ABSTRACT

STRIKWERDA, SARAH. Optimal Control in Fluid Flows through Deformable Porous Media. (Under the direction of Lorena Bociu).

Poro-elasticity refers to fluid flow through deformable porous media such as soil or biological tissues. Poro-elastic systems have been studied extensively due to their important applications in petroleum engineering [7, 22] as well as biological applications: fluid flows through cartilages [23, 36, 46, 52, 54, 33], bones [55], and blood flow through human tissues [19, 20]. More recently, poro-vsico-elastic models have been studied [44, 10] since biological tissues have both elastic and visco-elastic effects [57].

We study optimal control problems constrained by poro-elastic or poro-visco-elastic PDE systems. There are distinct differences in the well-posedness of the poro-elastic and poro-visco-elastic models. Therefore, we study these two scenarios separately. In both cases, we focus on the problem of maintaining the solid displacement and fluid pressure close to desired/target values, motivated by applications in biomedicine, using the sources present in the system as control variables. One application of interest is perfusion in the optic nerve head in the eye and its connection with the development of glaucoma.

Glaucoma is the leading cause of irreversible blindness worldwide [32]. The reasons for the development and progression of the disease are not fully understood [58]. It is known that a tissue at the base of the optic nerve head called the lamina cribrosa plays an important role in glaucoma [19, 28, 37]. Mathematically, we can study the dynamics in the lamina cribrosa using the poro-elastic or poro-visco-elastic model which has state variables representing solid displacement and fluid pressure which are both related to the development of glacuoma [1, 2]. Therefore, understanding how to control these states could lead to breakthroughs in the treatment of glaucoma.

Here we show existence and uniqueness of optimal controls and derive necessary optimality conditions. The necessary optimality conditions are derived through adjoint PDE systems. We show existence and uniqueness of solutions to the adjoint equations using semi-group theory for implicit operators and Rothe's method. The analysis done on these problems is complemented by a numerical investigation where we approximate optimal controls. To approximate optimal controls, we use dual-mixed hybridized finite elements method to approximate a solution to the poro-elastic and poro-visco-elastic systems and apply a quasi-Newton method to minimize the cost functional.

Furthermore, we use a recently proposed optimization algorithm [35] to investigate finding an optimal control where the cost-function contains a total variation term and controls are taken to be piecewise constant functions. In this problem, we focus on seeking a control that is an optimal point in a ball in $L^1(0,T)$ centered around the control. When analyzing the asymptotics of this algorithm, the authors in [35] made an assumption on the second derivative of the cost functional. We work to show when that assumption is satisfied or violoated in optimal control problems contrained by poro-elastic and poro-visco-elastic systems. In the cases where the assumption does not hold, the algorithm is run on a sequence of problems where the cost functional is regularized through convolution operator of the control with a standard mollifier with support parameter ϵ. Γ-convergence was shown for the altered optimization problems when ϵ was driven to 0. Preliminary computational results suggest that running the SLIP algorithm on the regularized problem for small values of ϵ improved results when the assumption on the second derivative was not satisfied.

Optimal Control in Fluid Flows through Deformable Porous Media

by
Sarah Strikwerda

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APPROVED BY:

Patrick L. Combettes	Khai T. Nguyen
Ryan Murray	Lorena Bociu Chair of Advisory Committee

BIOGRAPHY

The author was born in Holland, Michigan to Tom and Karen Strikwerda. She grew up with her older siblings Emily and Mark while attending Holland Christian Schools. She always enjoyed learning math from Emily, Tom who was a seventh-grade math teacher, and from her teachers in school.

She attended Calvin College with the intention of becoming a secondary math teacher where she would be able to share her enjoyment of math with students. However, after realizing she wanted to continue learning math, she participated in undergraduate research and switched her math education major to a math major.

In 2018, the author graduated Calvin College and began a PhD program at North Carolina State University. Early on in her time at North Carolina State University, she decided to work with Dr. Lorena Bociu. While pursuing her PhD she enjoyed many social activities with her housemates, friends from her PhD program, and friends from church. Additionally, she mentored middle school and high school students weekly from her church. Following her time in Raleigh, she plans to complete a postdoc at the University of Pennsylvania in Philadelphia.

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INTRODUCTION

1.1 Glaucoma

Glaucoma is a group of eye diseases that leads to damage to the optic nerve (through retinal ganglion cell loss) and ultimately, vision loss. Glaucoma is the second leading cause of blindness worldwide [34], but unfortunately its cause and treatment currently remain unclear. However, it is strongly believed that the biomechanics of the Lamina Cribrosa (LC) inside the optic nerve head plays an important role in the development and progression of glaucoma [28, 19, 37]. The LC is a thin, porous tissue at the base of the optic nerve head in the eye, formed by a multilayered network of collagen fibers that insert into the scleral canal wall. It allows passage of the central retinal vessels, and the retinal ganglion cell axons, which relay visual information from the retina to the brain. One of the main functions of the LC is to stabilize the pressure difference between the intraocular pressure (IOP) in the intraocular space and the retrolaminar tissue pressure (RLTp) in the optic nerve canal.

Many studies have indicated that chronical IOP elevation induces significant structural changes in the Lamina Cribrosa [8, 17, 25, 26, 27]. More specifically, being structurally

weaker than the much thicker and denser sclera, the LC is more sensitive to changes in the IOP and tends to react to increased pressure through posterior displacement. This is thought to be one of the causes of nerve damage in glaucoma, as the displacement of the LC causes the pores to deform and pinch the traversing nerve fibers and blood vessels. More specifically, being structurally weaker than the much thicker and denser sclera, the LC is more sensitive to changes in the IOP and tends to react to increased pressure through posterior displacement.

To date, elevated IOP is the only treatable risk factor for glaucoma, but there is significant evidence that other factors might be involved in the disease. In fact, many individuals with elevated IOP never develop glaucoma [30], while many patients continue to progress to blindness despite IOP within target levels [47]. Hence, to further understand the pathogenesis of glaucoma, improve diagnosis, and enable novel means for preventing or treating glaucoma, it is of interest to understand the effects of IOP, RLTp, and blood pressure on the optic nerve head, and in particular on the biomechanics-hemodynamics of the LC. The LC can be modeled through the poro-visco-elastic model (1.2.1).

1.2 Poro-Elastic and Poro-Visco-Elastic Models

Poro-elasticity refers to fluid flow through deformable porous media. One example of porous media is soil. Historically, the theoretical development of porous media flows has been driven by petroleum engineering [7, 22]. More recently, modeling approaches as those used to describe fluid flow through rocks and soil consolidation have been applied to the modeling of fluid flow through biological tissues, with applications spanning from bioengineering [21, 33, 38] to physiology [19, 20, 31]. Examples of applications include fluid flow inside cartilages, bones, and engineered tissue scaffold [33, 36, 54, 46, 52, 23], and blood flow through tissues in the human body, like the brain, the liver, and the eye [45, 53, 55, 19]. Terzaghi initiated the study of poro-elasticity by developing a one-dimensional model to describe soil consolidation in 1925. In the 1940s, Biot generalized this work to three dimensions and time dependent loads. Biot laid the mathematical foundation for the study of fluids flowing through porous media [7].

The poro-elastic system is characterized by equations describing the balance of linear momentum for the fluid-solid mixture and the balance of mass for the fluid. These two equations are coupled through a permeability tensor. In general, the permeability tensor,

$\mathbf{K}$, is dependent on porosity. We say $\mathbf{K} = k(\phi)\mathbf{I}$ where ϕ is porosity and k is a functional dependent on pore shape and dimension. Additionally, the poro-elasticity can be studied with compressible constituents or incompressible constituents. In the compressible case, fluid content is dependent on the divergence of mechanical displacement and fluid pressure, while in the incompressible case fluid content is only dependent on the divergence of the mechanical displacement. By incompressible constituents, we mean that the solid and fluid phases can't undergo volume changes at the microscale, so fluid content is equivalent to solid dilation (the divergence of mechanical displacement). Incompressible constituents are assumed for biological applications since tissues have a mass density similar to the mass density of water [22].

Most of the theoretical work related to poro-elastic systems has been done under the assumption of *constant permeability* ($\mathbf{K} = k\mathbf{I}$, with $k =$ constant), which yields a linear coupled system [3, 59, 41, 48]. In 1977, Auriault and Sanchez-Palencia studied the poro-elastic system with compressible constituents. They showed that with no sources and homogeneous Dirichlet boundary conditions a unique solution existed [3]. Zenisek followed this work in 1984 using Rothe's method to show well-posedness of solutions in the incompressible case with mixed boundary conditions [59]. Showalter proved similar results using semi-group theory for implicit operators and then worked with Su to study the well-posedness of a nonlinear Biot model with permeability depending nonlinearly on dilation [48, 49, 51, 50]. The analysis was performed with homogeneous Dirichlet boundary conditions and compressible constituents. Cao, Chen and Meir also studied a nonlinear version of the poro-elasticity in 2013, but with permeability depending nonlinear on fluid content [18]. Cao, Chen and Meir used compressible constituents and homogeneous Dirichlet boundary conditions. The strategy in [18] relies on Rothe's method, uses the simplified structure of a pressure-to-dilation operator introduced in [48], and takes advantage of compressibility and full elliptic regularity for the solid displacement. Existence was shown; however uniqueness was only shown for solutions satisfying certain assumptions. The result presented requires k to be Lipschitz continuous and $c_0\kappa_*/(C_p\kappa^* C_B) > 1$ where c_0 is the constant from the definition of the fluid content, C_p is the Poincare constant and C_B is the continuity constant for the pressure to dilation map. Additionally, the uniqueness result requires $\nabla p \in L^\infty(\Omega)$ with $\|\nabla p\|_{L^\infty(\Omega)} < C$ for all $t \in [0, T]$. When these assumptions are satisfied, p is unique in the set of functions $\{w \in L^2(0, T; \mathbf{V}) \mid \nabla p \in L^\infty(\Omega),\ \|\nabla p\|_{L^\infty(\Omega)} \leq C\}$. In [10], Bociu et. al. provide the first result for existence of weak solutions for nonlinear poro-elastic model with incompressible constituents and with permeability depending nonlinearly on solid dilation, and

mixed boundary conditions with no simplifying assumptions on the domain boundary. This work extends all previous results on analysis of poro-elastic models [59, 41, 48, 49, 18]. The theoretical results in [10] are complemented with numerical simulations based on a novel dual mixed hybridized finite element discretization. When the data are sufficiently regular, the simulations show that the solutions satisfy the energy estimates predicted by the theoretical analysis. Interestingly, the simulations also show that the Darcy velocity and the related fluid energy might become unbounded if the data do not enjoy the time regularity required by the theory.

In [10], the poro-visco-elastic case is analyzed for the first time as well. This model is useful in biological applications and have been widely used to analyze in vitro creep and stress relaxation experiments on the tissues under confined and unconfined compression tests [52]. Mechanical properties of biological tissues differ significantly from those of rocks. In particular, most biological tissues are composed by both elastin and collagen, and therefore the deformable matrix through which the fluid flows exhibits both elastic and visco-elastic behaviors. As material properties and volume fractions of elastic and collagen vary in age, health and disease, their influence on the physical system is a crucial part in the investigation of these biological fluid-mixture problems. Therefore, the authors of [10] complement their analysis and numerics of the incompressible poro-elastic system by also studying the poro-visco-elastic system. The model couples the viscoelastic solid matrix with the interstitial fluid flow through the permeability tensor which is dependent on solid dilation.

Both the poro-elastic model and poro-vsico-elastic model provided below describe the motion of a fluid-solid mixture under the assumption of full saturation, negligible inertia, small deformations and incompressible constituents. The assumption of full saturation means that any space in the domain that the solid does not occupy is filled with fluid. The PDE comes from ensuring the balance of mass for the fluid and linear momentum for the fluid-solid mixture. The following equations must hold in $\Omega \times (0, T)$:

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \mathbf{v} = S, \qquad \nabla \cdot \mathbf{T}(\mathbf{u}, p) + \mathbf{F} = \mathbf{0}, \tag{1.2.1a}$$

$$\zeta = \nabla \cdot \mathbf{u}, \qquad \mathbf{v} = -\mathbf{K}\nabla p, \tag{1.2.1b}$$

$$\mathbf{T}(\mathbf{u}, p) = \mathbf{T}_e + \delta \mathbf{T}_v - p\mathbf{I}, \qquad \mathbf{T}_e = 2\mu_e \epsilon(\mathbf{u}) + \lambda_e \mathrm{tr}(\epsilon(\mathbf{u}))\mathbf{I}, \qquad \mathbf{T}_v = 2\mu_v \epsilon(\mathbf{u}_t) + \lambda_v \mathrm{tr}(\epsilon(\mathbf{u}_t))\mathbf{I}, \tag{1.2.1c}$$

where ζ is the fluid content, $\mathbf{v}$ is the discharge (or Darcy) velocity, p is the Darcy pressure, $\mathbf{u}$ is the solid displacement, $\mathbf{T}$ is the total stress tensor, $\epsilon(\mathbf{w}) = (\nabla\mathbf{w} + \nabla\mathbf{w}^T)/2$ is the linearized strain tensor. Equations (1.2.1a) express the balance of mass and linear momentum, whereas Equations (1.2.1b)-(1.2.1c) are the constitutive equations that are necessary to close the system. In the general Biot model, the fluid content is given by $\zeta = c_0 p + \alpha\nabla\cdot\mathbf{u}$, c_0 being the constrained specific storage coefficient and α the Biot-Willis coefficient. In the case of incompressible mixture components, as often assumed in biological tissue modeling, we have that $c_0 = 0$ and $\alpha = 1$ [22], and therefore the fluid content equals the solid dilation, i.e., $\zeta = \nabla\cdot\mathbf{u}$, see Equations (1.2.1b). The discharge velocity $\mathbf{v}$ and the Darcy pressure p are related via the permeability tensor $\mathbf{K} = k(\nabla\cdot\mathbf{u})\mathbf{I}$. The permeability is a nonlinear function of porosity, $\nabla\cdot\mathbf{u}$, which represents the volume fraction of fluid within the medium. We assume that there exists constants κ_* and κ^* such that $0 < \kappa_* \le k(s) \le \kappa^* < \infty,\ \forall s \in \mathbb{R}$. Many studies assume k to be a constant or a function of $(\mathbf{x}, t)$ instead of porosity, thereby leading to a linear coupling between the equations for linear momentum and mass balance. Similarly to the nonlinear case, we have the assumption

Assumption 1.2.1. *For all $(x, t) \in \Omega \times [0, T]$, we assume that there exists constants κ_* and κ^* such that*

$$0 < \kappa_* \le k(\mathbf{x}, t) \le \kappa^*.$$

In the constitutive equation for the total stress tensor $\mathbf{T}$ given in Eq. (1.2.1c), the behavior of the solid component within the medium is characterized by the Lamé elastic parameters λ_e and μ_e and viscoelastic parameters λ_v and μ_v as in [10]. The extent to which the model includes viscoelastic effects is represented by the parameter $\delta > 0$ with $\delta = 0$ corresponding to the purely elastic case.

In [10] and the rest of this dissertation, we consider a bounded domain which would makes sense for biological applications. The boundary, $\partial\Omega$, is broken up into Γ_N and Γ_D. The intersection of the closure of Γ_N and Γ_D is not necessarily empty. On Γ_N, there are Neumann boundary conditions and on Γ_D there are Dirichlet boundary conditions on mechanical displacement. Furthermore, Γ_D is broken up into $\Gamma_{D,v}$, which has a Dirichlet boundary condition on fluid velocity, $\mathbf{v}$, and $\Gamma_{D,p}$ which has a Dirichlet boundary condition on fluid pressure. The boundary conditions are as follows: For $\mathbf{g}$ and ψ given functions of space and time, we assume

$$\mathbf{T}\mathbf{n} = \mathbf{g} \text{ and } \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_N \times (0, T), \tag{1.2.2}$$

$$\mathbf{u} = \mathbf{0} \text{ and } p = 0 \text{ on } \Gamma_{D,p} \times (0, T),\ \mathbf{u} = \mathbf{0} \text{ and } \mathbf{v}\cdot\mathbf{n} = \psi \text{ on } \Gamma_{D,v} \times (0, T), \tag{1.2.3}$$

where $\mathbf{n}$ is the outward unit normal vector to the surface boundary. Note that Γ_N is assumed to be an impermeable surface where the total stress is prescribed. In comparison, $\Gamma_{D,p}$ and $\Gamma_{D,v}$ are permeable clamped surfaces, on which either the pressure or the normal velocity are prescribed.

Finally, the system has an initial condition:

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\ \text{for}\ \mathbf{x}\in\Omega,\ \ \text{with}\ \ \mathbf{u}_0(\mathbf{x})\ \ \text{given. We also define}\ \ d_0(\mathbf{x})=\nabla\cdot\mathbf{u}_0(\mathbf{x}). \tag{1.2.4}$$

Since the time derivative of $\mathbf{u}$ only shows up on the divergence of $\mathbf{u}$ in the poro-elastic case (since $\delta=0$ in this case), the correct initial condition to use is $\nabla\cdot\mathbf{u}=d_0$. To ensure well-posedness, we still assume $d_0=\nabla\cdot\mathbf{u}_0$ in this case.

There are distinct differences in the well-posedness of the poro-elastic and poro-visco-elastic models. Structural visco-elasticity plays a critical role in determining the regularity requirements for both volumetric and boundary forces as well as for the corresponding solution. The numerical investigation in [10] also provided evidence of blow-up in the energy associated with Darcy pressure in the purely elastic case if the data is not sufficiently smooth in time. This result suggests that sudden changes in stress conditions combined with the absence of structural viscoelasticity could lead to microstructural damage. Additionally, sensitivity analysis revealed that the solution of the PVE model is more sensitive to the boundary data when the viscoelastic effects are not present [5] and showed that the solution is consistently more sensitive to boundary flux than boundary traction [40]. However, both studies proved that mechanical displacement was not as sensitive to the boundary data as the Darcy pressure. Therefore, when studying an optimal control problem subject to this coupled system, one should look for a different way to control/optimize the mechanical displacement.

1.3 Well-posedness of model

As usual, $H^s(D)$ represents the Sobolev space of order s defined on a domain D, while $H_0^s(D)$ is the closure of $C_0^\infty(D)$ in the $H^s(D)$ norm, denoted by $\|\cdot\|_{H^s(D)}$. We use $\mathbf{H}^s(D)$ to denote $(H^s(D))^3$ and $\mathbf{L}^2(D)$ to denote $(L^2(D))^3$. Unless otherwise specified, $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and inner-product, respectively, taken in $(L^2(D))^n$ where n is clear by the context. Additionally, $\langle\cdot,\cdot\rangle$ will be used to denote the $L^2(\partial D)$ inner-product on a portion of the boundary which will be denoted in a subscript, e.g. $\langle\mathbf{u},\mathbf{w}\rangle_{\Gamma_N}$.

The primary spaces in our analysis are

$$\mathbf{V} \equiv (H^1_{\Gamma_D}(\Omega))^3, \qquad V \equiv H^1_{\Gamma_{D,p}}(\Omega), \qquad \mathbb{V} \equiv \mathbf{V} \times V,$$

for displacement $\mathbf{u}$ and the pressure p, respectively. Note that the functional spaces are of the form $H^1_{\Gamma_*}(\Omega) = \{f \in H^1(\Omega) \,:\, \gamma[f]\Big|_{\Gamma_*} = 0\}$. For sake of exposition, we take the Lamé parameters in the elasticity term normalized to unity (the analysis follows similarly if the Lamé parameters were not normalized to unity), and we define the bilinear form associated with the elasticity operator by

$$a(\mathbf{u},\mathbf{w}) = (\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{w}) + (\nabla\mathbf{u} : \nabla\mathbf{w}) + (\nabla\mathbf{u} : (\nabla\mathbf{w})^T), \tag{1.3.1}$$

where $\nabla\mathbf{u}$ stands for the Jacobian matrix of $\mathbf{u}$ and the Frobenius inner product of two matrices is given by

$$(A,B) = \int_\Omega \mathbf{A}..\mathbf{B}\, d\Omega = \int_\Omega (A_{ij}B_{ij}) d\Omega.$$

We note that the bilinear form $a(\cdot,\cdot)$ defines an equivalent inner product on $\mathbf{V}$, due to Assumptions on the Domain described below). The inner product for V is inherited from $H^1(\Omega)$. Additionally, when simplified boundary conditions are used, we will use the Hilbert space

$$L^2_0(\Omega) = \{\mu \in L^2(\Omega) : \int_\Omega \mu d\,x = 0\}.$$

We recall here the existence and uniqueness of weak solution results provided in [10]. When $\delta > 0$, higher regularity can be shown on $\mathbf{u}$ since more time derivatives on $\mathbf{u}$ are present in the system. Therefore, we define the weak solutions for the two systems differently. In [10], k is a nonlinear function of $(\nabla\cdot\mathbf{u})$. However, the results also hold when k is a function of $\mathbf{x}$ and t or a constant.

Definition 1. *[Poro-elastic Weak Solution] A solution to* (1.2.1a)–(1.2.4) *is represented by the pair of functions* $\mathbf{u} \in L^2(0,T;\mathbf{V})$ *and* $p \in L^2(0,T;V)$ *such that:*
(a) for any $\mathbf{w} \in L^2(0,T;\mathbf{V})$*,* $q \in L^2(0,T;V)$*, and* $f \in C_0^\infty(0,T)$*, the following variational formulations are satisfied:*

$$\int_0^T a(\mathbf{u},\mathbf{w}) f(t)\, d\,t - \int_0^T (p, \nabla\cdot\mathbf{w}) f(t)\, d\,t = \int_0^T \langle \mathbf{g},\mathbf{w}\rangle_{\Gamma_N} f(t)\, d\,t + \int_0^T (\mathbf{F},\mathbf{w}) f(t)\, d\,t \tag{1.3.2}$$

$$\int_0^T (k\nabla p, \nabla q) f(t)\, dt - \int_0^T (\nabla \cdot \mathbf{u}, q) f'(t)\, dt = -\int_0^T \langle \psi, q \rangle_{\Gamma_{D,v}} f(t)\, dt + \int_0^T (S, q) f(t)\, dt \tag{1.3.3}$$

(b) for every $q \in V$, the term $(\nabla \cdot \mathbf{u}(t), q)_{L^2(\Omega)}$ uniquely defines an absolutely continuous function on $[0,T]$ and the initial condition $(\nabla \cdot \mathbf{u}(0), q) = (d_0, q)$ is satisfied.

Definition 2. *[Poro-visco-elastic Weak Solution] A weak solution to (1.2.1a)-(1.2.4) is represented by the pair of functions $\mathbf{u} \in H^1(0,T;\mathbf{V})$ and $p \in L^2(0,T;V)$ such that:*
(a) for any $\mathbf{w} \in L^2(0,T;\mathbf{V})$ and $w \in L^2(0,T;V)$, the following variational formulations are satisfied:

$$\delta \int_0^T a(\mathbf{u}_t, \mathbf{w})\, dt + \int_0^T a(\mathbf{u}, \mathbf{w})\, dt - \int_0^T (p, \nabla \cdot \mathbf{w})\, dt = \int_0^T \langle \mathbf{g}, \mathbf{w} \rangle_{\Gamma_N}\, dt + \int_0^T (\mathbf{F}, \mathbf{w})\, dt \tag{1.3.4}$$

$$\int_0^T (k(\mathbf{x},t)\nabla p, \nabla w)\, dt + \int_0^T (\nabla \cdot \mathbf{u}_t, w)\, dt = -\int_0^T \langle \psi, w \rangle_{\Gamma_{D,\mathbf{v}}}\, dt + \int_0^T (S, w)\, dt \tag{1.3.5}$$

(b) for every $w \in V$, the term $(\nabla \cdot \mathbf{u}(t), w)$ uniquely defines an absolutely continuous function on $[0,T]$ and the initial condition $(\nabla \cdot \mathbf{u}(0), w) = (\nabla \cdot \mathbf{u}_0, w)$ is satisfied.

Definition 3. *[Energy and Data] Energy functionals for solutions and data are defined as follows:*

$$E(\mathbf{u}(t)) \equiv \frac{1}{2}\left[||\nabla \cdot \mathbf{u}(t)||^2 + ||\nabla \mathbf{u}||^2 + (\nabla \mathbf{u} : \nabla \mathbf{u}^T) \right], \qquad E(p(t)) \equiv (k(\mathbf{x},t)\nabla p, \nabla p)$$
$$DATA\Big|_\delta^T \equiv \int_0^T \left[||\mathbf{g}(t)||^2_{\mathbf{L}^2(\Gamma_N)} + ||\psi(t)||^2_{L^2(\Gamma_{D,\mathbf{v}})} + ||S(t)||^2_{L^2(\Omega)} + ||\mathbf{F}(t)||^2_{\mathbf{L}^2(\Omega)} \right] \tag{1.3.6}$$

Assumption 1.3.1. *[Assumptions on the Domain]*

1. *We assume that Γ_D and $\Gamma_{D,p}$ are sets of positive measure. Thus Korn's Inequality for $\mathbf{u}$ and Poincaré's Inequality for p are satisfied:*

$$E(\mathbf{u}(t)) \geq c\, ||\mathbf{u}(t)||^2_{\mathbf{H}^1(\Omega)}, \qquad ||v||_{L^2(\Omega)} \leq C_P ||\nabla v||_{\mathbf{L}^2(\Omega)}, \ \ \forall v \in V.$$

2. *We assume the boundary Γ is such that either $\overline{\Gamma_N} \cap \overline{\Gamma_D} = \emptyset$, or Γ_N and Γ_D do not meet tangentially and the supremum of their dihedral angles is strictly less than π.*

Remark 1.3.1. The second assumption is related to the associated elasticity system

$$\begin{cases} -\nabla\cdot\left(2\mu_e\epsilon(\mathbf{u})+\lambda_e(\nabla\cdot\mathbf{u})\mathbf{I}\right)=-\nabla p-\mathbf{F} & \text{on } \Omega \\ \mathbf{u}=\mathbf{0} & \text{on } \Gamma_D \\ \mathbf{T}_e(\mathbf{u})\mathbf{n}=\mathbf{g}+p\mathbf{n} & \text{on } \Gamma_N \end{cases} \tag{1.3.7}$$

It is known that given $p\in V$, $\mathbf{g}\in\mathbf{L}^2(\Gamma_N)$, and $\mathbf{F}\in\mathbf{L}^2(\Omega)$, then there exists a unique solution $\mathbf{u}\in\mathbf{V}$. If in addition, $\overline{\Gamma_N}\cap\overline{\Gamma_D}=\emptyset$, then $\mathbf{u}\in\mathbf{H}^2(\Omega)\cap\mathbf{V}$, and thus $\nabla\cdot\mathbf{u}\in H^1(\Omega)$ [48]. Alternatively, we have the following result from [39]:

Lemma 1.3.1. *Consider the Lamé system* (1.3.7) *in polyhedral domains, with given* $p\in V$, $\mathbf{F}\in\mathbf{L}^2(\Omega)$, *and* $\mathbf{g}\in\mathbf{H}^{1/2}(\Gamma_N)$. *Assume the following geometrical assumptions on the boundary* Γ: Γ_N *and* Γ_D *do not meet tangentially and the supremum of their dihedral angles is strictly less than* π. *Then one obtains the solution regularity* $\mathbf{u}\in\mathbf{H}^{3/2+\epsilon}(\Omega)$ *for some* $\epsilon>0$.

Theorem 1.3.2 (Poro-Elastic: Existence of a Weak Solution [10])**.** *Under Assumptions on the Domain (1.3.1), with* $\mathbf{u}_0\in\mathbf{V}$, *and*

$$\mathbf{F}\in\ \mathbf{H}^1\left(0,T;\mathbf{L}^2(\Omega)\right),\ S\in L^2(0,T;L^2(\Omega)),\ \mathbf{g}\in\ \mathbf{H}^1\left(0,T;\mathbf{H}^{1/2}(\Gamma_N)\right),\ \psi\in\ L^2\left(0,T;L^2(\Gamma_{D,\mathbf{v}})\right),$$

there exists a weak solution $(\mathbf{u},p)$ *satisfying Definition 1 and the following energy estimate:*

$$\sup_{t\in[0,T]}E(\mathbf{u}(t))+\int_0^T\Big[E(p(t))+E(\mathbf{u}(t))\Big]\,dt\le C_1\Big[E(\mathbf{u}(0))+DATA\Big|_0^T\Big]e^{C_2T}.$$

Theorem 1.3.3 (Poro-Visco-Elastic: Existence of Weak Solution [10])**.** *Under Assumptions on the Domain (1.3.1), with* $\mathbf{u}_0\in\mathbf{V}$, *and*

$$\mathbf{F}\in\ \mathbf{L}^2\left(0,T;\mathbf{L}^2(\Omega)\right),\ S\in L^2(0,T;L^2(\Omega)),\ \mathbf{g}\in\ \mathbf{L}^2\left(0,T;\mathbf{H}^{1/2}(\Gamma_N)\right),\ \psi\in\ L^2\left(0,T;L^2(\Gamma_{D,\mathbf{v}})\right),$$

there exists a weak solution $(\mathbf{u},p)$ *satisfying Definition 2 and the following energy estimate:*

$$\begin{aligned}&\sup_{t\in[0,T]}E(\mathbf{u}(t))+\int_0^T\Big[E(p(t))+E(\mathbf{u}(t))+\delta E(\mathbf{u}_t(t))\Big]\,dt\\&\le C_1\Big[E(\mathbf{u}(0))+\left(\frac{1}{1+\delta}\right)DATA\Big|_\delta^T\Big]\exp\left(\frac{C_2T}{1+\delta}\right).\end{aligned} \tag{1.3.8}$$

Remark 1.3.2. We note the need for higher time regularity on $\mathbf{F}$ and $\mathbf{g}$ in the poro-elastic

case (where the time regularity of the solution is L^2) than in the poro-visco-elastic case (where $\mathbf{u}$ has H^1 time regularity).

Remark 1.3.3. We point out that the existence result in [10] is obtained in the more general case where the permeability function k is nonlinearly dependent on the structural dilation $\nabla\cdot\mathbf{u}$, with lower and upper constant bounds.

Also, in [10] it was shown that (1.3.8) holds for any $\mathbf{u}$, p satisfying Definition 2. Therefore, this energy estimate can be utilized to show uniqueness of a solution when the system is linear (k is a constant or a function of $\mathbf{x}$ and t). Assume $(\mathbf{u}_1, p_1)$ and $(\mathbf{u}_2, p_2)$ both satisfy Definition 2. Then due to the linearity of the system, $(\mathbf{u}_1-\mathbf{u}_2, p_1-p_2)$ satisfies Definition 2 with all sources and the initial condition equal to zero. Hence, (1.3.8) implies

$$\begin{aligned}\sup_{t\in[0,T]}\Big(E(\mathbf{u}_1(t)-\mathbf{u}_2(t))+\int_0^T\Big[E(p_1(t)-p_2(t))+E(\mathbf{u}_1(t)-\mathbf{u}_2(t))\\ +\delta E\left(\frac{d}{dt}\mathbf{u}_1(t)-\frac{d}{dt}\mathbf{u}_2(t)\right)\Big]\Big)dt\le 0.\end{aligned}\tag{1.3.9}$$

Notice that

$$E(\mathbf{u}_1(t)-\mathbf{u}_2(t))=\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{\mathbf{V}}^2\ge 0,$$

$$\delta E\left(\frac{d}{dt}\mathbf{u}_1(t)-\frac{d}{dt}\mathbf{u}_2(t)\right)=\|\frac{d}{dt}\mathbf{u}_1(t)-\frac{d}{dt}\mathbf{u}_2(t)\|_{\mathbf{V}}^2\ge 0,$$

$$\text{and }\ E(p_1(t)-p_2(t))=(k(\mathbf{x},t)(\nabla p_1(t)-\nabla p_2(t)),\nabla p_1(t)-\nabla p_2(t)).$$

Due to (1.2),

$$(k(\mathbf{x},t)(\nabla p_1(t)-\nabla p_2(t)),\nabla p_1(t)-\nabla p_2(t))\ge\kappa_*(\nabla p_1(t)-p_2(t),\nabla p_1(t)-p_2(t)).$$

Applying these facts to (1.3.9) yields

$$\int_0^T\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{\mathbf{V}}^2dt+\kappa_*\int_0^T\|p_1(t)-p_2(t)\|_V^2dt\le 0.\tag{1.3.10}$$

Hence, $\mathbf{u}_1=\mathbf{u}_2$ in $L^2(0,T;\mathbf{V})$ and $p_1=p_2$ in $L^2(0,T;V)$.

It is noteworthy that in the elastic case, when $\delta=0$, an estimate like (1.3.8) was only shown for the solutions constructed in the proof of existence. Therefore, from this work we cannot conclude that any weak solution in the elastic case satisfies an energy estimate like (1.3.8). However, uniqueness of a solution when k is constant can still be shown [59].

Let $(\mathbf{u}_1, p_1)$ and $(\mathbf{u}_2, p_2)$ both satisfy Defintion 1. Then we have that for any $\mathbf{w} \in L^2(0,T;\mathbf{V})$, $q \in L^2(0,T;V)$, and $f \in C_0^\infty(0,T)$ the following variational formulations are satisfied:

$$\int_0^T a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{w}) f(t)\, dt - \int_0^T (p_1 - p_2, \nabla \cdot \mathbf{w}) f(t)\, dt = 0 \tag{1.3.11}$$

$$\int_0^T (k\nabla p_1 - k\nabla p_2, \nabla q) f(t)\, dt - \int_0^T (\nabla \cdot (\mathbf{u}_1 - \mathbf{u}_2), q) f'(t)\, dt = 0 \tag{1.3.12}$$

From (1.3.11) we see that for almost every $t \in [0,T]$,

$$a(\mathbf{u}_1(t) - \mathbf{u}_2(t), \mathbf{w}) - (p_1(t) - p_2(t), \nabla \cdot \mathbf{w}) = 0.$$

We can set $\mathbf{w}$ to be $\mathbf{u}_1(t) - \mathbf{u}_2(t)$ and integrate from 0 to t^*. Hence, we have

$$\int_0^{t^*} a(\mathbf{u}_1(t) - \mathbf{u}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t)) dt - \int_0^{t^*} (p_1(t) - p_2(t), \nabla \cdot \mathbf{u}_1(t) - \mathbf{u}_2(t)) dt = 0 \tag{1.3.13}$$

Integrating the term with a time derivative in (1.3.12) by parts, we have that

$$\int_0^T (k\nabla p_1 - k\nabla p_2, \nabla q) f(t) dt + \int_0^T (\nabla \cdot (\mathbf{u}_1 - \mathbf{u}_2), q)_t f(t) dt = 0$$

Hence, for almost every $t \in [0,T]$, we see

$$(k\nabla p_1(t) - k\nabla p_2(t), q) + \frac{d}{ds}(\nabla \cdot (\mathbf{u}_1 - \mathbf{u}_2)(s), q) = 0 \tag{1.3.14}$$

Using the fact that $\nabla \cdot (\mathbf{u}_1(0) - \mathbf{u}_2(0)) = 0$ and integrating (1.3.14) from 0 to τ, we have

$$-(\nabla \cdot (\mathbf{u}_1 - \mathbf{u}_2)(\tau), q) = \int_0^\tau (k\nabla p_1(t) - k\nabla p_2(t), q) dt.$$

We will now let $q = p_1(\tau) - p_2(\tau)$ to obtain

$$-(\nabla \cdot (\mathbf{u}_1 - \mathbf{u}_2)(\tau), p_1(\tau) - p_2(\tau)) = \int_0^\tau (k\nabla p_1(t) - k\nabla p_2(t), \nabla p_1(\tau) - \nabla p_2(\tau)) dt.$$

We define $z(\tau)=\int_0^\tau \nabla p_1(t)-\nabla p_2(t)dt$. Then $z'(\tau)=\nabla p_1(\tau)-\nabla p_2(\tau)$, so

$$\int_0^\tau (k\nabla p_1(t)-k\nabla p_2(t),\nabla p_1(\tau)-\nabla p_2(\tau))dt=(kz(\tau),z'(\tau))=\frac{d}{d\tau}\frac{(kz(\tau),z(\tau))}{2} \tag{1.3.15}$$

Remark 1.3.4. Note that (1.3.15) is only possible if k is independent of t. If k was a function of t, then we could not pull it out of the time integral.

Therefore, we have that $-(\nabla\cdot(\mathbf{u}_1-\mathbf{u}_2)(\tau),p_1(\tau)-p_2(\tau))=\frac{d}{d\tau}\frac{(kz(\tau),z(\tau))}{2}$ which we will integrate from 0 to t^*. Recalling (1.3.13), we see

$$\int_0^{t^*} a(\mathbf{u}_1(t)-\mathbf{u}_2(t),\mathbf{u}_1(t)-\mathbf{u}_2(t))dt+\frac{(kz(t^*),z(t^*))}{2}=0 \tag{1.3.16}$$

Neither of the terms on the lefthand side could be negative, so they must both equal zero. Setting t^* to T, we see $\|\mathbf{u}_1-\mathbf{u}_2\|_{L^2(0,T;\mathbf{V})}=0$. Thus, $\int_0^T(\nabla\cdot(\mathbf{u}_1-\mathbf{u}_2),q)f'(t)dt=0$, so (1.3.12) implies $\|p_1-p_2\|_{L^2(0,T;V)}=0$. Therefore, there can only be one solution when k is a constant or a function of x.

More recently, uniqueness has been studied for the poro-elastic model with permeability, k is a function of $(\mathbf{x},t)$ [13]. To achieve this result simpler boundary conditions were used. Namely,

$$\mathbf{u}=0 \text{ on } \Gamma, \quad p=0 \text{ on } \Gamma_D, \quad k(\mathbf{x},t)\nabla p=0 \text{ on } \Gamma_N. \tag{1.3.17}$$

The result also shows higher regularity for $\mathbf{u}$

Theorem 1.3.4 (Existence of a Weak Solution [13])**.** *Let the permeability $k(\mathbf{x},t)$ satisfy Assumption 1.2.1. Let $d_0\in L^2_0(\Omega)$, $\mathbf{F}\in H^1(0,T;\mathbf{V}')$, and $S\in L^2(0,T;V')$. Then there exists a unique weak solution satisfying Definition 1 (with $\mathbf{g}=0$, $\psi=0$, $k=k(\mathbf{x},t)$)) such that $\mathbf{u}\in C([0,T];\mathbf{V})$, $p\in C([0,T];L^2_0(\Omega))\cap L^2(0,T;V)$, and*

$$\|\mathbf{u}\|^2_{L^\infty(0,T;\mathbf{V})}+\|p\|^2_{L^2(0,T;V)}+\|[\nabla\cdot\mathbf{u}]_t\|^2_{L^2(0,T;V')}\le C\left(\|d_0\|^2_{L^2(\Omega)}+\|S\|^2_{L^2(0,T;V')}+\|\mathbf{F}\|^2_{H^1(0,T;\mathbf{V}')}\right). \tag{1.3.18}$$

The proof for uniqueness in this case relied on proving an a priori bound. Recall that in the visco-elastic case this came from using $\mathbf{u}_t$ as a test function which is not possible in the poro-elastic case. However, one can use the time derivative of the convolution of $\mathbf{u}$ with a mollifier as a test function. This strategy is used to show uniqueness.

1.4 Sensitivity Analysis

Sensitivity analysis is the first step towards optimization and control problems associated with these fluid-solid mixture models. Numerical sensitivity analysis on the 1D poro-visco-elastic models from [57] with respect to boundary data, which are the main drivers of the system, was performed in [4, 5]. This was done using a complex step method where the derivative is approximated by the following:

$$\frac{\partial f}{\partial x} \approx \frac{Im[f(x+ih)]}{h}.$$

The authors compared the results obtained in the elastic case vs. viscoelastic case, as it is known that structural viscosity of biological tissues decreases with age and disease. Key observations: (1) Solution $(\mathbf{u}, p, \mathbf{v})$ is more sensitive to boundary traction $\mathbf{g}$ in the elastic case ($\delta = 0$) than in the viscoelastic scenario ($\delta > 0$). This could explain why in the theoretical results provided in [10], the boundary source, $\mathbf{g}$, was required to have higher time regularity in order to obtain solution $(\mathbf{u}, p) \in L^2(0, T; \mathbf{L}^2(\Omega) \times L^2(\Omega))$, and with appropriate energy estimate in terms of data, in the purely elastic case ($\delta = 0$); (2) Effects of the boundary source $\mathbf{g}$ are most significant for the discharge velocity $\mathbf{v}$, especially in the purely elastic case. This agrees with the numerical investigation in [10] which hinted that the fluid energy (dependent on the discharge velocity) becomes unbounded as the boundary source of traction loses H^1-smoothness in time, and viscoelasticity is no longer present; (3) The sensitivity of the solution with respect to the boundary source ψ was computed as well. Interestingly, the fluid-dynamical variables $\mathbf{v}$ and p appear to be more sensitive to changes in $\mathbf{g}$ than to changes in ψ. This suggests that, To control fluid velocity and pressure, it would be much more effective to act on the boundary conditions for the solid structure, namely the traction $\mathbf{g}$, rather than on the boundary conditions for the velocity itself, namely ψ; (4) Solid displacement $\mathbf{u}$ is the least sensitive to changes in $\mathbf{g}$ and ψ. This finding shows that small changes in the solid displacement may actually correspond to big changes in fluid velocity and pressure, thereby suggesting that monitoring the sole solid displacement might not be indicative of the fluid-dynamical state inside the medium.

In addition, in [12] the authors performed sensitivity analysis on the 1D dimensionless poro-elastic ($\delta = 0$) and poro-visco-elastic ($\delta > 0$) solutions with respect to the boundary traction and their dependence on the dimensionless parameters identified in [57]. The results in [12] consistently show that the maximum magnitude of the sensitivities (for all

three variables: solid displacement, fluid pressure and fluid velocity) is largest when viscoelasticity is not present and gets smaller monotonically with respect to the dimensionless parameter η associated to viscoelasticity (but also dependent on the value of the permeability and the length of the domain). However, the magnitudes of the sensitivities were not always monotonic with respect to η. The numerical results in [12] correspond closely with results typically observed in creep tests for poro-visco-elastic materials. The lag in the solid displacement due to the inclusion of structural viscosity is indicative of the increase in time that it takes for the solid to reach an equilibrium after an applied load.

CHAPTER

2

OPTIMAL CONTROL PROBLEM

We seek to solve an optimal control problem using either $\mathbf{F}, S, \mathbf{g}$ or ψ as the control. The variables we would like to control are the solid displacement $\mathbf{u}$ and the Darcy fluid pressure p which we will drive towards target $\mathbf{u}_d$ and p_d, respectively. Additionally, we add on a cost for the control itself which is regulated by the parameter λ. We denote our control with q and we will let

$$Q = H^1(0,T;\mathbf{L}^2(\Omega)),\ L^2(0,T;\mathbf{L}^2(\Omega)),\ L^2(0,T;L^2(\Omega)),\ H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N)),\\ L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N)),\ \text{or}\ L^2(0,T;L^2(\Omega)) \tag{2.0.1}$$

depending on if $\mathbf{F},\ S,\ \mathbf{g}$, or ψ is the control and if $\delta > 0$ or $\delta = 0$. In any case, we will denote the inner product as $(\cdot,\cdot)_Q$ and the norm as $\|\cdot\|_Q$. Our cost functional takes the following form:

$$J(\mathbf{u},p,q) = \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{2}\|p-p_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\|q\|^2_Q \tag{2.0.2}$$

where $\mathbf{u}_d$ and p_d are given variables.

We seek to minimize the cost functional subject to the state system (1.2.1)-(1.2.4) and to the control constraints $q \in Q_{ad} := \{q \in Q | q_l \le q \le q_u\}$. Note that q_l could possibly equal

$-\infty$ and q_u could possibly equal ∞.

Remark 2.0.1. This cost functional is written in a general form. However, this represents four different problems for the poro-visco-elastic case where $\mathbf{k}$ is a function of $\mathbf{x}$ and t, four problems for the poro-elastic case where k is a constant, and two problems in the poro-elastic case with simpler boundary conditions and k is a function of $\mathbf{x}$ and t

We note that for the Lamina Cribrosa application this optimal control problem would lead to controlling the structural displacement $\mathbf{u}$ within the Lamina Cribrosa and the fluid pressure p. Since the Lamina Cribrosa supports the Retinal Ganglion Cell Axons which run from the eye to the brain and changes in the structural displacement are indicative of strains experienced by the retinal-ganglion cells, the structural displacement should be studied and controlled. Additionally, fluid pressure, p, in the Lamina Cribrosa is indicative of tissue perfusion which can lead to glaucoma. Hence, we also work to control fluid pressure. We also note that a useful target $\mathbf{u}_d$ is available from experimental and clinical studies. However, p_d will have to be estimated based on the solution computed for IOP at healthy baseline values.

2.1 Well-posedness of the Optimal Control Problem

We will show well-posedness of the ten different optimal control problems mentioned in Remark 2.0.1. To do so, we will use a control-to-state operator that varies in each of these cases. The control-to-state operator can be defined as follows:

Definition 4. *[Control-to-State Operator] Let $G : Q_{ad} \to L^2(0,T;\mathbb{V})$ be the control-to-state operator that maps q to $y = (\mathbf{u}, p)$, where $(\mathbf{u}, p)$ is the weak solution to (1.2.1a)-(1.2.4) (or with the simpler boundary conditions provided in* (1.3.17) *in the poro-elastic case with k as a function of $\mathbf{x}$ and t) with the initial condition, boundary and interior sources all set to zero except the control q.*

In all the cases, the control-to-state operator satisfies the following properties:

Lemma 2.1.1. *The control-to-state operator $G : Q_{ad} \to L^2(0,T;\mathbb{V})$ introduced in Definition 4 is linear, continuous, and injective.*

Proof. Linearity of G is immediate from the linearity of the state system and the fact that the initial condition, boundary and interior sources except the control were set to zero.

Therefore, we will focus our attention on continuity. From Theorem 1.3.2 and Theorem 1.3.3, we have that a weak solution to (1.2.1a)-(1.2.4) satisfies the energy inequality

$$\sup_{t\in[0,T]} E(\mathbf{u}(t)) + \int_0^T [E(p(t)) + E(\mathbf{u}(t))]dt \leq C(T)[E(\mathbf{u}(0)) + DATA\Big|_0^T]e^{C_2 T}$$

Since $\mathbf{u}(0) = 0$ when defining the control-to-state operator, then $E(\mathbf{u}(0)) = 0$.

Using the calculus in an abstract space theorem from [24], we have

$$\sup_{[0,T]}\Big[\|\mathbf{F}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{g}(t)\|^2_{\mathbf{L}^2(\Gamma_N)}\Big] \leq C\left(\|\mathbf{F}(t)\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{g}\|^2_{H^1(0,T;\mathbf{L}^2(\Gamma_N))}\right).$$

combining this with (1.3.6), we obtain

$$\sup_{t\in[0,T]} E(\mathbf{u}(t)) + \int_0^T E(p(t)) + E(\mathbf{u}(t))dt \leq C\Big(\|\mathbf{F}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|S\|^2_{L^2(0,T;L^2(\Omega))} + \|\mathbf{g}\|^2_{H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N)} + \|\psi\|^2_{L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))}\Big).$$

Noticing that $E(\mathbf{u}(t)) = \|\mathbf{u}(t)\|^2_{\mathbf{V}} > 0$, and using the fact that for the control-to-state operator, we have set all sources to zero except the control, the estimate becomes

$$\int_0^T [E(p(t)) + E(\mathbf{u}(t))]dt \leq C\left(\|q\|^2_Q\right)$$

From Assumption 1.3.1, we have $E(p(t)) = k\|\nabla v\|^2_{L^2(\Omega)} \geq C\|v\|^2_V$. Therefore,

$$\|\mathbf{u}(t)\|^2_{L^2(0,T;\mathbf{V})} + \|p(t)\|^2_{L^2(0,T;V)} \leq C\|q\|^2_Q \;\Rightarrow\; \|G(q)\|_{L^2(0,T;\mathbb{V})} \leq C\|q\|_Q,$$

showing that $G : Q_{ad} \to L^2(0,T;\mathbb{V})$ is continuous.

Finally, we will show G is injective. It is sufficient to show if $G(q) = 0$, then $q = 0$. If $G(q) = 0$, then $\mathbf{u}$ and p are both equal to zero. Plugging 0 in for $\mathbf{u}$ and p in (1.3.4) and (1.3.5) yields

$$\int_0^T (\mathbf{F},\mathbf{w})dt + \int_0^T \langle \mathbf{g},\mathbf{w}\rangle dt + \int_0^T (S,w)dt - \int_0^T \langle \psi, w\rangle_{\Gamma_D} dt = 0 \quad \forall (\mathbf{w},w) \in L^2(0,T;\mathbb{V}) \tag{2.1.1}$$

Recalling that in the definition of the control-to-state operator all sources except the control are set to 0, we have the following results:

Case 1: *S is the control and $L^2(0,T;L^2(\Omega))$ is the control space.* (2.1.1) implies

$$\int_0^T (S,w)dt = 0 \ \ \forall w \in L^2(0,T;V) \tag{2.1.2}$$

Since $L^2(0,T;V)$ is dense in $L^2(0,T;L^2(\Omega))$, (2.1.2) holds for all w in $L^2(0,T;L^2(\Omega))$. Therefore, $S=0$ in $L^2(0,T;L^2(\Omega))$.

Case 2: **F** *is the control and $L^2(0,T;\mathbf{L}^2(\Omega))$ is the control space.* This follows the same as Case 1.

Case 3: **F** *is the control and $H^1(0,T;\mathbf{L}^2(\Omega))$ is the control space.* By the same reasoning as Case 1 and 2, if $\int_0^T(\mathbf{F},\mathbf{w})dt=0$ for all $\mathbf{w}\in L^2(0,T;\mathbf{V})$, we have $\mathbf{F}=0$ in $L^2(0,T;\mathbf{L}^2(\Omega))$. However, that implies $\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}=0$ $a.e.$. Since $\mathbf{F}\in H^1(0,T;L^2(\Omega))$, the function $t\mapsto\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}$ is continuous. Therefore, $\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}=0$ for all $t\in[0,T]$ which implies $\mathbf{F}=0$ in $H^1(0,T;\mathbf{L}^2(\Omega))$.

Case 4: *ψ is the control and $L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$ is the control space.* (2.1.1) implies

$$\int_0^T \langle\psi,w\rangle dt = 0 \ \ \forall w \in L^2(0,T;V) \tag{2.1.3}$$

Recall that the trace map from $H^1(\Omega)$ is surjective onto $H^{1/2}(\Gamma)$. Therefore, (2.1.3) holds for all $w\in L^2(0,T;H^{1/2}(\Gamma_{D,\mathbf{v}}))$, which implies $\psi=0$ in $L^2(0,T;H^{1/2}(\Gamma_{D,\mathbf{v}}))$. Hence, $\psi=0$ in $L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$.

Case 5: **g** *is the control and $H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))$ is the control space.* Similarly,

$$\int_0^T \langle\mathbf{g},\mathbf{w}\rangle_{\Gamma_N} dt = 0 \ \ \forall w \in L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))$$

implies $\mathbf{g}=0$ in $L^2(0,T;\mathbf{H}^{1/2})$. Therefore, like in Case 3, $\mathbf{g}=0$ in $H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))$.

Case 6: **g** *is the control and $L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))$ is the control space.* Satisfied by conclusion of Case 5. □

Since $L^2(0,T;\mathbb{V})$ is continuously embedded in $L^2(0,T;\mathbf{L}^2(\Omega)\times L^2(\Omega))$, we let $Y=L^2(0,T;\mathbf{L}^2(\Omega)\times L^2(\Omega))$ and consider $G:Q_{ad}\to Y$. G is still linear and continuous. However, expanding the range of G allows us to consider an expanded domain of G^* [56].

The optimal control problem (2.0.2) can now be written equivalently as

$$\min_{q\in Q_{ad}} J(q), \ \text{ with } \ J(q)=\frac{1}{2}\|G(q)+y_0-y_d\|_Y^2+\frac{\lambda}{2}\|q\|_Q^2,$$

where $y_d = (\mathbf{u}_d, p_d)$ and y_0 is the weak solution to (1.2.1a)-(1.2.4) with the control set to zero and the initial condition, boundary and interior sources set as desired. For the sake of exposition, we let $\tilde{y} = y_d - y_0$ and define $(\tilde{\mathbf{u}}, \tilde{p}) = \tilde{y}$. Then the optimal control problem becomes

$$\min_{q \in Q_{ad}} J(q), \text{ with } J(q) = \frac{1}{2}\|G(q) - \tilde{y}\|_Y^2 + \frac{\lambda}{2}\|q\|_Q^2. \tag{2.1.4}$$

Now we note the following properties of the cost functional $J(q)$.

Lemma 2.1.2. *$J(q)$ is continuous and strictly convex for any $\lambda \geq 0$.*

Proof. The continuity of $J(q)$ follows immediately from the fact that G is continuous. Since G is linear and norms are convex, we have $\frac{1}{2}\|G(q) - \tilde{y}\|$ is convex. Also, since $f(x) = \|x\|^2$ is strictly convex, we have $\frac{\lambda}{2}\|q\|_Q^2$ is strictly convex. Invoking the injectivity of G, we know $G(q_1) - \tilde{y} \neq G(q_2) - \tilde{y}$. Therefore, $\frac{1}{2}\|G(q) - \tilde{y}\|^2$ is strictly convex. Hence, $J(q)$ will also be strictly convex. □

Using the fact that $J(q)$ is continuous and strictly convex, we prove that the optimal control problem (2.1.4) has a unique solution, for all choices of control q. The results are presented in the following theorems. Below we show the proof of existence and uniqueness of an optimal control when $q = \psi$ is the control variable. We omit the proofs for the other cases, as they are very similar to the proof of Theorem 2.1.12.

Theorem 2.1.3. *Let $\delta > 0$, k be a function of x and t, $\mathbf{F} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g} \in L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))$, $\psi \in L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$, and initial condition $\mathbf{u}_0 \in \mathbf{V}$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{S} \in L^2(0,T;L^2(\Omega))$ that solves (2.1.4) subject to (1.2.1a)-(1.2.4).*

Theorem 2.1.4. *Let $\delta > 0$, k be a function of x and t, $\mathbf{F} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $S \in L^2(0,T;L^2(\Omega))$, $\psi \in L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$, and initial condition $\mathbf{u}_0 \in \mathbf{V}$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{\mathbf{g}} \in L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))$ that solves (2.1.4) subject to (1.2.1a)-(1.2.4).*

Theorem 2.1.5. *Let $\delta > 0$, k be a function of x and t, $S \in L^2(0,T;L^2(\Omega))$, $\mathbf{g} \in L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))$, $\psi \in L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$, and initial condition $\mathbf{u}_0 \in \mathbf{V}$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{\mathbf{F}} \in L^2(0,T;\mathbf{L}^2(\Omega))$ that solves (2.1.4) for $\lambda > 0$ subject to (1.2.1a)-(1.2.4).*

Theorem 2.1.6. *Let $\delta > 0$, k be a function of x and t, $\mathbf{F} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $S \in L^2(0,T;L^2(\Omega))$, $\mathbf{g} \in L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))$, and initial condition $\mathbf{u}_0 \in \mathbf{V}$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{\psi} \in L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$ that solves (2.1.4) subject to (1.2.1a)-(1.2.4).*

Theorem 2.1.7. *Let $\delta = 0$, k be a function of x and t, $\mathbf{F} \in H^1(0,T;\mathbf{V}')$, and $d_0 \in L^2_0(\Omega)$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{S} \in L^2(0,T : L^2(\Omega))$ that solves (2.1.4) subject to (1.2.1a),* (1.3.17)*, and* (1.2.4)*.*

Theorem 2.1.8. *Let $\delta = 0$, k be a function of x and t, $S \in L^2(0,T : V')$, and $d_0 \in L^2_0(\Omega)$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{\mathbf{F}} \in H^1(0,T;\mathbf{V}') \cap L^2(0,T;L^2(\Omega))$ that solves (2.1.4) subject to (1.2.1a),* (1.3.17)*, and* (1.2.4)*.*

Theorem 2.1.9. *Let $\delta = 0$, k be a constant, $\mathbf{F} \in H^1(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g} \in H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))$, $\psi \in L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$, and initial condition $\mathbf{u}_0 \in \mathbf{V}$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{S} \in L^2(0,T;L^2(\Omega))$ that solves (2.1.4) subject to (1.2.1a)-(1.2.4).*

Theorem 2.1.10. *Let $\delta = 0$, k be a constant, $\mathbf{F} \in H^1(0,T;\mathbf{L}^2(\Omega))$, $S \in L^2(0,T;L^2(\Omega))$, $\psi \in L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$, and initial condition $\mathbf{u}_0 \in \mathbf{V}$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{\mathbf{g}} \in H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))$ that solves (2.1.4) subject to (1.2.1a)-(1.2.4).*

Theorem 2.1.11. *Let $\delta = 0$, k be a constant, $S \in L^2(0,T;L^2(\Omega))$, $\mathbf{g} \in H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))$, $\psi \in L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$, and initial condition $\mathbf{u}_0 \in \mathbf{V}$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{\mathbf{F}} \in H^1(0,T;\mathbf{L}^2(\Omega))$ that solves (2.1.4) for $\lambda > 0$ subject to (1.2.1a)-(1.2.4).*

Theorem 2.1.12. *Let $\delta = 0$, k be a constant, $\mathbf{F} \in H^1(0,T;\mathbf{L}^2(\Omega))$, $S \in L^2(0,T;L^2(\Omega))$, $\mathbf{g} \in H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))$, and initial condition $\mathbf{u}_0 \in \mathbf{V}$. Let $[\lambda > 0]$ or $[\lambda = 0$ and $q_l \neq -\infty$, and $q_u \neq \infty]$. Then there exists a unique optimal control $\bar{\psi} \in L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$ that solves (2.1.4) subject to (1.2.1a)-(1.2.4).*

Proof. First we note that the problem can be reduced to a minimization problem over a bounded set. If $\lambda = 0$, we already have the assumption that (2.1.4) is minimized over a bounded set. Let us denote the bounded set by Q_b. If $\lambda > 0$, we have the following argument. Let $\psi_0 \in L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))$. If $\|\psi\|^2 > \frac{2}{\lambda} J(\psi_0)$, then

$$J(\psi) = \frac{1}{2}\|G(\psi) - \tilde{y}\|_Y^2 + \frac{\lambda}{2}\|\psi\|^2_{L^2(0,T;L^2(\Gamma_{D,\mathbf{v}}))} > \frac{1}{2}\|G(\psi) - \tilde{y}\|_Y^2 + J(\psi_0) \geq J(\psi_0). \tag{2.1.5}$$

Let $\psi_0 = 0$. Then 2.1.5 yields $\|\bar{\psi}\|^2 \leq \frac{1}{\lambda}\|\tilde{y}\|_Y^2$. Therefore the problem reduces to minimizing J over the set $Q_b := \{\psi \in L^2(0,T;L^2(\Omega)) \mid \|\psi\|^2 \leq \frac{1}{\lambda}\|\tilde{y}\|_Y^2\}$. If $0 \notin Q_{ad}$, we can choose an alternative $\psi_0 \in Q_{ad}$ and the argument follows similarly.

Next we show that a solution exists for (2.1.4). Since $J(q) \geq 0$, there exists

$$j := \inf_{q \in Q_b} J(q).$$

Hence, there is a sequence $\{q_n\}_{n=1}^{\infty} \subset Q_b$ such that $J(q_n) \to j$ as $n \to \infty$. Since Q is reflexive, and Q_b is closed, bounded, and convex, then Q_b is weakly sequentially compact. Therefore, there exists a subsequence of $\{q_n\}_{n=1}^{\infty}$, which for convenience we still denote by $\{q_n\}_{n=1}^{\infty}$, that converges weakly to some $\bar{q} \in Q_b$. The cost functional J is weakly lower semicontinuous, since it is a continuous and convex functional. Therefore, we have that

$$\liminf_{n\to\infty} J(q_n) \geq J(\bar{q}).$$

Hence we obtain that $j \geq J(\bar{q})$, but by the definition of j, we have $J(\bar{q}) = j$. Therefore $\bar{q}$ is the optimal control.

Finally, we prove uniqueness of optimal control. Assume two solutions $\bar{q}, \bar{r}$ exist for (2.1.4). By the strict convexity of J, we have

$$J(\mu\bar{q} + (1-\mu)\bar{r}) < \mu J(\bar{q}) + (1-\mu)J(\bar{r}) = j. \tag{2.1.6}$$

Given the convexity of Q_b, $\mu\bar{q} + (1-\mu)\bar{r} \in Q_b$, so (2.1.6) implies $\bar{q}$ and $\bar{r}$ are not optimal controls. This is a contradiction, and thus the optimal control is unique. □

2.2 Necessary Optimality Condition with Adjoint Operator

To derive a first order necessary optimality condition (NOC), we first use the adjoint operator of G. However, since the adjoint operator of G can be difficult to work with computationally, we will derive the adjoint system, investigate the well-posedness of the adjoint PDE system, and use it to characterize the optimal control. Since $G : Q_{ad} \to Y$ is linear, then $G - \tilde{y}$ is Frechét differentiable. Furthermore, $\|\cdot\|^2$ is differentiable on every Hilbert space. Therefore, using the chain rule, we have that the cost functional $J : Q_{ad} \to \mathbb{R}$ is Frechét differentiable.

We can compute its directional Gâteaux derivative as follows.

$$\lim_{t\to 0^+}\frac{1}{t}(J(q+th)-J(q))=\lim_{t\to 0^+}\frac{1}{2t}(\|G(q+th)-\tilde{y}\|_Y^2-\|G(q)-\tilde{y}\|_Y^2+\lambda\|q+th\|_Q^2-\lambda\|q\|_Q^2)$$
$$=(G(q)-\tilde{y},G(h))_Y+\lambda(q,h)_Q$$

Therefore, we have shown

$$J'(q)=G^*(G(q)-\tilde{y})+\lambda q.$$

We state the first characterization of the optimal control $\bar{q}$:

Theorem 2.2.1. *$\bar{q}$ is the optimal control to (2.1.4) if and only if*

$$(G^*(G\bar{q}-\tilde{y}),q-\bar{q})_Y+(\lambda\bar{q},q-\bar{q})_Q\geq 0\quad\forall q\in Q. \tag{2.2.1}$$

Proof. Let $\bar{q}\in Q$ be the optimal control and $q\in Q$ be arbitrary. Let $t\in(0,1]$. Then

$$J(\bar{q}+t(q-\bar{q}))\geq J(\bar{q})\Rightarrow\frac{1}{t}[J(\bar{q}+t(q-\bar{q}))-J(\bar{q})]\geq 0\ \ \forall t\in(0,1].$$

Taking t to 0, we have $J'(\bar{q})(q-\bar{q})\geq 0$. Therefore, if $\bar{q}$ is optimal, (2.2.1) is satisfied.

Now we assume that (2.2.1) is satisfied. Since J is convex, we have

$$J(\bar{q}+t(q-\bar{q}))=J(tq+(1-t)\bar{q})\leq tJ(q)+(1-t)J(\bar{q}).$$

This implies that

$$J(\bar{q}+t(q-\bar{q}))-J(\bar{q})\leq t(J(q)-J(\bar{q})).$$

Dividing by t and taking the limit as $t\to 0$, we obtain

$$J'(q)(q-\bar{q})\leq J(q)-J(\bar{q}),\ \ \forall\ q\in Q.$$

Since the left-hand side is greater than zero, then for all $q\in Q$, $J(q)\geq J(\bar{q})$ and thus $\bar{q}$ is the optimal control. □

This result is a necessary optimality condition. However, it makes use of the adjoint operator of the control-to-state operator. We do not have a good understanding of this operator at this point and instead favor a necessary optimality condition that uses an adjoint PDE system. Due to the regularity differences in the sources and mechanical displacement

in the poro-elastic and poro-visco-elastic cases, we now conduct our analysis in the poro-elastic case where k is a constant and the poro-visco-elastic case separately. A necessary optimality condition using an adjoint equation in the poro-elastic case where k is a function of $\mathbf{x}$ and t will also be different from an analysis perspective and is part of ongoing work that will not be included in this thesis.

2.3 Poro-visco-elastic, k as a function of x and t, NOC

2.3.1 Deriving the Adjoint Equation

We define, formally, the Lagrangian function $\mathscr{L}(\mathbf{u}, p, q, \mathbf{m}, h) : H^1(0,T;\mathbf{V}) \times L^2(0,T;V) \times Q \times L^2(0,T;\mathbf{V}) \times L^2(0,T;V) \to \mathbb{R}$ be given by

$$\begin{aligned}\mathscr{L}(\mathbf{u},p,q,\hat{\mathbf{m}},\hat{h}) = J(\mathbf{u},p,q) &+ \int_0^T (\nabla\cdot\mathbf{T}(\mathbf{u},p),\mathbf{m}_1)dt + \int_0^T (\mathbf{F},\mathbf{m}_1)dt - \int_0^T (\nabla\cdot\mathbf{u}_t,h_1)dt \\ &+ \int_0^T (\nabla\cdot(k(\mathbf{x},t)\nabla p),h_1)dt + \int_0^T (S,h_1)dt - (\nabla\cdot\mathbf{u}(0),h_2) + (d_0,h_2) + \int_0^T \langle \mathbf{g},\mathbf{m}_2\rangle_{\Gamma_N} dt \\ &- \int_0^T \langle \mathbf{T}(\mathbf{u},p)\mathbf{n},\mathbf{m}_2\rangle_{\Gamma_N} dt - \int_0^T \langle \mathbf{u},\mathbf{m}_3\rangle_{\Gamma_D} dt - \int_0^T \langle k(\mathbf{x},t)\nabla p\cdot\mathbf{n},h_3\rangle_{\Gamma_N} dt \\ &- \int_0^T \langle p,h_4\rangle_{\Gamma_{D,p}} dt - \int_0^T \langle k(\mathbf{x},t)\nabla p\cdot\mathbf{n},h_5\rangle_{\Gamma_{D,v}} dt - \int_0^T \langle \psi,h_5\rangle_{\Gamma_{D,v}} dt,\end{aligned}$$

where the Lagrange multipliers $\mathbf{m}_1$, h_1 are functions defined on $\Omega\times(0,T)$, h_2 is a function defined on Ω, and $\mathbf{m}_2$, $\mathbf{m}_3$, h_3, h_4, h_5 are functions defined on parts of $\Gamma\times(0,T)$, and they are expressed in $\mathscr{L}$ as the vectors $\hat{\mathbf{m}}$ and $\hat{h}$. For simplicity, we will let

$$\sigma = \mathbf{T}_e(\mathbf{m}) + \delta\mathbf{T}_v(\mathbf{m}).$$

Moreover, we recall that the control q will be taken to be one of the sources $\mathbf{F}$, S, $\mathbf{g}$, or ψ. In the formal Lagrangian, F, S, g, and ψ are all given except the choice of control which is equivalent to q. Assuming sufficient smoothness on $\hat{\mathbf{m}}$ and $\hat{h}$ and integrating by parts, we

have the following equalities:

$$\begin{aligned}
&\int_0^T (\nabla\cdot\mathbf{T}(\mathbf{u},p),\mathbf{m}_1)dt = \int_0^T (\mathbf{u},\nabla\cdot\sigma(\mathbf{m}_1))dt - \int_0^T \langle\sigma(\mathbf{m}_1)\mathbf{n},\mathbf{u}\rangle_\Gamma dt \\
&-\delta\int_0^T (\mathbf{u},\nabla\cdot\sigma(\mathbf{m}_1)_t)dt + \delta(\mathbf{u}(T),\nabla\cdot\sigma(\mathbf{m}_1(T))) - \delta(\mathbf{u}(0),\nabla\cdot\sigma(\mathbf{m}_1(0))) \\
&+\delta\int_0^T \langle\sigma(\mathbf{m}_1)_t\mathbf{n},\mathbf{u}\rangle_\Gamma dt - \delta\langle\sigma(\mathbf{m}_1)(T)\mathbf{n},\mathbf{u}(T)\rangle_\Gamma + \delta\langle\sigma(\mathbf{m}_1(0)),\mathbf{u}(0)\rangle_\Gamma \\
&+\int_0^T (p,\nabla\cdot\mathbf{m}_1)dt + \int_0^T \langle\sigma(\mathbf{u})\mathbf{n},\mathbf{m}_1\rangle_\Gamma dt - \delta\int_0^T \langle\sigma(\mathbf{u})\mathbf{n},\mathbf{m}_{1t}\rangle_\Gamma dt \\
&+\delta\langle\sigma(\mathbf{u}(T))\mathbf{n},\mathbf{m}_1(T)\rangle_\Gamma - \delta\langle\sigma(\mathbf{u}(0))\mathbf{n},\mathbf{m}_1(0)\rangle_\Gamma - \int_0^T \langle p,\mathbf{m}_1\mathbf{n}\rangle_\Gamma dt
\end{aligned}$$

$$\begin{aligned}
-\int_0^T (\nabla\cdot\mathbf{u}_t,h_1)dt = &-\int_0^T (\mathbf{u},\nabla h_{1t})dt + (\mathbf{u}(T),\nabla h_1(T)) - (\mathbf{u}(0),\nabla h_1(0)) + \int_0^T \langle\mathbf{u}\cdot\mathbf{n},h_{1t}\rangle_\Gamma dt \\
&-\langle\mathbf{u}(T)\cdot\mathbf{n},h_1(T)\rangle_\Gamma + \langle\mathbf{u}(0)\cdot\mathbf{n},h_1(0)\rangle_\Gamma
\end{aligned}$$

$$\begin{aligned}
\int_0^T (\nabla\cdot(k(\mathbf{x},t)\nabla p),h_1)dt = &\int_0^T (p,\nabla\cdot(k(\mathbf{x},t)\nabla h_1)dt + \int_0^T \langle k(\mathbf{x},t)\nabla p\cdot n,h_1\rangle_\Gamma dt \\
&-\int_0^T \langle p,k(\mathbf{x},t)\nabla h_1\cdot\mathbf{n}\rangle_\Gamma dt
\end{aligned}$$

$$\begin{aligned}
-\int_0^T \langle\mathbf{T}(\mathbf{u},p)\mathbf{n},\mathbf{m}_2\rangle_{\Gamma_N} dt = &-\int_0^T \langle\sigma(\mathbf{u}),\mathbf{m}_2\rangle_{\Gamma_N} dt + \delta\int_0^T \langle\sigma(\mathbf{u}),\mathbf{m}_{2t}\rangle_{\Gamma_N} dt \\
&-\delta\langle\sigma(\mathbf{u}(T)),\mathbf{m}_2(T)\rangle_{\Gamma_N} + \delta\langle\sigma(\mathbf{u}(0)),\mathbf{m}_2(0)\rangle_{\Gamma_N} + \int_0^T \langle p,\mathbf{m}_2\cdot\mathbf{n}\rangle_{\Gamma_N} dt
\end{aligned}$$

Therefore, the following equality holds

$$\mathscr{L}(\mathbf{u},p,q,\hat{\mathbf{m}},\hat{h}) = J(\mathbf{u},p,q) + \int_0^T (\mathbf{u},\nabla\cdot\sigma(\mathbf{m}_1))dt - \int_0^T \langle\sigma(\mathbf{m}_1)\mathbf{n},\mathbf{u}\rangle_\Gamma dt$$
$$-\delta\int_0^T (\mathbf{u},\nabla\cdot\sigma(\mathbf{m}_1)_t)dt + \delta(\mathbf{u}(T),\nabla\cdot\sigma(\mathbf{m}_1(T))) - \delta(\mathbf{u}(0),\nabla\cdot\sigma(\mathbf{m}_1(0)))$$
$$+\delta\int_0^T \langle\sigma(\mathbf{m}_1)_t\mathbf{n},\mathbf{u}\rangle_\Gamma dt - \delta\langle\sigma(\mathbf{m}_1)(T)\mathbf{n},\mathbf{u}(T)\rangle_\Gamma + \delta\langle\sigma(\mathbf{m}_1(0)),\mathbf{u}(0)\rangle_\Gamma + \int_0^T (p,\nabla\cdot\mathbf{m}_1)dt$$
$$+\int_0^T \langle\sigma(\mathbf{u})\mathbf{n},\mathbf{m}_1\rangle_\Gamma dt - \delta\int_0^T \langle\sigma(\mathbf{u})\mathbf{n},\mathbf{m}_{1t}\rangle_\Gamma dt + \delta\langle\sigma(\mathbf{u}(T))\mathbf{n},\mathbf{m}_1(T)\rangle_\Gamma - \delta\langle\sigma(\mathbf{u}(0))\mathbf{n},\mathbf{m}_1(0)\rangle_\Gamma$$
$$-\int_0^T \langle p,\mathbf{m}_1\cdot\mathbf{n}\rangle_\Gamma dt + \int_0^T (\mathbf{F},\mathbf{m}_1)dt - \int_0^T (\mathbf{u},\nabla h_{1t})dt + (\mathbf{u}(T),\nabla h_1(T)) - (\mathbf{u}(0),\nabla h_1(0))$$
$$+\int_0^T \langle\mathbf{u}\cdot\mathbf{n},h_{1t}\rangle_\Gamma dt - \langle\mathbf{u}(T)\cdot\mathbf{n},h_1(T)\rangle_\Gamma + \langle\mathbf{u}(0)\cdot\mathbf{n},h_1(0)\rangle_\Gamma + \int_0^T (p,\nabla\cdot(k(\mathbf{x},t)\nabla h_1)dt$$
$$+\int_0^T \langle k(\mathbf{x},t)\nabla p\cdot\mathbf{n},h_1\rangle dt - \int_0^T \langle p,k(\mathbf{x},t)\nabla h_1\cdot\mathbf{n}\rangle_\Gamma dt + \int_0^T (S,h_1)dt + (\mathbf{u}(0),\nabla h_2)$$
$$-\langle\mathbf{u}(0)\cdot\mathbf{n},h_2\rangle_\Gamma - \int_0^T \langle\sigma(\mathbf{u}),\mathbf{m}_2\rangle_{\Gamma_N} dt + \delta\int_0^T \langle\sigma(\mathbf{u}),\mathbf{m}_{2t}\rangle_{\Gamma_N} dt - \delta\langle\sigma(\mathbf{u}(T)),\mathbf{m}_2(T)\rangle_{\Gamma_N}$$
$$+\delta\langle\sigma(\mathbf{u}(0)),\mathbf{m}_2(0)\rangle_{\Gamma_N} + \int_0^T \langle p,\mathbf{m}_2\cdot\mathbf{n}\rangle_{\Gamma_N} dt + \int_0^T \langle\mathbf{g},\mathbf{m}_2\rangle_{\Gamma_N} dt - \int_0^T \langle\mathbf{u},\mathbf{m}_3\rangle_{\Gamma_D} dt$$
$$-\int_0^T \langle k(\mathbf{x},t)\nabla p\cdot\mathbf{n},h_3\rangle_{\Gamma_N} dt - \int_0^T \langle p,h_4\rangle_{\Gamma_{D,p}} dt - \int_0^T \langle k(\mathbf{x},t)\nabla p\cdot\mathbf{n},h_5\rangle_{\Gamma_{D,v}} dt$$
$$-\int_0^T \langle\psi,h_5\rangle_{\Gamma_{D,v}} dt. \tag{2.3.1}$$

By Lagrange principle, the optimal solution $(\bar{\mathbf{u}},\bar{p},\bar{q})$ and the multipliers $\hat{\mathbf{m}}$ and $\hat{h}$ should satisfy the optimality conditions associated with the unconstrained problem

$$\min\mathscr{L}(\mathbf{u},p,q,\hat{\mathbf{m}},\hat{h}) \quad \text{with} \quad q\in Q_{ad}.$$

Now we will take the derivative of the Lagrangian with respect to $\mathbf{u}$ and the derivative

with respect to p.

$$
\begin{aligned}
D_{\mathbf{u}}\mathscr{L}(\mathbf{u},p,q,\hat{\mathbf{m}},\hat{h})\mathbf{v} &= \int_0^T (\mathbf{u}-\mathbf{u}_d,\mathbf{v})dt + \int_0^T (\mathbf{v},\nabla\cdot\sigma(\mathbf{m}_1))dt - \int_0^T \langle\sigma(\mathbf{m}_1)\mathbf{n},\mathbf{v}\rangle_\Gamma dt \\
&-\delta\int_0^T (\mathbf{v},\nabla\cdot\sigma(\mathbf{m}_1)_t)dt + \delta(\mathbf{v}(T),\nabla\cdot\sigma(\mathbf{m}_1(T))) - \delta(\mathbf{v}(0),\nabla\cdot\sigma(\mathbf{m}_1(0))) \\
&+\delta\int_0^T \langle\sigma(\mathbf{m}_1)_t\mathbf{n},\mathbf{v}\rangle_\Gamma dt - \delta\langle\sigma(\mathbf{m}_1)(T)\mathbf{n},\mathbf{v}(T)\rangle_\Gamma + \delta\langle\sigma(\mathbf{m}_1(0)),\mathbf{v}(0)\rangle_\Gamma + \int_0^T \langle\sigma(\mathbf{v})\mathbf{n},\mathbf{m}_1\rangle_\Gamma dt \\
&-\delta\int_0^T \langle\sigma(\mathbf{v})\mathbf{n},\mathbf{m}_{1\,t}\rangle_\Gamma dt + \delta\langle\sigma(\mathbf{v}(T))\mathbf{n},\mathbf{m}_1(T)\rangle_\Gamma - \delta\langle\sigma(\mathbf{v}(0))\mathbf{n},\mathbf{m}_1(0)\rangle_\Gamma - \int_0^T (\mathbf{v},\nabla h_{1\,t})dt \\
&+(\mathbf{v}(T),\nabla h_1(T)) - (\mathbf{v}(0),\nabla h_1(0)) + \int_0^T \langle\mathbf{v}\cdot\mathbf{n},h_{1\,t}\rangle_\Gamma dt - \langle\mathbf{v}(T)\cdot\mathbf{n},h_1(T)\rangle_\Gamma + \langle\mathbf{v}(0)\cdot\mathbf{n},h_1(0)\rangle_\Gamma \\
&+(\mathbf{v}(0),\nabla h_2) - \langle\mathbf{v}(0)\cdot\mathbf{n},h_2\rangle_\Gamma - \int_0^T \langle\sigma(\mathbf{v}),\mathbf{m}_2\rangle_{\Gamma_N} dt + \delta\int_0^T \langle\sigma(\mathbf{v}),\mathbf{m}_{2\,t}\rangle_{\Gamma_N} dt \\
&-\delta\langle\sigma(\mathbf{v}(T)),\mathbf{m}_2(T)\rangle_{\Gamma_N} + \delta\langle\sigma(\mathbf{v}(0)),\mathbf{m}_2(0)\rangle_{\Gamma_N} - \int_0^T \langle\mathbf{v},\mathbf{m}_3\rangle_{\Gamma_D} dt
\end{aligned}
\tag{2.3.2}
$$

$$
\begin{aligned}
D_p\mathscr{L}(\mathbf{u},p,q,\hat{\mathbf{m}},\hat{h})r &= \int_0^T (p-p_d,r)dt + \int_0^T (r,\nabla\cdot\mathbf{m}_1)dt - \int_0^T \langle r,\mathbf{m}_1\cdot\mathbf{n}\rangle dt \\
&+\int_0^T (r,\nabla\cdot(k(\mathbf{x},t)\nabla h_1)dt + \int_0^T \langle k(\mathbf{x},t)\nabla r\cdot\mathbf{n},h_1\rangle_\Gamma dt - \int_0^T \langle r,k(\mathbf{x},t)\nabla h_1\cdot\mathbf{n}\rangle_\Gamma dt \\
&+\int_0^T \langle r,\mathbf{m}_2\cdot\mathbf{n}\rangle_{\Gamma_N} dt - \int_0^T \langle k(\mathbf{x},t)\nabla r\cdot\mathbf{n},h_3\rangle_{\Gamma_N} dt - \int_0^T \langle r,h_4\rangle_{\Gamma_{D,p}} dt \\
&-\int_0^T \langle k(\mathbf{x},t)\nabla r\cdot\mathbf{n},h_5\rangle_{\Gamma_{D,v}} dt
\end{aligned}
\tag{2.3.3}
$$

The derivative of $\mathscr{L}$ with respect to $\mathbf{u}$ and p will vanish at the optimal point $(\bar{\mathbf{u}},\bar{p},\bar{q})$. If we let

$$
\mathbf{m} := \mathbf{m}_1,\ h := h_1,\ \mathbf{m_2} = \mathbf{m}|_{\Gamma_N}\ \ \mathbf{m}_3 = (-\sigma(\mathbf{m}) + \delta\sigma(\mathbf{m}_1)_t + h_t\mathbf{I})\mathbf{n})|_{\Gamma_D},
$$

$$
\nabla h_2 = -\delta\nabla\cdot\sigma(\mathbf{m}_1(0)) + \nabla h(0)\ \ h_3 = h|_{\Gamma_N}\ \ h_4 = -k(\mathbf{x},t)\nabla h\cdot\mathbf{n}|_{\Gamma_{D,p}}\ \ h_5 = h|_{\Gamma_{D,v}},
$$

we obtain the following adjoint system:

$$\begin{cases} -\delta\nabla\cdot\sigma(\mathbf{m}_t)+\nabla\cdot\sigma(\mathbf{m})-\nabla h_t=-(\bar{\mathbf{u}}-\tilde{\mathbf{u}}) & \text{in } \Omega\times(0,T) \\ -\nabla\cdot\mathbf{m}-\nabla\cdot(k(\mathbf{x},t)\nabla h)=\bar{p}-\tilde{p} & \text{in } \Omega\times(0,T) \\ -k(\mathbf{x},t)\nabla h\cdot n=0 & \text{on } (\Gamma_N\cup\Gamma_{D,v})\times(0,T) \\ (\delta\sigma(\mathbf{m}_t)-\sigma(\mathbf{m})+h_t\mathbf{I})n=0 & \text{on } \Gamma_N\times(0,T) \\ h=0 & \text{on } \Gamma_{D,p}\times(0,T) \\ \mathbf{m}=0 & \text{on } \Gamma_D\times(0,T) \\ \nabla h(T)+\delta\nabla\cdot\sigma(\mathbf{m}(T))=0 & \text{in } \Omega \end{cases} \tag{2.3.4}$$

Remark 2.3.1. The proper definition of the Lagrangian function $\mathscr{L}$ is given below. For $\mathbf{m}\in L^2(0,T;\mathbf{V})$ and $h\in L^2(0,T;V)$, let the Lagrangian function $\mathscr{L}(\mathbf{u},p,q,\mathbf{m},h):H^1(0,T;\mathbf{V})\times L^2(0,T;V)\times Q\times L^2(0,T;\mathbf{V})\times L^2(0,T;V)\to\mathbb{R}$ be given by

$$\begin{aligned} \mathscr{L}(\mathbf{u},p,q,\mathbf{m},h)&=J(\mathbf{u},p,q)-\Big(\delta\int_0^T a(\mathbf{u}_t,\mathbf{m})\,dt \\ &+\int_0^T a(\mathbf{u},\mathbf{m})\,dt-\int_0^T(p,\nabla\cdot\mathbf{m})\,dt-\int_0^T\langle\mathbf{g},\mathbf{m}\rangle_{\Gamma_N}\,dt-\int_0^T(\mathbf{F},\mathbf{m})\,dt\Big) \\ &-\Big(\int_0^T(k(\mathbf{x},t)\nabla p,\nabla h)\,dt+\int_0^T(\nabla\cdot\mathbf{u}_t,h)\,dt+\int_0^T\langle\psi,h\rangle_{\Gamma_{D,\mathbf{v}}}\,dt-\int_0^T(S,h)\,dt\Big) \end{aligned} \tag{2.3.5}$$

2.3.2 Well-posedness of Adjoint System for Poro-visco-elastic Case

In the well-posedness analysis of the adjoint system, we will call $\mathbf{z}_1=\bar{\mathbf{u}}-\tilde{\mathbf{u}}$, $z_2=\bar{p}-\tilde{p}$ and we will let the terminal condition be the more general case where $\nabla h(T)+\delta\nabla\cdot\sigma(\mathbf{m}(T))=\nabla\mathbf{z}_3+\delta\nabla\cdot\sigma(\mathbf{m}(T))$. We define a weak solution for the adjoint system (2.3.4) as follows.

Definition 5. *[Weak Solution for Adjoint System] A weak solution to (2.3.4) is represented by the pair of functions* $\mathbf{m}\in L^2(0,T;\mathbf{V})$ *and* $h\in L^2(0,T;V)$ *such that:*
(a) for any $\mathbf{w}\in\mathbf{V}$, $w\in V$, *and* $f\in C_0^\infty(0,T)$ *the following variational formulations are satisfied:*

$$\delta\int_0^T a(\mathbf{m},\mathbf{w})f'(t)dt+\int_0^T a(\mathbf{m},\mathbf{w})f(t)\,dt+\int_0^T(h,\nabla\cdot\mathbf{w})f'(t)\,dt=\int_0^T(\mathbf{z}_1,\mathbf{w})f(t)\,dt \tag{2.3.6}$$

$$\int_0^T (k(\mathbf{x},t)\nabla h, \nabla w) f(t)\, dt - \int_0^T (\nabla\cdot\mathbf{m}, w) f(t)\, dt = \int_0^T (z_2, w) f(t)\, dt \tag{2.3.7}$$

(b) for every $\mathbf{w}\in\mathbf{V}$*, the term* $(h(t),\nabla\cdot\mathbf{w})+\delta a(\mathbf{m}(t),\mathbf{w})$ *uniquely defines an absolutely continuous function on* $[0,T]$ *and the terminal condition* $(h(T),\nabla\cdot\mathbf{w})+\delta a(\mathbf{m}(T),\mathbf{w})=\mathbf{z}_3$ *is satisfied.*

The adjoint system (2.3.4) is a linear, weakly coupled system. Once reversed in time, the system is as follows:

$$\begin{cases} \delta\nabla\cdot\sigma(\mathbf{m}_t)+\nabla\cdot\sigma(\mathbf{m})+\nabla h_t = -\mathbf{z}_1 & \text{in } \Omega\times(0,T) \\ -\nabla\cdot(k(\mathbf{x},t)\nabla h)-\nabla\cdot\mathbf{m} = z_2 & \text{in } \Omega\times(0,T) \\ -k(\mathbf{x},t)\nabla h\cdot n = 0 & \text{on } (\Gamma_N\cup\Gamma_{D,v})\times(0,T) \\ (\sigma(\mathbf{m}_t)+\sigma+h_t I)n = 0 & \text{on } \Gamma_N\times(0,T) \\ h = 0 & \text{on } \Gamma_{D,p}\times(0,T) \\ \mathbf{m} = 0 & \text{on } \Gamma_D\times(0,T) \\ \nabla h(\cdot,0)+\delta\nabla\cdot\sigma(\cdot,0) = \mathbf{z}_3 & \text{in } \Omega \end{cases} \tag{2.3.8}$$

Reversing the system in time simplifies the use of known existence techniques since there is now an initial condition instead of a terminal condition. We can show existence of a solution to the adjoint system reversed in time (2.3.8) using operator theory. A weak solution to the adjoint system reversed in time has the following definition:

Definition 6. *[Weak Solution for Reversed Adjoint Equation] A weak solution to (2.3.8) is represented by the pair of functions* $\mathbf{m}\in L^2(0,T;\mathbf{V})$ *and* $h\in L^2(0,T;V)$ *such that:*
(a) for any $\mathbf{w}\in\mathbf{V}$*,* $w\in V$*, and* $f\in C_0^\infty(0,T)$ *the following variational formulations are satisfied:*

$$-\delta\int_0^T a(\mathbf{m},\mathbf{w})f'(t)dt + \int_0^T a(\mathbf{m},\mathbf{w})f(t)\, dt - \int_0^T (h,\nabla\cdot\mathbf{w})f'(t)\, dt = \int_0^T (\mathbf{z}_1,\mathbf{w})f(t)\, dt \tag{2.3.9}$$

$$\int_0^T (k(\mathbf{x},t)\nabla h,\nabla w)f(t)\, dt - \int_0^T (\nabla\cdot\mathbf{m}, w)f(t)\, dt = \int_0^T (z_2, w)f(t)\, dt \tag{2.3.10}$$

(b) for every $\mathbf{w}\in\mathbf{V}$*, the term* $(h(t),\nabla\cdot\mathbf{w})+\delta a(\mathbf{m}(t),\mathbf{w})$ *uniquely defines an absolutely continuous function on* $[0,T]$ *and the terminal condition* $(h(T),\nabla\cdot\mathbf{w})+\delta a(\mathbf{m}(T),\mathbf{w})=(\mathbf{z}_3,\mathbf{w})$ *is satisfied.*

Now we introduce the following operators.

1. The elasticity operator $\mathscr{E}$:
   First, consider the system
   $$\begin{cases} \nabla \cdot \sigma = -\mathbf{z}_1 & \text{in } \Omega \\ \mathbf{m} = 0 & \text{on } \Gamma_D \\ \sigma n = 0 & \text{on } \Gamma_N \end{cases}$$
   Let $\mathbf{v} \in \mathbf{V}$ then we have
   $$\begin{aligned} \int_\Omega (\nabla \cdot \sigma) \cdot \mathbf{v} d\Omega &= \int_\Omega -z_1 \cdot \mathbf{v} d\Omega \\ \int_\Gamma \sigma n \mathbf{v} h \Gamma - \int_\Omega (\sigma, \nabla \mathbf{v}) d\Omega &= \int_\Omega -z_1 \cdot \mathbf{v} d\Omega \\ \int_\Omega (\sigma, \nabla \mathbf{v}) d\Omega &= \int_\Omega z_1 \cdot \mathbf{v} d\Omega \\ a(\mathbf{m}, \mathbf{v}) &= \int_\Omega z_1 \cdot \mathbf{v} d\Omega \end{aligned}$$
   Now, we denote the operator $\mathscr{E}(\mathbf{m}) = a(\mathbf{m}, \cdot)$. Hence, $\mathscr{E} : \mathbf{V} \to \mathbf{V}'$. Since we are taking $a(\cdot, \cdot)$ to be the inner product on $\mathbf{V}$, it is clear $\mathscr{E}$ is coercive.
   Let $w \in \mathbf{V}'$. Then by Lax-Milgram there exists a $\mathbf{u} \in \mathbf{V}$ such that $a(\mathbf{u}, \mathbf{v}) = w(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}$. Hence, $\mathscr{E}\mathbf{u} = w$. Therefore, the range of $\mathscr{E}$ is $\mathbf{V}'$.

2. The family of $A(t)$ operators:
   Consider the following:
   $$\begin{cases} -\nabla \cdot (k(\mathbf{x}, t) \nabla h) = z_2 & \text{in } \Omega \\ h = 0 & \text{on } \Gamma_{D,p} \\ k(\mathbf{x}, t) \nabla h \cdot n = 0 & \text{on } \Gamma_N \cup \Gamma_{D,v} \end{cases}$$

Let $w \in V$. Then multiplying by w and integrating we have the following:

$$\int_\Omega (-\nabla \cdot (k(\mathbf{x},t)\nabla h)w)d\Omega = \int_\Omega z_2 w\, d\Omega$$
$$-\int_\Gamma k(\mathbf{x},t)w\nabla h \cdot n\, d\Gamma + \int_\Omega k(\mathbf{x},t)\nabla h \cdot \nabla w\, d\Omega = \int_\Omega z_2 w\, d\Omega$$
$$\int_\Omega k(\mathbf{x},t)\nabla h \cdot \nabla w\, d\Omega = \int_\Omega z_2 w\, d\Omega$$

We define

$$A(t): V \to V' \text{ by } A(t)(h)(w) = \int_\Omega k(\mathbf{x},t)\nabla h \cdot \nabla w\, d\Omega$$

Using (1.2), we see that $(A(t)(h),h) \geq \kappa_* \|\nabla h\|^2_{L^2(\Omega)} \geq 0$. Therefore, A is monotone. Furthermore, Friedrichs's inequality, (1.2) and Assumption 1.3.1, yields

$$\kappa_* \|\nabla h\|^2_{L^2(\Omega)} \geq C\|h\|_V \quad \forall h \in V.$$

Hence, $A(t)$ is coercive for all $t \in [0,T]$. Therefore, if $z_2 \in V'$ and $t \in [0,T]$ then by Lax-Milgram, a unique solution h will exist such that $A(t)(h) = z_2$, so the range of $A(t) = V'$. Also note, the kernel of $A(t)$ is the kernel of ∇ Hence, any function that is constant almost everywhere is in the kernel of $A(t)$. Finally, by the definition, we clearly see that $A(t)$ is self-adjoint.

3. The coupling terms:
   Consider the problem

$$\nabla h = 0 \text{ on } \Omega \text{ and } h = 0 \text{ on } \Gamma_N$$

Note that if $h \in V$, then $\nabla h = 0$ implies $h = 0$ on Γ by Poincare's inequality and trace theorem. Also, we have

$$\begin{aligned} 0 &= \int_0^T (\nabla h, \mathbf{v}) d\,t \\ &= -\int_0^T (h, \nabla \cdot \mathbf{v}) d\,t + \int_0^T \langle h\mathbf{I} \cdot \mathbf{n}, \mathbf{v} \rangle d\,t = 0 \\ &= -\int_0^T (h, \nabla \cdot \mathbf{v}) d\,t \end{aligned}$$

Therefore,

$$\int_0^T (h, \nabla \cdot \mathbf{v}) d\,t = 0.$$

We define $\overline{\nabla} : V \to \mathbf{V}'$ by $\overline{\nabla}(h)(\mathbf{v}) = \int_\Omega h \nabla \cdot \mathbf{v} d\Omega$. Similarly, define $\overline{\nabla \cdot} : \mathbf{V} \to V'$ by $\overline{\nabla \cdot} \mathbf{m}(\upsilon) = \int_\Omega \nabla \cdot \mathbf{m} \upsilon \Omega$. With these definitions we can immediately see that $\overline{\nabla}$ is the adjoint of $\overline{\nabla \cdot}$.

Now that we have defined these operators, we see that Definition 6 is equivalent to

Definition 7. *[Weak Solution for Reversed Adjoint Equation] A weak solution to (3.1.1) is represented by the pair of functions* $\mathbf{m} \in L^2(0,T;\mathbf{V})$ *and* $h \in L^2(0,T;V)$ *such that:*
(a) The following equations are satisfied:

$$\delta \frac{d}{d\,t} \mathscr{E}(\mathbf{m}(t)) + \mathscr{E}(\mathbf{m}(t)) + \frac{d}{d\,t} \overline{\nabla} h(t) = -\mathbf{z}_1(t) \ \in L^2(0,T;\mathbf{V}') \tag{2.3.11}$$

$$A(t)(h(t)) - \overline{\nabla \cdot} \mathbf{m}(t) = z_2(t) \ \in L^2(0,T;V') \tag{2.3.12}$$

(b) The map $t \mapsto \delta \mathscr{E}(\mathbf{m}(t)) + \overline{\nabla} h(t)$ *is absolutely continuous in* $\mathbf{V}'$ *on* $[0,T]$ *and the initial condition* $\delta \mathscr{E}(\mathbf{m}(0)) + \overline{\nabla}(h(0)) = \mathbf{z_3}$ *is satisfied.*

We have the following theorem

Theorem 2.3.1. *Given* $\mathbf{z}_1 \in L^2(0,T;\mathbf{L}^2(\Omega))$, $z_2 \in L^2(0,T;L^2(\Omega))$ *and* $\mathbf{z}_3 \in \mathbf{V}'$. *There exists a solution* $(\mathbf{m}, h)$ *that satisfies Definition 7 for all* $(\mathbf{w}, w) \in L^2(0,T;\mathbf{V} \times V)$ *and* $\mathbf{v} \in \mathbf{V}$.

We will first introduce a new variable $\mathbf{l}$ such that

$$\mathbf{l} = \int_0^t \mathbf{m}(s) d\,s + \mathbf{y}(t) \tag{2.3.13}$$

where

$$-\mathscr{E}(\mathbf{y}(t)) = \int_0^t \mathbf{z}_1(s) d\,s + \delta \mathscr{E}(\mathbf{m}(0)) + \overline{\nabla} h(0) \tag{2.3.14}$$

We notice that

$$\mathscr{E}(\mathbf{l}(t)) = \int_0^t \mathscr{E}(\mathbf{m}(s)) d\,s - \int_0^t \mathbf{z}_1(s) d\,s - \delta \mathscr{E}(\mathbf{m}(0)) - \overline{\nabla} h(0)$$

$$\frac{d}{d\,t} \mathscr{E}(\mathbf{l}(t)) = \mathscr{E}(\mathbf{m}(t)) - \mathbf{z}_1(t) \tag{2.3.15}$$

Now we will integrate (2.3.11) from 0 to t to get

$$\delta\mathscr{E}(\mathbf{m}(t))-\delta\mathscr{E}(\mathbf{m}(0))+\int_0^t \mathscr{E}(\mathbf{m}(s))ds+\overline{\nabla}h(t)-\overline{\nabla}h(0)=\int_0^t \mathbf{z}_1(s)ds$$

Now rewriting this in terms of $\mathbf{l}$, we have

$$\mathscr{E}(\mathbf{m}(t))+\mathscr{E}(\mathbf{l}(t))+\overline{\nabla}h(t)=0 \tag{2.3.16}$$

Putting (2.3.12), (2.3.15), and (2.3.16) together we have the following system

$$\left(\frac{d}{dt}\begin{bmatrix}0&0&0\\0&0&0\\0&0&\mathscr{E}\end{bmatrix}+\begin{bmatrix}\mathscr{E}&\overline{\nabla}&\mathscr{E}\\-\overline{\nabla\cdot}&A&0\\-\mathscr{E}&0&0\end{bmatrix}\right)\begin{bmatrix}\mathbf{m}\\h\\\mathbf{l}\end{bmatrix}=\begin{bmatrix}0\\z_2\\\mathbf{z}_1\end{bmatrix} \tag{2.3.17}$$

We define $\mathbb{V}=\mathbf{V}\times V\times\mathbf{V}$. Now let $\mathscr{B},\mathscr{A}:\mathbb{V}\to\mathbb{V}'$ be defined by

$$\mathscr{B}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&\mathscr{E}\end{bmatrix}\text{ and }\mathscr{A}=\begin{bmatrix}\mathscr{E}&\overline{\nabla}&\mathscr{E}\\-\overline{\nabla\cdot}&A&0\\-\mathscr{E}&0&0\end{bmatrix}$$

Because $\mathscr{E}$ is coercive, we have $\mathscr{B}$ is self-adjoint and monotone. Since $\mathscr{A}$ and $\mathscr{B}$ are time invariant they are both clearly a regular family and in $L^\infty(0,T)$ for all $w,v\in\mathbb{V}$. Recalling that $\overline{\nabla}$ is the adjoint of $\overline{\nabla\cdot}$ and $\mathscr{E}$ is self-adjoint, we have

$$\begin{aligned}\langle\mathscr{A}(\mathbf{m},h,\mathbf{l}),(\mathbf{m},h,\mathbf{l})\rangle_{\mathbb{V}',\mathbb{V}}&=\mathscr{E}(\mathbf{m})\mathbf{m}+\langle\overline{\nabla}h,\mathbf{m}\rangle_{\mathbf{V}',\mathbf{V}}+\mathscr{E}(\mathbf{l})\mathbf{m}+\langle\overline{\nabla\cdot}\mathbf{m},h\rangle_{V',V}+A(h)h-\mathscr{E}(\mathbf{m})\mathbf{l}\\&=\mathscr{E}(\mathbf{m})\mathbf{m}+A(h)h\geq 0\end{aligned}$$

Therefore, $\mathscr{A}$ is monotone, and

$$\mathscr{A}(\mathbf{m},h,\mathbf{l})(\mathbf{m},h,\mathbf{l})+\mathscr{B}(\mathbf{m},h,\mathbf{l})(\mathbf{m},h,\mathbf{l})=C(\|\mathbf{m}\|_{\mathbf{V}}^2+\|h\|_V+\|\mathbf{l}\|_{\mathbf{V}}^2)$$

Given these properties, we will show existence of a solution to (2.3.17) with the initial condition

$$\mathscr{B}(0)(\mathbf{m}(0),h(0),\mathbf{l}(0))=\mathscr{B}(0)(\mathbf{m}_0,h_0,\mathbf{l}_0).$$

We let $\mathbf{l}_0$ be defined such that $\mathscr{E}(\mathbf{l}_0)=\mathbf{z}_3$. Note that since $\operatorname{Rg}\mathscr{E}=\mathbf{V}'$, we can find $\mathbf{l}_0$ such that

$\mathscr{E}(\mathbf{l}_0) = \mathbf{z}_3$ for any $\mathbf{z}_3 \in \mathbf{V}'$. Having shown the above properties on $\mathscr{A}$ and $\mathscr{B}$ above, we use the following proposition found in Showalter's book called Monotone Operator Theory for Linear Degenerate Equations (see [50] page 115):

Proposition 2.3.2. *Let* $\mathbf{V}$ *be a separable Hilbert space and* $\mathscr{A}(t) \in \mathscr{L}(\mathbf{V}, \mathbf{V}')$ *such that*

$$\mathscr{A}(\cdot)u(v) \in L^{\infty}(0,T) \quad \forall\, u, v \in \mathbf{V}.$$

Also, let $\mathscr{B}(t) \in \mathscr{L}(\mathbf{V}, \mathbf{V}')$ *such that* $\mathscr{B}(t)$ *is a regular family of self-adjoint operators and* $\mathscr{B}(\cdot)u(v) \in L^{\infty}(0,T)\ \forall\, u, v \in \mathbf{V}$. *Further, let* $\mathscr{B}(0)$ *be monotone. Let* $z_3 \in \mathbf{V}$, $\mathbf{z}_1 \in L^2(0,T,\mathbf{V}')$. *If there exists* $\lambda, c > 0$ *such that*

$$2\mathscr{A}(t)v(v) + \lambda\mathscr{B}(t)v(v) + \mathscr{B}'(t)v(v) \geq c\|v\|^2 \ \ \forall\, v \in \mathbf{V}, 0 \leq t \leq T \tag{2.3.18}$$

then there exists $u \in \mathbf{V}$ *such that*

$$\begin{gathered} \frac{d}{dt}(\mathscr{B}(t)u(t)) + \mathscr{A}(t)u(t) = \mathbf{z}(t) \ \ in\,(L^2(0,T;\mathbf{V}))' \\ (\mathscr{B}u)(0) = \mathscr{B}(0)u_0 \end{gathered} \tag{2.3.19}$$

where $\mathscr{B}(u)(t)$ *is absolutely continuous. Furthermore, the solution satisfies*

$$\|u\|_{L^2(0,T;\mathbf{V})} \leq C(\lambda, c)\Big(\|\mathbf{z}\|^2_{L^2(0,T;\mathbf{V}')} + \mathscr{B}(0)u_0(u_0)\Big)^{1/2} \tag{2.3.20}$$

Further, if $\mathscr{B}(t)$ *is monotone,* $\mathscr{A}(t)$ *is a regular family of self-adjoint operators, and there is a pair of numbers* $\lambda, c > 0$ *such that*

$$\mathscr{A}(t)v(v) + \lambda\mathscr{B}(t)v(v) \geq c\|v\|^2 \quad \forall\, v \in \mathbf{V}, a.e.\, t \in [0,T]$$

then this solution is unique.

To use this theorem to show uniqueness, we would need to show $\mathscr{A}$ is self-adjoint.

However, this is not the case as seen here:

$$\begin{aligned}\langle \mathscr{A}(\mathbf{m},h,\mathbf{l}),(\mathbf{w},g,\mathbf{j})\rangle_{\mathbb{V}',\mathbb{V}} &= \mathscr{E}(\mathbf{m})\mathbf{w}+\langle\overline{\nabla}h,\mathbf{w}\rangle_{\mathbf{V}',\mathbf{V}}+\mathscr{E}(\mathbf{l})\mathbf{w}-\langle\overline{\nabla\cdot}\mathbf{m},g\rangle_{V',V}+A(h)g-\mathscr{E}(\mathbf{m})\mathbf{j}\\ &= \mathscr{E}(\mathbf{w})\mathbf{m}=\langle\overline{\nabla\cdot}\mathbf{w},h\rangle_{V,V'}+\mathscr{E}(\mathbf{w})\mathbf{l}-\langle\overline{\nabla}g,\mathbf{m}\rangle_{\mathbf{V}',\mathbf{V}}+A(g)h-\mathscr{E}(\mathbf{j})\mathbf{m}\\ &= \left\langle\begin{bmatrix}\mathscr{E} & -\overline{\nabla} & -\mathscr{E}\\ \overline{\nabla\cdot} & A & 0\\ \mathscr{E} & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{w}\\ g\\ \mathbf{j}\end{bmatrix},\begin{bmatrix}\mathbf{m},h,\mathbf{l}\end{bmatrix}\right\rangle_{\mathbb{V}',\mathbb{V}}\end{aligned}$$

Thus for our case, Theorem 2.3.2 gives existence (but not uniqueness) of $(\mathbf{u},p,\mathbf{l})\in\mathbf{V}\times V\times\mathbf{V}$ such that

$$\mathscr{E}(\mathbf{m})+\overline{\nabla}h+\mathscr{E}(\mathbf{l})=0, \tag{2.3.21}$$

$$\overline{\nabla\cdot}\mathbf{m}+A(h)=z_2, \tag{2.3.22}$$

$$\frac{d}{dt}\mathscr{E}(\mathbf{l})-\mathscr{E}(\mathbf{m})=\mathbf{z}_1, \tag{2.3.23}$$

$$\mathscr{E}(\mathbf{l})(0)=\mathscr{E}(\mathbf{l}_0)=\mathbf{z}_3. \tag{2.3.24}$$

Let $f\in C_0^\infty(0,T)$. We multiply (2.3.21) by $f'(t)$ and integrate from 0 to T to see

$$\int_0^T\mathscr{E}(\mathbf{m})f'(t)dt+\int_0^T\overline{\nabla}hf'(t)dt+\int_0^T\mathscr{E}(\mathbf{l})f'(t)dt=0. \tag{2.3.25}$$

We similarly multiply (2.3.23) by f(t) and integrate from 0 to T to see

$$\int_0^T\frac{d}{dt}\mathscr{E}(\mathbf{l})f(t)dt-\int_0^T\mathscr{E}(\mathbf{m})f(t)dt=\int_0^T\mathbf{z}_1f(t)dt. \tag{2.3.26}$$

Integrating the first term by parts gives

$$-\int_0^T\mathscr{E}(\mathbf{l})f'(t)dt-\int_0^T\mathscr{E}(\mathbf{m})f(t)dt=\int_0^T\mathbf{z}_1f(t)dt. \tag{2.3.27}$$

Combining (2.3.27) with (2.3.25), we see

$$\int_0^T\mathscr{E}(\mathbf{m})f'(t)dt+\int_0^T\overline{\nabla}hf'(t)dt-\int_0^T\mathscr{E}(\mathbf{m})f(t)dt=\int_0^T\mathbf{z}_1f(t)dt. \tag{2.3.28}$$

Therefore,

$$\frac{d}{dt}(\mathscr{E}(\mathbf{m})+\overline{\nabla}h)=-\mathscr{E}(\mathbf{m}(t))+\mathbf{z}_1(t). \tag{2.3.29}$$

and $(\mathscr{E}(\mathbf{m})+\overline{\nabla}h)\in H^1(0,T;\mathbf{V}')$. We know $\mathscr{E}(\mathbf{m}(t))\in L^2(0,T;\mathbf{V}')$ because $\mathbf{m}\in L^2(0,T;\mathbf{V})$ and the operator $\mathscr{E}$ is continuous. Similarly, since all of the operators we used are continuous, all of the terms in (2.3.22) and (2.3.29) are in $L^2(0,T)$. Therefore, we have that for all $(\mathbf{w},w)\in L^2(0,T;\mathbf{V}\times V)$ the following holds:

$$\int_0^T(\mathscr{E}(\mathbf{m})+\overline{\nabla}h)'\mathbf{w}dt+\int_0^T\mathscr{E}(\mathbf{m})\mathbf{w}dt=\int_0^T\mathbf{z}_1\mathbf{w}dt \tag{2.3.30}$$

$$\int_0^T(\overline{\nabla}\cdot\mathbf{m})w\,dt+\int_0^T A(h)w\,dt=\int_0^T z_2 w\,dt \tag{2.3.31}$$

Also, for all $\mathbf{v}\in\mathbf{V}$,

$$\mathscr{E}(\mathbf{m}(0))\mathbf{v}+\overline{\nabla}h(0)\mathbf{v}=(\mathbf{z}_3,\mathbf{v}) \tag{2.3.32}$$

Therefore, we have proven Theorem 2.3.1. Additionally, $\hat{\mathbf{m}}(t)=\mathbf{m}(T-t)$ and $\hat{h}(t)=h(T-t)$ will satisfy the following theorem:

Theorem 2.3.3. *Given $\mathbf{z}_1\in L^2(0,T;\mathbf{L}^2(\Omega))$, $z_2\in L^2(0,T;L^2(\Omega))$ and $\mathbf{z}_3\in\mathbf{L}^2(\Omega)$ such that there exists $\mathbf{l}_0$ where $\mathscr{E}(\mathbf{l}_0)=\mathbf{z}_3$. There exists a solution $(\mathbf{m},h)$ that satisfies Definition 5 for all $(\mathbf{w},w)\in L^2(0,T;\mathbf{V}\times V)$ and $\mathbf{v}\in\mathbf{V}$.*

Theorem 2.3.4. *There can only be one unique $(\mathbf{m},h)$ that satisfies Definition 5 and the solution satisfies the following:*

$$\|\mathbf{m}\|_{L^2(0,T;\mathbf{V})}+\|h\|_{L^2(0,T;V)}\leq C(T,\delta)\Big(\|\mathbf{z}_1\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}+\|z_2\|^2_{L^2(0,T;L^2(\Omega))}+\|z_3\|^2_{L^2(\Omega)}\Big) \tag{2.3.33}$$

Proof. Assume $(\mathbf{m},h)$ satisfies Definition 5. Then for almost every $t\in[0,T]$,

$$-\frac{d}{dt}((\delta\mathscr{E}(\mathbf{m})+\overline{\nabla}),\mathbf{w})_{\mathbf{V}',\mathbf{V}}+(\mathscr{E}(\mathbf{m}),\mathbf{w})dt=(\mathbf{z}_1,\mathbf{w})$$

Integrating from t to T and using $\mathbf{m}(t)$ as the test function, we have

$$(\delta\mathscr{E}(\mathbf{m}(t))+\overline{\nabla}h(t),\mathbf{m}(t))+\int_t^T(\mathscr{E}(\mathbf{m}(\tau)),\mathbf{m}(t))d\tau=\int_t^T(\mathbf{z}_1(\tau),\mathbf{m}(t))d\tau+(\mathbf{z}_3,\mathbf{m}(t)) \tag{2.3.34}$$

By plugging $h(t)$ in as the test function in the second variational equality, we have for almost

every $t \in [0,T]$

$$-(\overline{\nabla} h(t), \mathbf{m}(t)) + A(h(t))(h(t)) = (z_2, h(t)).$$

Adding this together with (2.3.34), we have

$$\begin{aligned} &\delta \mathscr{E}(\mathbf{m}(t))(\mathbf{m}(t)) + A(h(t))(h(t)) + \int_t^T \mathscr{E}(\mathbf{m}(\tau))\mathbf{m}(t) d\tau \\ &= \int_t^T (\mathbf{z}_1(\tau), \mathbf{m}(t)) d\,t\,\tau + (\mathbf{z}_3, \mathbf{m}(t)) + (z_2, h(t)). \end{aligned} \tag{2.3.35}$$

Recall $\mathscr{E}(\mathbf{m}(\tau))\mathbf{m}(t) = a(\mathbf{m}(\tau), \mathbf{m}(t))$.

$$\begin{aligned} a(\mathbf{m}(\tau), \mathbf{m}(t)) &= (\nabla \cdot \mathbf{m}(\tau), \nabla \cdot \mathbf{m}(t)) + (\nabla \mathbf{m}(\tau) + (\nabla \mathbf{m}(\tau))^T, \nabla \mathbf{m}(t)) \\ &= (\nabla \cdot \mathbf{m}(\tau), \nabla \cdot \mathbf{m}(t)) + \frac{1}{2}\Big((\nabla \mathbf{m}(\tau), \nabla \mathbf{m}(t)) + ((\nabla \mathbf{m}(\tau))^T, \nabla (\mathbf{m}(t))^T) \\ &+ ((\nabla \mathbf{m}(\tau))^T, \nabla \mathbf{m}(t)) + (\nabla \mathbf{m}(\tau), (\nabla \mathbf{m}(t))^T) \Big) \\ &= (\nabla \cdot \mathbf{m}(\tau), \nabla \cdot \mathbf{m}(t)) + \frac{1}{2}(\nabla \mathbf{m}(\tau) + (\nabla \mathbf{m}(\tau))^T, \nabla \mathbf{m}(t) + (\nabla \mathbf{m}(t))^T) \end{aligned}$$

Therefore, set $y_1 = \int_t^T \nabla \cdot \mathbf{m}(\tau) d\tau$ and $y_2 = \int_t^T \nabla \mathbf{m}(\tau) + (\nabla \mathbf{m}(\tau))^T d\tau$. Then $y_1'(t) = -\nabla \cdot \mathbf{m}(t)$ and $y_2'(t) = -\nabla \mathbf{m}(t) + (\nabla \mathbf{m}(t))^T$. Hence,

$$\int_t^T a(\mathbf{m}(\tau), \mathbf{m}(t)) d\tau = -(y_1'(t), y_1(t)) - (y_2'(t), y_2(t)) = -\frac{1}{2}\frac{d}{d\,t}\|y_1(t)\|^2 - \frac{1}{2}\frac{d}{d\,t}\|y_2(t)\|^2 \tag{2.3.36}$$

Therefore, integrating (2.3.35) from 0 to T, we have

$$\begin{aligned} &\delta \int_0^T \mathscr{E}(\mathbf{m}(t))\mathbf{m}(t) d\,t + \int_0^T A(h(t))h(t) d\,t - \frac{1}{2}\|y_1(T)\|^2 - \frac{1}{2}\|y_2(T)\|^2 + \frac{1}{2}\|y_1(0)\|^2 + \frac{1}{2}\|y_2(0)\|^2 \\ &= \int_0^T \int_t^T (\mathbf{z}_1(\tau), \mathbf{m}(t)) d\tau + \int_0^T (\mathbf{z}_3, \mathbf{m}(t)) d\,t + \int_0^T (z_2, h(t)) d\,t. \end{aligned} \tag{2.3.37}$$

Using Young's inequality and Korn's inequality, we have

$$\begin{aligned}\int_0^T\int_t^T(\mathbf{z}_1(\tau),\mathbf{m}(t))d\tau dt &\le \int_0^T\int_t^T\frac{\|\mathbf{z}_1(\tau)\|^2}{2\epsilon}d\tau dt+\int_0^T\int_t^T\frac{\epsilon\|\mathbf{m}(t)\|^2}{2}d\tau dt\\ &=\int_0^T\int_0^\tau\frac{\|\mathbf{z}_1(\tau)\|^2}{2\epsilon}dt\,d\tau+\int_0^T\int_t^T\frac{\epsilon\|\mathbf{m}(t)\|^2}{2}d\tau dt\\ &=\int_0^T\tau\frac{\|\mathbf{z}_1(\tau)\|^2}{2\epsilon}d\tau+\int_0^T\int_t^T\frac{\epsilon\|\mathbf{m}(t)\|^2}{2}d\tau dt\\ &\le\frac{T}{2\epsilon}\|\mathbf{z}_1\|^2_{L^2(0,T;L^2(\Omega))}+\frac{\epsilon T}{2}\|\mathbf{m}(t)\|^2_{L^2(0,T;L^2(\Omega))}\\ &\le\frac{T}{2\epsilon}\|\mathbf{z}_1\|^2_{L^2(0,T;L^2(\Omega))}+\frac{\epsilon CT}{2}\mathscr{E}(\mathbf{m}(t),\mathbf{m}(t)).\end{aligned}$$

Similarly, we have

$$\int_0^T(\mathbf{z}_3,\mathbf{m}(t))dt\le\int_0^T\frac{\|z_3\|^2}{2\epsilon}dt+\int_0^T\epsilon\frac{\|\mathbf{m}(t)\|^2}{2}dt\le T\|z_3\|^2_{L^2(\Omega)}+\frac{\epsilon}{2}\|\mathbf{m}(t)\|^2.$$

Using Young's inequality and Poincare's inequality we have

$$\int_0^T(z_2,h(t))dt\le\frac{\|z_2\|_{L^2(0,T;L^2(\Omega))}}{2\epsilon}+\frac{C\epsilon}{2}\|h\|^2_{L^2(0,T;V)}.$$

Therefore, noticing $y_1(T)=y_2(T)=0$ and setting ϵ to be small enough, we have

$$\delta\int_0^T\mathscr{E}(\mathbf{m}(t))\mathbf{m}(t)dt+\int_0^T A(h(t))h(t)dt\le C(T)\Big(\|\mathbf{z}_1\|^2_{L^2(0,T;L^2(\Omega))}+\|z_2\|^2_{L^2(0,T;L^2(\Omega))}+\|z_3\|^2_{L^2(\Omega)}\Big)\tag{2.3.38}$$

Therefore, (2.3.33) is satisfied.

Since (2.3.4) is linear, if there are two solutions $(\mathbf{m}_1,h_1)$ and $(\mathbf{m}_2,h_2)$ then $(\mathbf{m}_1-\mathbf{m}_2,h_1-h_2)$ satisfies (2.3.4) with all sources and the terminal condition equal to zero. Since (2.3.33) holds for all solutions, $\mathbf{m}_1-\mathbf{m}_2=0$ and $h_1-h_2=0$. Hence, there is a unique solution. □

2.3.3 Deriving the Necessary Optimality Condition for the Poro-Visco-Elastic Model

First, we provide an identity which is essential in the derivation of the first order necessary optimality conditions for the optimal control.

Lemma 2.3.5. *Let* $(\mathbf{u}, p)$ *be the solution to the weak solution to the poro-elastic system (1.2.1a)-(1.2.4) with* $\mathbf{u}_0 = 0$. *Let* $(\mathbf{m}, h)$ *be the weak solution to the adjoint equation (3.1.1) with* $z_3 = 0$ *and* $\mathbf{z}_4 = 0$.

$$\int_0^T \langle \mathbf{g}, \mathbf{m} \rangle_{\Gamma_N} dt + \int_0^T (\mathbf{F}, \mathbf{m}) dt + \int_0^T \langle \psi, h \rangle_{\Gamma_{D,v}} dt + \int_0^T (S, h) dt = \int_0^T (\mathbf{z}_1, \mathbf{u}) dt + \int_0^T (z_2, p) dt$$

Proof. First, we will reformulate the weak solution to the adjoint equation. Let $(\mathbf{m}, h)$ be the weak solution to the adjoint equation (3.1.1). Notice,

$$(h(t), \nabla \cdot \mathbf{w}) \leq \|h(t)\|_{L^2(\Omega)} \|\mathbf{w}\|_{\mathbf{V}}$$

Hence, by Reisz-Representation Theorem, there exists $R(h_\nabla)(t) \in \mathbf{V}$ such that for all $\mathbf{w} \in \mathbf{V}$

$$(h(t), \nabla \cdot \mathbf{w})_{L^2(\Omega)} = (R(h_\nabla)(t), \mathbf{w})_{\mathbf{V}} \text{ and } \|R(h_\nabla)(t)\|_{\mathbf{V}} \leq \|h(t)\|_{L^2(\Omega)}$$

Therefore, $R(h_\nabla)(t) \in L^2(0, T; \mathbf{V})$.

Furthermore, for any $\mathbf{w} \in \mathbf{V}$ and $f \in C_0^\infty(0, T)$, we have

$$\begin{aligned} &\int_0^T a(\mathbf{m}(t), \mathbf{w}) f'(t) dt + \int_0^T a(\mathbf{m}(t), \mathbf{w}) f(t) dt + \int_0^T (R(h_\nabla)(t), \mathbf{w})_{\mathbf{V}} f'(t) dt \\ &= \int_0^T (\mathbf{z}_1(t), \mathbf{w}) f(t) dt. \end{aligned}$$

Now we will define the following functionals $F_i(t) : V \to \mathbb{R}$:

$$F_1(t) : \mathbf{w} \mapsto -(\mathbf{z}_1(t), \mathbf{w})_{L^2(\Omega)}$$

$$F_2(t) : \mathbf{w} \mapsto a(\mathbf{m}(t), \mathbf{w})$$

Both of these functionals are linear. We also have the following bounds:

$$|F_1(t)\mathbf{w}| \le \|z_1(t)\|_{(L^2(\Omega))^3}\|\mathbf{w}\|_{(L^2(\Omega))^3} \le \|z_1(t)\|_{\mathbf{V}}\|\mathbf{w}\|_{\mathbf{V}}$$

$$|F_2(t)\mathbf{w}| \le 2\|\nabla\mathbf{m}(t)\|_{(L^2(\Omega))^9}\|\nabla\mathbf{w}\|_{(L^2(\Omega))^9} \le 2\|\mathbf{m}(t)\|_{\mathbf{V}}\|\mathbf{w}\|_{\mathbf{V}}$$

This implies that $F_i(t) \in \mathbf{V}'$ for every t and there is some constant $C > 0$ such that

$$\|F_1(t)\|_{\mathbf{V}'} + \|F_2(t)\|_{\mathbf{V}'} \le C(\|z_1(t)\|_{\mathbf{V}} + \|\mathbf{m}(t)\|_{\mathbf{V}})$$

Since $\mathbf{z}_1(t), \mathbf{m}(t) \in L^2(0,T;\mathbf{V})$, then we obtain that $F(t) = F_1(t) + F_2(t) \in L^2(0,T;\mathbf{V}')$. Now using Bochner's Theorem and (2.4.4) we have

$$\begin{aligned}\left(-\int_0^T (\mathbf{m}(t) + R(h_\nabla)(t))f'(t)dt, \mathbf{w}\right)_{\mathbf{V}} &= -\int_0^T (\mathbf{m}(t) + R(h_\nabla)(t))f'(t), \mathbf{w})_{\mathbf{V}}dt \\ &= \int_0^T (F(t)f(t), \mathbf{w})_{\mathbf{V}',\mathbf{V}}dt = \left(\int_0^T F(t)f(t)dt, \mathbf{w}\right)_{\mathbf{V}',\mathbf{V}}\end{aligned}$$

Hence, we have that in the space of $\mathbf{V}'$,

$$-\int_0^T (\mathbf{m}(t) + R(h_\nabla)(t))f'(t)dt = \int_0^T F(t)f(t)dt, \quad \forall f \in C_0^\infty(0,T)$$

Therefore, by definition of distributional derivatives for vector valued functions, $\frac{d}{dt}(\mathbf{m}(t) + R(h_\nabla)(t)) = F(t) \in L^2(0,T;\mathbf{V}')$.

Additionally, (2.4.4) can be written as

$$\int_0^T a(\mathbf{m},\mathbf{w})f(t)dt - \int_0^T \left(\frac{d}{dt}(\mathbf{m} + R(h_\nabla)), \mathbf{w}\right)_{\mathbf{V}',\mathbf{V}} f(t)dt = \int_0^T (\mathbf{z}_1, \mathbf{w})f(t)dt \tag{2.3.39}$$

However, since $\{\mathbf{w}f(t) | \mathbf{w} \in \mathbf{V},\ f \in C_0^\infty(0,T)\}$ is dense in $L^2(0,T;\mathbf{V})$, we can let $\mathbf{u}$ and p be test functions in (2.4.5) and (2.4.51) to obtain

$$\begin{aligned}\int_0^T a(\mathbf{m},\mathbf{u})dt - \int_0^T \left(\frac{d}{dt}(\mathbf{m} + R(h_\nabla)), \mathbf{u}\right)_{\mathbf{V}',\mathbf{V}} dt - \int_0^T (\nabla\cdot\mathbf{m}, p)dt + \int_0^T (k\nabla h, \nabla p)dt \\ = \int_0^T (\mathbf{z}_1, \mathbf{u})dt + \int_0^T (z_2, p)dt.\end{aligned}$$

Since $\mathbf{u} \in H^1(0,T;\mathbf{V})$, we can integrate the second term by parts.

$$\int_0^T a(\mathbf{m},\mathbf{u})dt + \int_0^T (\mathbf{m}+R(h_\nabla),\mathbf{u}_t)_{\mathbf{V}}dt - (\mathbf{m}(T)+R(h_\nabla)(T),\mathbf{u}(T))_{\mathbf{L}^2(\Omega)}$$
$$+(\mathbf{m}(0)+R(h_\nabla)(0),\mathbf{u}(0))_{\mathbf{L}^2(\Omega)} - \int_0^T (\nabla\cdot\mathbf{m},p)dt + \int_0^T (k\nabla h,\nabla p)dt = \int_0^T (\mathbf{z}_1,\mathbf{u})dt \quad (2.3.40)$$
$$+\int_0^T (z_2,p)dt$$

Recall that for all $\mathbf{w}\in\mathbf{V}$, $\delta a(\mathbf{m}(T),\mathbf{w})+(h(T),\nabla\cdot\mathbf{w}) = \delta a(\mathbf{z}_4,\mathbf{w})+(z_3,\nabla\cdot\mathbf{w})$. Since $z_3=0$ and $\mathbf{z}_4=0$, $(z_3,\nabla\cdot\mathbf{w})=0$ and $\delta a(\mathbf{z}_4,\mathbf{w})=0$. Therefore, $(\delta\mathbf{m}(T)+R(h_\nabla)(T),\mathbf{w})_{\mathbf{V}} = \delta a(\mathbf{m}(T),\mathbf{w})+(h,\nabla\cdot\mathbf{w})=0$. Hence, $\mathbf{m}(T)+R(h_\nabla)(T)=0$. Also, we assumed $\mathbf{u}_0=0$. Therefore, the temporal boundary terms in (2.4.52) are equal to 0, and using the definition of $R(h_\nabla)$, we have

$$-\delta\int_0^T a(\mathbf{m},\mathbf{u}_t)dt + \int_0^T a(\mathbf{m},\mathbf{u})dt + \int_0^T (h,\nabla\cdot\mathbf{u}_t)dt - \int_0^T (\nabla\cdot\mathbf{m},p)dt + \int_0^T (k\nabla h,\nabla p)dt$$
$$= \int_0^T (\mathbf{z}_1,\mathbf{u})dt + \int_0^T (z_2,p)dt \quad (2.3.41)$$

Let $(\mathbf{m},h)$ be the test functions used in the weak form of the poro-elastic system. We obtain

$$\int_0^T a(\mathbf{u},\mathbf{m})dt - \int_0^T (p,\nabla\cdot\mathbf{m})dt + \int_0^T (k\nabla p,\nabla h)dt + \int_0^T (\nabla\cdot\mathbf{u}_t,h)dt$$
$$= \int_0^T \langle g,\mathbf{m}\rangle_{\Gamma_N}dt + \int_0^T (F,\mathbf{m})dt + \int_0^T \langle\psi,h\rangle_{\Gamma_{D,\mathbf{v}}}dt + \int_0^T (S,h)dt. \quad (2.3.42)$$

Combining (2.4.53) and (2.4.54), we obtain the desired equality.

$$\int_0^T \langle g,\mathbf{m}\rangle_{\Gamma_N}dt + \int_0^T (F,\mathbf{m})dt + \int_0^T \langle\psi,h\rangle_{\Gamma_{D,\mathbf{v}}}dt + \int_0^T (S,h)dt = \int_0^T (\mathbf{z}_1,\mathbf{u})dt + \int_0^T (z_2,p)dt$$

□

Theorem 2.3.6. *If $\bar{q}\in Q$ is the optimal control, then there exists a solution $(\mathbf{m},h)$ to the*

adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}} - \tilde{\mathbf{u}}$, $z_2 = \bar{p} - \tilde{p}$, $z_3 = 0$, *and* $\mathbf{z}_4 = 0$ *which satisfies*

$$\begin{cases} \int_0^T (q - \bar{q}, h + \lambda \bar{q})_Q \geq 0 \ \ \forall q \in Q_{ad} & \text{when } S \text{ or } \psi \text{ is used as the control variable} \\ \int_0^T (q - \bar{q}, \mathbf{m} + \lambda \bar{\mathbf{q}})_Q \geq 0 \ \ \forall q \in Q_{ad} & \text{when } \mathbf{F} \text{ or } \mathbf{g} \text{ is used as the control variable} \end{cases} \tag{2.3.43}$$

Conversely, let $\bar{q} \in Q_{ad}$ *with associated state* $(\bar{\mathbf{u}}, \bar{p})$. *Let* $(\mathbf{m}, h)$ *be the solution to the adjoint system (3.1.1) with* $\mathbf{z}_1 = \bar{\mathbf{u}} - \tilde{\mathbf{u}}$, $z_2 = \bar{p} - \tilde{p}$, $z_3 = 0$, $\mathbf{z}_4 = 0$ *and if (2.4.56) is satisfied, then* $\bar{q}$ *is a solution to the optimal control problem (2.0.2).*

Proof. Let $\bar{q} \in Q$ be the optimal control. Then $G(\bar{q}) = (\bar{\mathbf{u}}, \bar{p}) \in H^1(0, T; \mathbf{V}) \times L^2(0, T; L^2(\Omega))$. Since $\bar{q}$ is optimal, (2.2.1) yields:

$$(G\bar{q} - \tilde{y}, Gq - G\bar{q})_Y + (\lambda \bar{q}, q - \bar{q})_Q \geq 0 \ \ \forall q \in Q_{ad},$$

which is equivalent to

$$\int_0^T (\bar{\mathbf{u}} - \tilde{\mathbf{u}}, \mathbf{u} - \bar{\mathbf{u}}) dt + \int_0^T (\bar{p} - \tilde{p}, p - \bar{p}) dt + (\lambda \bar{q}, q - \bar{q})_Q \geq 0 \ \ \forall q \in Q_{ad}$$

Therefore,

$$\int_0^T (\mathbf{z}_1, \mathbf{u} - \bar{\mathbf{u}}) dt + \int_0^T (z_2, p - \bar{p}) dt + (\lambda \bar{q}, q - \bar{q})_Q \geq 0 \ \ \forall q \in Q_{ad} \tag{2.3.44}$$

Recall $G(q - \bar{q}) = (\mathbf{u} - \bar{\mathbf{u}}, p - \bar{p})$ corresponds to the solution of the poro-elastic system where all sources and initial conditions are zero except the control $q - \bar{q}$. There exists $(\mathbf{m}, h)$ satisfying the adjoint solution with $\mathbf{z}_1 = \bar{\mathbf{u}} - \tilde{\mathbf{u}}$, $z_2 = \bar{p} - \tilde{p}$, $z_3 = 0$, and $\mathbf{z}_4$. Therefore, applying Lemma 2.4.8, we obtain

$$(q - \bar{q}, h)_Q = \int_0^T (\mathbf{z}_1, \mathbf{u} - \bar{\mathbf{u}}) + \int_0^T (z_2, p - \bar{p}) \ \ \forall q \in Q_{ad}$$

if S or ψ is used as the control variable and

$$(q - \bar{q}, \mathbf{m})_Q = \int_0^T (\mathbf{z}_1, \mathbf{u} - \bar{\mathbf{u}}) + \int_0^T (z_2, p - \bar{p}) \ \ \forall q \in Q_{ad}$$

if $\mathbf{F}$ or $\mathbf{g}$ is used as the control variable. Combining this with inequality (2.4.57), we obtain

(2.4.56).

Now assume $G(\bar{q}) = (\bar{\mathbf{u}}, \bar{p})$ and that there exists a solution $(\mathbf{m}, h)$ to the adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}} - \tilde{\mathbf{u}}$, $z_2 = \bar{p} - \tilde{p}$, $z_3 = 0$ that satisfies (2.4.56). Then using the equality we get from Lemma 2.4.8 and the definition of $\mathbf{z}_1$ and z_2, we have

$$(G\bar{q} - \tilde{y}, Gq - G\bar{q})_Y + (\lambda\bar{q}, q - \bar{q})_Q \geq 0 \ \ \forall q \in Q_{ad}$$

Therefore $\bar{q}$ is the optimal control. □

2.4 Poro-elastic, k constant, NOC

2.4.1 Adjoint Equation

The adjoint is characterized by a PDE system found with the Lagrange Multiplier Method. The Lagrangian is given by

Lagrangian =Cost Function- (LHS of PDE, Multipliers)+ (RHS of the PDE, Multipliers)

For the Optimal Control Problem defined in (2.0.2) this is the Lagrangian

$$\begin{aligned}
\mathscr{L}(\mathbf{u}, p, q, \mathbf{m}, h) = J(\mathbf{u}, p, q) &+ \int_0^T (\nabla \cdot \mathbf{T}(\mathbf{u}, p), \mathbf{m}_1) d\,t + \int_0^T (\mathbf{F}, \mathbf{m}_1) d\,t - \int_0^T (\nabla \cdot \mathbf{u}_t, h_1) d\,t \\
&+ \int_0^T (k\Delta p, h_1) d\,t + \int_0^T (S, h_1) d\,t - (\nabla \cdot \mathbf{u}(0), h_2) + (d_0, h_2) - \int_0^T \langle \mathbf{T}(\mathbf{u}, p)\mathbf{n}, \mathbf{m}_2 \rangle_{\Gamma_N} d\,t \\
&+ \int_0^T \langle \mathbf{g}, \mathbf{m}_2 \rangle_{\Gamma_N} d\,t - \int_0^T \langle \mathbf{u}, \mathbf{m}_3 \rangle_{\Gamma_D} d\,t - \int_0^T \langle \nabla p \cdot \mathbf{n}, h_3 \rangle_{\Gamma_N} d\,t - \int_0^T \langle p, h_4 \rangle_{\Gamma_{D,p}} d\,t \\
&+ \int_0^T \langle k\nabla p \cdot \mathbf{n}, h_5 \rangle_{\Gamma_{D,v}} d\,t - \int_0^T \langle \psi, h_5 \rangle_{\Gamma_{D,v}} d\,t
\end{aligned} \tag{2.4.1}$$

Taking the derivative of the Lagrangian with respect to $(\mathbf{u}, p)$ and setting it equal to zero

gives the weak formulation of the adjoint equation.

$$
\begin{aligned}
D_{\mathbf{u}}\mathscr{L}(\mathbf{u},p,q,\mathbf{m},h)\mathbf{v} &= \int_0^T (\mathbf{u}-\mathbf{u}_d,\mathbf{v})dt + \int_0^T (\mathbf{v},\nabla\cdot\sigma(\mathbf{m}_1))dt - \int_0^T \langle \mathbf{v},\sigma(\mathbf{m}_1)\mathbf{n}\rangle_\Gamma dt \\
&+ \int_0^T \langle \sigma(\mathbf{v})\mathbf{n},\mathbf{m}_1\rangle_\Gamma dt - \int_0^T (\mathbf{v},\nabla(h_1)_t)dt + (\mathbf{v}(T),\nabla h_1(T)) - (\mathbf{v}(0),\nabla h_1(0)) \\
&+ \int_0^T \langle \mathbf{v}\cdot\mathbf{n},(h_1)_t\rangle_\Gamma dt - \langle \mathbf{v}(T)\cdot\mathbf{n},h_1(T)\rangle_\Gamma + \langle \mathbf{v}(0)\cdot\mathbf{n},h_1(0)\rangle_\Gamma + (\mathbf{v}(0),\nabla h_2) \\
&- \langle \mathbf{v}(0)\cdot\mathbf{n},h_2\rangle_\Gamma - \int_0^T \langle \sigma(\mathbf{v})\mathbf{n},\mathbf{m}_2\rangle_{\Gamma_N} dt - \int_0^T \langle \mathbf{v},\mathbf{m}_3\rangle_{\Gamma_D} dt = 0
\end{aligned}
\tag{2.4.2}
$$

$$
\begin{aligned}
D_p\mathscr{L}(\mathbf{u},p,q,\mathbf{m},h)r &= \int_0^T (p-p_d,r)dt + \int_0^T (r,\nabla\cdot\mathbf{m}_1)dt - \int_0^T \langle r,\mathbf{m_1}\cdot\mathbf{n}\rangle_\Gamma dt \\
&+ \int_0^T (kr,\Delta h_1)dt - \int_0^T \langle kr,\nabla h_1\cdot\mathbf{n}\rangle_\Gamma dt + \int_0^T \langle k\nabla r\cdot\mathbf{n},h_1\rangle_\Gamma dt + \int_0^T \langle r,\mathbf{m}_2\cdot\mathbf{n}\rangle_{\Gamma_N} dt \\
&- \int_0^T \langle \nabla r\cdot\mathbf{n},h_3\rangle_{\Gamma_N} dt - \int_0^T \langle r,h_4\rangle_{\Gamma_{D,p}} dt + \int_0^T \langle k\nabla r\cdot\mathbf{n},h_5\rangle_{\Gamma_{D,v}} dt = 0
\end{aligned}
\tag{2.4.3}
$$

Now we will take $\mathbf{v}\in C_0^\infty(0,T;(C_0^\infty(\Omega))^3)$ and $r = L^2(0,T;C_0^\infty(\Omega))$ to see the following

$$\int_0^T (\mathbf{u}-\mathbf{u}_d,\mathbf{v})dt + \int_0^T (\mathbf{v},\nabla\cdot\sigma(\mathbf{m}_1))dt - \int_0^T (\mathbf{v},\nabla(h_1)_t)dt = 0$$

$$\int_0^T (p-p_d,r)dt + \int_0^T (r,\nabla\cdot\mathbf{m}_1)dt + \int_0^T (kr,\Delta h_1)dt = 0$$

Because of the density of $C_0^\infty(0,T;C_0^\infty(\Omega))$ in $L^2(0,T;L^2(\Omega))$ we have that provided sufficient regularity

$$-\nabla\cdot\sigma(\mathbf{m}_1) + \nabla(h_1)_t = \mathbf{u}-\mathbf{u}_d$$

$$-\nabla\cdot\mathbf{m}_1 - k\Delta h_1 = p - p_d$$

Now if we assume $\mathbf{v}|_\Gamma = 0$ and $r|_\Gamma = 0$, but $\nabla\mathbf{v}\mathbf{n}\neq 0$ and $\nabla r\cdot\mathbf{n}\neq 0$, then we have

$$\int_0^T \langle \sigma(\mathbf{v})\mathbf{n},\mathbf{m}_1\rangle_\Gamma dt - \int_0^T \langle \sigma(\mathbf{v})\mathbf{n},\mathbf{m}_2\rangle_{\Gamma_N} dt = 0$$

$$\int_0^T \langle k\nabla r\cdot\mathbf{n}, h_1\rangle_\Gamma dt - \int_0^T \langle \nabla r\cdot\mathbf{n}, h_3\rangle_{\Gamma_N} dt + \int_0^T \langle k\nabla r\cdot\mathbf{n}, h_5\rangle_{\Gamma_{D,v}} dt = 0$$

Hence, $\mathbf{m}_1 = 0$ on Γ_D and $\mathbf{m}_1 = \mathbf{m_2}$ on Γ_N. Additionally, $h_1 = 0$ on $\Gamma_{D,p}$, $kh_1 = h_3$ on Γ_N and $h_1 = h_5$ on $\Gamma_{D,v}$. Now assuming $\mathbf{v}$ and r are not restricted on the spacial boundary, we have

$$-\int_0^T \langle \mathbf{v}, \sigma(\mathbf{m}_1)\mathbf{n}\rangle_\Gamma dt + \int_0^T \langle \mathbf{v}, (h_1)_t\mathbf{n}\rangle dt - \int_0^T \langle \mathbf{v}, \mathbf{m}_3\rangle_{\Gamma_D} dt = 0$$

$$-\int_0^T \langle r, \mathbf{m}_1\cdot\mathbf{n}\rangle_\Gamma dt - \int_0^T \langle kr, \nabla h_1\cdot\mathbf{n}\rangle_\Gamma dt + \int_0^T \langle r, \mathbf{m}_2\cdot\mathbf{n}\rangle_{\Gamma_N} dt - \langle r, h_4\rangle_{\Gamma_{D,p}} dt = 0$$

Therefore, on Γ_D, $\mathbf{m}_3 = (-\sigma(\mathbf{m}_1) + (h_1)_t\mathbf{I})\mathbf{n}$ and on Γ_N, $(-\sigma(\mathbf{m}_1) + (h_1)_t\mathbf{I})\mathbf{n} = 0$. In the second equation, the terms containing $\mathbf{m}_1$ and $\mathbf{m}_2$ cancel out because $\mathbf{m}_1 = \mathbf{m}_2$ on Γ_N and $\mathbf{m}_1 = 0$ on Γ_D. Therefore, we have $h_4 = -k\nabla h_1\cdot\mathbf{n}$ on $\Gamma_{D,p}$ and $-k\nabla h_1\cdot\mathbf{n} = 0$ on $\Gamma_N \cup \Gamma_{D,v}$. Next, if we assume that $\mathbf{v}(0) \neq 0$, but $\mathbf{v}|_\Gamma = 0$ then we have

$$(\mathbf{v}(0), \nabla h_2) = (\mathbf{v}(0), \nabla h_1(0))$$

Therefore, $\nabla h_2 = \nabla h_1(0)$. Now assuming $\mathbf{v}|_\Gamma \neq 0$, we have

$$\langle \mathbf{v}(0)\cdot\mathbf{n}, h_1(0)\rangle - \langle \mathbf{v}(0)\cdot\mathbf{n}, h_2\rangle_\Gamma$$

Hence, $h_1(0) = h_2|_\Gamma$. Finally, if we remove the assumption that $\mathbf{v}(T) = 0$ and put the assumption that $\mathbf{v}|_\Gamma = 0$, we see

$$(\mathbf{v}(T), \nabla h_1(T)) = 0$$

Therefore, $\nabla h_1(T) = 0$. Additionally, $h = 0$ on $\Gamma_{D,p}$, so we also have that $h_1(T)|_\Gamma = 0$ which will satisfy the last constraint necessary for $D_{\mathbf{u}}\mathscr{L}(\mathbf{u}, p, q, \mathbf{m}, h)\mathbf{v} = 0$ for all $\mathbf{v} \in L^2(0, T; H^1(\Omega))$. Putting all of this together, we can define $\mathbf{m} := \mathbf{m}_1$ and $h := h_1$. Then

$$\mathbf{m_2} = \tau(\mathbf{m})|_{\Gamma_N} \quad \mathbf{m}_3 = (-\sigma(\mathbf{m}) + h_t\mathbf{I})\mathbf{n})|_{\Gamma_D}$$

$$h_2 = h(0) \quad h_3 = \tau(kh)|_{\Gamma_N} \quad h_4 = \tau(-k\nabla h\cdot\mathbf{n})|_{\Gamma_{D,p}} \quad h_5 = \tau(h)|_{\Gamma_{D,v}}$$

Then we have the following system for $\mathbf{m}$ and h:

$$\begin{cases} -\nabla\cdot\sigma(\mathbf{m})+\nabla h_t=\mathbf{u}-\mathbf{u}_d & \Omega\times(0,T)\\ -\nabla\cdot\mathbf{m}-k\Delta h=p-p_d & \Omega\times(0,T)\\ \mathbf{m}=0 & \Gamma_D\times(0,T)\\ h=0 & \Gamma_{D,p}\times(0,T)\\ (-\sigma(\mathbf{m})+h_t\mathbf{I})\mathbf{n}=0 & \Gamma_N\times(0,T)\\ -k\nabla h\cdot n=0 & (\Gamma_N\cup\Gamma_{D,v})\times(0,T)\\ \nabla h(T)=0 & \Omega \end{cases}$$

We notice that this system has implicit evolution on h which is a scalar-valued function, while there is no implicit evolution on $\mathbf{m}$ which is a vector-valued function. This is different from the original system which had implicit evolution on $\mathbf{u}$ which is a vector-valued function and did not have implicit evolution on p which is a scalar-valued function.

Adjoint Equation: Well-posedness with Rothe's method

When using the adjoint equation to form necessary optimality conditions, we will set $\mathbf{z}_1=\mathbf{u}-\mathbf{u}_d$, $z_2=p-p_d$, and $z_3=0$. Therefore in (3.1.1), we consider the following regularity for the data: $\mathbf{z}_1\in L^2(0,T;\mathbf{V})$, $z_2\in L^2(0,T;V)$, and $z_3\in V$.

Now we introduce the notion of a weak solution to the adjoint system (3.1.1).

Definition 8. *[Weak Solution for Adjoint Equation] A weak solution to (3.1.1) is represented by the pair of functions* $\mathbf{m}\in L^2(0,T;\mathbf{V})$ *and* $h\in L^2(0,T;V)$ *such that:*
(a) for any $\mathbf{w}\in\mathbf{V}$, $w\in V$, *and* $f\in C_0^\infty(0,T)$ *the following variational formulations are satisfied:*

$$\int_0^T a(\mathbf{m},\mathbf{w})f(t)\,dt+\int_0^T(h,\nabla\cdot\mathbf{w})f'(t)\,dt=\int_0^T(\mathbf{z}_1,\mathbf{w})f(t)\,dt \tag{2.4.4}$$

$$\int_0^T(k\nabla h,\nabla w)f(t)\,dt-\int_0^T(\nabla\cdot\mathbf{m},w)f(t)\,dt=\int_0^T(z_2,w)f(t)\,dt \tag{2.4.5}$$

(b) for every $\mathbf{w}\in\mathbf{V}$, *the term* $(h(t),\nabla\cdot\mathbf{w})$ *uniquely defines an absolutely continuous function on* $[0,T]$ *and the terminal condition* $(h(T),\nabla\cdot\mathbf{w})=(z_3,\nabla\cdot\mathbf{w})$ *is satisfied.*

We have the following theorem for the well-posedness of adjoint system (3.1.1).

Theorem 2.4.1. *Let $\mathbf{z}_1 \in L^2(0,T;\mathbf{V})$, $z_2 \in H^1(0,T;L^2(\Omega))$, and $z_3 \in V$, then there exists a unique solution $(\mathbf{m},h)$ that satisfies Definition 8. Furthermore, we have the following estimate*

$$\|\mathbf{m}\|^2_{L^2(0,T;\mathbf{V})}+\|h\|^2_{L^2(0,T;V)} \leq C\left(\|\mathbf{z}_1\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}+\int_0^T \|(z_2)_t\|^2_{L^2(\Omega)}dt+\sup_{t\in[0,T]}\|z_2(x,t)\|^2_{L^2(\Omega)}+\|z_3\|^2_V\right)$$

Proof. We prove existence and uniqueness of a solution $(\mathbf{m},h)$ to (3.1.1) by discretizing in time.

Step 1: Solving the discretized in time problem

Let us partition the time interval $[0,T]$ into r subintervals. Let $\Delta t = T/r$ and $t_i = i\Delta t$. Then define

$$\mathbf{z}_1^i \equiv \frac{1}{\Delta t}\int_{t_i}^{t_{i+1}} \mathbf{z}_1(x,t)dt \quad \forall 0 \leq i < r$$

and

$$z_2^i \equiv z_2(x,t_i) \quad \forall 0 \leq i < r$$

We want to show that there exists $\mathbf{m}^i \in \mathbf{V}$ and $h^i \in V$ for $0 \leq i < r$ such that

$$\Delta t\ a(\mathbf{m}^i,\mathbf{w})-(h^{i+1}-h^i,\nabla\cdot\mathbf{w})=\Delta t(\mathbf{z}_1^i,\mathbf{w}) \quad \forall \mathbf{w}\in\mathbf{V} \tag{2.4.6}$$

$$-(\nabla\cdot\mathbf{m}^i,w)+(k\nabla h^i,\nabla w)=(z_2^i,w) \ \forall w\in V \tag{2.4.7}$$

$$h^r = z_3 \text{ in } V \tag{2.4.8}$$

We define $\mathscr{F}:\mathbb{V}\times\mathbb{V}\to\mathbb{R}$ as follows:

$$\mathscr{F}\left(\begin{bmatrix}\mathbf{m}^i\\ h^i\end{bmatrix},\begin{bmatrix}\mathbf{w}\\ w\end{bmatrix}\right)=\Delta t\, a(\mathbf{m}^i,\mathbf{w})+(h^i,\nabla\cdot\mathbf{w})-(\nabla\cdot\mathbf{m}^i,w)+(k\nabla h^i,\nabla w) \tag{2.4.9}$$

where h^{i+1} is given. Using Assumption 1.3.1, we have the following estimate:

$$\begin{aligned}\mathscr{F}\left(\begin{bmatrix}\mathbf{m}^i\\ h^i\end{bmatrix},\begin{bmatrix}\mathbf{m}^i\\ h^i\end{bmatrix}\right)&=\Delta t\, a(\mathbf{m}^i,\mathbf{m}^i)+(h^i,\nabla\cdot\mathbf{m}^i)-(\nabla\cdot\mathbf{m}^i,h^i)+(k\nabla h^i,\nabla h^i)\\ &=\Delta t\|\mathbf{m}^i\|^2_{\mathbf{V}}+k\|h^i\|^2_V \geq C(\Delta t,k)(\|\mathbf{m}^i\|^2_{\mathbf{V}}+\|h^i\|^2_V)\end{aligned}$$

Therefore, $\mathscr{F}$ is coercive.

Notice, using Young's inequality, when a, b, c, and d are all non-negative real numbers,

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\leq 2\sqrt{a+b+c+d}. \tag{2.4.10}$$

This estimate along with Cauchy-Schwarz yields

$$\begin{aligned}
\mathscr{F}\left(\begin{bmatrix}\mathbf{m}^i\\ h^i\end{bmatrix},\begin{bmatrix}\mathbf{w}\\ w\end{bmatrix}\right) &\leq \Delta t\|\mathbf{m}^i\|_{\mathbf{V}}\|\mathbf{w}\|_{\mathbf{V}}+\|h^i\|\|\nabla\cdot\mathbf{w}\|+\|\nabla\cdot\mathbf{m}^i\|\|w\|+k\|\nabla h\|\|\nabla w\|\\
&\leq C(\Delta t,k)(\|\mathbf{m}^i\|_{\mathbf{V}}\|\mathbf{w}\|_{\mathbf{V}}+\|h^i\|_V\|\mathbf{w}\|_{\mathbf{V}}+\|\mathbf{m}\|_{\mathbf{V}}\|w\|_V+\|h\|_V\|w\|_V)\\
&\leq C(\Delta t,k)(\sqrt{\|\mathbf{m}^i\|_{\mathbf{V}}^2\|\mathbf{w}\|_{\mathbf{V}}^2}+\sqrt{\|h^i\|_V^2\|\mathbf{w}\|_{\mathbf{V}}^2}+\sqrt{\|\mathbf{m}\|_{\mathbf{V}}^2\|w\|_V^2}+\sqrt{\|h\|_V^2\|w\|_V^2})\\
&\leq C(\Delta t,k)(\sqrt{\|\mathbf{m}^i\|_{\mathbf{V}}^2\|\mathbf{w}\|_{\mathbf{V}}^2+\|h^i\|_V^2\|\mathbf{w}\|_{\mathbf{V}}^2+\|\mathbf{m}\|_{\mathbf{V}}^2\|w\|_V^2+\|h\|_V^2\|w\|_V^2})\\
&= C(\Delta t,k)\left\|\begin{bmatrix}\mathbf{m}^i\\ h^i\end{bmatrix}\right\|_{\mathbb{V}}\left\|\begin{bmatrix}\mathbf{w}\\ w\end{bmatrix}\right\|_{\mathbb{V}}.
\end{aligned}$$

Let $\mathscr{G}:\mathbb{V}\to\mathbb{R}$ be defined as

$$\mathscr{G}\left(\begin{bmatrix}\mathbf{w}\\ w\end{bmatrix}\right)=(h^{i+1},\nabla\cdot\mathbf{w})+\Delta t(\mathbf{z}_1^i,\mathbf{w})+(z_2^i,w) \tag{2.4.11}$$

where $h^{i+1},\mathbf{z}_1^i$, and z_2 are given. Using Cauchy-Schwarz's inequality and (2.4.10), we see

$$\begin{aligned}
\left|\mathscr{G}\left(\begin{bmatrix}\mathbf{w}\\ w\end{bmatrix}\right)\right| &\leq \|h^{i+1}\|\|\mathbf{w}\|_{\mathbf{V}}+\Delta t\|\mathbf{z}_1^i\|\|\mathbf{w}\|_{\mathbf{V}}+\|z_2^i\|\|w\|_V\\
&\leq C(h^{i+1},\Delta t,\mathbf{z}_1,z_2)\left\|\begin{bmatrix}\mathbf{w}\\ w\end{bmatrix}\right\|_{\mathbb{V}}
\end{aligned}$$

Since $\mathscr{F}$ is continuous and coercive, and $\mathscr{G}$ is continuous, we know from Lax-Milgram, that there exists a unique solution

$$\mathscr{F}\left(\begin{bmatrix}\mathbf{m}^i\\ h^i\end{bmatrix},\begin{bmatrix}\mathbf{w}\\ w\end{bmatrix}\right)=\mathscr{G}\left(\begin{bmatrix}\mathbf{w}\\ w\end{bmatrix}\right)\quad\forall(\mathbf{w},w)\in\mathbb{V} \tag{2.4.12}$$

and

$$\sqrt{\|\mathbf{m}^i\|_{\mathbf{V}}^2+\|h^i\|_V^2}\leq C(h^{i+1},\Delta t,\mathbf{z}_1,z_2)/C(\Delta t,k). \tag{2.4.13}$$

Therefore, (2.4.12) implies $\mathbf{m}^i$ and h^i satisfies (2.4.6) and (2.4.7). However, the bound in (2.4.13) degenerates as $\Delta \to 0$. Therefore, we only rely on Lax-Milgram To show existence of a discrete solution for each value of Δt. The passage to the limit in time relies on the next step of the proof where we derive a uniform bound over all choices of Δt.

Step 2: Uniform bounds

Plugging $\mathbf{m}^i$ in as the test function in (2.4.6) and $h^i - h^{i+1}$ in as a test function in (2.4.7), we have

$$\Delta t\, a(\mathbf{m}^i, \mathbf{m}^i) - (h^{i+1} - h^i, \nabla \cdot \mathbf{m}^i) = \Delta t(\mathbf{z}_1^i, \mathbf{m}^i). \tag{2.4.14}$$

$$-(\nabla \cdot \mathbf{m}^i, h^i - h^{i+1}) + (k\nabla h^i, h^i - h^{i+1}) = (z_2^i, h^i - h^{i+1}). \tag{2.4.15}$$

Adding the (2.4.14) and (2.4.15) together we have

$$\Delta t\, a(\mathbf{m}^i, \mathbf{m}^i) + (k\nabla h^i, \nabla h^i - \nabla h^{i+1}) = \Delta t(\mathbf{z}_1^i, \mathbf{m}^i) + (z_2^i, h^i - h^{i+1}). \tag{2.4.16}$$

Notice

$$\begin{aligned} 2(k\nabla h^i, \nabla h^i - \nabla h^{i+1}) &= 2(k\nabla h^i, \nabla h^i) - 2(k\nabla h^i, \nabla h^{i+1}) + (k\nabla h^{i+1}, \nabla h^{i+1}) - (k\nabla h^{i+1}, \nabla h^{i+1}) \\ &= (k\nabla h^i, \nabla h^i) - (k\nabla h^{i+1}, \nabla h^{i+1}) + (k(\nabla h^i - \nabla h^{i+1}), \nabla h^i - \nabla h^{i+1}). \end{aligned}$$

Therefore, (2.4.16) becomes

$$\begin{aligned} &\frac{1}{2}\left(2\Delta t\, a(\mathbf{m}^i, \mathbf{m}^i) + (k\nabla h^i, \nabla h^i) + (k(\nabla h^{i+1} - \nabla h^i), \nabla h^{i+1} - \nabla h^i)\right) \\ &= \Delta t(\mathbf{z}_1^i, \mathbf{m}^i) + (z_2^i, h^i - h^{i+1}) + \frac{1}{2}(k\nabla h^{i+1}, \nabla h^{i+1}) \end{aligned}$$

We now take the sum from $i = l$ to $r-1$ where $0 \le l \le r-1$ and we subtract $\sum_{i=l+1}^{r-1}(k\nabla h^i, \nabla h^i)$ from both sides to attain

$$\begin{aligned} &\frac{1}{2}\left(2\Delta t \sum_{i=l}^{r-1} a(\mathbf{m}^i, \mathbf{m}^i) + (k\nabla h^l, \nabla h^l) + \sum_{i=l}^{r-1}(k(\nabla h^{i+1} - \nabla h^i, \nabla h^{i+1} - \nabla h^i))\right) \\ &= \sum_{i=l}^{r-1} \Delta t(\mathbf{z}_1^i, \mathbf{m}^i) + \sum_{i=l}^{r-1}(z_2^i, h^i - h^{i+1}) + \frac{1}{2}(k\nabla h^r, \nabla h^r). \end{aligned}$$

Using Cauchy-Schwarz, Assumption 1.3.1, Holder's inequality and Young's inequality

we have

$$|\Delta t(\mathbf{z}_1^i,\mathbf{m}^i)| = |\Delta t(\frac{1}{\Delta t}\int_{t_i}^{t_{i+1}} \mathbf{z}_1^i\, dt, \mathbf{m}^i)| \le \|\int_{t_i}^{t_{i+1}} \mathbf{z}_1\, dt\|_{L^2(\Omega)}\|\mathbf{m}^i\|_{L^2(\Omega)}$$
$$\le (\Delta t)^{1/2}\|\mathbf{z}_1\|_{L^2((t_i,t_{i+1});L^2(\Omega))}\|\mathbf{m}^i\|_{L^2(\Omega)} \le C_\epsilon\|\mathbf{z}_1\|^2_{L^2((t_i,t_{i+1});L^2(\Omega))} + \epsilon\Delta t\, a(\mathbf{m}^i,\mathbf{m}^i).$$

Using the definition of z_2^i, discrete integration by parts, Cauchy-Schwarz, Young's inequality and Bochner's Theorem, we have

$$\sum_{i=l}^{r-1}(z_2^i, h^i - h^{i+1}) = \sum_{i=l+1}^{r-1}(z_2^i - z_2^{i-1}, h^i) + (z_2^l, h^l) - (z_2^{r-1}, h^r)$$
$$\le \sum_{i=l+1}^{r-1}\|z_2^i - z_2^{i-1}\|\|h^i\| + \frac{1}{2\epsilon}\|z_2^l\|^2 + \frac{\epsilon}{2}\|h^l\|^2 + \frac{1}{2\epsilon}\|z_2^{r-1}\|^2 + \frac{\epsilon}{2}\|h^r\|^2$$
$$= \sum_{i=l+1}^{r-1}\|\int_{t_{i-1}}^{t_i}(z_2)_t\, dt\|\|h^i\| + \frac{1}{2\epsilon}\|z_2^l\|^2 + \frac{\epsilon}{2}\|h^l\|^2 + \frac{1}{2\epsilon}\|z_2^{r-1}\|^2 + \frac{\epsilon}{2}\|h^r\|^2$$
$$\le \sum_{i=l+1}^{r-1}(\Delta t)^{1/2}\|(z_2)_t\|_{L^2(t_{i-1},t_i;L^2(\Omega))}\|h^i\| + \frac{1}{2\epsilon}\|z_2^l\|^2 + \frac{\epsilon}{2}\|h^l\|^2 + \frac{1}{2\epsilon}\|z_2^{r-1}\|^2 + \frac{\epsilon}{2}\|h^r\|^2$$
$$\le \int_{t_l}^{T}\|(z_2)_t\|^2 dt + \sum_{i=l+1}^{r-1}\frac{\Delta t}{2}\|h^i\|^2 + \frac{1}{2\epsilon}\|z_2^l\|_2 + \frac{\epsilon}{2}\|h^l\|^2 + \frac{1}{2\epsilon}\|z_2^{r-1}\|^2 + \frac{\epsilon}{2}\|h^r\|^2.$$

Putting all of this together, we have

$$\frac{1}{2}\Big(2\Delta t\sum_{i=l}^{r-1}a(\mathbf{m}^i,\mathbf{m}^i) + (k\nabla h^l, \nabla h^l) + \sum_{i=l}^{r-1}(k(\nabla h^{i+1} - \nabla h^i, \nabla h^{i+1} - \nabla h^i))\Big)$$
$$\le C_\epsilon\sum_{i=l}^{r-1}\|\mathbf{z}_1\|^2_{L^2((t_i,t_{i+1});L^2(\Omega))} + \sum_{i=l}^{r-1}\epsilon\Delta t\, a(\mathbf{m}^i,\mathbf{m}^i) + \int_{t_l}^{T}\|(z_2)_t\|^2 dt + \sum_{i=l+1}^{r-1}\frac{\Delta t}{2}\|h^i\|^2 + \frac{1}{2\epsilon}\|z_2^l\|^2$$
$$+ \frac{\epsilon}{2}\|h^l\|^2 + \frac{1}{2\epsilon}\|z_2^{r-1}\|^2 + \frac{\epsilon}{2}\|h^r\|^2 + \frac{1}{2}(k\nabla h^r, \nabla h^r).$$

By the definition of h^r, we know $\|h^r\|_V \le \|z_3\|_V$. Therefore, letting ϵ be small enough,

we achieve the bound

$$\begin{aligned}&\frac{1}{2}\Big(2\Delta t\sum_{i=l}^{r-1}a(\mathbf{m}^i,\mathbf{m}^i)+(k\nabla h^l,\nabla h^l)+\sum_{i=l}^{r-1}(k(\nabla h^{i+1}-\nabla h^i,\nabla h^{i+1}-\nabla h^i))\Big)\\&\leq C\Big(\int_0^T\|\mathbf{z}_1\|^2_{L^2(\Omega)}dt+\int_0^T\|(z_2)_t\|^2_{L^2(\Omega)}dt+\max_{[0,T]}\|z_2\|^2_{L^2(\Omega)}+\|z_3\|^2_V\Big)+\sum_{i=l+1}^{r-1}\frac{\Delta t}{2}\|h^i\|^2.\end{aligned}\tag{2.4.17}$$

Noticing that all the terms on the left-hand side of this equation are positive and applying Poincaré's inequality with Poincaré constant C_p, we have

$$\begin{aligned}(\nabla h^l,\nabla h^l)\leq C(T,k)\Big(&\int_0^T\|\mathbf{z}_1\|^2_{L^2(\Omega)}dt+\int_0^T\|(z_2)_t\|^2_{L^2(\Omega)}dt\\&+\sup_{t\in[0,T]}\|z_2(x,t)\|^2_{L^2(\Omega)}+\|z_3\|^2_V\Big)+\frac{C_P\Delta t}{2k}\sum_{i=l+1}^{r-1}(\nabla h^i,\nabla h^i)\end{aligned}$$

Let

$$D=C(T,k)\Big(\int_0^T\|\mathbf{z}_1\|^2_{\mathbf{L}^2(\Omega)}dt+\int_0^T\|(z_2)_t\|^2_{L^2(\Omega)}dt+\sup_{t\in[0,T]}\|z_2(x,t)\|^2_{L^2(\Omega)}+\|z_3\|^2_V\Big).\tag{2.4.18}$$

Then

$$(\nabla h^l,\nabla h^l)\leq D+\frac{C_P\Delta t}{2k}\sum_{i=l+1}^{r-1}(\nabla h^i,\nabla h^i)$$

Let $\hat{h}^l=h^{r-l}$. Then $h^l=\hat{h}^{r-l}$ and we have the following

$$\begin{aligned}(\nabla\hat{h}^{r-l},\nabla\hat{h}^{r-l})&\leq D+\frac{C_P\Delta t}{2k}\sum_{i=l+1}^{r-1}(\nabla h^i,\nabla h^i)\\&=D+\frac{C_P\Delta t}{2k}\sum_{i=l+1}^{r-1}(\nabla\hat{h}^{r-i},\nabla\hat{h}^{r-i})\\&=D+\frac{C_P\Delta t}{2k}\sum_{j=1}^{r-l-1}(\nabla\hat{h}^j,\nabla\hat{h}^j).\end{aligned}$$

Hence,

$$(\nabla\hat{h}^l,\nabla\hat{h}^l)\leq D+\frac{C_P\Delta t}{2k}\sum_{j=1}^{l-1}(\nabla\hat{h}^j,\nabla\hat{h}^j).$$

Now applying discrete Gronwall inequality and using the fact that $\Delta t = T/r$, we see

$$(\nabla \hat{h}^l, \nabla \hat{h}^l) \leq D \exp\left(\sum_{i=j+1}^{l-1} \frac{C_P \Delta t}{2k} \right) \leq D \exp\left(r \frac{C_P \Delta t}{2k} \right) = D \exp\left(\frac{C_P T}{2k} \right)$$

which implies

$$(\nabla h^l, \nabla h^l) = (\nabla \hat{h}^{r-l}, \nabla \hat{h}^{r-l}) \leq D \exp\left(\frac{C_P T}{2k} \right). \tag{2.4.19}$$

Therefore,

$$\Delta t \sum_{i=0}^{r-1} (\nabla h^i, \nabla h^i) \leq D \exp\left(\frac{C_P T}{2k} \right) T. \tag{2.4.20}$$

Subbing (2.4.20) into (2.4.17), we see

$$\Delta t \sum_{i=0}^{r-1} a(\mathbf{m}^i, \mathbf{m}^i) \leq D + D \exp\left(\frac{C_P T}{2k} \right) \tag{2.4.21}$$

$$\sum_{i=0}^{r-1} (\nabla h^{i+1} - \nabla h^i, \nabla h^{i+1} - \nabla h^i) \leq D + D \exp\left(\frac{C_P T}{2k} \right). \tag{2.4.22}$$

Step 3: Limit passage in time. Now we define approximate solutions as follows:

$$\mathbf{m}^{[r]} \equiv \mathbf{m}^i \text{ in } [t_i, t_{i+1}), \;\; h^{[r]} \equiv h^i \text{ in } [t_i, t_{i+1}) \text{ for } 0 \leq i \leq r-1 \text{ and } h^{[r]}(T) = h^r.$$

$$(h^{[r]})^+_{\Delta t} = \frac{h^{i+1} - h^i}{\Delta t} \text{ in } [t_i, t_{i+1}) \text{ for } 0 \leq i \leq r-1.$$

Finally, we define

$$z_2^{[r]} \equiv z_2^i \text{ in } [t_i, t_{i+1}) \text{ for } 0 \leq i \leq r-1.$$

Using (2.4.20) and (2.4.21), we have the following inequalities:

$$\int_0^T (\nabla h^{[r]}, \nabla h^{[r]}) dt = \Delta t \sum_{i=0}^{r-1} (\nabla h^i, \nabla h^i) \leq CD$$

$$\int_0^T a(\mathbf{m}^{[r]}, \mathbf{m}^{[r]}) dt = \Delta t \sum_{i=0}^{r-1} a(\mathbf{m}^i, \mathbf{m}^i) \leq CD.$$

Therefore, $h^{[r]} \rightharpoonup h$ and $\mathbf{m}^{[r]} \rightharpoonup \mathbf{m}$ in $L^2(0,T;V)$ and $L^2(0,T;\mathbf{V})$, respectively. Also,

$$\int_0^T a(\mathbf{m},\mathbf{m})dt \leq CD, \tag{2.4.23}$$

$$\int_0^T (\nabla h, \nabla h)dt \leq CD. \tag{2.4.24}$$

Let $f \in C_0^\infty([0,T])$. Then let $f^i \equiv f(t_i)$ for $1 \leq i \leq r$. Then define

$$f^{[r]} \equiv f^i \text{ in } [t_i, t_{i+1}) \ \forall 0 \leq i \leq r-1$$

$$(f^{[r]})^+_{\Delta t} = \frac{f^i - f^{i-1}}{\Delta t} \text{ in } [t_i, t_{i+1}) \ \forall 1 \leq i \leq r-1$$

By Taylor's theorem, we have for all $f \in C_0^\infty([0,\infty])$

$$\|f^{[r]} - f\|_{L^2(0,T)} \leq C[\Delta t] \text{ and } \|(f^{[r]})^+_{\Delta t} - f'\|_{L^2(0,T)} \leq C[\Delta t].$$

We multiply (2.4.6) and (2.4.7), by f^i and sum from $i=0$ to $r-1$ to obtain

$$\sum_{i=0}^{r-1} \Delta t\, a(\mathbf{m}^i, \mathbf{w}) f^i - \sum_{i=0}^{r-1} (h^{i+1} - h^i, \nabla \cdot \mathbf{w}) f^i = \sum_{i=0}^{r-1} \Delta t (\mathbf{z}_1^i, \mathbf{w}) f^i$$

$$-\sum_{i=0}^{r-1} (\nabla \cdot \mathbf{m}^i, w) f^i + \sum_{i=0}^{r-1} (k \nabla h^i, \nabla w) f^i = \sum_{i=0}^{r-1} (z_2^i, w) f^i$$

Notice that

$$\sum_{i=0}^{r-1} (h^{i+1} - h^i, \nabla \cdot \mathbf{w}) f^i = -(h^0, \nabla \cdot \mathbf{w}) f^0 + \sum_{i=1}^{r-1} (h^i, \nabla \cdot \mathbf{w})(f^{i-1} - f^i) + (h^r, \nabla \cdot \mathbf{w}) f^{r-1}$$

Using these expressions, we have the equalities

$$\int_0^T a(\mathbf{m}^{[r]},\mathbf{w})f^{[r]}dt+(h(0),\nabla\cdot\mathbf{w})f(0)+\int_{\Delta t}^T(h^{[r]},\nabla\cdot\mathbf{w})(f^{[r]})^+_{\Delta t}dt-(h(T),\nabla\cdot\mathbf{w})f(T-\Delta t)$$

$$=\int_0^T(\mathbf{z}_1,\mathbf{w})f^{[r]}dt-\int_0^T(\nabla\cdot\mathbf{m}^{[r]},w)f^{[r]}dt+\int_0^T(k\nabla h^{[r]},\nabla w)f^{[r]}dt \tag{2.4.25}$$

$$=\int_0^T(z_2^{[r]},w)f^{[r]}dt \tag{2.4.26}$$

Since $f\in C_0^\infty(0,T)$, the terms with $f(0)$ are zero. We set $h^{[r]}(T)=h^r$ and $h^r=z_3$ for all Δt. Therefore,

$$\lim_{\Delta t\to 0}(h^{[r]}(T),\nabla\cdot\mathbf{w})f(T-\Delta t)=(z_3,\nabla\cdot\mathbf{w})f(T)=0 \tag{2.4.27}$$

We have the following equality

$$\int_0^T a(\mathbf{m}^{[r]},\mathbf{w})(f^{[r]})dt=\Big\{\int_0^T a(\mathbf{m}^{[r]},\mathbf{w})f^{[r]}dt-\int_0^T a(\mathbf{m},\mathbf{w})f^{[r]}dt\Big\}+\Big\{\int_0^T a(\mathbf{m},\mathbf{w})f^{[r]}dt$$
$$-\int_0^T a(\mathbf{m},\mathbf{w})f\,dt\Big\}+\int_0^T a(\mathbf{m},\mathbf{w})f\,dt \tag{2.4.28}$$

Now we observe that for $\mathbf{v}\in L^2(0,T;\mathbf{V})$, the following is a continuous linear functional on $L^2(0,T)$,

$$F_1(f):=\int_0^T a(\mathbf{v},\mathbf{w})(f)dt.$$

Also, when we fix $f^{[r]}(t)\in L^2(0,T)$, the following is a continuous linear functional on $L^2(0,T;\mathbf{V})$

$$F_2(\mathbf{v}):=\int_0^T a(\mathbf{v},\mathbf{w})(f)dt.$$

The continuity of both the linear functionals follows from Cauchy-Schwarz,

$$\|F(\mathbf{v})\|\le\|\mathbf{v}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}\|\|f\|_{L^2(0,T)}$$

Therefore, both sets of brackets in (2.4.28) go to 0 as $r \to \infty$. Hence,

$$\lim_{\Delta t \to 0} \int_0^T a(\mathbf{m}^{[r]}, \mathbf{w})(f^{[r]}) d\,t = \int_0^T a(\mathbf{m}, \mathbf{w}) f\, d\,t. \tag{2.4.29}$$

Similarly,

$$\lim_{\Delta t \to 0} \int_{\Delta t}^T (h^{[r]}, \nabla \cdot \mathbf{w})(f^{[r]})^+_{\Delta t} d\,t = \int_0^T (h, \nabla \cdot \mathbf{w}) f' d\,t \tag{2.4.30}$$

$$\lim_{\Delta t \to 0} \int_0^T (\mathbf{z}_1, \mathbf{w}) f^{[r]} d\,t = \int_0^T (\mathbf{z}_1, \mathbf{w}) f\, d\,t \tag{2.4.31}$$

$$\lim_{\Delta t \to 0} \int_0^T (\nabla \cdot \mathbf{m}^{[r]}, w) f^{[r]} d\,t = \int_0^T (\nabla \cdot \mathbf{m}, w) f\, d\,t \tag{2.4.32}$$

$$\lim_{\Delta t \to 0} \int_0^T (k \nabla h^{[r]}, \nabla w) f^{[r]} d\,t = \int_0^T (k \nabla h^{[r]}, \nabla w) f\, d\,t \tag{2.4.33}$$

$$\lim_{\Delta t \to 0} \int_0^T (z_2^{[r]}, w) f^{[r]} d\,t = \int_0^T (z_2, w) f\, d\,t \tag{2.4.34}$$

Therefore 2.4.26, 2.4.27, and 2.4.29-2.4.34 imply

$$\int_0^T a(\mathbf{m}, \mathbf{w}) f\, d\,t + \int_0^T (h, \nabla \cdot \mathbf{w}) f' d\,t = \int_0^T (\mathbf{z}_1, \mathbf{w}) f\, d\,t \tag{2.4.35}$$

$$-\int_0^T (\nabla \cdot \mathbf{m}, w) f\, d\,t + \int_0^T (k \nabla h, \nabla w) f\, d\,t = \int_0^T (z_2, w) f\, d\,t \tag{2.4.36}$$

Step 4: Recovering the Terminal Condition

Let $\mathbf{w} \in \mathbf{V}$. We define the following functions:

$$G(t) = -(h(t), \nabla \cdot \mathbf{w})$$

$$H(t) = (\mathbf{z}_1(t), \mathbf{w}) - a(\mathbf{m}(t), \mathbf{w})$$

$$F(t) = -\int_t^T H(\tau) d\,\tau.$$

$F(t)$ is absolutely continuous on $[0, T]$ and satisfies $F'(t) = H(t)$ $a.e.$ $(0, T)$.

Given $f \in C_0^\infty(0,T)$, from (2.4.35) we have

$$\int_0^T (G-F)f'(t)dt = \int_0^T G(t)f'(t)+F'(t)f(t)dt = \int_0^T G(t)f'(t)+H(t)f(t)dt$$
$$= -\int_0^T (h,\nabla\cdot\mathbf{w})f'(t)dt + \int_0^T (\mathbf{z}_1,\mathbf{w})f(t)dt - \int_0^T a(\mathbf{m},\mathbf{w})f(t)dt = 0.$$

This implies that

$$G(t)-F(t) = c. \tag{2.4.37}$$

Returning to (2.4.26) with $f \in C^\infty([0,T])$, $f(0)=0,\ f(T)=1$, and completing the limit in passage, we have

$$\int_0^T a(\mathbf{m},\mathbf{w})f(t)dt + \int_0^T (h,\nabla\cdot\mathbf{w})f'(t)dt = \int_0^T (\mathbf{z}_1,\mathbf{w})dt + (z_3,\nabla\cdot\mathbf{w}).$$

Therefore,

$$-\int_0^T H(t)f(t)dt - \int_0^T G(t)f'(t)dt = (z_3,\nabla\cdot\mathbf{w}).$$

Integrating the second integral by parts and applying (2.4.37), we have

$$-\int_0^T H(t)f(t)dt + \int_0^T F'(t)f(t)dt - (F(t)+c)f(t)|_0^T = (z_3,\nabla\cdot\mathbf{w}).$$

Recalling $F'(t)=H(t)$, $F(T)=0$, $f(T)=1$ and $f(0)=0$, we have

$$-c = (z_3,\nabla\cdot\mathbf{w}).$$

Therefore,

$$-(z_3,\nabla\cdot\mathbf{w}) + (h(t),\nabla\cdot\mathbf{w}) = \int_t^T H(\tau)d\tau.$$

Setting $t=T$, we have

$$(z_3,\nabla\cdot\mathbf{w}) = (h(T),\nabla\cdot\mathbf{w}).$$

Step 5: Uniqueness. For almost every $t\in[0,T]$ and $\mathbf{w}\in\mathbf{V}$,

$$a(\mathbf{m}(t),\mathbf{w}) - \frac{d}{dt}(h,\nabla\cdot\mathbf{w}) = (\mathbf{z}_1,\mathbf{w}) \tag{2.4.38}$$

Integrating from t to T and letting $\mathbf{w} = \mathbf{m}(t)$, we have

$$\int_t^T a(\mathbf{m}(\tau), \mathbf{m}(t))d\tau + (h(t), \nabla \cdot \mathbf{m}(t)) = \int_t^T (\mathbf{z}_1(\tau), \mathbf{m}(t))d\tau + (h(T), \nabla \cdot \mathbf{m}(t)).$$

By plugging $h(t)$ into (2.4.36) we see that for almost every $t \in [0, T]$,

$$-(\nabla \cdot \mathbf{m}(t), h(t)) + (k\nabla h(t), \nabla h(t)) = (z_2(t), h(t)).$$

Hence,

$$\int_t^T a(\mathbf{m}(\tau), \mathbf{m}(t))d\tau + (k\nabla h(t), \nabla h(t)) = \int_t^T (\mathbf{z}_1(\tau), \mathbf{m}(t))d\tau + (h(T), \nabla \cdot \mathbf{m}(t)) + (z_2, h(t)) \tag{2.4.39}$$

Recall that

$$\begin{aligned} a(\mathbf{m}(\tau), \mathbf{m}(t)) &= (\nabla \cdot \mathbf{m}(\tau), \nabla \cdot \mathbf{m}(t)) + (\nabla \mathbf{m}(\tau) + (\nabla \mathbf{m}(\tau))^T : \nabla \mathbf{m}(t)) \\ &= (\nabla \cdot \mathbf{m}(\tau), \nabla \cdot \mathbf{m}(t)) + \frac{1}{2}\Big((\nabla \mathbf{m}(\tau) : \nabla \mathbf{m}(t)) + ((\nabla \mathbf{m}(\tau))^T : \nabla (\mathbf{m}(t))^T) \\ &+ ((\nabla \mathbf{m}(\tau))^T : \nabla \mathbf{m}(t)) + (\nabla \mathbf{m}(\tau) : (\nabla \mathbf{m}(t))^T)\Big) \\ &= (\nabla \cdot \mathbf{m}(\tau), \nabla \cdot \mathbf{m}(t)) + \frac{1}{2}(\nabla \mathbf{m}(\tau) + (\nabla \mathbf{m}(\tau))^T : \nabla \mathbf{m}(t) + (\nabla \mathbf{m}(t))^T) \end{aligned}$$

Therefore we set $y_1(t) = \int_t^T \nabla \cdot \mathbf{m}(\tau)d\tau$ and $y_2(t) = \frac{1}{2}\int_t^T \nabla \mathbf{m}(\tau) + (\nabla \mathbf{m}(\tau))^T d\tau$.

Then $y_1'(t) = -\nabla \cdot \mathbf{m}(t)$ and $y_2'(t) = -\nabla \mathbf{m}(t) - (\nabla \mathbf{m}(t))^T$. Clearly, $y_1(t)$ and $y_2(t)$ are absolutely continuous, and their derivatives belong to $L^2(0, T; L^2(\Omega))$. Therefore,

$$y_1 \in H^1(0, T; L^2(\Omega)) \quad \text{and} \quad y_2 \in H^1(0, T; (L^2(\Omega))^9).$$

Additionally,

$$\int_t^T a(\mathbf{m}(\tau), \mathbf{m}(t))d\tau = -(y_1'(t), y_1(t)) - (y_2'(t) : y_2(t)) = -\frac{1}{2}\frac{d}{dt}\|y_1(t)\|^2 - \frac{1}{2}\frac{d}{dt}\|y_2(t)\|^2$$

Therefore, integrating (2.4.39) from 0 to T, we have

$$-\frac{1}{2}\|y_1(T)\|^2-\frac{1}{2}\|y_2(T)\|^2+\frac{1}{2}\|y_1(0)\|^2+\|y_2(0)\|^2+\int_0^T(k\nabla h(t),\nabla h(t))dt$$
$$=\int_0^T\int_t^T(\mathbf{z}_1(\tau),\mathbf{m}(t))d\tau dt+\int_0^T(z_3,\nabla\cdot\mathbf{m}(t))dt+\int_0^T(z_2,h(t))dt$$

However, $y_1(T)=y_2(T)=0$. Therefore, we have

$$\int_0^T(k\nabla h(t),\nabla h(t))\leq\int_0^T\int_t^T(\mathbf{z}_1(\tau),\mathbf{m}(t))d\tau dt+\int_0^T(z_3,\nabla\cdot\mathbf{m}(t))dt+\int_0^T(z_2,h(t))dt \tag{2.4.40}$$

Therefore, if there were two solutions $(\mathbf{m}_1,h_1)$ and $(\mathbf{m}_2,h_2)$ to the adjoint system then $(\mathbf{m}_1-\mathbf{m}_2,h_1-h_2)$ would be a solution to the adjoint system with the sources set to zero. Inequality (2.4.40) implies that $\|h_1-h_2\|_{L^2(0,T;V)}=0$. However, if $\mathbf{z}_1=0$ and $h_1-h_2=0$ in $L^2(0,T;V)$, then (2.4.38) implies

$$\int_0^T a(\mathbf{m}_1-\mathbf{m}_2,\mathbf{w})f\,dt=0\ \ \forall\mathbf{w}\in\mathbf{V}\ \ \forall f\in C_0^\infty(0,T)$$

However, since $\{\mathbf{w}f\ |\mathbf{w}\in\mathbf{V},\ f\in C_0^\infty(0,T)\}$ is dense in $L^2(0,T;\mathbf{V})$, we have that $\|\mathbf{m}_1-\mathbf{m}_2\|_{L^2(0,T;\mathbf{V})}=0$. Therefore, the solution to the adjoint system is unique.

Step 6: Bounds on Solution

Notice that the bounds on the solution can be obtained from (2.4.18), (2.4.23), and (2.4.24).

□

Adjoint Equation: Well-posedness with operator theory for implicit evolution equations

To prove existence and uniqueness of a weak solution using operator theory, we will observe the system reversed in time. In this case we have the following system and definition of a

weak solution:

$$\begin{cases} \nabla\cdot\sigma+\nabla h_t=-\mathbf{z}_1 & \text{in } \Omega\times(0,T) \\ -\nabla\cdot\mathbf{m}-k\Delta h=z_2 & \text{in } \Omega\times(0,T) \\ \mathbf{m}=0 & \Gamma_D\times(0,T) \\ h=0 & \Gamma_{D,p}\times(0,T) \\ (-\sigma(\mathbf{m})+h_t\mathbf{I})\mathbf{n}=0 & \Gamma_N\times(0,T) \\ -k\nabla h\cdot n=0 & (\Gamma_N\cup\Gamma_{D,v})\times(0,T) \\ \nabla h(T)=\nabla z_3 & \Omega \end{cases} \tag{2.4.41}$$

Definition 9. *[Weak Solution for Reversed Adjoint Equation] A solution to (2.4.41) is represented by the pair of functions* $\mathbf{m}\in L^2(0,T;\mathbf{V})$ *and* $h\in L^2(0,T;V)$ *such that:*
(a) for any $\mathbf{w}\in L^2(0,T;\mathbf{V})$ *and* $w\in L^2(0,T;V)$*, the following variational formulations are satisfied:*

$$\int_0^T a(\mathbf{m},\mathbf{w})\,dt-\int_0^T(\nabla h_t,\mathbf{w})_{\mathbf{V}',\mathbf{V}}\,dt=\int_0^T(\mathbf{z}_1,\mathbf{w})\,dt \tag{2.4.42}$$

$$\int_0^T(k\nabla h,\nabla w)\,dt-\int_0^T(\nabla\cdot\mathbf{m},w)\,dt=\int_0^T(z_2,q)\,dt \tag{2.4.43}$$

(b) for every $q\in V$*, the term* $(\nabla h(t),\mathbf{w})_{L^2(\Omega)}$ *uniquely defines an absolutely continuous function on* $[0,T]$ *and the terminal condition* $(\nabla h(0),\mathbf{w})=(\nabla z_3,q)$ *is satisfied.*

We now define the following operators which will be used to show existence and uniqueness of a solution to the adjoint equation.

1. The $\mathscr{E}$ operator:
   First, consider the system

$$\begin{cases} \nabla\cdot\sigma=-\mathbf{z}_1 & \text{in } \Omega \\ \mathbf{m}=0 & \text{on } \Gamma_D \\ \sigma n=0 & \text{on } \Gamma_N \end{cases}$$

   Remark: In the case $\overline{\Gamma_N}\cap\overline{\Gamma_D}\neq\emptyset$, Lemma 1.3.1 yields that $\mathbf{m}\in H^2(\Omega)$. This fact along with the fact that $\mathbf{z}_1\in L^2(0,T;\mathbf{L}^2(\Omega))$, yields $\nabla\cdot\sigma\in L^2(0,T;L^2(\Omega))$.

Let $\mathbf{v} \in \mathbf{V}$ then we have

$$\begin{aligned}
\int_\Omega (\nabla \cdot \sigma) \cdot \mathbf{v} d\Omega &= \int_\Omega -z_1 \cdot \mathbf{v} d\Omega \\
\int_\Gamma \sigma n \mathbf{v} h \Gamma - \int_\Omega (\sigma, \nabla \mathbf{v}) d\Omega &= \int_\Omega -z_1 \cdot \mathbf{v} d\Omega \\
\int_\Omega (\sigma, \nabla \mathbf{v}) d\Omega &= \int_\Omega z_1 \cdot \mathbf{v} d\Omega \\
a(\mathbf{m}, \mathbf{v}) &= \int_\Omega z_1 \cdot \mathbf{v} d\Omega
\end{aligned}$$

Now, we denote the operator $\mathscr{E}(\mathbf{m}) = a(\mathbf{m}, \cdot)$. Hence, $\mathscr{E} : \mathbf{V} \to \mathbf{V}'$. Since we are taking $a(\cdot, \cdot)$ to be the inner product on $\mathbf{V}$, it is clear $\mathscr{E}$ is coercive.

Now we will show that the range of $\mathscr{E}$ is $\mathbf{V}'$. Let $w \in \mathbf{V}'$. Then by Lax-Milgram there exists a $\mathbf{u} \in \mathbf{V}$ such that $a(\mathbf{u}, \mathbf{v}) = w(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}$. Hence, $\mathscr{E}\mathbf{u} = w$. Therefore, the range of $\mathscr{E}$ is $\mathbf{V}'$.

2. The A operator:
   Consider the following:

$$\begin{cases} -k\Delta h = z_2 & \text{in } \Omega \\ h = 0 & \text{on } \Gamma_{D,p} \\ k\nabla h \cdot n = 0 & \text{on } \Gamma_N \cup \Gamma_{D,v} \end{cases}$$

   Let $w \in V$. Then multiplying by w and integrating we have the following:

$$\begin{aligned}
\int_\Omega (-k\Delta h w) d\Omega &= \int_\Omega z_2 w d\Omega \\
-\int_\Gamma k w \nabla h \cdot n d\Gamma + \int_\Omega k\nabla h \cdot \nabla w d\Omega &= \int_\Omega z_2 w d\Omega \\
\int_\Omega k\nabla h \cdot \nabla w d\Omega &= \int_\Omega z_2 w d\Omega
\end{aligned}$$

   We define

$$A : V \to V' \text{ by } A(h)(w) = \int_\Omega k\nabla h \cdot \nabla w d\Omega$$

We see that $(A(h),h) = k\|\nabla h\|^2_{L^2(\Omega)} \geq 0$. Therefore, A is monotone. Furthermore, Friedrichs's inequality and assumption 1.3.1, yields

$$k\|\nabla h\|^2_{L^2(\Omega)} \geq C\|h\|_V \ \ \forall h \in V.$$

Hence, A is coercive. Therefore, if $z_2 \in L^2(\Omega)$ then by Lax-Milgram, a unique solution h will exist such that $A(h) = z_2$, so the range of $A = V'$ and A is injective. Also note, the kernel of A is the kernel of ∇ Hence, any function that is constant almost everywhere is in the kernel of A. Finally, by the definition, we clearly see that A is self-adjoint.

3. The coupling terms:
   Consider the problem
   $$\nabla h = 0 \text{ on } \Omega \text{ and } \ h = 0 \text{ on } \Gamma_N$$
   Note that if $h \in V$, then $\nabla h = 0$ implies $h = 0$ on Γ by Poincare's inequality and trace theorem. Also, we have
   $$\begin{aligned} 0 &= \int_0^T (\nabla h, \mathbf{v}) d\,t \\ &= -\int_0^T (h, \nabla \cdot \mathbf{v}) d\,t + \int_0^T \langle h\mathbf{I}\cdot\mathbf{n}, \mathbf{v}\rangle d\,t = 0 \\ &= -\int_0^T (h, \nabla \cdot \mathbf{v})) d\,t \end{aligned}$$
   Therefore,
   $$\int_0^T (h, \nabla \cdot \mathbf{v} d\,t = 0.$$
   We define $\overline{\nabla} : V \to \mathbf{V}'$ by $\overline{\nabla}(h)(\mathbf{v}) = \int_\Omega h\nabla\cdot\mathbf{v} d\Omega$. Similarly, define $\overline{\nabla\cdot} : \mathbf{V} \to V'$ by $\overline{\nabla\cdot}\mathbf{m}(v) = \int_\Omega \nabla\cdot\mathbf{m}\, v\Omega$. With these definitions we can immediately see that $\overline{\nabla}$ is the adjoint of $\overline{\nabla\cdot}$.

Then Definition (9) can be written as the following:

$$\begin{aligned} \mathscr{E}(\mathbf{m}(t)) + \frac{d}{d\,t}\overline{\nabla} h(t) &= \mathbf{z}_1(t) \\ A(h(t)) - \overline{\nabla\cdot}\mathbf{m} &= z_2(t) \\ \overline{\nabla} h(0) &= \overline{\nabla} z_3 \end{aligned} \tag{2.4.44}$$

Now, assume $z_2(t) \in L^2(\Omega)$ for all $t \in (0,T)$ and let $\gamma(t) \in V$ be defined such that $A(\gamma(t)) = z_2$. Then we can replace $h(t)$ by $h(t)+\gamma(t)$. To obtain:

$$\begin{aligned} \mathscr{E}(\mathbf{m}(t)) + \frac{d}{dt}\overline{\nabla}(h(t)+\gamma(t)) &= \mathbf{z}_1(t) \\ A(h(t)) - \overline{\nabla}\cdot\mathbf{m} &= 0 \\ \overline{\nabla}h(0) + \overline{\nabla}\gamma(0) &= \overline{\nabla}z_3. \end{aligned}$$

or equivalently,

$$\begin{aligned} \mathscr{E}(\mathbf{m}(t)) + \frac{d}{dt}\overline{\nabla}h(t) &= \mathbf{z}_1(t) - \overline{\nabla}\gamma_t(t) \\ A(h(t)) - \overline{\nabla}\cdot\mathbf{m} &= 0 \\ \overline{\nabla}h(0) &= \overline{\nabla}(z_3 - \gamma(0)). \end{aligned}$$

Hence, if we replace $\mathbf{z}_1 - \overline{\nabla}\gamma_t(t)$ by $\mathbf{z}_1$ and $\overline{\nabla}(z_3-\gamma(0))$ by z_3, we have

$$\begin{aligned} \mathscr{E}(\mathbf{m}(t)) + \frac{d}{dt}\overline{\nabla}h(t) &= \mathbf{z}_1(t) \\ A(h(t)) - \overline{\nabla}\cdot\mathbf{m} &= 0 \\ \overline{\nabla}h(0) &= \overline{\nabla}z_3 \end{aligned} \tag{2.4.45}$$

where the first equation holds in $\mathbf{V}'$ and the second equation holds in V'. Note that initial condition remains the same.

Now we define an operator B on $\mathbf{V}$ as

$$B(\mathbf{m}) = \overline{\nabla}h(t), \text{ where } A(h(t)) = \overline{\nabla}\cdot\mathbf{m} \text{ in } V', \text{ and } h \in V$$

Proposition 2.4.2. *$B : \mathbf{V} \to \mathbf{V}'$ is well-defined.*

Proof. Let $\mathbf{m} \in \mathbf{V}$, then $\overline{\nabla}\cdot\mathbf{m} \in V'$. Since $\operatorname{Rg} A = V'$, there exists $h \in V$ such that $A(h(t)) = \overline{\nabla}\cdot\mathbf{m}$. Finally, $\overline{\nabla}h \in \mathbf{V}'$, so $B : \mathbf{V} \to \mathbf{V}'$. □

Now (2.4.45) becomes

$$\begin{aligned} \mathscr{E}(\mathbf{m}(t)) + \frac{d}{dt}B(\mathbf{m}(t)) &= \mathbf{z}_1(t) \\ B(\mathbf{m}(0)) &= B(\mathbf{m}_0) \end{aligned} \tag{2.4.46}$$

where $\mathbf{m}_0$ satisfies $B(\mathbf{m}_0) = \overline{\nabla} z_3$. Note that this implies $\overline{\nabla} z_3 \in Rg(B)$. If we set our original z_3 to be zero, then this will only require $-\gamma(0) \in Rg(B)$. However, since $A(\gamma(0)) = z_2(0)$ and $B(\mathbf{m}_0) = \overline{\nabla}\gamma(0)$ if and only if $\overline{\nabla\cdot}\mathbf{m}_0 = A(\gamma(0)) = z_2(0)$, then we must require $z_2(0) \in Rg\overline{\nabla\cdot}$.

Lemma 2.4.3. *A^{-1} is self-adjoint i.e. $(w, A^{-1}(v))_{V',V} = (v, A^{-1}(w))_{V',V}$.*

Proof. Let $p,\ w \in V' = Rg(A)$, and let R denote the Reisz-map. Since $Rg(A) = V'$, there exists a $\tilde{p}$ such that

$$\int_\Omega \nabla\tilde{p}\nabla r\, dx = (R(p), r)_V \quad \forall r \in V.$$

Similarly, there exists $\tilde{w}$ such that

$$\int_\Omega \nabla\tilde{w}\nabla r\, dx = (R(w), r)_V \quad \forall r \in V.$$

If we let $r = \tilde{w}$ in the first equation and $r = \tilde{p}$ in the second equation, we obtain

$$(R(p), \tilde{w})_V = \int_\Omega \nabla\tilde{p}\nabla\tilde{w}\, dx = (R(w), \tilde{p})_V$$

Using this information, we have

$$(w, A^{-1}(p))_{V',V} = (R(w), \tilde{p})_V = (R(p), \tilde{w})_V = (p, A^{-1}(w))_{V',V}$$

Therefore, A^{-1} is self-adjoint. □

Theorem 2.4.4. *The operator B is linear, self-adjoint, continuous, and monotone.*

Proof. Linearity can be seen clearly. Let $A(h) = \overline{\nabla\cdot}v$ and $A(g) = \overline{\nabla\cdot}w$. Invoking the fact that $\overline{\nabla}$ and $\overline{\nabla\cdot}$ are adjoints of each other and A^{-1} is self-adjoint, we have the following:

$$\begin{aligned}
(Bv, w)_{\mathbf{V}',\mathbf{V}} &= (\overline{\nabla}h, w)_{\mathbf{V}',\mathbf{V}} \\
&= (\overline{\nabla\cdot}w, h)_{V',V} \\
&= (\overline{\nabla\cdot}w, A^{-1}(\overline{\nabla\cdot}v))_{V',V} \\
&= (\overline{\nabla\cdot}v, A^{-1}(\overline{\nabla\cdot}w))_{V',V} \\
&= (\overline{\nabla\cdot}v, g)_{V',V} \\
&= (\overline{\nabla}g, v)_{\mathbf{V}',\mathbf{V}} \\
&= (Bw, v)_{\mathbf{V}',\mathbf{V}}
\end{aligned}$$

Therefore, B is self-adjoint. Recall, adjoint's are necessarily closed operators. Hence, B self-adjoint implies that B is closed. Therefore, by the closed graph theorem, B is continuous. Finally,

$$(B v, v)_{\mathbf{V}',\mathbf{V}} = (\overline{\nabla\cdot} v, h)_{V',V} = (A(h), h)_{V',V} = k\int_\Omega \nabla h\cdot\nabla h\, dx = k\|\nabla h\|_{L^2(\Omega)} \geq 0$$

Therefore, B is monotone. □

We now are ready to invoke the following well-posedness result, which can be found in [50, p. 115] by identifying $\mathscr{A} = \mathscr{E}$ and $\mathscr{B} = B$.

Proposition 2.4.5. *Let* $\mathbf{V}$ *be a separable Hilbert space and* $\mathscr{A}(t)\in\mathscr{L}(\mathbf{V},\mathbf{V}')$ *such that*

$$\mathscr{A}(\cdot)u(v)\in L^\infty(0,T)\quad \forall\, u, v\in\mathbf{V}.$$

Also, let $\mathscr{B}(t)\in\mathscr{L}(\mathbf{V},\mathbf{V}')$ *such that* $\mathscr{B}(t)$ *is a regular family of self-adjoint operators and* $\mathscr{B}(\cdot)u(v)\in L^\infty(0,T)\ \forall\, u, v\in\mathbf{V}$. *Further, let* $\mathscr{B}(0)$ *be monotone. Let* $z_3\in\mathbf{V}$, $\mathbf{z}_1\in L^2(0,T,\mathbf{V}')$. *If there exists* $\lambda, c>0$ *such that*

$$2\mathscr{A}(t)v(v)+\lambda\mathscr{B}(t)v(v)+\mathscr{B}'(t)v(v)\geq c\|v\|^2\ \ \forall v\in\mathbf{V}, 0\leq t\leq T \tag{2.4.47}$$

then there exists $u\in\mathbf{V}$ *such that*

$$\begin{gathered}\frac{d}{dt}(\mathscr{B}(t)u(t))+\mathscr{A}(t)u(t)=\mathbf{z}_1(t)\ \ in\,(L^2(0,T;\mathbf{V}))'\\ (\mathscr{B}u)(0)=\mathscr{B}(0)u_0\end{gathered} \tag{2.4.48}$$

where $\mathscr{B}(u)(t)$ *is absolutely continuous. Furthermore, the solution satisfies*

$$\|u\|_{L^2(0,T;\mathbf{V})}\leq C(\lambda,c)\Big(\|\mathbf{z}_1\|^2_{L^2(0,T;\mathbf{V}')}+\mathscr{B}(0)u_0(u_0)\Big)^{1/2} \tag{2.4.49}$$

Further, if $\mathscr{B}(t)$ *is monotone,* $\mathscr{A}(t)$ *is a regular family of self-adjoint operators, and there is a pair of numbers* $\lambda, c>0$ *such that*

$$\mathscr{A}(t)v(v)+\lambda\mathscr{B}(t)v(v)\geq c\|v\|^2\quad\forall v\in\mathbf{V}, a.e.\, t\in[0,T]$$

then this solution is unique.

Let the V in this proposition be $\mathbf{V}$, and $\mathscr{A}(t) = \mathscr{E}$ for all $t \in [0,T]$. Let $\mathscr{B}(t) = B$ for all $t \in [0,T]$ Let $f = \mathbf{z}_1$ and $u_0 = z_3$. Since $\mathscr{E}$ and B do not change with respect to time, applying them to u, v in $\mathbf{V}$ leaves us with a constant function which is clearly in $L^\infty(0,T)$. Also, this means they are trivially regular families of operators.

We will need to require $\mathbf{z_1} \in \mathbf{L^2(0,T;V')}$ and $z_3 \in \mathbf{V}$. Recall, that we replaced $\mathbf{z}_1(t)$ with $\mathbf{z}_1(t) - \overline{\nabla}\gamma_t(t)$ where $A(\gamma(t)) = z_2$. If we require $z_2(t) \in L^2(\Omega)$ for all t then $\gamma(t) \in V$ for all t. Therefore, $\overline{\nabla}\gamma(t) \in \mathbf{V}'$ for all t. This implies, if we require $z_2 \in H^1(0,T;L^2(\Omega))$ then $\gamma \in H^1(0,T;V)$ and $\overline{\nabla}\gamma \in H^1(0,T;\mathbf{V}')$, so $\overline{\nabla}\gamma_t \in L^2(0,T;\mathbf{V}')$. Hence, in order for $z_1 - \overline{\nabla}\gamma_t \in L^2(0,T;\mathbf{V}')$, we need $z_2 \in H^1(0,T;L^2(\Omega))$.

We already showed that $\mathscr{E}$ is self-adjoint and coercive and B is self-adjoint and monotone. Since $\mathscr{E}$ is coercive, B is monotone, and $B'(t) = 0$, the inequalities are satisfied for any $\lambda > 0$ and c being the same constant used in the coercivity constant we have on $\mathscr{E}$. Hence, applying the above theorems, we have a unique solution, $\mathbf{m} \in L^2(0,T;\mathbf{V})$ to the problem

$$\mathscr{E}(\mathbf{m}(t)) + \frac{d}{dt}(B\mathbf{m})(t) = z_1(t) \ \in (L^2(0,T;\mathbf{V}))'$$

$$(B\mathbf{m})(0) = B(0)u_0$$

Therefore,

$$\mathscr{E}(\mathbf{m}(t)) + \frac{d}{dt}(\overline{\nabla}h(t)) = z_1(t) \ \in (L^2(0,T;\mathbf{V}))'$$

$$A(h(t)) - \overline{\nabla}\cdot\mathbf{m} = 0 \ \in V'$$

The initial condition is equivalent to:

$$(h(0), \nabla\cdot\mathbf{w})_{L^2(\Omega)} = (u_0, \nabla\cdot\mathbf{w})_{L^2(\Omega)} \quad \forall \mathbf{w} \in \mathbf{V}$$

Or equivalently,

$$-(\nabla h(0), \mathbf{w})_{L^2(\Omega)} + (h(0), \mathbf{w}\cdot n)_{L^2(\Gamma)} = -(\nabla u_0, \mathbf{w})_{L^2(\Omega)} + (u_0, \mathbf{w}\cdot n)_{L^2(\Gamma)}$$

which will be satisfied when $\nabla h(0) = \nabla z_3$. We can also write this problem in another way. First, we define $W^{1,2}(0,T;\mathbf{V}) = \{f : [0,T] \to \mathbf{V} | g \in L^2(0,T;\mathbf{V}), f(t) = f(0) + \int_0^T g(s)ds\}$. Then

if for all $\mathbf{w} \in L^2(0,T;\mathbf{V}) \cap W^{1,2}(0,T;\mathbf{V})$ with $\mathbf{w}(T) = 0$

$$-\int_0^T B(\mathbf{m}(t))(\mathbf{w}'(t))dt + \int_0^T \mathscr{E}(\mathbf{m}(t))(\mathbf{w}(t))dt = \int_0^T \mathbf{z}_1(t)(\mathbf{w}(t))dt + Bu_0(\mathbf{w}(0))$$

Hence,

$$-\int_0^T (h(t), \nabla\cdot\mathbf{w}'(t))dt + \int_0^T a(\mathbf{m},\mathbf{w})dt = \int_0^T (\mathbf{z}_1,\mathbf{w})dt + (z_3, \nabla\cdot\mathbf{w}(0))$$

Notice that $\mathbf{v}f(t) \in L^2(0,T;\mathbf{V}) \cap W^{1,2}(0,t\mathbf{V})$ when $\mathbf{v} \in \mathbf{V}$ and $f \in C_0^\infty(0,T)$ and in that case we have

$$\int_0^T a(\mathbf{m},\mathbf{v})f(t)dt - \int_0^T (h(t), \nabla\cdot\mathbf{v})f'(t) = \int_0^T (\mathbf{z}_1,\mathbf{v})dt$$

All of this yields the following result:

Theorem 2.4.6. *Let $\mathbf{z}_1 \in L^2(0,T;\mathbf{V}')$, $z_2 \in H^1(0,T;L^2(\Omega))$, $\int_\Omega z_2(0)d\Omega = 0$, and $z_3 \in \mathbf{V}$. Then the reversed adjoint system has a weak solution $(\hat{\mathbf{m}}, \hat{h}) \in L^2(0,T;\mathbf{V}) \times L^2(0,T;V)$ such that for all $\mathbf{w} \in \mathbf{V}$, $f \in C_0^\infty(0,T)$ and $w \in L^2(0,T;V)$:*

1. $\int_0^T a(\hat{\mathbf{m}},\mathbf{w})f(t)dt - \int_0^T (\hat{h}, \nabla\cdot\mathbf{w})f'(t)dt = \int_0^T (\mathbf{z}_1,\mathbf{w})f(t)dt$
2. $-\int_0^T (\nabla\cdot\hat{\mathbf{m}}, w)dt + \int_0^T (k\nabla\hat{h}, \nabla w)dt = \int_0^T (z_2, w)dt$
3. *For every $w \in \mathbf{V}$, the initial condition $(\nabla\hat{h}(0),\mathbf{w}) = (\nabla z_3,\mathbf{w})$ is satisfied.*

Reversing the system yields the following result:

Theorem 2.4.7. *Let $\mathbf{z}_1 \in L^2(0,T;\mathbf{V})$, $z_2 \in H^1(0,T;L^2(\Omega))$, $\int_\Omega z_2(T)d\Omega = 0$, and $z_3 \in \mathbf{V}$. Then the adjoint system has a weak solution $(\mathbf{m}, h) \in L^2(0,T;\mathbf{V}) \times L^2(0,T;V)$ such that for all $\mathbf{w} \in \mathbf{V}$, $f \in C_0^\infty(0,T)$, and $w \in L^2(0,T;V)$, we have*

1. $\int_0^T a(\mathbf{m},\mathbf{w})f(t)dt + \int_0^T (h, \nabla\cdot\mathbf{w})f'(t)dt = \int_0^T (\mathbf{z}_1,\mathbf{w})f(t)dt$
2. $-\int_0^T (\nabla\cdot\mathbf{m}, w)dt + \int_0^T (k\nabla h, \nabla w)dt = \int_0^T (z_2, w)dt$
3. *For every $w \in \mathbf{V}$, the terminal condition $(\nabla h(T),\mathbf{w}) = (\nabla z_3,\mathbf{w})$ is satisfied.*

Proof. Let $\tau \in [0,T]$ and define $\hat{\mathbf{m}}(\tau) = \mathbf{m}(T-\tau)$, $\hat{\mathbf{w}}(\tau) = \mathbf{w}(T-\tau)$, $\hat{h}(\tau) = h(T-\tau)$, $\hat{w} = w(T-\tau)$, $\hat{z}_1(\tau) = z_1(T-\tau)$, and $\hat{z}_2(\tau) = z_2(T-\tau)$. Notice using the change of variables $t = T-\tau$

we have

$$\int_0^T a(\mathbf{m},\mathbf{w})d\,t = -\int_T^0 a(\hat{\mathbf{m}},\hat{\mathbf{w}})d\,\tau$$
$$= \int_0^T a(\mathbf{m},\hat{\mathbf{w}})d\,\tau$$

Also with the same change of variables,

$$-\int_0^T (h_t,\nabla\cdot\mathbf{w})_{V',V}\,d\,t = \int_T^0 (h_t,\nabla\cdot\mathbf{w})_{V',V}\,d\,\tau$$
$$= \int_0^T (h_\tau,\nabla\cdot\mathbf{w})_{V',V}\,d\,\tau$$

This time the sign between the second and third expression didn't change because we changed the bounds on the integral and the variable that the derivative of h was taken with respect to. Similarly, we can complete a change of variables, so that the weak form becomes:

$$\int_0^T a(\mathbf{m},\mathbf{w})d\,t - \int_0^T (h_t,\nabla\cdot\mathbf{w})_{V',V}\,d\,t = \int_0^T (z_1,\mathbf{w})d\,t$$
$$-\int_0^T (\nabla\cdot\mathbf{m},w)d\,t + \int_0^T (k\nabla h,\nabla w)d\,t = \int_0^T (z_2,w)d\,t$$

For every $w\in\mathbf{V}$, the terminal condition $(\nabla h(T),\mathbf{w}) = (\nabla z_3,\mathbf{w})$ and $\langle h(T),\mathbf{w}\cdot n\rangle_{L^2(\Gamma)} = \langle z_3,\mathbf{w}\cdot n\rangle_{L^2(\Gamma)}$ is satisfied. Existence and uniqueness of a solution $(\hat{\mathbf{m}},\hat{h})$ is given to us by the last theorem. Hence, $\mathbf{m}(x,t) = \hat{\mathbf{m}}(T-t)$ and $h(x,t) = \hat{h}(T-t)$ is the desired solution. □

2.4.2 First Order Necessary Optimality Conditions for Poro-elastic Case

First, we provide an identity which is essential in the derivation of the first order necessary optimality conditions for the optimal control.

Lemma 2.4.8. *Let* $(\mathbf{u},p)$ *be the solution to the weak solution to the poro-elastic system (1.2.1)-(1.2.4) with* $\mathbf{u}_0 = 0$. *Let* $(\mathbf{m},h)$ *be the weak solution to the adjoint equation (3.1.1) with* $z_3 = 0$.

Assume $\mathbf{u} \in H^1(0,T;\mathbf{V})$. *Then the following identity holds:*

$$\int_0^T \langle \mathbf{g}, \mathbf{m} \rangle_{\Gamma_N} dt + \int_0^T (\mathbf{F}, \mathbf{m}) dt - \int_0^T \langle \psi, h \rangle_{\Gamma_{D,\mathbf{v}}} dt + \int_0^T (S, h) dt = \int_0^T (\mathbf{z}_1, \mathbf{u}) dt + \int_0^T (z_2, p) dt \tag{2.4.50}$$

Proof. First we will reformulate the weak solution to the adjoint equation. Let $(\mathbf{m}, h)$ be the weak solution to the adjoint equation (3.1.1). Notice,

$$(h, \nabla \cdot \mathbf{w}) \leq \|h\|_{L^2(\Omega)} \|\mathbf{w}\|_{\mathbf{V}}$$

Hence, by Reisz-Representation Theorem, there exists $R(h_\nabla) \in \mathbf{V}$ such that for all $\mathbf{w} \in \mathbf{V}$

$$(h, \nabla \cdot \mathbf{w})_{L^2(\Omega)} = (R(h_\nabla), \mathbf{w})_{\mathbf{V}} \text{ and } \|R(h_\nabla)\|_{\mathbf{V}} \leq \|h\|_{L^2(\Omega)}$$

Therefore, $R(h_\nabla) \in L^2(0,T;\mathbf{V})$.

Furthermore, for any $\mathbf{w} \in \mathbf{V}$ and $f \in C_0^\infty(0,T)$, we have

$$\int_0^T a(\mathbf{m}, \mathbf{w}) f(t) dt + \int_0^T (R(h_\nabla), \mathbf{w})_{\mathbf{V}} f'(t) dt = \int_0^T (\mathbf{z}_1, \mathbf{w}) f(t) dt.$$

Now we will define the following functionals $F_i(t) : V \to \mathbb{R}$:

$$F_1(t) : \mathbf{w} \mapsto -(\mathbf{z}_1, \mathbf{w})_{L^2(\Omega)}$$

$$F_2(t) : \mathbf{w} \mapsto a(\mathbf{m}, \mathbf{w})$$

Both of these functionals are linear. We also have the following bounds:

$$|F_1(t)\mathbf{w}| \leq \|z_1\|_{(L^2(\Omega))^3} \|\mathbf{w}\|_{(L^2(\Omega))^3} \leq \|z_1\|_{\mathbf{V}} \|\mathbf{w}\|_{\mathbf{V}}$$

$$|F_2(t)\mathbf{w}| \leq 2\|\nabla \mathbf{m}\|_{(L^2(\Omega))^9} \|\nabla \mathbf{w}\|_{(L^2(\Omega))^9} \leq 2\|\mathbf{m}\|_{\mathbf{V}} \|\mathbf{w}\|_{\mathbf{V}}$$

This implies that $F_i(t) \in \mathbf{V}'$ for every t and there is some constant $C > 0$ such that

$$\|F_1(t)\|_{\mathbf{V}'} + \|F_2(t)\|_{\mathbf{V}'} \leq C(\|z_1\|_{\mathbf{V}} + \|\mathbf{m}\|_{\mathbf{V}})$$

Since $\mathbf{z}_1, \mathbf{m} \in L^2(0,T;\mathbf{V})$, then we obtain that $F(t) = F_1(t) + F_2(t) \in L^2(0,T;\mathbf{V}')$. Now using

Bochner's Theorem and (2.4.4) we have

$$\left(-\int_0^T R(h_\nabla)f'(t)dt,\mathbf{w}\right)_{\mathbf{V}} = -\int_0^T (R(h_\nabla)f'(t),\mathbf{w})_{\mathbf{V}}dt$$
$$= \int_0^T (F(t)f(t),\mathbf{w})_{\mathbf{V}',\mathbf{V}}dt = \left(\int_0^T F(t)f(t)dt,\mathbf{w}\right)_{\mathbf{V}',\mathbf{V}}$$

Hence, we have that in the space of $\mathbf{V}'$,

$$-\int_0^T R(h_\nabla)f'(t)dt = \int_0^T F(t)f(t)dt, \quad \forall f \in C_0^\infty(0,T)$$

Therefore, by definition of distributional derivatives for vector valued functions, $(R(h_\nabla))_t = F(t) \in L^2(0,T;\mathbf{V}')$.

Additionally, (2.4.4) can be written as

$$\int_0^T a(\mathbf{m},\mathbf{w})f(t)dt - \int_0^T ((R(h_\nabla))_t,\mathbf{w})_{\mathbf{V}',\mathbf{V}}f(t)dt = \int_0^T (\mathbf{z}_1,\mathbf{w})f(t)dt \tag{2.4.51}$$

However, since $\{\mathbf{w}f(t)|\mathbf{w}\in\mathbf{V},\ f\in C_0^\infty(0,T)\}$ is dense in $L^2(0,T;\mathbf{V})$, we can let $\mathbf{u}$ and p be test functions in (2.4.5) and (2.4.51) to obtain

$$\int_0^T a(\mathbf{m},\mathbf{u})dt - \int_0^T ((R(h_\nabla))_t,\mathbf{u})_{\mathbf{V}',\mathbf{V}}dt - \int_0^T (\nabla\cdot\mathbf{m},p)dt + \int_0^T (k\nabla h,\nabla p)dt$$
$$= \int_0^T (\mathbf{z}_1,\mathbf{u})dt + \int_0^T (z_2,p)dt.$$

Since $\mathbf{u}\in H^1(0,T;\mathbf{V}')$, we can integrate the second term by parts.

$$\int_0^T a(\mathbf{m},\mathbf{u})dt + \int_0^T (R(h_\nabla),\mathbf{u}_t)_{\mathbf{V},\mathbf{V}'}dt - (R(h_\nabla)(T),\mathbf{u}(T))_{\mathbf{L}^2(\Omega)} + (R(h_\nabla)(0),\mathbf{u}(0))_{\mathbf{L}^2(\Omega)}$$
$$-\int_0^T (\nabla\cdot\mathbf{m},p)dt + \int_0^T (k\nabla h,\nabla p)dt = \int_0^T (\mathbf{z}_1,\mathbf{u})dt + \int_0^T (z_2,p)dt \tag{2.4.52}$$

Recall that for all $\mathbf{w}\in\mathbf{V}$, $(h(T),\nabla\cdot\mathbf{w}) = (z_3,\nabla\cdot\mathbf{w})$. Since $z_3 = 0$, $(z_3,\nabla\cdot\mathbf{w}) = 0$. Hence, $R(h_\nabla)(T) = 0$. Also, we assumed $\mathbf{u}_0 = 0$. Therefore, the temporal boundary terms in (2.4.52) are equal to

0, and using the definition of $R(h_\nabla)$, we have

$$\begin{aligned}\int_0^T a(\mathbf{m},\mathbf{u})dt + \int_0^T (h,\nabla\cdot\mathbf{u}_t)dt - \int_0^T (\nabla\cdot\mathbf{m},p)dt + \int_0^T (k\nabla h,\nabla p)dt \\ = \int_0^T (\mathbf{z}_1,\mathbf{u})dt + \int_0^T (z_2,p)dt\end{aligned} \tag{2.4.53}$$

Let $(\mathbf{m},h)$ be the test functions used in the weak form of the poro-elastic system. We obtain

$$\begin{aligned}\int_0^T a(\mathbf{u},\mathbf{m})dt - \int_0^T (p,\nabla\cdot\mathbf{m})dt + \int_0^T (k\nabla p,\nabla h)dt + \int_0^T (\nabla\cdot\mathbf{u}_t,h)dt \\ = \int_0^T \langle g,\mathbf{m}\rangle_{\Gamma_N}dt + \int_0^T (F,\mathbf{m})dt + \int_0^T \langle\psi,h\rangle_{\Gamma_{D,\mathbf{v}}}dt + \int_0^T (S,h)dt.\end{aligned} \tag{2.4.54}$$

Combining (2.4.53) and (2.4.54), we obtain the desired equality.

$$\int_0^T \langle g,\mathbf{m}\rangle_{\Gamma_N}dt + \int_0^T (F,\mathbf{m})dt - \int_0^T \langle\psi,h\rangle_{\Gamma_{D,\mathbf{v}}}dt + \int_0^T (S,h)dt = \int_0^T (\mathbf{z}_1,\mathbf{u})dt + \int_0^T (z_2,p)dt$$

□

Lemma 2.4.9. *Let $(\mathbf{u},p)$ be the weak solution to the poro-elastic equation (1.2.1)-(1.2.4). If $p\in H^1(0,T;V)$, then $\mathbf{u}\in H^1(0,T;\mathbf{V})$.*

Proof. We know $(\mathbf{u},p)$ satisfies (1.3.7). Differentiating this equation, we have

$$\begin{cases}-\nabla\cdot\left(2\mu_e\epsilon(\mathbf{u_t})+\lambda_e(\nabla\cdot\mathbf{u_t})\mathbf{I}\right)=-\nabla p_t-\mathbf{F}_t & \text{on } \Omega\\ \mathbf{u}_t=\mathbf{0} & \text{on } \Gamma_D\\ \mathbf{T}_e(\mathbf{u}_t)\mathbf{n}=\mathbf{g}_t+p_t\mathbf{n} & \text{on } \Gamma_N\end{cases}$$

Since $\mathbf{g}\in H^1(0,T;\mathbf{L}^2(\Gamma_N))$, $\mathbf{F}\in H^1(0,T;\mathbf{L}^2(\Omega))$, and $p_t\in L^2(0,T;V)$, then by Lemma 1.3.1 we have $\mathbf{u}_t\in\mathbf{H}^1(\Omega)\cap\mathbf{V}$. □

First, we present the result on the first order necessary optimality conditions in the case when the control $q=S$ or $q=\psi$. We have the following theorem.

Theorem 2.4.10. *Let $\tilde{p} \in H^1(0,T;L^2(\Omega))$. If $\bar{\psi} \in L^2(0,T;L^2(\Gamma_{D.v}))$ is the optimal control and the associated state has the additional regularity $\bar{p} \in H^1(0,T;L^2(\Omega))$, then there exists a solution $(\mathbf{m},h)$ to the adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}} - \tilde{\mathbf{u}}$, $z_2 = \bar{p} - \tilde{p}$, $z_3 = 0$ which satisfies*

$$\int_0^T (\psi - \bar{\psi}, -h + \lambda\bar{\psi})_{L^2(0,T;L^2(\Gamma_{D,v}))} \geq 0 \ \ \forall \psi \in Q_{ad}. \tag{2.4.55}$$

Conversely, let $\bar{\psi}$ with associated state $(\bar{\mathbf{u}},\bar{p})$ such that $\nabla \cdot \mathbf{u}_t \in L^2(0,T;L^2(\Omega))$. If there exists a solution $(\mathbf{m},h)$ to the adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}} - \tilde{\mathbf{u}}$, $z_2 = \bar{p} - \tilde{p}$, $z_3 = 0$ and if (2.4.55) is satisfied, then $\bar{\psi}$ is a solution to the optimal control problem (2.0.2).

The result when S is used as the control is similar. However, there is a sign change since the sign in front of the ψ term and the S term in (2.4.50) is different.

Theorem 2.4.11. *Let $\tilde{p} \in H^1(0,T;L^2(\Omega))$. If $\bar{S} \in Q$ is the optimal control and the associated state has the additional regularity $\bar{p} \in H^1(0,T;L^2(\Omega))$, then there exists a solution $(\mathbf{m},h)$ to the adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}} - \tilde{\mathbf{u}}$, $z_2 = \bar{p} - \tilde{p}$, $z_3 = 0$ which satisfies*

$$\int_0^T (S - \bar{S}, h + \lambda\bar{S})_Q \geq 0 \ \ \forall S \in Q_{ad}. \tag{2.4.56}$$

Conversely, let $\bar{S}$ with associated state $(\bar{\mathbf{u}},\bar{p})$ such that $\nabla \cdot \mathbf{u}_t \in L^2(0,T;L^2(\Omega))$. If there exists a solution $(\mathbf{m},h)$ to the adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}} - \tilde{\mathbf{u}}$, $z_2 = \bar{p} - \tilde{p}$, $z_3 = 0$ and if (2.4.56) is satisfied, then $\bar{S}$ is a solution to the optimal control problem (2.0.2).

Proof. Let $\tilde{p} \in H^1(0,T;L^2(\Omega))$ and let $\bar{q} \in Q_{ad}$ be the optimal control. Assume $(\bar{\mathbf{u}},\bar{p}) \in L^2(0,T;\mathbf{V}) \times H^1(0,T;L^2(\Omega))$. Since $\bar{q}$ is optimal, (2.2.1) yields:

$$(G\bar{S} - \tilde{y}, GS - G\bar{S})_Y + (\lambda\bar{S}, S - \bar{S})_{L^2(0,T;L^2(\Omega))} \geq 0 \ \ \forall S \in Q_{ad},$$

which is equivalent to

$$\int_0^T (\bar{\mathbf{u}} - \tilde{\mathbf{u}}, \mathbf{u} - \bar{\mathbf{u}}) d\,t + \int_0^T (\bar{p} - \tilde{p}, p - \bar{p}) d\,t + (\lambda\bar{S}, S - \bar{S})_{L^2(0,T;L^2(\Omega))} \geq 0 \ \ \forall S \in Q_{ad}$$

Therefore,

$$\int_0^T (\mathbf{z}_1, \mathbf{u} - \bar{\mathbf{u}}) d\,t + \int_0^T (z_2, p - \bar{p}) d\,t + (\lambda\bar{S}, S - \bar{S})_Q \geq 0 \ \ \forall S \in Q_{ad} \tag{2.4.57}$$

Recall $G(S-\bar{S}) = (\mathbf{u}-\bar{\mathbf{u}}, p-\bar{p})$ corresponds to the solution of the poro-elastic system where all sources and initial conditions are zero except the control $S-\bar{S}$. Due to the regularity assumptions on $\tilde{p}$ and $\bar{p}$, there exists $(\mathbf{m}, h)$ satisfying the adjoint solution with $\mathbf{z}_1 = \bar{\mathbf{u}}-\tilde{\mathbf{u}}$, $z_2 = \bar{p}-\tilde{p}$ and $z_3 = 0$. Therefore, applying Lemmas 2.4.9 and 2.4.8, we obtain

$$(S-\bar{S}, h)_{L^2(0,T;L^2(\Omega))} = \int_0^T (\mathbf{z}_1, \mathbf{u}-\bar{\mathbf{u}}) + \int_0^T (z_2, p-\bar{p}) \ \ \forall S \in Q_{ad}$$

Combining this with inequality (2.4.57), we obtain (2.4.56).

Now assume $G(\bar{q}) = (\bar{\mathbf{u}}, \bar{p})$ and that there exists a solution $(\mathbf{m}, h)$ to the adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}}-\tilde{\mathbf{u}}$, $z_2 = \bar{p}-\tilde{p}$, $z_3 = 0$ that satisfies (2.4.56). Then using the equality we get from Lemma 2.4.8 and the definition of $\mathbf{z}_1$ and z_2, we have

$$(G\bar{q}-\tilde{y}, Gq-G\bar{q})_Y + (\lambda\bar{q}, q-\bar{q})_Q \geq 0 \ \ \forall q \in Q_{ad}$$

Therefore $\bar{q}$ is the optimal control. □

Now we focus on the case when the control q is either $\mathbf{F}$ or $\mathbf{g}$, where the control spaces have higher regularity in time. We have the following two results.

Theorem 2.4.12. *Let $\tilde{p} \in H^1(0,T;L^2(\Omega))$. If $\bar{\mathbf{F}} \in H^1(0,T;\mathbf{L}^2(\Omega))$ is the optimal control and the associated state has the additional regularity $\bar{p} \in H^1(0,T;L^2(\Omega))$, then there exists a solution $(\mathbf{m}, h)$ to the adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}}-\tilde{\mathbf{u}}$, $z_2 = \bar{p}-\tilde{p}$, $z_3 = 0$ which satisfies*

$$(\mathbf{F}-\bar{\mathbf{F}}, \mathbf{m})_{L^2(0,T;\mathbf{L}^2(\Omega))} + \lambda(\mathbf{F}-\bar{\mathbf{F}}, \bar{\mathbf{F}})_{H^1(0,T;\mathbf{L}^2(\Omega))} \geq 0 \ \ \forall \mathbf{F} \in Q_{ad}. \tag{2.4.58}$$

Conversely, let $\bar{\mathbf{F}}$ with associated state $(\bar{\mathbf{u}}, \bar{p})$ such that $\nabla \cdot \mathbf{u}_t \in L^2(0,T;L^2(\Omega))$. If there exists a solution $(\mathbf{m}, h)$ to the adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}}-\tilde{\mathbf{u}}$, $z_2 = \bar{p}-\tilde{p}$, $z_3 = 0$ and if (2.4.58) is satisfied, then $\bar{\mathbf{F}}$ is a solution to the optimal control problem (2.0.2).

Theorem 2.4.13. *Let $\tilde{p} \in H^1(0,T;L^2(\Omega))$. If $\bar{\mathbf{g}} \in H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))$ is the optimal control and the associated state has the additional regularity $\bar{p} \in H^1(0,T;L^2(\Omega))$, then there exists a solution $(\mathbf{m}, h)$ to the adjoint system (3.1.1) with $\mathbf{z}_1 = \bar{\mathbf{u}}-\tilde{\mathbf{u}}$, $z_2 = \bar{p}-\tilde{p}$, $z_3 = 0$ which satisfies*

$$(\mathbf{g}-\bar{\mathbf{g}}, \mathbf{m})_{L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))} + \lambda(\mathbf{g}-\bar{\mathbf{g}}, \bar{\mathbf{g}})_{H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))} \geq 0 \ \ \forall \mathbf{g} \in Q_{ad}. \tag{2.4.59}$$

Conversely, let $\bar{\mathbf{g}}$ *with associated state* $(\bar{\mathbf{u}}, \bar{p})$ *such that* $\nabla \cdot \mathbf{u}_t \in L^2(0,T;L^2(\Omega))$. *If there exists a solution* $(\mathbf{m}, h)$ *to the adjoint system (3.1.1) with* $\mathbf{z}_1 = \bar{\mathbf{u}} - \tilde{\mathbf{u}}$, $z_2 = \bar{p} - \tilde{p}$, $z_3 = 0$ *and if (2.4.59) is satisfied then* $\bar{\mathbf{g}}$ *is a solution to the optimal control problem (2.0.2).*

The proofs follow similarly to the proof of Theorem 2.4.11.

CHAPTER

3

NUMERICAL RESULTS

3.1 Approximating the Optimal Control

The Dual-Hybridized Finite Element Method was used to approximate a solution poro-elastic and poro-visco-elastic models in [10]. We used the same method to also approximate solutions to the poro-elastic adjoint equation and the poro-visco-elastic adjoint equation.

3.1.1 1-D Model

Recall the Adjoint Equation:

$$\begin{cases} \nabla\cdot\left[\mu_e\epsilon(\mathbf{m})+\lambda_e(\nabla\cdot\mathbf{m})\mathbf{I}-\delta\mu_v\epsilon(\mathbf{m}_t)-\delta\lambda_v(\nabla\cdot\mathbf{m}_t)\mathbf{I}\right]-\nabla h_t=-\mathbf{z}_1 & \Omega\times(0,T) \\ -\nabla\cdot\mathbf{m}-\nabla\cdot(k(\mathbf{x},t)\nabla h)=z_2 & \Omega\times(0,T) \\ -k(\mathbf{x},t)\nabla h\cdot n=0 & \text{on } (\Gamma_N\cup\Gamma_{D,v})\times(0,T) \\ (\sigma(\mathbf{m}_t)+\sigma(\mathbf{m})+h_t I)n=0 & \text{on } \Gamma_N\times(0,T) \\ h=0 & \text{on } \Gamma_{D,p}\times(0,T) \\ \mathbf{m}=0 & \text{on } \Gamma_D\times(0,T) \\ \nabla h(T)+\delta\nabla\cdot\sigma(\mathbf{m}(T))=\nabla z_3+\delta\nabla\cdot\sigma(\mathbf{z}_4) & \Omega \end{cases}$$

where $\epsilon(\mathbf{m})=(\nabla\mathbf{m}+\nabla\mathbf{m}^T)/2$. We will write this as a 1-D model with Lamé parameters set to unity, $\Omega=(x_{\text{start}},x_{\text{end}})$ with boundary $\Gamma_N\cup\Gamma_{D,p}\cup\Gamma_{D,v}=\partial\Omega=\{x_{\text{start}},x_{\text{end}}\}$. Similarly, the time domain will be $(t_{\text{start}},t_{\text{end}})$. The equations will be written as follows:

$$\frac{\partial\gamma}{\partial x}=-z_1 \quad \Omega\times(t_{\text{start}},t_{\text{end}}) \tag{3.1.1}$$

$$-\frac{\partial w}{\partial x}-\frac{\partial m}{\partial x}=z_2 \quad \Omega\times(t_{\text{start}},t_{\text{end}}) \tag{3.1.2}$$

$$\frac{\partial h(T)}{\partial x}+\delta\frac{\partial m}{\partial x}=\frac{\partial z_3}{\partial x} \quad \Omega \tag{3.1.3}$$

$$\gamma n=0 \quad wn=0 \quad \text{on } \Gamma_N\times(t_{\text{start}},t_{\text{end}}) \tag{3.1.4}$$

$$m=0 \quad h=0 \quad \text{on } \Gamma_{D,p}\times(t_{\text{start}},t_{\text{end}}) \tag{3.1.5}$$

$$m=0 \quad w=0 \quad \text{on } \Gamma_{D,v}\times(t_{\text{start}},t_{\text{end}}) \tag{3.1.6}$$

with constitutive equations, without normalizing the Lamé parameters, as follows:

$$\gamma=2\mu_e\frac{\partial m}{\partial x}-\rho-\frac{\partial}{\partial t}\left[\delta 2\mu_v\frac{\partial m}{\partial x}-\delta\frac{\lambda_v}{\lambda_e}\rho+h\right] \tag{3.1.7}$$

$$\frac{\rho}{\lambda_e}+\frac{\partial m}{\partial x}=0 \tag{3.1.8}$$

$$w=-k\frac{\partial h}{\partial x} \tag{3.1.9}$$

Note that our choice for γ and w simplifies the boundary conditions.

3.1.2 Temporal semi-discretization

We will approximate a solution depending only on the spatial variable on discrete time intervals $[t_{i-1}, t_i])$ for $i = 1, \ldots, r$ where r is a finite number and $\cup_{i=1\ldots r}[t_{i-1}, t_i) = [t_{\text{start}}, t_{\text{end}})$. For any function, we will let $\mathscr{W}^i := \lim_{t \to t_i -} \mathscr{W}(x,t)$. We seek to find a solution

$$(\gamma^{i-1}, w^{i-1}, \rho^{i-1}, m^{i-1}, h^{i-1}) \quad \text{when given } (w^i, \rho^i, m^i, h^i)$$

to the following equations:

$$\frac{\partial \gamma^{i-1}}{\partial x} = -z_1^{i-1} \tag{3.1.10}$$

$$-\frac{\partial w^{i-1}}{\partial x} + \frac{\rho^{i-1}}{\lambda_e} = z_2^{i-1} \tag{3.1.11}$$

$$\gamma^{i-1} = 2\mu_e \frac{\partial m^{i-1}}{\partial x} - \rho^{i-1} - \frac{\delta}{\Delta t}\left[2\mu_v \frac{\partial m^i}{\partial x} - 2\mu_v \frac{\partial m^{i-1}}{\partial x} - \frac{\lambda_v}{\lambda_e}\rho^i + \frac{\lambda_v}{\lambda_e}\rho^{i-1}\right] - \frac{1}{\Delta t}h^i + \frac{1}{\Delta t}h^{i-1} \tag{3.1.12}$$

$$\frac{\rho^{i-1}}{\lambda_e} + \frac{\partial m^{i-1}}{\partial x} = 0 \tag{3.1.13}$$

$$w^{i-1} = -k\frac{\partial h^{i-1}}{\partial x} \tag{3.1.14}$$

for all $x \in \Omega$ with the following boundary conditions:

$$\gamma^{i-1} n = 0 \quad w^{i-1} n = 0 \quad \text{on } \Gamma_N \tag{3.1.15}$$

$$m^{i-1} = 0 \quad h^{i-1} = 0 \quad \text{on } \Gamma_{D,p} \tag{3.1.16}$$

$$m^{i-1} = 0 \quad w^{i-1} n = 0 \quad \text{on } \Gamma_{D,v} \tag{3.1.17}$$

3.1.3 Dual Mixed Hybridized Finite Element Discretization's

We discretize $[x_{\text{start}}, x_{\text{end}}]$ into $n > 0$ intervals. Then $\eta := (x_{\text{end}} - x_{\text{start}})/n$. Let $x_k = x_{\text{start}} + \eta k$ and $K_k = (x_{k-1}, x_k)$ for all $k = 1, \ldots n$. Let $\mathbb{P}_0(X)$ represent the set of constant functions on X and $\mathbb{P}_1(X)$ represent the set of linear functions on X. Then we will use the following finite

element spaces:

$$U_\eta = \{u_\eta \in L^2(\Omega) \text{ such that } u_\eta|_{K_k} \in \mathbb{P}_0(K_k) \forall K_k,\ 1 \le k \le n\} \tag{3.1.18}$$

$$V_\eta = \{v_\eta \in L^2(\Omega) \text{ such that } v_\eta|_{K_k} \in \mathbb{P}_1(K_k) \forall K_k,\ 1 \le k \le n\} \tag{3.1.19}$$

$$M_\eta = \{\mu_\eta : \{x_k\}_{1 \le k \le n} \to \mathbb{R}\} \tag{3.1.20}$$

$$M^m_{\eta,0} = \{\mu_\eta \in M_\eta \text{ such that } \mu_\eta = 0 \text{ on } \Gamma_D\} \tag{3.1.21}$$

$$M^h_{\eta,0} = \{\mu_\eta \in M_\eta \text{ such that } \mu_\eta = 0 \text{ on } \Gamma_{D,p}\} \tag{3.1.22}$$

Now if we integrate (3.1.10)-(3.1.17) over Ω against test functions $\xi \in U_\eta$ and $\tau \in V_\eta$, we have the following:

$$\int_\Omega \frac{\partial \gamma^{i-1}}{\partial x} \xi \, d\Omega = -\int_\Omega z_1^{i-1} \, d\Omega \tag{3.1.23}$$

$$-\int_\Omega \frac{\partial w^{i-1}}{\partial x} \xi \, d\Omega + \int_\Omega \frac{\rho^{i-1}}{\lambda_e} \xi \, d\Omega = \int_\Omega z_2^{i-1} \xi \, d\Omega \tag{3.1.24}$$

$$\begin{aligned}\int_\Omega \gamma^{i-1} \tau \, d\Omega = c_m \int_\Omega \frac{\partial m^{i-1}}{\partial x} \tau \, d\Omega - c_\rho \int_\Omega \rho^{i-1} \tau \, d\Omega + \frac{\delta}{\Delta t} \int_\Omega 2\mu_\nu \frac{\partial m^i}{\partial x} \tau \, d\Omega - \frac{\delta}{\Delta t} \frac{\lambda_\nu}{\lambda_e} \int_\Omega \rho^i \tau \, d\Omega \\ - \frac{1}{\Delta t} \int_\Omega h^i \tau \, d\Omega + \frac{1}{\Delta t} \int_\Omega h^{i-1} \tau \, d\Omega\end{aligned} \tag{3.1.25}$$

$$\int_\Omega \frac{\rho^{i-1}}{\lambda_e} \tau \, d\Omega + \int_\Omega \frac{\partial m^{i-1}}{\partial x} \tau \, d\Omega = 0 \tag{3.1.26}$$

$$\int_\Omega w^{i-1} \tau \, d\Omega = -k \int_\Omega \frac{\partial h^{i-1}}{\partial x} \tau \, d\Omega \tag{3.1.27}$$

$$\int_{\Gamma_N} \gamma^{i-1} n \mu \, d\Gamma_N = 0 \quad \forall \mu \in M^m_{\eta,0} \tag{3.1.28}$$

$$\int_{\Gamma_N} w^{i-1} n \mu \, d\Gamma_N = 0 \quad \forall \mu \in M^h_{\eta,0} \tag{3.1.29}$$

Recall that u^{i-1} and h^{i-1} both belong to U_η. Therefore, their derivatives are zero a.e. Hence, to make sense of these terms we will define $\hat{u}_k^{i-1}$ and $\hat{h}_k^{i-1}$ for $k = 0, \dots n$ which will represent u and h on the spatial nodes. Then we will integrate by parts on K_k for $k = 1 \dots n$ and apply

(3.1.13) to (3.1.25). This yields the following:

$$\int_\Omega \gamma^{i-1}\tau\,d\Omega = -c_m\int_\Omega \frac{\partial\tau}{\partial x}m^{i-1}d\Omega + c_m\sum_{k=1}^{n}\int_{\partial K_k}\tau m^{i-1}n\,dx - c_\rho\int_\Omega \rho^{i-1}\tau\,d\Omega$$
$$+\frac{\delta}{\Delta t}\int_\Omega 2\mu_v\left(-\frac{\rho^i}{\lambda_e}\right)\tau\,d\Omega - \frac{\delta}{\Delta t}\frac{\lambda_v}{\lambda_e}\int_\Omega \rho^i\tau\,d\Omega - \frac{1}{\Delta t}\int_\Omega h^i\tau\,d\Omega + \frac{1}{\Delta t}\int_\Omega h^{i-1}\tau\,d\Omega \tag{3.1.30}$$

$$\int_\Omega \frac{\rho^{i-1}}{\lambda_e}\tau\,d\Omega - \int_\Omega \frac{\partial\tau}{\partial x}m^{i-1}d\Omega + \sum_{k=1}^{n}\int_{\partial K_k}\tau m^{i-1}n\,dx = 0 \tag{3.1.31}$$

$$\int_\Omega w^{i-1}\tau\,d\Omega = k\int_\Omega \frac{\partial\tau}{\partial x}h^{i-1}d\Omega - \sum_{k=1}^{n}\int_{\partial K_k}\tau h^{i-1}n\,dx \tag{3.1.32}$$

Alternatively, we could integrate (3.1.13) against $\xi\in U_\eta$. In this case, integrating by parts would yield:

$$\int_\Omega \frac{\rho^{i-1}}{\lambda_e}\xi\,d\Omega + \sum_{k=1}^{n}\int_{\partial K_k}\xi m^{i-1}n\,dx = 0. \tag{3.1.33}$$

We define the following bilinear forms:

$$A(J,\tau) = \sum_{k=1}^{n}\int_{K_k} J\tau\,dx \quad \forall (J,\tau)\in(V_\eta\times V_\eta)$$

$$B(q,J) = \sum_{k=1}^{n}\int_{K_k} q\frac{\partial J}{\partial x}dx \quad \forall (q,J)\in(U_\eta\times V_\eta)$$

$$C(\mu,J) = \sum_{k=1}^{n}\int_{\partial K_k} \mu J n_k\,ds \quad \forall (\mu,J)\in(M_\eta\times V_\eta)$$

$$D(q,\tau) = \sum_{k=1}^{n}\int_{K_k} q\tau\,dx \quad \forall (q,\tau)\in(U_\eta\times V_\eta)$$

$$G(\mu,\xi) = \sum_{k=1}^{n}\int_{\partial K_k} \xi\mu n_k\,ds \quad \forall (\xi,\mu)\in(U_\eta\times M_\eta).$$

Using these bilinear forms, we rewrite (3.1.23)-(3.1.24), (3.1.28)-(3.1.29) and (3.1.30)-(3.1.32), respectively,

$$B(\xi,\gamma^{i-1}) = -(z_1^{i-1},\xi) \quad \forall\xi\in U_\eta \tag{3.1.34}$$

$$-B(\xi, w^{i-1}) + \frac{1}{\lambda_e}(\rho^{i-1}, \xi)_{U_\eta} = (z_2^{i-1}, \xi) \quad \forall \xi \in U_\eta \tag{3.1.35}$$

$$C(\mu, \gamma) = 0 \quad \forall \mu \in M^m_{\eta,0} \tag{3.1.36}$$

$$C(\mu, w) = 0 \quad \forall \mu \in M^h_{\eta,0} \tag{3.1.37}$$

$$\begin{aligned} A(\gamma^{i-1}, \tau) = &- c_m B(m^{i-1}, \tau) + c_m \sum_{k=1}^{n} C(\hat{m}_k^{i-1}, \tau) - c_\rho D(\rho^{i-1}, \tau) - \frac{\delta}{\Delta t} \frac{2\mu_v + \lambda_v}{\lambda_e} D(\rho^i, \tau) \\ &- \frac{1}{\Delta t} D(h^i, \tau) + \frac{1}{\Delta t} D(h^{i-1}, \tau) \quad \forall \tau \in V_\eta \end{aligned} \tag{3.1.38}$$

$$A(w^{i-1}, \tau) - B(kh^{i-1}, \tau) + C(k\hat{h}^{i-1}, \tau) = 0 \quad \forall \tau \in V_\eta \tag{3.1.39}$$

where

$$c_m = 2\mu_e + \frac{\delta}{\Delta t} 2\mu_v \quad \text{and} \quad c_\rho = 1 + \frac{\delta \lambda_v}{\Delta t \lambda_e}.$$

Note that (3.1.36) and (3.1.37) represent the boundary conditions when γ and w are continuous. In this case, when we sum the boundary integrals on each spatial element as defined by C, the boundary conditions on all interior nodes of $(x_{\text{start}}, x_{\text{end}})$ will cancel out.

3.1.4 Static Condensation

We will now write these equations as a system of matrices and vectors. For variables in V_η, $\begin{bmatrix} a \\ b \end{bmatrix}$ represents $a\phi_1 + b\phi_2$ on K_k where ϕ_1 is a linear function that is 1 on x_{k-1} and 0 on x_k and ϕ_2 is a linear function that is 0 on x_{k-1} and 1 on x_k. For variables in U_η, variables can be thought of as a single constant on each interval. For variables in $M^m_{\eta,0}$ or $M^h_{\eta,0}$, variables can be represented by a 2×1 vector on each interval K_k since these functions are simply defined on the nodes.

Similarly, we will represent the bilinear forms as matrices where the entries correspond to the bilinear forms evaluated on the basis functions. More specifically, we will define the matrices as follows:

$$A := \begin{bmatrix} \int_{K_k} (\phi_1)^2 dx & \int_{K_k} \phi_1 \phi_2 dx \\ \int_{K_k} \phi_1 \phi_2 dx & \int_{K_k} (\phi_2)^2 dx \end{bmatrix}$$

$$B := \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$C := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D := \begin{bmatrix} \frac{\eta}{2} & \frac{\eta}{2} \end{bmatrix}$$

Following this same convention, we would have $G = \begin{bmatrix} -1 & 1 \end{bmatrix}$. However, because of how this matrix is used in (3.1.33), we define

$$G := \begin{bmatrix} -\frac{1}{\eta} & \frac{1}{\eta} \end{bmatrix}.$$

We have the following equations:

$$B\gamma_k^{i-1} = -\eta z_1^{i-1}(\bar{x}_k) \tag{3.1.40}$$

$$-Bw_k^{i-1} + \frac{\eta}{\lambda_e}\rho_k^{i-1} = \eta z_2^{i-1} \tag{3.1.41}$$

$$C\gamma = 0 \tag{3.1.42}$$

$$Cw = 0 \tag{3.1.43}$$

$$A\gamma_k^{i-1} + c_m B^T m_k^{i-1} - c_m C\hat{m}_k^{i-1} + c_\rho D\rho_k^{i-1} - \frac{1}{\Delta t} Dh_k^{i-1} = \frac{\delta}{\Delta t}\frac{2\mu_v + \lambda_v}{\lambda_e} D\rho_k^i - \frac{1}{\Delta t} Dh_k^i \tag{3.1.44}$$

$$Aw_k^{i-1} - kB^T h_k^{i-1} + kC\hat{h}_k^{i-1} = 0 \tag{3.1.45}$$

$$\rho_k^{i-1} = -\lambda_e G\hat{m}_k^{i-1} \tag{3.1.46}$$

3.1.5 Static Condensation

We will now eliminate all of the variables except $\hat{m}_k^{i-1}$ and $\hat{h}_k^{i-1}$. Reformulating (3.1.44), we see

$$\gamma_k^{i-1} = -c_m A^{-1} B^T m_k^{i-1} + c_m A^{-1} C\hat{m}_k^{i-1} - c_\rho A^{-1} D\rho_k^{i-1} + \frac{1}{\Delta t} A^{-1} Dh_k^{i-1} - \alpha_k^i \tag{3.1.47}$$

where $\alpha_k^i = -\frac{\delta}{\Delta t}\frac{2\mu_v + \lambda_v}{\lambda_e} A^{-1} D\rho_k^i + \frac{1}{\Delta t} A^{-1} Dh_k^i$. Applying (3.1.46), we have

$$\gamma_k^{i-1} = \mathscr{M}\hat{m}_k^{i-1} - c_m A^{-1} B^T m_k^{i-1} + \frac{1}{\Delta t} A^{-1} Dh_k^{i-1} - \alpha_k^i \tag{3.1.48}$$

where $\mathscr{M} = c_m A^{-1} C + \lambda_e c_\rho A^{-1} D G$. From (3.1.45), we have

$$w_k^{i-1} = -kA^{-1}\left[C\hat{h}_k^{i-1} - B^T h_k^{i-1}\right]. \tag{3.1.49}$$

We then substitute (3.1.46) and (3.1.49) into (3.1.41) which yields

$$kBA^{-1}\left[C\hat{h}_k^{i-1} - B^T h_k^{i-1}\right] - \eta G\hat{m}_k^{i-1} = \eta z_2^{i-1}(\bar{x}_k). \tag{3.1.50}$$

Defining the scalar $\mathscr{B}_p = kBA^{-1}B^T$ and rearranging shows

$$kBA^{-1}C\hat{h}_k^{i-1} - \eta G\hat{m}_k^{i-1} - \eta z_2^{i-1} = \mathscr{B}_p h_k^{i-1} \tag{3.1.51}$$

Therefore,

$$h_k^{i-1} = \mathscr{U}\hat{m}_k^{i-1} + \mathscr{V}\hat{h}_k^{i-1} + \mathscr{B}_p^{-1}\lambda_k^{i-1} \tag{3.1.52}$$

where $\mathscr{U} = -\mathscr{B}_p^{-1}\eta G$, $\mathscr{V} = \mathscr{B}_p^{-1}kBA^{-1}C$, and $\lambda_k^{i-1} = -\eta z_2^{i-1}(\bar{x}_k)$. Subbing (3.1.52) back into (3.1.49), we have

$$w_k^{i-1} = L_{hm}\hat{m}_k^{i-1} + L_{hh}\hat{h}_k^{i-1} + b_{h,k}^{i-1} \tag{3.1.53}$$

where

$$L_{hm} = kA^{-1}B^T\mathscr{U}, \quad L_{hh} = -kA^{-1}C + kA^{-1}B^T\mathscr{V}, \quad \text{and} \quad b_{h,k}^{i-1} = \mathscr{B}_p^{-1}kA^{-1}B^T\lambda_k^{i-1} \tag{3.1.54}$$

Plugging (3.1.48) and (3.1.52) into (3.1.40), we have

$$B\left[\mathscr{M}\hat{m}_k^{i-1} - c_m A^{-1}B^T m_k^{i-1} + \frac{1}{\Delta t}A^{-1}D(\mathscr{V}\hat{h}_k^{i-1} + \mathscr{U}\hat{m}_k^{i-1} + \mathscr{B}_p^{-1}\lambda^{i-1}) - \alpha_k^i\right] = -\eta z_1^{i-1}(\bar{x}_k) \tag{3.1.55}$$

which is equivalent to

$$\begin{aligned} m_k^{i-1} = {} & \mathscr{B}_u^{-1}B\mathscr{M}\hat{m}_k^{i-1} + \frac{\mathscr{B}_u^{-1}}{\Delta t}BA^{-1}D(\mathscr{V}\hat{h}_k^{i-1} + \mathscr{U}\hat{m}_k^{i-1} + \mathscr{B}_p^{-1}\lambda^{i-1}) - \mathscr{B}_u^{-1}B\alpha_k^i \\ & + \eta\mathscr{B}_u^{-1}z_1^{i-1}(\bar{x}_k) \end{aligned} \tag{3.1.56}$$

where $\mathscr{B}_u = c_m BA^{-1}B^T$. Hence, we can write

$$m_k^{i-1} = \tilde{\mathscr{U}}\hat{m}_k^{i-1} + \tilde{\mathscr{V}}\hat{h}_k^{i-1} + \mathscr{B}_u^{-1}\phi_k^{i-1} \tag{3.1.57}$$

where

$$\tilde{\mathscr{U}} = \mathscr{B}_u^{-1} B\mathscr{M} + \frac{\mathscr{B}_u^{-1}}{\Delta t} BA^{-1} D\mathscr{U}, \qquad \tilde{\mathscr{V}} = \frac{\mathscr{B}_u^{-1}}{\Delta t} BA^{-1} D\mathscr{V} \tag{3.1.58}$$

$$\phi_k^{i-1} = \frac{\mathscr{B}_p^{-1}}{\Delta t} BA^{-1} D\lambda^{i-1} - B\alpha_k^i + \beta_k^{i-1} \tag{3.1.59}$$

and $\beta_k^{i-1} = \eta z_1^{i-1}(\bar{x}_k)$. Therefore, from equation (3.1.48), (3.1.52), and (3.1.57) we have

$$\begin{aligned} \gamma_k^{i-1} = \mathscr{M}\hat{m}_k^{i-1} - c_m A^{-1} B^T (\tilde{\mathscr{U}}\hat{m}_k^{i-1} + \tilde{\mathscr{V}}\hat{h}_k^{i-1} + \mathscr{B}_u^{-1}\phi_k^{i-1}) \\ + \frac{1}{\Delta t} A^{-1} D(\mathscr{U}\hat{m}_k^{i-1} + \mathscr{V}\hat{h}_k^{i-1} + \mathscr{B}_p^{-1}\lambda_k^{i-1}) - \alpha_k^i \end{aligned} \tag{3.1.60}$$

Hence, we have

$$\gamma_k^{i-1} = L_{mm}\hat{m}_k^{i-1} + L_{mh}\hat{h}_k^{i-1} + b_{m,k}^{i-1} \tag{3.1.61}$$

where

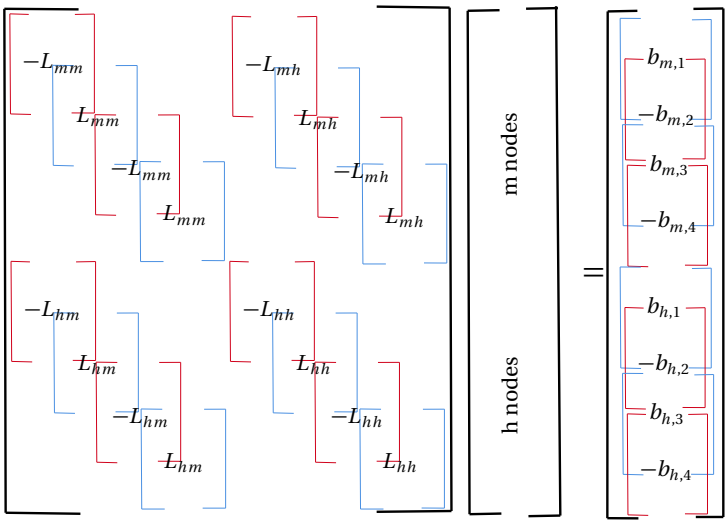
$$L_{mm} = \mathscr{M} - c_m A^{-1} B^T \tilde{\mathscr{U}} + \frac{1}{\Delta t} A^{-1} D\mathscr{U}, \quad L_{mh} = -c_m A^{-1} B^T \tilde{\mathscr{V}} + \frac{1}{\Delta t} A^{-1} D\mathscr{V}, \tag{3.1.62}$$

$$\text{and } b_{m,k}^{i-1} = -c_m A^{-1} B^T \mathscr{B}_u^{-1} \phi_k^{i-1} + \frac{1}{\Delta t} A^{-1} D\,\mathscr{B}_p^{-1}\lambda_k^{i-1} - \alpha_k^i. \tag{3.1.63}$$

3.1.6 The Algebraic System

We now invoke the fact that γ and w should be continuous over the whole time interval to develop and algebraic system. To ensure continuity, we require the second entry of γ_k^i to equal the first entry of γ_{k+1}^i and the second entry of w_k^i to equal the first entry of w_{k+1}^i for all $i = 1,\ldots,r$ and $k = 1,\ldots n-1$. From the above relations, we see that implies the second entry of $L_{mm}\hat{m}_k^{i-1} + L_{mh}\hat{h}_k^{i-1} + b_{m,k}^{i-1}$ must be equal to the first entry of $L_{mm}\hat{m}_{k+1}^{i-1} + L_{mh}\hat{h}_{k+1}^{i-1} + b_{m,(k+1)}^{i-1}$ and the second entry of $L_{hm}\hat{m}_k^{i-1} + L_{hh}\hat{h}_k^{i-1} + b_{h,k}^{i-1}$ must be equal to the first entry of $L_{hm}\hat{m}_{k+1}^{i-1} + L_{hh}\hat{h}_{k+1}^{i-1} + b_{h,(k+1)}^{i-1}$.

Consider the case where $n = 3$, then this information yields the system:



Finally, we change the entries of this system in the first row, middle two rows, and last row. For example, to impose the Dirichlet boundary conditions at the left-hand side of the boundary, we change the first row of the M to all zeros except the first entry. Additionally, we change the $n+2$ row to all zeros except the $n+2$nd entry. Then we will change the first entry of the vector b to zero and the $n+2$nd entry of the vector b to zero. Then we will solve this system for $\hat{m}$ and $\hat{h}$ and use these to solve for m, h, w, and γ.

3.1.7 Test Case of Adjoint Solver

Let us consider the case where $\Omega = (0, L)$, the time interval is $(0, T)$, $h = 0$, $m = 0$ at x_{start} and $\gamma n = 0$ $w n = 0$ at x_{end}. Finally, $m(x, T) = 0$ and $h(x, T) = 0$. Let $H_A = 2\mu_e + \lambda_e$, $H_v = 2\mu_v + \lambda_v$, and

$$z_1 := -H_A M_{ref} \chi''(x)\tau(t) + \delta H_v M_{ref} \chi''(x)\tau'(t) + H_{ref} \chi'(x)\tau'(t)$$

$$z_2 := kH_{ref} \chi''(x)\tau(t) - M_{ref} \chi'(x)\tau(t).$$

Then we have the following solution:

$$m = M_{ref} \chi(x)\tau(t), \quad h = H_{ref} \chi(x)\tau(t)$$

$$\rho = -\lambda_e M_{ref} \chi'(x)\tau(t), \quad w = -kH_{ref} \chi'(x)\tau(t),$$

and $\gamma = 2\mu_e M_{ref}\chi'(x)\tau(t) + \lambda_e M_{ref}\chi'(x)\tau(t) - \delta 2\mu_v M_{ref}\chi'(x)\tau'(t) - \delta\lambda_e M_{ref}\chi'(x)\tau'(t)$
$- H_{ref}\chi(x)\tau'(t)$

To ensure that the boundary conditions are met we need $\chi(L) = \chi'(L) = \chi(0) = 0$. To ensure the initial condition is met we need $\tau(T) = 0$. More specifically we will study the case where $T = 1$, $L = 1$, $\lambda_e = 1$, $\mu_e = 1$, $\lambda_v = 0.5$, $\mu_v = 0.5$, $M_{ref} = 0.1$, $H_{ref} = 0.1$, $\tau(t) = \sin^2(2\pi t)$, and $\chi(x) = x^2(1-x)^2$. Figure 3.1 shows the convergence rate for this test case.

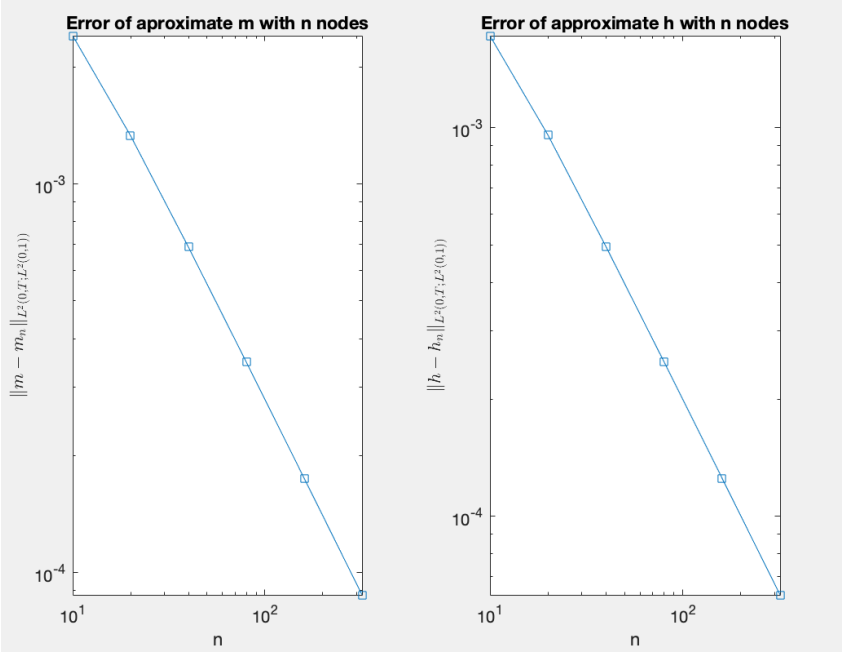
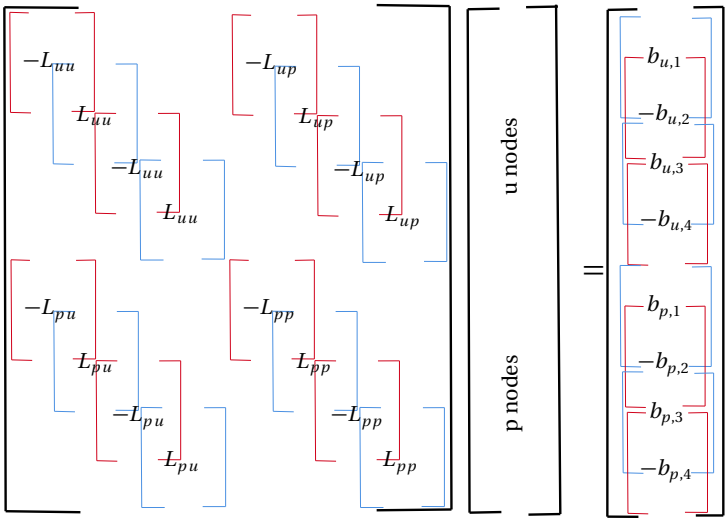


Figure 3.1: The $L^2(0,T;L^2(\Omega))$ error is shown for the approximate solutions (m_n, h_n) when n temporal nodes and n spatial nodes are used.

3.1.8 Solving for Derivative of Finite Dimensional Problem

Using the adjoint system to find the derivative of the finite dimensional problem is inaccurate. Instead, the derivative should be found directly. Using Dual-Hybridized Finite Element Method to approximate a solution to the poro-elastic or poro-visco-elastic model results in a linear system to solve for the next time step. We will let $y^i = [\hat{u}_1^i \dots \hat{u}_{n+1}^i \hat{p}_1^i \dots \hat{p}_{n+1}^i]^T$. Given y^i, we solve for y^{i+1} by solving $M y^{i+1} = b$ where M and b are derived from box 2×2

matrices and 2×1 vectors as follows:



Then the first row, middle two rows, and last row of the system are changed to reflect the spatial boundary conditions. Each sub-vector in b is dependent on y^i and the sources F and S. We will now separate these dependencies. We note that

$$b_{u,i} = r^i - m_u A^{-1} B^T \mathscr{B}_u^{-1} f^{i+1} - A^{-1} D \mathscr{B}_p^{-1} l^{i+1}$$

Note that r^i is not dependent on the sources. We will split l^{i+1} into l_q^{i+1} and l_u^{i+1} such that

$$l_q^{i+1} = hS^{i+1}(\bar{x}_k), \quad l_u^{i+1} = -\frac{h_k}{\lambda_e \Delta t} \mathscr{P}^i, \quad \text{and } l^{i+1} = l_q^{i+1} + l_u^{i+1}.$$

Then we split f^{i+1} such that

$$f_q^{i+1} = b^{i+1} - BA^{-1} D \mathscr{B}_p^{-1} l_q^{i+1}, \quad f_u^{i+1} = Br^i - BA^{-1} D \mathscr{B}_p^{-1} l_u^{i+1}, \quad \text{and } f^{i+1} = f_q^{i+1} + f_u^{i+1}.$$

Hence,

$$b_{u,i}^q = -m_u A^{-1} B^T \mathscr{B}_u^{-1} f_q^{i+1} - A^{-1} D \mathscr{B}_p^{-1} l_q^{i+1}, \quad b_{u,i}^u = r^i - m_u A^{-1} B^T \mathscr{B}_u^{-1} f_u^{i+1} - A^{-1} D \mathscr{B}_p^{-1} l_u^{i+1}$$

and $b_{u,i} = b^q_{u,i} + b^u_{u,i}$. Similarly,

$$b^q_{p,i} = kA^{-1}B^T\mathscr{B}_p^{-1}l^{i+1}_q, \quad b^u_{p,i} = kA^{-1}B^T\mathscr{B}_p^{-1}l^{i+1}_u, \quad \text{and } b_{p,i} = b^q_{p,i} + b^u_{p,i}.$$

Therefore, we let

$$\begin{aligned}\beta_u =& -\delta\frac{H_V}{\Delta t}A^{-1}DG - m_uA^{-1}B^T\mathscr{B}_u^{-1}B\delta\frac{H_V}{\Delta t}A^{-1}DG \\ &+ m_uA^{-1}B^T\mathscr{B}_u^{-1}BA^{-1}D\mathscr{B}_p^{-1}\frac{h}{\Delta t}G - A^{-1}D\mathscr{B}_p^{-1}\frac{h}{\Delta t}G\end{aligned} \tag{3.1.64}$$

and

$$\beta_p = kA^{-1}B^T\mathscr{B}_p^{-1}\frac{h}{\Delta t}G.$$

Additionally, we let

$$\beta_{u,F} = -m_uA^{-1}B^T\mathscr{B}_u^{-1}\begin{bmatrix}h/2 & h/2\end{bmatrix}$$

$$\beta_{u,S} = m_uA^{-1}B^T\mathscr{B}_u^{-1}BA^{-1}D\mathscr{B}_p^{-1}\begin{bmatrix}h/2 & h/2\end{bmatrix} - A^{-1}D\mathscr{B}_p^{-1}[h/2 h/2]$$

$$\beta_{p,S} = kA^{-1}B^T\mathscr{B}_p^{-1}\begin{bmatrix}h/2 & h/2\end{bmatrix}.$$

Therefore, we have the following equality:

$$\begin{bmatrix}b_{u,1}\\ b_{u,2}\\ b_{u,3}\\ b_{u,4}\\ b_{p,1}\\ b_{p,2}\\ b_{p,2}\\ b_{p,4}\end{bmatrix} = \begin{bmatrix}\beta_u & & & 0\\ \beta_u & \beta_u & & \\ \beta_u & \beta_u & \beta_u & \\ \beta_u & & & \\ \beta_p & & & 0\\ \beta_p & \beta_p & & \\ \beta_p & \beta_p & \beta_p & \\ \beta_p & & & \end{bmatrix}\begin{bmatrix}\\ \\ \\ \\ y^i\\ \\ \\ \\ \end{bmatrix} + \begin{bmatrix}\beta_{u,S} & & & \\ \beta_{u,S} & \beta_{u,S} & & \\ \beta_{u,S} & \beta_{u,S} & \beta_{u,S} & \\ \beta_{u,S} & & & \\ \beta_{p,S} & & & \\ \beta_{p,S} & \beta_{p,S} & & \\ \beta_{p,S} & \beta_{p,S} & \beta_{p,S} & \\ \beta_{p,S} & & & \end{bmatrix}\begin{bmatrix}S(x_0)\\ S(x_1)\\ \\ \vdots\\ \\ S(x_n)\end{bmatrix} + \begin{bmatrix}\beta_{u,F} & & & \\ \beta_{u,F} & \beta_{u,F} & & \\ \beta_{u,F} & \beta_{u,F} & \beta_{u,F} & \\ \beta_{u,F} & & & \\ & & & 0\\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix}F(x_0)\\ F(x_1)\\ \\ \vdots\\ \\ F(x_n)\end{bmatrix}$$

Hence, when given y^i we can solve for y^{i+1} by solving

$$My^{i+1} = \mathscr{B}y^i + \mathscr{B}_S \begin{bmatrix} S(x_0) \\ \vdots \\ S(x_n) \end{bmatrix} + \mathscr{B}_F \begin{bmatrix} F(x_0) \\ \vdots \\ F(x_n) \end{bmatrix} \tag{3.1.65}$$

3.1.9 Stationary Case

Considering the stationary case where F is used as the control, we have

$$My = \mathscr{B}_F F_{vec}.$$

We calculate the cost by calculating

$$J = \frac{1}{2}(y-y_d)^T(Trap_{double})(y-y_d) + \frac{\lambda}{2}F_{vec}^T(Trap)F_{vec}$$

where $Trap_{double}$ is a $2(n+1)\times 2(n+1)$ diagonal matrix with h on the diagonal, but $h/2$ in the first entry, the middle two entries and the last entry. Therefore, $y^T(Trap_{double})y$ calculates the integral of u^2 over $[0,L]$ and adding it to the integral of p^2 over $[0,L]$. Also, $Trap$ is a diagonal $(n+1)\times(n+1)$ matrix with h on the diagonal. Hence, $F_{vec}^T(Trap)F_{vec}$ calculates the integral of F^2 over $[0,L]$ using the trapezoidal method. We calculate

$$J'(F_{vec}) = \mathscr{B}_F^T * M^{-T} * Trap_{double}(M^{-1} * \mathscr{B}_F F_{vec} - y_d) + \lambda \text{Trap} F_{vec} = \mathscr{B}_F^T z + \lambda \text{Trap} F_{vec}$$

where $z = M^{-T} * Trap_{double}(M^{-1} * \mathscr{B}_F F_{vec} - y_d)$. In other words,

$$M^T z = Trap_{double}(M^{-1} * \mathscr{B}_F F_{vec} - y_d) = Trap_{double}(y - y_d)$$

3.1.10 Non-stationary Case

Let $W = M^{-1}\mathscr{B}$. Let y be a block vector where the ith block is y^i. Hence, y is a matrix that is the length of the number of spatial nodes plus one multiplied by the number of temporal

nodes. Then we can use (3.1.65) set up a system to solve for y.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} W \\ W^2 \\ \vdots \\ W^n \end{bmatrix} \begin{bmatrix} \\ y_0 \end{bmatrix} + \begin{bmatrix} M^{-1}\mathscr{B}_S & 0 & \dots \\ WM^{-1}\mathscr{B}_S & \ddots & \\ W^2M^{-1}\mathscr{B}_S & \ddots & \\ \vdots & \ddots & \\ W^nM^{-1}\mathscr{B}_S & & \end{bmatrix} \begin{bmatrix} S^{(1)} \\ \vdots \\ S^{(n)} \end{bmatrix} \tag{3.1.66}$$

For the optimal control problem, we set the initial condition equal to zero. Therefore, we have that the $y = Gq$ where q is the source. Therefore, we calculate

$$J'(q) = G^T * Trap_{double}(Gq - y_d) + \lambda Trapq = z + \lambda Trapq$$

where $z = G^T * Trap_{double}(y - y_d)$.

We can solve for z by solving for each time-step individually moving backwards in time. First, let $d = Trap_{double}(y - y_d)$. Then we will call $d^{(i)}$ the i^{th} block of d. Looking at what the transpose of G is, we can see $z^{(n)} = \mathscr{B}_S^T \bar{z}^{(n)}$ where $\bar{z}^{(n)} = M^{-T} d^{(n)}$. Then $z^{(i-1)} = \mathscr{B}_S^T \bar{z}^{(i-1)}$ where $\bar{z}^{(i-1)} = M^{-T}(d^{(i-1)} + \mathscr{B}^T \bar{z}^i)$.

When k is a function of x and t, $\mathscr{B}_S$ and $\mathscr{B}$ change at each time step. Therefore, we have the following system for y

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} M^{-1}\mathscr{B}_1 \\ M^{-1}\mathscr{B}_2M^{-1}\mathscr{B}_1 \\ \vdots \\ M^{-1}\mathscr{B}_nM^{-1}\mathscr{B}_{n-1}\dots M^{-1}\mathscr{B}_1 \end{bmatrix} \begin{bmatrix} \\ y_0 \end{bmatrix} + G \begin{bmatrix} S^{(1)} \\ \vdots \\ S^{(n)} \end{bmatrix} \tag{3.1.67}$$

where

$$G = \begin{bmatrix} M^{-1}\mathscr{B}_{S_1} & 0 & \dots & \\ M^{-1}\mathscr{B}_2M^{-1}\mathscr{B}_{S_1} & M^{-1}\mathscr{B}_{S_2} & 0 & \\ M^{-1}\mathscr{B}_3M^{-1}\mathscr{B}_2M^{-1}\mathscr{B}_{S_1} & M^{-1}\mathscr{B}_3M^{-1}\mathscr{B}_{S_2} & & \\ \vdots & \vdots & \ddots & \\ M^{-1}\mathscr{B}_nM^{-1}\mathscr{B}_{n-1}\dots M^{-1}\mathscr{B}_2M^{-1}\mathscr{B}_{S_1} & M^{-1}\mathscr{B}_nM^{-1}\mathscr{B}_{n-1}\dots\mathscr{B}_3M^{-1}\mathscr{B}_{S_2} & \dots & M^{-1}\mathscr{B}_{S_n} \end{bmatrix}.$$

Hence, we can solve for $z = G^T d$ by letting $z^{(n)} = \mathscr{B}_{S_n}^T \bar{z}^{(n)}$ where $\bar{z}^{(n)} = M^{-T} d^{(n)}$. Then

$z^{i-1} = \mathscr{B}_{S_{i-1}}^T \bar{z}_{i-1}$ where $\bar{z}_{i-1} = M^{-T}\mathscr{B}_i^T \bar{z}^{(i)} + M^{-T} d^{(i-1)}$.

3.1.11 Quasi-Newton Method

To approximate an optimal control, we used the fminunc function in MATLAB. This function is a quasi-Newton method. Newton's method finds the next iterate q_{n+1} by $q_n - (\nabla J(q_n))^{-1} J(q_n)$. Quasi-Newton methods use an approximation of $(\nabla J(q_n))^{-1}$.

Here we provide results for a stationary case. We will consider the case where $g = -0.3$, $\psi = -3$, $S = 0.3$, $u = p = 0$ at x_{start}, and $v = \sigma = 0$ at x_{end} where $x_{start} = 0$,and $x_{end} = 1$. Additionally, we set $k_{ref} = 1$, $\lambda_e = 1$ and $\mu_e = 1$. We will set $u_d = p_d = 1$ over the interval $[0,1]$. Then we use Gradient Descent to approximate the optimal control that minimizes the cost functional with $\lambda = 0.1$. The initial F is taken to be constantly 0.

The results are shown in Figures 3.2-3.4. We see that the cost functional seems to be minimized around 1.89. We can also see that the gradient of the cost functional is converging to zero. We additionally see that the shape of F, u, and p do not change significantly after the first step. We see some changes in the magnitude of the functions, with the most significant changes occurring in F.

Test Case 2 We also will show the results of a case non-stationary case. As before, $\Omega = (x_{start}, x_{end})$, and now additionally $t \in [0, T]$. We will minimize the cost functional

$$J(\mathbf{F}) = \frac{1}{2}\|\mathbf{u}_{\bar{F}} + \tilde{\mathbf{u}} - \mathbf{u}_d\|^2 + \frac{1}{2}\|p_{\bar{q}} + \tilde{p} - p_d\|^2 + 10^{-4}\|\mathbf{F}\|^2$$

where $\tilde{\mathbf{u}}$, $\tilde{p}$ is the solution to the PVE system with $\delta = 1$, $\mathbf{u} = p = 0$ when $x = 0$, and $\mathbf{Tn} = \mathbf{g}$ and $\nabla p = \psi$ at $x = 2$. We have set the non-control sources and initial condition to be

$$\begin{aligned}
&S(x,t) = 320\cos(80t)\cos(4x) + 160\sin(80t)\sin(4x)\\
&\psi(t) = -40\sin(80t)\cos(4x)\\
&\mathbf{g}(t) = 3\cos(8)\sin(80t) + 240\cos(8)\cos(80t) - 10\cos(8)\sin(80t)\\
&\mathbf{u}_0 = 0\\
&\mathbf{u}_d = \sin(80t)\sin(4x)\\
&p_d = 10\sin(80t)\sin(4x)
\end{aligned}$$

In Figure 3.5, we see the cost-functional decreasing as expected using the conjugate gradient method. After 4 iterations, the cost drops to 6.17067. The algorithm converges

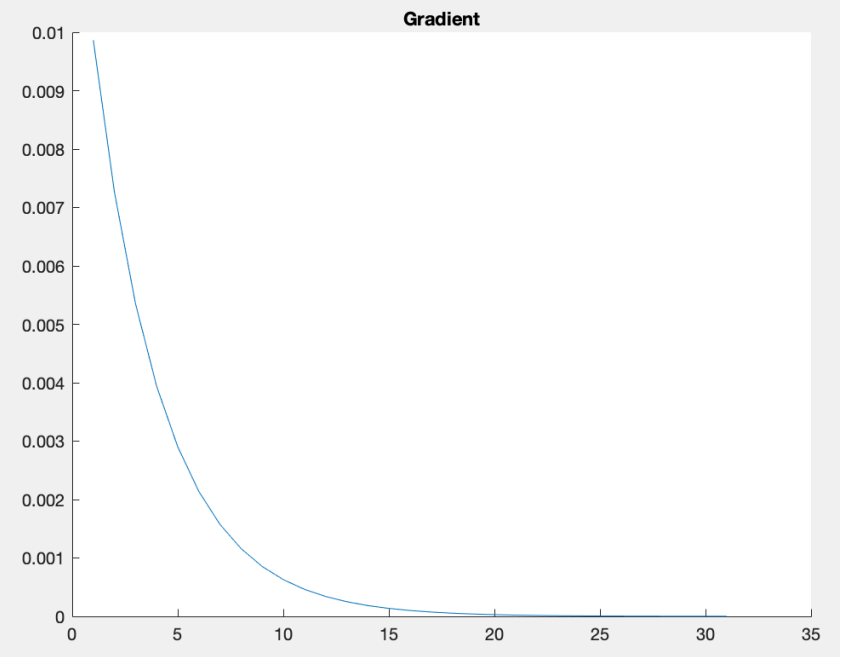


Figure 3.2: Over 31 iterations of gradient descent, we see the gradient of the cost function converging to zero.

to a cost around 2.40877, and we stop the algorithm after 96 iterations. We note that if $\mathbf{F} = 48\sin(4y)\sin(80t) + 3,840\sin(4y)\cos(80t) - 40\cos(4y)\sin(80t)$ (shown in Figure 3.6), then the desired $\mathbf{u}$ and desired p would be achieved. However, the L^2 norm of this function is quite high. Therefore, the cost-function is minimized with the $\mathbf{F}$ shown in Figure 3.7. We see that since most of the the local maximums and minimums are less extreme in the optimal $\mathbf{F}$, the L^2 norm would be much lower. We then look at the $\mathbf{u}$ produced by the optimal $\mathbf{F}$ on the left side of Figure 3.8. We see that the local maximum and minimums of $\mathbf{u}$ also are not as extreme as the desired $\mathbf{u}$, $\mathbf{u}_d$, shown on the right side of Figure 3.8. In Figure 3.9, we see that the p produced by the optimal $\mathbf{F}$ is exactly what was desired.

Remark 3.1.1. When running similar tests with other sources used as the control, both $\mathbf{u}$ and p are not achieved as desired.

This example demonstrates how the regularization term (the additional cost of the control) in the cost function can affect which control is defined as optimal. In the next section, we consider using a total variation term in addition or instead of the L^2 regularization term.

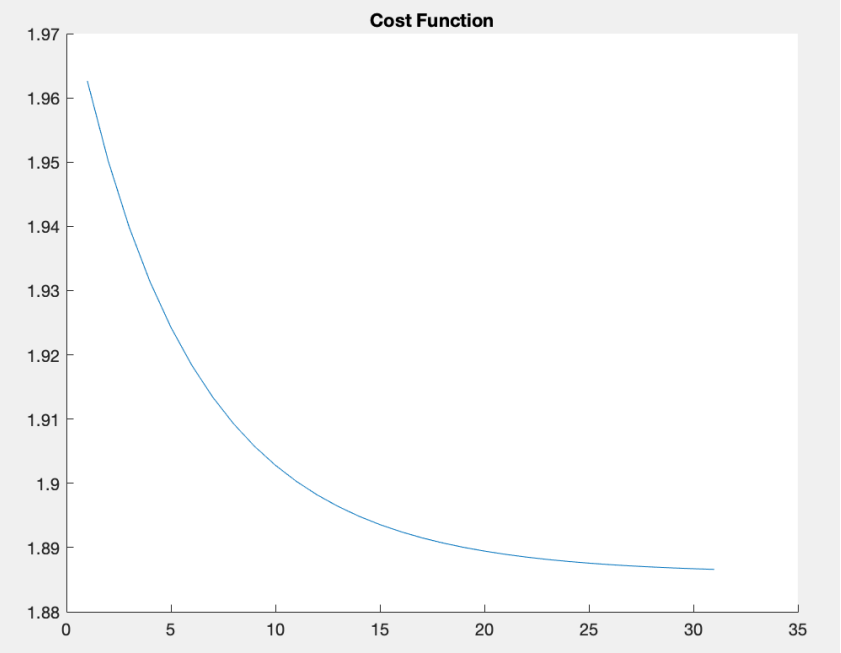


Figure 3.3: Over 31 iterations of gradient descent, we see the value of the cost functional decreasing and converging around 1.89.

3.2 SLIP Algorithm Applied to Poro-elastic and Poro-visco-elastic systems

3.2.1 Problem Statement

Let $\alpha > 0$ and $T > 0$. We are interested in solving the following optimization problem

$$\begin{aligned} &\min_{w \in L^2(0,T)} \; j(w) + \alpha \, \mathrm{TV}(w) \\ &\text{s.t.} \qquad w(t) \in W \subset \mathbb{Z} \text{ for almost all (a.a.) } t \in (0,T) \end{aligned} \tag{P}$$

where j is lower semi-continuous and bounded from below. Here TV represents the total variation. Total variation is defined for all functions in $L^1(0,T)$ as follows:

$$\mathrm{TV}(f) \coloneqq \sup \left\{ \int_0^T f(t)\phi'(t)\,\mathrm{d}t \;\middle|\; \phi \in C_c^1(0,T) \text{ and } \sup_{s \in (0,T)} |\phi(s)| \leq 1 \right\}.$$

Since $L^2(0,T) \subset L^1(0,T)$, we can define the total variation of all $w \in L^2(0,T)$. Note that this definition is equivalent to the following:

$$\mathrm{TV}(f) = \sup_{\mathscr{P}} \sum_{i=1}^{n_P} |f(x_i) - f(x_{i-1})|$$

where $\mathscr{P} = \{P = \{x_0, x_1, \ldots, x_{n_p}\}\}$ is the set of all partitions on $[0,T]$. Since the solution of (P) is integer-valued, $\mathrm{TV}(w)$ is only finite when w is piecewise constant functions with a finite number of jumps. We let $\mathrm{BV}(0,T)$ denote the set of $L^1(0,T)$ functions with finite total variation. A sequence $\{w_n\}_{n\geq 1} \subset \mathrm{BV}(0,T)$ is said to converge weakly-* to a function $w \in \mathrm{BV}(0,T)$ if $w^n \to w$ in $L^1(0,T)$ and $\limsup_{n\to\infty} \mathrm{TV}(w^n) < \infty$. Moreover, $\{w^n\}_{n\geq 1}$ is said to converge *strictly* to w if in addition $\mathrm{TV}(w^n) \to \mathrm{TV}(w)$. We define the subset of $\mathrm{BV}(0,T)$ that corresponds to the feasible set of the optimization problem (P) as

$$\mathrm{BV}_{\mathrm{W}}(0,T) = \{w \in \mathrm{BV}(0,T) \mid w(t) \in W\}.$$

3.2.2 Sequential Linear Integer Programming Algorithm

The goal of the SLIP algorithm is the find solutions v that are r-optimal.

Definition 10. *We say that $v \in \mathrm{BV}_{\mathrm{W}}(0,T)$ is r-optimal if*

$$j(v) + \alpha\,\mathrm{TV}(v) \leq j(w) + \alpha\,\mathrm{TV}(w) \quad \forall\, w \in \mathrm{BV}_{\mathrm{W}}(0,T) \text{ such that } \|v - w\|_{L^1(0,T)} \leq r.$$

The user of the algorithm starts by choosing a radius r, an initial guess w_0 and a stopping criteria parameter σ. The algorithm then solves a trust-region subproblem given by

$$\begin{aligned} \min_{d \in L^2(0,T)} \; & (\nabla j(w_0), d)_{L^2(0,T)} + \alpha\,\mathrm{TV}(w_0 + d) - \alpha\,\mathrm{TV}(w_0) =: \ell(w,d) \\ \text{s.t.} \quad & w_0(s) + d(s) \in W \text{ for a.a. } s \in (0,T), \\ & \|d\|_{L^1(0,T)} \leq r. \end{aligned} \tag{TR}$$

This trust-region subproblem can be solved since $\nabla j(w)$ is a linear functional. Note that ∇j is the Frechet derivative. Hence, $j(w+d) - j(w) - r(w,d) = (\nabla j(w), d)$ where $\frac{\|r(w,d)\|}{\|d\|} \to 0$ as $\|d\| \to 0$. If $\ell(w,d) = 0$, then 0 is a solution to the trust-region subproblem and the algorithm terminates with the prediction that w_0 solves (P). Otherwise, if $d \neq 0$ solves the trust-region subproblem, the algorithm decides if it should rerun the trust-region

subproblem around w_0+d. This decision is based on the ratio between the actual reduction ($j(w_0)+\alpha\,\mathrm{TV}(w_0)-j(w_0+d)-\alpha\,\mathrm{TV}(w_0+d)$) and the reduction $\ell(w,d)$ predicts. If the ratio is small then it is likely that the step taken is too large. Hence, the solution to the trust-region subproblem should not be used in the next step of the algorithm. Instead, the radius should be decreased, and the trust-region should be run again with a smaller radius.

Algorithm 1 Trust-region algorithm from [35]

Input: Initial guess w^0 (feasible for (P)), $\Delta^0 > 0$, $\sigma \in (0,1)$

for $n = 1, \ldots$ **do**
  $k \leftarrow 0$
  $\Delta^{n,0} \leftarrow \Delta^0$
  repeat
    $d^{n,k} \leftarrow$ minimizer of $\mathrm{TR}(w^{n-1}, \Delta^{n,k})$ ▷ Solve trust-region subproblem.
    if $\ell(w^{n-1}, d) = 0$ **then** ▷ Predicted reduction is zero ⇒ terminate.
      Terminate with solution w^{n-1}.
    else if $\frac{j(w^{n-1})+\alpha\,\mathrm{TV}(w^{n-1})-j(w^{n-1}+d^{n,k})-\alpha\,\mathrm{TV}(w^{n-1}+d^{n,k})}{\ell(w^{n-1},d^{n,k})} < \sigma$ **then** ▷ Reject step.
      $\Delta^{n,k+1} \leftarrow \Delta^{n,k}/2$
    else ▷ Accept step.
      $w^n \leftarrow w^{n-1} + d^{n,k}$
    end if
    $k \leftarrow k+1$
  until $\frac{j(w^{n-1})+\alpha\,\mathrm{TV}(w^{n-1})-j(w^{n-1}+d^{n,k})-\alpha\,\mathrm{TV}(w^{n-1}+d^{n,k})}{\ell(w^{n-1},d^{n,k})} \geq \sigma$
end for

The main known convergence result on Algorithm 1 to this point is that its iterates converge to so-called L-stationary points under a suitable regularity assumption on j [35]. A feasible point w is L-stationary if the objective (P) cannot be improved further by perturbing the locations of its jumps on $(0,T)$. Such perturbations leave the term $\alpha\,\mathrm{TV}(w)$ unchanged and only affect $j(w)$, yielding a condition on $\nabla j(w)$. In particular, the condition coincides with $\nabla j(w)(t_i) = 0$ for all jump locations t_i of w if $\nabla j(w)$ is a continuous function.

Assumption 3.2.1 (Assumption 4.1 in [35])**.** *Let $j : L^2(0,T) \to \mathbb{R}$ be twice Fréchet differen-*

tiable such that for all $w \in L^2(0,T)$

$$|\nabla^2 j(w)(\psi,\phi)| \leq C\|\psi\|_{L^1(0,T)}\|\phi\|_{L^1(0,T)}$$

holds for some $C>0$ *and all* $\psi,\phi \in L^2(0,T)$.

3.2.3 Input Regularization for SLIP Algorithm

Assumption 3.2.1 is at times too restrictive. The class of problems that SLIP can be applied to would be much larger if the assumption was replaced with the following:

Assumption 3.2.2. *Let* $j : L^2(0,T) \to \mathbb{R}$ *be twice Fréchet differentiable such that for all* $w \in L^2(0,T)$

$$|\nabla^2 j(w)(\psi,\phi)| \leq C\|\psi\|_{L^2(0,T)}\|\phi\|_{L^2(0,T)}$$

holds for some $C>0$ *and all* $\psi,\phi \in L^2(0,T)$.

When j satisfies Assumption 3.2.2, j composed with a convolution operator will satisfy Assumption 3.2.1 [11]. Let $K_\epsilon : L^1(0,T) \to L^2(0,T)$ be a bounded and linear convolution operator defined as $K_\epsilon(w) = \eta_\epsilon * w$ where η_ϵ is a Gaussian function given by

$$\eta_\epsilon(t) = \begin{cases} \frac{\kappa}{\epsilon} e^{-1/(1-\left|\frac{t}{\epsilon}\right|)} & \text{if } t \in (-\epsilon,\epsilon) \\ 0 & \text{otherwise.} \end{cases}$$

Note that κ is defined to ensure $\|\eta_\epsilon\|_{L^1(0,T)} = 1$. Then we define $j_\epsilon := j \circ K_\epsilon$. We will run the SLIP algorithm on j_ϵ and drive ϵ to zero.

3.2.4 SLIP applied to poro-elastic and poro-visco-elastic problem

To reduce the control problem to functions of one dimension, we will let $\chi(x)$ be fixed and insert $q = \chi(x)w(t)$ into (1.2.1a)-(1.2.4) in place of either S or ψ. Note that we don't use $\mathbf{g}$ or $\mathbf{F}$ as controls in this case since in the poro-elastic case, H^1 regularity in time is required in these cases. The only functions, $w(t) \in W$, in $H^1(0,T)$ are constant functions.

Similarly to the last chapter, we let $G : L^2(0,T) \to L^2(0,T;\mathbf{L}^2(\Omega)) \times L^2(0,T;L^2(\Omega))$ be defined by mapping $w(t) \in L^2(0,T)$ to the unique solution of (1.2.1a)-(1.2.4), embedded in $L^2(0,T;\mathbf{L}^2(\Omega)) \times L^2(0,T;L^2(\Omega))$, with all sources set to zero except q. Let $(\tilde{\mathbf{u}},\tilde{p})$ be the unique

solution (in the spaces provided in Theorem 1.3.2 and Theorem 1.3.3) to (1.2.1a)-(1.2.4) with the sources set as desired and $q = 0$. We introduce the control problem

$$\begin{aligned} \min_{w \in L^2(0,T)} \ & \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{2}\|p-p_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\|w\|^2_{L^2(0,T)} + \alpha\,\mathrm{TV}(w) \\ \text{s.t.} \quad & (\mathbf{u},p) = G(w) + (\tilde{\mathbf{u}},\tilde{p}), \\ & w(t) \in W \text{ for a.a. } t \in (0,T) \end{aligned} \tag{3.2.1}$$

for given $\mathbf{u}_d \in L^2(0,T;\mathbf{L}^2(\Omega))$, $p_d \in L^2(0,T;L^2(\Omega))$, $\chi \in L^2(\Gamma_{D,v})$ and $\lambda \geq 0$. We define

$$J(\mathbf{u},p,w) \coloneqq \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{2}\|p-p_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\|w\|^2_{L^2(0,T)} \tag{3.2.2}$$

for $(\mathbf{u},p,w) \in L^2(0,T;\mathbf{V}) \times L^2(0,T;V) \times L^2(0,T)$ and the reduced objective

$$j : L^2(0,T) \to \mathbb{R},\ \ j(w) \coloneqq J(G(w) + (\tilde{\mathbf{u}},\tilde{p}), w) \tag{3.2.3}$$

Next, we verify the applicability of the SLIP algorithm introduced in the previous section, by checking if Assumption 3.2.2 and Assumption 3.2.1 are satisfied.

3.2.5 Case 1: $\lambda > 0$

In this subsection, we assume that the regularization parameter λ in the cost-functional is greater than 0. We prove that in both poro-elastic and poro-visco elastic cases Assumption 3.2.2 is satisfied while the more stringent Assumption 3.2.1 is not satisfied.

Proposition 3.2.1. *Let $\xi \in L^2(0,T)$. Then it follows that the reduced objective $j : L^2(0,T) \to \mathbb{R}$ is twice continuously differentiable at ξ. Moreover, $|\nabla^2 j(w)(\xi,\varphi)| = |(G(\xi),G(\varphi)) + \lambda(\xi,\varphi)|$ and there exists $C > 0$ such that*

$$|\nabla^2 j(w)(\xi,\varphi)| \leq C\|\xi\|_{L^2(0,T)}\|\varphi\|_{L^2(0,T)} \tag{3.2.4}$$

for all $w \in L^2(0,T)$ and all $\xi, \varphi \in L^2(0,T)$ i.e. the Hessian is continuous on $L^2(0,T) \times L^2(0,T)$. When $\lambda > 0$, Assumption 3.2.1 does not hold.

Proof. Let $\tilde{y} = (\mathbf{u}_d - \tilde{\mathbf{u}}, p_d - \tilde{p})$. Based on the definition (3.2.3) of the reduced functional j,

we have that

$$\nabla j(q) = (G(\cdot), G(q) - \tilde{y}) + \lambda(\cdot, q) \text{ and } \nabla^2 j(q)(\xi, \varphi) = (G(\varphi), G(\xi)) + \lambda(\varphi, \xi)$$

Furthermore, we can estimate the Hessian of j as follows:

$$\begin{aligned}
&\left|\nabla^2 j(q)(\xi, \varphi)\right| = |(G(\varphi), G(\xi)) + \lambda(\varphi, \xi)| \\
&\leq \|G(\varphi)\|_{L^2(0,T;\mathbf{L}^2(\Omega)\times L^2(\Omega))} \|G(\xi)\|_{L^2(0,T;\mathbf{L}^2(\Omega)\times L^2(\Omega))} + \lambda \|\varphi\|_{L^2(0,T)} \|\xi\|_{L^2(0,T)} \leq (C^2 + \lambda)\|\varphi\| \|\xi\|,
\end{aligned} \tag{3.2.5}$$

which provides the desired estimate (3.2.4). Additionally, we note that

$$\lambda \|\varphi\|^2_{L^2(\Omega)} \leq \lambda(\varphi, \varphi) + (G(\varphi), G(\varphi)) = \nabla^2 j(q)(\varphi, \varphi) \quad \text{for all } \varphi \in L^2(0, T). \tag{3.2.6}$$

Consider the sequence $\phi_n(t) = n\left(\frac{t}{T}\right)^n \in L^2(0, T)$. Note that

$$\|\phi_n\|^2_{L^1(0,T)} = \left(\int_0^T \left| n\left(\frac{t}{T}\right)^n \right| dt \right)^2 = \frac{n^2 T^2}{(n+1)^2} \tag{3.2.7}$$

$$\lambda \|\phi_n\|^2_{L^2(0,T)} = \lambda \int_0^T \left| n\left(\frac{t}{T}\right)^n \right|^2 dt = \frac{\lambda n^2 T}{2n+1}. \tag{3.2.8}$$

Combining (3.2.6) with (3.2.7) and (3.2.8) we obtain

$$\frac{|\nabla^2 j(w)(\phi_n, \phi_n)|}{\|\phi_n\|^2_{L^1(0,T)}} \geq \frac{\lambda \|\phi_n\|^2_{L^2(\Omega)}}{\|\phi_n\|^2_{L^1(\Omega)}} = \frac{\lambda n^2 T}{2n+1} \frac{(n+1)^2}{n^2 T^2} = \lambda \frac{(n+1)^2}{(2n+1)T} \to \infty \quad \text{as } n \to \infty.$$

Hence, Assumption 3.2.1 does not hold. □

3.2.6 Case 2: $\lambda = 0$

Now we consider the case $\lambda = 0$. For the following proofs and numerical results, we consider the problem on one spatial dimension i.e. $\Omega = (0, L)$. Following [57, 9], the one dimensional

formulation of the poro-visco-elastic systems is given by

$$H_v \frac{\partial^3 u}{\partial x^2 \partial t} + H_e \frac{\partial^2 u}{\partial x^2} - \frac{\partial p}{\partial x} = 0 \qquad \forall (x,t) \in [0,L] \times [0,T] \tag{3.2.9}$$

$$\frac{\partial^2 u}{\partial x \partial t} - k \frac{\partial^2 p}{\partial x^2} = S \qquad \forall (x,t) \in [0,L] \times [0,T] \tag{3.2.10}$$

$$u(0,t) = p(0,t) = 0 \qquad \forall t \in [0,T] \tag{3.2.11}$$

$$-k \frac{\partial p}{\partial x}(L,t) = \psi(t) \qquad \forall t \in [0,T] \tag{3.2.12}$$

$$H_v \frac{\partial^2 u}{\partial x \partial t}(L,t) + H_e \frac{\partial u}{\partial x}(L,t) - p(L,t) = 0 \qquad \forall t \in [0,T] \tag{3.2.13}$$

$$\frac{\partial u}{\partial x}(x,0) = 0 \qquad \forall x \in [0,L] \tag{3.2.14}$$

where $H_e = \lambda_e + 2\mu_e$ and $H_v = \delta(\lambda_v + 2\mu_v)$. When $F = 0$, we know $p(x,t) = H_e \frac{\partial u}{\partial x} + H_v \frac{\partial^2 u}{\partial x \partial t} + h(t)$. When $g = 0$, applying (3.2.13) shows $h(t)$ must be equal to 0. Therefore, when $F = 0$ and $g = 0$ we have

$$p(x,t) = H_e \frac{\partial u}{\partial x} + H_v \frac{\partial^2 u}{\partial x \partial t}. \tag{3.2.15}$$

When $\lambda = 0$, we will see that Assumption 3.2.1 holds in the poro-elastic case, but not the poro-visco-elastic case. We will study these two cases separately.

Poro-elastic Case

We will first consider the case when ψ is used as the control. Then we will study the case where the control $w(t)$ enters the system in the source S, i.e. $S(x,t) = \chi(x)w(t)$.

Theorem 3.2.2. *Let $\lambda = 0$ and $\delta = 0$. Let $k(\mathbf{x}, t) = k$ where k is a positive constant and let G map a control ψ to the state (u,p) that satisfies* (3.2.9)-(3.2.14) *with all sources set to zero except $\psi(t)$. Then Assumption 3.2.1 holds with $w = \psi$.*

Proof. Recall $\nabla^2 j w(\psi_n, \psi_n) = (G(\psi_n), G(\psi_n))$. We will first calculate $G(\psi)$ when $\psi \in C_0^1(0,T)$. Then we will use the Bounded Linear Extension Theorem to show Assumption 3.2.1 holds. Note that ψ is only a function of t because we are considering the one-dimensional case where Γ_N is only one point. Since $\delta = 0$, $H_v = 0$. Plugging (3.2.15) into the PDE, we see that

$p(x,t)$ needs to satisfy

$$\begin{aligned} \frac{\partial p}{\partial t} - kH_e \frac{\partial^2 p}{\partial x^2} &= 0 && \forall (x,t) \in [0,L] \times [0,T] \\ p(0,t) = 0, \quad -k\frac{\partial p}{\partial x}(L,t) &= \psi(t) && \forall t \in [0,T] \\ p(x,0) &= 0 && \forall x \in [0,L]. \end{aligned}$$

Let $w = p + \frac{x}{k}\psi(t)$. Then we see

$$\begin{aligned} \frac{\partial w}{\partial t} - kH_e \frac{\partial^2 w}{\partial x^2} &= \frac{x}{k}\psi'(t) && \forall (x,t) \in [0,L] \times [0,T] \\ w(0,t) = 0, \quad -k\frac{\partial w}{\partial x}(L,t) &= 0 && \forall t \in [0,T] \\ w(x,0) &= 0 && \forall x \in [0,L]. \end{aligned}$$

Let $\lambda_n = \frac{(2n-1)\pi}{2L}$ Note that $(\sqrt{2/L}\sin(\lambda_n x))_{n\in\mathbb{N}}$ is a complete orthonormal basis. Let

$$w(x,t) = \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}\sin(\lambda_n x).$$

Then we see $w(x,t)$ satisfies all boundary conditions, and we have

$$\sum_{n=1}^{\infty} f_n'(t)\sqrt{2/L}\sin(\lambda_n x) + \lambda_n^2 kH_e \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}\sin(\lambda_n x) = \frac{x}{k}\psi'(t)$$

and

$$\sum_{n=1}^{\infty} f_n(0)\sqrt{2/L}\sin(\lambda_n x) = 0.$$

Multiplying both sides of these equations by $\sqrt{2/L}\sin(\lambda_n x)$, integrating these equations from 0 to L, and setting $c_n = (\frac{x}{k}, \sqrt{2/L}\sin(\lambda_n x))_{L^2(0,L)} = \frac{(-1)^n\sqrt{2L}}{k\lambda_n^2}$, we have

$$f_n'(t) + kH_e\lambda_n^2 f_n(t) = c_n\psi'(t) \quad \text{and} \quad f_n(0) = 0.$$

Therefore, $f_n(t) = e^{-kH_e\lambda_n^2 t}\int_0^t c_n e^{kH_e\lambda_n^2\xi}\psi'(\xi)d\xi$. Hence, we have

$$w(x,t) = \sum_{n=1}^{\infty} e^{-kH_e\lambda_n^2 t}\int_0^t c_n e^{kH_e\lambda_n^2\xi}\psi'(\xi)d\xi\sqrt{2/L}\sin(\lambda_n x),$$
$$p(x,t) = \sum_{n=1}^{\infty} e^{-kH_e\lambda_n^2 t}\int_0^t c_n e^{kH_e\lambda_n^2\xi}\psi'(\xi)d\xi\sqrt{2/L}\sin(\lambda_n x) - \frac{x}{k}\psi(t).$$

Using (3.2.15) and recalling that $H_v = 0$, we anti-differentiate p with respect to x and enforcing the boundary condition $u(x,0) = 0$, to see

$$u(x,t) = \sum_{n=1}^{\infty}\frac{e^{-kH_e\lambda_n^2 t}}{H_e\lambda_n}\int_0^t c_n e^{kH_e\lambda_n^2\xi}\psi'(\xi)d\xi\sqrt{2/L}(1-\cos(\lambda_n x)) - \frac{x^2}{2kH_e}\psi(t).$$

Using the fact that $c_n = (\frac{x}{k}, \sqrt{2/L}\sin(\lambda_n x))_{L^2(\Omega)}$, $(\sqrt{2/L}\sin(\lambda_n x))_{n\in\mathbb{N}}$ is an orthonormal sequence, Lebesgue Dominated Convergence Theorem, and Parseval's equality we have

$$\begin{aligned}
&\|p\|^2_{L^2(0,T;L^2(0,L))}\\
&= \int_0^T \sum_{n=1}^{\infty} c_n^2\left[\left(e^{-kH_e\lambda_n^2 t}\int_0^t e^{kH_e\lambda_n^2\xi}\psi'(\xi)d\xi\right)^2 - 2e^{-kH_e\lambda_n t}\int_0^t e^{kH_e\lambda_n^2\xi}\psi'(\xi)d\xi\psi(t)\right]\\
&+ \frac{x^2}{k^2}\psi^2(t)dt\\
&= \int_0^T \sum_{n=1}^{\infty} c_n^2\left[\left(e^{-kH_e\lambda_n^2 t}\int_0^t e^{kH_e\lambda_n^2\xi}\psi'(\xi)d\xi\right)^2 - 2e^{-kH_e\lambda_n t}\int_0^t e^{kH_e\lambda_n^2\xi}\psi'(\xi)d\xi\psi(t)\right.\\
&\left.+\psi^2(t)\right]dt
\end{aligned}$$

Integrating by parts, using $\psi(0)=0$ we see

$$\begin{aligned}
\|p\|^2_{L^2(0,T;L^2(0,L))} &= \int_0^T \sum_{n=1}^{\infty} c_n^2 \Bigg[\left(e^{-kH_e\lambda_n^2 t}\left(\psi(t)e^{kH_e\lambda_n^2 t} - \int_0^t \psi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2\xi} d\xi\right)\right)^2 \\
&-2e^{-kH_e\lambda_n^2 t}\left(\psi(t)e^{kH_e\lambda_n^2 t} - \int_0^t \psi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2\xi} d\xi\right)\psi(t) + \psi^2(t)\Bigg] dt \\
&= \int_0^T \sum_{n=1}^{\infty} c_n^2 \Bigg[\psi^2(t) - 2\psi(t)\int_0^t \psi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2(\xi-t)} d\xi + \left(\int_0^t \psi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2(\xi-t)} d\xi\right)^2 \\
&-2\psi^2(t) + 2\psi(t)\int_0^t \psi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2(\xi-t)} d\xi + \psi^2(t)\Bigg] dt \\
&= \int_0^T \sum_{n=1}^{\infty} c_n^2 \left(\int_0^t \psi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2(\xi-t)} d\xi\right)^2 dt \\
&\le \int_0^T \sum_{n=1}^{\infty} c_n^2 k^2 H_e^2 \lambda_n^4 \left(\int_0^T \psi(\xi) e^{-kH_e\lambda_n^2(t-\xi)} d\xi\right)^2 dt \\
&\le \int_0^T \sum_{n=1}^{\infty} H_e^2 \left(\int_0^T \psi(\xi) e^{-kH_e\lambda_n^2(t-\xi)} d\xi\right)^2 dt
\end{aligned}$$

Recall, Young's inequality for convolution: Given $f \in L^p(0,T)$ and $g \in L^q(0,T)$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$ with $1 \le p,q,r \le \infty$, $\|f*g\|_{L^r(0,T)} \le C\|f\|_{L^p(0,T)}\|g\|_{L^q(0,T)}$. Using $p=1$, $q=2$, and $r=2$, we have

$$\begin{aligned}
\|p\|^2_{L^2(0,T;L^2(\Omega))} &\le \sum_{n=1}^{\infty} H_e^2 \|e^{-kH_e\lambda_n^2(\cdot)}\|^2_{L^2(0,T)} \|\psi\|^2_{L^1(0,T)} \le \sum_{n=1}^{\infty} \frac{H_e^2}{2kH_e\lambda_n^2} \|\psi(\xi)\|^2_{L^1(0,T)} \\
&= \frac{H_e}{2k}\|\psi\|^2_{L^1(0,T)} \sum_{n=1}^{\infty} \frac{4L^2}{(2n-1)^2\pi^2} = \frac{H_e L^2}{4k}\|\psi\|^2_{L^1(0,T)}
\end{aligned}$$

Additionally, integrating (3.2.15) with respect to x, we have

$$\|u\|^2_{L^2(0,T;L^2(0,L))} = \int_0^T \int_0^L \left(\int_0^x p(\xi,t) d\xi\right)^2 dx\, dt \le \frac{H_e L^3}{4k}\|\psi\|_{L^1(0,T)}$$

Therefore, for all $\psi \in C_0^1(0,T)$, we have $\|G(\psi)\|^2_{L^2(0,T;L^2(0,L)\times L^2(0,L))} \le C\|\psi\|_{L^1(0,T)}$. Hence, $G|_{C_0^1(0,T)}$ is bounded and linear. Using the Bounded Linear Extension theorem, we have that the extension $G : L^1(0,T) \to L^2(0,T;L^2(0,L)\times L^2(0,L))$ is a bounded linear functional. Hence,

the desired bound holds. □

Now we consider the poro-elastic case with $\lambda = 0$ where the control is used in the source S. Let $\chi(x)$ be set and consider the case where G maps $s \in L^2(0,T)$ to the unique solution (u,p) to (1.2.1a)-(1.2.4) with all sources and the initial condition set to zero except $S := \chi(x)s(t)$. In this case, the process of finding $p(x,t)$ is similar to the process of finding $w(x,t)$ in the proof of Theorem 3.2.2. Hence, $G(s) = (u,p)$ where

$$p(x,t) = \sum_{n=1}^{\infty} e^{-kH_e\lambda_n^2 t} \int_0^t d_n e^{kH_e\lambda_n^2 \xi} s(\xi) d\xi \sqrt{2/L} \sin(\lambda_n x),$$

and applying (3.2.15)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{e^{-kH_e\lambda_n^2 t}}{H_e\lambda_n} \int_0^t d_n e^{kH_e\lambda_n^2 \xi} s(\xi) d\xi \sqrt{2/L} \sin(\lambda_n x)$$

where $d_n = (\chi(x), \sqrt{2/L}\sin(\lambda_n x))_{L^2(0,L)}$. The proof for showing this solution also satisfies Assumption 3.2.1 follows similarly to the proof of Theorem 3.2.2.

Poro-visco-elastic Case

We show that Assumption 3.2.1 is not satisfied in the poro-visco-elastic case, for both choices of controls ψ and S.

Theorem 3.2.3. *Let $\lambda = 0$ and $\delta > 0$. Let $k(\mathbf{x},t) = k$ where k is a positive constant and let G map a control ψ to the state (u,p) that satisfies* (3.2.9)-(3.2.14) *with all sources set to zero except $\psi(t)$. Then Assumption 3.2.1 does not hold.*

Proof. We will proceed with this proof by first finding $G(\psi) = (u,p)$ when $\psi(t) \in C_0^1(0,T)$. We want to show that there exists $(\varphi_m)_{m\in\mathbb{N}}$ such that

$$\frac{(G(\varphi_m), G(\varphi_m))}{\|\varphi_m\|^2_{L^1(0,T)}} = \frac{\|p_m\|^2_{L^2(0,T)} + \|u_m\|^2_{L^2(0,T)}}{\|\varphi_m\|^2_{L^1(0,T)}} \to \infty \text{ as } m \to \infty.$$

Plugging (3.2.15) into (3.2.9)-(3.2.14), we see that $u(x,t)$ needs to satisfy

$$\begin{aligned}
\frac{\partial^2 u}{\partial x \partial t} - kH_v \frac{\partial^4 u}{\partial x^3 \partial t} - kH_e \frac{\partial^3 u}{\partial x^3} &= 0 && \forall (x,t) \in [0,L) \times [0,T] \\
u(0,t) &= 0 && \forall t \in [0,T] \\
H_v \frac{\partial^2 u}{\partial x \partial t}(0,t) + H_e \frac{\partial u}{\partial x}(0,t) &= 0 && \forall t \in [0,T] \\
-kH_v \frac{\partial^3 u}{\partial x^2 \partial t}(L,t) - kH_e \frac{\partial^2 u}{\partial x^2}(L,t) &= \psi(t) && \forall t \in [0,T] \\
\frac{\partial u}{\partial x}(x,0) &= 0 && \forall x \in [0,L]
\end{aligned}$$

Let $y(x,t) = u(x,t) + \frac{x^2}{2L}\Psi(t)$ where

$$\Psi(t) = \frac{1}{kH_v} e^{-H_e/H_v t} \int_0^t \psi(\xi) e^{H_e/H_v \xi} d\xi.$$

Then $y(x,t)$ satisfies

$$\begin{aligned}
\frac{\partial^2 y}{\partial x \partial t} - kH_v \frac{\partial^4 y}{\partial x^3 \partial t} - kH_e \frac{\partial^3 y}{\partial x^3} &= \frac{x}{L}\Psi'(t) && \forall (x,t) \in [0,L] \times [0,T] \\
y(0,t) &= 0 && \forall t \in [0,T] \\
H_v \frac{\partial^2 y}{\partial x \partial t}(0,t) + H_e \frac{\partial y}{\partial x}(0,t) &= 0 && \forall t \in [0,T] \\
-kH_v \frac{\partial^3 y}{\partial x^2 \partial t}(L,t) - kH_e \frac{\partial^2 y}{\partial x^2}(L,t) &= 0 && \forall t \in [0,T] \\
\frac{\partial y}{\partial x}(x,0) &= 0 && \forall x \in [0,L]
\end{aligned}$$

Let $y(x,t) = \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}(1-\cos(\lambda_n x))$ where $\lambda_n = \frac{(2n-1)\pi}{2L}$. Notice that $y(0,t) = 0$, $\frac{\partial y}{\partial x}(0,t) = 0$, and $\frac{\partial^2 y}{\partial x^2}(L,t) = 0$. By plugging $y(x,t)$ into the first line of the PDE, we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} f_n'(t)\lambda_n \sqrt{2/L}\sin(\lambda_n x) + \lambda_n^3 kH_v \sum_{n=1}^{\infty} f_n'(t)\sqrt{2/L}\sin(\lambda_n x) \\
&+ \lambda_n^3 kH_e \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}\sin(\lambda_n x) = \frac{x}{L}\Psi'(t)
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} f_n(0)\sqrt{2/L}\sin(\lambda_n x)=0$$

Note that $\left(\sqrt{2/L}\sin(\lambda_n x)\right)_{n\in\mathbb{N}}$ is a complete orthonormal basis. Hence, multiplying both sides of these equations by $\sqrt{2/L}\sin(\lambda_n x)$, integrating these equations from 0 to L, setting $c_n=(\frac{x}{L},\sqrt{2/L}\sin(\lambda_n x))_{L^2(0,L)}=4\sqrt{2L}(-1)^{n+1}/((2n-1)^2\pi^2)$ we have

$$(\lambda_n+\lambda_n^3 kH_v)f_n'(t)+kH_e\lambda_n^3 f_n(t)=c_n\Psi'(t)\quad\text{and}\quad f_n(0)=0.$$

Therefore, $f_n(t)=\frac{1}{\lambda_n+\lambda_n^3 kH_v}e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{(kH_e\lambda_n^2\gamma_n\xi)}c_n\Psi'(\xi)d\xi$ where $\gamma_n=\frac{1}{1+\lambda_n^2 kH_v}$. Hence,

$$y(x,t)=\sum_{n=1}^{\infty}\frac{1}{\lambda_n+\lambda_n^3 kH_v}e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}c_n\Psi'(\xi)d\xi\sqrt{2/L}(1-\cos(\lambda_n x))$$
$$u(x,t)=\sum_{n=1}^{\infty}\frac{\gamma_n}{\lambda_n}e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}c_n\Psi'(\xi)d\xi\sqrt{2/L}(1-\cos(\lambda_n x))-\frac{x^2}{2L}\Psi(t)$$

and from (3.2.15),

$$p(x,t)=-H_e\frac{x}{L}\Psi(t)-H_v\frac{x}{L}\Psi'(t)+\sum_{n=1}^{\infty}c_n\lambda_n\sqrt{2/L}\sin(\lambda_n x)\Big(-kH_eH_v\lambda_n\gamma_n^2e^{-kH_e\gamma_n\lambda_n^2 t}$$
$$\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi+\frac{H_v\gamma_n}{\lambda_n}\Psi'(t)+\frac{H_e\gamma_n}{\lambda_n}e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi\Big)$$
$$=-\frac{x}{L}(H_e\Psi(t)+H_v\Psi'(t))+\sum_{n=1}^{\infty}c_n\sqrt{2/L}\lambda_n\sin(\lambda_n x)\Big(\frac{H_e\gamma_n^2}{\lambda_n}e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi$$
$$+\frac{H_v\gamma_n}{\lambda_n}\Psi'(t)\Big).$$

We notice

$$\Psi'(t)=\frac{1}{kH_v}\left(\frac{-H_e}{H_v}e^{-H_e/H_v t}\int_0^t\psi(\xi)e^{H_e/H_v\xi}d\xi+\psi(t)\right).\tag{3.2.16}$$

Hence,

$$p(x,t)=-\frac{x}{Lk}\psi(t)$$
$$+\sum_{n=1}^{\infty}c_n\sqrt{2/L}\sin(\lambda_n x)\left(H_e\gamma_n^2e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi+H_v\gamma_n\Psi'(t)\right).$$

Therefore,

$$\|p\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \sum_{n=1}^{\infty} c_n^2 \left(H_e \gamma_n^2 e^{-kH_e\lambda_n^2\gamma_n t} \int_0^t e^{kH_e\lambda_n^2\gamma_n \xi} \Psi'(\xi) d\xi + H_v \gamma_n \Psi'(t) \right)^2$$
$$-2\frac{c_n^2}{k}\psi(t)\left(H_e \gamma_n^2 e^{-kH_e\lambda_n^2\gamma_n t} \int_0^t e^{kH_e\lambda_n^2\gamma_n \xi} \Psi'(\xi) d\xi + H_v \gamma_n \Psi'(t) \right) + \frac{c_n^2}{k^2}\psi^2(t) dt.$$

Recalling (3.2.16), we see that when $\psi = \varphi_m = e^{mt}$, we have

$$\begin{aligned}
\Psi'(t) &= \frac{1}{kH_v}\left(-\frac{H_e}{H_v} e^{-H_e/H_v t} \int_0^t e^{m\xi} e^{H_e/H_v \xi} d\xi + e^{mt} \right) \\
&= \frac{1}{kH_v}\left(-\frac{H_e}{H_v}\left(\frac{e^{mt}}{m + H_e/H_v} - \frac{e^{-H_e/H_v t}}{m + H_e/H_v} \right) + e^{mt} \right) \\
&= \frac{1}{kH_v}\left(\frac{H_e e^{-H_e/H_v t}}{mH_v + H_e} - \frac{H_e e^{mt}}{mH_v + H_e} + e^{mt} \right) \\
&= \frac{1}{kH_v}\left(\frac{H_e e^{-H_e/H_v t}}{mH_v + H_e} + \frac{H_v m e^{mt}}{H_v m + H_e} \right) \geq 0
\end{aligned} \tag{3.2.17}$$

Hence,

$$\begin{aligned}
&e^{-kH_e\lambda_n^2\gamma_n t} \int_0^t e^{kH_e\lambda_n^2\gamma_n \xi} \Psi'(\xi) d\xi \\
&= \frac{1}{kH_v(mH_v + H_e)}\left(\frac{H_e}{kH_e\lambda_n^2\gamma_n - H_e/H_v}\left(e^{-H_e/H_v t} - e^{-kH_e\lambda_n^2\gamma_n t} \right) \right. \\
&\left. + \frac{H_v m}{kH_e\lambda_n^2\gamma_n + m}\left(e^{mt} - e^{-kH_e\lambda_n^2\gamma_n t} \right) \right) \\
&= \frac{1}{kH_v(mH_v + H_e)}\left(\frac{H_v}{k\lambda_n^2\gamma_n H_v - 1}\left(e^{-H_e/H_v t} - e^{-kH_e\lambda_n^2\gamma_n t} \right) \right. \\
&\left. + \frac{H_v m}{kH_e\lambda_n^2\gamma_n + m}\left(e^{mt} - e^{-kH_e\lambda_n^2\gamma_n t} \right) \right)
\end{aligned}$$

Recall $\gamma_n = \frac{1}{1+\lambda_n^2 kH_v}$, so

$$\begin{aligned} k\lambda_n^2\gamma_n H_v - 1 &= \frac{k\lambda_n^2 H_v}{1+\lambda_n^2 kH_v} - \frac{1+\lambda_n^2 kH_v}{1+\lambda_n^2 kH_v} \\ &= \frac{-1}{1+\lambda_n^2 kH_v} = -\gamma_n \end{aligned}$$

and

$$-kH_e\lambda_n^2\gamma_n = -\frac{kH_e\lambda_n}{1+\lambda_n^2 kH_v} > -\frac{kH_e\lambda_n^2}{\lambda_n^2 kH_v} = -\frac{H_e}{H_v},$$

which implies $1 \geq e^{-kH_e\lambda_n^2\gamma_n t} - e^{-H_e/H_v t} \geq 0$ for $t \in [0,T]$. Hence,

$$\begin{aligned} &e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi \\ &\geq \frac{1}{kH_v(mH_v+H_e)}\left(\frac{H_v}{\gamma_n} + \frac{H_v m}{kH_e\lambda_n^2\gamma_n + m}\left(e^{mt} - e^{-kH_e\lambda_n^2\gamma_n t}\right)\right) \end{aligned} \tag{3.2.18}$$

Let $l > 0$ satisfy

$$\lambda_l^2 = \frac{(2l-1)^2\pi^2}{4L^2} > \frac{8}{kH_v}. \text{ Then } \gamma_l = \frac{1}{1+\lambda_l^2 kH_v} < \frac{1}{\lambda_l^2 kH_v} < \frac{k}{8}. \tag{3.2.19}$$

Notice that

$$\begin{aligned} \|p_m\|^2_{L^2(0,T;L^2(\Omega))} \geq \int_0^T &c_l^2\left(H_e\gamma_l^2 e^{-kH_e\lambda_l^2\gamma_l t}\int_0^t e^{kH_e\lambda_l^2\gamma_l\xi}\Psi'(\xi)d\xi + H_v\gamma_l\Psi'(t)\right)^2 \\ &-2\frac{c_l^2}{k}\psi(t)\left(H_e\gamma_l^2 e^{-kH_e\lambda_l^2\gamma_l t}\int_0^t e^{kH_e\lambda_l^2\gamma_l\xi}\Psi'(\xi)d\xi + H_v\gamma_l\Psi'(t)\right) + \frac{c_l^2}{k^2}\psi^2(t)dt \end{aligned}$$

Dropping the first term since it is positive and applying (3.2.18) and (3.2.17) we see

$$\begin{aligned} \|p_m\|^2_{L^2(0,T;L^2(\Omega))} \geq \int_0^T &-2\frac{c_l^2}{k}e^{mt}\frac{H_e\gamma_l^2}{kH_v(mH_v+H_e)}\left(\frac{H_v}{\gamma_l} + \frac{H_v m}{kH_e\lambda_l^2\gamma_l + m}\left(e^{mt} - e^{-kH_e\lambda_l^2\gamma_l t}\right)\right) \\ &-2\frac{c_l^2}{k}e^{mt}\frac{\gamma_l}{k}\left(\frac{H_e e^{-H_e/H_v t}}{mH_v+H_e} + \frac{H_v m e^{mt}}{H_v m + H_e}\right) + \frac{c_l^2}{k^2}e^{2mt}dt \end{aligned}$$

Recalling that $-(e^{mt} - e^{-kH_e\lambda_l^2\gamma_l t}) \geq -e^{mt}$, $-\left(\frac{H_e e^{-H_e/H_v t}}{mH_v+H+e} + \frac{H_v m e^{mt}}{H_v m+H_e}\right) \geq -(1+e^{mt})$, and $mH_v +$

$H_e \geq H_e$ we see

$$
\begin{aligned}
&\|p_m\|^2_{L^2(0,T;L^2(\Omega))} \\
&\geq \int_0^T -2\frac{c_l^2}{k}e^{mt}\frac{H_e\gamma_l^2}{kH_v(mH_v+H_e)}\left(\frac{H_v}{\gamma_l}+\frac{H_v m}{m}e^{mt}\right)-2\frac{c_l^2}{k}e^{mt}\frac{\gamma_l}{k}(1+e^{mt})+\frac{c_l^2}{k^2}e^{2mt}\,dt \\
&\geq \int_0^T -2\frac{c_l^2}{k}e^{mt}\left(\frac{H_e\gamma_l^2}{kH_vH_e}\frac{H_v}{\gamma_l}+\frac{H_e\gamma_l^2H_v}{kH_vH_e}e^{mt}\right)-2\frac{c_l^2}{k}e^{mt}\frac{\gamma_l}{k}(1+e^{mt})+\frac{c_l^2}{k^2}e^{2mt}\,dt \\
&\geq \int_0^T -2\frac{c_l^2}{k^2}e^{2mt}\left(\gamma_l+\gamma_l^2\right)-2\frac{c_l^2}{k^2}e^{2mt}2\gamma_l+\frac{c_l^2}{k^2}e^{2mt}\,dt
\end{aligned}
$$

Hence, applying (3.2.19), we see

$$
\begin{aligned}
\|p_m\|^2_{L^2(0,T;L^2(\Omega))} &\geq \int_0^T -2\frac{c_l^2}{k^2}e^{2mt}\left(\frac{1}{8}+\frac{1}{64}+\frac{1}{4}\right)+\frac{c_l^2}{k^2}e^{2mt}\,dt \\
&\geq \int_0^T \frac{7c_l^2}{32^2}e^{2mt}\,dt = \frac{7c_l^2}{64k^2m}\left(e^{2mT}-1\right).
\end{aligned}
\tag{3.2.20}
$$

Also, note $\|\varphi_m\|^2_{L^1(0,T)} = \frac{(e^{mt}-1)^2}{m^2}$. Therefore,

$$
\begin{aligned}
\frac{\nabla^2 j(w)(\varphi_m,\varphi_m)}{\|\varphi_m\|^2_{L^1(0,T)}} = \frac{(G(\varphi_m),G(\varphi_m))}{\|\varphi_m\|^2_{L^1(0,T)}} &= \frac{\|p_m\|^2_{L^2(0,T;L^2(0,L))}+\|u_m\|^2_{L^2(0,T;L^2(0,L))}}{\|\varphi_m\|^2_{L^1(0,T)}} \geq \frac{\|p_m\|^2_{L^2(0,T;L^2(0,L))}}{\|\varphi_m\|^2_{L^1(0,T)}} \\
&\geq \frac{\frac{7c_l}{64mk^2}(e^{2mT}-1)m^2}{(e^{mT}-1)^2} \to \infty \text{ as } m\to\infty.
\end{aligned}
$$

Therefore, Assumption 3.2.1 is not satisfied. □

We now consider the poro-visco-elastic case where the control is used as the time component of the source S, and show Assumption 3.2.1 is not satisfied.

Theorem 3.2.4. *Let $\lambda = 0$ and $\delta > 0$. Let $k(\mathbf{x},t) = k$ and $\chi(x) \in L^2(0,L)$. Define $G : L^2(0,T) \to L^2(0,T;L^2(0,L)\times L^2(0,L))$ to be the map that maps $s(t) \in L^2(0,T)$ to the (u,p) that satisfies* (3.2.9)-(3.2.14) *with all sources set to 0 except $S(x,t) = s(t)\chi(x)$. Then Assumption 3.2.1 does not hold.*

Proof. We will proceed with this proof by first computing $G(s) = (u,p)$ and then showing a lower estimate on $\|p\|^2_{L^2(0,T;L^2(\Omega))}$. Plugging (3.2.15) into the PDE we see $u(x,t)$ needs to

satisfy

$$\begin{aligned}
\frac{\partial^2 u}{\partial x \partial t} - kH_v \frac{\partial^4 u}{\partial x^3 \partial t} - kH_e \frac{\partial^3 u}{\partial x^3} &= S && \forall (x,t) \in [0,L) \times [0,T] \\
u(0,t) &= 0 && \forall t \in [0,T] \\
H_v \frac{\partial^2 u}{\partial x \partial t}(0,t) + H_e \frac{\partial u}{\partial x}(0,t) &= 0 && \forall t \in [0,T] \\
-kH_v \frac{\partial^3 u}{\partial x^2 \partial t}(L,t) - kH_e \frac{\partial^2 u}{\partial x^2}(L,t) &= 0 && \forall t \in [0,T] \\
\frac{\partial u}{\partial x}(x,0) &= 0 && \forall x \in [0,L]
\end{aligned}$$

Let $u(x,t) = \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}(1-\cos(\lambda_n x))$, where $\lambda_n = \frac{(2n-1)\pi}{2L}$. Notice that $u(0,t) = 0$, $\frac{\partial u}{\partial x}(0,t) = 0$, and $\frac{\partial^2 u}{\partial x^2}(L,t) = 0$, so the boundary conditions are satisfied. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} f_n'(t)\lambda_n \sqrt{2/L}\sin(\lambda_n x) + \lambda_n^3 kH_v \sum_{n=1}^{\infty} f_n'(t)\sqrt{2/L}\sin(\lambda_n x) + \lambda_n^3 kH_e \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}\sin(\lambda_n x) \\
&= s(t)\chi(x)
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} f_n(0)\sqrt{2/L}\sin(\lambda_n x) = 0.$$

Note $\left(\sqrt{2/L}\sin(\lambda_n x)\right)_{n\in\mathbb{N}}$ is a complete orthonormal basis. Hence, multiplying both sides of these equations by $\sqrt{2/L}\sin(\lambda_n x)$, integrating these equations from 0 to L, and setting $c_n = (\chi(x), \sqrt{2/L}\sin(\lambda_n x))_{L^2(0,L)}$, we have for all $n \in \mathbb{N}$

$$(\lambda_n + \lambda_n^3 kH_v) f_n'(t) + kH_e \lambda_n^3 f_n(t) = c_n s(t) \quad \text{and} \quad f_n(0) = 0.$$

Therefore, $f_n(t) = \frac{1}{\lambda_n + \lambda_n^3 kH_v} e^{-kH_e \lambda_n^2 t/(1+\lambda_n^2 kH_v)} \int_0^t c_n e^{kH_e \lambda_n^2 \xi/(1+\lambda_n^2 kH_v)} s(\xi) d\xi$. Hence,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n + \lambda_n^3 kH_v} e^{-kH_e \lambda_n^2 t/(1+\lambda_n^2 kH_v)} \int_0^t e^{kH_e \lambda_n^2 \xi/(1+\lambda_n^2 kH_v)} s(\xi) d\xi \sqrt{2/L}(1-\cos(\lambda_n x))$$

and (3.2.15) gives

$$
\begin{aligned}
p(x,t) &= \sum_{n=1}^{\infty} c_n \sqrt{2/L}\sin(\lambda_n x)\left(\frac{-kH_eH_v\lambda_n}{(1+\lambda_n^2 kH_v)^2}e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)}\int_0^t e^{kH_e\lambda_n^2\xi/(1+\lambda_n^2 kH_v)}s(\xi)d\xi\right.\\
&\left.+\frac{H_v}{\lambda_n+\lambda_n^3 kH_v}s(t)+\frac{H_e}{\lambda_n+\lambda_n^3 kH_v}e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)}\int_0^t e^{kH_e\lambda_n^2\xi/(1+\lambda_n^2 kH_v)}s(\xi)d\xi\right)\\
&= \sum_{n=1}^{\infty} c_n \sqrt{2/L}\sin(\lambda_n x)\left(\left(\frac{-kH_eH_v\lambda_n^2}{\lambda_n(1+\lambda_n^2 kH_v)^2}+\frac{H_e(1+\lambda_n^2 kH_v)}{\lambda_n(1+\lambda_n^2 kH_v)^2}\right)e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)}\right.\\
&\left.\int_0^t e^{kH_e\lambda_n^2\xi/(1+\lambda_n^2 kH_v)}s(\xi)d\xi+\frac{H_v}{\lambda_n+\lambda_n^3 kH_v}s(t)\right)\\
&= \sum_{n=1}^{\infty} c_n \sqrt{2/L}\sin(\lambda_n x)\left(\frac{H_e}{\lambda_n(1+\lambda_n^2 kH_v)^2}e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)}\int_0^t e^{kH_e\lambda_n^2\xi/(1+\lambda_n^2 kH_v)}s(\xi)d\xi\right.\\
&\left.+\frac{H_v}{\lambda_n+\lambda_n^3 kH_v}s(t)\right)
\end{aligned}
$$

Therefore, when $s(t)$ is strictly non-negative,

$$\|p\|^2_{L^2(0,T;L^2(\Omega))} \geq \sum_{n=1}^{\infty}\frac{c_n^2 H_v^2}{(\lambda_n+\lambda_n^3 kH_v)^2}\|s\|^2_{L^2(0,T)} \geq C\|s\|^2_{L^2(0,T)}$$

for some $C>0$. Therefore, using $\varphi_m = m\left(\frac{t}{T}\right)^m$ (as was done in the proof of Assumption 3.2.1), we have

$$\frac{\nabla^2 j(w)(\varphi_m,\varphi_m)}{\|\varphi_m\|^2_{L^1(0,T)}} = \frac{(G(\varphi_m),G(\varphi_m)}{\|\varphi_m\|^2_{L^1(0,T)}} \geq \frac{\|p_m\|_{L^2(0,T;L^2(\Omega))}}{\|\varphi_m\|^2_{L^1(0,T)}} \geq \frac{C\|\varphi_m\|_{L^2(0,T)}}{\|\varphi_m\|^2_{L^1(0,T)}} \to \infty \text{ as } m\to\infty.$$

Therefore, Assumption 3.2.1 is not satisfied. □

The original SLIP algorithm was run as well as the SLIP algorithm on $j\circ K_\epsilon$ where ψ was used as the control or q satisfying $S = q(t)\chi(x)$ was used as the control in both the elastic and visco-elastic cases. Preliminary results seem to suggest that the SLIP algorithm was sufficient when Assumption 3.2.1 was satisfied and that running the SLIP algorithm on $j\circ K_\epsilon$ improved results when Assumption 3.2.1 was not satisfied.

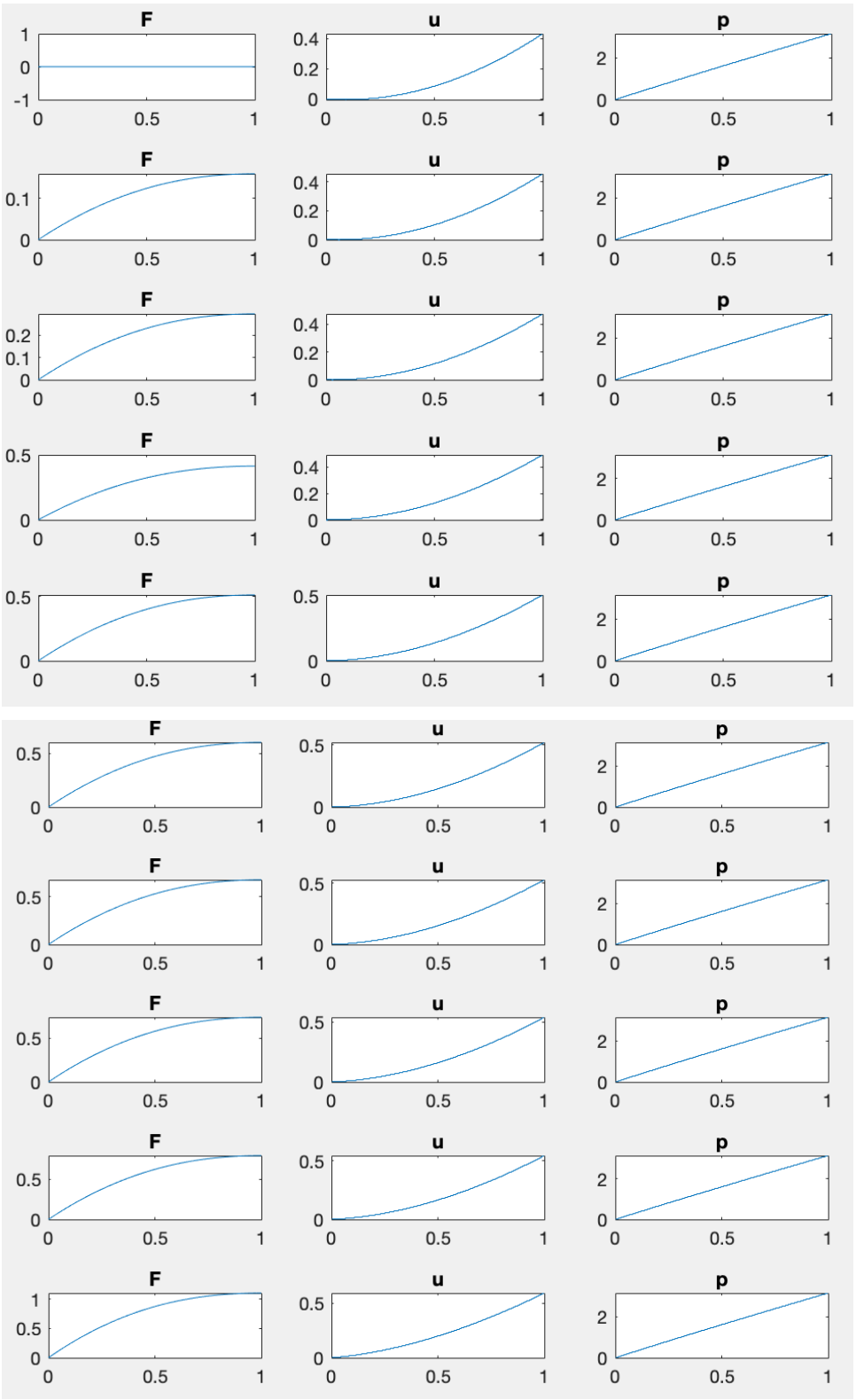


Figure 3.4: We see the first 9 iterations of F, u, and p plotted. The last 3 plots in the figure on the right are the F, u, and p from the last iteration.

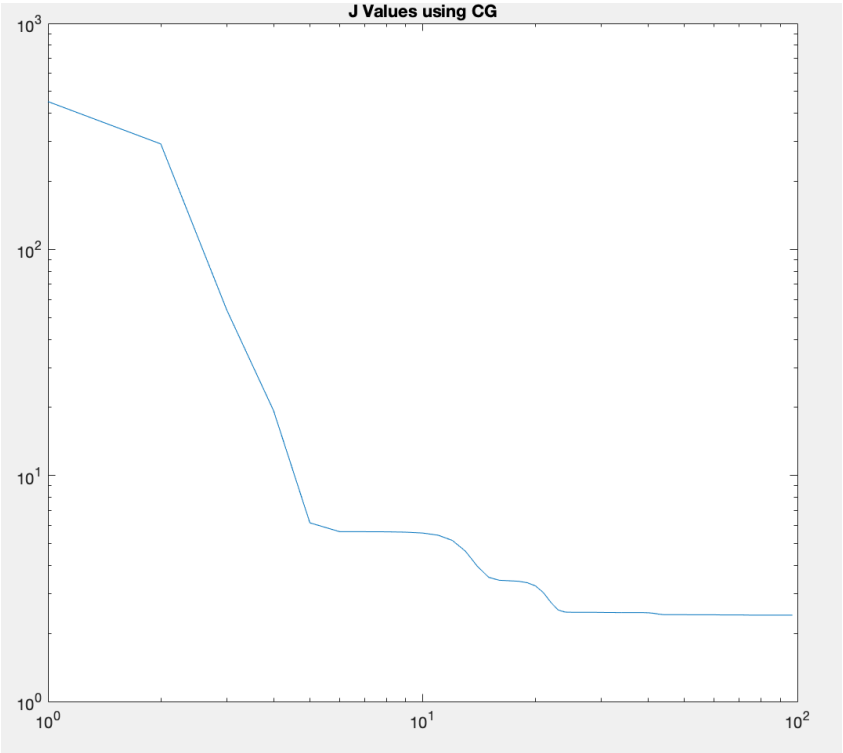


Figure 3.5: We see the cost functional decreasing rapidly and then seeming to converge on a log-log plot.

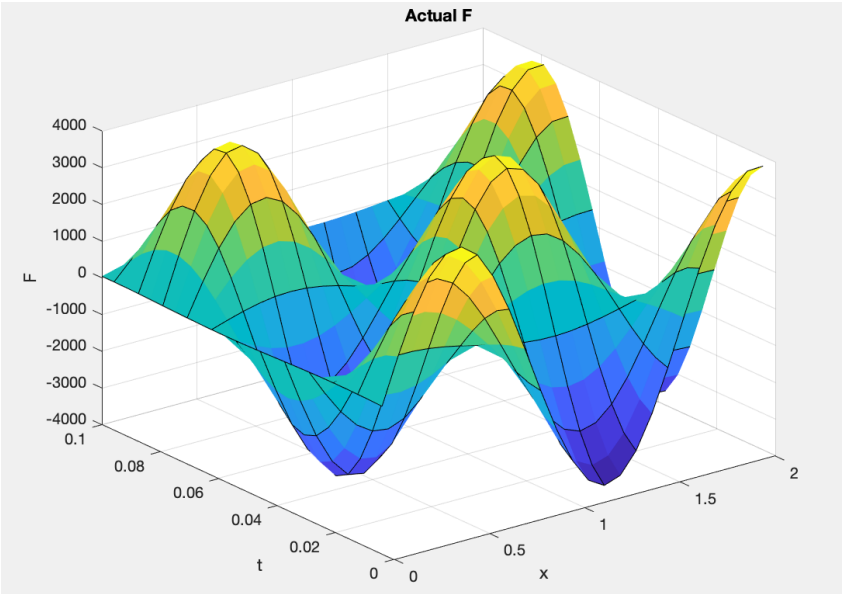


Figure 3.6: The **F** needed to achieve the desired **u** and p is shown. The **F** oscillates somewhere within -4000 to 4000.

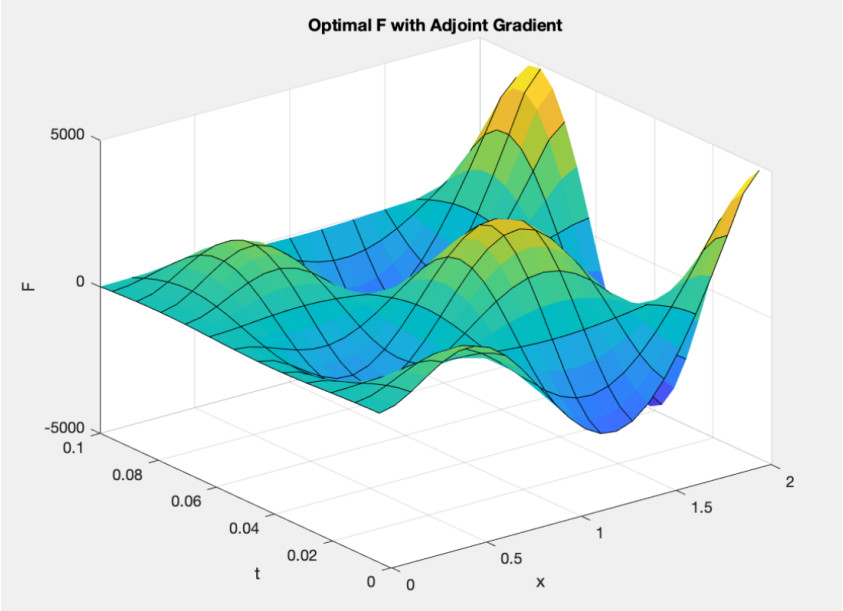


Figure 3.7: The function **F** shown is the optimal control to Test Case 2. It oscillates within -5000 and 5000 with the oscillations becoming more extreme as x increases.

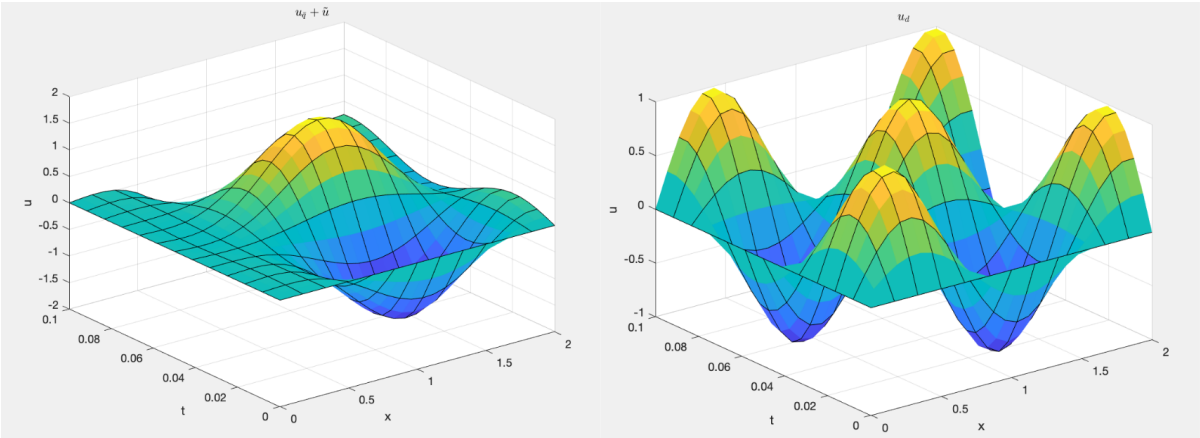


Figure 3.8: The function **u** satisfying the poro-visco-elastic system with the optimal **F** is shown on the left side and the desired $\mathbf{u}_d$ is shown on the right side. We see that the **u** found by the optimization scheme does not oscillate as largely near the boundary of the space-time domain.

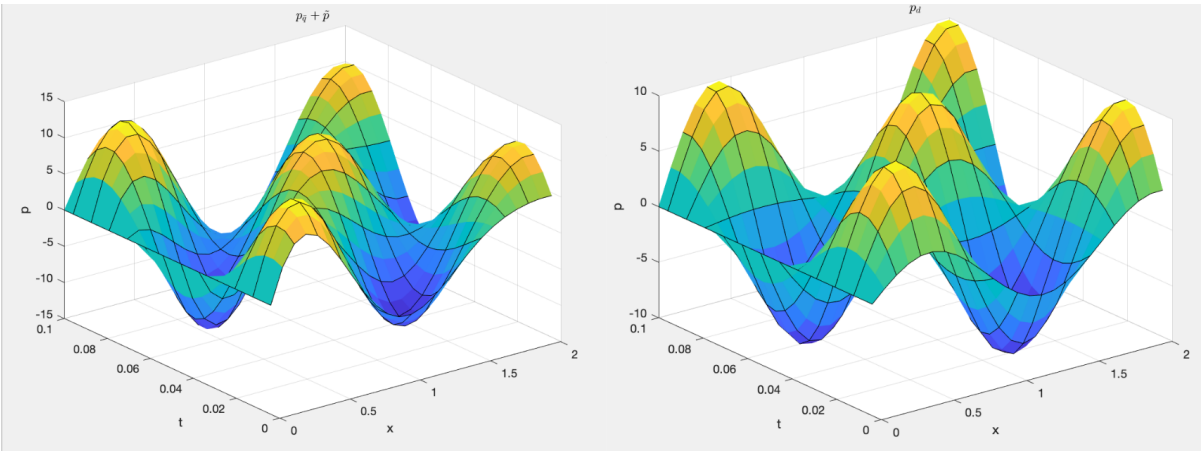


Figure 3.9: The function p satisfying the poro-visco-elastic system with the optimal $\mathbf{F}$ is shown on the left side and the desired p, p_d, is shown on the right side. We see that they are the same.

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