
3. $[M_{i+1}, H], [H, M_{i+1}] \subseteq M_i$ for all $i \in \mathbb{N}$

When this is true we write $H \leq^\omega A$.

Lemma 3.2.3. *Let H be a subalgebra of Leibniz algebra A. H is ω-step weakly ascendant if and only if (H,A) is a N_∞-pair.*

Proof. Assume H is ω-step weakly ascendant. Then we have a weakly ascending chain $M_0 \subseteq \cdots \subseteq M_\omega$. Let $x \in A$. Then $x \in M_i$ for some $i \in \mathbb{N}$. Since $H \subseteq M_j$ for all $j \in \mathbb{N}$, $\{x, {}_iH\} \subseteq H$ and (H,A) is a N_∞-pair.

Now assume (H,A) is a N_∞-pair. We will construct a weakly ascending chain of subspaces. Put $M_i = \{a \in A | \{a, {}_iH\} \subseteq H\}$ for $0 \leq i \in \mathbb{N}$. Then

1. $M_\omega = \bigcup_{i=1}^{\infty} M_i$
2. $M_0 = \{a \in A | \{a, {}_0H\} \subseteq H\} = H$
3. $M_\omega = \{a \in A | \{a, {}_iH\} \subseteq H \text{ for some } i \in \mathbb{N}\} = A$
4. $[M_{j+1}, H] \subseteq \{M_{j+1}, {}_1H\} = M_j$
5. $[H, M_{j+1}] \subseteq \{M_{j+1}, {}_1H\} = M_j$.

Thus $(M_i)_{i\leq n}$ is a weakly ascending chain for H in A so H is ω-step weakly ascendant in A. □

Lemma 3.2.4. *Let A be a Leibniz algebra over a field.*

a) If H is an ω-step weakly ascendant subalgebra of A and K is a subalgebra of A then $H \cap K$ is ω-step weakly ascendant in K.

b) If H is an ω-step weakly ascendant subalgebra of A and K is an ideal of A then $H + K$ is ω-step weakly ascendant in A.

c) Let f be a homomorphism of A onto $\overline{A}$, H be a subalgebra of A and $\overline{H}$ be a subalgebra of $\overline{A}$. If $H \leq^\omega A$ then $f(H) \leq^\omega \overline{A}$. If $\overline{H} \leq^\omega \overline{A}$ then $f^{-1}(\overline{H}) \leq^\omega A$.

Proof. Let $M_0 \subseteq \cdots \subseteq M_\omega$ be the weakly ascending chain of H in A.

a) Consider the chain $(M_0 \cap K) \subseteq \cdots \subseteq (M_\omega \cap K)$. $(M_0 \cap K) = H \cap K$. $(M_\omega \cap K) = A \cap K = A$. $\bigcup_{i=1}^{\infty}(M_i \cap K) = \left(\bigcup_{i=1}^{\infty} M_i\right) \cap K = M_\omega \cap K = K$. Let $x \in M_{i+1} \cap K$ and $y \in H \cap K$. Then $[x,y] \in M_i$ since $x \in M_{i+1}$ and $y \in H$ and $[x,y] \in K$ since $x,y \in K$. Thus $[x,y] \in M_i \cap K$ which implies that $[M_{i+1} \cap K, H \cap K] \subseteq M_i \cap K$. Similarly $[H \cap K, M_{i+1} \cap K] \subseteq M_i \cap K$. Thus

$(M_0 \cap K) \subseteq \cdots \subseteq (M_\omega \cap K)$ is a weakly ascending chain and $H \cap K$ is ω-step weakly ascendant in A.

b) Consider the chain $(M_0 + K) \subseteq \cdots \subseteq (M_\omega + K)$. $(M_0 + K) = H + K$. $(M_\omega + K) = A + K = A$. $\bigcup_{i=1}^{\infty}(M_i + K) = \left(\bigcup_{i=1}^{\infty} M_i\right) + K = M_\omega + K = M_\omega$. Let $m \in M_{i+1}, h \in H, k \in K$ and $j \in K$. Then

$$[m+k, h+j] = \underbrace{[m,h]}_{\in M_i} + \underbrace{[m,j]}_{\in K} + \underbrace{[k,h]}_{\in K} + \underbrace{[k,j]}_{\in K} \in M_i + K$$

which implies that $[M_{i+1} + K, H + K] \subseteq M_i + K$. Similarly $[H + K, M_{i+1} + K] \subseteq M_i + K$. Thus $(M_0 + K) \subseteq \cdots \subseteq (M_\omega + K)$ is a weakly ascending chain and $H + K$ is ω-step weakly ascendant in A.

c) $f(M_0) \subseteq \cdots \subseteq f(M_\omega)$ is a weakly ascending chain for $f(H)$ in $\overline{A}$. Let $\overline{M_0} \subseteq \cdots \subseteq \overline{M_\omega}$ be a weakly ascending chain for $\overline{H}$ in $\overline{A}$. Then $f^{-1}(M_0) \subseteq \cdots \subseteq f^{-1}(M_\omega)$ is a weakly ascending chain for $f^{-1}(\overline{H})$ in A.

□

Lemma 3.2.5. *Let A be a Leibniz algebra over a field such that $A = H + K$ with H a subalgebra of A, and K and abelian ideal of A. Then H is ω-step weakly ascendant if and only if H is ω-ascendant in A.*

Proof. If $H \lhd^\omega A$ then $H \leq^\omega A$.

Assume that $H \leq^\omega A$ and let $M_0 \subseteq \cdots \subseteq M_\omega$ be the weakly ascending chain for H in A. Then for all $i \in \mathbb{N}$ $[M_i, H], [H, M_i] \subseteq M_i$ and $M_i = M_i \cap (H + K) = H + (M_i \cap K)$. Then

$$\begin{aligned}
[M_i, M_{i+1}] &= [H + (M_i \cap K), H + (M_i \cap K)] \\
&\subseteq H^2 + [H, M_{i+1} \cap K] + [M_i \cap K] + K^2 \\
&\subseteq H + (M_i \cap K) \\
&= M_i
\end{aligned}$$

Similarly $[M_{i+1}, M_i] \subseteq M_i$. Hence M_i is a subalgebra of A and $M_i \lhd M_{i+1}$ for all $i \in \mathbb{N}$. Thus H is ω-ascendant in A. □

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