This research was supported in part by the National Institutes of Health, Institute of General Medical Sciences, Grant No. 12868-04, and partially by the National Science Foundation, Grant No. GP-5790.

ASYMPTOTICALLY OPTIMAL NONPARAMETRIC TESTS FOR MISCELLANEOUS PROBLEMS OF LINEAR REGRESSION

by

Malay Ghosh
Department of Statistics

University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 634

JULY 1969

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

It is a pleasure and privilege to acknowledge my sincerest gratitude to Professor P. K. Sen, my adviser, for proposing the problems in this dissertation and making numerous suggestions and comments during its progress.

I wish also to thank Professor H. A. David, Professor Wassily Hoeffding and Professor N. L. Johnson for going through the manuscript and offering many helpful suggestions. Thanks are also due to Professor G. E. Simmons for some useful discussions and Professor W. H. Graves, the other member of my doctoral committee.

I wish to acknowledge the Department of Statistics for financial assistance during the first two years of my graduate program and the Department of Biostatistics and the National Institutes of Health for financial assistance in the third and final year of my graduate studies.

I wish also to thank Professor James E. Grizzle of the Department of Biostatistics for his never-failing help during my graduate studies and Professor H. K. Nandi of the University of Calcutta for stimulating my interest in Mathematical Statistics. I also owe a deep sense of gratitude to my parents and grandparents for their active co-operation and enthusiasm, but for which I would not have reached this stage of my academic career.

Finally, thankful appreciation is due to Mrs. Bobbie Wallick and Mrs. Mary Flanagan for their careful and patient typing of the manuscript.

ABBREVIATIONS AND NOTATIONS

Abbreviations

r.v.	random variable
d.f.	distribution function
i.i.d.	independent and identically distributed
i.i.d.r.v.	independent and identically distributed random variables
p.d.	positive definite
a.e.	almost everywhere
U.M.P.	uniformly most powerful
C.L.T.	Central Limit Theorem

Notations

$N_p(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$	p variate normal with mean vector $\underset{\sim}{\mu}$ and dispersion matrix $\underset{\sim}{\Sigma}$
χ_p^2	Central chi-square with p degrees of freedom
$\chi_p^2(\eta)$	Noncentral chi-square with p degrees of freedom and non-centrality parameter η
$a_\nu \sim b_\nu (b_\nu \neq 0)$	$a_\nu / b_\nu \to 1$ as $\nu \to \infty$
identity matrix of order p	I_p (unless otherwise mentioned)

<u>Notations</u> (cont.)

$$\begin{pmatrix} a_{11} & \cdots & a_{1q} \\ \cdots & \cdots & \cdots \\ a_{p1} & \cdots & a_{pq} \end{pmatrix} = A = ((a_{ij}))_{p \times q}$$

ABSTRACT

The following four problems are considered in this dissertation.

The first problem concerns testing the hypothesis of no regression in a multiple (linear) regression model with stochastic predictors. A class of permutationally (conditionally) distribution-free tests for the above problem has been proposed and its asymptotic properties together with asymptotic relative efficiency results have been studied.

The second problem deals with testing the hypothesis of no regression in a multiple (linear) regression model with non-stochastic predictors, but with grouped data. Permutationally distribution-free tests, their asymptotic properties and some specific applications of such tests have been studied.

The third problem deals with the same model, but we are interested in testing that the intercept is zero along with the hypothesis of no regression. Tests analogous to 'signed-rank tests' have been used here with an interesting application to the problem of 'paired comparison.'

Finally, the above findings have been generalized to a multivariate set up.

CHAPTER I

INTRODUCTION AND SUMMARY

1.1. A Review of the Literature.

Hájek (1962) considered the following problem: Having observed $X_i = \alpha + \beta c_i + \sigma\varepsilon_i$, we test the hypothesis

$$H_0 : \beta = 0 \text{ against the alternatives } \beta > 0, \tag{1.1.1}$$

where α and $\sigma(>0)$ are nuisance parameters, β the parameter under test, c_i are known constants and $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d.r.v. A class of genuinely distribution-free rank-statistics has been proposed by Hájek for the above testing problem. The asymptotic approach of Hajek consists in regarding the given testing problem as a member of a sequence $\{H_{0\nu}, H_\nu; \nu \geq 1\}$ of similar testing problems. In this sequence, the ν^{th} testing problem concerns N_ν observations with $N_\nu \to \infty$ as $\nu \to \infty$. As a rule, $H_{0\nu}$ depends on ν through N_ν only, i.e., $H_{0\nu} \equiv H_0(N_\nu)$, (e.g. we might assume $H_{0\nu} \equiv H_0$, H_0 being applied to $N = N_\nu$ observations), while H_ν depends on some parameters $c_{\nu i}$ $(i = 1, 2, \ldots, N_\nu)$ in addition. More specifically, he has considered a sequence of random vectors $\underset{\sim}{X}_\nu = (X_{\nu 1}, \ldots, X_{\nu N_\nu})'$, where the $X_{\nu i}$'s are independent and

$$\Pr(X_{\nu i} \leq x) = F_{\nu i}(x) = F(\sigma^{-1}(x - \alpha - \beta c_{\nu i})), \tag{1.1.2}$$

$i = 1, 2, \ldots, N_\nu$, $\nu \geq 1$.

In the above, F is a known distribution function, $c_{\nu i}$ $(i = 1, 2,\ldots, N_\nu)$ are known constants, α and $\sigma(> 0)$ are nuisance parameters and β is the parameter under test. The testing problem (1.1.1) has been considered. The asymptotic theory has been studied by using the "contiguity" lemmas of LeCam (see e.g. [19] or [25]; see section 1.3 for definition of contiguity). It has been shown that the test one would have obtained by using the Fundamental Lemma of Neyman and Pearson (see e. g. [26], p. 65) is asymptotically power-equivalent to Hajek's rank order tests under certain assumptions on the $c_{\nu i}$'s and F.

Assume that

(1.1.3) the density $f(x) = F'(x)$ exists and

$$\int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x)\,dx < \infty .$$

Let the constants $c_{\nu i}$ $(i = 1, 2,\ldots, N_\nu)$ satisfy a condition (known in the literature as Noether condition, and, henceforth, referred to as such)

$$\text{(1.1.4)} \qquad \lim_{\nu\to\infty}\left\{ \max_{1\le i\le N_\nu} (c_{\nu i} - \bar{c}_\nu)^2 \Big/ \sum_{i=1}^{N_\nu} (c_{\nu i} - \bar{c}_\nu)^2 \right\} = 0,$$

where $\bar{c}_\nu = N_\nu^{-1} \sum_{i=1}^{N_\nu} c_{\nu i}$. Further the $c_{\nu i}$'s satisfy the boundedness condition

$$\text{(1.1.5)} \qquad \sup_{\nu} \sum_{i=1}^{N_\nu} (c_{\nu i} - \bar{c}_\nu)^2 < \infty .$$

Let F^{-1} denote the inverse of F and let

$$(1.1.6) \qquad \phi(u) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}, \quad 0 < u < 1.$$

Then, $\int_0^1 \phi^2(u)\,du = \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x)\,dx < \infty.$

We call ϕ a "score" function.

Now, let $R_{\nu i}$ be the rank of $X_{\nu i}$ in the ordered sample $V_{\nu 1} \le \ldots \le V_{\nu N_\nu}$, i.e.

$$(1.1.7) \qquad X_{\nu i} = V_{\nu R_{\nu i}} \text{ with probability 1, } 1 \le i \le N_\nu,\ \nu \ge 1.$$

Further, let $a_\nu(i)$, $1 \le i \le N_\nu$ be some arbitrary scores satisfying

$$(1.1.8) \qquad \lim_{\nu\to\infty} \int_0^1 [a_\nu(1 + [uN_\nu]) - \phi(u)]^2\,du = 0,$$

[x] being the largest integer less than or equal to x. Let the rank-order statistics S_ν be expressed as

$$(1.1.9) \qquad S_\nu = \sum_{i=1}^{N_\nu} (c_{\nu i} - \bar{c}_\nu) a_\nu(R_{\nu i}), \ \nu \ge 1.$$

Let ε_ν and $Q_\nu(\beta, \sigma)$ be the size and power of test based on critical region

$$(1.1.10) \qquad S_\nu > K_\varepsilon d_\nu,\ \nu \ge 1,$$

where K_ε is the upper 100ε% point of a standard normal variable and

$$(1.1.11) \qquad d_\nu^2 = \sum_{i=1}^{N_\nu} (c_{\nu i} - \bar{c}_\nu)^2 \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x)\,dx.$$

The main theorem of Hájek is as follows:

THEOREM 1.1. Under the model (1.1.2) and under the assumptions (1.1.3) - (1.1.8), the critical region (1.1.10) provides an asymptotically UMP test of limiting size ε for (1.1.1), and the asymptotic power equals $1 - \Phi(K_\varepsilon - \frac{\beta}{\sigma} d_\nu)$, where, $\Phi(x) = (\sqrt{2\pi})^{-1} \int_{-\infty}^{x} \exp.(-\frac{t^2}{2})dt$, and K_ε and d_ν are defined in (1.1.10) and (1.1.11) respectively.

If however, the true d.f. is G instead of F (where G satisfies similar regularity conditions as F), Hájek's test is no longer asymptotically UMP, but the A.R.E. of the proposed test relative to the UMP test has been obtained. It turns out to be the square of the correlation coefficient between "true score" and the "assumed score" i.e.

$$\rho^2 = \frac{(\int_0^1 \psi(u)\phi(u)du)^2}{(\int_0^1 \psi^2(u)du)(\int_0^1 \phi^2(u)du)},$$

where $\psi(u) = \dfrac{-g'(G^{-1}(u))}{g(G^{-1}(u))}, \quad 0 < u < 1,$

$g(x) = G'(x), \int_{-\infty}^{\infty} [g'(x)/g(x)]^2 \, g(x)dx < \infty.$

The notation X for 'dependent' variable (i.e. 'predictand') has been used all through except in Chapter II, where the $c_{\nu i}$ are stochastic. We have used the notation Y for 'dependent' variable in that chapter and X's for 'independent' variables, (i.e. 'predictors').

A situation likely to arise in practice is when $\underset{\sim}{X}_\nu$ is not observable, but we have a finite (or a countable) set of class-intervals

(1.1.12) $$I_j : a_j < x \le a_{j+1}, \quad j = \ldots, -2, 1, 0, 1, 2, \ldots$$

[where $-\infty < \ldots < a_{-1} < a_0 < a_1 < \ldots < \infty$ is any (finite or countable) set of ordered points on the (extended) real line $[-\infty, \infty]$], and the observable stochastic vector is $\underset{\sim}{X}_\nu^* = (X_{\nu 1}^*, \ldots, X_{\nu N_\nu}^*)'$, where

(1.1.13) $$X_{\nu i}^* = \sum_{j=-\infty}^{\infty} I_j Z_{\nu ij}$$

and

(1.1.14) $$Z_{\nu ij} = \begin{cases} 1 & \text{if } X_{\nu i} \in I_j \\ 0 & \text{otherwise} , \end{cases}$$

$(i = 1, 2, \ldots, N_\nu, \quad \nu \ge 1; \quad j = \ldots, -2, -1, 0, 1, 2, \ldots)$.

The model, the testing problem and the assumptions are the same as before.

In practice, we may very often come across situations when data are collected in several groups or ordered categories, thus, rendering any analysis suitable for ungrouped data, impossible. Even in studying the height distribution of a large number of individuals, it is very often convenient to record the observations classified into several groups.

Another example may be, when we want to compare the efficacies of several contraceptive devices. A number of couples are studied for a certain length of time, and the distributions of the conceptions are compared. By nature, the exact time of conception cannot be studied. Rather, the cycle in which the conception takes place, can be recorded, though the actual distribution of the conceptions can be taken to be absolutely continuous. Hence, the data can be recorded only in terms of intervals of approximate 4-weekly cycle-lengths.

In the case of grouped data, Sen (1967) has obtained a class of permutationally (conditionally) distribution-free tests for the testing problem in (1.1.1). He has also obtained a class of asymptotically UMP parametric tests under the same conditions as of Hájek (1962) and has established the asymptotic power equivalence of the two kinds of tests under "contiguous" alternatives. Sen has also obtained an A.R.E. expression similar to that of Hájek when the true d.f. is G instead of F. He has also shown that the loss of efficiency due to grouping can be made arbitrarily small provided the probabilities of observations belonging to class intervals $\{I_j\}$ can be made arbitrarily small. Throughout our investigation, we have taken $N_\nu = \nu$, $\nu \geq 1$.

1.2. A Summary of the Results in Chapters II-VI.

The results of Hájek and Sen, described in the earlier section, have been extended in several directions. In Chapter II, we have extended the findings of Hájek to the multiple (linear) regression model with stochastic predictors including the situations where the predictors carry only partial information, and, may be referred to as 'partially informed stochastic predictors.' For example, we may have the general linear model $Y_{\nu i} = \beta_0 + \beta_1 X_{1\nu i} + \cdots + \beta_p X_{p\nu i} + \sigma\varepsilon_{\nu i}$, $i = 1, 2, \ldots, \nu$, $\nu \geq 1$, where the observations $(Y_{\nu i}, X_{1\nu i}, \ldots, X_{p\nu i})$, $i = 1, 2, \ldots, \nu$, $\nu \geq 1$ may not be available, but only the ranks on these are available, ranking being done independently within each Y_ν or $X_{k\nu}$. We want to test

$$(1.2.1) \qquad H_0 : \underset{\sim}{\beta} = (\beta_1, \ldots, \beta_p)' = \underset{\sim}{0} ,$$

against the alternatives $\underset{\sim}{\beta} \neq \underset{\sim}{0}$. A class of permutationally (conditionally) distribution-free tests has been obtained for the above problem which is unconditionally distribution-free for the particular case $p = 1$. This class of tests is asymptotically optimal in the sense of Wald [34]. The A.R.E. results are given and in the particular case $p = 1$, the expression for the A.R.E. turns out to be the product of the squared correlation coefficients between the "true score" and "assumed score" for each of Y and X.

In Chapter III, the results of Sen have been generalized to the multiple linear regression model. From this chapter onwards, we have worked with grouped data. Sen's model is s simple linear regression model where the hypothesis of no regression has been tested against two-sided alternatives. Unlike his case, since we have here two-sided alternatives, we have not been able to get a UMP test, but a test which is an "asymptotically most stringent test"; (for definition, see Wald [34]). This has been achieved by showing the "asymptotic equivalence" (definition given in Section 1.3) of the proposed test statistic to the likelihood ratio test statistic, (see [34] or [35]).

Consequently, other optimal properties of the likelihood ratio test as proved by Wald [34], hold for our test as well.

Regarding the mathematical tools used in this chapter, there is a basic difference from that of Sen. In proving many of the results in this chapter, we have avoided the "contiguity" argument.

The reason behind this is to extend the results to the multivariate case, where "contiguity" of the sequence of probability measures corresponding to the alternatives to the corresponding sequence for the null hypothesis, at least, does not seem apparent. A corollary to the lemma 1 of LeCam (see e.g. [19], p. 204) gives a sufficient condition under which "contiguity" holds. This does not hold in general in the multivariate case. We are not aware of any condition ensuring "contiguity" in the multivariate situation. In Sen's case, however, "contiguity" has been used as a major tool. Further, Sen has considered a countable set of class-intervals, whereas, we have for all essential purposes, considered a finite set of class-intervals, and ultimately have extended the results to the case of a countable set of class-intervals by using the "contiguity" argument. We have obtained the same efficiency expressions as in the case of Sen and consequently, his conclusions regarding the same hold in our case as well. Our results also include the $K(\geq 2)$-sample problem as a particular case.

In Chapter IV a similar problem has been considered, the only difference being that here we are interested in testing for the intercept along with the regression coefficients. More specifically, if we have a linear model,

$$X_{\nu i} = \beta_0 + \beta_1 c_{1\nu i} + \ldots + \beta_p c_{p\nu i} + \sigma \varepsilon_{\nu i} \; (i = 1, \ldots, \nu;\ \nu \geq 1) ,$$

the $c_{\nu i}$'s being known constants, we have tested $H_0 : (\beta_1, \ldots, \beta_p) = (0, \ldots, 0)$ in Chapter III against all possible alternatives, whereas we have tested

$H_0 : (\beta_0, \beta_1, \ldots, \beta_p) = (0, 0, \ldots, 0)$ against all possible alternatives in Chapter IV. The parametric test procedures for the two problems are exactly the same, but the permutation test procedures vary in the two cases, since in Chapter III, statistics analogous to rank statistics have been considered, whereas in ChapterIV statistics analogous to signed-rank statistics have been considered. A detailed explanation of these has been given in Chapter IV.

As an interesting application of the test procedure considered in this chapter, we have been able to find a test for the paired-comparison problem as considered by Mehra and Puri [28]. It has been shown that the two tests are asymptotically power-equivalent when each pair of treatments is compared the same number of times. But if that is not so (and, obviously, the number of treatments compared is ≥ 3), the two tests may not be asymptotically power-equivalent. A counterexample has been obtained to this effect, and, in fact, in the particular example considered, our test is asymptotically more powerful than that of Mehra and Puri. This shows that the test considered by Mehra and Puri (which seems rather arbitrary apart from the fact that the test statistic is asymptotically a chi-square) need not be asymptotically optimal in the sense of Wald [34], whereas our statistic can claim that property. Moreover, it seems that our statistic is computationally simpler than its rival.

The findings of Chapter III and IV have been extended to the multivariate case in Chapter V. Our attempts seem to be the first in this respect, since very little seems to have been said or done regarding the general linear regression problem for multivariate grouped data even under the normal theory set-up. We may, however, remark that unlike the univariate case, the asymptotic optimality in the sense of Wald cannot be claimed in the multivariate case in a quite general sense.

In Chapter VI, a specific example has been considered. This, however, illustrates the test procedure developed in Chapter III, although, as we shall see later on in Chapter V, the multivariate extension of such tests is quite simple, even in practice.

A general manner of presentation which has been followed in Chapters III-V is this: In section 1, We have introduced the problem and have given the basic notations and assumptions. In section 2, we have derived a class of parametric tests and have established the asymptotic optimality (if any) of such tests. In section 3, we have developed a class of permutationally distribution-free non-parametric tests. The remaining few sections deal with miscellaneous remarks, derivation of the A.R.E. results and applications to specific cases.

1.3. Some Definitions and well-known Results.

In this section, we give some definitions and well-known results which have been used repeatedly in the subsequent chapters.

i) Contiguity (Definition: Hâjek [17]).

Let $\{P_\nu\}$ and $\{Q_\nu\}$ be two sequences of probability measures defined on a sequence of measurable spaces $\{\mathcal{X}_\nu, \mathcal{A}_\nu\}$, $\nu \geq 1$. If for any sequence of events $A_\nu \in \mathcal{A}_\nu$, $P_\nu(A_\nu) \to 0$ entails $Q_\nu(A_\nu) \to 0$, we say that the probability measures Q_ν are contiguous to the probability measures P_ν. Contiguity implies that any sequence of r.v.'s Y_ν converging to Y in P_ν-probability converges to Y in Q_ν-probability as well.

ii) Uniformly Continuous Integrals (Definition: Loève [27]).

Consider the probability space $(\Omega, \mathcal{A}, P)$. Let $\{X_\nu, \nu \geq 1\}$, be a sequence of r.v.'s (measurable functions) on this space. Then the integrals of the r.v.'s X_ν are said to be uniformly continuous, if, for any $A \in \mathcal{A}$, $\int_A |X_\nu| dP \to 0$ uniformly in ν as $P(A) \to 0$; in other words, for every $\varepsilon > 0$, there exists a δ_ε independent of ν such that $\int_A |X_\nu| dP < \varepsilon$ for any set A with $P(A) < \delta_\varepsilon$.

iii) Uniformly Integrable (Definition: Loève [27]).

Let $B_\nu = [|X_\nu| \geq a]$, X_ν's being r.v.'s as in (ii). We say the r.v.'s $|X_\nu|$ are uniformly integrable, if $\int_{B_\nu} |X_\nu| dP \to 0$ uniformly in ν, as $a \to \infty$.

NOTE: The r.v.'s X_ν are uniformly integrable if, and only if, their integrals are uniformly continuous and uniformly bounded.

iv) Kronecker or Direct Product (Definition).

Let $A = ((a_{ij}))$ be a $p \times q$ matrix and $\beta = ((b_{\alpha\beta}))$ be a $r \times s$ matrix. Then, the Kronecker or Direct Product of A and B denoted by $A \otimes B$, is given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1q}B \\ \cdots & \cdots & \cdots \\ a_{p1}B & \cdots & a_{pq}B \end{bmatrix}$$

v) $O_p(f(\nu))$ (Definition).

Consider a sequence of r.v. 's $\{X_\nu, \nu \geq 1\}$. We write $X_\nu = O_p(f(\nu))$ (X_ν is of probability order $O(f(\nu))$), if, for each $\varepsilon > 0$, there exists an $A_\varepsilon > 0$, such that $Pr\cdot\{|X_\nu| \leq A_\varepsilon f(\nu)\} \geq 1 - \varepsilon$, for all values of ν.

vi) $o_p(f(\nu))$ (Definition).

Consider a sequence of r.v. 's $\{X_\nu, \nu \geq 1\}$. We write $X_\nu = o_p(f(\nu))$, if for every $\varepsilon > 0$, $\lim_{\nu\to\infty} Pr\cdot\{[|X_\nu|/|f(\nu)|) \leq \varepsilon\} = 1$, (i.e. $X_\nu/f(\nu)$ converges in probability to zero as $\nu \to \infty$).

vii) Asymptotically Equivalent (Definition).

We say two sequences $\{X_\nu\}$ and $\{Y_\nu\}$ of r.v.'s are "asymptotically equivalent" if $X_\nu - Y_\nu \to 0$ in probability as $\nu \to \infty$, and $\{X_\nu\}$ or $\{Y_\nu\}$ has a non-degenerate limiting distribution.

viii) Asymptotic Normality (Definition).

We say a sequence $\underset{\sim}{X}_\nu = (X_{1\nu},\ldots, X_{p\nu})'$ of r.v. 's is asymptotically $N_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, where $\underset{\sim}{\Sigma}$ is a p.d. matrix if, and only if, for every vector $\underset{\sim}{a} = (a_1,\ldots, a_p)' \neq \underset{\sim}{0}$, with finite and real elements, $(\underset{\sim}{a}'\underset{\sim}{X}_\nu - \underset{\sim}{a}'\underset{\sim}{\mu})/[\underset{\sim}{a}'\underset{\sim}{\Sigma}\underset{\sim}{a}]^{\frac{1}{2}}$ is asymptotically $N_1(0, 1)$.

ix) Bonferroni Inequality (see e.g. Feller [12]).

Let $(\mathcal{X}, \mathcal{A}, P)$ be a measure space, and let $A_i \in \mathcal{A}$ $(i = 1, 2,\ldots, r)$. Let $S_1 = \sum_{i=1}^{r} P(A_i)$,

$$S_2 = \sum\sum_{1\leq i_1<i_2\leq r} P(A_{i_1} \cap A_{i_2}), \ldots,$$

$$S_m = \sum_{1\leq i_1<i_2<\ldots<i_m\leq r} \cdots \sum P(A_{i_1} \cap A_{i_2} \cap\ldots\cap A_{i_m}),$$

. . . .

$$S_r = P(A_1 \cap A_2 \cap\ldots\cap A_r).$$

Then the first set of Bonferroni Inequalities says that the probability P_m that m or more of the events $A_1, A_2, \ldots, A_r$ occur simultaneously, satisfies the inequality

$$S_m - S_{m+1} \leq P_m \leq S_m \quad (m = 1, 2, \ldots, r - 1).$$

x) <u>A Convergence Theorem (see Crámer [11], p. 254)</u>.

If $\{X_\nu\}$, $\{Y_\nu\}$ and $\{Z_\nu\}$ are sequences of r.v.'s $a, b \neq 0$ are numbers such that X_ν converges in law to X as $\nu \to \infty$, Y_ν and Z_ν converge in probabilities to a and b respectively as $\nu \to \infty$, then $Z_\nu X_\nu + Y_\nu$ converges in law to $bX + a$. In particular, if X_ν converges in law to X as $\nu \to \infty$, $X'_\nu - X_\nu \to 0$ in probability as $\nu \to \infty$, X'_ν converges in law to X as $\nu \to \infty$.

xi) <u>Slutsky's Theorem (see Cramer [11], p. 255)</u>.

If $\xi_{1\nu}, \xi_{2\nu}, \ldots, \xi_{p\nu}$ are sequences of r.v.'s converging in probability to the constants $c_1, c_2, \ldots, c_p$ respectively, any rational function $R(\xi_{1\nu}, \ldots, \xi_{p\nu})$ converges in probability to the constant $R(c_1, \ldots, c_p)$, provided that the latter is finite. It follows that any power $R^k(\xi_{1\nu}, \ldots, \xi_{p\nu})$ with $k > 0$ converges in probability to $R^k(c_1, \ldots, c_p)$.

CHAPTER II

RANK ORDER TESTS FOR REGRESSION WITH

PARTIALLY INFORMED STOCHASTIC PREDICTORS

2.1. Introduction, Notations and Assumptions.

Consider a (double) sequence of stochastic matrices $\underset{\sim}{Z}_\nu = (\underset{\sim}{Z}_{\nu 1},\ldots, \underset{\sim}{Z}_{\nu\nu})$, $1 \le \nu < \infty$, where $\underset{\sim}{Z}'_{\nu i} = (Y_{\nu i}, \underset{\sim}{X}'_{\nu i}) = (Y_{\nu i}, X^{(1)}_{\nu i},\ldots, X^{(p)}_{\nu i})$, $(p \ge 1)$ are i.i.d. random vectors having an absolutely continuous d.f. $F_\nu(y, \underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^p$, the p-dimensional real space. We assume that the p-variate marginal d.f. $F_{01}(\underset{\sim}{x}) = F_\nu(\infty, \underset{\sim}{x})$ does not depend on ν and possesses a continuous density function $f_{01}(\underset{\sim}{x})$. We denote the conditional density function of $Y_{\nu i}$ given $\underset{\sim}{X}_{\nu i} = \underset{\sim}{x}$ by $f_\nu(y|\underset{\sim}{x})$. We also assume that $f_\nu(y|\underset{\sim}{x}) = f_{20}[(y - \beta_0 - \nu^{-\frac{1}{2}} \underset{\sim}{\beta}'\underset{\sim}{x})/\sigma]$, where β_0 and $\sigma(> 0)$ are nuisance parameters and $\underset{\sim}{\beta} = (\beta_1,\ldots, \beta_p)'$ is the parameter (vector) under test. Thus the density function corresponding to the d.f. $F_\nu(y, \underset{\sim}{x})$ is given by

$$(2.1.1)\qquad f_\nu(y|\underset{\sim}{x}) = f_{01}(\underset{\sim}{x})f_{20}([y - \beta_0 - \nu^{-\frac{1}{2}} \underset{\sim}{\beta}'\underset{\sim}{x}]/\sigma),\ \nu \ge 1.$$

We want to test the null hypothesis of no regression i.e.

$$(2.1.2)\qquad H_0 : \underset{\sim}{\beta} = \underset{\sim}{0} \text{ against the alternatives } \underset{\sim}{\beta} \ne \underset{\sim}{0}.$$

It may be noted that under H_0, $f_\nu(y,\underset{\sim}{x}) = f(y,\underset{\sim}{x}) = f_{01}(\underset{\sim}{x})f_{20}(\sigma^{-1}(y - \beta_0))$, for all $\nu \geq 1$ and thus $Y_{\nu i}$ and $\underset{\sim}{X}_{\nu i}$ are stochastically independent for all $i = 1, 2, \ldots, \nu$ and all $\nu \geq 1$. We shall use the notations $\overline{X}_\nu^{(k)} = \nu^{-1} \sum_{i=1}^{\nu} X_{\nu i}^{(k)}$ $(k = 1, \ldots, p)$ and $\overline{Y}_\nu = \nu^{-1} \sum_{i=1}^{\nu} Y_{\nu i}$. The following assumptions are made:

(2.1.3) $f_{20}'(y) = (d/dy)f_{20}(y)$ exists and $I(f_{20}) = \int_{\infty}^{\infty} (f_{20}'/f_{20})^2 f_{20}dx < \infty$;

(2.1.4) $E(X_{\nu 1}^{(k)}) = 0$ (assumed without any loss of generality) and

$E|X_{\nu 1}^{(k)}|^{4+\delta} < c < \infty$, for some $\delta > 0$ and for all $k = 1, \ldots, p$, $\nu \geq 1$;

(2.1.5) $\underset{\sim}{\Sigma} = \mathrm{Var}(\underset{\sim}{X}_{\nu 1})$ is p.d.

Let

(2.1.6) $\phi(u) = - f_{20}'(F_{20}^{-1}(u))/f_{20}(F_{20}^{-1}(u)), \quad 0 < u < 1,$

where $F_{20}^{-1}(u) = \inf\{x : F_{20}(x) \geq u\}$;

(2.1.7) $\phi^*(u) = - g'(G^{-1}(u))/g(G^{-1}(u)), \quad 0 < u < 1,$

where G is a d.f. with density function g satisfying (2.1.3); thus

(2.1.8) $I(g) = \int_0^1 \phi^{*2}(u)du = \int_{\infty}^{\infty} (g'/g)^2 \, gdx < \infty.$

We introduce a class of score functions $a_\nu(i)$, $i = 1, 2, \ldots, \nu$, satisfying

(2.1.9) $\lim_{\nu\to\infty} \int_0^1 \{a_\nu(1 + [u\nu]) - \phi^*(u)\}^2 \, du = 0,$

It may be noted that under H_0, $f_\nu(y,\underset{\sim}{x}) = f(y,\underset{\sim}{x}) = f_{01}(\underset{\sim}{x})f_{20}(\sigma^{-1}(y - \beta_0))$, for all $\nu \geq 1$ and thus $Y_{\nu i}$ and $\underset{\sim}{X}_{\nu i}$ are stochastically independent for all $i = 1, 2, \ldots, \nu$ and all $\nu \geq 1$. We shall use the notations $\overline{X}_\nu^{(k)} = \nu^{-1} \sum_{i=1}^{\nu} X_{\nu i}^{(k)}$ $(k = 1, \ldots, p)$ and $\overline{Y}_\nu = \nu^{-1} \sum_{i=1}^{\nu} Y_{\nu i}$. The following assumptions are made:

(2.1.3) $f_{20}'(y) = (d/dy)f_{20}(y)$ exists and $I(f_{20}) = \int_{\infty}^{\infty} (f_{20}'/f_{20})^2 f_{20}dx < \infty$;

(2.1.4) $E(X_{\nu 1}^{(k)}) = 0$ (assumed without any loss of generality) and

$E|X_{\nu 1}^{(k)}|^{4+\delta} < c < \infty$, for some $\delta > 0$ and for all $k = 1, \ldots, p$, $\nu \geq 1$;

(2.1.5) $\underset{\sim}{\Sigma} = \mathrm{Var}(\underset{\sim}{X}_{\nu 1})$ is p.d.

Let

(2.1.6) $$\phi(u) = - f_{20}'(F_{20}^{-1}(u))/f_{20}(F_{20}^{-1}(u)), \quad 0 < u < 1,$$

where $F_{20}^{-1}(u) = \inf\{x : F_{20}(x) \geq u\}$;

(2.1.7) $$\phi^*(u) = - g'(G^{-1}(u))/g(G^{-1}(u)), \quad 0 < u < 1,$$

where G is a d.f. with density function g satisfying (2.1.3); thus

(2.1.8) $$I(g) = \int_0^1 \phi^{*2}(u)du = \int_{\infty}^{\infty} (g'/g)^2 \, gdx < \infty.$$

We introduce a class of score functions $a_\nu(i)$, $i = 1, 2, \ldots, \nu$, satisfying

(2.1.9) $$\lim_{\nu\to\infty} \int_0^1 \{a_\nu(1 + [u\nu]) - \phi^*(u)\}^2 \, du = 0,$$

where [x] denotes the highest integer less than or equal to x. In particular, we can take the following types of score functions as suggested by Hájek [19]:

(a) $a_\nu(i) = E[\phi^*(U_\nu^{(i)})]$, $i = 1,\ldots, \nu$, $\nu \geq 1$;

(b) $a_\nu(i) = \phi^*(i/(\nu + 1))$, $i = 1,\ldots, \nu$, $\nu \geq 1$;

(c) $a_\nu(i) = \nu\int_{(i-1)/\nu}^{i/\nu} \phi^*(u)\,du$, $i = 1,\ldots, \nu$, $\nu \geq 1$, where

$U_{\nu 1}^{(1)} \ \ldots \ U_{\nu\nu}^{(\nu)}$

denote ordered statistics in a sample $U_{\nu 1},\ldots, U_{\nu\nu}$ of size ν from a uniform distribution on (0, 1), $U_{\nu i}$'s being distributed independently of $X_{\nu i}^{(k)}$'s ($i = 1,\ldots, \nu$; $k = 1,\ldots, p$). Let $F_k(x)$ be the marginal distribution of $X_{\nu 1}^{(k)}$ and $F_{k,k'}(x, y)$ the joint bivariate marginal distribution of $X_{\nu 1}^{(k)}$ and $X_{\nu 1}^{(k')}$.

Consider another sequence of random vectors $\{\underset{\sim}{W}_\nu\}$, where, $\underset{\sim}{W}_\nu = (W_\nu^{(1)},\ldots, W_\nu^{(p)})'$ is assumed to be distributed independently of $\underset{\sim}{Y}_\nu = (Y_{\nu 1},\ldots, Y_{\nu\nu})'$. Also, assume that the conditions (2.1.4) and (2.1.5) are satisfied with $\underset{\sim}{W}_\nu$ replacing $\underset{\sim}{X}_{\nu 1}$. Let F_k^* denote the d.f. of $W_{\nu i}^{(k)}$ and $\psi_k^*(u) = F_k^{*-1}(u)$, $k = 1, 2,\ldots, p$. Consider another class of "score" functions $b_{k\nu}(i)$, ($i = 1, 2,\ldots, \nu$, $\nu \geq 1$; $k = 1, 2,\ldots, p$) satisfying

$$(2.1.10) \qquad \lim_{\nu\to\infty} \int_0^1 \{b_{k\nu}(1 + [u\nu]) - \psi_k^*(u)\}^2 \, du = 0.$$

In section 2.2, a class of permutationally (conditionally) invariant distribution-free tests has been proposed and studied. In section 2.3 the asymptotic permutation distribution of the proposed class of test statistics (under H_0) has been developed. The asymptotic non-null distributions have been obtained in section 2.4 by using the contiguity lemmas of LeCam (see e.g. [19] or [25]). In section 2.5 the optimality properties of the proposed tests have been developed under the conditions of Wald and the ARE results of the allied tests (in Pitman's sense) have been derived.

In the remainder of this section, we prove a lemma which will be used repeatedly in the subsequent sections.

LEMMA 2.1.1. Let $\{X_{\nu i}, i = 1,\ldots, \nu; \nu \geq 1\}$ be a double sequence of r.v.'s such that for each ν, $X_{\nu 1},\ldots, X_{\nu\nu}$ are independently distributed with d.f. $F_{\nu i}(x)$ for which $E|X_{\nu i}|^{2+\delta} < c < \infty$ for some $\delta > 0$ and all $i = 1,\ldots, \nu$. Then, $\overline{X}_\nu - E(\overline{X}_\nu) \to 0$ a.e. as $\nu \to \infty$.

The proof of the above lemma is based on the following result:

LEMMA 2.1.2. Let $X_{\nu 1}, X_{\nu 2},\ldots, X_{\nu\nu}$ be samples from distributions with d.f. $F_{\nu 1}(x), F_{\nu 2}(x),\ldots, F_{\nu\nu}(x)$ respectively. If there exists η (> 2) such that $E|X_{\nu i}|^\eta < c < \infty$, for all $i = 1, 2,\ldots, \nu$, $\nu \geq 1$, then given any $\varepsilon > 0$, $P\{|\overline{X}_\nu - \overline{\mu}_\nu| > \varepsilon\} < K/(\varepsilon^\eta \nu^{\eta/2})$ for all $\nu > \nu_0$ and some positive K, where $\overline{X}_\nu = \nu^{-1} \cdot \sum_{i=1}^{\nu} X_{\nu i}$, $\mu_\nu = E(\overline{X}_\nu)$.

Proof. The proof of the lemma 2.1.2 follows along the lines of Brillinger [8] (who however considered the case of i.i.d.r.v. $X_1, X_2,\ldots$ with $E|X_1|^\eta < \infty$), and hence is omitted.

Proof of LEMMA 2.1.1. Given any $\varepsilon > 0$, let A_ν denote the event "$|\overline{X}_\nu - E(\overline{X}_\nu)| > \varepsilon$." Let C be the event that "infinitely many A_ν

happen." Using lemma 1.1 we get,

$$P\{|\overline{X}_\nu - E(\overline{X}_\nu)| > \varepsilon\} \le K/(\varepsilon^{2+\delta}\,\nu^{1+\frac{\delta}{2}}), \text{ for all } \nu > \nu_0,$$

so that

$$\sum_{\nu=1}^{\infty} P(A_\nu) \le \sum_{\nu=1}^{\nu_0} P(A_\nu) + (K/\varepsilon^{2+\delta}) \sum_{\nu=\nu_0+1}^{\infty} \nu^{-1-\frac{\delta}{2}} < \infty.$$

Using now the well-known Borel-Cantelli lemma (see e.g. [27]) we get $P(C) = 0$ i.e. A_ν can happen only "finitely many times." Hence

$$\overline{X}_\nu - E(\overline{X}_\nu) \to 0 \quad \text{a.e. as } \nu \to \infty.$$

2.2. <u>A class of conditonally distribution-free rank-order tests.</u>

We rank the observations within each row of the stochastic matrix $\underset{\sim}{Z}_\nu$, ranking being done independently between rows. Let $R^{(0)}_{\nu i}$ be the rank of $Y_{\nu i}$ among $Y_{\nu 1}, \ldots, Y_{\nu\nu}$ and $R^{(k)}_{\nu i}$ the rank of $X^{(k)}_{\nu i}$ among $X^{(k)}_{\nu 1}, \ldots, X^{(k)}_{\nu\nu}$ $(k = 1, \ldots, p;\ \nu \ge 1)$; we define the <u>collection</u> (-rank) matrix by

$$(2.2.1) \qquad \underset{\sim}{R}_\nu = \begin{bmatrix} R^{(0)}_{\nu 1} & \cdots & R^{(0)}_{\nu\nu} \\ R^{(1)}_{\nu 1} & \cdots & R^{(1)}_{\nu\nu} \\ \cdots & \cdots & \cdots \\ R^{(p)}_{\nu 1} & \cdots & R^{(p)}_{\nu\nu} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{R}^{(0)}_\nu \\ \underset{\sim}{R}^{(1)}_\nu \\ \vdots \\ \underset{\sim}{R}^{(p)}_\nu \end{bmatrix}$$

where $\underset{\sim}{R}^{(k)}_\nu = (R^{(k)}_{\nu 1}, \ldots, R^{(k)}_{\nu\nu})$, $k = 0, 1, \ldots, p$. Each row of the above stochastic matrix is a random permutation of the integers $1, 2, \ldots, \nu$.

Let

$$(2.2.2)\quad S_{k\nu} = \nu^{-1/2} \sum_{i=1}^{\nu} [b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}][a_\nu(R_{\nu i}^{(0)}) - \bar{a}_\nu],\ k = 1,\dots, p,$$

where $\bar{b}_{k\nu} = \nu^{-1} \sum_{i=1}^{\nu} b_{k\nu}(i)$, $(k = 1,\dots, p)$ and $\bar{a}_\nu = \nu^{-1} \sum_{i=1}^{\nu} a_\nu(i)$.

The test statistic to be proposed is a quadratic form in the $S_{k\nu}$'s.

The nonparametric methods developed in this paper are useful when the $X_{\nu i}^{(k)}$'s are partially informed, i.e., they are not observable, but only the ranks on them are available. Our techniques are equally applicable when we use mixed rank statistics, i.e., using the observations $X_{\nu i}^{(k)}$'s as such instead of rank-scores on them.

The test criterion is formulated by using the rank-permutation principle of Chatterjee and Sen (1964), (also discussed in Puri and Sen (1966)).

Since each row of $\underset{\sim}{R}_\nu$ is a random permutation of the numbers 1, 2,..., ν, $\underset{\sim}{R}_\nu$ is a random matrix having $(\nu!)^{p+1}$ possible realizations. Now, as has been noted after (2.1.2), under the null hypothesis the Y_ν's are distributed independently of the $X_{k\nu}$'s $(k = 1,\dots, p)$. By assumption, the Y_ν's are identically distributed. Hence, under H_0, the $\nu!$ possible permutations of Y_ν's are independent of the permutations of $X_{k\nu}$'s and are equiprobable among themselves, each permutation having the probability $(\nu!)^{-1}$. So, if we delete $\underset{\sim}{R}_\nu^{(0)}$ in $\underset{\sim}{R}_\nu$, we are left with a rank matrix involving ranks of the $X_{k\nu}$'s which we denote by $\underset{\sim}{R}_{\nu(0)}$. Two collection matrices, say, $\underset{\sim}{R}_{\nu(0)}$ and $\underset{\sim}{R}^*_{\nu(0)}$ are said to be permutationally equivalent when it is possible to arrive at one from the other by only a number of interchanges of its columns. So if $\underset{\sim}{R}^*_{\nu(0)}$ is a matrix having the same column vectors as $\underset{\sim}{R}_{\nu(0)}$, but the columns are so arranged that the first row of $\underset{\sim}{R}^*_{\nu(0)}$ consists of the numbers 1, 2,..., ν in the natural order, i.e.

$$(2.2.3) \qquad \underset{\sim}{R}^*_{\nu(0)} = \begin{bmatrix} 1 & 2 & \cdots & \nu \\ R^{*(2)}_{\nu 1} & R^{*(2)}_{\nu 2} & \cdots & R^{*(2)}_{\nu\nu} \\ \cdots & \cdots & \cdots & \cdots \\ R^{*(p)}_{\nu 1} & R^{*(p)}_{\nu 2} & \cdots & R^{*(p)}_{\nu\nu} \end{bmatrix},$$

then $\underset{\sim}{R}^*_{\nu(0)}$ is permutationally equivalent to $\underset{\sim}{R}_{\nu(0)}$. Thus corresponding to each $\underset{\sim}{R}^*_{\nu(0)}$, there will be a set $\Sigma(\underset{\sim}{R}^*_{\nu(0)})$ of $\nu!$ realizations of $\underset{\sim}{R}_{\nu(0)}$ such that any member of the set is permutationally equivalent to $\underset{\sim}{R}^*_{\nu(0)}$. The probability distribution of $\underset{\sim}{R}_{\nu(0)}$ over the $(\nu!)^p$ possible realizations will depend on $F_{01}(\underset{\sim}{x})$ even under H_0 in (2.1.2), (unless $F_{01}(\underset{\sim}{x}) = \prod_{k=1}^{p} F_k(x_k)$); thus rank-statistics $\underset{\sim}{S}_\nu = (S_{1\nu},\ldots, S_{p\nu})'$ are, in general, not distribution-free under H_0. However, given a particular set $\Sigma(\underset{\sim}{R}^*_{\nu(0)})$ (of $\nu!$ realizations), the conditonal distribution of $\underset{\sim}{R}_{\nu(0)}$ over the $\nu!$ permutations of the columns of $\underset{\sim}{R}^*_{\nu(0)}$ would be uniform under H_0, i.e.

$$(2.2.4) \quad P\{\underset{\sim}{R}_{\nu(0)} = \underset{\sim}{r}_{\nu(0)} \mid \Sigma(\underset{\sim}{R}^*_{\nu(0)}),\ H_0\} = (\nu!)^{-1} \text{ for all } \underset{\sim}{r}_{\nu(0)} \in \Sigma(\underset{\sim}{R}^*_{\nu(0)})$$

irrespective of $F_{01}(\underset{\sim}{x})$. Let $\mathcal{P}_\nu$ denote the permutational (conditional) probability measure generated by the conditional law in (2.2.4). Then, after some algebraic computations, we get the following results:

$$(2.2.5) \qquad E(S_{k\nu} | \mathcal{P}_\nu) = 0, \quad k = 1,\ldots, p;$$

$$(2.2.6) \quad V_{kk',\nu} = \mathrm{Cov}\,(S_{k\nu}, S_{k'\nu} | \mathcal{P}_\nu)$$

$$= (\nu/(\nu-1))\{\nu^{-1} \sum_{i=1}^{\nu} (b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu})(b_{k'\nu}(R_{\nu i}^{(k')}) - \bar{b}_{k'\nu})\}\cdot$$

$$\{\nu^{-1} \sum_{i=1}^{\nu} (a_\nu(i) - \bar{a}_\nu)^2\}, \quad (k, k' = 1,\ldots, p).$$

We propose the following test statistic:

$$(2.2.7) \qquad M_\nu = \underset{\sim}{S}'_\nu \, \underset{\sim}{V}^*_\nu \, \underset{\sim}{S}_\nu ,$$

where $\underset{\sim}{V}^*_\nu$ is a generalized inverse of $\underset{\sim}{V}_\nu = ((V_{kk'\nu}))$. We shall show in the next section that under the assumptions (2.1.3), (2.1.4), (2.1.9) and (2.1.10), $\underset{\sim}{V}_\nu$ converges in probability to the matrix $\underset{\sim}{\Sigma}_0 \, I(g)$, where

$$(2.2.8) \qquad \underset{\sim}{\Sigma}_0 = ((\sigma^0_{kk'}))$$

with

$$\sigma_{kk} = \int_0^1 \psi_k^{*2}(u)\,du, \quad k = 1,\ldots, p, \quad \text{and}$$

$$\sigma_{kk'} = \int_0^1 \int_0^1 \psi_k^*(u)\psi_{k'}^*(u')\,dH_{kk'}(u,u'), \quad 1 \le k \ne k' \le p,$$

$H_{k,k'}(u,u')$ being the bivariate joint d.f. of $F_k(X_{\nu 1}^{(k)})$ and $F_{k'}(X_{\nu 1}^{(k')})$, $1 \le k \ne k' \le p$. The following test procedure is proposed based on the critical function $\zeta_1(\underset{\sim}{Z}_\nu)$:

$$(2.2.9) \qquad \zeta_1(\underset{\sim}{Z}_\nu) = \begin{cases} 1 & , \text{ if } M_\nu > M_{\nu,\varepsilon} \\ \delta_{1\nu\varepsilon} & , \text{ if } M_\nu = M_{\nu,\varepsilon} \\ 0 & , \text{ if } M_\nu < M_{\nu,\varepsilon} \end{cases} ,$$

where $M_{\nu,\varepsilon}$ and $\delta_{1\nu\varepsilon}$ are so chosen that $E_{\mathcal{P}_\nu}(\zeta_1(\underset{\sim}{Z}_\nu)) = \varepsilon$. This implies that $E(\zeta_1(\underset{\sim}{Z}_\nu)|H_0) = \varepsilon$, $E(\cdot|H_0)$ standing for expectation under H_0, i.e. $\zeta_1(\underset{\sim}{Z}_\nu)$ is a similar size ε test. However, this procedure requires evaluation of $\nu!$ possible realizations of M_ν. The task becomes prohibitively laborious for large ν and this leads us to the study of the asymptotic distribution of the permutation test statistic M_ν.

2.3. Asymptotic permutation distribution of M_ν.

We shall now prove a theorem which will be used in deriving the asymptotic permutation distribution of M_ν. This theorem generalizes Theorem 4.2 of Puri and Sen (1966) in the sense that unlike Puri and Sen we do not have to assume the existence of the derivatives of the score functions $\phi^*(u)$ or $\psi_k^*(u)$'s and the boundedness conditions thereon. In proving the result all we need to assume is the square integrability of ϕ^* and the uniform boundedness of the $(2 + \delta)^{\text{th}}$ order moments of $W_{\nu i}^{(k)}$'s $(i = 1,\ldots, \nu, \; k = 1,\ldots, p)$ for some $\delta > 0$; the latter implies the conditions $|\psi_k^*(u)| \le [u(1 - u)]^{-1/2+\delta'}$ for some appropriate $\delta' > 0$ as assumed in [29].

<u>THEOREM 2.3.1.</u> <u>Under</u> (2.1.8) $\underset{\sim}{V}_\nu \to I(g)\underset{\sim}{\Sigma}_0$ <u>in probability as</u> $\nu \to \infty$.

<u>Proof.</u> $\nu^{-1} \sum_{i=1}^{\nu} [a_\nu(i) - \bar{a}_\nu]^2 = \nu^{-1} \sum_{i=1}^{\nu} a_\nu^2(i) - \bar{a}_\nu^2$. Now,

$$(2.2.9) \qquad \zeta_1(\underset{\sim}{Z}_\nu) = \begin{cases} 1 & , \text{ if } M_\nu > M_{\nu,\varepsilon} \\ \delta_{1\nu\varepsilon} & , \text{ if } M_\nu = M_{\nu,\varepsilon} \\ 0 & , \text{ if } M_\nu < M_{\nu,\varepsilon} \end{cases} ,$$

where $M_{\nu,\varepsilon}$ and $\delta_{1\nu\varepsilon}$ are so chosen that $E_{\mathcal{P}_\nu}(\zeta_1(\underset{\sim}{Z}_\nu)) = \varepsilon$. This implies that $E(\zeta_1(\underset{\sim}{Z}_\nu)|H_0) = \varepsilon$, $E(\cdot|H_0)$ standing for expectation under H_0, i.e. $\zeta_1(\underset{\sim}{Z}_\nu)$ is a similar size ε test. However, this procedure requires evaluation of $\nu!$ possible realizations of M_ν. The task becomes prohibitively laborious for large ν and this leads us to the study of the asymptotic distribution of the permutation test statistic M_ν.

2.3. Asymptotic permutation distribution of M_ν.

We shall now prove a theorem which will be used in deriving the asymptotic permutation distribution of M_ν. This theorem generalizes Theorem 4.2 of Puri and Sen (1966) in the sense that unlike Puri and Sen we do not have to assume the existence of the derivatives of the score functions $\phi^*(u)$ or $\psi_k^*(u)$'s and the boundedness conditions thereon. In proving the result all we need to assume is the square integrability of ϕ^* and the uniform boundedness of the $(2+\delta)^{\text{th}}$ order moments of $W_{\nu i}^{(k)}$'s $(i = 1,\ldots, \nu,\ \ k = 1,\ldots, p)$ for some $\delta > 0$; the latter implies the conditions $|\psi_k^*(u)| \le [u(1-u)]^{-1/2+\delta'}$ for some appropriate $\delta' > 0$ as assumed in [29].

THEOREM 2.3.1. Under (2.1.8) $\underset{\sim}{V}_\nu \to I(g)\underset{\sim}{\Sigma}_0$ in probability as $\nu \to \infty$.

Proof. $\nu^{-1} \sum_{i=1}^{\nu} [a_\nu(i) - \bar{a}_\nu]^2 = \nu^{-1} \sum_{i=1}^{\nu} a_\nu^2(i) - \bar{a}_\nu^2$. Now,

$\bar{a}_\nu = \int_0^1 a_\nu(1 + [u\nu])du = \int_0^1 [a_\nu(1 + [u\nu]) - \phi^*(u)]du$, (since $\int_0^1 \phi^*(u)du = 0$). Using now (2.1.9) and the Schwarz inequality, we get, $\bar{a}_\nu \to 0$ as $\nu \to \infty$. Further,

$$\nu^{-1} \sum_{i=1}^{\nu} a_\nu^2(i) = \int_0^1 a_\nu^2(1 + [u\nu])du = \int_0^1 [a_\nu(1 + [u\nu]) - \phi^*(u)]^2 du$$
$$+ 2\int_0^1 \phi^*(u)[a_\nu(1 + [u\nu]) - \phi^*(u)]du + \int_0^1 \phi^{*2}(u)du$$

Using now, (2.1.9) and the fact that

$$\int_0^1 |\phi^*(u)\{a_\nu(1 + [u\nu]) - \phi^*(u)\}|du \le I^{\frac{1}{2}}(g)\{\int_0^1 [a_\nu(1 + [u\nu]) - \phi^*(u)]^2 du\}^{\frac{1}{2}}$$

with $I(g) < \infty$, we get, $\nu^{-1} \sum_{i=1}^{\nu} a_\nu^2(i) \to I(g)$ as $\nu \to \infty$. Thus,

$$(2.3.1) \qquad \nu^{-1} \sum_{i=1}^{\nu} [a_\nu(i) - \bar{a}_\nu]^2 \to I(g) \text{ as } \nu \to \infty.$$

Proceeding exactly in the same way and using (2.1.10), we can show that

$$(2.3.2) \quad \nu^{-1} \sum_{i=1}^{\nu} [b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}]^2 = \nu^{-1} \sum_{i=1}^{\nu} [b_{k\nu}(i) - \bar{b}_{k\nu}]^2 \to \sigma_{kk}^0$$

as $\nu \to \infty$, $(k = 1,\dots, p)$.

Using (2.2.6), (2.3.1) and (2.3.2) we get,

$$(2.3.3) \qquad V_{kk,\nu} \to \sigma_{kk}^0 I(g) \quad \text{as } \nu \to \infty.$$

For $k \neq k'$, we get from (2.2.6),

$$V_{kk'\nu} = (\nu/(\nu-1))\{\nu^{-1}\sum_{i=1}^{\nu} b_{k\nu}(R_{\nu i}^{(k)})b_{k'\nu}(R_{\nu i}^{(k')}) - \bar{b}_{k\nu}\bar{b}_{k'\nu}\} .$$

$$\{\nu^{-1}\sum_{i=1}^{\nu} a_\nu^2(i) - \bar{a}_\nu^2\} .$$

For proving the theorem, all we have to show now is that

$$(2.3.4) \quad \nu^{-1}\sum_{i=1}^{\nu} b_{k\nu}(R_{\nu i}^{(k)})b_{k'\nu}(R_{\nu i}^{(k')}) \to \sigma_{kk'}^0 \text{ in probability.}$$

We introduce the following notations:

$$(2.3.5) \quad H_{kk'\nu}(u, v) = \nu^{-1}[\# \text{ of}\{F_k(X_{\nu i}^{(k)}), F_{k'}(X_{\nu i}^{(k')})\} \le (u, v)];$$

$$(2.3.6) \quad H_{k\nu}(u) = \nu^{-1}[\# \text{ of } F_k(X_{\nu i}^{(k)}) \le u] ;$$

$$(2.3.7) \quad C_{k\nu}(i/(\nu+1)) = b_{k\nu}(i); \quad (i = 1,\dots, \nu;\ 1 \le k \ne k' \le p).$$

Hence,

$$(2.3.8) \quad \nu^{-1}\sum_{i=1}^{\nu} b_{k\nu}(R_{\nu i}^{(k)})b_{k'\nu}(R_{\nu i}^{(k')})$$

$$= \int_0^1\int_0^1 C_{k\nu}(\nu H_{k\nu}(u)/(\nu+1))C_{k'\nu}(\nu H_{k'\nu}(v)/(\nu+1))dH_{kk'\nu}(u,v)$$

$$= \int_0^1\int_0^1 [C_{k\nu}(\nu H_{k\nu}(u)/(\nu+1)) - \psi_k^*(\nu H_{k\nu}(u)/(\nu+1))]C_{k'\nu}(\nu H_{k'\nu}(v)/(\nu+1))dH_{kk'\nu}(u,v)$$

$$+ \int_0^1\int_0^1 [C_{k'\nu}(\nu H_{k'\nu}(v)/(\nu+1)) - \psi_{k'}^*(\nu H_{k'\nu}(v)/(\nu+1))]\psi_k^*(\nu H_{k\nu}(u)/(\nu+1))dH_{kk'\nu}(u,v)$$

$$+ \int_0^1\int_0^1 [\psi_k^*(\nu H_{k\nu}(u)/(\nu+1)) - \psi_k^*(u)]\psi_{k'}^*(H_{k'\nu}(\nu)/(\nu+1))dH_{kk'\nu}(u,v)$$

$$+ \int_0^1\int_0^1 [\psi^*_{k'}(\nu H_{k'\nu}(v)/(\nu+1)) - \psi^*_{k'}(v)]\psi^*_k(u)\,dH_{kk'\nu}(u,v)$$

$$+ \int_0^1\int_0^1 \psi^*_k(u)\psi^*_{k'}(v)\,dH_{kk'\nu}(u,v)$$

$$= \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)}, \text{ (say)}.$$

$$(2.3.9) \quad |\text{(I)}| \le \{\int_0^1 [C_{k\nu}(\nu H_{k\nu}(u)/(\nu+1)) - \psi^*_k(\nu H_{k\nu}(u)/(\nu+1))]^2 dH_{k\nu}(u)\}^{1/2}$$

$$\{\int_0^1 C^2_{k'\nu}(\nu H_{k'\nu}(v)/(\nu+1))\,dH_{k'\nu}(v)\}^{1/2}$$

$$= \{\nu^{-1}\sum_{i=1}^{\nu}[C_{k\nu}(R^{(k)}_{\nu i}/(\nu+1)) - \psi^*_k(R^{(k)}_{\nu i}/(\nu+1))]^2\}^{1/2} \cdot$$

$$\{\nu^{-1}\sum_{i=1}^{\nu} C^2_{k'\nu}(R^{(k')}_{\nu i}/(\nu+1))\}^{1/2}$$

$$= \{\nu^{-1}\sum_{i=1}^{\nu}[b_{k\nu}(i) - \psi^*_k(i/(\nu+1))]^2\}^{1/2}\{\nu^{-1}\sum_{i=1}^{\nu} b^2_{k\nu}(i)\}^{1/2}$$

Now,

$$\nu^{-1}\sum_{i=1}^{\nu}[b_{k\nu}(i) - \psi^*_k(i/(\nu+1))]^2 = \int_0^1 [b_{k\nu}(1+[u\nu]) - \psi^*_k((1+[u\nu])/(\nu+1))]^2 du$$

$$\le 2\int_0^1 [b_{k\nu}(1+[u\nu]) - \psi^*_k(u)]^2 du + 2\int_0^1 [\psi^*_k(1+[u\nu]/(\nu+1)) - \psi^*_k(u)]^2 du$$

Since $\psi^*_k(u)$ is a monotonically non-decreasing and square integrable function of u, using lemma a (p. 164) of [19], we get, $\int_0^1 [\psi^*_k((1+[u\nu])/(\nu+1)) - \psi^*_k(u)]^2 du \to 0$ as $\nu \to \infty$. Using now (2.1.10) which implies $\nu^{-1}\sum_{i=1}^{\nu} b^2_{k\nu}(i) \to \int_0^1 \psi^{*2}_k(u)\,du < \infty$, we get from (2.3.9), (I) $\to 0$ as $\nu \to \infty$. Similarly (II) $\to 0$ as $\nu \to \infty$.

(2.3.10):

$$|(\mathrm{III})| \le [\int_0^1 \{\psi_k^*(\nu H_{k\nu}(u)/(\nu+1) - \psi_k^*(u)\}^2 dH_{k\nu}(u)]^{\frac{1}{2}} [\int_0^1 \psi_{k'}^{*2}(\nu H_{k'\nu}(v)/(\nu+1)\, dH_{k'\nu}(v)]^{1/2}$$

But,

$$(2.3.11)\quad \int_0^1 \psi_{k'}^{*2}\ (\nu H_{k'\nu}(v)/(\nu+1))\, dH_{k'\nu}(v) \to \sigma^0_{k'k'} \quad \text{as } \nu \to \infty.$$

Also,

$$(2.3.12)\qquad E[\int_0^1 [\psi_k^*\ (\nu H_{k\nu}(u)/(\nu+1)) - \psi_k^*(u)]^2 dH_{k\nu}(u)]$$

$$= E[\nu^{-1} \sum_{i=1}^{\nu} \{\psi_k^*(R_{\nu i}^{(k)}/(\nu+1)) - \psi_k^*(U_{\nu i})\}^2]$$

$$= E[\nu^{-1} \sum_{i=1}^{\nu} (\{\psi_k^{**}(R_{\nu i}^{(k)}/(\nu+1)) - \psi_k^{**}(U_{\nu i})\} + \{\psi_k^{**}(U_{\nu i}) - \psi_k^*(U_{\nu i})\})^2$$

where $\psi_k^{**}(\lambda)$ is a function defined on $0 < \lambda \le 1$ by

$$(2.3.13)\quad \psi_k^{**}(\lambda) = \psi_k^*(i/(\nu+1)) \text{ for } (i-1)/\nu < \lambda \le i/\nu,\ (i = 1,\ldots,\nu)$$

so that $\psi_k^{**}(i/(\nu+1)) = \psi_k^*(i/(\nu+1))$ for all $i = 1,\ldots,\nu$. Then using the elementary inequality $(a + b)^2 \le 2(a^2 + b^2)$, we get from (2.3.12),

$$(2.3.14)\qquad E[\int_0^1 [\psi_k^*(\nu H_{k\nu}(u)/(\nu+1)) - \psi_k^*(u)]^2 dH_{k\nu}(u)]$$

$$\le 2\ E[\nu^{-1} \sum_{i=1}^{\nu} \{\psi_k^{**}(R_{\nu i}^{(k)}/(\nu+1)) - \psi_k^{**}(U_{\nu i})\}^2]$$

$$+ 2\ E[\nu^{-1} \sum_{i=1}^{\nu} \{\psi_k^{**}(U_{\nu i}) - \psi_k^*(U_{\nu i})\}^2]$$

$$= 2\, E[\psi_k^{**}(R_{\nu 1}^{(k)}/(\nu+1)) - \psi_k^{**}(U_{\nu 1})]^2$$

$$+ 2 \int_0^1 [\psi_k^{**}(u) - \psi_k^{*}(u)]^2 du$$

Now, $\int_0^1 [\psi_k^{**}(u) - \psi_k^{*}(u)]^2 du = \int_0^1 [\psi_k^{*}(1 + [u\nu]/(\nu+1)) - \psi_k^{*}(u)]^2 du.$

Since $\psi_k^*(u)$ is a monotonically non-decreasing and square integrable function of u, using lemma a, p. 164 of [19], $\int_0^1 [\psi_k^{**}(u) - \psi_k^{*}(u)]^2 du \to 0$ as $\nu \to \infty$. Again using lemma 2.1 of [16], we get,

(2.3.15) $E[\psi_k^{**}(R_{\nu 1}^{(k)}/(\nu+1)) - \psi_k^{**}(U_{\nu 1})]^2$

$$\leq 2\sqrt{2}\, (\nu^{-1/2} \max_{1 \leq i \leq \nu} |\psi_k^{**}(i/(\nu+1)) - \overline{\psi}_k^{**}|)(\nu^{-1} \sum_{i=1}^{\nu} (\psi_k^{**}(i/(\nu+1)) - \overline{\psi}_k^{**})^2)^{1/2},$$

where $\overline{\psi}_k^{**} = \nu^{-1} \sum_{i=1}^{\nu} \psi_k^{**}(i/(\nu+1)) = \nu^{-1} \sum_{i=1}^{\nu} \psi_k^{*}(i/(\nu+1)) \to \int_0^1 \psi_k^*(u)\, du = 0$

as $\nu \to \infty$. Also, $\nu^{-1} \sum_{i=1}^{\nu} \psi^{**2}(i/(\nu+1)) \to \sigma_{kk}^0 < \infty$ as $\nu \to \infty$, and

$$\nu^{-1/2} \max_{1 \leq i \leq \nu} |\psi_k^{**}(i/(\nu+1))| = \nu^{1/2} \max_{1 \leq i \leq \nu} \left|\int_{(i-1)/\nu}^{i/\nu} \psi_k^*(u)\, du\right|$$

$$\leq \max_{1 \leq i \leq \nu} \{\int_{(i-1)/\nu}^{i/\nu} \psi_k^{*2}(u)\, du\}^{1/2} \to 0$$

as $\nu \to \infty$ since $\sigma_{kk}^0 < \infty$. Now, from (2.3.15),

$$E[\Psi_k^{**}(R_{\nu 1}^{(k)}/(\nu+1)) - \psi_k^{**}(U_{\nu 1})]^2 \to 0 \quad \text{as } \nu \to \infty.$$

Thus, from (2.3.14),

(2.3.16) $E[\int_0^1 [\psi_k^*(\nu H_{k\nu}(u)/(\nu+1)) - \psi_k^*(u)]^2 \, dH_{k\nu}(u)] \to 0 \quad \text{as } \nu \to \infty.$

Using (2.3.11) and (2.3.16) with the fact that convergence in mean implies convergence in probability, we get from (2.3.10), (III) $\to 0$ in probability as $\nu \to \infty$. Similarly (IV) $\to 0$ in probability as $\nu \to \infty$. Again,

$$(V) = \nu^{-1} \sum_{i=1}^{\nu} \psi_k^*(F_k(X_{\nu i}^{(k)}))\ \psi_{k'}^*(F_{k'}(X_{\nu i}^{(k')})).$$

Since,

$$E|\psi_k^*(F_k(X_{\nu i}^{(k)}))\psi_{k'}^*(F_{k'}(X_{\nu i}^{(k')}))|^{1+\frac{\delta}{2}}$$

$$\le \{E|\psi_k^*(F_k(X_{\nu i}^{(k)}))|^{2+\delta}\ E|\psi_{k'}^*(F_{k'}(X_{\nu i}^{(k')}))|^{2+\delta}\}^{1/2} < \infty,$$

using Markov's weak law of large numbers, we get,

$$(V) \to E[\psi_k^*(F_k(X_{\nu i}^{(k)}))\ \psi_{k'}^*(F_{k'}(X_{\nu i}^{(k')}))] = \sigma_{kk'}^0$$

in probability as $\nu \to \infty$. Hence (2.3.4) is proved.

THEOREM 2.3.2. *Under the permutational model* $\mathcal{P}_\nu$ *and* (2.1.8) $\underset{\sim}{S}_\nu$ *is asymptotically* N_p $(\underset{\sim}{0}, I(g)\cdot\underset{\sim}{\Sigma}_0)$ *in probability*.

Proof. Let $\underset{\sim}{e} = (e_1, \ldots, e_p)' \ne \underset{\sim}{0}$ be a vector of finite real constants. Then

$$E(\underset{\sim}{e}'\underset{\sim}{S}_\nu|\mathcal{P}_\nu) = 0,\ E[(\underset{\sim}{e}'\underset{\sim}{S}_\nu)^2|\mathcal{P}_\nu] = (\underset{\sim}{e}'\underset{\sim}{V}_\nu\underset{\sim}{e})I(g). \tag{2.3.17}$$

We have to show that for every vector $\underset{\sim}{e}$ as above, $\underset{\sim}{e}'\underset{\sim}{S}_\nu$ is under $\mathcal{P}_\nu$ asymptotically normal $(0, (\underset{\sim}{e}'\underset{\sim}{\Sigma}_0\underset{\sim}{e})I(g))$ in probability. We have,

$$\underset{\sim}{e}'\underset{\sim}{S}_\nu = \sum_{i=1}^{\nu} \{\sum_{k=1}^{p} e_k(b_k(R_{\nu i}^{(k)}) - \bar{b}_{k\nu})\}\{a_\nu(R_{\nu i}^{(0)}) - \bar{a}_\nu\}\ . \tag{2.3.18}$$

Under p_ν, $\{\sum_{k=1}^{p} e_k(b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}),\ i = 1,\ldots,\nu\}$ is equiprobable over the $\nu!$ permutations of the columns of $R^*_{\nu(0)}$ while $\{a_\nu(i),\ i = 1,\ldots,\nu\}$ is equiprobable over the $\nu!$ permutations of the integers $1, 2,\ldots,\nu$ independently of the permutations of $\{\sum_{k=1}^{p} e_k(b_{k\nu}(R_{\nu i}^{(k)}))\ i = 1,\ldots,\nu\}$. Since $\underset{\sim}{V}_\nu \to \underset{\sim}{\Sigma}_0$ in probability as $\nu \to \infty$, to prove the theorem it is sufficient to show that the sequence $\{\sum_{k=1}^{p} e_k b_k(R_{\nu i}^{(k)})\}$ satisfies the Noether condition (see e.g. (16)) in probability and the sequence $\{a_\nu(i)\}$ satisfies the condition Q of Hájek (16), namely,

$$(2.3.19)\quad \lim_{\nu\to\infty} k_\nu/\nu = 0 \Rightarrow \lim_{\nu\to\infty} \frac{\max\limits_{1\le i_1\le\ldots\le i_{k\nu}\le\nu} \sum_{\alpha=1}^{k_\nu} [a_\nu(i_\alpha) - \bar{a}_\nu]^2}{\sum_{i=1}^{\nu} [a_\nu(i) - \bar{a}_\nu]^2} = 0\ ,$$

which implies the Noether condition, and then apply Theorem 4.2 of [16], p. 514. Let $a^2_{\nu(1)} \le a^2_{\nu(2)} \le \ldots \le a^2_{\nu(\nu)}$ be the ordered $a^2_\nu(i)$'s and let us define a step function $\eta_\nu(u)$ as

$$(2.3.20)\quad \eta_\nu(u) = a^2_\nu(i) \text{ for } (i-1)/\nu < u \le i/\nu,\ i = 1,\ldots,\nu.$$

Then $\nu^{-1} \max\limits_{1\le i_1\le\ldots\le i_{k_\nu}\nu} \sum_{\alpha=1}^{k} [a^2_\nu(i_\alpha)] = \int_{1-k_\nu/\nu}^{1} \eta_\nu(u)\,du$. But,

$$\int_0^1 \eta_\nu(u) = \nu^{-1} \sum_{i=1}^{\nu} a^2_\nu(i) \to I(g) < \infty \quad \text{as } \nu \to \infty.$$

Hence, $\int_{1-k_\nu/\nu}^{1} \eta_\nu(u)\,du \to 0$ as $\nu \to \infty$ since then $k_\nu/\nu \to 0$. Also $\bar{a}_\nu \to 0$ as $\nu \to \infty$. Hence, (2.3.21):

$$\frac{\max\limits_{1\le i_1<\ldots<i_{k_\nu}\le\nu} \sum\limits_{\alpha=1}^{k_\nu}[a_\nu(i_\alpha) - \bar{a}_\nu]^2}{\sum\limits_{i=1}^{\nu}[a_\nu(i) - \bar{a}_\nu]^2} \le \frac{2\nu^{-1} \max\limits_{1\le i_1<\ldots<i_{k_\nu}\le\nu} \sum\limits_{\alpha=1}^{k} a_\nu^2(i_\alpha) + 2\bar{a}_\nu^2}{\nu^{-1}\sum\limits_{i=1}^{\nu} a_\nu^2(i) - \bar{a}_\nu^2} \to 0$$

as $\nu \to \infty$. Putting in particular $k_\nu = 1$, it follows that the sequence $\{a_\nu(i)\}$ satisfies the Noether condition i.e.

$$\frac{\max\limits_{1\le i\le\nu} [a_\nu(i) - \bar{a}_\nu]^2}{\sum\limits_{i=1}^{\nu}[a_\nu(i) - \bar{a}_\nu]^2} \to 0 \text{ as } \nu \to \infty . \tag{2.3.22}$$

Also,

$$\begin{aligned} &\nu^{-1} \max_{1\le i\le\nu} \{\sum_{k=1}^{p} e_k[b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}]\}^2 \\ &\le \nu^{-1} \max_{1\le i\le\nu} p \sum_{k=1}^{p} e_k^2[b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}]^2 \\ &\le (p \sum_{k=1}^{p} e_k^2) \max_{1\le i\le\nu} \max_{1\le k\le p} \nu^{-1}[b_{k\nu}(i) - \bar{b}_{k\nu}]^2 \to 0 \end{aligned} \tag{2.3.23}$$

as $\nu \to \infty$ (the proof being similar to that in the case of $a_\nu(i)$'s). Further,

$$\nu^{-1}\sum_{i=1}^{\nu}\{\sum_{k=1}^{p} e_k[b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}]\}^2 = \underset{\sim}{e}'\underset{\sim}{V}_\nu\underset{\sim}{e} \to \underset{\sim}{e}'\underset{\sim}{\Sigma}_0\underset{\sim}{e} > 0 \tag{2.3.24}$$

in probability (using lemma 2.3.1 and the fact that $\underset{\sim}{\Sigma}_0$ is p.d.). Hence,

$$(2.3.25) \qquad \frac{\max\limits_{1\leq i\leq \nu} \{ \sum\limits_{k=1}^{p} e_k[b_{k\nu}(R_{\nu i}^{(k)}) - \overline{b}_{k\nu}]\}^2}{\sum\limits_{i=1}^{\nu} \{ \sum\limits_{k=1}^{p} e_k[b_{k\nu}(R_{\nu i}^{(k)}) - \overline{b}_{k\nu}]\}^2} \to 0 \text{ in probability.}$$

Hence the theorem. An immediate consequence of the above theorem is:

<u>THEOREM 2.3.3.</u> <u>Under</u> $\mathcal{P}_\nu$, M_ν <u>is asymptotically distributed as</u> χ_p^2.

2.4. <u>Asymptotic non-null distribution of</u> M_ν.

We first prove the following elementary lemma:

LEMMA 2.4.1. (i) $\nu^{-1/2} \max\limits_{1\leq i\leq \nu} |X_{\nu i}^{(k)}| \to 0$ <u>a.e. as</u> $\nu \to \infty$ <u>for all</u>

$k = 1, \ldots, p$.

(ii) $\nu^{-1} \sum\limits_{i=1}^{\nu} (X_{\nu i}^{(k)} - \overline{X}_\nu^{(k)})^2 \to E(X_{\nu i}^{(k)})^2$ <u>a.e. as</u> $\nu \to \infty$,

<u>for all</u> $k = 1, \ldots, p$.

<u>Proof</u>. We first prove (ii). We can write,

$$(2.4.1) \qquad \nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)} - \overline{X}_\nu^{(k)})^2 = \nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)})^2 - \overline{X}_\nu^{(k)2} .$$

Since $(X_{\nu 1}^{(k)})^2, \ldots, (X_{\nu\nu}^{(k)})^2$ are i.i.d. with

$$E|(X_{\nu 1}^{(k)})^2|^{2+\frac{\delta}{2}} = E|X_{\nu 1}^{(k)}|^{4+\delta} < c < \infty ,$$

using lemma 2.1.1,

$$(2.4.2) \qquad E[\nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)})^2] \to E(X_{\nu 1}^{(k)})^2 \quad \text{a.e.} \quad \text{as } \nu \to \infty.$$

[Note that the p-variate marginal d.f. of $\underset{\sim}{X}_{\nu 1}$ does not depend on ν].

Again, $E|X_{\nu 1}^{(k)}|^{2+\delta} \leq (E|X_{\nu 1}^{(k)}|^{4+\delta})^{\frac{2+\delta}{4+\delta}} < c^{\frac{2+\delta}{4+\delta}} < \infty$. So,

$$(2.4.3) \qquad \bar{X}_{\nu}^{(k)} \to E(X_{\nu 1}^{(k)}) = 0 \text{ a.e. as } \nu \to \infty .$$

(ii) follows now from (2.4.1) - (2.4.3).

To prove (i), we first recall two equivalent forms of Noether condition given by Hoeffding (1951). We state these in a slightly different form convenient for our purpose:

$$(2.4.4) \qquad (\max_{1\leq i\leq \nu} (X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)})^2)/(\sum_{i=1}^{\nu} (X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)})^2) \to 0 \text{ a.e. as } \nu \to \infty$$

$$\iff \lim_{\nu\to\infty}(\sum_{i=1}^{\nu} |X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)}|^r)/[\sum_{i=1}^{\nu} (X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)})^2]^{r/2} = 0$$

a.e. for some $r > 2$.

The right hand expression in (2.4.4) can be rewritten as

$$(2.4.5) \qquad \lim_{\nu\to\infty} \nu^{1-\frac{r}{2}} \{\nu^{-1} \sum_{i=1}^{\nu} |X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)}|^r\}/\{\nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)})^2\}^{r/2} = 0$$

a.e. for some $r > 2$.

We shall prove first (2.4.5). Taking $r = 2 + \delta'$ where $(0 < \delta' < \delta/2)$ and using an elementary inequality, we get,

$$(2.4.6) \qquad \nu^{-1} \sum_{i=1}^{\nu} |X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)}|^{2+\delta'} \leq 2^{1+\delta'}(\nu^{-1} \sum_{i=1}^{\nu} |X_{\nu i}^{(k)}|^{2+\delta'} + |\bar{X}_{\nu}^{(k)}|^{2+\delta'})$$

Taking now, $\delta'' = \dfrac{\delta - 2\delta'}{2 + \delta'}$, we have,

$$(E|X_{\nu i}^{(k)}|^{2+\delta'})^{2+\delta''} \le E|X_{\nu i}^{(k)}|^{(2+\delta')(2+\delta'')} = E|X_{\nu i}^{(k)}|^{4+\delta} < c < \infty.$$

Hence, by lemma 2.1.1,

$$(2.4.7) \qquad \nu^{-1}\sum_{i=1}^{\nu}|X_{\nu i}^{(k)}|^{2+\delta'} \to E|X_{\nu 1}^{(k)}|^{2+\delta'} \quad \text{a.e. as } \nu \to \infty.$$

Also $\overline{X}_{\nu}^{(k)} \to 0$ a.e. as $\nu \to \infty$. Using now (ii), (2.4.6) and (2.4.7), we get (2.4.5). (i) follows immediately from (2.4.7) since

$$(\nu^{-\frac{1}{2}} \max_{1\le i\le \nu} |X_{\nu i}|)^{2+\delta'} \le \nu^{-1-\frac{\delta'}{2}} \sum_{i=1}^{\nu} |X_{\nu i}|^{2+\delta'}.$$

Consider a sequence of measure spaces $(\mathcal{X}_\nu, \mathcal{A}_\nu, \mu_\nu)$, $\nu \ge 1$, a typical element of $\mathcal{X}_\nu$ being $\underset{\sim}{z}_\nu$, where

$$\underset{\sim}{z}_\nu = (y_{\nu 1}, x_{\nu 1}^{(1)}, \ldots, x_{\nu 1}^{(p)}, \ldots, y_{\nu\nu}, x_{\nu\nu}^{(1)}, \ldots, x_{\nu\nu}^{(p)})' .$$

Now take a particular alternative, say $\beta_0 = \beta_0^*$, $\underset{\sim}{\beta} = \underset{\sim}{\beta}^* = (\beta_{10}^*, \ldots, \beta_{p0}^*)'$, $\sigma = \sigma_0^*$ and associate with it the particular hypothesis $\beta_0 = \beta_0^*$, $\underset{\sim}{\beta} = \underset{\sim}{0}$, $\sigma = \sigma_0^*$. Denote the distribution of $\underset{\sim}{Z}_\nu$ under the alternative and the hypothesis by Q_ν and P_ν respectively, where $dQ_\nu = q_\nu d\mu_\nu$, $dP_\nu = p_\nu d\mu_\nu$. Let the respective densities for $\underset{\sim}{Z}_{\nu i}$ under the alternative and the hypothesis be $q_{\nu i}(\underset{\sim}{x})$ and $p_{\nu i}(\underset{\sim}{x})$, $\underset{\sim}{x} = (x_1, \ldots, x_p)'$, $i = 1, \ldots, \nu$. Let

$$(2.4.8) \qquad r_{\nu i}(\underset{\sim}{x}) = \begin{cases} q_{\nu i}(\underset{\sim}{x})/p_{\nu i}(\underset{\sim}{x}) & \text{if } p_{\nu i}(\underset{\sim}{x}) \quad 0 \\ 0 & \text{otherwise} \end{cases}$$

Consider the following two statistics:

$$(2.4.9) \qquad W_\nu = 2\sum_{i=1}^{\nu} (r_{\nu i}^{1/2} - 1) ;$$

(2.4.10) $$L_\nu = \prod_{i=1}^{\nu} r_{\nu i} .$$

We introduce the notations:

(2.4.11) $$\gamma_k = \beta^*_{k0}/\sigma^*_0 \quad (k = 1,\ldots, p),\ \underset{\sim}{\gamma}' = (\gamma_1,\ldots, \gamma_p) ,$$

$$Y'_{\nu i} = (Y_{\nu i} - \beta^*_0)/\sigma^*_0 ,\ h_{\nu i} = \nu^{-\frac{1}{2}} \underset{\sim}{\gamma}'\underset{\sim}{X}_{\nu i}\ (i = 1,\ldots, \nu).$$

The asymptotic distribution of M_ν will be obtained by using the contiguity lemmas of LeCam (see [19] or [25]). We need show that the distributions Q_ν are contiguous to the distributions P_ν. We adopt a procedure analogous to that of Hájek [17]. The following lemmas will be needed:

LEMMA 2.4.2. Under the given model and (2.1.3)-(2.1.5), $E(W_\nu | P_\nu) \to -\frac{1}{4} I(f_{20})(\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma})$ as $\nu \to \infty$.

Proof. Write $s_{20}(x) = f_{20}^{1/2}(x)$. Using the notations in (2.4.11), we get from (2.4.9),

(2.4.12) $$W_\nu = 2 \sum_{i=1}^{\nu} [\{s_{20}(Y'_{\nu i} - h_{\nu i})/s_{20}(Y'_{\nu i})\} - 1].$$

Use the Bonferroni inequality, the fact that p is fixed (independent of ν) and lemma 2.4.1 to get

(2.4.13) $$\max_{1\le i\le \nu} |h_{\nu i}| \to 0 \quad \text{a.e. as } \nu \to \infty.$$

Now, as in [17], we can write after some algebraic simplifications,

$$E(W_\nu | P_\nu, h_{\nu 1},\ldots, h_{\nu\nu}) = - \sum_{i=1}^{\nu} h^2_{\nu i}\, g_0(h_{\nu i}) ,$$

where $g_0(h_{\nu i}) = \int_{-\infty}^{\infty} [\{s_{20}(y - h_{\nu i}) - s_{20}(y)\}/h_{\nu i}]\, dy$. But, we may note that since $\max_{1\le i\le \nu} |h_{\nu i}| \to 0$ a.e. as $\nu \to \infty$, for fixed $h_{\nu i}(\omega)$'s, ω's being points in measurable sets in the basic probability space, proceeding as in [17], $g_0(h_{\nu i}) \to \frac{1}{4} I(f_{20})$ a.e. (i.e. except for a set of points ω of zero measure). Also,

$$\sum_{i=1}^{\nu} h_{\nu i}^2 = \sum_{k=1}^{p} \sum_{k'=1}^{p} \gamma_k \gamma_{k'} (\nu^{-1} \sum_{i=1}^{\nu} X_{\nu i}^{(k)} X_{\nu i}^{(k')}) \to \underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma} \quad \text{a.e.},$$

since

$$E(\nu^{-1} \sum_{i=1}^{\nu} X_{\nu i}^{(k)} X_{\nu i}^{(k')}) = \sigma_{kk'},\ k,\ k' = i,\ldots,\ p, \text{ and,}$$

$$E|X_{\nu 1}^{(k)} X_{\nu 1}^{(k')}|^{2+\delta/2} \le [E|X_{\nu 1}^{(k)}|^{4+\delta}\ E|X_{\nu 1}^{(k')}|^{4+\delta}]^{1/2} \le c < \infty,$$

so that lemma 2.1.1 can be applied. Thus to prove the lemma, all we have to show is that the random variables $\sum_{i=1}^{\nu} h_{\nu i}^2\, g_0(h_{\nu i})$ have (i) 'uniformly continuous' integrals and (ii) the integrals are 'uniformly bounded' (see chapter I), and then apply the L_r-convergence theorem given in Loève [27, p. 163] with $r = 1$. Since it can be shown that (see e.g. [19]), $|g_0(h_{\nu i})| \le \frac{1}{4} I(f_{20})$, uniformly in $i = 1,\ldots,\nu, \nu \ge 1$ and $E(\sum_{i=1}^{\nu} h_{\nu i}^2) = \underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma}$, $E|(\sum_{i=1}^{\nu} h_{\nu i} g_0(h_{\nu i}^2))| \le \frac{1}{4} I(f_{20}) \underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma}$. (i) follows, since for any measurable set A, with I_A being defined as

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \varepsilon A \\ 0 & \text{otherwise}, \end{cases}$$

$$E(\sum_{i=1}^{\nu} h_{\nu i}^2 g_0(h_{\nu i}) I_A) \le \frac{1}{4} I(f_{20}) [E(\sum_{i=1}^{\nu} h_{\nu i}^2)^2]^{1/2} (EI_A)^{1/2}.$$

$$(\sum_{i=1}^{\nu} h_{\nu i}^2)^2 = [\sum_{k=1}^{p} \sum_{k'=1}^{p} \gamma_k \gamma_{k'} \frac{1}{\nu} \sum_{i=1}^{\nu} X_{\nu i}^{(k)} X_{\nu i}^{(k')}]^2$$

$$\leq (\max_{1\leq k\leq p} \gamma_k^2)^2 (\sum_{k=1}^{p} \sum_{k'=1}^{p} \nu^{-1} \sum_{i=1}^{\nu} |X_{\nu i}^{(k)} X_{\nu i}^{(k')}|)^2$$

$$\leq p^4 (\max_{1\leq k\leq p} \gamma_k^2)^2 \sum_{k=1}^{p} \sum_{k'=1}^{p} \{\nu^{-1} \sum_{i=1}^{\nu} |X_{\nu i}^{(k)} X_{\nu i}^{(k')}|\}^2$$

$$\leq p^4 (\max_{1\leq k\leq p} \gamma_k^2)^2 \sum_{k=1}^{p} \sum_{k'=1}^{p} [\nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)})^2 (X_{\nu i}^{(k')})^2$$

Hence, $E(\sum_{i=1}^{\nu} h_{\nu i}^2)^2$

$$\leq p^4 (\max_{1\leq k\leq p} \gamma_k^2)^2 \sum_{k=1}^{p} \sum_{k'=1}^{p} \nu^{-1} \sum_{i=1}^{\nu} [E(X_{\nu i}^{(k)})^4 E(X_{\nu i}^{(k')})^4$$

$$\leq c^{4/(4+\delta)} p^6 (\max_{1\leq k\leq p} \gamma_k^2)^2 .$$

Thus, $E(\sum_{i=1}^{\nu} h_{\nu i}^2 g_0(h_{\nu i}) I_A)$

$$\leq \frac{1}{4} I(f_{20}) c^{2/(4+\delta)} p^3 (\max_{1\leq k\leq p} \gamma_k^2)(E(I_A))^{1/2} \to 0$$

uniformly in ν as $E(I_A) = P(A) \to 0$. Q.E.D.

Let $\underset{\sim}{T}_\nu = (T_{1\nu}, \ldots, T_{p\nu})'$, where

$$T_{k\nu} = \nu^{-\frac{1}{2}} \sum_{i=1}^{\nu} X_{\nu i}^{(k)} \phi(F_{20}(Y'_{\nu i}))$$

$$= \nu^{-\frac{1}{2}} \sum_{i=1}^{\nu} X_{\nu i}^{(k)} (- f'_{20}(Y'_{\nu i})/f_{20}(Y'_{\nu i}))$$

$$= - 2 \nu^{-\frac{1}{2}} \sum_{i=1}^{\nu} X_{\nu i}^{(k)} [s'_{20}(Y'_{\nu i})/s_{20}(Y'_{\nu i})], \quad k = 1, 2, \ldots, p.$$

Proceeding along Hájek [17] and using the technique of conditional expectation as in the preceeding lemma, we get,

LEMMA 2.4.3. Under the given model and the assumptions (2.1.3)-(2.1.5) $\mathrm{Var}(W_\nu - \gamma'\underset{\sim}{T}_\nu | P_\nu) \to 0$ as $\nu \to \infty$.

Using lemmas 2.4.2 and 2.4.3, it now follows that

$$E[(W_\nu - \gamma'\underset{\sim}{T}_\nu + \tfrac{1}{4} I(f_{20})\gamma'\underset{\sim}{\Sigma}\gamma)^2 | P_\nu] \to 0$$

which implies

(2.4.14) $\quad W_\nu - \gamma'\underset{\sim}{T}_\nu + \frac{1}{4} I(f_{20})\gamma'\underset{\sim}{\Sigma}\gamma \to 0$ in P_ν-probability.

LEMMA 2.4.4. Under P_ν, the given model (2.1.1) and (2.1.3)-(2.1.5), is asymptotically multinormal $(\underset{\sim}{0}, I(f_{20})\underset{\sim}{\Sigma})$ a.e.

Proof. We have to show that for every vector $\underset{\sim}{e} = (e_1, \ldots, e_p)' \neq 0$ of finite real constants, $\underset{\sim}{e}'\underset{\sim}{T}_\nu$ is asymptotically normal $(0, I(f_{20})\underset{\sim}{e}'\underset{\sim}{\Sigma}\underset{\sim}{e})$ a.e. We can write,

(2.4.15) $$\underset{\sim}{e}'\underset{\sim}{T}_\nu = \nu^{-1/2} \sum_{i=1}^{\nu} \left(\sum_{k=1}^{p} e_k X_{\nu i}^{(k)}\right) \phi(F_{20}(Y'_{\nu i})).$$

(2.4.16) $\quad E[\phi(F_{20}(Y'_{\nu i})) | P_\nu] = 0,\ E[\phi^2(F_{20}(Y'_{\nu i})) | P_\nu] = I(f_{20}) < \infty.$

Also, $\max\limits_{1 \le i \le \nu} \left| \sum_{k=1}^{p} e_k X_{\nu i}^{(k)} / \nu^{1/2} \right| \to 0$ as $\nu \to \infty$ from lemma 2.4.1 (i) and

$\sum_{i=1}^{\nu} \{\sum_{k=1}^{p} e_k X_{\nu i}^{(k)}\}^2 \to \underset{\sim}{e}'\underset{\sim}{\Sigma}\underset{\sim}{e}$ a.e. as $\nu \to \infty$ from lemma 2.4.1 (ii). So,

$\max\limits_{1 \le i \le \nu} (\sum_{k=1}^{p} e_k X_{\nu i}^{(k)})^2 / \sum_{i=1}^{\nu} \{\sum_{k=1}^{p} e_k X_{\nu i}^{(k)}\}^2 \to 0$ a.e. as $\nu \to \infty$. i.e.

(2.4.17) the sequence $\{\sum_{k=1}^{p} e_k X_{\nu i}^{(k)}\}$ satisfies the Noether condition a.e.

Under P_ν, $Y'_{\nu 1},\ldots, Y'_{\nu\nu}$, $\sum_{k=1}^{p} e_k X_{\nu 1}^{(k)},\ldots, \sum_{k=1}^{p} e_k X_{\nu\nu}^{(k)}$ are mutually independent. $\phi(F_{20}(Y'_{\nu 1})),\ldots, \phi(F_{20}(Y'_{\nu\nu}))$ (all identically distributed under the hypothesis) are Borel-measurable functions of $Y'_{\nu 1},\ldots, Y'_{\nu\nu}$ respectively. Now, using (2.4.16) and (2.4.17), conditional on the $X_{\nu i}^{(k)}$'s fixed, $\underset{\sim}{e}'\underset{\sim}{T}_\nu$ is asymptotically

$$N_1(0,[\underset{\sim}{\gamma}'((\nu^{-1}\sum_{i=1}^{\nu} X_{\nu i}^{(k)}\ X_{\nu i}^{(k')}))\underset{\sim}{\gamma}]I(f_{20}))\ .$$

Since, $((\nu^{-1}\sum_{i=1}^{\nu} X_{\nu i}^{(k)}\ X_{\nu i}^{(k')})) \to \underset{\sim}{\Sigma}$ a.e. as $\nu \to \infty$, $\underset{\sim}{e}'\underset{\sim}{T}_\nu$ is unconditionally asymptotically $N_1(0,(\underset{\sim}{e}'\underset{\sim}{\Sigma}\underset{\sim}{e})I(f_{20}))$ a.e. Q.E.D.

Using the above lemma, we get from (2.4.14),

(2.4.18) W_ν is asymptotically $N_1(-\frac{1}{4}I(f_{20})\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma},\ I(f_{20})\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma})$ a.e.

We next prove the following lemma:

LEMMA 2.4.5. <u>Under</u> (2.1.1) <u>and</u> (2.1.3)-(2.1.5),

$$\lim_{\nu\to\infty}\ \max_{1\le i\le\nu}\ P_\nu(|\{q_{\nu i}(\underset{\sim}{Z}_{\nu i})/p_{\nu i}(\underset{\sim}{Z}_{\nu i})\} - 1| > \varepsilon) = 0.$$

<u>Proof</u>. $P_\nu(|(q_{\nu i}(\underset{\sim}{Z}_{\nu i})/p_{\nu i}(\underset{\sim}{Z}_{\nu i})) - 1| > \varepsilon)$

$$\le \varepsilon^{-1}\ E[|\{q_{\nu i}(\underset{\sim}{Z}_{\nu i})/p_{\nu i}(\underset{\sim}{Z}_{\nu i})\} - 1||P_\nu]$$

$$= \varepsilon^{-1}\ E[|h_{\nu i}|\int_{-\infty}^{\infty}|\{f_{20}(y-h_{\nu i}) - f_{20}(y)\}/h_{\nu i}|dy]\ .$$

But, it can be shown after some elementary algebra that

$$\int_{-\infty}^{\infty}|\{f_{20}(y - h_{\nu i}) - f_{20}(y)\}/h_{\nu i}|dy \le I^{1/2}(f_{20}),\ (i = 1, 2,\ldots, \nu).$$

Hence, $P_\nu\ (|\{q_{\nu i}(\underset{\sim}{Z}_{\nu i})/p_{\nu i}(\underset{\sim}{Z}_{\nu i})\} - 1| > \varepsilon)$

$\le \varepsilon^{-1}\ I^{1/2}\ (f_{20})E|h_{\nu i}|$. So,

$$\text{(2.4.19)} \qquad \max_{1\le i\le\nu} P_\nu(|\{q_{\nu i}(\underset{\sim}{Z}_{\nu i})/p_{\nu i}(\underset{\sim}{Z}_{\nu i})\} - 1| > \varepsilon)$$

$$\le \varepsilon^{-1}\ I^{1/2}\ (f_{20}) \max_{1\le i\le\nu} E(|h_{\nu i}|)$$

$$\le \varepsilon^{-1}\ I^{1/2}\ (f_{20})\ E(\max_{1\le i\le\nu}|\ h_{\nu i}|)\ .$$

We know, $\max\limits_{1\le i\le\nu}|h_{\nu i}| \to 0$ a.e. as $\nu \to \infty$, and

$$\max_{1\le i\le\nu}|h_{\nu i}| \le \nu^{-\frac{1}{2}}\{\sum_{k=1}^{p} \max_{1\le i\le\nu}|X_{\nu i}^{(k)}|\}(\max_{1\le k\le p}|\gamma_k|)\ .$$

But, $\displaystyle\int_{A_k} \nu^{-\frac{1}{2}} \max_{1\le i\le\nu}|X_{\nu i}^{(k)}(\omega)|dP_{0k}(\omega)$

$$= \int \nu^{-\frac{1}{2}}(\max_{1\le i\le\nu}|X_{\nu i}^{(k)}(\omega)|I_{A_k}(\omega))dP_{0k}(\omega)\ ,$$

(where $(\Omega_k,\ \ _k, P_{0k})$ is the measure space, $A_k \varepsilon\ \ _k$ and

$$I_{A_k}(\omega) = \begin{cases} 1, \text{ if } \omega\ \varepsilon\ A_k \\ 0, \text{ if } \omega\ \varepsilon\ \Omega_k - A_k) \end{cases}$$

$$\le \nu^{-\frac{1}{2}}[\int(\max_{1\le i\le\nu}|X_{\nu i}^{(k)}(\omega)|)^2 dP_{0k}(\omega)]^{1/2}\ [\int I_{A_k}(\omega)dP_{0k}(\omega)]^{1/2}$$

$$= \nu^{-1}\ \{E[\max_{1\le i\le\nu}|X_{\nu i}^{(k)}|^2]\}^{1/2}\ P_{0k}^{\ 1/2}\ (A_k)\ .$$

Since $X_{\nu 1}^{(k)},\dots, X_{\nu\nu}^{(k)}$ are i.i.d.r.v., using a well-known result (see e.g. Hartley and David [21]),

$$E[\max_{1\le i\le \nu} |X_{\nu i}^{(k)}|^2] \le [\nu/(2\nu-1)^{1/2}][E(X_{\nu i}^{(k)})^4]^{1/2}$$

Hence,

$$\int_{A_k} \nu^{-\frac{1}{2}} \max_{1\le i\le \nu} |X_{\nu i}^{(k)}(\omega)|\, dP_{0k}(\omega)$$

$$\le [1/(2\nu-1)\ E(X_{\nu i}^{(k)})^4]^{1/4}\ P_{0k}^{1/2}\ (A_k)$$

$$\le c^{1/(4+\delta)}\ P_{0k}^{1/2}\ (A_k) \to 0$$

uniformly in ν as $P_{0k}(A_k) \to 0$. Hence, integrals of r.v. $\max_{1\le i\le \nu} X_{\nu i}^{(k)}$ are uniformly continuous for each $k = 1,\ldots, p$ and using (2.4.20) and the L_r-convergence theorem of Loève [27], p. 163), with $r = 1$, we get,

$$E(\max_{1\le i\le \nu} |h_{\nu i}|) \to 0 \quad \text{as } \nu \to \infty . \tag{2.4.21}$$

The lemma now follows from (2.4.19).

We now state LeCam's second lemma (see [19], p. 205) which is an immediate consequence of (2.4.18) and lemma 2.4.5.

LEMMA 2.4.6. The statistic $\log L_\nu$ satisfies

(i) $\log L_\nu - W_\nu + \frac{1}{4} I(f_{20})(\chi'\Sigma\chi) \to 0$ in P_ν-probability.

(ii) $\log L_\nu$ is asymptotically normal $(-\frac{1}{2} I(f_{20})(\chi'\Sigma\chi),\ I(f_{20})(\chi'\Sigma\chi))$ a.e.

(iii) the distributions Q_ν are contiguous to the distributions P_ν.

We are now in a position to derive the asymptotic non-null distribution of $\underset{\sim}{S}_\nu$ by using LeCam's third lemma (see [19], p. 208). We first define the following statistics:

$$(2.4.22)\quad S^*_{k\nu} = \nu^{-1/2}\sum_{i=1}^{\nu}[b_{k\nu}(R^{(k)}_{\nu i}) - \bar{b}_{k\nu}]\phi^*(F_{20}(Y'_{\nu i})),\ k = 1,\dots, p;$$

$$(2.4.23)\quad T^*_{k\nu} = \nu^{-1/2}\sum_{i=1}^{\nu}\psi^*_k(U_{\nu i})\phi^*(F_{20}(Y'_{\nu i})),\ k = 1,\dots, p.$$

Then,

$$(2.4.24)\ E[(S^*_{k\nu} - T^*_{k\nu})^2|P_\nu]$$

$$= E[(\nu^{-1/2}\sum_{i=1}^{\nu}\{b_{k\nu}(R^{(k)}_{\nu i}) - \bar{b}_{k\nu} - \psi^*_k(U_{\nu i})\}\phi^*(F_{20}(Y'_{\nu i})))^2|P_\nu\}$$

$$= \nu^{-1}\sum_{i=1}^{\nu}E[b_{k\nu}(R^{(k)}_{\nu i}) - \bar{b}_{k\nu} - \psi^*_k(U_{\nu i})]^2\ I(g)$$

$$\le [2\ \nu^{-1}\sum_{i=1}^{\nu}E[b_{k\nu}(R^{(k)}_{\nu i}) - \psi^*_k(U_{\nu i})]^2 + 2\ \bar{b}^2_{k\nu}]\ I(g)$$

But $I(g) < \infty$, $\bar{b}_{k\nu} \to 0$ as $\nu \to \infty$ and proceeding as in Theorem 1.5. a (p. 157) and 1.6. a (p. 163) of [19] $E[b_{k\nu}(R^{(k)}_{\nu i}) - \psi^*_k(U_{\nu i})]^2 \to 0$ as $\nu \to \infty$, for all $i = 1,\dots, \nu$. Hence, from (2.4.24),

$$(2.4.25)\quad E[(S^*_{k\nu} - T^*_{k\nu})^2|P_\nu] \to 0 \quad \text{as } \nu \to \infty, \text{ for all } k = 1,\dots, p.$$

Hence, $S^*_{k\nu} - T^*_{k\nu} \to 0$ in P_ν-probability (and hence in Q_ν-probability because of contiguity) for all $k = 1,\dots, p$.
Using Bonferroni inequality, we get,

$$(2.4.26)\quad \underset{\sim}{S}^*_\nu - \underset{\sim}{T}^*_\nu \to \underset{\sim}{0} \text{ in } P_\nu\text{-probability (and hence in } Q_\nu\text{-probability)},$$

where $\underset{\sim}{S}^*_\nu = (S^*_{1\nu},\dots, S^*_{p\nu})'$ and $\underset{\sim}{T}^*_\nu = (T^*_{1\nu},\dots, T^*_{p\nu})'$. Further,

$$S_{k\nu} - S^*_{k\nu} = \nu^{-1/2} \sum_{i=1}^{\nu} [b_{k\nu}(R^{(k)}_{\nu i}) - \overline{b}_{k\nu}][a_\nu(R^{(0)}_{\nu i}) - \phi^*(F_{20}(Y'_{\nu i}))],$$

$$k = 1,\ldots, p.$$

Hence,

(2.4.27) $E[(S_{k\nu} - S^*_{k\nu})^2|P_\nu]$

$$= E[\nu^{-1} \sum_{i=1}^{\nu} (b_{k\nu}(R^{(k)}_{\nu i}) - \overline{b}_{k\nu})^2(a_\nu(R^{(0)}_{\nu i}) - \phi^*(U^*_{\nu i}))^2$$

$$+ \nu^{-1} \sum_{i\neq i'} (b_{k\nu}(R^{(k)}_{\nu i}) - \overline{b}_{k\nu})(b_{k\nu}(R^{(k)}_{\nu i'}) - \overline{b}_{k\nu}).$$

$$(a_\nu(R^{(0)}_{\nu i}) - \phi^*(U^*_{\nu i}))(a_\nu(R^{(0)}_{\nu i'}) - \phi^*(U^*_{\nu i'}))|P_\nu],$$ (writing $U^*_{\nu i}$ for $F_{20}(Y'_{\nu i})$).

$$= \nu^{-1} \sum_{i=1}^{\nu} E(b_{k\nu}(R^{(k)}_{\nu i}) - \overline{b}_{k\nu})^2\ E(a_\nu(R^{(0)}_{\nu i}) - \phi^*(U^*_{\nu i}))^2$$

$$+ \nu^{-1} \sum_{i\neq i'} E\{(b_{k\nu}(R^{(k)}_{\nu i}) - \overline{b}_{k\nu})(b_{k\nu}(R^{(k)}_{\nu i'}) - \overline{b}_{k\nu})\}.$$

$$E[(a_\nu(R^{(0)}_{\nu 1}) - \phi^*(U^*_{\nu 1}))(a_\nu(R^{(0)}_{\nu 2}) - \phi^*(U^*_{\nu 2}))]$$

$$\leq \nu^{-1} \sum_{i=1}^{\nu} (b_{k\nu}(i) - \overline{b}_{k\nu})^2\{E[a_\nu(R^{(0)}_{\nu 1}) - \phi^*(U^*_{\nu 1})]^2$$

$$+ (E\ [a_\nu(R^{(0)}_{\nu 1}) - \phi^*(U^*_{\nu 1})]^2\ E[a_\nu(R^{(0)}_{\nu 2}) - \phi^*(U^*_{\nu 2})]^2)^{1/2}\}.$$

But,

$$\nu^{-1}\sum_{i=1}^{\nu} [b_{k\nu}(i) - \overline{b}_{k\nu}]^2 \to \sigma^0_{kk} < \infty \text{ and } E[a_\nu(R^{(0)}_{\nu i}) - \phi^*(U^*_{\nu i})]^2 \to 0$$

for all $i = 1,\ldots, \nu$. Hence, from (2.4.27),

(2.4.28) $E[(S_{k\nu} - S^*_{k\nu})^2|P_\nu] \to 0$ as $\nu \to \infty$ for all $k - 1,\ldots, p$.

(2.4.29) $\underset{\sim}{S}_\nu - \underset{\sim}{S}_\nu^* \to \underset{\sim}{0}$ in P_ν (and hence in Q_ν)-probability.

(2.4.26) and (2.4.29) give,

(2.4.30) $\underset{\sim}{S}_\nu - \underset{\sim}{T}_\nu^* \to \underset{\sim}{0}$ in P_ν (and hence in Q_ν)-probability.

From (2.4.14) and lemma 2.4.6 (i),

$$\log L_\nu - \underset{\sim}{\gamma}'\underset{\sim}{T}_\nu + \frac{1}{2} I(f_{20})\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma} \to 0$$

in P_ν-probability (also used in proving lemma 2.4.6 (ii)). Hence, for any vector $\underset{\sim}{e} \neq \underset{\sim}{0}$ of finite real constants, $(\log L_\nu, \underset{\sim}{e}'\underset{\sim}{S}_\nu)$ has asymptotically the same distribution as of

$$(\underset{\sim}{\gamma}'\underset{\sim}{T}_\nu - \frac{1}{2} I(f_{20})\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma},\ \underset{\sim}{e}'\underset{\sim}{S}_\nu).$$

Using (2.4.30), the latter has the same asymptotic distribution as of

$$(\underset{\sim}{\gamma}'\underset{\sim}{T}_\nu - \frac{1}{2} I(f_{20})\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma},\ \underset{\sim}{e}'\underset{\sim}{T}_\nu^*)..$$

Let

(2.4.31) $$\underset{\sim}{\Sigma}_{00} = \int_0^1 \int_0^1 \psi_k^*(u)\psi_{k'}(u')\,dH_{kk'}(u,u'),\quad \psi_k(u) = F_k^{-1}(u),$$

$$(k,\ k' = 1,\dots,\ p).$$

We prove the following lemma:

LEMMA 2.4.7. <u>Under</u> P_ν, <u>the model</u> (2.1.1) <u>and the assumptions</u> (2.1.3)-(2.1.5), (2.1.8)-(2.1.10), $(\underset{\sim}{\gamma}'\underset{\sim}{T}_\nu - \frac{1}{2} I(f_{20})\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma},\ \underset{\sim}{e}'\underset{\sim}{T}_\nu^*)$ is <u>asymptotically</u> $N_2(-\frac{1}{2} I(f_{20})(\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma}),\ 0,\ I(f_{20})(\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma}),\ (\underset{\sim}{e}'\underset{\sim}{\Sigma}_0\underset{\sim}{e})I(g),$ $(\underset{\sim}{e}'\underset{\sim}{\Sigma}_{00}\underset{\sim}{\gamma})\int_0^1 \phi(u)\phi^*(u)du)$ <u>a.e.</u>

<u>Proof</u>. We have to show that for every $(a,b) \neq (0,0)$, where a, b are finite and real constants, $a\underset{\sim}{\gamma}'\underset{\sim}{T}_\nu + b\underset{\sim}{e}'\underset{\sim}{T}_\nu^*$ is asymptotically $N_1(0,\ a^2(\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma})I(f_{20}) + b^2(\underset{\sim}{e}'\underset{\sim}{\Sigma}_0\underset{\sim}{e})I(g) + 2\,ab(\underset{\sim}{e}'\underset{\sim}{\Sigma}_{00}\underset{\sim}{\gamma})\int_0^1 \phi(u)\phi^*(u)du)$ a.e.

Now, $a\underset{\sim}{\gamma}'\underset{\sim\nu}{T} + b\underset{\sim}{e}'\underset{\sim\nu}{T}^*$

$$= \nu^{-\frac{1}{2}} \sum_{i=1}^{\nu} \Big[\sum_{k=1}^{p} a\gamma_k \psi_k(U_{\nu i}) \phi(F_{20}(Y'_{\nu i}))$$

$$+ \sum_{k=1}^{p} b e_k \psi_k^*(U_{\nu i}) \phi^*(F_{20}(Y'_{\nu i}))\Big]$$

$$= \nu^{-\frac{1}{2}} \sum_{i=1}^{\nu} (Z^*_{1\nu i} + Z^*_{2\nu i}), \text{ say,}$$

where,

$$Z^*_{1\nu i} = \nu^{-\frac{1}{2}} \sum_{k=1}^{p} a\gamma_k \psi_k(U_{\nu i}) \phi(F_{20}(Y'_{\nu i})),$$

$$Z^*_{2\nu i} = \nu^{-\frac{1}{2}} \sum_{k=1}^{p} b e_k \psi_k^*(U_{\nu i}) \phi^*(F_{20}(Y'_{\nu i})), \quad (i = 1, 2, \ldots, \nu).$$

Let $Z^*_{\nu i} = Z^*_{1\nu i} + Z^*_{2\nu i}$, $i = 1, 2, \ldots, \nu$.

We may also note that $E[a\underset{\sim}{\gamma}'\underset{\sim\nu}{T} + b\underset{\sim}{e}'\underset{\sim\nu}{T}^*|H_0] = 0$ and

$\mathrm{Var}(a\underset{\sim}{\gamma}'\underset{\sim\nu}{T} + b\underset{\sim}{e}'\underset{\sim\nu}{T}^*|H_0) = a^2(\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma})I(f_{20}) + b^2(\underset{\sim}{e}'\underset{\sim}{\Sigma}_0\underset{\sim}{e})I(g)$

$+ 2\, ab(\underset{\sim}{e}'\underset{\sim}{\Sigma}_{00}\underset{\sim}{\gamma})\int_0^1 \phi(u)\phi^*(u)\,du$. As in [19] (pp. 217-218) all we need show now is that $\sum_{i=1}^{\nu} E[Z^{*2}_{\nu i}\, \chi_{Z^*_{\nu i}}(\delta)\,|P_\nu] \to 0$ as $\nu \to \infty$, where for any $\delta > 0$,

$$\chi_{Z^*_{\nu i}}(\delta) = \begin{cases} 1, & \text{if } |Z^*_{\nu i}| > \delta \\ 0, & \text{otherwise.} \end{cases}$$

But $\chi_{Z^*_{\nu i}(\delta)} \le \chi_{Z^*_{1\nu i}(\delta/2)} + \chi_{Z^*_{2\nu i}(\delta/2)}$.

Hence, $\sum_{i=1}^{\nu} E[Z^*_{\nu i}\ \chi_{Z^*_{\nu i}}(\delta)\,|P_\nu]$

$$\le 2\sum_{i=1}^{\nu} E[(Z^{*2}_{1\nu i} + Z^{*2}_{2\nu i})(\chi_{Z^*_{1\nu i}}(\delta/2) + \chi_{Z^*_{2\nu i}}(\delta/2))\,|P_\nu]$$

$$\le 4\sum_{i=1}^{\nu} E([Z^{*2}_{1\nu i}\ \chi_{Z^*_{1\nu i}}(\delta/2) + Z^{*2}_{2\nu i}\ \chi_{Z^*_{2\nu i}}(\delta/2)]\,|P_\nu).$$

So it is sufficient to show that $E[\sum_{i=1}^{\nu} Z^{*2}_{j\nu i}\ \chi_{Z^*_{j\nu i}}(\delta/2)\,|P_\nu] \to 0$ as $\nu \to \infty$ for $j = 1, 2$ (these conditions being equivalent to the Lindeberg condition of the classical Central Limit Theorem for $\sum_{i=1}^{\nu} Z^*_{1\nu i}$ and $\sum_{i=1}^{\nu} Z^*_{2\nu i}$). The proof is exactly the same as in [19, p. 218], and, hence, is omitted.

Using new LeCam's third lemma (see [19], P. 208) and lemma 2.4.6, we get, under Q_ν, $\underset{\sim}{e}'\underset{\sim}{T}^*_\nu$ is asymptotically

$$N_1((\underset{\sim}{e}'\underset{\sim}{\Sigma}_{00}\underset{\sim}{\gamma})\int_0^1 \phi(u)\phi^*(u)\,du,\ (\underset{\sim}{e}'\underset{\sim}{\Sigma}_0\underset{\sim}{e})I(g))\ \text{a.e.}$$

for every $\underset{\sim}{e} \neq \underset{\sim}{0}$ of finite real constants i.e. $\underset{\sim}{T}^*_\nu$ (and hence $\underset{\sim}{S}_\nu$) is under Q_ν asymptotically

$$N_p((\underset{\sim}{\Sigma}_{00}\underset{\sim}{\gamma})\int_0^1 \phi(u)\phi^*(u)\,du,\ \underset{\sim}{\Sigma}_0 I(g))\ \text{a.e.}$$

Also, from Theorem 2.3.1, $\underset{\sim}{V}_\nu \to \underset{\sim}{\Sigma}_0 I(g)$ in probability. Recalling definition of M_ν in (2.2.8) and using Slutsky's theorem, we get that under Q_ν, M_ν is distributed asymptotically as

$$\chi^2_p((\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}'_{00}\underset{\sim}{\Sigma}_0^{-1}\underset{\sim}{\Sigma}_{00}\underset{\sim}{\gamma})(\int_0^1 \phi(u)\phi^*(u)\,du)^2/I(g))\ \text{a.e.}$$

2.5 Aymptotic optimality of tests and A.R.E.

Let

$$\lambda_\nu = [L(\underset{\sim}{Z}_\nu|\underset{\sim}{\gamma})_{\underset{\sim}{\gamma}=\underset{\sim}{0}}/L(\underset{\sim}{Z}_\nu|\underset{\sim}{\gamma})_{\underset{\sim}{\gamma}=\hat{\underset{\sim}{\gamma}}_\nu}] , \tag{2.5.1}$$

($\hat{\underset{\sim}{\gamma}}_\nu = (\hat{\gamma}_{1\nu},\ldots, \hat{\gamma}_{p\nu})$ being the maximum likelihood estimator (m.l.e.) of $\underset{\sim}{\gamma}$), be the likelihood ratio test criterion for the testing problem considered here. Let

$$M_\nu^* = (\underset{\sim}{T}_\nu'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{T}_\nu)/I(f_{20}). \tag{2.5.2}$$

We make the assumptions on the likelihood function $L(\underset{\sim}{Z}_\nu|\underset{\sim}{\gamma})$ as in Wald [34], (given in details in chapter III, and, hence, are not restated here for the sake of brevity). Under these assumptions, and also under the assumptions (2.1.3)-(2.1.5), (2.1.8)-(2.1.10), we get the following result:

THEOREM 2.5.1. $M_\nu^* + 2\log_e\lambda_\nu \to 0$ <u>in</u> P_ν<u>-probability</u>.

<u>Proof</u>. Same as in Theorem 3.2.3 of chapter III with $\underset{\sim}{Z}_\nu$, $\underset{\sim}{\Sigma}$, $I(f_{20})$ and M_ν^* replacing $\underset{\sim}{X}_\nu^*$, $\underset{\sim}{\Lambda}_\nu$, A^2 (F, $\{I_j\}$) and S_ν respectively.

The theorem implies that $M_\nu^* + 2\log_e\lambda_\nu \to 0$ in Q_ν-probability (because of contiguity). So, if the assumptions (2.1.3)-(2.1.5) and (2.1.8) along with the regularity assumptions of Wald are satisfied, a test procedure, similar to the one we have considered, based on M_ν^* possesses the optimal properties of the likelihood ratio test as described in Theorem 3.2.2 of chapter III.

We might also remark that using LeCam's third lemma, it can be shown that, under the alternatives, M_ν^* is asymptotically

$\chi_p^2((\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma})I(f_{20}))$ a.e. If, under the same sequence of alternatives, two test statistics are asymptotically non-central chi-squares with the same degrees of freedom, it has been shown by Hannan [20] and Andrews [2] that the A.R.E. of one test procedure with respect to the other is given by the ratio of their non-centrality parameters. Using this, we get the A.R.E. of the test based on M_ν with respect to the one based on M_ν^* is given by

$$(2.5.3.) \qquad e = [(\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}_{00}'\underset{\sim}{\Sigma}_0^{-1}\underset{\sim}{\Sigma}_{00}\underset{\sim}{\gamma})/(\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma})]\rho_2^2 ,$$

where $\rho_2 = \int_0^1 \phi(u)\phi^*(u)du/[(\int_0^1 \phi^2(u)du)^{1/2} (\int_0^1 \phi^{*2}(u)du)^{1/2}]$.

We may also observe that in the particular case, $\phi^* = \phi$, $\psi_k^* = \psi_k$, for all $k = 1,\ldots, p$, $e = 1$ i.e. the test procedure based on the statistic M_ν as we have proposed is asymptotically optimal in Wald's sense.

We may note that the general efficiency expression depends on $\underset{\sim}{\gamma} = \sigma^{-1} \underset{\sim}{\tau}$. However, using a well-known theorem of Courant (see Bodewig [1956]), we get

$$(2.5.4) \qquad \max_{\underset{\sim}{\gamma}} e = \text{maximal eigenvalue of } \underset{\sim}{\Sigma}^{-1}\underset{\sim}{\Sigma}_{00}'\underset{\sim}{\Sigma}_0^{-1}\underset{\sim}{\Sigma}_{00} ;$$

$$\min_{\underset{\sim}{\gamma}} e = \text{minimal eigenvalue of } \underset{\sim}{\Sigma}^{-1}\underset{\sim}{\Sigma}_{00}'\underset{\sim}{\Sigma}_0^{-1}\underset{\sim}{\Sigma}_{00} .$$

It is interesting to observe that instead of considering the rank statistics $S_{k\nu}(k = 1,\ldots, p)$ as defined in (2.2.2) if we consider the mixed statistics

$$S_{k\nu}^0 = \nu^{-1/2} \sum_{i=1}^{\nu} X_{\nu i}^{(k)}[a_\nu(R_{\nu i}^{(0)}) - \bar{a}_\nu], \; k = 1,\ldots, p,$$

(which would be more logical to consider when the $X_{\nu i}^{(k)}$ $(i = 1,\ldots, \nu,$ $k = 1,\ldots, p)$ are observable), the appropriate statistic will be the corresponding quadratic form in $S^0_{k\nu}$'s and the efficiency expression will be

$$e' = \rho_2^2 . \tag{2.5.5}$$

Thus in the particular case when $\phi^* = \phi$, the resulting test is optimal according to criteria described earlier. However, as already mentioned, the mixed statistics cannot be used when the $X^{(k)}_{\nu i}$ are not observable. We may also remark that unlike M_ν, the statistic based on $X^{(k)}_{\nu i}$;s is not unconditionally distribution-free even when $p = 1$.

We next investigate some particular cases where the efficiency expression can be simplified considerably leading to some interesting results.

<u>Case I</u> <u>$p = 1$</u>. Here, $e = \rho_1^2 \rho_2^2$, where,

$$\rho_1 = \int_0^1 \psi_1(u)\psi_1^*(u)du/[(\int_0^1 \psi_1^2(u)du)^{1/2} (\int_0^1 \psi_1^{*2}(u)du)^{1/2}] . \tag{2.5.6}$$

We may note that this efficiency expression is the same as we would have obtained by an extension of Hájek's ideas to simple linear regression with stochastic predictors. The test statistic which we would consider in that case will be simply $S_{1\nu}$. This will be unconditionally distribution-free since in this situation under the hypothesis of no regression (or independence according to our model), the permutation distribution of $S_{1\nu}$ becomes identical with its unconditional distribution. For two-sided alternatives, the test procedure is optimal in Wald's sense. For one-sided alternatives, this will be asymptotically uniformly most powerful (UMP) as in

Hájek (1962) in case $\phi^* = \phi$ and $\psi_1^* = \psi_1$. It might also be remarked that the test considered here is an alternative approach to the study of independence of two random variables as has been considered by Bhuchongkul [5]. Bhuchongkul followed the Chernoff-Savage [10] approach in proving the asymptotic normality of her test statistic and as such had to put rather stringent conditions regardingthe existence of the derivatives of the score function and on its bounds. But unlike our case the asymptotic normality of her statistic follows even when regression is non-linear. However, her observation that the normal scores test is the locally most powerful rank test under the alternative that the variables under consideration have a bivariate normal distribution is equally valid in our case. In fact, in case of linear regression our result is slightly more general in the sense that whatever be the parent distribution, if $\psi_1^* = \psi_1$ and $\phi^* = \phi$, the resulting test as considered by us is asymptotically optimal. We give expressions for e in some particular cases:

TABLE 2.5.1. Showing some efficiency expressions.

$\phi(u) = \sigma_0^{-1}\Phi^{-1}(u)$, $\psi_1(u) = \sigma_1\Phi^{-1}(u)$, $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} \exp.(-t^2/2)dt$

ϕ^*	$2u-1$	$2u-1$	$\sigma_{00}^{-1}\,\Phi^{-1}(u)$	$\sigma_{00}^{-1}\,\Phi^{-1}(u)$
ψ_1^*	$u-\frac{1}{2}$	$\sigma_2\Phi^{-1}(u)$	$u-\frac{1}{2}$	$\sigma_2\Phi^{-1}(u)$
e	$(3/\pi)^2$	$3/\pi$	$3/\pi$	1

Case II. p = 2.

(i) $\psi_k^*(u) = u - \frac{1}{2}$ (k = 1, 2), $\psi_k(u) = \sigma_k \Phi^{-1}(u)$, (k = 1, 2)

$$\phi^*(u) = 2u - 1,\ \phi(u) = \sigma_0^{-1}\ \Phi^{-1}(u),\ \rho_{12} = \mathrm{corr}(X_{\nu 1}^{(1)}, X_{\nu 1}^{(2)}).$$

Here $\rho_2^2 = 3/\pi$ and we get after some simplifications,

$$\underset{\sim}{\Sigma}_{00} = (2\sqrt{\pi}\ )^{-1} \begin{bmatrix} \sigma_1 & \sigma_2\rho_{12} \\ \sigma_1\rho_{12} & \sigma_2 \end{bmatrix}$$

$$\underset{\sim}{\Sigma}_0 = \begin{bmatrix} 1/12 & (1/2\pi)\sin^{-1}(\rho_{12}/2) \\ (1/2\pi)\sin^{-1}(\rho_{12}/2) & 1/12 \end{bmatrix}$$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

using these, we get after some lengthy algebra,

$$e = (3/\pi)(\underset{\sim}{\delta}'\underset{\sim}{\Gamma}\underset{\sim}{\delta})/(\underset{\sim}{\delta}'\underset{\sim}{\Gamma}_0\underset{\sim}{\delta}),$$

where,

$$\underset{\sim}{\delta}' = (\gamma_1\sigma_1,\ \gamma_2\sigma_2),\ \underset{\sim}{\Gamma}_0 = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix} \text{ and } \underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}_0 \begin{bmatrix} \pi/3 & 2\sin^{-1}(\rho_{12}/2) \\ 2\sin^{-1}(\rho_{12}/2) & \pi/3 \end{bmatrix}^{-1} \underset{\sim}{\Gamma}_0$$

Then,

$$e_M = \max_{\underset{\sim}{\delta}} e = (3/\pi)^2(1 + \rho_{12})/(1 + (6/\pi)\sin^{-1}(\rho_{12}/2)),\ 0 \le \rho_{12} < 1$$

$$= (3/\pi)^2(1 - \rho_{12})/(1-(6/\pi)\sin^{-1}(\rho_{12}/2)),\ -1 < \rho_{12} \le 0$$

$$e_M = \min_{\underset{\sim}{\delta}} e = (3/\pi)^2(1 - \rho_{12})/(1-(6/\pi)\sin^{-1}(\rho_{12}/2)),\ 0 \le \rho_{12} < 1$$

$$= (3/\pi)^2(1 + \rho_{12})/(1+(6/\pi)\sin^{-1}(\rho_{12}/2)),\ -1 < \rho_{12} \le 0$$

We can draw the following conclusions as in Bickel [6].

(1) $.912 \leq e_M \leq .917$. As a function of ρ, e_M is symmetric about $\rho = 0$, concave for $0 \leq \rho < 1$ and $-1 < \rho \leq 0$, attains its minimum of $(3/\pi)^2$ at $\rho = 0$, approaches it as $|\rho| \to 1$ and attains its maximum of .917 at $= .78$.

(2) e_m considered as a function of ρ is symmetric about 0 and decreases from a maximum of $(3/\pi)^2$ at $\rho = 0$ to an infimum of $(3/\pi)\sin(\pi/3) = .827$ as $\rho \to 1$.

(ii) $\psi_k^*(u) = u - \frac{1}{2}$, $(k = 1, 2)$, $\psi_k(u) = \sigma_k\ \Phi^{-1}(u)$, $(k = 1, 2)$

$$\phi^*(u) = \sigma_{00}^{-1}\ \Phi^{-1}(u),\ \phi(u) = \sigma_0^{-1}\ \Phi^{-1}(u)\ .$$

Here $\rho_2^2 = 1$ and so $e = \rho_1^2$. Hence for e_M, e_m we get exactly the same expressions as in Bickel (1965) and hence, all his conclusions hold in this case.

(iii) $\psi_k^*(u) = \sigma_k^*\ \Phi^{-1}(u)$, $\psi_k(u) = \sigma_k\ \Phi^{-1}(u)$,

$$\phi^*(u) = 2u - 1,\ \phi(u) = \sigma_0^{-1}\ \Phi^{-1}(u)\ .$$

Here, $\rho_2^2 = 3/\pi$ and $\rho_1^2 = 1$. Hence, $e = 3/\pi = .955$.

(iv) $\psi_k^*(u) = \sigma_k^*\ \Phi^{-1}(u)$, $\psi_k(u) = \sigma_k\ \Phi^{-1}(u)$, $(k = 1, 2)$,

$$\phi^*(u) = \sigma_{00}^{-1}\ \Phi^{-1}(u),\ \phi(u) = \sigma_0^{-1}\ \Phi^{-1}(u).\quad \text{Then, } e = 1.$$

It can be shown in the general situation that if $\psi_k^*(u) = \sigma_k^*\ \Phi^{-1}(u)$, $\psi_k(u) = \sigma_k\ \Phi^{-1}(u)$, $(k = 1,\dots, p)$, then $\rho_1^2 = 1$ so that $e = \rho_2^2$. Also, when, $\psi_k^*(u) = u - \frac{1}{2}$, $\psi_k(u) = \sigma_k \, \Phi^{-1}(u)$, $(k = 1,\ldots, p)$, it can be found as in the case $p = 2$,

$$e = \rho_2^2 \, (\underset{\sim}{\delta}'\underset{\sim}{\Gamma}\underset{\sim}{\delta})/(\underset{\sim}{\delta}'\underset{\sim}{\Gamma}_0\underset{\sim}{\delta}) \, , \tag{2.5.7}$$

where $\underset{\sim}{\delta} = (\delta_1,\ldots,\delta_p)' = (\gamma_1\sigma_1,\ldots, \gamma_k\sigma_k)'$, $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}_0\underset{\sim}{V}^{-1}\underset{\sim}{\Gamma}_0$, where, $\underset{\sim}{\Gamma}_0 = ((\rho_{kk'}))$, $\underset{\sim}{V} = ((2 \sin^{-1} (\frac{1}{2}\rho_{kk'})))$. We shall next show that $\underset{\sim}{\Gamma}_0^{-1} - \underset{\sim}{V}^{-1}$ is positive definite. To prove this, it is sufficient to show that $\underset{\sim}{\tau}'\underset{\sim}{\Gamma}_0\underset{\sim}{\tau} \le \underset{\sim}{\tau}'\underset{\sim}{V}\underset{\sim}{\tau}$ for all $\underset{\sim}{\tau} \ne \underset{\sim}{0}$ (see e.g. Fraser [13], p. 55).

$$\text{But } \underset{\sim}{\tau}'\underset{\sim}{V}\underset{\sim}{\tau} - \underset{\sim}{\tau}'\underset{\sim}{\Gamma}_0\underset{\sim}{\tau} = \sum_{j,j'=1}^{p} \tau_j\tau_{j'}[2 \sin^{-1}(\rho_{jj'}/2) - \rho_{jj'}]$$

$$= \sum_{j,j'=1}^{p} \tau_j\tau_{j'} \sum_{n=2}^{\infty} c_n \, \rho_{jj'}^{2n-1} \, ,$$

where c_n are positive constants and are the same for all $j, j' = 1,\ldots, p$. Thus

$$\underset{\sim}{\tau}'\underset{\sim}{V}\underset{\sim}{\tau} - \underset{\sim}{\tau}'\underset{\sim}{\Gamma}_0\underset{\sim}{\tau} = \sum_{n=2}^{\infty} c_n \sum_{j,j'=1}^{p} \tau_j\tau_{j'} \, \rho_{jj'}^{2n-1} \, . \tag{2.5.8}$$

Using now a result of I. Schur (see [4], p. 96) we get that if $A = ((a_{jj'}))$ and $B = ((b_{jj'}))$ are positive definite matrices, $C = ((c_{jj'})) = ((a_{jj'}b_{jj'}))$ is positive definite. Hence, since $((\rho_{jj'}))$ is positive definite, $((\rho_{jj'}^r))$ is positive definite for all $r = 1, 2, \ldots$. Thus from (2.5.8),

$$\underset{\sim}{\tau}'\underset{\sim}{V}\underset{\sim}{\tau} - \underset{\sim}{\tau}'\underset{\sim}{\Gamma}_0\underset{\sim}{\tau} > 0.$$

Since $\underset{\sim}{\Gamma}_0^{-1} - \underset{\sim}{V}^{-1}$ is positive definite,

$$\underset{\sim}{\delta}'\underset{\sim}{\Gamma}_0(\underset{\sim}{\Gamma}_0^{-1} - \underset{\sim}{V}^{-1})\underset{\sim}{\Gamma}_0\underset{\sim}{\delta} > 0, \text{ for all } \underset{\sim}{\delta} \neq \underset{\sim}{0}$$

i.e. $\underset{\sim}{\delta}'\underset{\sim}{\Gamma}_0\underset{\sim}{\delta} > \underset{\sim}{\delta}'\underset{\sim}{\Gamma}\underset{\sim}{\delta}$. Hence, from (2.5.7), $e < \rho_2^2$.

CHAPTER III

NONPARAMETRIC TESTS FOR GROUPED DATA IN THE UNIVARIATE LINEAR REGRESSION MODEL

3.1. Introduction, Notations and Assumptions.

Consider a (double) sequence of random vectors $\underset{\sim}{X}_\nu = (X_{\nu 1}, \dots, X_{\nu\nu})'$, $\nu \geq 1$, consisting of ν independent r.v.'s, where $X_{\nu i}$ has a continuous d.f. $F_{\nu i}(x)$ given by

$$F_{\nu i}(x) = F(\sigma^{-1}[x - \beta_0 - \underset{\sim}{\beta}'\underset{\sim}{c}_{\nu i}]),\ i = 1, 2, \dots, \nu,\ \nu \geq 1, \tag{3.1.1.}$$

where $\underset{\sim}{\beta}' = (\beta_1, \dots, \beta_p)$ is the parameter (vector) of interest, β_0 and $\sigma(>0)$ are real (nuisance) parameters and $\underset{\sim}{c}_{\nu i} = (c_{1\nu i}, \dots, c_{p\nu i})'$, $i = 1, 2, \dots, \nu$, are vectors of known constants satisfying (without any loss of generality)

$$\sum_{i=1}^{\nu} c_{k\nu i} = 0, \quad k = 1, 2, \dots, p. \tag{3.1.2}$$

Define

$$c^*_{k\nu i} = c_{k\nu i}/C_{k\nu}, \text{ where } C^2_{k\nu} = \sum_{i=1}^{\nu} c^2_{k\nu i}, \quad k = 1, 2, \dots, p. \tag{3.1.3}$$

We assume

$$\max_{1\leq i\leq\nu} |c^*_{k\nu i}| = o(1), \text{ for all } k = 1, \dots, p. \tag{3.1.4}$$

Let

$$(3.1.5)\qquad \underset{\sim}{\Lambda}_\nu = \Big(\Big(\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i}\Big)\Big) \text{ and } \underset{\sim}{\Lambda} = ((\lambda_{kk'})).$$

We assume that $\underset{\sim}{\Lambda}_\nu$ and $\underset{\sim}{\Lambda}$ are positive definite, and $\underset{\sim}{\Lambda}_\nu \to \underset{\sim}{\Lambda}$ as $\nu \to \infty$. Further, we assume that $f(x) = F'(x)$ exists and

$$(3.1.6)\qquad A^2(F) = \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x)\,dx < \infty.$$

We consider the measure spaces $(\mathfrak{X}_\nu, \mathcal{A}_\nu, \mu_\nu)$, a typical point of the space $\mathfrak{X}_\nu$ being denoted by $\underset{\sim}{x}_\nu = (x_1,\ldots, x_\nu)$, $\nu \geq 1$. Our problem is to test the null hypothesis

$$(3.1.7)\qquad H_0\colon \underset{\sim}{\beta} = \underset{\sim}{0} \quad \text{(i.e. no regression) against } \underset{\sim}{\beta} \neq \underset{\sim}{0}.$$

$H_0 \equiv H_{0\nu}$ depends on ν only through the number of observations, while the sequence of alternatives (as will be considered) will depend on the parameters $c_{k\nu i}$ $(i = 1, 2,\ldots, \nu;\ k = 1,\ldots, p)$ in addition. More specifically, we consider the sequence of alternatives

$$(3.1.8)\qquad H_\nu\colon \beta_k \equiv \beta_{k\nu} = \tau_k/C_{k\nu},\quad k = 1,\ldots, p,$$

where $\underset{\sim}{\tau} = (\tau_1,\ldots, \tau_p)'$ is a vector with fixed real elements. Let $\{P_\nu\}$ and $\{Q_\nu\}$ be two sequences of probability measures (associated with the sequences of null and alternative hypotheses respectively) defined on the sequences of measurable spaces $\{\mathfrak{X}_\nu, \mathcal{A}_\nu\}$, where $dP_\nu = p_\nu d\mu_\nu$, $dQ_\nu = q_\nu d\mu_\nu$, $\nu \geq 1$.

Introduce the likelihood ratio $L_{0\nu} \equiv L_{0\nu}(\underset{\sim}{X}_\nu)$ as follows:

$$(3.1.9) \qquad L_{0\nu}(\underset{\sim}{X}_\nu) = \begin{cases} q_\nu(\underset{\sim}{x}_\nu)/p_\nu(\underset{\sim}{x}_\nu), & \text{if } p_\nu(\underset{\sim}{x}_\nu) > 0 \\ 1 & \text{, if } p_\nu(\underset{\sim}{x}_\nu) = q_\nu(\underset{\sim}{x}_\nu) = 0 \\ \infty & \text{, if } p_\nu(\underset{\sim}{x}_\nu) = 0 < q_\nu(\underset{\sim}{x}_\nu) \ . \end{cases}$$

We are concerned with the situation when $\underset{\sim}{X}_\nu$ is not observable. We have a finite (or countable) set of class-intervals

$$(3.1.10) \qquad I_j : a_j < x \le a_{j+1} \ , \ j = -\infty, \ldots, -1, 0, 1, \ldots, \infty,$$

[where, $-\infty \le \ldots < a_{-1} < a_0 < a_1 < \ldots \le \infty$ is any (finite or countable) set of ordered points on the (extended) real line $[-\infty, \infty]$]. The observable stochastic vector is $\underset{\sim}{X}^*_\nu = (X^*_{\nu 1}, \ldots, X^*_{\nu\nu})'$, where

$$(3.1.11) \qquad X^*_{\nu i} = \sum_{j=-\infty}^{\infty} I_j Z_{\nu ij}, \ i = 1, 2, \ldots, \nu,$$

and

$$(3.1.12) \qquad Z_{\nu ij} = \begin{cases} 1, & \text{if } X_{\nu i} \in I_j \\ 0, & \text{otherwise,} \end{cases}$$

for all $i = 1, 2, \ldots, \nu$, $\nu \ge 1$, $j = -\infty, \ldots, -1, 0, 1, \ldots, \infty$.

We shall first consider the case when we have a finite set of class intervals $I_{-s_1}, I_{-s_1+1}, \ldots, I_{s_2}$, $(a_{-s_1} = -\infty, a_{s_2+1} = \infty)$ so that $X^*_{\nu i} = \sum_{j=-s_1}^{s_2} I_j Z_{\nu ij}$ $(i = 1, 2, \ldots, \nu; \nu \ge 1)$. This is always possible by amalgamating classes on either end. Also, if $s_1 + s_2 = \ell$, since there are only a finite number of class-intervals, we can relabel the suffixes of the I_j's by replacing j by $s_1 + j$ $(j = -s_1, \ldots, s_2)$. Thus the relabelled class-intervals are

$I_0, I_1, \ldots, I_\ell$ $(a_0 = -\infty,\ a_{\ell+1} = \infty)$, so that $X^*_{\nu i} = \sum_{j=0}^{\ell} I_j Z_{\nu ij}$ $(i = 1, 2, \ldots, \nu,\ \nu \geq 1)$. Later on, however, the results are extended to the case of a countable set of class-intervals.

In the present chapter, we have considered some permutationally distribution-free tests for the above problem. It has been shown that under a set of mild regularity conditions, these are asymptotically optimal in a certain sense.

The assumption that rank $\underset{\sim}{\Delta}_\nu = p$ is made without any loss of generality, since, otherwise, a reparametrization in (3.1.1) will lead to a lower order $\underset{\sim}{\Delta}_\nu$ which will be of full rank.

3.2. Asymptotically Optimal Parametric Test.

Define

(3.2.1) $F_j = F((a_j - \beta_0)/\sigma)$, $j = 0, 1, \ldots, \ell + 1$;

(3.2.2) $P_j = F_{j+1} - F_j$, $j = 0, 1, \ldots, \ell$;

(3.2.3) $\phi(u) = -f'(F^{-1}(u))/f(F^{-1}(u))$, $0 < u < 1$;

(3.2.4) $\Delta_j = \int_{F_j}^{F_{j+1}} \phi(u)\,du \Big/ \int_{F_j}^{F_{j+1}} du$, when $P_j \neq 0$

$= 0$, otherwise.

We can rewrite Δ_j as

(3.2.5) $\Delta_j = P_j^{-1}[f(a_j - \beta_0)/\sigma) - f((a_{j+1} - \beta_0)/\sigma)]$, $j = 0, 1, \ldots, \ell$,

when $P_j \neq 0$. Let

(3.2.6) $J = \{j \mid j \in [0, 1, \ldots, \ell],\ P_j > 0\}$;

(3.2.7) $$A^2(F,\{I_j\}) = \sum_{j=0}^{\ell} \Delta_j^2 P_j = \sum_{j\varepsilon J} \Delta_j^2 P_j \ ;$$

Hence,

(3.2.8) $$0 < A^2(F,\{I_j\}) = \sum_{j\varepsilon J} [\int_{F_j}^{F_{j+1}} \phi(u)du]^2 / \int_{F_j}^{F_{j+1}} du \leq A^2(F) < \infty.$$

Assume,

(3.2.9) $$\max_{0\leq j\leq \ell} P_j < 1.$$

Let,

(3.2.10) $$\underset{\sim}{\gamma} = (\gamma_1, \ldots, \gamma_p)' = (\tau_1/\sigma, \ldots, \tau_p/\sigma)' = \underset{\sim}{\tau}/\sigma \ ;$$

$$h_{\nu i} = \sum_{k=1}^{p} c^*_{k\nu i} \tau_k \quad (i = 1, 2, \ldots, \nu).$$

Then,

(3.2.11) $$\Pr(X_{\nu i} \varepsilon I_j | H_\nu) = F[\{(a_{j+1} - \beta_0)/\sigma\} - h_{\nu i}] - F[\{(a_j - \beta_0)/\sigma\} - h_{\nu i}] \ .$$

Using Taylor expansion, (3.2.1), (3.2.2) and (3.2.4), we get now, for all $j \varepsilon J$, $i = 1, 2, \ldots, \nu$,

(3.2.12) $$\Pr(X_{\nu i} \varepsilon I_j | H_\nu) = P_j[1 + h_{\nu i}\Delta_j + h_{\nu i}\theta_j g_j(h_{\nu i})] = P_{\nu ij} \text{ (say)},$$

where $0 < \theta_j < 1$ and,

(3.2.13) $$g_j(h_{\nu i}) = P_j^{-1} \int_0^{h_{\nu i}} [f'(\{(a_{j+1} - \beta_0)/\sigma\} - \theta_j y) - f'(\{(a_j - \beta_0)/\sigma\} - \theta_j y)]dy$$

$$= o(1), \text{ uniformly in } j \varepsilon J, \quad i = 1, 2, \ldots, \nu.$$

Fix first α and σ, but not γ_k's $(k = 1, 2, \ldots, p)$. Then the likelihood function of $\underset{\sim}{X}_\nu^*$ under H_ν is given by

$$(3.2.14)\quad L(\underset{\sim}{X}_\nu^* | \underset{\sim}{\gamma}) = \prod_{i=1}^{\nu} \{ \sum_{j=0}^{\ell} Z_{\nu ij}(F(a'_{j+1} - h_{\nu i}) - F(a'_j - h_{\nu i})) \} ,$$

where $a'_j = (a_j - \beta_0)/\sigma$, $(j = 0, 1, \ldots, \ell)$.

Let

$$(3.2.15)\quad T_{k\nu} = [(\partial/\partial\gamma_k)\log L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})]_{\underset{\sim}{\gamma}=\underset{\sim}{0}}$$

$$= [(\partial/\partial\gamma_k)\{ \sum_{i=1}^{\nu} \log[\sum_{j=0}^{\ell} Z_{\nu ij}(F(a'_{j+1} - h_{\nu i}) - F(a'_j - h_{\nu i}))]\}]_{\underset{\sim}{\gamma}=\underset{\sim}{0}}$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i} \frac{\sum_{j=0}^{\ell} Z_{\nu ij}\Delta_j P_j}{\sum_{j=0}^{\ell} Z_{\nu ij} P_j}$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j=0}^{\ell} Z_{\nu ij}\Delta_j , \quad k = 1, 2, \ldots, p,$$

(noting that $Z_{\nu ij}Z_{\nu ij'} = 0$ for all $j \neq j'$, $0 \leq j, j' \leq \ell$, $i = 1, 2, \ldots, \nu$, $\nu \geq 1$).

We fix first α and σ. For each fixed α and σ, it can be shown that a quadratic form involving the statistics $T_{k\nu}$'s is asymptotically equivalent to the likelihood ratio test criterion (to be defined later). Also, we shall see in section 2.3 that if we use a quadratic form involving permutation test statistics, it is asymptotically equivalent to the likelihood ratio test criterion for each fixed α and σ. Since the permutation test-statistics and hence, the quadratic form in them is asymptotically distribution-free, the test has the correct limiting size for all possible particular hypothesis; so the test statistic is indeed "asymptotically equivalent" to the likelihood ratio test criterion.

As we have already mentioned, the test statistic we are going to consider will be an appropriate quadratic form in the statistics $T_{k\nu}$'s $(k = 1, 2, \dots, p)$. Apart from the intuitive appeal of these statistics being co-ordinatewise extensions of Sen's [33] statistic, the rationale behind using this quadratic form (to be proposed later) is that it can be proved to be "asymptotically equivalent" to the 'likelihood ratio test criterion', and, hence, (under regularity assumptions of Wald [34] enjoys the asymptotic optimal properties of the latter.

To formulate the optimal parametric test procedure, we need the following lemmas:

LEMMA 3.2.1. <u>Under</u> H_0, <u>the model</u> (3.1.1) <u>and the assumptions</u> (3.1.2)-(3.1.6), $\underset{\sim}{T}_\nu = (T_{1\nu}, \dots, T_{p\nu})'$ <u>is asymptotically</u> $N_p(\underset{\sim}{0}, \underset{\sim}{\Lambda} A^2(F, \{I_j\}))$.

<u>Proof</u>. It is sufficient to show that for any vector $\underset{\sim}{e} = (e_1, \dots, e_p)' \neq 0$ with fixed and real elements, $\underset{\sim}{e}'\underset{\sim}{T}_\nu$ is asymptotically $N_1(0, (\underset{\sim}{e}'\underset{\sim}{\Lambda}\underset{\sim}{e})A^2(F, \{I_j\}))$. We can write,

$$\underset{\sim}{e}'\underset{\sim}{T}_\nu = \sum_{i=1}^{\nu} m_{\nu i} W_{\nu i}, \tag{3.2.16}$$

where, $m_{\nu i} = \sum_{k=1}^{p} e_k c^*_{k\nu i}$, $W_{\nu i} = \sum_{j=0}^{\ell} \Delta_j Z_{\nu ij}$, $i = 1, 2, \dots, \nu$. Then, since, $E(Z_{\nu ij} | P_\nu) = P_j$, $0 \le j \le \ell$, $i = 1, 2, \dots, \nu$, $\nu \ge 1$, it is immediate to check that $E(W_{\nu i}|P_\nu) = \int_0^1 \phi(u)du = 0$,

$\mathrm{Var}(W_{\nu i}|P_\nu) = A^2(F,\{I_j\})$, $i = 1,\ldots, \nu$, $\nu \geq 1$. Hence,

$E(\underset{\sim}{e}'\underset{\sim}{T}_\nu|P_\nu) = 0$, $\mathrm{Var}(\underset{\sim}{e}'\underset{\sim}{T}_\nu|P_\nu) = (\underset{\sim}{e}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{e})A^2(F,\{I_j\}) \sim (\underset{\sim}{e}'\underset{\sim}{\Lambda}\underset{\sim}{e})A^2(F,\{I_j\})$.

Under H_0, $W_{\nu 1},\ldots, W_{\nu\nu}$ are i.i.d.r.v. with $0 < \mathrm{Var}(W_{\nu i}) < \infty$, $i = 1,\ldots, \nu$. Using now a theorem given in p. 153 of [19], to prove the lemma, all we have to show is that the coefficients $m_{\nu i}(i = 1, 2,\ldots, \nu)$ in (3.2.16) satisfy the Noether condition given in chapter II. This is satisfied since

$$\max_{1\leq i\leq \nu} |m_{\nu i}| \leq (\max_{1\leq i\leq \nu} \max_{1\leq k\leq p} |c^*_{k\nu i}|)(\sum_{k=1}^{p} |e_k|) = o(1),$$

using (3.1.3) and $\sum_{i=1}^{\nu} m^2_{\nu i} = \underset{\sim}{e}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{e} \sim \underset{\sim}{e}'\underset{\sim}{\Lambda}\underset{\sim}{e}$ as $\nu \to \infty$.

The next lemma due to Hájek [17] ensures the contiguity of the sequence of probability measures $\{Q_\nu\}$ to the sequence of probability measures $\{P_\nu\}$.

LEMMA 3.2.2. <u>Under</u> H_0, <u>the model</u> (3.1.1) <u>and the assumptions</u> (3.1.2)-(3.1.6), $L_{0\nu}(\underset{\sim}{X}_\nu)$ <u>is asymptotically</u> $N_1(-\frac{1}{2}(\underset{\sim}{\gamma}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma})A^2(F), (\underset{\sim}{\gamma}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma})A^2(F))$.

REMARK. Hájek obtained the result under the model (3.1.1) with $p = 1$ (i.e. case of simple regression). Hájek's result can be trivially extended to our model once we observe that $h_{\nu i} = o(1)$, $i = 1, 2,\ldots, \nu$.

Next we prove a result which gives the asymptotic distribution of $\underset{\sim}{T}_\nu$ under the sequence of alternatives H_ν. Because of 'contiguity' it is possible to use LeCam's third lemma (see e.g. [17]

or [19]) in proving the result. However, we present an alternative proof with a view to extending it to the multivariate situation.

LEMMA 3.2.3. <u>Under the sequence of alternatives</u> H_ν, <u>the model</u> (3.1.1) <u>and the assumptions</u> (3.1.2)-(3.1.6), $\underset{\sim}{T}_\nu$ <u>is asymptotically</u> $N_p((\underset{\sim}{\Lambda}\underset{\sim}{\gamma})A^2(F,\{I_j\}), \underset{\sim}{\Lambda}A^2(F,\{I_j\}))$.

<u>Proof</u>. We introduce $\underset{\sim}{e}$, $m_{\nu i}$, $W_{\nu i}$ $(i = 1, 2,\ldots, \nu)$ as in lemma 3.2.1. It is sufficient to show that $\underset{\sim}{e}'\underset{\sim}{T}_\nu$ is asymptotically $N_1((\underset{\sim}{e}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma})A^2(F,\{I_j\}), (\underset{\sim}{e}'\underset{\sim}{\Lambda}\underset{\sim}{e})A^2(F,\{I_j\}))$. Now

$$(3.2.17)\quad E(W_{\nu i}|H_\nu) = \sum_{j\varepsilon J} \Delta_j P_j(1 + h_{\nu i}\Delta_j + h_{\nu i}\theta_j g_j(h_{\nu i}))$$

$$= h_{\nu i}A^2(F,\{I_j\}) + o(h_{\nu i}) .$$

$$(3.2.18)\quad E(W_{\nu i}^2|H_\nu) = \sum_{j\varepsilon J} \Delta_j^2 P_j(1 + h_{\nu i}\Delta_j + h_{\nu i}\theta_j g_j(h_{\nu i}))$$

$$= A^2(F,\{I_j\}) + O(h_{\nu i}) .$$

$$(3.2.19)\quad E(|W_{\nu i}|^3|H_\nu) \le \sum_{j\varepsilon J} |\Delta_j|^3 P_j(1 + h_{\nu i}\Delta_j + h_{\nu i}\theta_j g_j(h_{\nu i}))$$

$$= \sum_{j\varepsilon J} |\Delta_j|^3 P_j + O(h_{\nu i}) .$$

In deriving the above results we have made repeated use of the fact that for any positive integer r,

$$\sum_{j\varepsilon J} |\Delta_j|^r P_j \le (\max_{j\varepsilon J} P_j^{1-r}) \sum_{j\varepsilon J} |\Delta_j|^r P_j^r \le (\max_{j\varepsilon J} P_j^{1-r})(\sum_{j\varepsilon J} |\Delta_j| P_j)^r$$

and $\max_{j\varepsilon J} P_j^{1-r} < \infty$ independently of any $i = 1,\ldots, \nu$ and

$$\sum_{j\varepsilon J} |\Delta_j| P_j \le \int_0^1 |\phi(u)| du \le [\int_0^1 \phi^2(u) du]^{1/2} = A(F).$$

Thus,

$$\sum_{i=1}^{\nu} E(|m_{\nu i} W_{\nu i}|^3 | H_\nu) \le \sum_{i=1}^{\nu} |m_{\nu i}|^3 (\sum_{j\varepsilon J} |\Delta_j|^3 P_j + O(h_{\nu i})) = o(1),$$

since, $\max_{1\le i\le \nu} |m_{\nu i}| = o(1)$, and $\sum_{i=1}^{\nu} m_{\nu i}^2 = O(1)$.

Hence,

$$(3.2.20) \qquad \sum_{i=1}^{\nu} E(|m_{\nu i}(W_{\nu i} - E(W_{\nu i})|^3 | H_\nu)$$

$$\le 4 \sum_{i=1}^{\nu} |m_{\nu i}|^3 [E(|W_{\nu i}|^3 | H_\nu) + |E(W_{\nu i}|H_\nu)|^3] = o(1),$$

since $E(W_{\nu i}|H_\nu) = O(h_{\nu i})$, $i = 1, 2, \ldots, \nu$.

Further,

$$(3.2.21) \qquad \sum_{i=1}^{\nu} \mathrm{Var}[m_{\nu i} W_{\nu i} | H_\nu] = \sum_{i=1}^{\nu} m_{\nu i}^2 [A^2(F,\{I_j\}) + O(h_{\nu i})]$$

$$= (\underset{\sim}{e}'\Lambda_\nu \underset{\sim}{e}) A^2(F,\{I_j\}) + o(1) \sim (\underset{\sim}{e}'\underset{\sim}{\Lambda}\underset{\sim}{e}) A^2(F,\{I_j\})$$

and

$$(3.2.22) \qquad E[\sum_{i=1}^{\nu} m_{\nu i} W_{\nu i} | H_\nu] = (\sum_{i=1}^{\nu} m_{\nu i} h_{\nu i}) A^2(F,\{I_j\}) + o(1)$$

$$= (\underset{\sim}{e}'\Lambda_\nu \underset{\sim}{\gamma}) A^2(F,\{I_j\}) + o(1) \sim (\underset{\sim}{e}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma}) A^2(F,\{I_j\}) .$$

Using (3.2.20) and (3.2.21) we get the conditions of Liapounov (see e.g. [11], p. 215) under which the central limit theorem holds, are satisfied. The result follows now on using (3.2.21) and (3.2.22).

Define the statistic

$$(3.2.23)\qquad S_\nu = \underset{\sim}{T}'_\nu(\underset{\sim}{\Lambda}_\nu A^2(F,\{I_j\}))^{-1}\underset{\sim}{T}_\nu = A^{-2}(F,\{I_j\})\underset{\sim}{T}'_\nu\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{T}_\nu \ .$$

We propose the following parametric test procedure for testing H_0 against the alternatives given in (3.1.7) based on the critical function $\psi_1(\underset{\sim}{X}^*_\nu)$:

$$(3.2.24)\qquad \psi_1(\underset{\sim}{X}^*_\nu) = \begin{cases} 1 \ , & \text{if } S_\nu > S_{\nu,\varepsilon} \\ \delta_{1\nu\varepsilon}, & \text{if } S_\nu = S_{\nu,\varepsilon} \\ 0 & \text{if } S_\nu < S_{\nu,\varepsilon} \ , \end{cases}$$

where $S_{\nu,\varepsilon}$ and $\delta_{1\nu\varepsilon}$ are chosen in such a way that

$$E[\psi_1(\underset{\sim}{X}^*_\nu)|P_\nu] = \varepsilon, \quad 0 < \varepsilon < 1,$$

ε being the desired level of significance of the test. We may note that the above test is a similar size ε test. Further, under the model (3.1.1) and the assumptions (3.1.2)-(3.1.6), S_ν is distributed asymptotically under H_0 as χ^2_p and under the alternatives as $\chi^2_p(\eta)$, where

$$(3.2.25)\qquad \eta = (\underset{\sim}{\gamma}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma})A^2(F,\{I_j\}) \ .$$

Noting also that $\delta_{1\nu\varepsilon} \to 0$ and $S_{\nu,\varepsilon} \to \chi^2_{p,\varepsilon}$, where $\chi^2_{p,\varepsilon}$ is the upper $100\varepsilon\%$ point of a central chi-square distribution with p degrees of freedom, we can easily obtain the following theorem:

THEOREM 3.2.1. <u>Under the model</u> (3.1.1) <u>and the assumptions</u> (3.1.2)-(3.1.6), <u>the asymptotic power of the test procedure described in</u> (3.2.24) <u>is given by</u>

$$P[\chi_p^2(\eta) \geq \chi_{p,\varepsilon}^2] \ ,$$

where the non-centrality parameter η is defined in (3.2.25).

Let λ_ν be the likelihood ratio test criterion for the testing problem considered in (3.1.7) i.e. $\lambda_\nu = (L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma}))_{\underset{\sim}{\gamma}=\underset{\sim}{0}} / (L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma}))_{\underset{\sim}{\gamma}=\hat{\underset{\sim}{\gamma}}_\nu}$ where $\hat{\underset{\sim}{\gamma}}_\nu = (\hat{\gamma}_{1\nu}, \ldots, \hat{\gamma}_{p\nu})'$ is the maximum likelihood estimator (m.ℓ.e.) of $\underset{\sim}{\gamma}$. Our next objective is to show $S_\nu + 2\log_e \lambda_\nu \to 0$ in P_ν-probability, and hence also in Q_ν-probability because of the contiguity of the Q_ν-probability measure to the P_ν-probability measure. We make the following assumptions on the likelihood function $L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})$ as in Wald [34].

ASSUMPTION I. Denote by D_ν the set of all sample points E_ν for which the m.ℓ.e. $\hat{\underset{\sim}{\gamma}}_\nu$ exists and the second order partial derivatives $(\partial^2/\partial\gamma_k\partial\gamma_{k'})L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})$ are continuous functions of $\gamma_1, \ldots, \gamma_p$. It is assumed that

(3.2.26) $$\lim_{\nu\to\infty} P(D_\nu|\underset{\sim}{\gamma}) = 1, \text{ uniformly in } \underset{\sim}{\gamma} \ .$$

If for a sample point E_ν, there exists several maximum likelihood estimators, we can select one of them by some given rule. Hence, we shall consider $\hat{\gamma}_\nu$ as a single valued function of E_ν defined for all points of D_ν.

ASSUMPTION II. For any positive ε,

(3.2.27) $$\lim_{\nu\to\infty} P[|\hat{\underset{\sim}{\gamma}}_\nu - \underset{\sim}{\gamma}| < \varepsilon|\underset{\sim}{\gamma}] = 1 \ ,$$

uniformly in $\underset{\sim}{\gamma}$, where $|\hat{\underset{\sim}{\gamma}}_\nu - \underset{\sim}{\gamma}| = (\sum_{k=1}^{p} (\hat{\gamma}_{k\nu} - \gamma_k)^2)^{\frac{1}{2}}$.

Write, $L(\underset{\sim}{X}_{\nu}^{*}|\underset{\sim}{\gamma}) = \prod_{i=1}^{\nu} L(X_{\nu i}^{*}|\underset{\sim}{\gamma})$. For any x, any $\delta(> 0)$ and any $\underset{\sim}{\gamma}_0 = (\gamma_{01},\ldots, \gamma_{0p})'$, let

(3.2.28) $\psi_{kk',i}(x,\underset{\sim}{\gamma}_0,\delta) = \underset{\underset{\sim}{\gamma}}{\text{g.}\ell\text{.b.}}\ (\partial^2/\partial\gamma_k\partial\gamma_{k'})\log L(X_{\nu i}^{*}|\underset{\sim}{\gamma})$,

and

(3.2.29) $\phi_{kk',i}(x,\underset{\sim}{\gamma}_0,\delta) = \underset{\underset{\sim}{\gamma}}{\ell\text{.u.b.}}\ (\partial^2/\partial\gamma_k\partial\gamma_{k'})\log L(X_{\nu i}^{*}|\underset{\sim}{\gamma})$,

where $|\underset{\sim}{\gamma} - \underset{\sim}{\gamma}_0| \leq \delta$.

ASSUMPTION III.

(a) For any sequences $\{\underset{\sim}{\gamma}_{1\nu}\}$, $\{\underset{\sim}{\gamma}_{2\nu}\}$ and $\{\delta_\nu\}$ for which

$\lim_{\nu\to\infty} \underset{\sim}{\gamma}_{1\nu} = \lim_{\nu\to\infty} \underset{\sim}{\gamma}_{2\nu} = \underset{\sim}{\gamma}$ and $\lim_{\nu\to\infty} \delta_\nu = 0$, we have,

(3.2.30) $\lim_{\nu\to\infty} E_{\underset{\sim}{\gamma}_{1\nu}} \psi_{kk',i}(x, \underset{\sim}{\gamma}_{2\nu}, \delta_\nu) = \lim_{\nu\to\infty} E_{\underset{\sim}{\gamma}_{1\nu}} \phi_{kk',i}(x, \underset{\sim}{\gamma}_{2\nu}, \delta_\nu)$

$= E_{\underset{\sim}{\gamma}}[(\partial^2/\partial\gamma_k\partial\gamma_{k'})\log L(X_{\nu i}^{*}|\underset{\sim}{\gamma})]$,

uniformly in $\underset{\sim}{\gamma}$, where E stands for expectation.

(b) There exists an $\varepsilon > 0$ such that

$E_{\underset{\sim}{\gamma}_1} [\psi_{kk',i}(x, \underset{\sim}{\gamma}_2, \delta)]^2 = E_{\underset{\sim}{\gamma}_1} [\phi_{kk',i}(x, \underset{\sim}{\gamma}_2, \delta)]^2$

are bounded functions of $\underset{\sim}{\gamma}_1$, $\underset{\sim}{\gamma}_2$ and δ in the domain D_ε defined by the inequalities $|\underset{\sim}{\gamma}_1 - \underset{\sim}{\gamma}_2| \leq \varepsilon$ and $|\delta| \leq \varepsilon$.

(c) The greatest lower bound with respect to $\underset{\sim}{\gamma}$ of the absolute value of the determinant of the matrix

(3.2.31) $E_{\underset{\sim}{\gamma}}(((-\partial^2/\partial\gamma_k\partial\gamma_{k'})\log L(\underset{\sim}{X}_{\nu}^{*}|\underset{\sim}{\gamma})))$ is positive.

ASSUMPTION IV.

(3.2.32) $E_{\gamma}[(\partial/\partial\gamma_k)L(X^*_{\nu i}|\gamma)] = E_{\gamma}[(\partial^2/\partial\gamma_k\partial\gamma_{k'})L(X^*_{\nu i}|\gamma)] = 0$,

for all $k, k' = 1, \ldots, p$; $i = 1, \ldots, \nu$.

ASSUMPTION V. There exists an $\eta > 0$ such that

(3.2.33) $E_{\gamma}|(\partial/\partial\gamma_k)\log L(X^*_{\nu i}|\gamma)|^{2+\eta}$ are bounded functions of γ ,

for all $k = 1, \ldots, p$; $i = 1, \ldots, \nu$.

Let the surface $S_c(\gamma)$ be defined by

(3.2.34) $\gamma'((E_{\gamma}(-\partial^2/\partial\gamma_k\partial\gamma_{k'})\log L(X^*_{\nu}|\gamma))\gamma = c$

and the weight function $\eta(\gamma)$ be defined by

(3.2.35) $\eta(\gamma) = \lim_{\rho\to 0} [A\{\omega'(\gamma, \rho)\}/A\{\omega(\gamma, \rho)\}]$,

where for any γ and any $\rho > 0$,

(3.2.36) $\omega(\gamma, \rho) = \{\tilde{\gamma}: |\tilde{\gamma} - \gamma| \leq \rho\}$,

$\tilde{\gamma}$ lies on the same surface $S_c(\gamma)$ as γ, $\omega'(\gamma, \rho)$ is the image of $\omega(\gamma, \rho)$ by transformation $\gamma^* = B_{\gamma}\gamma$ and B_{γ} is a non-singular matrix such that

(3.2.37) $B'_{\gamma}B_{\gamma} = ((E_{\gamma}(-\partial^2/\partial\gamma_k\partial\gamma_{k'})\log L(X^*_{\nu}|\gamma)))$,

and for any set ω, $A(\omega)$ denotes the $(p - 1)$ dimensional area of ω.

THEOREM 3.2.2. (Wald). Let $S_c(\gamma)$ and $\eta(\gamma)$ be defined as in (3.2.34) and (3.2.35). Then for testing H_0 as defined in (3.1.7), under the

Assumptions I - V, the likelihood ratio test

(a) has asymptotically the best average power with respect to the surfaces $S_c(\gamma)$ and weight function $\eta(\gamma)$;

(b) has asymptotically best constant power on the surfaces $S_c(\gamma)$;

(c) is an asymptotically most stringent test.

We next prove the following theorem which establishes the "asymptotic equivalence" of the statistic S_ν to the likelihood ratio test statistic $-2 \log_e \lambda_\nu$.

THEOREM 3.2.3. Under (3.1.1)-(3.1.7), $S_\nu + 2 \log_e \lambda_\nu \to 0$ in P_ν-probability.

Proof. We have,

$$(3.2.38) \quad -2 \log_e \lambda_\nu = -2[(\log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma}))_{\underset{\sim}{\gamma}=\underset{\sim}{0}} - (\log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma}))_{\underset{\sim}{\gamma}=\hat{\underset{\sim}{\gamma}}_\nu}].$$

By Taylor expansion,

$$(3.2.39) \quad \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma}) = (\log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma}))_{\underset{\sim}{\gamma}=\hat{\underset{\sim}{\gamma}}_\nu}$$

$$+ \sum_{k=1}^{p} (\gamma_k - \hat{\gamma}_{k\nu})\left(\frac{\partial \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial \gamma_k}\right)_{\underset{\sim}{\gamma}=\hat{\underset{\sim}{\gamma}}_\nu}$$

$$+ \frac{1}{2} \sum_{k=1}^{p} \sum_{k'=1}^{p} (\gamma_k - \hat{\gamma}_{k\nu})(\gamma_{k'} - \hat{\gamma}_{k'\nu})\left(\frac{\partial^2 \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial \gamma_k \partial \gamma_{k'}}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_\nu^*},$$

where $\underset{\sim}{\gamma}_\nu^*$ lies in the p-dimensional rectangle $[\underset{\sim}{\gamma}, \hat{\underset{\sim}{\gamma}}_\nu]$. Since,

$$\left.\frac{\partial \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial \gamma_k}\right|_{\underset{\sim}{\gamma}=\hat{\underset{\sim}{\gamma}}_\nu} = 0$$

for all $k = 1, 2, \ldots, p$, we have from (3.2.38) and (3.2.39),

$$(3.2.40) \quad -2\ \log_e \lambda_\nu = \hat{\gamma}_\nu'\left(\left(-\frac{\partial^2 \log_e L(\underset{\sim}{X}_\nu^*|\gamma)}{\partial\gamma_k \partial\gamma_{k'}}\right)\right)_{\gamma=\gamma_\nu^*} \hat{\gamma}_\nu \ .$$

But, we can write,

$$\log L(\underset{\sim}{X}_\nu^*|\gamma) = \sum_{i=1}^{\nu} \log[Z_{\nu i0} F(a_1' - h_{\nu i})$$

$$+ \sum_{j=1}^{\ell-1} Z_{\nu ij}\{F(a_{j+1}' - h_{\nu i}) - F(a_j' - h_{\nu i}) + Z_{\nu i} (1 - F(a_\ell' - h_{\nu i}))].$$

Hence,

$$\frac{\partial \log L(\underset{\sim}{X}_\nu^*|\gamma)}{\partial\gamma_k} = \sum_{i=1}^{\nu} c_{k\nu i}^* \frac{\sum_{j=0}^{\ell} Z_{\nu ij}\{f(a_j' - h_{\nu i}) - f(a_{j+1}' - h_{\nu i})\}}{\sum_{j=0}^{\ell} Z_{\nu ij}\{F(a_{j+1}' - h_{\nu i}) - F(a_j' - h_{\nu i})\}}$$

$f(a_0' - h_{\nu i}) = f(a_{\ell+1}' - h_{\nu i}) = 0$. Hence,

$$\frac{\partial^2 \log L(\underset{\sim}{X}_\nu^*|\gamma)}{\partial\gamma_k \partial\gamma_{k'}}$$

$$= \sum_{i=1}^{\nu} c_{k\nu i}^* c_{k'\nu i}^* \frac{Z_{\nu i0} f'(a_1' - h_{\nu i}) + \sum_{j=1}^{\ell-1} Z_{\nu ij}\{f'(a_{j+1}' - h_{\nu i}) - f'(a_j' - h_{\nu i})\} + Z_{\nu i\ell}(-f'(a_\ell' - h_{\nu i}))}{\sum_{j=0}^{\ell} Z_{\nu ij}\{F(a_{j+1}' - h_{\nu i}) - F(a_j' - h_{\nu i})\}}$$

$$- \sum_{i=1}^{\nu} c_{k\nu i}^* c_{k'\nu i}^* \frac{\sum_{j=0}^{\ell} Z_{\nu ij}\{f(a_j' - h_{\nu i}) - f(a_{j+1}' - h_{\nu i})\}^2}{\sum_{j=0}^{\ell} Z_{\nu ij}\{F(a_{j+1}' - h_{\nu i}) - F(a_j' - h_{\nu i})\}^2} \ .$$

Under H ,

$$(3.2.41)\quad E_{\underset{\sim}{\gamma}=\underset{\sim}{0}}\left[\frac{\partial^2 \log L(\underset{\sim}{X}_\nu^*|\gamma)}{\partial\gamma_k\partial\gamma_{k'}}\right]$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i}E\left\{\left[\frac{\sum_{j=0}^{\ell} Z_{\nu ij}\{f'(a'_{j+1}) - f'(a'_j)\}}{\sum_{j=0}^{\ell} Z_{\nu ij}\{F(a'_{j+1}) - F(a'_j)\}}\right] \Big| P_\nu\right\}$$

$$- \sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i}E(\sum_{j=0}^{\ell} Z_{\nu ij}\Delta_j^2|P_\nu)$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i} \sum_{j=0}^{\ell} \{f'(a'_{j+1}) - f'(a'_j)\}$$

$$- (\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i})A^2(F,\{I_j\}) = - (\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i})A^2(F,\{I_j\}) .$$

The above expression could also be obtained from the relation

$$E_{\underset{\sim}{\gamma}=\underset{\sim}{0}}\left(-\frac{\partial^2 \log L}{\partial\gamma_k\partial\gamma_{k'}}\right) = E_{\underset{\sim}{\gamma}=\underset{\sim}{0}}\left[\left(\frac{\partial \log L}{\partial\gamma_k}\right)\left(\frac{\partial \log L}{\partial\gamma_{k'}}\right)\right]$$

which holds under the stated regularity assumptions of Wald.

Now,

$$(3.2.42)\quad E_{\underset{\sim}{\gamma}=\underset{\sim}{0}}\left[\frac{\partial^2 \log L(\underset{\sim}{X}_\nu^*|\gamma)}{\partial\gamma_k\partial\gamma_{k'}} + (\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i})A^2(F,\{I_j\})\right]^2$$

$$= E[\{\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i} \frac{\sum_{j=0}^{\ell} Z_{\nu ij}\{f'(a'_{j+1}) - f'(a'_j)\}}{\sum_{j=0}^{\ell} Z_{\nu ij}P_j}$$

$$- \sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i} \sum_{j=0}^{\ell} \Delta_j^2(Z_{\nu ij} - P_j)\}^2|P_\nu]^2$$

$$\leq 2 \sum_{i=1}^{\nu} c^{*2}_{k\nu i} c^{*2}_{k'\nu i} \sum_{j\varepsilon J} P_j^{-1}\{f'(a'_{j+1}) - f'(a'_j)\}^2$$

$$+ 2 \sum_{i=1}^{\nu} c^{*2}_{k\nu i} c^{*2}_{k'\nu i} E[\{\sum_{j=0}^{\ell} \Delta_j^2 (Z_{\nu ij} - P_j)\}^2 | P_\nu]$$

$$\leq 2(\max_{j\varepsilon J} P_j^{-1})(\max_{1\leq i\leq \nu} c^{*2}_{k'\nu i}) \sum_{j\varepsilon J} \{f'(a'_{j+1}) - f'(a'_j)\}^2$$

$$+ 2 \;(\max_{1\leq i\leq \nu} c^{*2}_{k\nu i}) \sum_{j=0}^{\ell} \Delta_j^4 P_j(1 - P_j) \;.$$

But, we have already seen that $|\Delta_j|$ is finite for all $j = 0, 1, \ldots, \ell$.

Further, $\sum_{j\varepsilon J} \{f'(a'_{j+1}) - f'(a'_j)\}^2 < \infty$, since the summation involves only a finite number of terms.

Using (3.1.4) and the above, we get now from (3.2.42),

$$E_{\underset{\sim}{\gamma}=\underset{\sim}{0}} \left[\frac{\partial^2 \log L(\underset{\sim}{X}^*_\nu | \underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_{k'}} + (\sum_{i=1}^{\nu} c^*_{k\nu i} c^*_{k'\nu i}) A^2(F,\{I_j\})\right]^2 = o(1) \;.$$

Thus, under H_0,

$$\frac{\partial^2 \log L(\underset{\sim}{X}^*_\nu | \underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_{k'}} + (\sum_{i=1}^{\nu} c^*_{k\nu i} c^*_{k'\nu i}) A^2(F,\{I_j\}) = o_p(1) \;,$$

for all $k, k' = 1, 2, \ldots, p$.

Also, using properties of maximum likelihood estimators, we can show in this case, under H_0, $\hat{\underset{\sim}{\gamma}}_\nu$ is asymptotically

$$N_p\left(\underset{\sim}{0}, \left(\left(E_{\underset{\sim}{\gamma}=\underset{\sim}{0}}\left(-\frac{\partial^2 \log L(\underset{\sim}{X}^*_\nu | \underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_{k'}}\right)\right)\right)^{-1}\right)$$

i.e. $N_p(\underset{\sim}{0}, \underset{\sim}{\Lambda}_\nu^{-1} \; A^{-2}(F,\{I_j\}))$. Hence, $\hat{\gamma}_{k\nu} = O_p(1)$, for all $k = 1,2,\ldots,p$.

Also, we have

$$\left(\frac{\partial^2 \log L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_{k'}}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_\nu^*} - \left(\frac{\partial^2 \log L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_{k'}}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{0}}$$

converges to zero in P_ν-probability for all k, k' = 1, 2,..., p.

Now, from (3.2.40), we get,

$$(3.2.43) \qquad -2\log_e\lambda_\nu - (\hat{\underset{\sim}{\gamma}}_\nu' \underset{\sim}{\Lambda}_\nu \hat{\underset{\sim}{\gamma}}_\nu) A^2(F,\{I_j\}) \to 0 \quad \text{in } P_\nu\text{-probability.}$$

But, again,

$$(3.2.44) \quad 0 = \left.\frac{\partial \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial\gamma_k}\right|_{\underset{\sim}{\gamma}=\hat{\underset{\sim}{\gamma}}_\nu}$$

$$= \left.\frac{\partial \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial\gamma_k}\right|_{\underset{\sim}{\gamma}=\underset{\sim}{0}} + \sum_{k'=1}^{p} \hat{\gamma}_{k'\nu} \left.\frac{\partial^2 \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_{k'}}\right|_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_\nu^{**}}$$

where $\underset{\sim}{\gamma}_\nu^{**}$ lies in the p-dimensional rectangle $[\underset{\sim}{0}, \hat{\underset{\sim}{\gamma}}_\nu]$. Thus,

$$(3.2.45) \quad T_{k\nu} = -\hat{\underset{\sim}{\gamma}}_\nu' \left(\frac{\partial^2 \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_1} \cdots \frac{\partial^2 \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_p}\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_\nu^{**}},$$

so that

$$(3.2.46) \quad \underset{\sim}{T}_\nu' = \hat{\underset{\sim}{\gamma}}_\nu' \left(\!\!\left(-\frac{\partial^2 \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_{k'}}\right)\!\!\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_\nu^{**}}$$

But,

$$\left(\!\!\left(-\frac{\partial^2 \log_e L(\underset{\sim}{X}_\nu^*|\underset{\sim}{\gamma})}{\partial\gamma_k \partial\gamma_{k'}}\right)\!\!\right)_{\underset{\sim}{\gamma}=\underset{\sim}{\gamma}_\nu^{**}} - \underset{\sim}{\Lambda}_\nu A^2(F,\{I_j\})$$

also converges in P_ν-probability to zero. Hence,

$$(3.2.47)\quad S_\nu - (\hat{\gamma}_\nu' \Lambda_\nu \hat{\gamma}_\nu) A^2(F,\{I_j\})$$

$$= (T_\nu' \Lambda_\nu^{-1} T_\nu) A^{-2}(F,\{I_j\}) - (\hat{\gamma}_\nu' \Lambda_\nu \hat{\gamma}_\nu) A^2(F,\{I_j\})$$

$$= \hat{\gamma}_\nu' \left(\left(- \frac{\partial^2 \log_e L(X_\nu^* | \gamma)}{\partial\gamma_k \partial\gamma_{k'}} \right)\right)_{\gamma=\gamma_\nu^{**}} (\Lambda_\nu^{-1} A^{-2}(F,\{I_j\}).$$

$$\left(\left(- \frac{\partial^2 \log_e L(X_\nu^* | \gamma)}{\partial\gamma_k \partial\gamma_{k'}} \right)\right)_{\gamma=\gamma_\nu^{**}} \hat{\gamma}_\nu$$

$$- \hat{\gamma}_\nu' (\Lambda_\nu A^2(F,\{I_j\}))(\Lambda_\nu A^2(F,\{I_j\}))^{-1} (\Lambda_\nu A^2(F,\{I_j\})\hat{\gamma}_\nu \to 0$$

in P_ν-probability by Slutsky's theorem. The theorem now follows from (3.2.43) and (3.2.47).

Note that, since $\{Q_\nu\}$ is contiguous to $\{P_\nu\}$, $S_\nu + 2\log_e\lambda_\nu \to 0$ in Q_ν-probability as well.

The above theorem implies that S_ν and $-2\log_e\lambda_\nu$ have asymptotically the same distribution under H_0 and also, under the sequence $\{H_\nu\}$ of alternatives. Thus, under the given model and the assumptions, the test procedure based on S_ν possesses the asymptotic optimal properties of the likelihood ratio test as given in Theorem 3.2.2.

3.3. Asymptotically Optimal Nonparametric Test.

Define as in Sen [33],

$$(3.3.1)\quad \sum_{i=1}^{\nu} Z_{\nu ij} = \nu_j \text{ for } j = 0, 1, \ldots, \ell;\ \nu = \sum_{j=0} \nu_j ;$$

$$(3.3.2)\quad F_{\nu,0} = 0,\ F_{\nu,j+1} = \sum_{m=0}^{j} \nu_m/\nu,\quad j = 0, 1, \ldots, \ell .$$

Define

$$(3.3.3) \qquad \hat{\Delta}_{\nu j} = \int_{F_{\nu,j}}^{F_{\nu,j+1}} \phi(u)\,du \Big/ \int_{F_{\nu,j}}^{F_{\nu,j+1}} du, \text{ for } \nu_j > 0, = 0 \text{ if } \nu_j = 0$$

($j = 0.\ 1, \ldots, \ell$). Let

$$(3.3.4) \qquad U_{k\nu} = \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j=0}^{\ell} \hat{\Delta}_{\nu j} Z_{\nu ij}, \quad k = 1, \ldots, p,$$

and $U_\nu = (U_{1\nu}, \ldots, U_{p\nu})'$;

$$(3.3.5) \qquad A^2(F_\nu, \{I_j\}) = \sum_{j=0}^{\ell} \hat{\Delta}^2_{\nu j}(\nu_j/\nu).$$

We propose the test statistic

$$(3.3.6) \qquad M_\nu = (\underset{\sim}{U}'_\nu \underset{\sim}{\Lambda}^{-1}_\nu \underset{\sim}{U}_\nu) A^{-2}(F_\nu, \{I_j\}) .$$

The rationale behind using this test statistic is as follows: We may first note that (as in Sen [33]) the distribution of S_ν will depend on the unknown Δ_j ($j = 0, 1, \ldots, \ell$) even under H_o in (3.1.7). This is because we are dealing with grouped data. However, S_ν provides a distribution-free test under the same permutation argument as of Sen. We repeat this briefly for the sake of completeness. Under H_0 in (3.1.7) $X^*_{\nu i}$ ($i = 1, 2, \ldots, \nu$) are i.i.d.r.v. and, hence, the joint distribution of $\underset{\sim}{X}^*_\nu$ remains invariant under any permutation of its ν arguments. A permutational probability measure denoted by $\mathcal{P}_\nu$ is defined on the set of $\nu!$ equiprobable points (actually there are $\nu!/\prod_{j=0} \nu_j!$ distinct equally likely permutations of $\underset{\sim}{X}^*_\nu$) on the ν dimensional real space $\mathbb{R}^\nu$. Under $\mathcal{P}_\nu$, all the $\nu!$

equally likely permutations have the common probability $(\nu!)^{-1}$. Also ν_j in (3.3.1) and $\hat{\Delta}_{\nu j}$ in (3.3.3) are permutation-invariant for all $j = 0, 1, \ldots, \ell$. We can easily verify that

(3.3.7) $E(Z_{\nu ij}|\mathcal{P}_\nu) = \nu_j/\nu \; (i = 1, 2, \ldots, \nu;\; j = 0, 1, \ldots, \ell)$;

(3.3.8) $E(Z_{\nu ij}Z_{\nu ij'}|\mathcal{P}_\nu) = 0 \; (i = 1, 2, \ldots, \nu;\; j, j' = 0, 1, \ldots, \ell,\; j \neq j')$;

(3.3.9) $E(Z_{\nu ij}Z_{\nu i'j'}|\mathcal{P}_\nu) = (\nu_j(\nu_{j'} - \delta_{jj'}))/(\nu(\nu - 1))$,

$(j, j' = 0, 1, \ldots, \ell;\; i, i' = 1, 2, \ldots, \nu,\; i \neq i')$, where $\delta_{jj'}$ are Kronecker deltas. Using these, we get,

(3.3.10) $E(U_{k\nu}|\mathcal{P}_\nu) = \sum_{i=1}^{\nu} c^*_{k\nu i}(\int_0^1 \phi(u)du) = 0,\; k = 1, 2, \ldots, p$;

(3.3.11) $\mathrm{Cov.}(U_{k\nu}, U_{k'\nu}|\mathcal{P}_\nu) = (\nu/(\nu-1))(\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i})A^2(F\;,\{I_j\})$.

Thus,

(3.3.12) $E(\underset{\sim}{U}_\nu|\mathcal{P}_\nu) = \underset{\sim}{0},\; \mathrm{Var}(\underset{\sim}{U}_\nu|\mathcal{P}_\nu) = (\nu/(\nu - 1))A^2(F_\nu, \{I_j\})\underset{\sim}{\Lambda}_\nu$.

Let

(3.3.13) $M_\nu = A^{-2}(F_\nu, \{I_j\})\underset{\sim}{U}'_\nu\underset{\sim}{\Lambda}^{-1}\underset{\sim}{U}_\nu$.

The following nonparametric test is proposed:

$$\psi_2(\underset{\sim}{X}^*_\nu) = \begin{cases} 1 & \text{if } M_\nu > M_{\nu,\varepsilon} \\ \delta_{2\nu\varepsilon} & \text{if } M_\nu = M_{\nu,\varepsilon} \\ 0 & \text{if } M_\nu < M_{\nu,\varepsilon} \end{cases},$$

$M_{\nu,\varepsilon}$ and $\delta_{2\nu\varepsilon}$ being so chosen as $E[\psi_2(\underset{\sim}{X}^*_\nu)|\mathcal{P}_\nu] = \varepsilon$, the level of significance. This implies $E[\psi_2(\underset{\sim}{X}^*_\nu)|P_\nu] = \varepsilon$ i.e. $\psi_2(\underset{\sim}{X}^*_\nu)$ is a similar size ε test.

We may remark again in this chapter the limitations of the permutation test statistic. The task of finding the percentage points of the actual permutation distribution becomes tremendous even with moderately large sample sizes. This leads to the study of the large-sample distribution of the permutation test statistic.

Next we prove a lemma which is due to Sen [33]. An alternative proof is, however, proposed with a view to extending it to the multivariate situation. Sen's method consists in proving the result under the null hypothesis and then extending it to the sequence of 'contiguous' alternatives. Our proof involves a direct argument and does not appear to contiguity.

LEMMA 3.3.1. Under the assumption (3.1.6), $A^2(F_\nu,\{I_j\}) \to A^2(F,\{I_j\})$ as $\nu \to \infty$ either in P_ν- (or in Q_ν-) probability.

Proof. First we prove the result under P_ν-probability. Write $A^2(F_\nu,\{I_j\}) = \sum_{j\in J} \hat{\Delta}^2_{\nu j}(\nu_j/\nu) + \sum_{j\notin J} \hat{\Delta}^2_{\nu j}(\nu_j/\nu)$. Now, for any $j \notin J$,

$$(3.3.15)\quad \hat{\Delta}^2_{\nu j}(\nu_j/\nu) \le \int_{F_{\nu,j}}^{F_{\nu,j+1}} \phi^2(u)\,du = \int_{F_{\nu,j}}^{F_{\nu,j+1}} \phi^2(u)\,du - \int_{F_j}^{F_{j+1}} \phi^2(u)\,du$$

(since $P_j = 0$ for $j \notin J$).

$$= \int_{F_{j+1}}^{F_{\nu,j+1}} \phi^2(u)\,du - \int_{F_j}^{F_{\nu,j}} \phi^2(u)\,du\ .$$

But using a well-known result, under H_0, $F_{\nu,j} - F_j = O_p(\nu^{-\frac{1}{2}})$ for each fixed j. Since $\int_0^1 \phi^2(u)\,du = A^2(F) < \infty$, we have

$$\int_{F_j}^{F_{\nu,j}} \phi^2(u)du = o_p(1) \text{ for all } j = 0, 1,\ldots, \ell.$$

Hence,

(3.3.16) $$\sum_{j \notin J} \hat{\Delta}^2_{\nu j}(\nu_j/\nu) = o_p(1) ,$$

since the summation involves only a finite number of terms. Also,

(3.3.17) $$\sum_{j\varepsilon J} \hat{\Delta}^2_{\nu j}(\nu_j/\nu)$$

$$= \sum_{j\varepsilon J} (\hat{\Delta}^2_{\nu j} - \Delta^2_j)(\nu_j/\nu) + \sum_{j\varepsilon J} \Delta^2_j((\nu_j/\nu) - P_j) + \sum_{j\varepsilon J} \Delta^2_j P_j .$$

Now, under P_ν, $(\nu_j/\nu) - P_j = (F_{\nu,j+1} - F_{j+1}) - (F_{\nu,j} - F_j) = O_p(\nu^{-\frac{1}{2}})$, for all $j = 0, 1,\ldots, \ell$. (Using Chebyshev inequality, it follows that for each fixed j, $F_{\nu,j} - F_j = O_p(\nu^{-\frac{1}{2}})$).

As already observed in Section 3.2,

$|\Delta_j| \le (\max_{j\varepsilon J} P_j^{-1})A(F)$ for all $j \varepsilon J$, and we get,

(3.3.18) $$\sum_{j\varepsilon J} \Delta^2_j((\nu_j/\nu) - P_j) = o_p(1) .$$

Further, for any $j \varepsilon [0, 1,\ldots, \ell]$, given any $\varepsilon > 0$,

(3.3.19) $$\Pr\cdot[(\nu_j/\nu)|\hat{\Delta}^2_{\nu j} - \Delta^2_j| > \varepsilon \,|P_\nu]$$

$$= \Pr\cdot[(\nu_j/\nu)|\hat{\Delta}^2_{\nu j} - \Delta^2_j| > \varepsilon,\ \nu_j > 0|P_\nu]$$

$$+ \Pr\cdot[(\nu_j/\nu)|\hat{\Delta}^2_{\nu j} - \Delta^2_j| > \varepsilon,\ \nu_j = 0|P_\nu]$$

$$= \Pr[(\nu_j/\nu)|\hat{\Delta}^2_{\nu j} - \Delta^2_j| > \varepsilon,\ \nu_j > 0|P_\nu]$$

But for any $j \varepsilon J$ and $\nu_j > 0$, we can write

$$(\hat{\Delta}^2_{\nu j} - \Delta^2_j)(\nu_j/\nu)$$

$$= (\nu_j/\nu)^{-1}\left[\int_{F_{\nu,j}}^{F_{\nu,j+1}} \phi(u)\,du\right]^2 - (\nu_j/\nu)P_j^{-2}\left[\int_{F_j}^{F_{j+1}} \phi(u)\,du\right]^2$$

$$= \left[\int_{F_{\nu,j}}^{F_{\nu,j+1}} \phi(u)\,du\right]^2 (\nu/(\nu_j P_j))(P_j - \frac{\nu_j}{\nu})$$

$$+ P_j^{-1}\left\{\left(\int_{F_{\nu,j}}^{F_{\nu,j+1}} \phi(u)\,du\right)^2 - \left(\int_{F_j}^{F_{j+1}} \phi(u)\,du\right)^2\right\}$$

$$+ P_j^{-2}(P_j - \frac{\nu_j}{\nu})\left(\int_{F_j}^{F_{j+1}} \phi(u)\,du\right)^2 .$$

Hence, for $j \varepsilon J$ and $\nu_j > 0$, after a few steps, we get.

$$|\hat{\Delta}^2_{\nu j} - \Delta^2_j|(\nu_j/\nu)$$

$$\leq P_j^{-1}\ A^2(F)|P_j - \frac{\nu_j}{\nu}|$$

$$+ P_j^{-1}\left(\int_{F_{\nu,j}}^{F_{\nu,j+1}} |\phi(u)|\,du + \int_{F_j}^{F_{j+1}} |\phi(u)|\,du\right)\left(\left|\int_{F_{\nu,j}}^{F_j} \phi(u)\,du - \int_{F_{\nu,j+1}}^{F_{j+1}} \phi(u)\,du\right|\right)$$

$$+ P_j^{-1}|P_j - \frac{\nu_j}{\nu}|A^2(F)$$

$$\le 2P_j^{-1}A^2(F)|(\nu_j/\nu) - P_j| + 2P_j^{-1}A(F)[|F_{\nu,j} - F_j|^{1/2} + |F_{\nu,j+1} - F_{j+1}|^{1/2}].$$

Using the above inequality, given any $\varepsilon > 0$, we get from (3.3.19),

(3.3.20) $$\Pr\cdot[(\nu_j/\nu)|\hat{\Delta}^2_{\nu j} - \Delta^2_j| > \varepsilon|P_\nu]$$

$$\le \Pr\cdot[2P_j^{-1}A^2(F)|(\nu_j/\nu) - P_j| > \frac{\varepsilon}{3}, \nu_j > 0|P_\nu]$$

$$+ \Pr\cdot[2P_j^{-1}A(F)|F_{\nu,j} - F_j|^{1/2} > \frac{\varepsilon}{3}, \nu_j > 0|P_\nu]$$

$$+ \Pr\cdot[2P_j^{-1}A(F)|F_{\nu,j+1} - F_{j+1}|^{1/2} > \frac{\varepsilon}{3}, \nu_j > 0|P_\nu]$$

$$\le \Pr\cdot[|(\nu_j/\nu) - P_j| > \frac{\varepsilon P_j}{6A^2(F)}|P_\nu]$$

$$+ \Pr\cdot[|F_{\nu,j} - F_j| > \frac{\varepsilon^2 P_j^2}{36A^2(F)}|P_\nu]$$

$$+ \Pr\cdot[|F_{\nu,j+1} - F_{j+1}| > \frac{\varepsilon^2 P_j^2}{36A^2(F)}|P_\nu] = o(1) .$$

(3.3.16), (3.3.17), (3.3.18), (3.3.20) show that

(3.3.21) $$\sum_{j=0}^{\ell} \hat{\Delta}^2_{\nu j}(\nu_j/\nu) \to \sum_{j\varepsilon J} \Delta_j^2 P_j = A^2(F,\{I_j\}) \text{ in } P_\nu\text{-probability.}$$

To prove that the result holds in Q_ν-probability, we proceed as follows:

Let

(3.3.22) $$F_{\nu i,j} = F[\sigma^{-1}(a_j - \beta_0) - h_{\nu i}],\ i = 1, 2,\dots, \nu,\ j = 0, 1,\dots,\ell.$$

From (3.3.12), $P_{\nu ij} = P_j + o(1)$, uniformly in $i = 1, 2, \ldots, \nu$, $j = 0, 1, \ldots, \ell$. Let

(3.3.23) $$H^*(x) = \nu^{-1} \sum_{i=1}^{\nu} F_{\nu i}(x) \ ;$$

(3.3.24) $$H^*_{\nu,j} = \nu^{-1} \sum_{i=1}^{\nu} F_{\nu i,j} \ (j = 0, 1, \ldots, \ell) \ ;$$

(3.3.25) $$\Delta_{\nu j0} = \int_{H^*_{\nu,j}}^{H^*_{\nu,j+1}} \phi(u)\,du \Big/ \int_{H^*_{\nu,j}}^{H^*_{\nu,j+1}} du, \text{ if } H^*_{\nu,j+1} - H^*_{\nu,j} > 0,$$

$$= 0, \text{ otherwise.}$$

Let, $H^{**}_{\nu,j} = H^*_{\nu,j+1} - H^*_{\nu,j}$ $(j = 0, 1, \ldots, \ell)$. Define

(3.3.26) $$A^2(F^*_{0\nu}, \{I_j\}) = \sum_{j=0}^{\ell} \Delta^2_{\nu j0}\, H^{**}_{\nu,j} \ .$$

Arguing as before, it can be shown now that

(3.3.27) $$A^2(F_\nu, \{I_j\}) - A^2(F^*_{0\nu}, \{I_j\}) \to 0 \text{ in } Q_\nu\text{-probability.}$$

$$A^2(F^*_{0\nu}, \{I_j\}) - A^2(F, \{I_j\})$$

$$= \sum_{j=0}^{\ell} (\Delta^2_{\nu j0} - \Delta^2_j) H^{**}_{\nu j} + \sum_{j=0}^{\ell} \Delta^2_j (H^{**}_{\nu j} - P_j) \ .$$

But, $H^{**}_{\nu j} - P_j = \nu^{-1} \sum_{i=1}^{\nu} (P_{\nu ij} - P_j) = o(1)$, for all $j = 0, 1, \ldots, \ell$, and $\sum_{j=0}^{\ell} \Delta^2_j (H^{**}_{\nu j} - P_j) = \sum_{j \in J} \Delta^2_j (H^{**}_{\nu j} - P_j)$. Using now the remark immediately before (3.3.18), $\sum_{j=0}^{\ell} \Delta^2_j (H^{**}_{\nu j} - P_j) = o(1)$. Also,

$\sum_{j=0}^{\ell} (\Delta^2_{\nu j0} - \Delta^2_j)H^{**}_{\nu j} = \sum_{j\varepsilon J} (\Delta^2_{\nu j0} - \Delta^2_j)H^{**}_{\nu j} + \sum_{j\notin J} (\Delta^2_{\nu j0} - \Delta^2_j)H^{**}_{\nu j}$. For $j \notin J$,

$\Delta_j = 0$ and as in (3.3.15) we can show that $\sum_{j\notin J} (\Delta^2_{\nu j0} - \Delta^2_j)H^{**}_{\nu j} = o(1)$.

Proceeding now as in (3.3.20) it can be shown that

$\sum_{j\varepsilon J} (\Delta^2_{\nu j0} - \Delta^2_j)H^{**}_{\nu j} = o(1)$. Thus

(3.3.28) $A^2(F^*_{0\nu},\{I_j\}) - A^2(F,\{I_j\}) \to 0$ in Q_ν-probability.

(3.3.27) and (3.3.28) prove the result. Q.E.D.

Define

(3.3.29) $$W^*_{\nu i} = \sum_{j=0}^{\ell} \hat{\Delta}_{\nu j} Z_{\nu ij}, \quad i = 1, 2, \ldots, \nu.$$

Under $\mathcal{P}_\nu$, $\hat{\Delta}_{\nu j}$ are all invariant, while $Z_{\nu ij}$ are stochastic. ν_j of $W^*_{\nu i}$ are equal to $\hat{\Delta}_{\nu j}$ ($j = 0, 1, \ldots, \ell$). Also $U_{k\nu} = \sum_{i=1}^{\nu} c^*_{k\nu i} W^*_{\nu i}$, $k = 1, 2, \ldots, p$. The following theorem gives the permutation limit distribution of $\underset{\sim}{U}_\nu$.

THEOREM 3.3.1. <u>Under $\mathcal{P}_\nu$ and (3.1.1)-(3.1.6), $\underset{\sim}{U}_\nu$ is asymptotically</u> $N_p(\underset{\sim}{0},\underset{\sim}{\Lambda}\ A^2(F,\{I_j\}))$ <u>in probability</u>.

<u>Proof</u>. It is sufficient to show that for any $\underset{\sim}{d} = (d_1, \ldots, d_p)' \neq \underset{\sim}{0}$, $d_1, \ldots, d_p$ real and fixed, $\underset{\sim}{d}'\underset{\sim}{U}_\nu$ is asymptotically $N_1(0, (\underset{\sim}{d}'\underset{\sim}{\Lambda}\underset{\sim}{d})A^2(F,\{I_j\}))$ under $\mathcal{P}_\nu$ in probability. Write,

(3.3.30) $$\underset{\sim}{d}'\underset{\sim}{U}_\nu = \sum_{i=1}^{\nu} e_{\nu i} W^*_{\nu i} ,$$

where $e_{\nu i} = \sum_{k=1}^{p} d_k c^*_{k\nu i}$ ($i = 1, 2, \ldots, \nu$). Using (3.3.10) and (3.3.11), (3.3.31) $E(\underset{\sim}{d}'\underset{\sim}{U}_\nu|\mathcal{P}_\nu) = 0$, $\mathrm{Var}(\underset{\sim}{d}'\underset{\sim}{U}_\nu|\mathcal{P}_\nu) = (\nu/(\nu - 1))(\underset{\sim}{d}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{d})A^2(F_\nu,\{I_j\})$.

To prove the theorem, we shall make use of the Wald-Wolfowitz-Noether-Hájek permutational central limit theorem again. We shall show that the sequence $\{e_{\nu i}\}$ satisfies the Noether condition, while the sequence $\{W_{\nu i}\}$ satisfies the condition Q of Hájek (see chapter II) in probability. We note first that $\bar{e}_\nu = 0$ and since ν_j of the $W^*_{\nu i}$ are equal to $\hat{\Delta}_{\nu j}$, $j = 0, 1, \ldots, \ell$, $\overline{W}^*_\nu = \nu^{-1} \sum_{j=0}^{\ell} \nu_j \hat{\Delta}_{\nu j} = 0$. Since,

$$\max_{1\le i\le \nu} |e_{\nu i}| \le \left(\max_{1\le i\le \nu}\ \max_{1\le k\le p} |c^*_{k\nu i}|\right) \sum_{k=1}^{p} |d_k| = o(1) \quad \text{and}$$

$$\sum_{i=1}^{\nu} e^2_{\nu i} = \underset{\sim}{d}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{d} \sim \underset{\sim}{d}'\underset{\sim}{\Lambda}\underset{\sim}{d} \text{ (non-zero and finite). So,}$$

$$\max_{1\le i\le \nu} e^2_{\nu i} / \sum_{i=1}^{\nu} e^2_{\nu i} = o(1),$$ i.e. the sequence $\{e_{\nu i}\}$ satisfies the Noether condition. As in chapter II we can show that

$$\max_{1\le i_1<\ldots<i_{k_\nu}\le \nu} \nu^{-1} \sum_{\alpha=1}^{k_\nu} W^{*2}_{\nu i_\alpha} \to 0 \text{ in } \mathcal{P}_\nu \text{ probability and}$$

$$\nu^{-1} \sum_{i=1}^{\nu} W^{*2}_{\nu i} = A^2(F_\nu,\{I_j\}) \to A^2(F,\{I_j\}) \text{ in probability by lemma 3.3.1.}$$

Hence, the theorem.

Next we prove that the test procedure based on $\psi_2(\underset{\sim}{X}^*_\nu)$ is power-equivalent to the one based on $\psi_1(\underset{\sim}{X}^*_\nu)$. To prove this, we need the following lemma in addition to lemma 3.3.1.

LEMMA 3.3.2. Under (3.1.1)-(3.1.6), $\underset{\sim}{U}_\nu - \underset{\sim}{T}_\nu \to \underset{\sim}{0}$ in P_ν- (or in Q_ν-) probability as $\nu \to \infty$.

Proof. It is sufficient to show that $U_{k\nu} - T_{k\nu} \to 0$ in probability for each $k = 1,\ldots, p$. We have,

$$(3.3.32)\quad U_{k\nu} - T_{k\nu} = \sum_{j=0}^{\ell} (\hat{\Delta}_{\nu j} - \Delta_j) \sum_{i=1}^{\nu} c^*_{k\nu i} Z_{\nu ij} .$$

But,

$$E(\sum_{i=1}^{\nu} c^*_{k\nu i} Z_{\nu ij} | Q_\nu) = \sum_{i=1}^{\nu} c^*_{k\nu i} (P_j + h_{\nu i}\Delta_j P_j + h_{\nu i} P_j \theta_j g_j(h_{\nu i}))$$

$$= \Delta_j P_j \sum_{i=1}^{\nu} c^*_{k\nu i} h_{\nu i} + P_j \theta_j \sum_{i=1}^{\nu} c^*_{k\nu i} h_{\nu i} g_j(h_{\nu i}) = O(1),$$

since $\sum_{i=1}^{\nu} c^*_{k\nu i} h_{\nu i} = \sum_{k'=1}^{p} \gamma_{k'} (\sum_{i=1}^{\nu} c^*_{k\nu i} c^*_{k'\nu i}) = O(1)$, and $g_j(h_{\nu i}) = o(1)$ uniformly in $0 \le j \le \ell$, $1 \le i \le \nu$.

$$\mathrm{Var}(\sum_{i=1}^{\nu} c^*_{k\nu i} Z_{\nu ij} | Q_\nu) = \sum_{i=1}^{\nu} c^{*2}_{k\nu i} \mathrm{Var}(Z_{\nu ij} | Q_\nu)$$

$$= \sum_{i=1}^{\nu} c^{*2}_{k\nu i} P_{\nu ij}(1 - P_{\nu ij}) \le 1/4 \sum_{i=1}^{\nu} c^{*2}_{k\nu i} = 1/4.$$ Then, using Chebyshev inequality, we can conclude, for each $j (0 \le j \le \ell)$, $\sum_{i=1}^{\nu} c^*_{k\nu i} Z_{\nu ij} = O_p(1)$ under the sequence of alternatives H_ν. The same conclusion follows under H_0, since $E(\sum_{i=1}^{\nu} c^*_{k\nu i} Z_{\nu ij} | P_\nu) = 0$, and

$\mathrm{Var}(\sum_{i=1}^{\nu} c^*_{k\nu i} Z_{\nu ij} | P_\nu) = P_j(1 - P_j)$, $0 \le j \le \ell$. Next we have to show that for each $j = 0, 1,\ldots, \ell$, $|\hat{\Delta}_{\nu j} - \Delta_j| = o_p(1)$ either in P_ν- or in Q_ν- probability. We first prove the result under H_0. For $j \notin J$, $P_j = 0$, $\Delta_j = 0$. Then,

$$\Pr\cdot(\hat{\Delta}_{\nu j} \neq 0) \leq \Pr\cdot(\nu_j \neq 0) = 1 - \Pr\cdot(\nu_j = 0) = 1 - (1 - P_j)^\nu = 0,$$

that is, $\hat{\Delta}_{\nu j} - \Delta_j = o_p(1)$.

For $j \varepsilon J$, given any $\varepsilon > 0$,

$$\Pr\cdot(|\hat{\Delta}_{\nu j} - \Delta_j| > \varepsilon | P_\nu) \tag{3.3.33}$$

$$= \Pr\cdot(|\hat{\Delta}_{\nu j} - \Delta_j| > \varepsilon,\ \nu_j > 0 | P_\nu)$$

$$+ \Pr\cdot(|\hat{\Delta}_{\nu j} - \Delta_j| > \varepsilon,\ \nu_j = 0 | P_\nu)\ .$$

But,

$$\Pr\cdot(|\hat{\Delta}_{\nu j} - \Delta_j| > \varepsilon,\ \nu_j = 0) \leq \Pr\cdot(\nu_j = 0) = (1 - P_j)^\nu \to 0 \tag{3.3.34}$$

as $\nu \to \infty$ since $P_j > 0$.

Again for $j \varepsilon J$, $\nu_j > 0$,

$$\hat{\Delta}_{\nu j} = (\nu/\nu_j)\left[\int_{F_{\nu,j}}^{F_j} \phi(u)\,du + \int_{F_j}^{F_{j+1}} \phi(u)\,du + \int_{F_{j+1}}^{F_{\nu,j+1}} \phi(u)\,du\right]$$

Hence,

$$\hat{\Delta}_{\nu j} - \Delta_j = \left(\frac{\nu}{\nu_j} - \frac{1}{P_j}\right) \int_{F_j}^{F_{j+1}} \phi(u)\,du + \frac{\nu}{\nu_j} \int_{F_{\nu,j}}^{F_j} \phi(u)\,du + \frac{\nu}{\nu_j} \int_{F_{\nu,j+1}}^{F_{j+1}} \phi(u)\,du\ .$$

Thus,

$$|\hat{\Delta}_{\nu j} - \Delta_j| \leq \left|\frac{\nu}{\nu_j} - \frac{1}{P_j}\right| A(F) P_j^{1/2} + \frac{\nu}{\nu_j} A(F) |F_{\nu,j} - F_j|^{1/2}$$

$$+ \frac{\nu}{\nu_j} |F_{\nu,j+1} - F_{j+1}|^{1/2} A(F).$$

Hence,

$$(3.3.35)\quad \Pr\cdot(|\hat{\Delta}_{\nu j} - \Delta_j| > \varepsilon,\ \nu_j > 0|P_\nu)$$

$$\leq \Pr\cdot[|\frac{\nu}{\nu_j} - \frac{1}{P_j}|A(F)P_j^{1/2} > \varepsilon/3,\ \nu_j > 0|P_\nu]$$

$$+ \Pr[\frac{\nu}{\nu_j} A(F)|F_{\nu,j} - F_j|^{1/2} > \varepsilon/3,\ \nu_j > 0|P_\nu]$$

$$+ \Pr[\frac{\nu}{\nu_j} A(F)|F_{\nu,j+1} - F_{j+1}|^{1/2} A(F) > \varepsilon/3,\ \nu_j > 0|P_\nu]$$

$$\leq \Pr\cdot[|\frac{\nu}{\nu_j} - \frac{1}{P_j}| > \frac{\varepsilon}{3A(F)} P_j^{-\frac{1}{2}}|P_\nu]$$

$$+ \Pr\cdot[(\frac{\nu}{\nu_j})^2|F_{\nu,j} - F_j| > \frac{\varepsilon^2}{9A^2(F)}|P_\nu]$$

$$+ \Pr\cdot[(\frac{\nu}{\nu_j})^2|F_{\nu,j+1} - F_{j+1}| > \frac{\varepsilon^2}{9A^2(F)}|P_\nu]$$

But, since, under P_ν, $(\nu_j/\nu) - P_j = 0_p(\nu^{-\frac{1}{2}})$, $F_{\nu,j} - F_j = 0_p(\nu^{-\frac{1}{2}})$ for $j = 0, 1, \ldots, \ell$, $(\nu/\nu_j) - \frac{1}{P_j} = 0_p(\nu^{-\frac{1}{2}})$ for $j \varepsilon J$.

Hence each term on the right hand side of (3.3.35) can be made arbitrarily small for $j \varepsilon J$. Using this along with (3.3.34), we get from (3.3.33), $\hat{\Delta}_{\nu j} - \Delta_j = o_p(1)$ for $j \varepsilon J$. We have already shown, $\hat{\Delta}_{\nu j} - \Delta_j = o_p(1)$ for $j \varepsilon J$. Thus, the lemma is proved under H_0.

Under $\{H_\nu\}$, we prove the result by writing $\hat{\Delta}_{\nu j} - \Delta_j = \hat{\Delta}_{\nu j} - \Delta_{\nu j0} + \Delta_{\nu j0} - \Delta_j$, and then showing $\hat{\Delta}_{\nu j} - \Delta_{\nu j0} = o_p(1)$, $\Delta_{\nu j0} - \Delta_j = o(1)$. The proof is analogous to the one earlier and hence details are omitted. We may, however, point out for later reference that in this case $F_{\nu,j} - H^*_{\nu,j} = o_p(1)$, and $H^*_{\nu,j} - F_j = o(1)$, each for all $j = 0, 1, \ldots, \ell$ so that $F_{\nu,j} - F_j = o_p(1)$ for all $j = 0, 1, \ldots, \ell$. Hence, $(\nu_j/\nu) - P_j = o_p(1)$, for all $j = 0, 1, \ldots, \ell$. Using the above lemma, it now follows that under H_0 and also under the sequence of alternatives $\{H_\nu\}$, $\underset{\sim}{U}_\nu$ and $\underset{\sim}{T}_\nu$ have asymptotically the same distribution. This along with lemma 3.3.1 implies that the quadratic forms M_ν and S_ν as defined in (3.3.13) and (3.2.23) respectively have asymptotically the same distribution. We conclude that under the sequence of alternatives $\{H_\nu\}$, M_ν is distributed asymptotically as $\chi_p^2(\eta)$, η being defined in (3.2.25).

3.4. Extension to the case of countable set of class intervals.

Suppose, instead of a finite set of class intervals, we have a countable set of class intervals

$$I_j = [b_j, b_{j+1}), \; j = \ldots, -2, -1, 0, 1, 2, \ldots$$

We introduce the notations

$$(3.4.1) \qquad F'_{\nu,j} = F(\sigma^{-1}(b_j - \beta_0)), \; j = \ldots, -2, -1, 0, 1, 2, \ldots .$$

$$(3.4.2) \qquad P_{j0} = F'_{\nu,j+1} - F'_{\nu,j}, \; j = \ldots, -2, -1, 0, 1, 2, \ldots$$

$$(3.4.3) \qquad \Delta_{j0} = \int_{F'_{\nu,j}}^{F'_{\nu,j+1}} \phi(u)\,du \Big/ \int_{F'_{\nu,j}}^{F'_{\nu,j+1}} du \quad \text{if } P_{j0} \neq 0, \; = 0, \text{ otherwise},$$

$(j = \ldots, -2, -1, 0, 1, 2, \ldots)$. Let

$$(3.4.4) \qquad Z'_{\nu ij} = \begin{cases} 1 & \text{if } X_{\nu i} \in I_j \\ 0 & \text{otherwise} \end{cases}, \; j = \ldots, -2, -1, 0, 1, 2, \ldots$$

Consider the statistics

$$(3.4.5)\qquad T_{k\nu 0} = \sum_{i=1}^{\nu} c^{*}_{k\nu i} \sum_{j=-\infty}^{\infty} \Delta_{j0} Z'_{\nu ij}, \qquad k = 1, 2,\ldots, p.$$

Our objective is to show that in this case, if classes are coalesced on either end, thus reducing the problem to the case of finite number of class-intervals, and consider test procedures developed earlier, by proper choice of the terminal class intervals, the resulting test procedure will be asymptotically power equivalent to the one based on the statistics $T_{k\nu 0}(k = 1, 2,\ldots, p)$, (i.e. one which uses all the class intervals). More specifically, if we introduce the statistics

$$(3.4.6)\qquad T_{k\nu 1} = \sum_{i=1}^{\nu} c^{*}_{k\nu i} \sum_{j=-s_1}^{s_2} \Delta_j Z_{\nu ij} \quad (k = 1, 2,\ldots, p),$$

where $\Delta_j = \Delta_{j0} \quad (j = -s_1 + 1,\ldots, s_2 - 1)$

$$\Delta_{s_2} = \sum_{j=s_2}^{\infty} \Delta_{j0} P_{j0} \Big/ \sum_{j=s_2}^{\infty} P_{j0},$$

$$\Delta_{-s_1} = \sum_{j=-\infty}^{-s_1} \Delta_{j0} P_{j0} \Big/ \sum_{j=-\infty}^{-s_1} P_{j0},$$

$$Z_{\nu ij} = Z'_{\nu ij} \quad (j = -s_1 + 1,\ldots, s_2 - 1),$$

$$Z_{\nu is_2} = \sum_{j=s_2}^{\infty} Z'_{\nu ij}, \quad Z_{\nu i(-s_1)} = \sum_{j=-\infty}^{s_1} Z'_{\nu ij},$$

we want to show that $T_{k\nu 0} - T_{k\nu 1} \to 0$ in P_ν- (and because of contiguity in Q_ν-) probability for all $k = 1, 2,\ldots, p$. It is sufficient to show that

$$(3.4.7)\qquad E[(T_{k\nu 0} - T_{k\nu 1})^2 | P_\nu] \to 0, \text{ for all } k = 1, 2,\ldots, p.$$

Now,

$$(3.4.8)\quad T_{k\nu 0} - T_{k\nu 1}$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i}[(\sum_{j=s_2}^{\infty} \Delta_{j0} Z'_{\nu ij} - \Delta_{s_2} Z_{\nu is_2})$$

$$+ (\sum_{j=-\infty}^{-s_1} \Delta_{j0} Z'_{\nu ij} - \Delta_{(-s_1)} Z_{\nu i(-s_1)})]$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i}[\sum_{j=s_2}^{\infty} (\Delta_{j0} - \Delta_{s_2}) Z_{\nu ij} + \sum_{j=-\infty}^{-s_1} (\Delta_{j0} - \Delta_{(-s_1)}) Z_{\nu ij}] .$$

Then,

$$(3.4.9)\quad E(T_{k\nu 0} - T_{k\nu 1} | P_\nu) = \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j=s_2}^{\infty} (\Delta_{j0} - \Delta_{s_2}) P_{j0}$$

$$+ \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j=-\infty}^{-s_1} (\Delta_{j0} - \Delta_{(-s_1)}) P_{j0} = 0 ;$$

$$(3.4.10)\quad E[(T_{k\nu 0} - T_{k\nu 1})^2 | P_\nu]$$

$$= \sum_{i=1}^{\nu} c^{*2}_{k\nu i} E[\{\sum_{j=s_2}^{\infty} (\Delta_{j0} - \Delta_{s_2}) Z_{\nu ij} + \sum_{j=-\infty}^{-s_1} (\Delta_{j0} - \Delta_{(-s_1)}) Z_{\nu ij}\}^2 | P_\nu]$$

$$\le 2 \sum_{i=1}^{\nu} c^{*2}_{k\nu i} \sum_{j=s_2}^{\infty} (\Delta_{j0} - \Delta_{s_2})^2 P_{j0}$$

$$+ 2 \sum_{i=1}^{\nu} c^{*2}_{k\nu i} \sum_{j=-\infty}^{-s_1} (\Delta_{j0} - \Delta_{(-s_1)})^2 P_{j0}$$

$$= 2 \sum_{i=1}^{\nu} c^{*2}_{k\nu i} [\sum_{j=s_2}^{\infty} \Delta^2_{j0} P_{j0} - \Delta^2_{s_2} P_{s_2} + \sum_{j=-\infty}^{-s_1} \Delta^2_{j0} P_{j0} - \Delta^2_{(-s_1)} P_{(-s_1)}]$$

(where, $P_{s_2} = \sum_{j=s_2}^{\infty} P_{j0}$, $P_{(-s_1)} = \sum_{j=-\infty}^{-s_1} P_{j0}$)

$$\leq 2 \sum_{i=1}^{\nu} c_{k\nu i}^{*2} [\sum_{j=s_2}^{\infty} \Delta_{j0}^2 P_{j0} + \sum_{j=-\infty}^{-s_1} \Delta_{j0}^2 P_{j0}].$$

Since, $\sum_{j=-\infty}^{\infty} \Delta_{j0}^2 P_{j0} \leq A^2(F) < \infty$, uniformly in j, given any $\varepsilon > 0$,

we can find s_1 and s_2 such that

$$\sum_{j=s_2}^{\infty} \Delta_{j0}^2 P_{j0} < \varepsilon/2, \quad \sum_{j=-\infty}^{-s_1} \Delta_{j0}^2 P_{j0} < \varepsilon/2 ,$$

so that from (3.4.10),

$E[(T_{k\nu 0} - T_{k\nu 1})^2 | P_\nu] < 2\varepsilon$. This proves (3.4.7).

Since, $\sum_{j=0}^{\infty} \Delta_{j0}^2 P_{j0} \leq A^2(F) < \infty$, uniformly in j, given any $\varepsilon > 0$, we can find an ℓ such that $\sum_{j=\ell}^{\infty} \Delta_{j0}^2 P_{j0} < \varepsilon$. This proves (3.4.7).

3.5. Asymptotic Relative Efficiency (ARE).

Suppose the true distribution function is G(x) instead of F(x) and let $g(x) = G'(x)$. Let

(3.5.1) $$A^2(G) = \int_{-\infty}^{\infty} [g'(x)/g(x)]^2 g(x)dx < \infty.$$

Further, we define,

(3.5.2) $$G_j = G[\sigma^{-1}(a_j - \beta_0)], \quad j = 0, 1, \ldots, \ell + 1;$$

(3.5.3) $$P_j^* = G_{j+1} - G_j, \quad j = 0, 1, \ldots, \ell;$$

(3.5.4) $$\phi^*(u) = -g'(G^{-1}(u))/g(G^{-1}(u)), \quad 0 < u < 1 ;$$

$$(3.5.5)\qquad \Delta_j^* = \int_{G_j}^{G_{j+1}} \phi^*(u)\,du/P_j^*, \text{ if } P_j^* > 0, = 0, \text{ otherwise},$$

$$(j = 0, 1, \ldots, \ell).$$

$$(3.5.6)\qquad \Delta_j^{**} = \int_{G_j}^{G_{j+1}} \phi(u)\,du/P_j^*\ ,\quad j = 0, 1, \ldots, \ell;$$

$$(3.5.7)\qquad A^2(G,\{I_j\}) = \sum_{j=0}^{\ell} \Delta_j^{*2} P_j^*\ ;$$

$$(3.5.8)\qquad B^2(F, \{I_j\}) = \sum_{j=0}^{\ell} \Delta_j^{**2}\, P_j^*\ ;$$

$$(3.5.9)\qquad C(F, G,\{I_j\}) = \sum_{j=0}^{\ell} \Delta_j^* \Delta_j^{**} P_j^*\ ;$$

$$(3.5.10)\quad \rho(F, G,\{I_j\}) = C(F, G,\{I_j\})/[A(G, \{I_j\})B(F,\{I_j\})];$$

$$(3.5.11)\qquad k_j^*(h_{\nu i}) = P_j^{*-1} \int_0^{h_{\nu i}} [g'\{(a_{j+1} - \beta_0)/\sigma - \Theta_j y\}$$

$$- g'\{(a_j - \beta_0)/\sigma - \Theta_j y\}]dy\ ,$$

where $0 < \theta_j < 1$ for all $j \in J^* = \{j | 0 \le j \le \ell,\ P_j^* > 0\}$,

$i = 1, 2, \ldots, \nu$. Define the statistics

$$(3.5.12)\qquad T_{k\nu}^* = \sum_{i=1}^{\nu} c_{k\nu i}^* \sum_{j=0}^{\ell} Z_{\nu ij} \Delta_j^*,\ k = 1, 2, \ldots, p.$$

Let $\underset{\sim}{T}_\nu^* = (T_{1\nu}^*, \ldots, T_{p\nu}^*)'$. Then,

$$(3.5.13)\qquad E(\underset{\sim}{T}_\nu^* | P_\nu) = \underset{\sim}{0},\ \mathrm{Var}(\underset{\sim}{T}_\nu^* | P_\nu) = \underset{\sim}{\Lambda}_\nu A^2(G,\{I_j\})\ .$$

Let,

$$(3.5.14)\qquad S_\nu^* = A^{-2}(G,\{I_j\})\underset{\sim}{T}_\nu^{*\prime}\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{T}_\nu^* .$$

Further, let,

$$(3.5.15)\qquad T_{k\nu}^{**} = \sum_{i=1}^{\nu} c_{k\nu i}^* \sum_{j=0}^{\ell} Z_{ij}\Delta_j^{**}, \quad k = 1, 2, \ldots, p,$$

$$\underset{\sim}{T}_\nu^{**} = (T_{1\nu}^{**}, \ldots, T_{p\nu}^{**})' ;$$

$$(3.5.16)\qquad S_\nu^{**} = (\underset{\sim}{T}_\nu^{**}\underset{\sim}{\Lambda}_\nu^{-1}\underset{\sim}{T}_\nu^{**})B^{-2}(F, \{I_j\}) .$$

Then,

$$(3.5.17)\qquad E(\underset{\sim}{T}_\nu^{**}|P_\nu) = \underset{\sim}{0},\ \mathrm{Var}(\underset{\sim}{T}_\nu^{**}|P_\nu) = \underset{\sim}{\Lambda}_\nu B^2(F,\{I_j\}) .$$

If the permutation test procedure as described in Section 3.3 is now used, it is asymptotically power-equivalent to the test procedure (based on the critical function $\psi_3(\underset{\sim}{X}_\nu^*)$) defined as follows:

$$\psi_3(\underset{\sim}{X}_\nu^*) = \begin{cases} 1 & , \text{ if } S_\nu^{**} > S_{\nu,\varepsilon}^{**} \\ \delta_{3\nu\varepsilon}, & \text{ if } S_\nu^{**} = S_{\nu,\varepsilon}^{**} \\ 0 & , \text{ if } S_\nu^{**} < S_{\nu,\varepsilon}^{**} \end{cases}$$

where $S_{\nu,\varepsilon}^{**}$ and $\delta_{3\nu\varepsilon}$ are so chosen that $E[\psi_3(\underset{\sim}{X}_\nu^*)|P_\nu] = \varepsilon$.

We may note that for any $\underset{\sim}{e} \neq \underset{\sim}{0}$ with real and fixed elements,

$$(3.5.19)\qquad E(\underset{\sim}{e}'\underset{\sim}{T}_\nu^{**}|P_\nu) = \sum_{i=1}^{\nu} \left(\sum_{k=1}^{p} e_k c_{k\nu i}^*\right) \sum_{j=0}^{\ell} \Delta_j^{**}\, E(Z_{\nu ij}|H_\nu)$$

$$= \sum_{i=1}^{\nu} \left(\sum_{k=1}^{p} e_k c_{k\nu i}^*\right) \sum_{j=0}^{\ell} \Delta_j^{**} p_j^*(1+ h_{\nu i}\Delta_j^* + h_{\nu i}\Theta_j k_j^*(h_{\nu i}))$$

$$= (\sum_{k=1}^{p} e_k \sum_{i=1}^{\nu} c^*_{k\nu i} h_{\nu i}) \sum_{j=0}^{\ell} \Delta^*_j \Delta^{**}_j P^*_j + o(1)$$

$$= (\underset{\sim}{e}'\underset{\sim}{\Lambda}_\nu \underset{\sim}{\gamma}) C(F, G, \{I_j\}) + o(1) \sim (\underset{\sim}{e}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma}) C(F, G, \{I_j\}) .$$

$$(3.5.20) \quad \mathrm{Var}(\underset{\sim}{e}'\underset{\sim}{T}^{**}_\nu | P_\nu) = \sum_{i=1}^{\nu} (\sum_{k=1}^{p} e_k c^*_{k\nu i})^2 \sum_{j=0}^{\ell} \Delta^{**2}_j E(Z_{\nu ij} | H_\nu)$$

$$= \sum_{i=1}^{\nu} (\sum_{k=1}^{p} e_k c^*_{k\nu i})^2 \sum_{j=0}^{\ell} \Delta^{**2}_j P^*_j + o(1)$$

$$= (\underset{\sim}{e}'\underset{\sim}{\Lambda}_\nu \underset{\sim}{e}) B^2(F, \{I_j\}) + o(1) \sim (\underset{\sim}{e}'\underset{\sim}{\Lambda}\underset{\sim}{e}) B^2(F, \{I_j\}) .$$

It can be proved in the same way as lemma 3.2.3 that under the sequence of alternatives $\{H_\nu\}$, the model (3.1.1) and the assumptions (3.1.2)-(3.1.6) with G and g replacing F and f respectively, $\underset{\sim}{T}^{**}_\nu$ is asymptotically $N_p((\underset{\sim}{\Lambda}\underset{\sim}{\gamma}) C(F, G, \{I_j\}), \underset{\sim}{\Lambda} B^2(F, \{I_j\}))$. Hence, S^{**}_ν is, under H_0, distributed asymptotically as $\chi^2_p(\eta'_0)$, where,

$$(3.5.21) \qquad \eta'_0 = (\underset{\sim}{\gamma}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma}) C^2(F, G, \{I_j\}) / B^2(F, \{I_j\}) .$$

Also, $\underset{\sim}{T}^*_\nu$ is under H_ν asymptotically $N_p((\underset{\sim}{\Lambda}\underset{\sim}{\gamma}) A^2(G, \{I_j\}), \underset{\sim}{\Lambda} A^2(G, \{I_j\}))$. Hence, S^*_ν (which provides now the asymptotically optimal test in the sense described in section 3.2) is asymptotically $\chi^2_p(\eta_0)$ with

$$(3.5.22) \qquad \eta_0 = (\underset{\sim}{\gamma}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma}) A^2(G, \{I_j\}) .$$

Using the definition of ARE as given by Hannan [20] or Andrews [2] as the ratio of non-centrality parameters the ARE of the permutation test procedure as given in (3.3.14) relative to the asymptotically optimal test as given in (3.2.24) with S^*_ν replacing S_ν is given by

$$(3.5.23) \qquad e_{F,G,\{I_j\}} = \eta_0'/\eta_0 = \rho^2(F,\ G,\ \{I_j\})\ .$$

The above expression turns out to be the same as in Sen [33]. We may note that the expression for ARE depends on the true distribution, the assumed distribution as also on the grouping structure. However, this is unavoidable and the optimum way of grouping in a particular context will depend heavily on the parent distribution.

In case of ungrouped data where the $X_{\nu i}$'s are observable, while the model (3.1.1) and the assumptions (3.1.2)-(3.1.6) remain the same, for testing H_0 against $\{H_\nu\}$, one proceeds on lines analogous to that of Hájek [17]. Hence, if $S_{\nu 0}$ denotes the test statistic for ungrouped data, in the situation where both the true and assumed distribution functions are F, $S_{\nu 0}$ is asymptotically $\chi_p^2(\eta_{00})$ with $\eta_{00} = (\underset{\sim}{\gamma}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma})A^2(F)$. In that case the relative loss of efficiency due to grouping will be

$$(3.5.24) \quad (\eta_{00} - \eta)/\eta_{00} = \sum_{j=0}^{\ell} \int_{F_j}^{F_{j+1}} [\phi(u) - \Delta_j]^2 du \Big/ \int_0^1 \phi^2(u)\,du$$

$$= 1 - \frac{A^2(F,\{I_j\})}{A^2(F)}\ .$$

We can determine the class-intervals in such a way that given any $\varepsilon \to 0$, $\max_j P_j = \max_j (F_{j+1} - F_j) < \varepsilon$. This can be achieved independently of ν. Thus, the loss of efficiency can also be made arbitrarily small by proper choice of class-intervals.

Again, in situations where the true distribution function G and the assumed distribution function F differ, the relative loss of efficiency is

$$(3.5.25)\quad [(\gamma'\Lambda\gamma)\rho^2 A^2(G) - \eta_0']/(\gamma'\Lambda\gamma)\rho^2 A^2(G)$$

$$= 1 - [\rho(F,\ G,\{I_j\})/\rho]^2[A(G,\{I_j\})/A(G)]^2,$$

which also can be made arbitrarily small as in the preceding case. Also, the expression (3.5.25) may be greater than, equal to or less than (3.5.24) with G replacing F, depending on $\rho(F,\ G,\{I_j\})$ and ρ.

We give expressions for $\phi(u)$ and Δ_j in cases of some well-known distributions:

TABLE 3.5.1. Some expressions for $\phi(u)$ and Δ_j.

$(a_j' = \sigma^{-1}(a_j - \beta_0))$

Density	f(x)	-f'(x)/f(x)	$\phi(u)$	Δ_j (when $P_j \neq 0$), $j=0,1,\ldots,\ell$.
Normal	$(2\pi)^{-\frac{1}{2}}\exp.(-\frac{x^2}{2})$	x	$\Phi^{-1}(u)$	$[f(a_j')-f(a_{j+1}')]/[\Phi(a_{j+1}')-\Phi(a_j')]$
Double Exponential	$\frac{1}{2}\exp.(-\lvert x\rvert)$	sgn x	sgn(2u-1)	-1 if $a_{j+1}' \leq 0$, $+1$ if $a_j' \geq 0$, $(e^{a_j'}-e^{-a_{j+1}'})/(2-e^{a_j'}-e^{-a_{j+1}'})$ if $a_j'<0<a_{j+1}'$.
Logistic	$e^{-x}(1+e^{-x})^{-2}$	$(1-e^{-x})(1+e^{-x})^{-1}$	2u-1	$(1+e^{-a_j'})^{-1}+(1+e^{-a_{j+1}'})^{-1}-1$.

From the above table, it is clear that the Normal Score Test, Sign Test or Wilcoxon Test are asymptotically optimal according as the parent distribution is Normal, Double Exponential or Logistic respectively.

3.6 Applications.

The model considered includes as particular case the p-sample ($p \geq 2$) problem. Consider the situation when the $X_{\nu i}$ ($i = 1, 2,\ldots, \nu$) are from p populations $\Pi_1,\ldots, \Pi_p$ and out of the ν observations $n_{k\nu}$ are from population Π_k ($k = 1, 2,\ldots, p$). Note that $\nu = \sum_{k=1}^{p} n_{k\nu}$. Assume

$$(3.6.1) \quad \lim_{\nu\to\infty} n_{k\nu}/\nu = \pi_k \quad (0 < \pi_k < 1, \text{ for all } k = 1, 2,\ldots, p; \quad \sum_{k=1}^{p} \pi_k = 1).$$

The regression constants $c_{k\nu i}$ are given in this case by

$$(3.6.2) \quad c_{k\nu i} = \begin{cases} 1 - (n_{k\nu}/\nu) & \text{if the } i^{th} \text{ observation is from } k^{th} \text{ sample,} \\ -(n_{k\nu}/\nu) & \text{otherwise,} \end{cases}$$

($i = 1,\ldots, \nu$; $k = 1, 2,\ldots, p$). Then,

$$\sum_{i=1}^{\nu} c_{k\nu i} = 0, \quad \sum_{i=1}^{\nu} c_{k\nu i}^2 = n_{k\nu}(\nu - n_{k\nu})/\nu ,$$

$$\sum_{i=1}^{\nu} c_{k\nu i} c_{k'\nu i} = -(n_{k\nu} n_{k'\nu}/\nu), \text{ for all } k, k' = 1,\ldots, p.$$

Hence,

$$(3.6.3) \qquad \underset{\sim}{\Lambda}_{\nu} = \begin{bmatrix} 1 & -\left[\frac{n_{1\nu}n_{2\nu}}{(\nu-n_{1\nu})(\nu-n_{2\nu})}\right]^{1/2} & \cdots & -\left[\frac{n_{1\nu}n_{p\nu}}{(\nu-n_{1\nu})(\nu-n_{p\nu})}\right]^{1/2} \\ -\left[\frac{n_{1\nu}n_{2\nu}}{(\nu-n_{1\nu})(\nu-n_{2\nu})}\right]^{1/2} & 1 & \cdots & -\left[\frac{n_{2\nu}n_{p\nu}}{(\nu-n_{2\nu})(\nu-n_{p\nu})}\right]^{1/2} \\ \cdots & \cdots & \cdots & \cdots \\ -\left[\frac{n_{1\nu}n_{p\nu}}{(\nu-n_{1\nu})(\nu-n_{p\nu})}\right]^{1/2} & -\left[\frac{n_{2\nu}n_{p\nu}}{(\nu-n_{2\nu})(\nu-n_{p\nu})}\right]^{1/2} & \cdots & 1 \end{bmatrix}$$

Under (3.6.1), $\underset{\sim}{\Lambda}_{\nu} \to \underset{\sim}{\Lambda}$, where,

$$\underset{\sim}{\Lambda} = \begin{bmatrix} 1 & -\left[\frac{\pi_1\pi_2}{(1-\pi_1)(1-\pi_2)}\right]^{1/2} & \cdots & -\left[\frac{\pi_1\pi_p}{(1-\pi_1)(1-\pi_p)}\right]^{1/2} \\ -\left[\frac{\pi_1\pi_2}{(1-\pi_1)(1-\pi_2)}\right]^{1/2} & 1 & \cdots & -\left[\frac{\pi_2\pi_p}{(1-\pi_2)(1-\pi_p)}\right]^{1/2} \\ \cdots & \cdots & \cdots & \cdots \\ -\left[\frac{\pi_1\pi_p}{(1-\pi_1)(1-\pi_p)}\right]^{1/2} & -\left[\frac{\pi_2\pi_p}{(1-\pi_2)(1-\pi_p)}\right]^{1/2} & \cdots & 1 \end{bmatrix}$$

$\underset{\sim}{\Lambda}_{\nu}$ is a $p \times p$ matrix of rank $p - 1$. Writing

$$\underset{\sim}{\Lambda}_{\nu} = \begin{bmatrix} \underset{(p-1)\times(p-1)}{\underset{\sim}{\Lambda}_{11\nu}} & \underset{(p-1)\times 1}{\underset{\sim}{\Lambda}_{12\nu}} \\ \underset{1\times(p-1)}{\underset{\sim}{\Lambda}_{21\nu}} & \Lambda_{22\nu} \end{bmatrix} ,$$

We notice that $\underset{\sim}{\Lambda}_{11\nu}$ is of full rank. Let $\underset{(p-1)\times p}{\underset{\sim}{D}} = (I_{p-1}\underset{\sim}{0})$, where I_{p-1} is the identity matrix of order (p-1). $\underset{\sim}{D}\underset{\sim}{U}_\nu$ has asymptotically the same distribution as $\underset{\sim}{D}\underset{\sim}{T}_\nu$ under H_0 and also under the sequence of alternatives $\{H_\nu\}$. Thus, under H_ν, $\underset{\sim}{D}\underset{\sim}{U}_\nu$ is asymptotically $N_p((\underset{\sim}{D}\underset{\sim}{\Lambda}\underset{\sim}{\gamma})A^2(F,\{I_j\}),\ (\underset{\sim}{D}\underset{\sim}{\Lambda}\underset{\sim}{D}')A^2(F,\{I_j\}))$. But $\underset{\sim}{D}\underset{\sim}{\Lambda}\underset{\sim}{D}' = \underset{\sim}{\Lambda}_{11}$ where

$$\underset{\sim}{\Lambda} = \begin{bmatrix} \underset{(p-1)\times(p-1)}{\underset{\sim}{\Lambda}_{11}} & \underset{(p-1)\times 1}{\underset{\sim}{\Lambda}_{12}} \\ \underset{1\times(p-1)}{\underset{\sim}{\Lambda}_{21}} & \Lambda_{22} \end{bmatrix},$$

and thus by reparametrization get a non-singular distursion matrix for transformed variables. Since, $\underset{\sim}{U}'_\nu\underset{\sim}{\Lambda}^*_\nu\underset{\sim}{U}_\nu = (\underset{\sim}{D}_\nu\underset{\sim}{U}_\nu)'\underset{\sim}{\Lambda}^{-1}_{11\nu}(\underset{\sim}{D}_\nu\underset{\sim}{U}_\nu)$, where $\underset{\sim}{\Lambda}^*_\nu$ is a generalized inverse of $\underset{\sim}{\Lambda}_\nu$, M_ν remains invariant under reparametrization. $\underset{\sim}{\Lambda}^{-1}_{11\nu}$ can be obtained by using a formula given in Rao [32], p. 20. In the case of ungrouped data, the p-sample problem was considered by Puri [28(a)] under essentially more stringent conditions.

We might remark that in the p-sample problem, Basu [3] has considered the censored case where only $\nu^*(<\nu)$ of the ordered variables in the combined sample are observable, while $\nu^*/\nu \to p$, $(0 < p < 1)$ as $\nu \to \infty$. A similar problem follows as a special case of ours when $I_0: x < x_0$, while probabilities P_j $(j = 1,\ldots, \ell)$ of belonging to $I_1, I_2,\ldots, I_\ell$ are sufficiently small. However, while in Basu's case ν^* is given, while the corresponding truncation point is random, in our case, the truncation points are fixed, while ν_j $(j = 0, 1,\ldots,\ell)$ are random. In spite of these basic differences, the A.R.E.'s of the two tests will be the same, and further, we can study the optimality properties of Basu's test according to the criteria described earlier. As such, for the logistic distribution, Basu's test is optimal. Similar comparison can be made in particular with the test proposed by Gastwirth [14] for the two sample problem in the censored case by taking $p = 2$. (See also [33]).

The above analysis is also applicable when the observable r.v.'s are grouped in several ordered categories (the case of Categorical Data) if the underlying parent distribution is continuous.

3.7. Concluding Remarks.

The present chapter includes, as a particular case (when $p = 1$), Sen's model where the null hypothesis has been tested against one-sided alternatives. This also includes a similar two sample problem as considered by Gastwirth [15]. Unlike the case of one-sided alternatives, we do not get here an asymptotically most powerful test, but a test asymptotically optimal in Wald's sense. Also when the class intervals are sufficiently small, the efficiency of the M_ν-test reduces to that of a similar test (for ungrouped data) considered by Jogdeo [24]. We may also remark that unlike Hájek [17] and Sen [33] we need only impose the Noether condition on the $c_{k\nu i}$'s, but no other assumption like $\sum_{i=1}^{\nu} c_{k\nu i}^2 = O(1)$, for all $k = 1,\dots, p$. This helps avoiding some artificiality in taking the regression constants $c_{k\nu i} = o(1)$ $(k = 1,\dots, p;\ i = 1,\dots, \nu;\ \nu \geq 1)$; here, for example, we may consider the case

$c_{k\nu i} = i - \frac{\nu+1}{2}$ $(k = 1,\ldots, p;\ i = 1,\ldots, \nu;\ \nu \geq 1)$. On the other hand, we have considered a sequence $\underset{\sim}{\beta}_\nu$ of alternatives which are "local," while Hájek and Sen have regarded the regression coefficient β as fixed and still, in effect, have considered local alternatives by imposing the condition $\sum_{i=1}^{\nu} c_{\nu i}^2 = O(1)$ along with the Noether condition.

CHAPTER IV

ON TESTING THE IDENTITY OF MULTIPLE REGRESSION SURFACES FOR GROUPED DATA

4.1. Notations and Assumptions.

Consider a double sequence of i.i.d.r.v. $\underset{\sim}{X}_\nu = (X_{\nu 1},\ldots, X_{\nu\nu})'$, $\nu \geq 1$, satisfying the model (3.1.1) and the assumptions (3.1.2)-(3.1.6). In this case, however, we want to test

$$(4.1.1) \qquad H_0 : \underset{\sim}{\beta}^* = (\beta_0, \beta_1, \ldots, \beta_p)' = \underset{\sim}{0} .$$

We assume in addition,

$$(4.1.2) \qquad F(x) + F(-x) = 1, \text{ for all real } x.$$

The usefulness of the symmetry assumption about F will be revealed in section 4.3. A particular situation where this model is realistic is as follows:

Consider a double sequence of i.i.d. stochastic vectors $\underset{\sim}{X}_{\nu i} = (X_{\nu i}^{(1)}, X_{\nu i}^{(2)})'$, $i = 1,\ldots, \nu$, $\nu \geq 1$, satisfying

$$(4.1.3) \qquad X_{\nu i}^{(\ell)} = \sum_{k=0}^{p} \beta_k^{(\ell)} \, c_{k\nu i} + \sigma\varepsilon_{\nu i}^{(\ell)} ,$$

$\ell = 1, 2$; $i = 1, 2,\ldots, \nu$, $\nu \geq 1$, where, $c_{\nu i} = (c_{0\nu i},\ldots, c_{p\nu i})'$, $i = 1, 2,\ldots, \nu$ with $c_{0\nu i} = 1$ $(i = 1, 2,\ldots, \nu)$, $\nu \geq 1$, are vectors of known constants, $\sigma(>0)$ is a nuisance parameter and $\beta_k^{(\ell)}$, $k = 0, 1, \ldots, p$, $\ell = 1, 2$, are unknown regression parameters. We assume as in [30], $(\varepsilon_{\nu i}^{(1)}, \varepsilon_{\nu i}^{(2)})$ are either interchangeable or their joint distribution is diagonally symmetric about (0, 0). We want to test

$$(4.1.4) \qquad H_0^{\prime}: \beta_k^{(1)} = \beta_k^{(2)}, \text{ for all } k = 0, 1, \ldots, p,$$

against the alternatives that at least one of the (p + 1) equalities in (4.1.2) is not true. Let $\beta_k = \beta_k^{(1)} - \beta_k^{(2)}$, $k = 0, 1, \ldots, p$. Then $H_0^{\prime} \equiv H_0$ given in (4.1.1). Writing $X_{\nu i} = X_{\nu i}^{(1)} - X_{\nu i}^{(2)}$, $\varepsilon_{\nu i} = \varepsilon_{\nu i}^{(1)} - \varepsilon_{\nu i}^{(2)}$, $i = 1, 2, \ldots, \nu$, $\nu \geq 1$, the model can be rewritten as

$$(4.1.5) \qquad X_{\nu i} = \beta_0 + \sum_{k=1}^{p} \beta_k c_{k\nu i} + \sigma\varepsilon_{\nu i}, \; i = 1, \ldots, \nu, \; \nu \geq 1.$$

If $\varepsilon_{\nu i}$ (which are i.i.d.) have a common d.f.F, (3.1.1) and (4.1.2) follow. Note that, $C_{0\nu}^2 = \sum_{i=1}^{\nu} c_{0\nu i}^2 = \nu$. So, $c_{0\nu i}^* = c_{0\nu i}/C_{0\nu} = \nu^{-\frac{1}{2}}$, for all $i = 1, 2, \ldots, \nu$.
$\sum_{i=1}^{\nu} c_{0\nu i}^* c_{k\nu i}^* = \sum_{i=1}^{\nu} c_{k\nu i}^* = 0$, $k = 1, 2, \ldots, p$. We introduce the notations

$$(4.1.6) \qquad \underset{\sim}{\Lambda}_{\nu 0} = \left(\left(\sum_{i=1}^{\nu} c_{k\nu i}^* c_{k'\nu i}^*\right)\right)_{k,k'=0,1,\ldots,p} = \begin{bmatrix} 1 & \underset{\sim}{0} \\ \underset{\sim}{0}' & \underset{\sim}{\Lambda}_\nu \end{bmatrix} \to \underset{\sim}{\Lambda}_0,$$

where,

$$(4.1.7)\qquad \underset{\sim}{A}_0 = \begin{bmatrix} 1 & \underset{\sim}{0} \\ & \\ \underset{\sim}{0}' & \underset{\sim}{A} \end{bmatrix}.$$

(3.1.5) implies $\underset{\sim}{A}_{\nu 0}$ and $\underset{\sim}{A}_0$ are positive definite. Let

$$(4.1.8)\qquad \phi(u) = -f'(F^{-1}(u))/f(F^{-1}(u)),\quad 0 < u < 1 ;$$

$$(4.1.9)\qquad \psi(u) = -f'(F^{-1}((1+u)/2))/f(F^{-1}((1+u)/2)),\ 0 < u < 1 ;$$

then,

$$(4.1.10)\qquad \int_0^1 \phi^2(u)\,du = \int_0^1 \psi^2(u)\,du = A^2(F) < \infty.$$

In our case, the $X_{\nu i}$ $(i = 1, \ldots, \nu)$ are not observable. We have a finite set of class intervals

$$(4.1.11)\qquad I_j : a_{j-\frac{1}{2}} \le x < a_{j+\frac{1}{2}}\ (j = -\ell, -\ell+1, \ldots, -1, 0, 1, \ldots, \ell-1, \ell)$$

[where $-\infty = a_{-\ell-\frac{1}{2}} < \ldots < a_{-\frac{1}{2}} < a_{\frac{1}{2}} < \ldots < a_{\ell+\frac{1}{2}} = \infty$ denote a finite set of ordered points on the real line with $a_{-j} = a_j$; $a_0 = 0$]. The observable stochastic vector is $\underset{\sim}{X}^*_\nu = (X^*_{\nu 1}, \ldots, X^*_{\nu\nu})'$, where

$$(4.1.12)\qquad X^*_{\nu i} = \sum_{-\ell}^{\ell} I_j Z_{\nu ij}\quad (i = 1, \ldots, \nu)$$

and

$$(4.1.13)\qquad Z_{\nu ij} = \begin{cases} 1 & \text{if } X_{\nu i} \in I_j \\ 0 & \text{otherwise} \end{cases},$$

for all $i = 1, 2, \ldots, \nu$, $\nu \ge 1$ $(j = -\ell, \ldots, -1, 0, 1, \ldots, \ell)$.

For the testing problem considered here, a class of permutationally (conditionally) distribution-free nonparametric tests has been proposed which is comparable to one in the previous chapter where the problem was one of testing the hypothesis $H_0:\beta_k = 0$ $(k = 1, 2,\ldots, p)$, while β_0 and σ were nuisance parameters. As it has been remarked already, in chapter III, statistics analogous to rank statistics were considered, while here statistics analogous to signed-rank statistics have been used. We may also comment at this stage that in what follows, the assumption (3.1.2) namely $\sum_{i=1}^{\nu} c^*_{k\nu i} = 0$ $(k = 1, 2,\ldots, p)$ is unnecessary, and the same results without any modification will hold once we assume that $\underset{\sim}{\Lambda}_{\nu 0}$ and $\underset{\sim}{\Lambda}_0$ are positive definite. This assumption, however, can be made without any loss of generality as already observed in chapter III.

4.2. Asymptotically Optimal Parametric Test.

We define,

(4.2.1) $F_j = F(a_{j-\frac{1}{2}}/\sigma)$, $j = -\ell,\ldots, -1, 0, 1,\ldots, \ell + 1$;

(4.2.2) $P_j = F_{j+1} - F_j$, $j = -\ell,\ldots, -1, 0, 1,\ldots, \ell$;

(4.2.3) $F^*_j = F_j - F_{-j+1} = 2F_j - 1$ $(j = 1, 2,\ldots, \ell + 1)$,

so that, $P^*_j = F^*_{j+1} - F^*_j = 2P_j$ $(j = 0, 1,\ldots, \ell)$.

(4.2.4) $$\Delta_j = \int_{F_j}^{F_{j+1}} \phi(u)\,du/P_j, \text{ if } P_j \neq 0 \qquad j = -\ell,\ldots, -1, 0, 1,\ldots, \ell.$$
$$= 0, \text{ otherwise}$$

$$(4.2.5)\quad \Delta_j^* = \int_{F_j^*}^{F_{j+1}^*} \psi(u)\,du/P_j^* \text{ , if } P_j^* \neq 0 \qquad j = 1, 2, \ldots, \ell.$$

$$= 0, \text{ otherwise}$$

When $P_j \neq 0$, we can write,

$$(4.2.6)\quad \Delta_j = [f(a_{j-\frac{1}{2}}/\sigma) - f(a_{j+\frac{1}{2}}/\sigma)]/P_j,\ j = -\ell, \ldots, -1, 0, 1, \ldots, \ell.$$

It follows immediately that,

$$(4.2.7)\quad P_{-j} = P_j,\ \Delta_{-j} = \Delta_j \quad (j = 0, 1, \ldots, \ell).$$

So, $\Delta_0 = 0$. Also, if $P_j \neq 0$,

$$(4.2.8)\quad \Delta_j^* = \left[2\int_{(1+F_j^*)/2}^{(1+F_{j+1}^*)/2} \phi(u)\,du \Bigg/ 2\int_{(1+F_j^*)/2}^{(1+F_{j+1}^*)/2} du \right].$$

$$= \int_{F_j}^{F_{j+1}} \phi(u)\,du \Bigg/ \int_{F_j}^{F_{j+1}} du = \Delta_j \quad (j = 1, 2, \ldots, \ell).$$

Let,

$$(4.2.9)\quad A_0^2(F,\{I_j\}) = \sum_{j=-\ell}^{\ell} \Delta^2{}_j P_j = 2\sum_{j=1}^{\ell} \Delta_j^2 P_j = 2\sum_{j\varepsilon J_+} \Delta_j^2 P_j = 2\sum_{j\varepsilon J_+} \Delta_j^{*2} P_j \text{ ,}$$

where $J_+ = \{j \,\varepsilon\, (1, 2, \ldots, \ell) | P_j > 0\}$. Hence,

$$(4.2.10)\quad 0 < A_0^2(F,\{I_j\}) = 2\sum_{j\varepsilon J_+} \left[\int_{F_j}^{F_{j+1}} \phi(u)\,du \qquad \int_{F_j}^{F_{j+1}} du \right]^2$$

$$\leq \sum_{j=-\ell}^{\ell} \int_{F_j}^{F_{j+1}} \phi^2(u)\,du = A^2(F) < \infty.$$

Assume,

(4.2.11) $$\max_{j=0,1,\ldots,\ell} P_j < 1.$$

Consider the sequence of alternatives,

(4.2.12) $$H_\nu : \beta_k = \beta_{k\nu} = \tau_k / C_{k\nu} \quad (k = 0, 1, \ldots, p) .$$

Let,

(4.2.13) $$\gamma_k = \sigma^{-1} \tau_k \quad (k = 0, 1, \ldots, p) \text{ and let } \gamma^* = (\gamma_0, \ldots, \gamma_p)';$$

(4.2.14) $$h_{\nu i} = \sigma^{-1} \sum_{k=0}^{p} c^*_{k\nu i} \tau_k \quad (i = 1, 2, \ldots, \nu).$$

Then the likelihood function of $\underset{\sim}{X}^*_\nu$ under H_ν is given by

(4.2.15) $$L(\underset{\sim}{X}^*_\nu | \gamma^*) = \prod_{i=1}^{\nu} \{ \sum_{j=-\ell}^{\ell} Z_{\nu ij} P(X_{\nu i} \in I_j) \}$$

$$= \prod_{i=1}^{\nu} \{ \sum_{j=-\ell}^{\ell} Z_{\nu ij} (F[(a_{j+1}/\sigma) - h_{\nu i}] - F[(a_j/\sigma) - h_{\nu i}]) \}$$

$$= \prod_{i=1}^{\nu} \{ \sum_{j=-\ell}^{\ell} Z_{\nu ij} P_j (1 + h_{\nu i}\Delta_j + h_{\nu i}\theta_j g_j(h_{\nu i})) \},$$

$$0 < \theta_j < 1,$$

where,

(4.2.16) $$g_j(h_{\nu i}) = P_j^{-1} \int_0^{h_{\nu i}} \{ f'[(a_{j+1}/\sigma) - \theta_j y] - f'[(a_j/\sigma) - \theta_j y] \} dy,$$

for $j \in J_+ \cup J_-$, where $J_- = \{ j \in (-1, -2, \ldots, -\ell) | P_j > o \}$.

Hence $g_j(h_{\nu i}) = o(1)$, uniformly in $j \in J_+ \cup J_-$, $i = 1, 2, \ldots, \nu$. Let

$$(4.2.17)\quad T_{k\nu 0} = [(\partial/\partial\gamma_k)\log L(\underset{\sim}{X}^*_\nu|\gamma)]_{\gamma=\underset{\sim}{0}}$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j=-\ell}^{\ell} Z_{\nu ij}\Delta_j \quad \text{(as in chapter III)}$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j=1}^{\ell} (Z_{\nu ij} - Z_{\nu i(-j)})\Delta_j,\ k = 0, 1, \ldots, p.$$

Let

$$(4.2.18)\quad U_{\nu ij} = \begin{cases} 1 & \text{if } |X_{\nu i}| \,\varepsilon\, I_j \text{ i.e. } X_{\nu i} \,\varepsilon\, I_j \cup I_{-j} \\ 0 & \text{otherwise} \end{cases}$$

$(i = 1,\ldots, \nu;\ j = 1, 2,\ldots, \ell)$.

Then,

$$(4.2.19)\quad Z_{\nu ij} - Z_{\nu i(-j)} = U_{\nu ij}\ \mathrm{sgn}(X_{\nu i}),$$

$j = 1, 2,\ldots, \ell,\ i = 1, 2,\ldots, \nu,$

where,

$$(4.2.20)\quad \mathrm{sgn}\, X_{\nu i} = \begin{cases} 1 & \text{if } X_{\nu i} > 0 \\ 0 & \text{if } X_{\nu i} = 0 \\ -1 & \text{if } X_{\nu i} < 0 \end{cases},\ i = 1, 2,\ldots, \nu.$$

Thus,

$$(4.2.21)\quad T_{k\nu 0} = \sum_{i=1}^{\nu} c^*_{k\nu i}\Big(\sum_{j=1}^{\ell} U_{\nu ij}\Delta_j\Big)\mathrm{sgn}\, X_{\nu i},\ k = 0, 1,\ldots, p.$$

Let

$$(4.2.22)\quad \underset{\sim}{T}_{\nu 0} = (T_{0\nu 0},\ldots, T_{p\nu 0})'.$$

Next we state three lemmas the proofs of which are exactly the same as lemmas 3.2.1 - 3.2.3.

LEMMA 4.2.1. Under H_0, the model (3.1.1) and the assumptions (3.1.2) - (3.1.6), $\underset{\sim}{T}_{\nu 0}$ is asymptotically $N_{p+1}(\underset{\sim}{0}, \underset{\sim}{\Lambda}_0 A_0^2(F,\{I_j\}))$.

LEMMA 4.2.2. Under H_0, the model (3.1.1) and the assumptions (3.1.2) - (3.1.6), $L_{0\nu}(\underset{\sim}{X}_\nu)$ is asymptotically $N_1(-\frac{1}{2}(\chi^{*\prime}\underset{\sim}{\Lambda}_0\chi^*)A^2(F), (\chi^{*\prime}\underset{\sim}{\Lambda}_0\chi^*)A^2(F))$.

LEMMA 4.2.3. Under the sequence of alternatives $\{H_\nu\}$, the model (3.1.1) and the assumptions (3.1.2) - (3.1.6), $\underset{\sim}{T}_{\nu 0}$ is asymptotically $N_{p+1}((\underset{\sim}{\Lambda}_0\chi^*)A_0^2(F,\{I_j\}), \underset{\sim}{\Lambda}_0 A_0^2(F,\{I_j\}))$.

If $\{P_\nu\}$ and $\{Q_\nu\}$ denote two sequences of probability measures as in chapter III, lemma 4.2.2 ensures that "Q_ν are contiguous to P_ν." Define now,

$$S_{\nu 00} = A_0^{-2}(F,\{I_j\})\underset{\sim}{T}'_{\nu 0}\underset{\sim}{\Lambda}_{\nu 0}^{-1}\underset{\sim}{T}_{\nu 0} \; . \tag{4.2.22}$$

The optimal parametric test procedure in the sense described in chapter III is the same as in (3.2.24) with $S_{\nu 00}$ replacing S_ν.

4.3. A Class of Nonparametric Tests.

We introduce the following notations:

$$\sum_{i=1}^{\nu} Z_{\nu ij} = \nu_j, \; (j = -\ell,\ldots, -1, 0, 1,\ldots, \ell). \tag{4.3.1}$$

So,

$$\sum_{i=1}^{\nu} U_{\nu ij} = \nu_j + \nu_{-j} \; (j = 1, 2,\ldots, \ell). \tag{4.3.2}$$

Let

$$F_{\nu,-\ell} = 0, \; F_{\nu,j+1} = \sum_{m=-\ell}^{j} \nu_m/\nu, \; j = -\ell,\ldots, -1, 0, 1,\ldots, \ell; \tag{4.3.3}$$

$$(4.3.4)\quad F^*_{\nu,j+1} = F_{\nu,j+1} - F_{\nu,-j} = \nu^{-1} \sum_{m=-j}^{j} \nu_m, \; j = 0, 1, 2, \ldots, \ell;$$

$$(4.3.5)\quad \hat{\Delta}^*_{\nu j} = \int_{F^*_{\nu,j}}^{F^*_{\nu,j+1}} \psi(u)\,du \Big/ \int_{F^*_{\nu,j}}^{F^*_{\nu,j+1}} du$$

$$= \int_{(1+F^*_{\nu,j})/2}^{(1+F^*_{\nu,j+1})/2} \phi(u)\,du \Big/ \int_{(1+F^*_{\nu,j})/2}^{(1+F^*_{\nu,j+1})/2} du\ ,$$

if $\nu_j + \nu_{-j} \neq 0$;

$$\hat{\Delta}^*_{\nu j} = 0, \text{ otherwise } (j = 1, 2, \ldots, \ell).$$

Let

$$(4.3.6)\quad U^*_{k\nu} = \sum_{i=1}^{\nu} c^*_{k\nu i}\Big(\sum_{j=1}^{\ell} U_{\nu ij}\hat{\Delta}^*_{\nu j}\Big)\operatorname{sgn} X_{\nu i}, \; k = 0, 1, \ldots, p,$$

and $\underset{\sim}{U}^* = (U^*_{0\nu}, \ldots, U^*_{p\nu})'$.

We may note that in the case $X_{\nu i} \in I_0$, we have no information regarding the value of sgn $X_{\nu i}$. This, however, does not affect the test procedure (based on the statistics $U^*_{k\nu}$, $k = 0, 1, \ldots, p$), since, if $X_{\nu i} \in I_0$, $U_{\nu ij} = 0$ for all $j = 1, 2, \ldots, \ell$ and, as such, contribution of such terms to the calculation of $U^*_{k\nu}$ $(k = 0, 1, \ldots, p)$ is zero.

Since, we are dealing with grouped data, even under H_0, the joint distribution of the $U^*_{k\nu}$'s will depend on the Δ^*_j's. But, using a permutation argument, we can get here a permutationally (conditionally) distribution-free test. Under H_0 in (4.1.1), $|X_{\nu i}|$, $i = 1, 2, \ldots, \nu$ are i.i.d.r.v. and so are $X^*_{\nu i0} = \sum_{j=1}^{\ell} I'_j U_{\nu ij}$,

$(I'_j = I_j \cup I_{-j},\ j = 1,\ldots, \ell,\ i = 1,\ldots, \nu)$. Defining now,

$W_{\nu i} = \sum_{j=1}^{\ell} U_{\nu ij}\hat{\Delta}^*_{\nu j}$ $(i = 1,\ldots, \nu)$, we see that $\underset{\sim}{W}_\nu = (W_{\nu 1},\ldots, W_{\nu\nu})'$ has a joint distribution which remains invariant under any permutation of its ν arguments under H_0. Further, under H_0, sgn $X_{\nu i}$ can assume two values +1 and -1 each with prob. 1/2 independently of the $W_{\nu i}$'s; sgn $X_{\nu 1},\ldots,$ sgn $X_{\nu\nu}$ are independently distributed. Use the notations $X^*_{\nu 0} = (X^*_{\nu 10},\ldots X^*_{\nu\nu 0})'$ and sgn X_ν = (sgn $X_{\nu 1},\ldots,$ sgn $X_{\nu\nu}$).

If we now consider a finite group G_ν of transformations $\{g_\nu\}$ which maps the sample space onto itself, where a typical transformation g_ν is such that

$$(4.3.7)\quad g_\nu\underset{\sim}{W}_\nu = ((-1)^{m_1} W_{\nu i_1},\ldots, (-1)^{m_\nu} W_{\nu i_\nu}),\ m_\ell = (0, 1),$$

$\ell = 1, 2,\ldots, \nu$ and $(i_1,\ldots, i_\nu)$ is any permutation of the integers $(1,\ldots, \nu)$, then, there is a set of $2^\nu\,\nu!$ transformations in G_ν. Under H_0, the conditional distribution over the set of $2^\nu\,\nu!$ realizations (generated by G_ν) is uniform, each having common conditional probability $(2^\nu\,\nu!)^{-1}$. We denote this conditional probability measure by $\mathcal{P}'_\nu$. We may note that under $\mathcal{P}'_\nu$, $F^*_{\nu,j}$'s and hence $\hat{\Delta}^*_{\nu j}$'s remain fixed. It is now easy to verify that

$$(4.3.8)\quad E(U^*_{k\nu}|\mathcal{P}'_\nu) = 0 \quad (k = 0, 1,\ldots, p)\ ;$$

$$E(U^*_{k\nu}U^*_{k'\nu}|\mathcal{P}'_\nu) = (\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i})A_0^2(F^*_\nu,\{I_j\})\ ,$$

$(k, k' = 0, 1,\ldots, p)$, where,

$$(4.3.9)\quad A_0^2(F^*_\nu,\{I_j\}) = \sum_{j=1}^{\ell} \hat{\Delta}^{*2}_{\nu j}(\nu_j + \nu_{-j})/\nu\ .$$

Consider the quadratic form,

$$(4.3.10)\quad M_\nu^* = (\underset{\sim}{U}_\nu^{*\prime}\underset{\sim}{A}_\nu^{-1}\underset{\sim}{U}_\nu^*)A_0^{-2}(F_\nu^*,\{I_j\}) .$$

As in the earlier case, the following nonparametric test is proposed:

$$(4.3.11)\quad \psi_4(\underset{\sim}{X}_{\nu 0}^*, \text{ sgn } \underset{\sim}{X}_\nu) = \begin{cases} 1 & , \text{ if } M_\nu^* > M_{\nu,\varepsilon} \\ \delta_{4\nu\varepsilon} & , \text{ if } M_\nu^* = M_{\nu,\varepsilon} \\ 0 & , \text{ if } M_\nu^* < M_{\nu,\varepsilon} \end{cases} ,$$

where $M_{\nu,\varepsilon}$ and $\delta_{4\nu\varepsilon}$ are chosen in such a way that $E[\psi_4(\underset{\sim}{X}_{\nu 0}^*, \text{ sgn } \underset{\sim}{X}_\nu)|\mathcal{P}_\nu'] = \varepsilon$, the level of significance. This implies that $E[\psi_4(\underset{\sim}{X}_{\nu 0}^*, \text{ sgn } \underset{\sim}{X}_\nu)|P_\nu] = \varepsilon$ i.e. $\psi_4(\underset{\sim}{X}_{\nu 0}^*, \text{ sgn } \underset{\sim}{X}_\nu)$ is a similar size ε test.

We may remark that it is necessary to introduce the "signed-score" statistics instead of the ordinary "score" statistics as in chapter 3 because here we are interested in testing for β_0 along with $\beta_1, \ldots, \beta_p$. Had permutation-test statistics as in the earlier chapter been introduced without consideration of signs, the one for β_0 would have reduced to a constant under the permutational probability measure $\mathcal{P}_\nu'$. A similar situation was faced by Adichie [1] and by Puri and Sen [30] in case of ungrouped data and they introduced "signed-rank" statistics instead of "rank"-statistics for similar testing problems.

To study the asymptotic permutation distribution of the test statistic, we first extend a result of Hájek [16] involving the asymptotic distribution of the "simple linear rank statistic" to that of the "simple linear signed-rank statistic." To formlate the theorem, we proceed as follows:

Consider double sequences $\{b_{\nu i},\ 1 \le i \le \nu,\ \nu \ge 1\}$ and $\{a_{\nu i},\ 1 \le i \ge \nu,\ \nu \ge 1\}$ of real numbers satisfying

$$(4.3.13)\quad \lim_{\nu\to\infty} \max_{1\le i\le \nu} b_{\nu i}^2 / \left(\sum_{i=1}^{\nu} b_{\nu i}^2\right) = 0,$$

$$(4.3.14)\quad \lim_{\nu\to\infty}[k_\nu/\nu = 0] \Rightarrow \lim_{\nu\to\infty} \frac{\max_{1\le i_1\le\dots<i_{k_\nu}\le\nu} \sum_{\alpha=1}^{k_\nu} (a_{\nu i_\alpha} - \overline{a}_\nu)^2}{\sum_{i=1}^{\nu} (a_{\nu i} - \overline{a}_\nu)^2} = 0.$$

Let $X_{\nu 1},\dots, X_{\nu\nu}$ be ν i.i.d.r.v. having a continuous d.f. $F(x)$ satisfying $F(x) + F(-x) = 1$ for all real x; also let $(R_{\nu 1},\dots, R_{\nu\nu})$ be a random vector which takes $\nu!$ possible permutations of $(1, 2,\dots, \nu)$ with equal probabilities and distributed independently of $(X_{\nu 1},\dots, X_{\nu\nu})$. Let

$$(4.3.15)\qquad S_\nu = \sum_{i=1}^{\nu} b_{\nu i} a_{\nu R_{\nu i}} \operatorname{sgn} X_{\nu i},$$

where sgn $X_{\nu i}$'s are defined in (4.2.20).

THEOREM 4.3.1. <u>Under</u> (4.3.13) - (4.3.14) <u>and the assumptions following it</u> S_ν <u>defined in</u> (4.3.15) <u>is asymptotically</u> $N_1(0,\ \nu^{-1} \sum_{i=1}^{\nu} b_{\nu i}^2 \sum_{i=1}^{\nu} a_{\nu i}^2)$ <u>a.e.</u>

<u>Proof</u>. It can be assumed without any loss of generality that $a_{\nu 1} \le a_{\nu 2} \le\dots\le a_{\nu\nu}$ and put $a(\lambda) = a_{\nu i}$ for $(i-1)/\nu < \lambda \le i/\nu$, $1 \le i \le \nu$. Let $T_\nu = \sum_{i=1}^{\nu} b_{\nu i} a_\nu(U_i) \operatorname{sgn} X_{\nu i}$, where $U_1, U_2,\dots$ is a sequence of independent random variables distributed independently of the $X_{\nu i}$'s, each U_i uniformly distributed over (0, 1], and the rank of U_i in the partial sequence $U_1,\dots, U_\nu$ is denoted by

$R'_{\nu i}$ $(1 \le i \le \nu,\ \nu \ge 1)$. Hence, the vector $(R'_{\nu 1}, \ldots, R'_{\nu\nu})$ satisfies the same conditions as $(R_{\nu 1}, \ldots, R_{\nu\nu})$ in the theorem, and we get,

$$(4.3.16)\quad E(T_\nu - S_\nu)^2 = \left(\sum_{i=1}^{\nu} b^2_{\nu i}\right) E[a_\nu(U_1) - a_\nu(R_{\nu 1}/\nu)]^2$$

$$\le \left(\sum_{i=1}^{\nu} b^2_{\nu i}\right) 2\sqrt{2}\,\left(\nu^{-\frac{1}{2}} \max_{1\le i\le\nu} |a_{\nu i} - \bar{a}_\nu|\right)\left(\nu^{-1}\sum_{i=1}^{\nu}(a_{\nu i} - \bar{a}^2_\nu)\right)^{1/2},$$

using lemma 2.1 of [16]. Also, $E(S_\nu) = 0$ and

$$\mathrm{Var}(S_\nu) = \nu^{-1} \sum_{i=1}^{\nu} b^2_{\nu i} \sum_{i=1}^{\nu} a^2_{\nu i}.$$

Hence,

$$(4.3.17)\quad E(T_\nu - S_\nu)^2/\mathrm{Var}(S_\nu) \le 2\sqrt{2}\left[\max_{1\le i\le\nu} |a_{\nu i} - \bar{a}_\nu| / \left(\sum_1^{\nu} a^2_{\nu i}\right)^{1/2}\right]$$

$$\le 2\sqrt{2}\left[\max_{1\le i\le\nu} |a_{\nu i} - \bar{a}_\nu| / \left(\sum_1^{\nu} (a_{\nu i} - \bar{a}_\nu)^2\right)^{1/2}\right]$$

$\to 0$ as $\nu \to \infty$ from (4.3.14) putting $k_\nu = 1$. To prove the theorem, it is sufficient to show that (i) $(T_\nu - E(T_\nu))/\sqrt{\mathrm{Var}(T_\nu)}$ is asymptotically $N_1(0, 1)$ a.e. and (ii) $(S_\nu - E(S_\nu))/\sqrt{\mathrm{Var}(S_\nu)} - (T_\nu - E(T_\nu))/\sqrt{\mathrm{Var}(T_\nu)} \to 0$ in probability. The proof which is based on (4.3.17) is given in Hájek [16].

To prove (i), we use Theorems 4.1 and 4.2 of Hájek [16]. We need observe only that the Lindeberg condition

$$\lim_{\nu\to\infty} \frac{1}{\mathrm{Var}(T_\nu)} \sum_{i=1}^{\nu} E[T^2_{\nu i} I_{(|T_{\nu i}|\ge\delta)}] = 0,$$

where $T_{\nu i} = (b_{\nu i} - \bar{b}_\nu) a_\nu(U_i) \operatorname{sgn} X_{\nu i}$ $(i = 1, 2, \ldots, \nu)$, and

$$I_{(|T_{\nu i}|\ge\delta)} = \begin{cases} 1 & \text{if } |T_{\nu i}| \ge \delta \\ 0 & \text{otherwise} \end{cases},$$

$1 \le i \le \nu$, is equivalent to the condition

$$\lim_{\nu\to\infty} \frac{1}{\mathrm{Var}(T'_\nu)} \sum_{i=1}^{\nu} E[T'^2_{\nu i}\, I_{(|T'_{\nu i}|\ge\delta)}] = 0,$$

where $T'_{\nu i} = \sum_{i=1}^{\nu} (b_{\nu i} - \bar{b}_\nu) a_\nu(U_i)$, $T'_\nu = \sum_{i=1}^{\nu} T'_{\nu i}$ and

$$I_{(|T'_{\nu i}|\ge\delta)} = \begin{cases} 1 & \text{if } |T'_{\nu i}|\ge\delta \\ 0 & \text{otherwise} \end{cases}.$$

The last assertion follows from the fact $|\mathrm{sgn}\ X_{\nu i}| = 1$ with probability 1 for all $i = 1, 2, \ldots, \nu$ and $\mathrm{sgn}\ X_{\nu 1}, \ldots, \mathrm{sgn}\ X_{\nu\nu}, U_1, \ldots, U_\nu$ are mutually independent.

The above theorem will be utilized in deriving the asymptotic permutation distribution of $\underset{\sim}{U}^*_\nu$. In order to prove the result, the following lemma is also needed.

LEMMA 4.3.1. <u>Under the assumption</u> (3.1.6) <u>and</u> (4.1.2) $A_0^2(F^*_\nu,\{I_j\}) \to A_0^2(F,\{I_j\})$ <u>as</u> $\nu \to \infty$ <u>either in</u> P_ν- (<u>or in</u> Q_ν-) <u>probability</u>.

<u>Proof</u>. $A_0^2(F^*_\nu,\{I_j\}) = \sum_{j=1}^{\ell} \hat{\Delta}^{*2}_{\nu j}(\nu_j + \nu_{-j})/\nu$. Noting that $(\nu_j + \nu_{-j})/\nu \to 2P_j$ in probability as $\nu \to \infty$, we get exactly as in lemma 3.3.1,

$$A_0^2(F^*_\nu,\{I_j\}) \to \sum_{j=1}^{\ell} \Delta^{*2}_j(2P_j) = \sum_{j=1}^{\ell} \Delta^2_j(2P_j) = A_0^2(F,\{I_j\}) \text{ in } P_\nu\text{- (or in}$$

Q_ν-) probability as $\nu \to \infty$.

THEOREM 4.3.2. <u>Under</u> $\mathcal{P}'_\nu$, (3.1.1) - (3.1.6) <u>and</u> (4.1.2) $\underset{\sim}{U}^*_\nu$ <u>is asymptotically</u> $N_{p+1}(\underset{\sim}{0}, \underset{\sim}{\Delta}_0 A_0^2(F,\{I_j\}))$ <u>in probability</u>.

<u>Proof</u>. Consider any linear combination $\underset{\sim}{d}'\underset{\sim}{U}^*_\nu$, $\underset{\sim}{d} = (d_0, \ldots, d_p)' \ne \underset{\sim}{0}$, $d_0, d_1, \ldots, d_p$ are fixed and finite.

$$\underset{\sim}{d}'\underset{\sim}{U}^*_\nu = \sum_{i=1}^{\nu} \left(\sum_{k=0}^{p} d_k c^*_{k\nu i}\right)\left(\sum_{j=1}^{\ell} U_{\nu ij}\hat{\Delta}^*_{\nu j}\right)\text{sgn } X_{\nu i} = \sum_{i=1}^{\nu} m_{\nu i}W_{\nu i} \text{ sgn } X_{\nu i} ,$$

where $m_{\nu i} = \sum_{k=0}^{p} d_k c^*_{k\nu i}$, $i = 1, 2, \ldots, \nu$. Now, $\max_{1\le i\le\nu} |m_{\nu i}| = o(1)$ and $\sum_{i=1}^{\nu} m^2_{\nu i} = O(1)$, (because of the assumptions (3.1.4) and (3.1.5)).

So the sequence $\{m_{\nu i}\}$ satisfies the Noether condition. Further, $\nu^{-1}\sum_{i=1}^{\nu} W^2_{\nu i} = A^2_0(F^*_\nu,\{I_j\}) \to A^2_0(F,\{I_j\})$ in probability. It can also be shown as in chapter II (see also chapter III) that

$$\max_{1\le i_1\le\ldots\le i_{k_\nu}\le\nu} \nu^{-1}\sum_{\alpha=1}^{k_\nu} W^2_{\nu i_\alpha} \to 0 \text{ in } \mathcal{P}'_\nu\text{-probability.}$$

So the sequence $\{W_{\nu i}\}$ satisfies Hajek's condition Q in probability (see [16], p. 519). Now, we refer to Theorem 4.3.1 to get the result.

LEMMA 4.3.2. Under (3.1.1) - (3.1.6) and (4.1.2) $\underset{\sim}{U}^*_\nu - \underset{\sim}{T}_{\nu 0} \to \underset{\sim}{0}$ in P_ν- (or in Q_ν-) probability.

Proof. Write

$$(4.3.18)\qquad U^*_{k\nu} - T_{k\nu 0} = \sum_{j=1}^{\ell}(\hat{\Delta}^*_{\nu j} - \Delta^*_j)\sum_{i=1}^{\nu} c^*_{k\nu i}U_{\nu ij}\text{sgn } X_{\nu i},\ k = 0,1,\ldots,p.$$

The rest of the proof is the same as in lemma 3.3.2 and, hence, is omitted.

From the above results, it is now obvious that M^*_ν and $S_{\nu 00}$ have asymptotically the same distribution under H_0 or $\{H_\nu\}$. It is immediate on the basis of lemmas 4.2.1 - 4.2.3 that M^*_ν (or $S_{\nu 00}$) is asymptotically χ^2_{p+1} under H_0 and $\chi^2_{p+1}(\zeta_0)$ under $\{H_\nu\}$, where,

$$(4.3.19)\qquad \zeta_0 = (\underset{\sim}{\gamma}^{*\prime}\underset{\sim}{\Lambda}_0\underset{\sim}{\gamma}^*)A^2_0(F,\{I_j\}) .$$

The extension of the above results to the case of countable set of class-intervals is similar as in chapter III since the sequence $\{Q_\nu\}$ of probability measures is contiguous to the sequence $\{P_\nu\}$ of probability measures from lemma 4.2.2. Again, in this case, since the asymptotically optimal parametric test is similar as in chapter III (involving only the additional statistic corresponding to β_0), the ARE of the proposed test procedure compared to the asymptotically optimal test procedure will be similar as $\rho^2(F, G, \{I_j\})$ given in (3.5.23) where $\rho^2(F, G, \{I_j\})$ is defined similarly as in section 3.5. The loss of efficiency expressions will also be similar as (3.5.24) or (3.5.25).

4.4. <u>An Important Application</u>.

The test procedure developed in earlier sections can be used in the particular case of 'paired comparison' problem. The problem (which has been considered by Mehra and Puri [28]) is as follows:

Consider p treatments in a sequence of experiments, the ν^{th} sequence yielding paired observations $(Y_{\nu km}, Y_{\nu k'm})$, $m = 1,2,\ldots,n_{\nu kk'}$, $1 \leq k < k' \leq p$, $= \sum\sum_{1 \leq k < k' \leq p} n_{\nu kk'}$. Assume that the $n_{\nu kk'}$ difference scores $Y_{\nu m}^{(k,k')} = Y_{\nu km} - Y_{\nu k'm}$, $(m = 1, 2,\ldots, n_{\nu kk'})$, have a common continuous d.f. $F_{\nu kk'}(x)$. This is the situation, for example if in the analysis of an incomplete block experiment, with each block of size two, one makes the assumption of additivity as in the usual analysis of variance model. We want to test the hypothesis

$$(4.4.1) \quad H_0 : F_{\nu kk'}(x) + F_{\nu kk'}(-x) = 1 \text{ and } F_{\nu kk'}(x) = F_{\nu k''k'''}(x)$$

for any two pairs (k, k') and (k'', k'''), where $1 \le k, k', k'', k''' \le p$, $k \ne k'$, $k'' \ne k'''$ against the sequence of alternatives

$$(4.4.2) \qquad F_{\nu kk'}(x) = F_\nu(x + \mu_{kk'}/\sqrt{\nu}\,),$$

for each pair (k, k'), $1 \le k < k' \le p$, where $\mu_{kk'}$ are certain constants not all zeroes satisfying $\mu_{kk'} = -\mu_{k'k}$ $(1 \le k \ne k' \le p)$.

The test statistic used by Mehra and Puri can be introduced as follows:

Let $\{J_{\nu i},\ 1 \le i \le \nu,\ \nu \ge 1\}$ be a double sequence of numbers satisfying either the assumptions (on the "score" functions) by Chernoff and Savage (1958) or by Hájek (1962). Let $R_{\nu,m}^{(k,k')}$ be the rank of $|Y_{\nu m}^{(k,k')}|$ when the $\nu = \sum\sum_{1 \le k < k' \le p} n_{\nu kk'}$ values of $|Y_{\nu m}^{(k,k')}|$, $m = 1, 2, \ldots, n_{\nu kk'}$; $1 \le k < k' \le p$, are arranged in ascending order of magnitude in a combined ranking. Define

$$\tau_\nu^{(k,k')} = \sum_{m=1}^{n_{\nu kk'}} J_\nu(R_{\nu,m}^{(k,k')}/(\nu + 1))\,\mathrm{sgn}\, Y_{\nu m}^{(k,k')},$$

where $J_\nu(u)$ is a step function defined over (0, 1] taking constant values $J_{\nu i}$ over the interval $((i - 1)/\nu, i/\nu]$, i.e. $J_\nu(u) = J_{\nu i} = J(i/(\nu + 1))$ for $(i - 1)/\nu < u \le i/\nu$. Then the test statistics are of the form

$$(4.4.3) \quad K_{\nu 0} = \sum_{k=1}^{p} \{ \sum_{k' \ne k} \tau_\nu^{(k,k')}/\sqrt{n_{\nu kk'}}\ \}^2 (\nu^{-1} \sum_{i=1}^{\nu} J_{\nu i}^2) p.$$

It has also been shown that under H_0 defined in (4.4.1) and under the conditions given in [28], $K_{\nu 0}$ is under H_0 asymptotically

χ^2_{p-1} and under the sequence of alternatives defined in (4.4.2) is asymptotically $\chi^2_{p-1}(\delta)$, where

$$(4.4.4) \quad \delta = p^{-1} \sum_{k=1}^{p} \{ \sum_{k' \neq k} (\sqrt{\rho_{kk'}}\, \mu_{kk'}) \}^2 ;$$

$\rho_{kk'} = \lim_{\nu\to\infty} n_{\nu kk'}/\nu$ is assumed to exist and is > 0 for each pair (k, k'), $(1 \le k < k' \le p)$. $\rho_{kk'} = \rho_{k'k}$ $(1 \le k \ne k' \le p)$.

Let $n_{\nu k} = \sum_{k'=1}^{p} n_{\nu kk'}$. To reduce the problem to our model, we first set

$$(4.4.5) \quad \frac{\mu_{kk'}}{\sqrt{\nu}} = \frac{\gamma_{k'}}{\sqrt{n_{\nu k'}}} - \frac{\gamma_k}{\sqrt{n_{\nu k}}}, \quad 1 \le k \ne k' \le p,$$

$\gamma = (\gamma_1, \ldots, \gamma_p)'$ being defined in (3.2.10). We pool all the observations $Y_{\nu m}^{(k,k')}$ $(m = 1, 2, \ldots, n_{\nu kk'}, 1 \le k < k' \le p)$ and denote this pooled set of observations by $X_{\nu 1}, \ldots, X_{\nu\nu}$. Then this testing problem belongs to our model with $\beta_0 = 0$, $c_{k\nu i} = +1$ if $X_{\nu i}$ is from a block where the k^{th} treatment is paired with a treatment k' $(k' = k + 1, \ldots, p)$, $c_{k\nu i} = -1$ if $X_{\nu i}$ is from a block where the k^{th} treatment is paired with a treatment k' $(k' = 1, 2, \ldots, k-1)$, $c_{k\nu i} = 0$, if $X_{\nu i}$ is not involved in the block at all. We consider the situation when the observations are classified in several groups. Define test statistics $T_{k\nu 0}$ as in (4.2.21) or $U^*_{k\nu}$ as in (4.3.6), $k = 1, 2, \ldots, p$. Then the quadratic form M^*_ν (or $S_{\nu 00}$) is asymptotically χ^2_{p-1} under H_0 and $\chi^2_{p-1}(\zeta)$ under the sequence of alternatives with

$$(4.4.6) \qquad \zeta = (\underset{\sim}{\gamma}'\underset{\sim}{\Lambda}\underset{\sim}{\gamma})A_0^2(F,\{I_j\}),$$

$\underset{\sim}{\Lambda}$ and $A_0^2(F,\{I_j\})$ being defined in (3.1.5) and (4.2.9) respectively. We may also mention here that since, under the model β_0 is absent, even in the process of finding permutationally distribution-free tests, statistics (as in chapter III), regardless of signs, may be used.

We may remark at this stage that the test proposed by Mehra and Puri, considered as such has some limitations owing to the fact that it allows a judge only to give two possible verdicts "better" or "worse" in comparing one treatment with another. However, if we want to allow greater flexibility in the judgment of a judge by making the judgment level classified into several ordered categories, the above test procedure will fail. For example, in comparing the performances of two musicians at a music competition, the performance of Musician 1 compared to that of Musician 2 may be classified as "Much Inferior," "Slightly Inferior," "More or less Parallel," "Slightly Better" and "Much Better." In such cases, it will not be possible to use the test of Mehra and Puri, while our test procedure can be used if these ordered categories are represented as class-intervals on the real axis symmetric about the origin.

It should be possible to introduce, however, statistics analogous to that of Mehra and Puri for grouped data by replacing "score" functions $J_\nu(R_{\nu,m}^{(k,k')}/(\nu + 1))$ by the corresponding "scores" for grouped data. We have already mentioned that their tests do not agree in general with our tests. To show this, we first define, statistics analogous to that of Mehra and Puri for grouped data. We have a finite set of ordered points on the (extended) real line giving a finite number of class-intervals in which the observations $Y_{\nu m}^{(k,k')}$ belong. These points may vary for different pairs (k, k'), $1 \leq k < k' \leq p$. For each pair (k, k') we define scores $\hat{\Delta}_{\nu j}^{*(k,k')}$ as in (4.3.5). Also, we define

$$(4.4.7) \qquad U_{\nu mj}^{(k,k')} = \begin{cases} 1 & \text{if } Y_{\nu m}^{(k,k')} \in I_j^{(k,k')} \\ 0 & \text{otherwise} \end{cases},$$

$(m = 1, 2, \ldots, n_{\nu kk'}$; $j = 0, 1, \ldots, \ell_{kk'}$ (say) $I_j^{(k,k')}$ being defined similar to (3.1.10) for each pair (k, k'), $1 \leq k < k' \leq p)$. Then the grouped data analogue of the statistics of Mehra and Puri will be

$$(4.4.8) \qquad \tau_\nu^{(k,k')} = \sum_{m=1}^{n_{\nu kk'}} \left(\sum_{j=0}^{\ell_{kk'}} \hat{\Delta}_{\nu j}^{*(k,k')} U_{\nu mj}^{(k,k')} \right) \operatorname{sgn} Y_{\nu m}^{(k,k')},$$

$1 \leq k < k' \leq p$. The appropriate quadratic form (see [28]) will be

$$(4.4.9) \qquad M_{\nu 0} = p \sum_{k=1}^{p} \Big\{ \sum_{k' \neq k} \tau_\nu^{(k,k')} / \sqrt{n_{\nu kk'}} \Big\}^2 \Big/$$

$$\Big\{ \nu^{-1} \sum\sum_{1 \leq k < k' \leq p} \sum_{j=0}^{\ell_{kk'}} (\hat{\Delta}_{\nu j}^{*(k,k')})^2 n_{\nu kk'} \Big\} .$$

It can be proved as in [28] that under H_0, $M_{\nu 0}$ is asymptotically χ_p^2, while under the sequence of alternatives, $M_{\nu 0}$ is asymptatically $\chi_p^2(\delta)$, where

$$(4.4.10) \quad \delta = \frac{A_0^2(F,\{I_j\})}{p} \sum_{k=1}^{p} \{ \sum_{k' \neq k} (\sqrt{\rho_{kk'}}\, \mu_{kk'}) \}^2 ,$$

$\rho_{kk'}$ $(1 \le k \ne k' \le p)$ being already defined in (4.4.4).

In the particular case when each pair of treatments is compared the same number of times i.e. $n_{\nu kk'} = \binom{p}{2}^{-1}\nu$, for all (k, k') such that $1 \le k < k' \le p$, we have $\rho_{kk'} = \binom{p}{2}^{-1}\nu$, $1 \le k < k' \le p$. From (4.4.5), $\mu_{kk'} = \binom{p}{2}^{\frac{1}{2}} (\gamma_{k'} - \gamma_k)$, $1 \le k < k' \le p$. Then from (4.4.10),

$$(4.4.11) \quad \delta = \frac{A_0^2(F,\{I_j\})}{p(p-1)} \sum_{k=1}^{p} \{ \sum_{k' \neq k} (\gamma_{k'} - \gamma_k) \}^2$$

$$= \frac{p}{p-1} A_0^2(F,\{I_j\}) \sum_{k=1}^{p} (\gamma_k - \overline{\gamma})^2 ,$$

where $\overline{\gamma} = p^{-1} \sum_{k=1}^{p} \gamma_k$. Also, in this case,

$$\sum_{i=1}^{\nu} c_{k\nu i} = \sum_{k'=1}^{p} n_{\nu kk'} = \nu\binom{p}{2}^{-1}(p - 1) = 2\nu/p, \quad k = 1, \ldots, p,$$

$$\sum_{i=1}^{\nu} c_{k\nu i}c_{k'\nu i} = - n_{\nu kk'} = - \nu\binom{p}{2}^{-1}, \quad 1 \le k \ne k' \le p.$$

So, $\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i} = - \frac{p}{2\nu} \cdot\nu \binom{p}{2}^{-1} = - (p - 1)^{-1}$, $1 \le k \ne k' \le p$.

$$\text{Hence, } \underset{\sim}{\Lambda} = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \rho & \rho & \cdots & 1 \end{bmatrix}, \text{ where } \rho = -(p - 1)^{-1} .$$

(Note that $\underset{\sim}{\Lambda}$ is not of full rank. But we can take a principal submatrix of order $(p - 1)$ which is of full rank).

From (4.4.4) we get,

$$(4.4.12)\quad \zeta = A_0^2(F,\{I_j\})\Big(\sum_{k=1}^{p} \gamma_k^2 - \frac{1}{p-1} \sum_{k\neq k'} \gamma_k\gamma_{k'}\Big)$$

$$= \frac{p\, A_0^2(F,\{I_j\})}{p-1} \sum_{k=1}^{p} (\gamma_k - \bar{\gamma})^2 \ .$$

(4.4.11) and (4.4.12) justify our assertion. If however all possible pairs of treatments are not compared the same number of times, the two test procedures are not necessarily asymptotically power equivalent since the noncentrality parameters ζ and δ may differ. The following example will justify the statement:

Consider the case $p = 3$, $n_{\nu 12} = n_{\nu 13} = \nu/6$, $n_{\nu 23} = 2\nu/3$.

Then, $\rho_{12} = \rho_{13} = 1/6$, $\rho_{23} = 2/3$; $\mu_{12} = \dfrac{\gamma_1}{\sqrt{2/6}} - \dfrac{\gamma_2}{\sqrt{5/6}} = \sqrt{6}\left(\dfrac{\gamma_1}{\sqrt{2}} - \dfrac{\gamma_2}{\sqrt{5}}\right)$,

$\mu_{13} = \sqrt{6}\left(\dfrac{\gamma_1}{\sqrt{2}} - \dfrac{\gamma_3}{\sqrt{5}}\right)$, $\mu_{23} = \sqrt{6}\left(\dfrac{\gamma_2}{\sqrt{5}} - \dfrac{\gamma_3}{\sqrt{5}}\right)$. Hence, we get, after some algebraic simplifications,

$$\delta = A_0^2(F,\{I_j\})\Big[\gamma_1^2 + \frac{14}{15}(\gamma_2^2 + \gamma_3^2) - \sqrt{\tfrac{2}{5}}\,(\gamma_1\gamma_2 + \gamma_1\gamma_3) - \frac{22}{15}\gamma_2\gamma_3\Big] \ .$$

Also, here,

$$\underset{\sim}{\Lambda} = \begin{bmatrix} 1 & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & 1 & -\frac{4}{5} \\ -\frac{1}{\sqrt{10}} & -\frac{4}{5} & 1 \end{bmatrix}$$

Hence, $\zeta = A_0^2(F,\{I_j\})[\gamma_1^2 + \gamma_2^2 + \gamma_3^2 - \sqrt{\frac{2}{5}}(\gamma_1\gamma_2 + \gamma_1\gamma_3) - \frac{8}{5}\gamma_2\gamma_3]$.

Thus, $\zeta - \delta = \frac{1}{15} A_0^2(F,\{I_j\})(\gamma_2 - \gamma_3)^2 > 0$, when $\gamma_2 \neq \gamma_3$.

The above example also shows that in this particular case our test procedure cannot be less powerful than that of [28]. We may, thus, conclude that the test proposed in [28] does not have the asymptotic optimality properties as described in section 3.2. Also, computationally, our test statistic seems to be simpler than its rival.

It is also obvious that the univariate one-sample location problem for grouped data follows as a particular case of our model, since then we have only to set $\beta_1 = \beta_2 = \ldots = \beta_p = 0$.

CHAPTER V

NONPARAMETRIC TESTS FOR GROUPED DATA IN THE

MULTIVARIATE LINEAR REGRESSION MODEL

5.1 Introduction, Notations and Assumptions.

Consider a double sequence of stochastic matrices $\underset{\sim}{X}_\nu = (\underset{\sim}{X}_{\nu 1}, \ldots, \underset{\sim}{X}_{\nu\nu})$, $\nu \geq 1$, where, $\underset{\sim}{X}'_{\nu i} = (X^{(1)}_{\nu i}, \ldots, X^{(q)}_{\nu i})$, $i = 1, 2, \ldots, \nu$ are independent stochastic vectors having continuous d.f.

(5.1.1) $F_{\nu i}(\underset{\sim}{x}') = F(\underset{\sim}{x}' - \underset{\sim}{\beta}'_0 - \underset{\sim}{c}'_{\nu i}\underset{\sim}{\beta}')$, $i = 1, \ldots, \nu$, $\nu \geq 1$,

and $\underset{\sim}{x} = (x_1, \ldots, x_q)'$, $\underset{\sim}{\beta}_0 = (\beta_{10}, \ldots, \beta_{q0})'$ is the nuisance parameter (vector), $\underset{\sim}{\beta} = \underset{q\times p}{((\beta_{sk}))}$ is the parameter (matrix) under test and $\underset{\sim}{c}_{\nu i} = (c_{1\nu i}, \ldots, c_{p\nu i})'$, $(i = 1, 2, \ldots, \nu, \nu \geq 1)$ are known regression constants (vectors). We want to test the null hypothesis of no regression, that is,

(5.1.2) $$H_0 : \underset{\sim}{\beta} = \underset{q\times p}{\underset{\sim}{0}},$$

against the set of alternatives $\underset{\sim}{\beta} \neq \underset{\sim}{0}$.

The notations and assumptions in (3.1.2)-(3.1.5) regarding the regression constants $c_{k\nu i}$ $(k = 1, 2, \ldots, p;\ i = 1, 2, \ldots, \nu,$ $\nu \geq 1)$ are the same in this case. Let the univariate marginal d.f.

of the s^{th} variate corresponding to the d.f. $F_{\nu i}(\underset{\sim}{x}')$ and $F(\underset{\sim}{x}')$ be $F_{\nu i,s}(x)$ and $F_s(x)$ respectively. ($s = 1,\ldots, q$; $i = 1, 2, \ldots, \nu$, $\nu \geq 1$). The corresponding bivariate joint d.f. for the s^{th} and s'^{th} variates are denoted respectively by $F_{\nu i,ss'}(x, y)$ and $F_{ss'}(x, y)$; ($1 \leq s \neq s' \leq q$; $i = 1, 2,\ldots, \nu$, $\nu \geq 1$). We also introduce the notation $H_\nu(\underset{\sim}{x}') = \nu^{-1} \sum_{i=1}^{\nu} F_{\nu i}(\underset{\sim}{x}')$, (the average q-variate d.f. for $\underset{\sim}{X}_{\nu 1},\ldots, \underset{\sim}{X}_{\nu\nu}$). The corresponding univariate marginal d.f. for the s^{th} variate and the bivariate joint d.f. for $(s,s')^{th}$ variates are denoted by $H_{\nu,s}(x)$ and $H_{\nu,ss'}(x, y)$ respectively; ($1 \leq s \neq s' \leq q$). Let

$$(5.1.3)\quad f_s(x) = F_s'(x) \text{ exist and } A^2(F_s) = \int_{-\infty}^{\infty} (f_s'/f_s)^2 f_s dx < \infty ,$$

$$\text{for all } s = 1, 2,\ldots, q.$$

Consider the sequence of alternatives

$$(5.1.4)\quad H_\nu : \underset{\sim}{\beta} = \underset{\sim}{\beta}_\nu = ((\tau_{sk}/C_{k\nu})) = \underset{q\times p}{\underset{\sim}{\tau}}\ \underset{p\times p}{\underset{\sim}{D}_\nu} , \quad \text{where}$$

$$\underset{\sim}{\tau} = \begin{bmatrix} \tau_{11} & \cdots & \tau_{1p} \\ & \cdots\cdots & \\ \tau_{q1} & \cdots & \tau_{qp} \end{bmatrix} , \quad D = \begin{pmatrix} 1/C_{1\nu} & & \underset{\sim}{0} \\ & \ddots & \\ \underset{\sim}{0} & & 1/C_{p\nu} \end{pmatrix} .$$

The elements of $\underset{\sim}{\tau}$ are assumed to be real and finite.

We are concerned with the situation when $\underset{\sim}{X}_\nu$ is not observable. We have q-dimensional rectangles in the real q-space $\mathbb{R}^q$ defined by

$$(5.1.5) \qquad \bigcap_{s=1}^{q} I_{j_s,s} = \bigcap_{s=1}^{q} \{x_s | a_{j_s,s} < x_s \leq a_{j_s+1,s}\} ,$$

$j_s = 0, 1, \ldots, \ell_s$, $s = 1, 2, \ldots, q$, (where $-\infty = a_{0,s} < a_{1,s}$ $\ldots < a_{\ell_s,s} < a_{\ell_s+1,s} = \infty$ are ordered points on the (extended) real line $[-\infty, \infty]$, $s = 1, 2, \ldots, q$). Let

$$(5.1.6) \qquad Z_{\nu i j_s,s} = \begin{cases} 1 & \text{if } X_{\nu i} \in I_{j_s,s} \\ 0 & \text{otherwise} \end{cases} ,$$

($j_s = 0, 1, \ldots, \ell_s$, $s = 1, \ldots, q$; $i = 1, \ldots, \nu$, $\nu \geq 1$).

The observable stochastic matrix is given by

$$(5.1.7) \qquad \underset{\sim}{X}_\nu^* = (\underset{\sim}{X}_{\nu 1}^*, \ldots, \underset{\sim}{X}_{\nu\nu}^*) , \; \nu \geq 1 ,$$

where $\underset{\sim}{X}_{\nu i}^* = ((X_{\nu i}^{(1)})^*, \ldots, (X_{\nu i}^{(q)})^*)'$, with

$$(5.1.8) \qquad (X_{\nu i}^{(s)})^* = \sum_{j_s=0}^{\ell_s} I_{j_s,s} Z_{\nu i j_s,s}$$

($s = 1, 2, \ldots, q$; $i = 1, 2, \ldots, \nu$, $\nu \geq 1$).

Under the model (5.1.1) it follows immediately that

$$(5.1.9) \qquad F_{\nu i,s}(x) = F_s(x - \beta_{s0} - \sum_{k=1}^{p} \beta_{sk} c_{k\nu i}) ,$$

and

$$(5.1.10) \qquad F_{\nu i,s,s'}(x, y) = F_{ss'}(x - \beta_{s0} - \sum_{k=1}^{p} \beta_{sk} c_{k\nu i} , \; y - \beta_{s'0} - \sum_{k=1}^{p} \beta_{s'k} c_{k\nu i}) ,$$

$1 \leq s \neq s' \leq q$; $i = 1, 2, \ldots, \nu$, $\nu \geq 1$.

Let

$$(5.1.11)\qquad F_{\nu i,j_s,s} = \Pr(X_{\nu i}^{(s)} \le a_{j_s,s} | H_\nu)$$

$$= F_s(a_{j_s,s} - \beta_{s0} - \sum_{k=1}^{p} \tau_{sk}\, c_{k\nu i}^{*}) ,$$

$$(5.1.12)\quad F_{\nu i,j_s,j_{s'},s,s'} = \Pr(X_{\nu i}^{(s)} \le a_{j_s,s},\ X_{\nu i}^{(s')} \le a_{j_{s'},s'} | H_\nu)$$

$$= F_{ss'}(a_{j_s,s} - \beta_{s0} - \sum_{k=1}^{p} \tau_{sk}\, c_{k\nu i}^{*},\ a_{j_{s'},s'} - \beta_{s'0} - \sum_{k=1}^{p} \tau_{s'k} c_{k\nu i}^{*}),$$

$(0 \le j_s \le \ell_s + 1,\ 0 \le j_{s'} \le \ell_{s'} + 1,\ 1 \le s \ne s' \le q,$

$i = 1, 2,\ldots, \nu,\ \nu \ge 1).$

Define

$$(5.1.13)\qquad H_{\nu,j_s,s} = \nu^{-1} \sum_{i=1}^{\nu} F_{\nu i,j_s,s} ,$$

$$(5.1.14)\qquad H_{\nu,j_s,j_{s'},s,s'} = \nu^{-1} \sum_{i=1}^{\nu} F_{\nu i,j_s,j_{s'},s,s'} ,$$

$(0 \le j_s \le \ell_s + 1,\ 0 \le j_{s'} \le \ell_{s'} + 1,\ 1 \le s \ne s' \le q,\ \nu \ge 1)$.

Also let,

$$(5.1.15)\quad F_{j_s,s} = F_s(a_{j_s,s} - \beta_{s0}) = \Pr(X_{\nu i}^{(s)} \le a_{j_s,s} | H_0) ,$$

for all $i = 1, 2,\ldots, \nu$;

$$(5.1.16)\quad F_{j_s,j_{s'},s,s'} = F_{ss'}(a_{j_s,s} - \beta_{s0},\ a_{j_{s'},s'} - \beta_{s'0})$$

$$= \Pr(X_{\nu i}^{(s)} \le a_{j_s,s},\ X_{\nu i}^{(s')} \le a_{j_{s'},s'} | H_0),\ \text{for all } i = 1,2,\ldots, \nu;$$

$(0 \le j_s \le \ell_s + 1,\ 0 \le j_{s'} \le \ell_{s'} + 1;\ 1 \le s \ne s' \le q)$.

Define now,

$$(5.1.17) \qquad P_{\nu i, j_s, s} = F_{\nu i, j_s+1, s} - F_{\nu i, j_s, s} ,$$

$$(5.1.18) \qquad P_{\nu i, j_s, s, j_{s'}, s'}$$

$$= F_{\nu i, j_s+1, j_{s'}+1, s, s'} - F_{\nu i, j_s, j_{s'}+1, s, s'}$$

$$- F_{\nu i, j_s, j_{s'}+1, s, s'} + F_{\nu i, j_s, j_{s'}, s, s'} ,$$

$(0 \le j_s \le \ell_s,\ 0 \le j_{s'} \le \ell_{s'},\ 1 \le s \ne s' \le q,\ 1 \le i \le \nu,\ \nu \ge 1)$,

and let,

$$(5.1.19) \qquad P_{j_s, s} = F_{j_s+1, s} - F_{j_s, s} ,$$

$$(5.1.20) \qquad P_{j_s, j_{s'}, s, s'}$$

$$= F_{j_s+1, j_{s'}+1, s, s'} - F_{j_s, j_{s'}+1, s, s'}$$

$$- F_{j_s+1, j_{s'}, s, s'} + F_{j_s, j_{s'}, s, s'}$$

$(0 \le j_s \le \ell_s,\ 0 \le j_{s'} \le \ell_{s'},\ 1 \le s \ne s' \le q)$.

The above notations will be used repeatedly in the subsequent sections. The "scale" parameters σ's have all been assumed to be 1 throughout this chapter.

5.2. A Class of Parametric Tests.

We introduce first the "score"-functions

$$(5.2.1) \qquad \phi_s(u) = -f_s'(F_s^{-1}(u))/f_s(F_s^{-1}(u)),\ 0 < u < 1,\ s = 1, 2, \ldots, q.$$

Let

$$(5.2.2)\quad \Delta_{j_s,s} = \int_{F_{j_s,s}}^{F_{j_s+1,s}} \phi(u)\,du \Bigg/ \int_{F_{j_s,s}}^{F_{j_s+1,s}} du, \text{ when } P_{j_s,s} \neq 0$$

$$= 0, \text{ otherwise.} \qquad (s = 1, 2, \ldots, q).$$

When $P_{j_s,s} \neq 0$, we can rewrite $\Delta_{j_s,s}$ as

$$(5.2.3)\quad \Delta_{j_s,s} = P_{j_s,s}^{-1}\ [f_s(a_{j_s,s} - \beta_{s0}) - f_s(a_{j_s+1,s} - \beta_{s0})].$$

Let

$$(5.2.4)\quad J_s = \{j_s | j_s \varepsilon [0, 1, \ldots, \ell_s],\ P_{j_s,s} > 0\},\ s = 1, 2, \ldots, q.$$

$$(5.2.5)\quad J_{ss'} = \{(j_s, j_{s'}) | j_s\ \varepsilon [0, 1, \ldots, \ell_s],\ j_{s'}\ \varepsilon [0, 1, \ldots, \ell_{s'}],$$

$$P_{j_s, j_{s'},s,s'} > 0\},\ 1 \leq s \neq s' \leq q$$

$$(5.2.6)\quad A^2(F_s, \{I_{j_s}\}) = A_{ss} = \sum_{j_s=0}^{\ell_s} \Delta^2_{j_s,s} P_{j_s,s} = \sum_{j_s \varepsilon J_s} \Delta^2_{j_s,s} P_{j_s,s};$$

$$(5.2.7)\qquad A(F_s, F_{s'}, \{I_{j_s}\}, \{I_{j_{s'}}\} = A_{ss'}$$

$$= \sum_{j_s=0}^{\ell_s} \sum_{j_{s'}=0}^{\ell_{s'}} \Delta_{j_s,s} \Delta_{j_{s'},s'} P_{j_s,j_{s'},s,s'}$$

$$= \sum\sum_{(j_s,j_{s'})\ \varepsilon J_{ss'}} \Delta_{j_s,s} \Delta_{j_{s'},s'} P_{j_s,j_{s'},s,s'}.$$

We may note that

$$(5.2.8)\qquad 0 < A^2_{ss'} < A_{ss}\ A_{s's'} < A^2(F_s)\ A^2(F_{s'}) < \infty$$

(similarly as (3.2.8)).

Let

(5.2.9) $$A = \underset{q\times q}{((A_{ss'}))}.$$

We introduce the statistics

(5.2.10) $$T_{\nu,sk} = \sum_{i=1}^{\nu} c^{*}_{k\nu i} \sum_{j_s=0}^{\ell_s} Z_{\nu i j_s, s}\, \Delta_{j_s, s}, \quad (s = 1,\dots,q,\ k = 1,\dots,p).$$

We define a test procedure based on the statistics $T_{\nu,sk}$'s $(s = 1,\dots, q,\ k = 1,\dots, p)$. These statistics are co-ordinate wise extensions of the statistics used in Chapter III. Moreover, unlike the univariate case, the test procedure we are going to propose here may not be asymptotically optimal (in Wald's sense) even under the regularity conditions given in Chapter III. Nevertheless, it seems to be the most natural one to consider because of its simplicity and mathematical tractability. We may remark also that our statistics are generalizations in case of grouped data for analogous statistics considered in case of ungrouped data by Puri and Sen [31]. It is, however, worth mentioning that the restriction on the "score function" ϕ is much less stringent in our case than that in the case of Puri and Sen who had to impose the conditions of Hajek [18] and Hoeffding [23] on ϕ. Let P_ν, Q_ν be respective distributions of $\underset{\sim}{X}^*_\nu$ under $H_0 \equiv H_{0\nu}$ and H_ν.

We get the following expressions after some algebraic simplifications:

(5.2.11) $$E(T_{\nu,sk}|P_\nu) = 0 ;$$

$$E(T_{\nu,sk}\, T_{\nu,sk'}|P_\nu) = (\sum_{i=1}^{\nu} c^*_{k\nu i}\, c^*_{k'\nu i})\, A_{ss} \;;$$

$$E(T_{\nu,sk}\, T_{\nu,s'k'}|P_\nu) = (\sum_{i=1}^{\nu} c^*_{k\nu i}\, c^*_{k'\nu i})\, A_{ss'} \;;$$

$(k, k' = 1,\ldots, p;\ 1 \leq s \neq s' \leq q)$. Write,

(5.2.12) $$\underset{\sim}{T}_\nu = \underset{q\times p}{((T_{\nu,sk}))} \;.$$

Then,

(5.2.13) $$E(\underset{\sim}{T}_\nu|P_\nu) = \underset{q\times p}{\underset{\sim}{0}} \;,\; \mathrm{Var}(\underset{\sim}{T}_\nu|P_\nu) = \underset{p\times p}{\underset{\sim}{\Lambda}_\nu} \otimes \underset{q\times q}{\underset{\sim}{A}}$$

$$\rightarrow \underset{p\times p}{\underset{\sim}{\Lambda}} \otimes \underset{q\times q}{A} \quad \text{as } \nu \rightarrow \infty \;.$$

We assume that

(5.2.14) $$A \text{ is p.d.}$$

Next we prove two lemmas:

LEMMA 5.2.1. <u>Under</u> P_ν, <u>the model</u> (5.1.1), <u>the assumptions</u> (3.1.2)-(3.1.5) <u>and</u> (5.1.3), <u>the joint distribution of the elements of</u> $\underset{\sim}{T}_\nu$ <u>is asymptotically</u> $N_{pq}(\underset{\sim}{0}, \underset{\sim}{\Lambda} \otimes \underset{\sim}{A})$.

Proof. Consider any linear combination $\sum_{s=1}^{q} \sum_{k=1}^{p} e_{\nu,sk}\, T_{\nu,sk}$, where $e_{\nu,sk}$ $(s = 1,\ldots, q,\ k = 1,\ldots, p)$ are fixed and real constants. We can write,

(5.2.15) $$\sum_{s=1}^{q} \sum_{k=1}^{p} e_{\nu,sk}\, T_{\nu,sk} = \sum_{i=1}^{\nu} W_{\nu i} \;,$$

where $W_{\nu i} = \sum_{s=1}^{q} \sum_{k=1}^{p} e_{\nu,sk}\, c^*_{k\nu i} \sum_{j_s=0}^{\ell_s} Z_{\nu i j_s, s}\, \Delta_{j_s, s}$ $(i = 1, 2,\ldots, \nu, \nu \geq 1)$.

Now, $E(W_{\nu i}|P_\nu) = 0$, $(i = 1, 2,\ldots, \nu)$.

$$\operatorname{Var}(W_{\nu i}|P_\nu) = \sum_{s,s'=1}^{q} \sum_{k,k'=1}^{p} e_{\nu,sk}\, e_{\nu,s'k'}\, c^*_{k\nu i}\, c^*_{k'\nu i}\, A_{ss'} < \infty$$

$$(i = 1, 2,\ldots, \nu).$$

Again, $|W_{\nu i}| \leq (\max\limits_{1\leq s\leq q} \max\limits_{1\leq k\leq p} |e_{\nu,sk}|) \sum\limits_{k=1}^{p} |c^*_{k\nu i}| \sum\limits_{s=1}^{q} \sum\limits_{j_s=0}^{\ell_s} |\Delta_{j_s,s}| Z_{\nu i j_s,s}$

Hence, writing $m_{\nu,pq} = \max\limits_{1\leq s\leq q} \max\limits_{1\leq k\leq p} |e_{\nu,sk}|$, we get,

$$(5.2.16) \quad |W_{\nu i}|^3 \leq m^3_{\nu,pq}\, 4^p \sum_{k=1}^{p} |c^*_{k\nu i}|^3 \left(\sum_{s=1}^{q} \sum_{j_s=0}^{\ell_s} |\Delta_{j_s,s}| Z_{\nu i j_s,s} \right)^3$$

$$\leq m^3_{\nu,pq}\, 4^{p+q} \sum_{k=1}^{p} |c^*_{k\nu i}|^3 \sum_{s=1}^{q} \left[\sum_{j_s=0}^{\ell_s} |\Delta_{j_s,s}| Z_{\nu i j_s,s} \right]^3$$

$$= m^3_{\nu,pq}\, 4^{p+q} \sum_{k=1}^{p} |c^*_{k\nu i}|^3 \sum_{s=1}^{q} \sum_{j_s=0}^{\ell_s} |\Delta_{j_s,s}|^3\, Z_{\nu i j_s,s}$$

Hence,

$$(5.2.17) \quad E(|W_{\nu i}|^3|P_\nu) \leq m^3_{\nu,pq}\, 4^{p+q} \sum_{k=1}^{p} |c^*_{k\nu i}|^3 \sum_{s=1}^{q} \sum_{j_s=0}^{\ell_s} |\Delta_{j_s,s}|^3\, P_{j_s,s}$$

$$\leq m^3_{\nu,pq} \left(\max_{1\leq s\leq q} \max_{j_s \varepsilon J_s} P^{-\frac{1}{2}}_{j_s,s} \right) 4^{p+q} \sum_{k=1}^{p} |c^*_{k\nu i}|^3 \sum_{s=1}^{q} \left(\int_0^1 |\phi^2_s(u)|\, du \right)^{\frac{3}{2}}$$

$$= m^3_{\nu,pq} \left(\max_{1\leq s\leq q} \max_{j_s \varepsilon J_s} P^{-\frac{1}{2}}_{j_s,s} \right) 4^{p+q} \sum_{k=1}^{p} |c^*_{k\nu i}|^3 \left(\sum_{s=1}^{q} A^3(F_s) \right).$$

Thus,

$$(5.2.18)\quad \sum_{i=1}^{\nu} E(|W_{\nu i}|^3|P_\nu) \le m^3_{\nu,pq} \left(\max_{1\le s\le q} \max_{j_s \varepsilon J_s} P_{j_s,s}^{-\frac{1}{2}} \right) 4^{p+q}$$

$$\left[\sum_{k=1}^{p} \max_{1\le i\le \nu} |c^*_{k\nu i}| \left(\sum_{s=1}^{q} A^3(F_s) \right) \right]$$

$$\le p\, m^3_{\nu,pq} \left(\max_{1\le s\le q} \max_{j_s \varepsilon J_s} P_{j_s,s}^{-\frac{1}{2}} \right) \max_{1\le k\le p} \max_{1\le i\le \nu} |c^*_{k\nu i}| \left(\sum_{s=1}^{q} A^3(F_s) \right)$$

$= o(1)$, from (3.1.4).

$$(5.2.19)\quad \sum_{i=1}^{\nu} E(W^2_{\nu i}|P_\nu) = \sum_{i=1}^{\nu} \mathrm{Var}(W_{\nu i}|P_\nu)$$

$$= \sum_{s,s'=1}^{q} \sum_{k,k'=1}^{p} e_{\nu,sk}\, e_{\nu,s'k'}\, A_{ss'} \left(\sum_{i=1}^{\nu} c^*_{k\nu i}\, c^*_{k'\nu i} \right)$$

$$= O(1)\ ,$$

using (3.1.5), (5.2.8) and the fact that $e_{\nu,sk}$ are finite and real elements, and also the summations involve a finite number of terms.

(5.2.17)-(5.2.19) validate Liapounov's criteria under which the classical C.L.T. holds. Now, using (5.2.13), the lemma is proved.

Q.E.D.

Let

$$(5.2.20)\quad h_{\nu si} = \sum_{k=1}^{p} \tau_{sk}\, c^*_{k\nu i} \quad (s = 1, 2,\ldots, q;\ i = 1, 2,\ldots, \nu,\ \nu \ge 1).$$

LEMMA 5.2.2. <u>Under</u> Q_ν, <u>the model</u> (5.1.1) <u>and the same assumptions as in lemma</u> 5.2.1., <u>the joint distribution of the elements of</u> $\underset{\sim}{T}_\nu$ <u>is asymptotically</u> $N_{pq}(\underset{\sim}{D}\ (A)\ \underset{\sim}{\tau}\ \underset{\sim}{\Lambda}\ ,\ \underset{\sim}{\Lambda} \otimes \underset{\sim}{A})$, <u>where</u>,

$$(5.2.21) \qquad \underset{\sim}{D}(A) = \begin{pmatrix} A_{11} & & \underset{\sim}{0} \\ & \ddots & \\ \underset{\sim}{0} & & A_{qq} \end{pmatrix}.$$

Proof. Using Taylor expansion, we get from (5.1.11), (5.1.15), (5.1.17) and (5.1.19),

$$(5.2.22) \quad P_{\nu i,j_s,s} = P_{j_s,s}[1 + \Delta_{j_s,s}\, h_{\nu si} + h_{\nu si}\, \theta_{j_s,s}\, g_{j_s,s}(h_{\nu si})] ,$$

for $j_s \varepsilon J_s$, where, $0 < \theta_{j_s,s} < 1$, and

$$(5.2.23) \quad g_{j_s,s}(h_{\nu si}) = P^{-1}_{j_s,s} \int_0^{h_{\nu si}} [f'_s(a_{j_s,s} - \beta_{s0} - \theta_{j_s,s}\, y) - f'_s(a_{j_s+1,s} - \beta_{s0} - \theta_{j_s,s}\, y)]\, dy$$

Now,

$$E(T_{\nu,sk}|Q_\nu) = \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j_s=0}^{\ell_s} \Delta_{j_s,s}\, P_{j_s,s}(1 + h_{\nu si}\Delta_{j_s,s} + h_{\nu si}\, \theta_{j_s,s}\, g_{j_s,s}(h_{\nu si}))$$

$$= (\sum_{i=1}^{\nu} c^*_{k\nu i}\, h_{\nu si})\, A_{ss} + o(1)$$

$$= \sum_{k'=1}^{p} \tau_{sk'}(\sum_{i=1}^{\nu} c^*_{k\nu i}\, c^*_{k'\nu i})\, A_{ss} + o(1),\ s = 1,.., q,\ k = 1,..., p.$$

Hence,

$$(5.2.24) \quad E(\underset{\sim}{T}_\nu|Q_\nu)$$

$$= \begin{bmatrix} \sum_{k=1}^{p} \tau_{1k}(\sum_{i=1}^{\nu} c^*_{1\nu i} c^*_{k\nu i})A_{11} & \cdots & \sum_{k=1}^{p} \tau_{1k}(\sum_{i=1}^{\nu} c^*_{p\nu i} c^*_{k\nu i})A_{11} \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{p} \tau_{qk}(\sum_{i=1}^{\nu} c^*_{1\nu i} c^*_{k\nu i})A_{qq} & \cdots & \sum_{k=1}^{p} \tau_{qk}(\sum_{i=1}^{\nu} c^*_{p\nu i} c^*_{k\nu i})A_{qq} \end{bmatrix}$$

$$+ \underset{\sim}{o}_1(1)$$

(where by $\underset{\sim}{o}_1(1)$ we mean a q×p matrix each element of which tends to zero as $\nu \to \infty$)

$$= \begin{bmatrix} A_{11} & \cdots & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdots & A_{qq} \end{bmatrix} \begin{bmatrix} \tau_{11} & \cdots & \tau_{1p} \\ \cdot & \cdot & \cdot \\ \tau_{q1} & \cdots & \tau_{qp} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{\nu} c_{1\nu i}^{*2} & \cdots & \sum_{i=1}^{\nu} c_{1\nu i}^{*} c_{p\nu i}^{*} \\ \cdot & \cdot & \cdot \\ \sum_{i=1}^{\nu} c_{1\nu i}^{*} c_{p\nu i}^{*} & \cdots & \sum_{i=1}^{\nu} c_{p\nu i}^{*2} \end{bmatrix} + \underset{\sim}{o}(1).$$

$$= \underset{\sim}{D}(A)\ \underset{\sim}{\tau}\ \underset{\sim}{\Lambda}_{\nu} + \underset{\sim}{o}(1) \to \underset{\sim}{D}(A)\ \underset{\sim}{\tau}\ \underset{\sim}{\Lambda} \text{ as } \nu \to \infty .$$

Also,

(5.2.25) $\quad \text{Cov.}\ (T_{\nu,sk}, T_{\nu,sk'} | Q_{\nu})$

$$= \text{Cov.}\ \left(\sum_{i=1}^{\nu} c_{k\nu i}^{*} \sum_{j_s=0}^{\ell_s} \Delta_{j_s,s} Z_{\nu i j_s,s},\ \sum_{i=1}^{\nu} c_{k'\nu i}^{*} \sum_{j_s=0}^{\ell_s} \Delta_{j_s,s} Z_{\nu i j_s,s} | Q_{\nu} \right)$$

$$= \sum_{i=1}^{\nu} c_{k'\nu i}^{*}\ c_{k'\nu i}^{*}\ \text{Var}\ \left(\sum_{j_s=0}^{\ell_s} Z_{\nu i j_s,s}\ \Delta_{j_s,s} | Q_{\nu} \right) .$$

But, $\text{Var}\ [(\sum_{j_s=0}^{\ell_s} \Delta_{j_s,s}\ Z_{\nu i j_s,s}) | Q_{\nu}]$

$$= E[(\sum_{j_s=0}^{\ell_s} Z_{\nu i j_s,s}\ \Delta_{j_s,s})^2 | Q_{\nu}]$$

$$- [\{E(\sum_{j_s=0}^{\ell_s} Z_{\nu i j_s,s}\ \Delta_{j_s,s})\}^2 | Q_{\nu}]$$

$$= \sum_{j_s=0}^{\ell_s} \Delta^2_{j_s,s} P_{j_s,s}(1 + h_{\nu si} \Delta_{j_s,s} + h_{\nu si} \theta_{j_s,s} g_{j_s}(h_{\nu si}))$$

$$- [h_{\nu si} A_{ss} + h_{\nu si} \sum_{j_s=0}^{\ell_s} \Delta_{j_s,s} P_{j_s,s} \theta_{j_s,s} g_{j_s}(h_{\nu si})]^2$$

$$= A_{ss} + O(h_{\nu i}) .$$

Hence, from (5.2.25),

(5.2.26) $\mathrm{Cov}\,(T_{\nu,sk}, T_{\nu,sk'}|Q_\nu)$

$$= (\sum_{i=1}^{\nu} c^*_{k\nu i} c^*_{k'\nu i}) A_{ss} + o(1),\ s = 1, 2,\ldots, q;\ k, k' = 1,\ldots,p.$$

Similarly,

(5.2.27) $\mathrm{Cov}\,(T_{\nu,sk}, T_{\nu,s'k'}|Q_\nu)$

$$= (\sum_{i=1}^{\nu} c^*_{k\nu i} c^*_{k'\nu i}) A_{ss'} + o(1),\ 1 \leq s \neq s' \leq q;\ k,k' = 1,\ldots, p.$$

Using (5.2.26) and (5.2.27) we get,

(5.2.28) $$\mathrm{Var}\,(\underset{\sim}{T}_\nu|Q_\nu) = \underset{\sim}{\Lambda}_\nu \otimes \underset{\sim}{A} + \underset{\sim}{o}_2(1) ,$$

(where by $\underset{\sim}{o}_2(1)$ we mean a square matrix of order pq each element of which tends to zero as $\nu \to \infty$).

Thus,

(5.2.29) $$\mathrm{Var}\,(\underset{\sim}{T}_\nu|Q_\nu) \to \underset{\sim}{\Lambda} \otimes \underset{\sim}{A} \text{ as } \nu \to \infty.$$

Let $W_{\nu i}$ ($i = 1, 2,\ldots, \nu$) be the same as in lemma 5.2.1. Using the inequality (5.2.16), (5.2.30) $E(|W_{\nu i}|^3|Q_\nu)$

$$\leq m^3_{\nu,pq}\ 4^{p+q} \sum_{k=1}^{p} |c^*_{k\nu i}|^3 \sum_{s=1}^{q} \sum_{j_s=0}^{\ell_s} |\Delta_{j_s,s}|^3\ P_{\nu i j_s,s}$$

$$= m^3_{\nu,pq}\ 4^{p+q} \sum_{k=1}^{p} |c^*_{k\nu i}|^3 \sum_{s=1}^{q} \sum_{j_s=0}^{\ell_s} |\Delta_{j_s,s}|^3 (P_{j_s,s} + o(1))$$

Using now (5.2.18) and (5.2.30), we get,

$$\text{(5.2.30(a))} \qquad \sum_{i=1}^{\nu} E(|W_{\nu i}|^3|Q_\nu) = o(1)\ .$$

Also, $E(W_{\nu i}|Q_\nu) = \sum_{s=1}^{q} \sum_{k=1}^{p} e_{\nu,sk}\ c^*_{k\nu i} \sum_{j_s=0}^{\ell_s} \Delta_{j_s,s}(P_{j_s,s} + o(1))$

$$= o(c^*_{k\nu i})\ .$$

Hence,

$$\text{(5.2.30(b))} \qquad \sum_{i=1}^{\nu} |E(W_{\nu i}|Q_\nu)|^3 = \sum_{i=1}^{\nu} o(c^{*\,3}_{k\nu i}) = o(1)\ .$$

Using the inequality $|W_{\nu i} - E(W_{\nu i})|^3 \leq 4(|W_{\nu i}|^3 + |E(W_{\nu i})|^3)$, we get now from (5.2.30(a)) and (5.2.30(b)),

$$\text{(5.2.31)} \qquad \sum_{i=1}^{\nu} E[|W_{\nu i} - E(W_{\nu i})|^3] = o(1)\ .$$

Similarly,

(5.2.32) $\sum_{i=1}^{\nu} \text{Var}\ (W_{\nu i}|Q_\nu)$

$$= \sum_{s,s'=1}^{q} \sum_{k,k'=1}^{p} e_{\nu,sk}\ e_{\nu,s'k'}\ A_{ss'}(\sum_{i=1}^{\nu} c^*_{k\nu i}\ c^*_{k'\nu i} + o(1))$$

$$= O(1)\ .$$

Using (5.2.24) and (5.2.29) and arguing as in the last part of lemma 5.2.1, we get the result. Q.E.D.

$\underset{\sim}{T}_\nu$ being a stochastic matrix, for actual test construction, it is more convenient to use a real valued function of $\underset{\sim}{T}_\nu$ as a test statistic. As in chapters II, III and IV, a suitable statistic will be a non-negative definite quadratic form in the pq elements of $\underset{\sim}{T}_\nu$ where the discriminant of the quadratic form is the inverse of the matrix $\underset{\sim}{E}_\nu = \underset{\sim}{A} \otimes \underset{\sim}{\Lambda}_\nu = ((e_{\nu,sk,s'k'}))$ (say). [We may note that according to our convention, Var $(\underset{\sim}{T}_\nu' | P_\nu) = \underset{\sim}{A} \otimes \underset{\sim}{\Lambda}_\nu$]. Consider the statistic

$$(5.2.33) \quad S_\nu = \sum_{s=1}^{q} \sum_{s'=1}^{q} \sum_{k=1}^{p} \sum_{k'=1}^{p} T_{\nu,sk} T_{\nu,s'k'} e_\nu^{sk,s'k'} ,$$

where $\underset{\sim}{E}_\nu^{-1} = ((e_\nu^{sk,s'k'})) = \underset{\sim}{A}^{-1} \otimes \underset{\sim}{\Lambda}_\nu^{-1} = ((a^{ss'} \lambda^{kk'}))$

A size-ε test procedure based on the critical function $\psi_1(\underset{\sim}{X}_\nu^*)$ is as follows:

$$(5.2.34) \qquad \psi_1(\underset{\sim}{X}_\nu^*) = \begin{cases} 1 & , \text{ if } S_\nu > S_{\nu,\varepsilon} \\ \delta_{1\nu\varepsilon} & , \text{ if } S_\nu = S_{\nu,\varepsilon} \\ 0 & , \text{ if } S_\nu < S_{\nu,\varepsilon} \end{cases} ,$$

where $S_{\nu,\varepsilon}$ and $\delta_{1\nu\varepsilon}$ are so chosen that $E[\psi_1(\underset{\sim}{X}_\nu^*) | P_\nu] = \varepsilon$. Thus, we get a similar size-ε test.

By virtue of lemmas 5.2.1 and 5.2.2 S_ν is asymptotically χ^2_{pq} under P_ν. Then $S_{\nu,\varepsilon} \to \chi^2_{pq,\varepsilon}$ (upper 100ε% point of χ^2_{pq}),

$\delta_{1\nu\varepsilon} \to 0$ as $\nu \to \infty$. The large-sample test procedure will be as follows:

$$\text{if } S_\nu \begin{cases} \geq \chi^2_{pq,\varepsilon}, & \text{reject } H_0 \text{ in (5.1.2)} \\ < \chi^2_{pq,\varepsilon}, & \text{accept } H_0 . \end{cases}$$

We may also observe that writing $\underset{\sim}{T}^*_\nu = (T_{\nu,11}, \ldots, T_{\nu,1p}, \ldots, T_{\nu,q1}, \ldots, T_{\nu,qp})'$, we get, $S_\nu = \underset{\sim}{T}^{*\prime}_\nu (\underset{\sim}{A} \otimes \underset{\sim}{\Lambda}_\nu)^{-1} \underset{\sim}{T}^*_\nu = \underset{\sim}{T}^{*\prime}_\nu (\underset{\sim}{A}^{-1} \otimes \underset{\sim}{\Lambda}^{-1}_\nu) \underset{\sim}{T}^*_\nu$.

From (5.2.24), we get,

$$(5.2.35) \quad E(\underset{\sim}{T}^*_\nu | Q_\nu) = \begin{bmatrix} \sum_{k=1}^{p} \tau_{1k} (\sum_{i=1}^{\nu} c^*_{1\nu i} c^*_{k\nu i}) A_{11} \\ \vdots \\ \sum_{k=1}^{p} \tau_{1k} (\sum_{i=1}^{\nu} c^*_{p\nu i} c^*_{k\nu i}) A_{11} \\ \vdots \\ \sum_{k=1}^{p} \tau_{qk} (\sum_{i=1}^{\nu} c^*_{1\nu i} c^*_{k\nu i}) A_{qq} \\ \vdots \\ \sum_{k=1}^{p} \tau_{qk} (\sum_{i=1}^{\nu} c^*_{p\nu i} c^*_{k\nu i}) A_{qq} \end{bmatrix}$$

+ a column vector with pq elements each of which $\to 0$ as $\nu \to \infty$.

We can write the first vector in (5.2.35) as

$$\underset{pq\times 1}{\underset{\sim}{B}_\nu} = \widetilde{\underset{\sim}{D}(A)} \, \widetilde{\underset{\sim}{\Lambda}_\nu} \, \underset{\sim}{\tau}^* ,$$

where $\underset{\sim}{\tau}^* = (\tau_{11}, \ldots, \tau_{1p}, \ldots, \tau_{q1}, \ldots, \tau_{qp})'$, $\widetilde{\underset{\sim}{D}(A)} = \underset{\sim}{A} \otimes I_p$ and $\widetilde{\underset{\sim}{\Lambda}_\nu} = I_q \otimes \underset{\sim}{\Lambda}_\nu$. Thus, $\widetilde{\underset{\sim}{D}(A)} \, \widetilde{\underset{\sim}{\Lambda}_\nu} = \underset{\sim}{A} \otimes \underset{\sim}{\Lambda}_\nu = \underset{\sim}{E}_\nu$ i.e. $\underset{\sim}{B}_\nu = \underset{\sim}{E}_\nu \underset{\sim}{\tau}^* \to \underset{\sim}{E}^* \underset{\sim}{\tau}^*$ as $\nu \to \infty$, where $\underset{\sim}{E}^* = \underset{\sim}{A} \otimes \underset{\sim}{\Lambda}$. Thus, $E(\underset{\sim}{T}^*_\nu | Q_\nu) \to \underset{\sim}{E}^* \underset{\sim}{\tau}^*$ as $\nu \to \infty$.

Similarly, from (5.2.29), Var $(\underset{\sim}{T}_\nu^* | Q_\nu)$ = Var $(\underset{\sim}{T}'_\nu | Q_\nu) \to \underset{\sim}{E}^*$ as $\nu \to \infty$.

Using lemma 5.2.2., we get now, under H_ν, S_ν is asymptotically $\chi_p^2(\eta)$, where

$$(5.2.36)\quad \eta = (\underset{\sim}{E}^* \underset{\sim}{\tau}^*)' \underset{\sim}{E}^{*-1} (\underset{\sim}{E}^* \underset{\sim}{\tau}^*) = \underset{\sim}{\tau}^{*\prime} \underset{\sim}{E}^* \underset{\sim}{\tau}^* = \underset{\sim}{\tau}^{*\prime} (\underset{\sim}{A} \otimes \underset{\sim}{\Lambda}) \underset{\sim}{\tau}^*$$

5.3. A Class of Nonparametric Tests.

The unconditional distribution of S_ν depends on the unknown $\Delta_{j_s,s}$ ($j_s = 0, 1, \ldots, \ell_s$, $s = 1, 2, \ldots, p$) even under H_0. However, under a permutation model, S_ν provides a distribution-free test. Since, the vectors $\underset{\sim}{X}_{\nu 1}, \ldots, \underset{\sim}{X}_{\nu\nu}$ are i.i.d. under H_0, they have a joint distribution which remains invariant under $\nu!$ possible permutations of the ν vectors. Thus, in the ν dimensional real space $\mathbb{R}^\nu$, we have a set of $\nu!$ permutationally equiprobable points. Let $\mathcal{P}_\nu$ denote the permutational probability measure defined on this set. Under $\mathcal{P}_\nu$, all the $\nu!$ equally likely realizations have common probability $(\nu!)^{-1}$.

We introduce the following notations:

$$(5.3.1)\quad \sum_{i=1}^{\nu} Z_{\nu i j_s, s} = \nu_{j_s,s} \quad ;$$

$$(5.3.2)\quad \sum_{i=1}^{\nu} Z_{\nu i j_s, s} Z_{\nu i j_{s'}, s'} = \nu_{j_s, j_{s'}, s, s'} \quad ;$$

$$(5.3.3)\quad F_{\nu,0,s} = 0,\ F_{\nu,j_s+1,s} = \nu^{-1} \sum_{m_s=0}^{j_s} \nu_{m_s,s}\ ;$$

$$(5.3.4)\quad \hat{\Delta}_{\nu,j_s,s} = \int_{F_{\nu,j_s,s}}^{F_{\nu,j_s+1,s}} \phi(u)\,du \Bigg/ \int_{F_{\nu,j_s,s}}^{F_{\nu,j_s+1,s}} du \text{ if } \nu_{j_s,s} \neq 0\ ,$$

$$= 0, \text{ otherwise.}$$

$(j_s = 0, 1, \ldots, \ell_s;\ j_{s'} = 0, 1, \ldots, \ell_{s'};\ 1 \le s \neq s' \le q)$.

We may note that $\nu_{j_s,s}$, $\nu_{j_s,j_{s'},s,s'}$, $\hat{\Delta}_{\nu,j_s,s}$

$(0 \le j_s \le \ell_s,\ 0 \le j_{s'} \le \ell_{s'},\ 1 \le s \neq s' \le q)$ remain unaffected by the permutation of the vectors $\underset{\sim}{X}^*_{\nu 1}, \ldots, \underset{\sim}{X}^*_{\nu\nu}$. It follows now immediately that

$$(5.3.5)\quad E(Z_{\nu i j_s,s} | \mathcal{P}_\nu) = \nu_{j_s,s}/\nu\ ;$$

$$(5.3.6)\quad E(Z_{\nu i j_s,s}\, Z_{\nu i j'_s,s} | \mathcal{P}_\nu) = 0\ ;$$

$$(5.3.7)\quad E(Z_{\nu i j_s,s}\, Z_{\nu i' j'_s,s} | \mathcal{P}_\nu) = \frac{\nu_{j_s,s}\,(\nu_{j'_s,s} - \delta_{j_s j'_s})}{\nu(\nu - 1)}\ ,$$

(where $0 \le j_s \neq j'_s \le \ell_s$, $1 \le s \le q$; $1 \le i \neq i' \le \nu$, $\nu \ge 1$; $\delta_{j_s j_{s'}}$ are Kronecker deltas). Also, we get after some simplifications,

$$(5.3.8)\quad E(Z_{\nu i j_s,s}\, Z_{\nu i j_{s'},s'} | \mathcal{P}_\nu) = \nu^{-1} \nu_{j_s,j_{s'},s,s'}\ ;$$

$$(5.3.9)\quad E(Z_{\nu i j_s,s}\, Z_{\nu i' j_{s'},s'} | \mathcal{P}_\nu)$$

$$= \nu^{-1}(\nu - 1)^{-1} \nu_{j_s,s} \nu_{j_{s'},s'} - \nu^{-1}(\nu - 1)^{-1} \nu_{j_s,j_{s'},s,s'}\ ;$$

$(0 \le j_s \le \ell_s,\ 0 \le j_{s'} \le \ell_{s'},\ 1 \le i \ne i' \le \nu,\ \nu \ge 1)$.

Define the permutation test statistics,

$$(5.3.10)\quad U_{\nu,sk} = \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j_s=0}^{\ell_s} Z_{\nu i j_s,s} \hat{\Delta}_{\nu j_s,s}\ ,$$

$(s = 1, 2\ldots, q;\ k = 1, 2,\ldots, p)$. Let

$$(5.3.11)\quad A_{\nu,ss} = \sum_{j_s=0}^{\ell_s} \hat{\Delta}^2_{\nu,j_s,s}(\nu_{j_s,s}/\nu)\ ,$$

$$(5.3.12)\quad A_{\nu,ss'} = \sum_{j_s=0}^{\ell_s} \sum_{j_{s'}=0}^{\ell_{s'}} \hat{\Delta}_{\nu,j_s,s}\hat{\Delta}_{\nu,j_{s'},s'}(\nu_{j_s,j_{s'},s,s'}/\nu)\ ,$$

$(1 \le s \ne s' \le q)$.

Now, after some lengthy algebraic manipulations, we get,

$$(5.3.13)\quad E(U_{\nu,sk}|\mathcal{P}_\nu) = 0;$$

$$(5.3.14)\quad E(U_{\nu,sk}U_{\nu,s'k'}|\mathcal{P}_\nu) = \nu(\nu-1)^{-1}\left(\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i}\right)A_{\nu,ss'}\ ;$$

$(1 \le k,\ k' \le p,\ 1 \le s,\ s' \le q)$.

Writing

$$(5.3.15)\quad \underset{\sim}{U}_\nu = ((U_{\nu,sk}))_{q\times p}\ ,$$

we get,

$$(5.3.16)\quad E(\underset{\sim}{U}_\nu|\mathcal{P}_\nu) = \underset{\sim}{0},\ \mathrm{Var}(\underset{\sim}{U}_\nu|\mathcal{P}_\nu) = \underset{\sim}{\Lambda}_\nu \otimes \underset{\sim}{A}_\nu\ ,$$

where $\underset{\sim}{A}_\nu = ((A_{\nu,ss'}))_{q\times q}$.

The following test statistic (which is a quadratic form in $U_{\nu,sk}$'s) is proposed:

$$(5.3.17)\quad M_\nu = \sum_{s=1}^{q}\sum_{s'=1}^{q}\sum_{k=1}^{p}\sum_{k'=1}^{p} U_{\nu,sk}U_{\nu,s'k'}K_\nu^{sk,s'k'} ,$$

where $\underset{\sim}{K}_\nu = \underset{\sim}{\Lambda}_\nu \otimes \underset{\sim}{A}_\nu$ and

$\underset{\sim}{K}_\nu^g = \underset{\sim}{\Lambda}_\nu^{-1} \otimes \underset{\sim}{A}_\nu^g = ((K_\nu^{sk,s'k'}))$ denotes a generalized inverse of $\underset{\sim}{K}_\nu$ ($\underset{\sim}{A}_\nu^g$ is a generalized inverse of $\underset{\sim}{A}_\nu$). We may also remark that $\underset{\sim}{A}_\nu$ (as we shall show later on) is p.d. in probability.

Now a (permutation) test procedure based on a critical function $\psi_2(\underset{\sim}{X}_\nu^*)$ can be formulated as follows:

$$(5.3.18)\quad \psi_2(\underset{\sim}{X}_\nu^*) = \begin{cases} 1 , & \text{if } M_\nu > M_{\nu,\varepsilon} \\ \delta_{2\nu\varepsilon}, & \text{if } M_\nu = M_{\nu,\varepsilon} \\ 0 , & \text{if } M_\nu < M_{\nu,\varepsilon} , \end{cases}$$

where $\delta_{2\nu\varepsilon}$ and $M_{\nu,\varepsilon}$ are chosen in such a way that $E[\psi_2(\underset{\sim}{X}_\nu^*)|\mathcal{P}_\nu] = \varepsilon$. Then, $E[\psi_2(\underset{\sim}{X}_\nu^*)|P_\nu] = \varepsilon$, so that we get a similar size ε test.

The limitations of the permutation test procedure for moderately large sample sizes have already been discussed in chapter II and reemphasized in chapter III.

We next want to find the asymptotic distribution of M_ν. With this end in view, we first prove the following lemma:

LEMMA 5.3.1. Under the model (5.1.1), the assumptions (3.1.2)-(3.1.5) and (5.1.3), $\underset{\sim}{A}_\nu \to \underset{\sim}{A}$ in P_ν- (or in Q_ν-) probability as $\nu \to \infty$.

Proof. Lemma 3.3.1 gives $A_{\nu,ss} \to A_{ss}$ in P_ν- (or in Q_ν-) probability as $\nu \to \infty$. (We can work with univariate marginal distributions in this case). So, only we need to show that the elements $A_{\nu,ss'} \to A_{ss'}$ (in P_ν- or in Q_ν-probability, where $1 \le s \ne s' \le q$) as $\nu \to \infty$. We can write

$$(5.3.19)\quad A_{\nu,ss'} - A_{ss'}$$

$$= \sum_{j_s=0}^{\ell_s} \sum_{j_{s'}=0}^{\ell_{s'}} [\hat{\Delta}_{\nu,j_s,s}\hat{\Delta}_{\nu,j_{s'},s'}(\nu_{j_s,j_{s'},s,s'}/\nu)$$

$$- \Delta_{j_s,s}\Delta_{j_{s'},s'}P_{j_s,j_{s'},s,s'}]$$

$$= \sum_{j_s=0}^{\ell_s} \sum_{j_{s'}=0}^{\ell_{s'}} (\hat{\Delta}_{\nu,j_s,s} - \Delta_{j_s,s})\hat{\Delta}_{\nu,j_{s'},s'}(\nu_{j_s,j_{s'},s,s'}/\nu)$$

$$+ \sum_{j_s=0}^{\ell_s} \sum_{j_{s'}=0}^{\ell_{s'}} (\hat{\Delta}_{\nu,j_{s'},s'} - \Delta_{j_{s'},s'})\Delta_{j_s,s}(\nu_{j_s,j_{s'},s,s'}/\nu)$$

$$+ \sum_{j_s=0}^{\ell_s} \sum_{j_{s'}=0}^{\ell_{s'}} \Delta_{j_s,s}\Delta_{j_{s'},s'}[(\nu_{j_s,j_{s'},s,s'}/\nu) - P_{j_s,j_{s'},s,s'}] .$$

Applying Cauchy-Schwarz inequality to the square of the first term on the right hand side of (5.3.19), we get,

$$(5.3.20)\quad [\sum_{j_s=0}^{\ell_s} \sum_{j_{s'}=0}^{\ell_{s'}} (\hat{\Delta}_{\nu,j_s,s} - \Delta_{j_s,s})\hat{\Delta}_{\nu,j_{s'},s'}(\nu_{j_s,j_{s'},s,s'}/\nu)]^2$$

$$\le [\sum_{j_s=0}^{\ell_s} (\hat{\Delta}_{\nu,j_s,s} - \Delta_{j_s,s})^2(\nu_{j_s,s}/\nu)]$$

$$\cdot [\sum_{j_{s'}=0}^{\ell_{s'}} \hat{\Delta}^2_{\nu,j_{s'},s'}(\nu_{j_{s'},s'}/\nu)]$$

$$\le [\sum_{j_s=0}^{\ell_s} (\hat{\Delta}_{\nu,j_s,s} - \Delta_{j_s,s})^2(\nu_{j_s,s}/\nu)]A^2(F_{s'})$$

Now, as in lemma 3.3.2, we can show here that $\hat{\Delta}_{\nu,j_s,s} - \Delta_{j_s,s} = o_p(1)$ (either under P_ν- or under Q_ν-) for each $j_s = 0, 1,\dots, \ell_s$. Since

$A^2(F_{s'}) < \infty$ and the summation of the right hand side of (5.3.20) involves only a finite number of terms, we get the square of the first term (and hence the first term) on the right hand side of (5.3.19) is $o_p(1)$. The same technique (i.e. Schwarz inequality) enables us to write

(5.3.21) $[2^{nd}$ term on the right hand side of $(5.3.19)]^2$

$$\le \left[\sum_{j_{s'}=0}^{\ell_{s'}} (\hat{\Delta}_{\nu,j_{s'},s'} - \Delta_{j_{s'},s'})^2 (\nu_{j_{s'},s'}/\nu)\right] \cdot \left[\sum_{j_s=0}^{\ell_s} \Delta^2_{j_s,s} (\nu_{j_s,s}/\nu)\right] .$$

As before, the first factor on the right hand side of (5.3.21) is $o_p(1)$ (either under P_ν- or Q_ν-probability). The second factor can be written as

$$\sum_{j_s=0}^{\ell_s} \Delta^2_{j_s,s} P_{j_s,s} + \sum_{j_s=0}^{\ell_s} \Delta^2_{j_s,s} [(\nu_{j_s,s}/\nu) - P_{j_s,s}] .$$

The first term in the above is uniformly bounded in j_s by $A^2(F_s)$, while we can show as in chapter III, $\nu_{j_s,s}/\nu - P_{j_s,s} = O_p(\nu^{-\frac{1}{2}})$ either in P_ν (or in Q_ν-probability). So, the second term on the right hand side of (5.3.19) is $o_p(1)$. Write the third term as

$$\sum\sum_{(j_s,j_{s'})\varepsilon J_{ss'}} \Delta_{j_s,s} \Delta_{j_{s'},s'} [(\nu_{j_s,j_{s'},s,s'}/\nu) - P_{j_s,j_{s'},s,s'}] .$$

Use the fact for each pair $(j_s, j_{s'}) \varepsilon J_{ss'}$, $\Delta_{j_s,s}$, $\Delta_{j_{s'},s'}$ are bounded valued, while $(\nu_{j_s,j_{s'},s,s'}/\nu) - P_{j_s,j_{s'},s,s'} \to 0$ either in P_ν- or in Q_ν-probability as $\nu \to \infty$. The last statement can be proved by writing

$$(\nu_{j_s,j_{s'},s,s'}/\nu) - P_{j_s,j_{s'},s,s'}$$

$$= (F_{\nu,j_s+1,j_{s'}+1,s,s'} - F_{j_s+1,j_{s'}+1,s,s'})$$

$$- (F_{\nu,j_s+1,j_{s'},s,s'} - F_{j_s+1,j_{s'},s,s'})$$

$$- (F_{\nu,j_s,j_{s'}+1,s,s'} - F_{j_s,j_{s'}+1,s,s'})$$

$$+ (F_{\nu,j_s,j_{s'},s,s'} - F_{j_s,j_{s'},s,s'}) ,$$

where

$$F_{\nu,j_s+1,j_{s'}+1,s,s'} = \nu^{-1} \sum_{m_s=0}^{j_s} \sum_{m_{s'}=0}^{j_{s'}} \nu_{m_s,m_{s'},s,s'},$$

($j_s = 0, 1, \ldots, \ell_s$; $j_{s'} = 0, 1, \ldots, \ell_s'$, $1 \leq s \neq s' \leq q$), and then shwoing each term in the above sum is $o_p(1)$ by using the Chebyshev inequality and the fact that under Q_ν, $H_{\nu,j_s,j_{s'},s,s'} - F_{j_s,j_{s'},s,s'} = o(1)$, ($j_s = 0, 1, \ldots, \ell_s+1$; $j_{s'} = 0, 1, \ldots, \ell_{s'}+1$; $1 \leq s \neq s' \leq q$). Hence the third term on the right hand side of (5.3.19) is $o_p(1)$, the summation involving only a finite number of terms.

COROLLARY 5.3.1. <u>Under the conditions of the above lemma</u> $\underset{\sim}{\Lambda}_\nu \otimes \underset{\sim}{A}_\nu \to \underset{\sim}{\Lambda} \otimes \underset{\sim}{A}$ <u>in</u> P_ν- <u>or in</u> Q_ν-<u>probability as</u> $\nu \to \infty$.

<u>Proof</u>. Write $\underset{\sim}{\Lambda}_\nu \otimes \underset{\sim}{A}_\nu - \underset{\sim}{\Lambda} \otimes \underset{\sim}{A} = \underset{\sim}{\Lambda}_\nu \otimes (\underset{\sim}{A}_\nu - \underset{\sim}{A}) + (\underset{\sim}{\Lambda}_\nu - \underset{\sim}{\Lambda}) \otimes \underset{\sim}{A}$. The elements of Λ_ν are all bounded above by 1, while

$\det(\underset{\sim}{A}) \leq \prod_{s=1}^{q} A_{ss} \leq \prod_{s=1}^{q} A^2(F_s) < \infty$. We get the result by using lemma 5.3.1 and the assumption that $\underset{\sim}{A}_\nu \to \underset{\sim}{A}$ as $\nu \to \infty$.

Our next theorem gives the permutation limit distribution of $\underset{\sim}{U}_\nu$.

THEOREM 5.3.1. Under the model (5.1.1), if the assumptions (3.1.2)-(3.1.5), (5.1.3) and (5.2.14) hold, then the joint distribution of the elements of $\underset{\sim}{U}_\nu$ is under $\mathcal{P}_\nu$ asymptotically $N_{pq}(\underset{\sim}{0}, \underset{\sim}{\Lambda} \otimes \underset{\sim}{A})$.

Proof. The proof proceeds along the lines of Puri and Sen [31]. For any given $\underset{\sim}{m} = (m_1, \ldots, m_p)'$, define,

$$\underset{\sim}{U}_{\nu 00} = \left(\sum_{k=1}^{p} m_k U_{\nu 1k}, \ldots, \sum_{k=1}^{p} m_k U_{\nu qk}\right)'.$$

We can write,

$$\sum_{k=1}^{p} m_k U_{\nu sk} = \sum_{i=1}^{\nu} \left(\sum_{k=1}^{p} m_k c^*_{k\nu i}\right)\left(\sum_{j_s=0}^{\ell_s} Z_{\nu i j_s, s} \hat{\Delta}_{\nu, j_s, s}\right)$$

$$= U_{\nu s 00} \text{ (say)}, \quad s = 1, 2, \ldots, q.$$

We have to show that $\underset{\sim}{U}_{\nu 00}$ is asymptotically $N_q(\underset{\sim}{0}, (\underset{\sim}{m}'\underset{\sim}{\Lambda}\underset{\sim}{m})\underset{\sim}{A})$. To prove this, we consider any linear combination $\underset{\sim}{e}'\underset{\sim}{U}_{\nu 00} = \sum_{s=1}^{q} e_s U_{\nu s 00}$, where, $\underset{\sim}{e} = (e_1, \ldots, e_q)'$.

$$\underset{\sim}{e}'\underset{\sim}{U}_{\nu 00} = \sum_{i=1}^{\nu} \left(\sum_{k=1}^{p} m_k c^*_{k\nu i}\right)\left(\sum_{s=1}^{q} e_s \sum_{j_s=0}^{\ell_s} Z_{\nu i j_s, s} \hat{\Delta}_{\nu, j_s, s}\right).$$

It is easy to see, in view of (3.1.4) and (3.1.5) that

$$\max_{1 \leq i \leq \nu} \left(\sum_{k=1}^{p} m_k c^*_{k\nu i}\right)^2 = o(1), \text{ and } \sum_{i=1}^{\nu}\left(\sum_{k=1}^{p} m_k c^*_{k\nu i}\right)^2 = \underset{\sim}{m}'\underset{\sim}{\Lambda}_\nu\underset{\sim}{m} \to \underset{\sim}{m}'\underset{\sim}{\Lambda}\underset{\sim}{m} \text{ as } \nu \to \infty,$$

i.e. $\sum_{i=1}^{\nu}(\sum_{k=1}^{p} m_k c^*_{k\nu i})^2 = O(1)$. Hence, the (double) sequence $\{\sum_{k=1}^{p} m_k c^*_{k\nu i}\}$ satisfies the Noether condition. Also,

$$\nu^{-1}\sum_{i=1}^{\nu}\{\sum_{s=1}^{q} e_s \sum_{j_s=0}^{\ell_s} Z_{\nu i j_s,s}\hat{\Delta}_{\nu,j_s,s}\}^2$$

$$= \sum_{s=1}^{q}\sum_{s'=1}^{q} e_s e_{s'} \sum_{j_s=0}^{\ell_s}\sum_{j_{s'}=0}^{\ell_{s'}} \hat{\Delta}_{\nu,j_s,s}\hat{\Delta}_{\nu,j_{s'},s'}(\nu_{j_s,j_{s'},s,s'}/\nu)$$

$\rightarrow \underset{\sim}{e}'\underset{\sim}{A}\underset{\sim}{e}$ in P_ν- (or in Q_ν-) probability as $\nu \rightarrow \infty$.

Also, we can show as in (2.3.19)-(2.3.21) that

$$\lim_{\nu\to\infty} k_\nu/\nu = 0 \Rightarrow \lim_{\nu\to\infty} \nu^{-1}[\max_{1\le i_1<\dots<i_{k_\nu}\le\nu} \sum_{\alpha=1}^{k_\nu} W_\nu^2(i_\alpha)] = 0,$$

where, $W_\nu(i) = \sum_{s=1}^{q} e_s \sum_{j_s=0}^{\ell_s} Z_{\nu i j_s,s}\hat{\Delta}_{\nu j_s,s}$,

so that $\overline{W}_\nu = \nu^{-1}\sum_{i=1}^{\nu} W_\nu(i) = \sum_{s=1}^{q} e_s \int_0^1 \phi_s(u)\,du = 0$.

Now, use Theorem 4.2 of [16] along with (5.3.16) to get the result.

Q.E.D.

The next lemma will establish the "asymptotic equivalence" of the permutation test statistics to the parametric test statistics.

LEMMA 5.3.2. <u>Under</u> (5.1.1), (3.1.2)-(3.1.5), (5.1.3) <u>and</u> (5.2.14),

$$\underset{\sim}{U}_\nu - \underset{\sim}{T}_\nu \rightarrow \underset{q\times p}{\underset{\sim}{0}}$$ <u>in</u> P_ν- (<u>or in</u> Q_ν-) <u>probability</u>.

<u>Proof</u>. Write, $U_{\nu,sk} - T_{\nu,sk} = \sum_{i=1}^{\nu} c^*_{k\nu i}\sum_{j_s=0}^{\ell_s}(\hat{\Delta}_{\nu,j_s,s} - \Delta_{j_s,s})Z_{\nu i j_s,s}$,

(s = 1, 2,..., q, k = 1, 2,..., p), and proceed as in lemma 3.3.2 to show $U_{\nu,sk} - T_{\nu,sk} \to 0$ in P_ν- or in Q_ν- probability for each pair (s, k), where s = 1, 2,..., q and k = 1, 2,..., p. Then using Bonferroni inequality, we get the result, Q.E.D.

As an immediate consequence of lemmas 5.3.1 and 5.3.2. we get the permutation test statistic M_ν defined in (5.3.17) is "asymptotically equivalent" to the parametric test statistic S_ν defined in (5.2.33). Thus M_ν and S_ν have asymptotically the same distribution either under H_0 or under H_ν. Hence the test procedure based on M_ν given in (5.3.18) is asymptotically power-equivalent to the one based on S_ν given in (5.2.34).

5.4. A Multivariate Generalization of Tests in Chapter IV.

We have the model (5.1.1) and we want to test in this section

$$(5.4.1) \qquad H_0: \underset{q\times(p+1)}{\underset{\sim}{\beta}^*} = \underset{q\times(p+1)}{\underset{\sim}{0}},$$

where,

$$(5.4.2) \qquad \underset{\sim}{\beta}^* = \begin{bmatrix} \beta_{10} & \beta_{11} & \cdots & \beta_{1p} \\ \beta_{20} & \beta_{21} & \cdots & \beta_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{q0} & \beta_{q1} & \cdots & \beta_{qp} \end{bmatrix},$$

against the set of alternatives $\underset{\sim}{\beta}^* \neq \underset{\sim}{0}$. The assumptions (3.1.2)-(3.1.5) hold in this case also, although we may remark as in chapter IV that the assumption (3.1.2) is unnecessary here once we assume that the matrices $\underset{\sim}{\Lambda}_{\nu 0} = \underset{(p+1)\times(p+1)}{\left(\left(\sum_{i=1}^{\nu} c^*_{k\nu i}\, c^*_{k'\nu i}\right)\right)}$ and $\underset{\sim}{\Lambda}_0 = \underset{(p+1)\times(p+1)}{((\lambda_{kk'}))}$ are p.d. where $\underset{\sim}{\Lambda}_{\nu 0} \to \underset{\sim}{\Lambda}_0$

as $\nu \to \infty$. We may note that as in chapter IV, $c_{0\nu i} = 1$, $C_{0\nu} = \nu$, $c^*_{0\nu i} = \nu^{-\frac{1}{2}}$.

In the present situation, we assume in addition that the q-variate d.f. F is diagonally symmetric about the zero vector, that is if $\underset{\sim}{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_p)'$ has d.f. F, so has $-\underset{\sim}{\varepsilon}$. This implies that any ε_i has marginally the same distribution as $-\varepsilon_i$ $(i = 1, 2, \ldots, p)$ and any pair $(\varepsilon_i, \varepsilon_j)$ has the same joint distribution as $(-\varepsilon_i, -\varepsilon_j)$, $1 \le i \ne j \le p$.

We keep the notations $F_{\nu i}(\underset{\sim}{x}')$, $F_{\nu i,s}(x)$, $F_{\nu i,ss'}(x,y)$, $F(\underset{\sim}{x}')$, $F_s(x)$, $F_{ss'}(x,y)$, $H_\nu(\underset{\sim}{x})$, $H_{\nu,s}(x)$, $H_{\nu,ss'}(x,y)$ $(1 \le s \ne s' \le q)$ same as in section 5.1. However, the sequence of alternatives to be considered here is as follows:

$$(5.4.3) \quad H_\nu : \underset{\sim}{\beta}^* = \underset{\sim}{\beta}^*_\nu = ((\tau_{sk}/C_{k\nu})) = \underset{\sim}{\tau}_0 \underset{\sim}{D}_\nu \ ,$$

$$\text{where } \underset{\sim}{D}_{\nu 0} = \begin{pmatrix} 1/C_{0\nu} & & & \underset{\sim}{0} \\ & 1/C_{1\nu} & & \\ & & \ddots & \\ \underset{\sim}{0} & & & 1/C_{p\nu} \end{pmatrix}, \quad C_{0\nu} = \nu^{1/2} \ ,$$

$$\underset{\sim}{\tau}_0 = \underset{q\times(p+1)}{((\tau_{sk}))} \ .$$

For notational convenience, we consider here q-dimensional rectangles in the real q-space $\mathbb{R}^q$ defined as

$$(5.4.4) \quad \bigcap_{s=1}^{q} I_{j_s,s} = \bigcap_{s=1}^{q} \{x_s \mid a_{j_s-\frac{1}{2},s} < x_s \le a_{j_s+\frac{1}{2},s}\} \ ,$$

$(j_s = -\ell_s, \ldots, -1, 0, 1, \ldots, \ell_s$,

$-\infty = a_{-\ell_s - \frac{1}{2}} < \ldots < a_{-\frac{1}{2}} < a_{\frac{1}{2}} < \ldots \; a_{\ell_s + \frac{1}{2}} = \infty$ denote a finite set of ordered points on the real line with $a_{-j_s,s} = -a_{j_s,s}$; let $a_{0,s} = 0$; $s = 1, 2, \ldots, q$). The observable stochastic matrix is

$$\text{(5.4.5)} \qquad \underset{\sim}{X}^*_\nu = (\underset{\sim}{X}^*_{\nu 1}, \ldots, \underset{\sim}{X}^*_{\nu\nu}), \; \nu \geq 1,$$

where $\underset{\sim}{X}^*_{\nu i} = ((X^{(1)}_{\nu i})^*, \ldots, (X^{(q)}_{\nu i})^*)'$, with

$$\text{(5.4.6)} \qquad (X^{(s)}_{\nu i})^* = \sum_{j_s=-\ell_s}^{\ell_s} I_{j_s,s} \, Z_{\nu i j_s,s} \; ,$$

$(s = 1, 2, \ldots, q;\ i = 1, \ldots, \nu,\ \nu \geq 1)$, and

$$\text{(5.4.7)} \qquad Z_{\nu i j_s,s} = \begin{cases} 1 \; , & \text{if } X^{(s)}_{\nu i} \in I_{j_s,s} \\ 0 \; , & \text{otherwise} \; , \end{cases}$$

$(j_s = -\ell_s, \ldots, -1, 0, 1, \ldots, \ell_s,\ s = 1, \ldots, q,\ i = 1, 2, \ldots, \nu,\ \nu \geq 1)$.

For the testing problem considered here, a class of permutationally distribution-free tests has been proposed which is a multivariate extension of the permutation argument in chapter IV. We shall discuss more about it later.

First, we consider a class of parametric tests. For this, we need introduce the following notations:

$$\text{(5.4.8)} \qquad F_{\nu i, j_s, s} = \Pr\cdot(X^{(s)}_{\nu i} \leq a_{j_s,s} \,|\, H_\nu)$$

$$= F_s(a_{j_s - \frac{1}{2}, s} - \sum_{k=0}^{p} \tau_{sk} c^*_{k\nu i})$$

$$(5.4.9)\quad F_{\nu i,j_s,j_{s'},s,s'} = \Pr\cdot(X_{\nu i}^{(s)} \le a_{j_s,s},\ X_{\nu i}^{(s')} \le a_{j_{s'},s'}|H_\nu)$$

$$= F_{ss'}(a_{j_s-\frac{1}{2},s} - \sum_{k=0}^{p} \tau_{sk}c^*_{k\nu i},\ a_{j_{s'}-\frac{1}{2},s'} - \sum_{k=0}^{p} \tau_{s'k}c^*_{k\nu i}),$$

$(-\ell_s \le j_s \le \ell_s+1;\ -\ell_{s'} \le j_{s'} \le \ell_{s'}+1;\ 1 \le s \ne s' \le q;\ \ i = 1, 2,\dots, \nu,$ $\nu \ge 1)$. Let

$$(5.4.10)\qquad H_{\nu,j_s,s} = \nu^{-1} \sum_{i=1}^{\nu} F_{\nu i,j_s,s}\ ;$$

$$(5.4.11)\qquad H_{\nu,j_s,j_{s'},s,s'} = \nu^{-1} \sum_{i=1}^{\nu} F_{\nu i,j_s,j_{s'},s,s'}$$

$(-\ell_s \le j_s \le \ell_s+1;\ -\ell_{s'} \le j_{s'} \le \ell_{s'}+1;\ 1 \le s \ne s' \le q;\ \nu \ge 1)$.

Also, let

$$(5.4.12)\quad F_{j_s,s} = F_s(a_{j_s-\frac{1}{2},s}) = \Pr\cdot(X_{\nu i}^{(s)} \le a_{j_s,s}|H_0);$$

$$(5.4.13)\quad F_{j_s,j_{s'},s,s'} = F_{ss'}(a_{j_s-\frac{1}{2},s},\ a_{j_{s'}-\frac{1}{2},s'})$$

$$= \Pr\cdot(X_{\nu i}^{(s)} \le a_{j_s,s},\ X_{\nu i}^{(s')} \le a_{j_{s'},s'}|H_0);$$

$(-\ell_s \le j_s \le \ell_s+1;\ -\ell_{s'} \le j_{s'} \le \ell_{s'}+1;\ 1 \le s \ne s' \le q)$.

Define

$$(5.4.14)\qquad P_{\nu i,j_s,s} = F_{\nu i,j_s+1,s} - F_{\nu i,j_s,s}\ ,$$

The notations (5.1.17)-(5.1.20) are the same except that j_s ranges from $-\ell_s$ to ℓ_s for all $s = 1, 2,\dots, q$.

The "score" functions $\phi_s(u)$ are the same as in (5.2.1). $\Delta_{j_s,s}$ are the same as in (5.2.2). We introduce another class of "score" functions $\psi_s(u)$ defined by

$$(5.4.15) \qquad \psi_s(u) = -\frac{f_s'(F_s^{-1}(\frac{1+u}{2}))}{f_s(F_s^{-1}(\frac{1+u}{2}))}, \ 0 < u < 1, \ 1 \le s \le q.$$

Further, let

$$(5.4.16) \qquad F^*_{j_s,s} = F_{j_s,s} - F_{-j_s+1,s} = 2F_{j_s,s} - 1 \, (j_s = 1, 2, \ldots, \ell_s+1),$$

$$(5.4.17) \qquad P^*_{j_s,s} = F^*_{j_s+1,s} - F^*_{j_s,s} = 2P_{j_s,s} \; (j_s = 1, 2, \ldots, \ell_s),$$

$$(5.4.18) \qquad \Delta^*_{j_s,s} = \int_{F^*_{j_s,s}}^{F^*_{j_s+1,s}} \psi(u)\,du \Big/ \int_{F^*_{j_s,s}}^{F^*_{j_s+1,s}} du$$

$$= \int_{\frac{1+F^*_{j_s,s}}{2}}^{\frac{1+F^*_{j_s+1,s}}{2}} \phi(u)\,du \Big/ \int_{F^*_{j_s,s}}^{F^*_{j_s+1,s}} du$$

$$= \Delta_{j_s,s}\,, \ j_s = 1, \ldots, \ell_s; \ 1 \le s \le q\,.$$

We may note that $\Delta_{-j_s,s} = -\Delta_{j_s,s}$, $P_{j_s,s} = P_{-j_s,s}$ $(j_s = 1, \ldots, \ell_s)$, $\Delta_{0,s} = 0$, for all $s = 1, \ldots, q$. Let

$$(5.4.19) \qquad A_0^2(F_s, \{I_{j_s}\}) = A_{0ss} = \sum_{j_s=-\ell_s}^{\ell_s} \Delta^2_{j_s,s} P_{j_s,s} = 2 \sum_{j_s=1}^{\ell_s} \Delta^2_{j_s,s} P_{j_s,s}$$

$$(5.4.20)\quad A_0(F_s,\ F_{s'},\{I_{j_s}\}\ ,\ \{I_{j_{s'}}\}) = A_{0ss'}$$

$$= \sum_{j_s=-\ell_s}^{\ell_s}\ \sum_{j_{s'}=-\ell_{s'}}^{\ell_{s'}} \Delta_{j_s,s}\Delta_{j_{s'},s'}P_{j_s,j_{s'},s,s'}\ .$$

Now, $P_{j_s,j_{s'},s,s'} = P_{(-j_s),(-j_{s'}),s,s'}$,

and, $P_{(-j_s),j_{s'},s,s'} = P_{(j_s),(-j_{s'}),s,s'}$

$(j_s = 0,\ 1,\ldots,\ \ell_s;\ j_{s'} = 0,\ 1,\ldots,\ \ell_{s'};\ 1 \le s \ne s' \le q)$.

Hence, from (5.4.20), we can rewrite

$$A_{0ss'} = 2\sum_{j_s=1}^{\ell_s}\ \sum_{j_{s'}=1}^{\ell_{s'}} \Delta_{j_s,s}\Delta_{j_{s'},s'}[P_{j_s,j_{s'},s,s'} - P_{j_s,(-j_{s'}),s,s'}]$$

$$(1 \le s \ne s' \le q)\ .$$

(5.4.21) Assume $A_0 = \underset{q(p+1)\times q(p+1)}{((A_{0ss'}))}$ is p.d.

We introduce the statistics

$$(5.4.22)\quad T_{\nu,sk0} = \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j_s=-\ell_s}^{\ell_s} Z_{\nu ij_s,s}\Delta_{j_s,s}$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i} \sum_{j_s=1}^{\ell_s}[Z_{\nu ij_s,s} - Z_{\nu i(-j_s),s}]\Delta_{j_s,s}$$

$$= \sum_{i=1}^{\nu} c^*_{k\nu i}\Big(\sum_{j_s=1}^{\ell_s} U_{\nu ij_s,s}\Delta_{j_s,s}\Big)\operatorname{sgn} X^{(s)}_{\nu i}\ ,$$

where,

$$(5.4.23)\quad U_{\nu ij_s,s} = \begin{cases} 1\ , & \text{if } |X^{(s)}_{\nu i}| \in I_{j_s,s} \\ 0\ , & \text{otherwise} \end{cases}\ ,$$

$(j_s = 1, 2, \ldots, \ell_s)$ and,

$$(5.4.24)\quad \operatorname{sgn} X_{\nu i}^{(s)} = \begin{cases} 1, & \text{if } X_{\nu i}^{(s)} > 0 \\ 0, & \text{if } X_{\nu i}^{(s)} = 0 \\ -1, & \text{if } X_{\nu i}^{(s)} < 0 \end{cases},$$

$(s = 1, 2, \ldots, q;\ k = 1, 2, \ldots, p;\ \nu \geq 1)$. Let P_ν and Q_ν denote respective distributions of $\underset{\sim}{X}_\nu^*$ under $H_0 \equiv H_{0\nu}$ and H_ν. The following two results (similar as lemma 5.2.1 and 5.2.2) are immediate:

LEMMA 5.4.1. <u>Under</u> P_ν, <u>the model</u> (5.1.1), <u>and the assumptions</u> (3.1.2)-(3.1.5), (5.1.3) <u>and the diagonal symmetry of the q-variate</u> <u>d.f.</u> F, <u>the joint distribution of the elements of</u>

$$(5.4.25)\qquad \underset{\sim}{T}_{\nu 0} = \underset{q\times(p+1)}{((T_{\nu,sk0}))}$$

<u>is asymptotically</u> $N_{(p+1)q}\ (\underset{\sim}{0}, \underset{\sim}{A}_0 \otimes \underset{\sim}{A}_0)$.

LEMMA 5.4.2. <u>Under</u> Q_ν, <u>the model</u> (5.1.1) <u>and the same assumptions as</u> <u>in lemma</u> 5.4.1, <u>the joint distribution of the elements of</u> $\underset{\sim}{T}_\nu$ <u>is</u> <u>asymptotically</u> $N_{(p+1)q}\ (\underset{\sim}{D}(A)\underset{\sim}{\tau}_0\underset{\sim}{A}_0, \underset{\sim}{A}_0 \otimes \underset{\sim}{A}_0)$, $\underset{\sim}{D}(A)$ <u>being defined in</u> (5.2.21).

We may remark at this stage that for testing the null hypothesis in (5.4.1), the test statistic we use is a quadratic form in the elements of $\underset{\sim}{T}_{\nu 0}$. The test statistic is

$$(5.4.26)\quad S_{\nu 0} = \sum_{s=1}^{q}\ \sum_{s'=1}^{q}\ \sum_{k=0}^{p}\ \sum_{k'=0}^{p} T_{\nu,sk0}T_{\nu,s'k'0}e_{\nu 0}^{sk,s'k'},$$

where, $\underset{\sim}{E}_{\nu 0} = \underset{\sim}{A} \otimes \underset{\sim}{A}_{\nu 0} = ((e_{\nu 0,sk,s'k'}))$ and $\underset{\sim}{E}_{\nu 0}^{-1} = ((e_{\nu 0}^{sk,s'k'}))$.

The test procedure based on the statistic $S_{\nu 0}$ is specified by the critical function $\psi_3(X_\nu^*)$, where

$$(5.4.27)\qquad \psi_3(\underset{\sim}{X}_\nu^*) = \begin{cases} 1 & , \text{ if } S_{\nu 0} > S_{\nu 0,\varepsilon} \\ \delta_{3\nu\varepsilon} & , \text{ if } S_{\nu 0} = S_{\nu 0,\varepsilon} \\ 0 & , \text{ if } S_{\nu 0} < S_{\nu 0,\varepsilon} \end{cases},$$

where $S_{\nu 0,\varepsilon}$, $\delta_{3\nu\varepsilon}$ are chosen in such a way that $E[\psi_3(\underset{\sim}{X}_\nu^*)|P_\nu] = \varepsilon$, so that the test is a similar size ε test.

From lemmas 5.4.1 and 5.4.2, it is easy to see also that under H_0, $S_{\nu 0}$ is asymptotically $\chi^2_{q(p+1)}$, while under H_ν, it is $\chi^2_{q(p+1)}(\eta_0)$, where,

$$(5.4.28)\qquad \eta_0 = \underset{\sim}{\tau}_{00}'(\underset{\sim}{A}_0 \otimes \underset{\sim}{A}_0)\underset{\sim}{\tau}_{00},$$

where $\underset{\sim}{\tau}_{00} = (\tau_{01},\ldots, \tau_{0p}, \tau_{10},\ldots, \tau_{1p},\ldots, \tau_{q0} \ldots \tau_{qp})'$.

However the above test is not distribution-free. To obtain a (conditionally) distribution-free test, we use once again a permutation argument. First, the following notations are introduced:

$$(5.4.29)\qquad X_{\nu i0}^{(s)} = \sum_{j_s=1}^{\ell_s} I'_{j_s,s} U_{\nu i j_s,s},$$

where $I'_{j_s,s} = I_{j_s,s} \cup I_{(-j_s),s}$,

$(j_s = 1, 2,\ldots, \ell_s;\ s = 1, 2,\ldots, q;\ i = 1, 2,\ldots, \nu,\ \nu \geq 1)$.

$$(5.4.30)\qquad \underset{\sim}{X}_{\nu 0}^* = \begin{bmatrix} X_{\nu 10}^{(1)} & \ldots & X_{\nu\nu 0}^{(1)} \\ \ldots & \ldots & \ldots \\ X_{\nu 10}^{(s)} & \ldots & X_{\nu\nu 0}^{(s)} \end{bmatrix};$$

(5.4.31) $$\operatorname{sgn} \underset{\sim}{X}^*_{\nu 0} = \begin{bmatrix} \operatorname{sgn} X^{(1)}_{\nu 1} & \cdots & \operatorname{sgn} X^{(1)}_{\nu\nu} \\ \cdots & \cdots & \cdots \\ \operatorname{sgn} X^{(q)}_{\nu 1} & \cdots & \operatorname{sgn} X^{(q)}_{\nu\nu} \end{bmatrix} ;$$

(5.4.32) $$\sum_{i=1}^{\nu} Z_{\nu i j_s, s} = \nu_{j_s, s} \quad (j_s = -\ell_s, \ldots, -1, 0, 1, \ldots, \ell_s);$$

(5.4.33) $$\sum_{i=1}^{\nu} Z_{\nu i j_s, s} Z_{\nu i j_{s'}, s'} = \nu_{j_s, j_{s'}, s, s'}$$

$$(-\ell_s \le j_s \le \ell_s;\ -\ell_{s'} \le j_{s'} \le \ell_{s'});$$

(5.4.34) $$F_{\nu, -\ell_s, s} = 0, \quad F_{\nu, j_s+1, s} = \nu^{-1} \sum_{m_s=-\ell_s}^{j_s} \nu_{m_s, s} ,$$

$$(-\ell_s \le j_s \le \ell_s)$$

(5.4.35) $$F_{\nu,\ -\ell_s,\ -\ell_{s'},\ s,\ s'} = 0 ;$$

$$F_{\nu, j_s+1, j_{s'}+1, s, s'} = \nu^{-1} \sum_{m_s=-\ell_s}^{j_s} \sum_{m_{s'}=-\ell_{s'}}^{j_{s'}} \nu_{m_s, m_{s'}, s, s'} ;$$

$$(-\ell_s \le j_s \le \ell_s;\ -\ell_{s'} \le j_{s'} \le \ell_{s'}) .$$

(5.4.36) $$F^*_{\nu, j_s+1, s} = F_{\nu, j_s+1, s} - F_{\nu, -j_s, s} \ ; \ (1 \le j_s \le \ell_s).$$

(5.4.37) $$F^*_{\nu, j_s+1, j_{s'}+1, s, s'}$$

$$= F_{\nu, j_s+1, j_{s'}+1, s, s'} - F_{\nu, j_s+1, (-j_{s'}), s, s'}$$

$$- F_{\nu, (-j_s), j_{s'}+1, s, s'} + F_{\nu, (-j_s), (-j_{s'}), s, s'} ,$$

$$(1 \le j_s \le \ell_s,\ 1 \le j_{s'} \le \ell_{s'};\quad s \ne s');$$

$$(5.4.38)\quad \hat{\Delta}^*_{\nu,j_s,s} = \int_{F^*_{\nu,j_s,s}}^{F^*_{\nu,j_s+1,s}} \psi(u)\,du \Big/ \int_{F^*_{\nu,j_s,s}}^{F^*_{\nu,j_s+1,s}} du$$

$$= \int_{(1+F^*_{\nu,j_s,s})/2}^{(1+F^*_{\nu,j_s+1,s})/2} \phi(u)\,du \Big/ \int_{(1+F^*_{\nu,j_s,s})/2}^{(1+F^*_{\nu,j_s+1,s})/2} du$$

$$\text{if } \nu_{j_s,s} + \nu_{(-j_s),s} \neq 0$$

$\hat{\Delta}^*_{\nu,j_s,s} = 0$, otherwise.

Consider the statistics,

$$(5.4.39)\quad U^*_{\nu,sk0} = \sum_{i=1}^{\nu} c^*_{k\nu i}\Big(\sum_{j_s=1}^{\ell_s} \hat{\Delta}^*_{\nu,j_s,s} U_{\nu i j_s,s}\Big) \operatorname{sgn} X^{(s)}_{\nu i}$$

($s = 1, 2,\ldots, q$, $k = 1, 2,\ldots, p$). The permutation test procedure, we are going to discuss below, will be based on $U^*_{\nu,sk0}$ ($s = 1,\ldots, q$; $k = 1,\ldots, p$).

Under H_0 in (5.4.1), $\underset{\sim}{X}_{\nu i0}$ ($i = 1, 2,\ldots, \nu$) are ν i.i.d. random vectors. Hence, the joint distribution of the elements of $\underset{\sim}{X}^*_{\nu 0}$ remains invariant under $\nu!$ possible permutations of its columns. Further, under H_0, there are 2^ν possible sign-changes of the columns of sgn $\underset{\sim}{X}^*_{\nu}$, each such sign change being independent of the permutations of the columns of $\underset{\sim}{X}^*_{\nu 0}$. Hence, if we denote

$$(5.4.40)\quad W^{(s)}_{\nu i} = \sum_{j_s=1}^{\ell_s} \hat{\Delta}^*_{\nu,j_s,s} U_{\nu i j_s,s}\ (i = 1,\ldots, \nu;\ s = 1,\ldots, q),$$

$$(5.4.41)\qquad \underset{\sim}{W}^*_\nu = \begin{bmatrix} W^{(1)}_{\nu 1} & \cdots & W^{(1)}_{\nu\nu} \\ \cdots & \cdots & \cdots \\ W^{(q)}_{\nu 1} & \cdots & W^{(q)}_{\nu\nu} \end{bmatrix},$$

then the distribution of $\underset{\sim}{W}^*_\nu$ remains invariant under any permutation of its columns, and, further, 2^ν possible sign changes of the columns of sgn $\underset{\sim}{X}^*_{\nu 0}$ do not affect $W^{(s)}_{\nu i}$ $(i = 1,\ldots, \nu;\ s = 1,\ldots, q)$.

If we now consider a finite group $\{G_\nu\}$ of transformations $\{g_\nu\}$ which maps the sample space onto itself, where a typical transformation g_ν is such that

$$(5.4.42)\qquad g_\nu \underset{\sim}{W}^*_\nu = \begin{bmatrix} (-1)^{m_1} W^{(1)}_{\nu i_1} & \cdots & (-1)^{m_\nu} W^{(1)}_{\nu i_\nu} \\ \cdots & \cdots & \cdots \\ (-1)^{m_1} W^{(q)}_{\nu i_1} & \cdots & (-1)^{m_\nu} W^{(q)}_{\nu i_\nu} \end{bmatrix},$$

where, $m_d = 0, 1;\ d = 1, 2,\ldots, \nu$ and $(i\ ,\ldots, i_\nu)$ is any permutation of the integers $(1,\ldots, \nu)$. Thus there is a set of $2^\nu\ \nu!$ transformations in G_ν. Under H_0, the conditional distribution over the set of $2^\nu\ \nu!$ realizations (generated by G_ν) is uniform, each having common conditional probability $(2^\nu\ \nu!)^{-1}$. We denote the conditional probability measure by $\mathcal{P}'_\nu$. Then,

$$(5.4.43)\qquad E(U^*_{\nu,sk0} | \mathcal{P}'_\nu) = 0$$

Let,

$$(5.4.44)\qquad A_{0\nu,ss} = \sum_{j_s=1}^{\ell_s} \hat{\Delta}^{*2}_{\nu,j_s,s}(\nu_{j_s,s} + \nu_{(-j_s),s})/\nu,\ s = 1, 2,\ldots, q;$$

$$(5.4.45)\quad A_{0\nu,ss'} = \sum_{j_s=1}^{\ell_s}\ \sum_{j_{s'}=1}^{\ell_{s'}} \hat{\Delta}^*_{\nu,j_s,s}\hat{\Delta}^*_{\nu,j_{s'},s'}[\nu_{j_s,j_{s'},s,s'}$$

$$- \nu_{j_s,-j_{s'},s,s'} - \nu_{-j_s,j_{s'},s,s'} + \nu_{-j_s,-j_{s'},s,s'}]/\nu,$$

$$1 \le s \ne s' \le q .$$

Using now the results,

$$E[U^2_{\nu ij_s,s}\ \mathrm{sgn}^2(X^{(s)}_{\nu i})|\mathcal{P}'_\nu] = [\nu_{j_s,s} + \nu_{(-j_s),s}]/\nu,$$

$$(1 \le j_s \le \ell_s,\ 1 \le s \le q)\ ,$$

$$E[U_{\nu ij_s,s}U_{\nu ij_{s'},s'}\mathrm{sgn}(X^{(s)}_{\nu i})\mathrm{sgn}(X^{(s')}_{\nu i})|\mathcal{P}'_\nu]$$

$$= [\nu_{j_s,j_{s'},s,s'} - \nu_{j_s,-j_{s'},s,s'} - \nu_{-j_s,j_{s'},s,s'} + \nu_{-j_s,-j_{s'},s,s'}]/\nu,$$

$$(1 \le j_s \le \ell_s,\ 1 \le j_{s'} \le \ell_{s'},\ 1 \le s \ne s' \le q),$$

$$E[U_{\nu ij_s,s}U_{\nu i'j_{s'},s'}\ \mathrm{sgn}(X^{(s)}_{\nu i})\mathrm{sgn}(X^{(s')}_{\nu i'})|\mathcal{P}'_\nu] = 0,$$

$$(1 \le j_s \le \ell_s,\ 1 \le j_{s'} \le \ell_{s'},\ 1 \le s,\ s' \le q),$$

we get,

$$(5.4.46)\quad E(U^*_{\nu,sk0}U^*_{\nu,s'k'0}|\mathcal{P}'_\nu) = (\sum_{i=1}^{\nu} c^*_{k\nu i}c^*_{k'\nu i})A_{0\nu,ss'}$$

$$(k,\ k' = 0,\ 1,\dots,\ p;\ s,s' = 1,\ 2,\dots,\ q)$$

Write

$$(5.4.47)\qquad \underset{\sim}{U}^*_{\nu 0} = \underset{q\times(p+1)}{((U^*_{\nu,sk0}))},$$

(5.4.48) $\underset{\sim}{A}_{0\nu} = \underset{q\times q}{((A_{0\nu,ss'}))}$,

We have,

(5.4.49) $E(\underset{\sim}{U}^*_{\nu 0}|\mathcal{P}'_\nu) = \underset{\sim}{0}$

(5.4.50) $\mathrm{Var}(\underset{\sim}{U}^*_{\nu 0}|\mathcal{P}'_\nu) = \underset{\sim}{\Lambda}_\nu \otimes \underset{\sim}{A}_{0\nu} = \underset{\sim}{K}_{0\nu}$ (say)

Again, assuming $\underset{\sim}{A}_{0\nu}$ is p.d. so that $\underset{\sim}{K}_{0\nu}$ is p.d. the test statistic we propose is given by

(5.4.51) $$M^*_{\nu 0} = \sum_{s=1}^{q} \sum_{s'=1}^{q} \sum_{k=0}^{p} \sum_{k'=0}^{p} U^*_{\nu,sk0} U^*_{\nu,sk'0} K^{sk,s'k'}_{0\nu} ,$$

where, $((K^{sk,s'k'}_{0\nu})) = K^g_{0\nu}$, a generalized inverse of $K_{0\nu}$. We can show, here, that $K_{0\nu}$ is p.d. in probability.

The following test is proposed based on the critical function $\psi_4(\underset{\sim}{X}^*_{\nu 0}, \text{sgn } \underset{\sim}{X}^*_\nu)$ given by

(5.4.52) $$\psi_4(\underset{\sim}{X}^*_{\nu 0}, \text{sgn } \underset{\sim}{X}^*_\nu) = \begin{cases} 1 & , \text{ if } M^*_{\nu 0} > M^*_{\nu 0,\varepsilon} \\ \delta_{4\nu\varepsilon} & , \text{ if } M^*_{\nu 0} = M^*_{\nu 0,\varepsilon} \\ 0 & , \text{ if } M^*_{\nu 0} < M^*_{\nu 0,\varepsilon} \end{cases}$$

where $M^*_{\nu 0,\varepsilon}$, $\delta_{4\nu\varepsilon}$ are so chosen that $E[\psi_4(\underset{\sim}{X}^*_{\nu 0}, \text{sgn}\underset{\sim}{X}^*_\nu)|\mathcal{P}'_\nu] = \varepsilon$. This implies that $E[\psi_4(\underset{\sim}{X}^*_{\nu 0}, \text{sgn } \underset{\sim}{X}^*_\nu)|P_\nu] = \varepsilon$, so that the test is a similar size ε test.

However, as in the earlier cases, in this case also, we need find the asymptotic distribution of $M^*_{\nu 0}$. With this end in view, we first prove the following lemma.

LEMMA 5.4.3. Under (5.1.1), (3.1.2)-(3.1.5), (5.1.3) along with the assumption of diagonal symmetry of F, we get,

$$\underset{\sim}{A}_{0\nu} \to \underset{\sim}{A}_0 \text{ in } P_\nu\text{- (and in } Q_\nu\text{-) probability.}$$

Proof. We can show $A_{0\nu,ss} \to A_{0ss}$ as $\nu \to \infty$ in P_ν- (and in Q_ν-) probability similarly as in lemma 4.3.1.

Now, proceeding as in lemma 5.3.1 we can show $A_{0\nu,ss'} \to A_{0ss'}$ as $\nu \to \infty$ $(1 \le s \ne s' \le q)$. Q.E.D.

We next find the permutation limit distribution of $\underset{\sim}{U}^*_{\nu 0}$.

THEOREM 5.4.1. Under the model (5.1.1), if the assumptions (3.1.2)-(3.1.5), (5.1.3), (5.2.14) along with the assumption of diagonal symmetry of F hold, then the joint distribution of the elements of $\underset{\sim}{U}^*_{\nu 0}$ is under $\mathcal{P}'_\nu$ asymptotically $N_{q(p+1)}$ $(\underset{\sim}{0}, \underset{\sim}{A}_0 \otimes \underset{\sim}{A}_0)$.

Proof. The proof follows by combining Theorem 5.3.1 and Theorem 4.3.2 and hence is omitted.

The next lemma establishes the "asymptotic equivalence" of the parametric test statistics $T_{\nu,sk0}$ to the permutation test-statistics $U^*_{\nu,sk0}$.

LEMMA 5.4.4. Under the same assumptions as in Theorem 5.3.1,

$$\underset{\sim}{U}^*_{\nu 0} - \underset{\sim}{T}_{\nu 0} \to \underset{\sim}{0}_{q\times(p+1)} \quad \text{in } P_\nu\text{- or in } Q_\nu\text{-probability.}$$

Proof. Write,

$$U^*_{\nu,sk0} - T_{\nu,sk0} = \sum_{i=1}^{\nu} c^*_{k\nu i}\Big[\sum_{j_s=1}^{\ell_s}(\hat{\Delta}^*_{\nu,j_s,s} - \Delta^*_{j_s,s})U_{\nu i j_s,s}\Big]\text{sgn}\, X^{(s)}_{\nu i}.$$

Using lemma 4.3.2 we can show that $U^*_{\nu,sk0} - T_{\nu,sk0} \to 0$ in P_ν- and in Q_ν-probability for each $s = 1, 2, \ldots, q$, $k = 0, 1, \ldots, p$. Using now the Bonferroni inequality, we get the result.

The above result along with lemmas 5.4.1 and 5.4.2 shows that under H_0, the joint distribution of the elements of $\underset{\sim}{U}^*_{\nu 0}$ is asymptotically $N_{q(p+1)}$ $(\underset{\sim}{0}, \underset{\sim}{\Lambda}_0 \otimes \underset{\sim}{A}_0)$, while under H_ν it is asymptotically $N_{q(p+1)}$ $(\underset{\sim}{D}(A)\underset{\sim}{\Sigma}_0\underset{\sim}{\Lambda}_0, \underset{\sim}{\Lambda}_0 \otimes \underset{\sim}{A}_0)$. Thus, $M^*_{\nu 0}$ is under H_0 asymptotically $\chi^2_{(p+1)q}$, while under H_ν it is asymptotically $\chi^2_{(p+1)q}(\eta_0)$, η_0 being defined in (5.4.28).

We may note that the above results can be extended to the case of a countable set of class-intervals by using co-ordinatewise contiguity.

5.5. Concluding Remarks and Scope for further Research.

Unlike tests in the univariate case, namely those in chapters III and IV, we cannot claim here asymptotic optimality from the point of view of Wald in a quite general sense. It has been shown by Puri and Sen [31] that when the derivative of the logarithm of the conditional density of one variable given the rest is a linear combination of the derivatives of the logarithms of the marginal densities, then the tests proposed by them in case of a similar problem for ungrouped data is asymptotically optimal according to Wald's criterion. The assumption holds in case of multinormal distribution or for any coordinate wise independent distribution. The latter conclusion holds true here as a trivial consequence of the results derived in the univariate case. But, the former conclusion is not apparent. Studies can be pursued along this line, but at the moment, we are not aware of any particular statistics which will give asymptotic optimality in our desired sense. Also, we comment as in Sen and Puri [31] that the model (5.1.1) can be made a little more general by taking

$$X_{\nu i}^{(s)} = \beta_{s0} + \sum_{k=1}^{p} \beta_{sk}\, c_{k\nu i}^{(s)} + \varepsilon_{\nu i}^{(s)}$$

$$(s = 1, 2, \ldots, q;\ i = 1, 2, \ldots, \nu,\ \nu \geq 1).$$

The model presents certain complications, since, then, the variance-covariance matrix of the statistics cannot be expressed as a Kronecker product of two non-negative definite matrices, thereby, rendering the analysis more difficult. However, in many actual situations, (e.g. the c-sample problem), such requirement is met.

A further scope for future research is to extend the findings of chapters III, IV and V to the situation where the predictors are stochastic. We have solved the problem only in the univariate case with ungrouped data in chapter II. A multivariate extension of those results for ungrouped data also seems feasible. Also, extending the findings of chapter II to the case of signed-ranked-statistics we can tackle the analysis of covariance problems in paired comparison designs.

Finally, we make the comment that the problem of tied ranks can be tackled very easily with the techniques developed earlier, if for tied observations, we use "scores" described in earlier chapters, the class-intervals being properly picked from the ranks of these observations, while for untied observations, we take class-intervals (depending on the ranks of such observations) with arbitrarily small probabilities of belonging to them.

CHAPTER VI

A NUMERICAL EXAMPLE

6.1. Introduction and the Data.

In this chapter, we consider an example to illustrate some of the test procedures developed in the earlier chapters. The example, we shall be using, will illustrate test procedures developed in Chapter III, although more general examples can be easily constructed.

The problem is to test the effectiveness of several contraceptive devices. 1500 married women in the age-group 15-25 are observed for a period of 36 months. Women are either subject to no contraceptive device, (hereafter, referred to as contraceptive 1 i.e. the 'control group') contraceptive 2 or contraceptive 3. There are 500 women in each group.

A 'conception delay' is defined as the exposure months preceding but not including the month of conception. We treat conception as a chance event and 'conception delay' as a random variable. An 'ovulatory cycle' is equated to a calender month. For simplicity it is assumed that the population is 'homogeneous' i.e. the monthly probability of conceiving remains constant from one month to another and for a homogeneous population conception delay is distributed geometrically.[1] The data shown in Table 6.1, though fictitious, corresponding to this fact, admitting, of course, sampling fluctuations. Conception delay is calculated by subtracting nine months from the interval between marriage and the first live-birth. There are 12 class intervals within each of which we know the number of observations belonging to the three groups earlier mentioned. Hence, the problem is a 3-sample grouped data problem.

Throughout, this chapter the suffix ν has been dropped, whenever possible, without ambiguity.

TABLE 6.1 showing distribution of conception delays for women subject to contraceptives 1, 2, or 3.

Conception Delay (in months)	Frequency of women subject to			Total Class Frequency
	Contra-ceptive 1	Contra-ceptive 2	Contra-ceptive 3	
0-1	145	140	128	413
1-2	107	102	94	303
2-3	76	65	69	210
3-4	45	56	60	161
4-5	42	43	40	125
5-6	28	23	25	76
6-7	16	23	18	57
7-8	17	18	17	52
8-9	6	8	15	29
9-10	6	5	12	23
10-11	3	3	5	11
11 & above	9	14	17	40
Total	500	500	500	1500

[1] see e.g. a paper by Mindel Sheps, "Uses of stochastic models in the evaluation of population policies. I. Theory and approaches to data analysis", Proceedings of the 5th Berkeley Symposium on Mathematical Statistics and Probability Vol. IV 115-136.

Let T_1, T_2, T_3 denote conception delays for women subject to contraceptives 1, 2 and 3 respectively. Let $P_1(t)$, $P_2(t)$, $P_3(t)$ denote corresponding d.f.'s. We assume $P_2(t) = P_1(\rho_2 t)$, $P_3(t) = P_1(\rho_3 t)$, $\rho_2, \rho_3 > 0$. Then $\log T_1$, $\log T_2 + \log \rho_2$ and $\log T_3 + \log \rho_3$ are identically distributed. In other words, we are assuming that the use of a contraceptive is effecting a shift in the scale of the distribution of conception delay for women who are not subject to any contraceptive device, so that when we take logarithm of this variable, there is a shift in location of the transformed variable. Assume $\Pr(\log T_1 \leq u) = F_1(u) = F(u - \beta_0')$. Hence, $\Pr(\log T_j \leq u) = F(u - \beta_0' + \log \rho_j)$, $j = 1, 2$. If now X denotes logarithm of conception delay and the 1500 r.v.'s are labelled as $X_1, X_2, \ldots, X_{1500}$, the d.f. of X_i (conception delay for i^{th} woman) can be written as

(6.1.1) $$\Pr(X_i \leq x) = F(x - \beta_0' - \beta_1 c_{1i}' - \beta_2 c_{2i}' - \beta_3 c_{3i}') ,$$

where, $$c_{ki}' = \begin{cases} 1, & \text{if the } i^{th} \text{ woman is subject to contraceptive } k \\ 0, & \text{otherwise,} \end{cases}$$

($k = 1, 2, 3$; $i = 1, 2, \ldots, 1500$). (6.1.1) can be rewritten as

(6.1.2) $$\Pr(X_i \leq x) = F(x - \beta_0 - \beta_1 c_{1i} - \beta_2 c_{2i} - \beta_3 c_{3i}) ,$$

where $$c_{ki} = \begin{cases} 2/3, & \text{if } i^{th} \text{ woman is subject ot contraceptive } k \\ -1/3, & \text{otherwise,} \end{cases}$$

($k = 1, 2, 3$; $i = 1, 2, \ldots, 1500$). Thus,

$$(6.1.3)\quad \sum_{i=1}^{1500} c_{ki} = 0,\ k = 1, 2, 3.$$

$$\sum_{i=1}^{1500} c_{ki}^2 = 500(2/3)^2 + 1000(-1/3)^2 = 1000/3$$

$$\sum_{i=1}^{1500} c_{ki}\, c_{k'i} = 1000(2/3)(-1/3) + 500(-1/3)(-1/3)$$

$$= -500/3,\ 1 \leq k \neq k' \leq 3.$$

So,

$$(6.1.4)\quad \underset{\sim}{\Lambda}_\nu = \begin{bmatrix} 1 & -.5 & -.5 \\ -.5 & 1 & -.5 \\ -.5 & -.5 & 1 \end{bmatrix} \quad \text{which is of rank 2.}$$

So, if we work with a principal submatrix of order 2 of $\underset{\sim}{\Lambda}_\nu$, it will be of full rank. Also $\max\limits_{1\leq i\leq 1500} c_{ki}^2 = 4/9$ and $\sum\limits_{i=1}^{1500} c_{ki}^2 = 1000/3$, for all k = 1, 2, 3. Hence,

$$(6.1.5)\quad \max_{1\leq i\leq 1500} c_{ki}^{*\,2} = 4/3000 = .001333\ .$$

(6.1.2) is the same as (3.1.1). Also, from, (6.1.3), (6.1.4) and the remarks below it, and (6.1.5), we find that (3.1.2)-(3.1.5) are satisfied. The underlying distribution F is assumed to be such that (3.1.6) is satisfied. We want to test

$$(6.1.6)\quad H_0 :\ \beta_1 = \beta_2 = \beta_3 = 0.$$

The usual chi-square statistic for testing the homogenity of several populations can be used in this case. In the next section, we compute the chi-square statistic, and the permutation test statistic defined in (3.3.6) using (1) Wilcoxon scores and (2) Normal scores. The number of observations being large, the asymptotic chi-square distribution of the permutation test statistic seems reasonable.

6.2. Calculation of actual test statistics and analysis of the data.

We calculate first the homogeneity chi-square statistic. Let f_{jk} denote the observed frequency for the j^{th} class interval and k^{th} group ($j = 0, 1, \ldots, 11$; $k = 1, 2, 3$). Let

$$f_{j0} = \sum_{k=1}^{3} f_{jk} \ (j = 0, 1, \ldots, 11); \ f_{0k} = \sum_{j=0}^{11} f_{jk} \ (k = 1, 2, 3).$$

Then, under H_0 in (6.1.6), the expected frequency for the j^{th} class-interval and k^{th} group is $f_{jk_0} = f_{j0} f_{0k}/1500$ ($j = 0, 1, \ldots 11$; $k = 1, 2, 3$). The homogeneity chi-square statistic is given by

$$(6.2.1) \qquad \chi^{(0)^2} = \sum_{j=0}^{11} \sum_{k=1}^{3} f_{jk}^2/f_{jk_0} - 1500 ,$$

which under H_0 is a central chi-square with $11\times 2 = 22$ degrees of freedom. The observed value of chi-square in this case turns out to be 18.6506. It is insignificant even at upper 25% level of significance. We can, thus, conclude that if a homogeneity chi-square test is used, the data bear no evidence to reject the null hypothesis.

Next, we proceed to calculate the test statistic given in (3.3.6) using Wilcoxon score. Then $\phi(u) = 2u - 1$,

$\hat{\Delta}_{\nu,j} = F_{\nu,j} + F_{\nu,j+1} - 1.$

TABLE 6.2. showing values of $\hat{\Delta}_{\nu j}$ for the Wilcoxon scores test.

t	j	frequency	cumulative frequency	$\nu\hat{\Delta}_{\nu j}$
0-1	0	413	413	-1087
1-2	1	303	716	- 371
2-3	2	210	926	142
3-4	3	161	1087	513
4-5	4	125	1212	799
5-6	5	76	1288	1000
6-7	6	57	1345	1133
7-8	7	52	1397	1242
8-9	8	29	1426	1323
9-10	9	23	1449	1375
10-11	10	11	1460	1409
above 11	11	40	1500	1460
Total		1500		

Using the formula (3.3.4), we get

(6.2.2) $\quad (U_1, U_2, U_3) = (-1.0619, -.1521, 1.2140).$

From (6.1.4) we find a generalized inverse of $\underset{\sim}{\Lambda}_\nu$ given by

(6.2.3) $$\underset{\sim}{\Lambda}_\nu^* = 4/3 \begin{bmatrix} 1 & .5 & 0 \\ .5 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4/3 & 2/3 & 0 \\ 2/3 & 4/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From (3.3.5),

(6.2.4) $$A^2(F_\nu, \{I_j\}) = .32202299 .$$

Hence, using (3.3.6) we get observed value of the chi-square statistic as $\chi^{(1)^2} = 5.4331.$

Under H_0, the statistic is distributed as χ_2^2. The theoretical upper 10% value of a χ_2^2 is 4.605 and upper 5% value of a χ_2^2 is 5.991. So, the observed statistic (using Wilcoxon score) is significant at 10% level of significance, but not at 5% level of significance. At any rate, we can see that in this case, it is better than the homogeneity chi-square statistic.

We next consider the normal scores test. In this case, $\phi(u) = \Phi^{-1}(u)$, where, $\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-t^2/2)\, dt$. Then,

$$(6.2.5) \qquad \hat{\Delta}_{\nu j} = (\nu/\nu_j)[\ \xi(\Phi^{-1}(F_{\nu,j})) - \xi(\Phi^{-1}(F_{\nu,j+1}))] \ ,$$

ν_j, $F_{\nu,j}$ and $\hat{\Delta}_{\nu j}$ $(j = 0, 1\ldots, 11)$ being defined in (3.3.1), (3.3.2) and (3.3.3) respectively, and $\xi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

TABLE 6.3. showing values of $\hat{\Delta}_{\nu j}$ for the Normal Scores Test.

j	$\hat{\Delta}_{\nu j}$	j	$\hat{\Delta}_{\nu j}$	j	$\hat{\Delta}_{\nu j}$
0	-1.21260679	4	.72908880	8	1.56503017
1	- .31894609	5	.96909789	9	1.73387609
2	.11955300	6	1.16507737	10	1.87671818
3	.44431388	7	1.36861240	11	2.31332213

Then, $(U_1, U_2, U_3) = (-1.9152, -.2940, 2.2092)$

$$A^2(F_\nu, \{I_j\}) = .91897472 \ .$$

Then, the observed value of the chi-square statistic (defined in (3.3.6)) is given by ${\chi^{(2)}}^2 = 6.2643$. Thus, the statistic used (a χ^2_2) is significant at 5% level of significance, and we may draw the conclusion, that, in this case the Normal Scores test is better than the Homogeneity chi-square test or the Wilcoxon test.

The above is an example where our proposed tests are much better than the classical tests available in the literature. The drawback with the homogeneity chi-square test is that too many degrees of freedom are associated with it, thus making it less likely to detect any heterogeneity among populations than the tests we have proposed. It is expected that in usual situations, the value of a chi-square per degree of freedom will be much less for a homogeneity chi-square than the one used by us. This suggests the superiority of our proposed tests to the classical tests.

The case of tied ranks for the p($\geq$ 2) sample problem has been treated in the null situation by Kruskal[2] and Kruskal and Wallis[3]. Also, asymptotic power properties of such tests can be treated very easily by using asymptotic distributions of U-statistics.[4] The treatment of ties in the "normal scores" case is not, however, available in the literature. Our proposed methods include that situation as a particular case.

---

[2]Kruskal, W. H. (1952). A nonparametric test for the several sample problem. Ann. Math. Statist. 23 525-540

[3]Kruskal, W. H. and Wallis, W. A. (1952). Use of ranks in one-criterion variance analysis. J. Amer. Statist. Assoc. 47 583-621

[4]Hoeffding, W. (1948). A class of statistics with asymptotic normal distribution. Ann. Math. Statist. 19 293-325

We might remark that in this case no attempt has been made to study the permutation distribution of the test statistics. The difficulties associated with their use in practice have already been mentioned in earlier chapters. However, attempts are being made to study the permutation distribution with specific scores and moderately large sample sizes.

REFERENCES

[1] ADICHIE, J. N. (1967). Asymptotic efficiency of a class of nonparametric tests for regression parameters. Ann. Math. Statist. 38 884-892.

[2] ANDREWS, F. C. (1954). Asymptotic behaviour of some rank tests for analysis of variance. Ann. Math. Statist. 25 724-735.

[3] BASU, A. P. (1967). On two K-sample rank tests for censored data. Ann. Math. Statist. 38 1520-1535.

[4] BELLMAN, R. (1960). Introduction to Matrix Analysis. McGraw Hill, New York.

[5] BHUCHONGKUL, S. (1964). A class of nonparametric tests for independence in bivariate populations. Ann. Math. Statist. 35 138-149.

[6] BICKEL, P. J. (1965). On some asymptotically nonparametric competitors of Hotelling's T^2. Ann. Math. Statist. 36 160-173.

[7] BODEWIG, E. (1956). Matrix Calculus. Interscience.

[8] BRILLINGER, D. R. (1962). A note on the rate of convergence of a mean. Biometrika 49 574-576.

[9] CHATTERJEE, S. K. and SEN, P. K. (1964). Nonparametric tests for the bivariate two sample location problem. Calcutta Statist. Asscn. Bull. 13 18-58.

[10] CHERNOFF, H. and SAVAGE, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. Ann. Math. Statist. 29 972-994.

[11] CRAMÉR, H. (1946). Mathematical Methods of Statistics. Princeton Univ. Press.

[12] FELLER, W. (1957]. An Introduction to Probability Theory and Its Applications Vol. 1. Wiley, New York.

[13] FRASER, D.A.S. (1957). Nonparametric Methods in Statistics. Wiley, New York.

[14] GASTWIRTH, J. L. (1965). Asymptotically most powerful tests for the two sample problem with censored data. Ann. Math. Statist. 1243-1247.

[15] GASTWIRTH, J. L. (1966). On robust procedures. J. Amer. Statist. Asscn. 61 929-948.

[16] HÁJEK, J. (1961). Some extensions of the Wald-Wolfowitz-Noether theorem. Ann. Math. Statist. 32 506-523.

[17] HÁJEK, J. (1962). Asymptotically most powerful rank order tests. Ann. Math. Statist. 33 1124-1147.

[18] HÁJEK, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. Ann. Math. Statist. 39 325-346.

[19] HÁJEK, J. and SIDÁK, Z. (1967). Theory of Rank Tests. Academic-Press, New York.

[20] HANNAN, E. J. (1956). The asymptotic power of tests based on multiple correlation. J. Roy. Statist. Soc. Ser. B 18 227-233.

[21] HARTLEY, H. O. and DAVID, H. A. (1954). Universal Bounds for Mean Range and Extreme Observation. Ann. Math. Statist. 25 85-99.

[22] HOEFFDING, W. (1951). A combinatorial central limit theorem. Ann. Math. Statist. 22 558-566.

[23] HOEFFDING. W. (1968). On the centering of a simple linear rank statistic. Inst. of Statist. Mimeo Series No. 585. University of North Carolina.

[24] JOGDEO, K. (1962). Unpublished doctoral dissertation. University of California, Berkeley.

[25] LeCAM, L. (1960). Locally asymptotic normal families of distribution. Univ. of California Publ. Statist. 3 No. 2 37-98.

[26] LEHMANN, E. L. (1959). Testing Statistical Hypotheses. Wiley, New York.

[27] LOÉVE, M. (1963). Probability Theory (3rd Edition) Van Nostrand, Princeton.

[28] MEHRA, K. L. and PURI, M. L. (1967). Multisample Analogoues of some one-sample tests. Ann. Math. Statist. 38 523-549.

[28] [a] PURI, M. L. (1964). Asymptotic efficiency of a class of c-sample tests. Ann. Math. Statist. 35 102-121.

[29] PURI, M. L. and SEN, P. K. (1966). On a class of multivariate multisample rank-order tests. Sankhya 28 353-376.

[30] PURI, M. L. and SEN, P. K. (1969). On a class of rank-order tests for the identity of two multiple regression surfaces. Zeit. Wahr. und Verw. Gebiete 12 1-8.

[31] PURI, M. L. and SEN, P. K. (1969). A class of rank-order tests for a general linear hypothesis (to appear in the August issue of Ann. Math. Statist. 1969).

[32] RAO, C. R. (1965). Linear Statistical Inference and Its Applications. Wiley, New York.

[33] SEN, P. K. (1967). Asymptotically most powerful rank-order tests for grouped data. Ann. Math. Statist. 38 1229-1239.

[34] WALD, A. (1943). Tests of statistical hypotheses concerning several parameters when the number of observations is large. Trans. Amer. Math. Soc. 54 426-482.

[35] WILKS, S. S. (1938). The large-sample distribution of the likelihood ratio for testing composite hypotheses. Ann. Math. Statist. 9 60-62.