Abstract

Wu, Ling. Classification of involutions of $\mathrm{SL}(n,k)$ and $\mathrm{SO}(2n+1,k)$ (Under the direction of Dr. Aloysius Helminck)

In this paper, we classify the involutions of $\mathrm{SL}(n,k)$ and $\mathrm{SO}(2n+1,k)$, where k is the complex numbers (algebraically closed field in general), real numbers, finite field and p-adic numbers. We did this in a couple of ways: directly and by using the characterization given in [Helm2000]. In the case of $\mathrm{SO}(2n+1,\mathbb{Q}_p)$, we restrict the classification of the involutions to those fields $k=\mathbb{Q}_p$ for which -1 is a square. We also identify those isomorphy classes whose fixed point groups are compact and prove that the others are not. The classification in this paper will be fundamental for the analysis of other cases such as $\mathrm{SO}(2n,k)$ and $\mathrm{SP}(n,k)$.

CLASSIFICATION OF INVOLUTIONS OF SL(N, K) AND SO($2N + 1, K$)

BY
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A DISSERTATION SUBMITTED TO THE GRADUATE FACULTY OF
NORTH CAROLINA STATE UNIVERSITY
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

RALEIGH
MAY 08, 2002

APPROVED BY:

CHAIR OF ADVISORY COMMITTEE

To my parents and my advisor

for their love and help

Biography

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Acknowledgements

Foremost, I would like to express my gratitude to my thesis advisor, Professor Aloysius Helminck for his invaluable assistance, guidance, encouragement and friendship throughout my Ph.D study and thesis project. It is an enjoyable experience working with him. He always visualized important longterm goals and also very open to suggestions and very flexible in dealing with my specific needs and style. And he has been excellent at motivating me throughout the whole ptoject. I'll never forget the good times I spent with him and his team.

Furthermore, I would like to thank all the committe members: Professor Luh, Professor Selgrade and Professor Stitzinger. The supports and encouragements I received from them go beyond this papers.

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Chapter 1

Introduction

1.1 Background and motivation

Symmetric spaces have been studied for over 100 years. Initially they were only studied over the real numbers, but in the last 15 to 20 years symmetric spaces over other fields have become of importance in other areas of mathematics as well. In the following we will give a brief introduction.

1.1.1 Symmetric bilinear forms and Reductive Symmetric Spaces

Let $V = k^n$ be a finite-dimensional vector space over a field k,

$$M_n(k) = M(n,k) = \{n \times n\text{-matrices with entries in } k\},$$

$$\mathrm{GL}(V) = \mathrm{GL}_n(k) = \mathrm{GL}(n,k) = \{A \in M_n(k) \mid \det(A) \neq 0\}$$

and

$$\mathrm{SL}(V) = \mathrm{SL}_n(k) = \mathrm{SL}(n,k) = \{A \in M_n(k) \mid \det(A) = 1\}.$$

Let B be a non-degenerate symmetric bilinear form on V, i.e. $B(x,y) = {}^T xMy$, where $M = M^T$ an invertible symmetric $n \times n$-matrix and $x, y \in V$. If $M = \mathrm{id}$, then B is the standard innerproduct on $\mathbb{R}^n$. For each $A \in M_n(k)$ its *adjoint* A' *with respect to* B is defined by

$$B(Ax, y) = B(x, A'y) \quad \text{for all} \quad x,\, y \in V.$$

Therefore

$${}^T x^T AMy = {}^T xMA'y \ \Rightarrow \ A' = M^{-1T} AM.$$

The adjoint defines a map of $M_n(k) \to M_n(k)$ and both $\mathrm{GL}_n(k)$ and $\mathrm{SL}_n(k)$ are invariant under this map. A matrix $A \in \mathrm{GL}_n(k)$ is called *B-orthogonal* if $A'A =$

$AA' = \mathrm{id}$, that is if $B(Ax, Ay) = B(x, y)$ for all $x, y \in V$. The orthogonal operators form an algebraic linear group, denoted by $\mathrm{O}(V, B)$. Let $\mathrm{SO}(V, B) = \mathrm{SL}_n(k) \cap \mathrm{O}(V, B)$. Consider the map $\tau : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$, defined by $\tau(A) = AA'$. Then

$$\mathrm{O} = \mathrm{O}(V, B) = \tau^{-1}(\mathrm{id}) = \{A \in \mathrm{GL}_n(k) \mid AA' = \mathrm{id}\}.$$

Let

$$X = X(V, B) = \mathrm{Im}(\tau) = \{AA' \mid A \in \mathrm{GL}_n(k)\}.$$

The space $X = X(V, B)$ is also called a *reductive symmetric space*. We will give a more general definition later.

1.1.2 Examples

(1). $G = \mathrm{SL}_2(\mathbb{R})$ and $B(x, y) = {}^T xy$, where $x, y \in V$. Then

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, hence

$$\mathrm{SO}(V, B) = \mathrm{O} \cap \mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \mid \phi \in \mathbb{R} \right\} = \mathrm{SO}(2, \mathbb{R}) \simeq S^1$$

and

$$X \cap \mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \mid \phi \in \mathbb{R} \text{ and } a \in \mathbb{R}^+ \right\}$$

since for all $A \in X$ there exists an orthonormal basis of V consisting of eigenvectors of A and all eigenvalues are positive.

(2). $G = \mathrm{SL}_2(\mathbb{R})$ and $B(x, y) = {}^T xMy$, where $x, y \in V$ and $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, hence

$$\mathrm{SO}(V, B) = \mathrm{O} \cap \mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a^2 - b^2 = 1 \right\} = \left\{ \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \mid \phi \in \mathbb{R} \right\}$$

which is non compact. In this case

$$X \cap \mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a^2 - b^2 & -ca + bd \\ ca - bd & d^2 - c^2 \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

consists of non-symmetric matrices. Contrary to the previous example this space contains both a compact ($\mathrm{SO}(2, \mathbb{R}) \simeq S^1$) and a noncompact part. This is an example of an affine symmetric space and in this case there exists an affine structure.

(3). $k = \mathbb{R}$ and $G = \mathrm{GL}_n(k)$ and $B(x, y) = x^T y$, where $x, y \in V$ (the standard inner product). In this case $A' = A^T$, so the self adjoint matrices are the exactly the symmetric matrices. If A is a real symmetric matrix, then there exists an orthonormal basis $\{e_i\}$ of V consisting of eigenvectors of A and all eigenvalues are real. Similarly any matrix AA^T is symmetric with positive real eigenvalues. So X is the set of symmetric matrices with positive real eigenvalues and all elements of X are semisimple. Note that if $A \mapsto \|A\|$ is the operator norm in $M_n(k)$ (i.e. $\|A\| = \max_{x \neq 0} \|A(x)\|/\|x\|$ with respect to the norm $\|x\| = (B(x, x))^{\frac{1}{2}}$ in V), then $\|A\| = 1$ for all $A \in \mathrm{O}$, hence O is compact. In fact it is a maximal compact subgroup of $\mathrm{GL}_n(k)$.

Theorem 1. *X is a closed submanifold of $\mathrm{GL}_n(k)$ and the mapping $\pi : \mathrm{O} \times X \to \mathrm{GL}_n(k)$ defined by $\pi : (h, x) \mapsto hx$ is a diffeomorphism.*

In this case θ is called the *Cartan involution* of $\mathrm{GL}_n(k)$ with respect to B and X is called a Riemannian symmetric space. So it has a pseudo-Riemannian structure and is totally geodesic.

1.1.3 Another way to look at it

Let $\theta : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$ be defined by $\theta(A) = (A')^{-1} = M^{-1T} A^{-1} M$. Then $\theta^2 = \mathrm{id}$, $\tau(A) = AA' = A\theta(A)^{-1}$ and

$$\mathrm{O} = \{A \in \mathrm{GL}_n(k) \mid \theta(A) = A\}$$

is the fixed point group of θ and

$$X = \{A\theta(A)^{-1} \mid A \in \mathrm{GL}_n(k)\} \subset \{A \in \mathrm{GL}_n(k) \mid \theta(A) = A^{-1}\}.$$

This leads to the following generalization of the reductive symmetric space X:

Definition 1. Let G be a reductive linear algebraic group defined over a field k of characteristic not 2, $\theta \in \mathrm{Aut}(G)$ an involution, i. e. $\theta^2 = \mathrm{id}$ and

$$\mathrm{O} = H = G^\theta = \{x \in G \mid \theta(x) \ = x\}$$

the fixed point group of θ. Let $\tau : G \to G$ be the map defined by $\tau(x) = x\theta(x)^{-1}$ and

$$X = \tau(G) = \left\{ x\theta(x)^{-1} \mid x \in G \right\}.$$

Then $X \simeq G/H$ is called a *reductive symmetric space*. X is also called a *symmetric k-variety*, especially when $k \neq \mathbb{R}$.

If $k = \mathbb{R}$, then X is also called an affine symmetric space. If moreover O is compact, then X is also called a Riemannian symmetric space. These symmetric spaces play an essential role in many areas of mathematics including mathematical physics, Lie theory, representation theory and differential geometry.

We note that with this definition every linear algebraic group itself is a reductive symmetric space.

Example 1. *Groups case:* Consider $G_1 = G \times G$ and $\theta(x,y) = (y,x)$, then $H = \{(x,x) \mid x \in G\} \simeq G$ embedded diagonally and $X = \{(x,x^{-1}) \mid x \in G\} \simeq G$ embedded anti-diagonally.)

As we can see from the definition, involutions plays an essential role in the theory of symmetric spaces.

1.1.4 Motivation and History of Symmetric Spaces

Reductive symmetric spaces occur in many areas of mathematics. Examples include geometry (see [DCP83, DCP85] and [Abe88]), singularity theory (see [LV83] and [HS90]) and the cohomology of arithmetic subgroups. (This involves a study of $\mathbb{Q}$-involutions, see [TW89]). However they are probably best known for their role in representation theory and harmonic analysis. In the following we will briefly explain why the reductive symmetric spaces are of importance in harmonic analysis.

Harmonic analysis

Harmonic analyses may be defined broadly as the attempt to decompose functions by superposition of some particularly simple functions, as in the theory of Fourier decompositions. To be more explicit:

Let X be a space *acted on* by a group G (typically a group of symmetries or a motion group): $G \times X \to X$. For example:

(1) $X = \mathbb{R}^2$ and $G = \{\text{translations}\}$ or

(2) $X = S^1$ and $G = \{\text{rotations}\}$.

Then G also acts on the functions on X:

$$g.f(x) = f(g^{-1}x), \quad x \in X, g \in G, f \in C_c(X).$$

Assume that the action of G on X leaves invariant a positive measure dx on X, i.e.

$$\int_X f(g.x)dx = \int_X f(x)dx, \quad f \in C_c(X), g \in G$$

This action of G on X defines a natural "unitary representation" λ (= left regular representation) of G on the Hilbert space $L^2(X, dx)$ of square integrable functions on X:

$$\lambda(g)f(x) = g.f(x) = f(g^{-1}x), \quad x \in X, g \in G, f \in C_c(X).$$

The aim of Harmonic analysis on X is to decompose this representation into irreducible subrepresentations. This decomposition is known as the abstract "Plancherel formula". The following example illustrates this decomposition.

Example 2. Take $X = G = S^1 = \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ unit circle (compact) and let dx be the Haar measure on X, normalized so that $\mathrm{vol}(X) = 1$. The Hilbert space $L^2(X, dx)$ is basically the space of periodic functions on the interval $[0, 2\pi]$:

$$f \in L^2(X, dx) : \quad f(t) = f(e^{i\phi}), \ \ t = e^{i\phi}, \phi \in [0, 2\pi]$$

Since f is periodic with period 2π, it has a Fourier series expansion (in periodic functions).

$$f(e^{i\phi}) = \sum_{n=-\infty}^{\infty} c_n e^{in\phi} = \sum_{n=-\infty}^{\infty} c_n \chi_n$$

where $\chi_n : G = X \to \mathbb{C}$ are defined by $\chi_n(x) = x^n$, with $x = e^{i\phi} \in X$ and

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) e^{-in\phi} d\phi.$$

The χ_n are the "unitary characters" of X. It follows that $L^2(X, dx)$ as a G-module has a decomposition in 1-dimensional G-modules:

$$L^2(X, dx) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}\chi_n.$$

The coefficients c_n are the classical Fourier coefficients of f. We obtained a "Fourier transform" for the functions on X. From classical analysis it is well known that such a transform is extremely useful in solving differential equations. In particular differential equations invariant under the group G, which is often a transformation group.

In general a study of the harmonic analysis of a space X is only a viable project if one has a lot of additional structure on both G and X. The reductive symmetric spaces have an extremely detailed fine structure what makes them particularly well suited for studying their harmonic analysis.

The representation theory and Plancherel formulas of reductive symmetric spaces over the real numbers has been studied extensively in the last few decades. The first breakthrough was made in the early fifties when Harish-Chandra commenced his study of general semisimple Lie groups. This finally led to the Plancherel formula (published 1976). Before that, in 1957, Harish-Chandra already found a Plancherel formula for any semisimple symmetric space G_k/H_k with H_k compact. (This case was simpler, since there are no discrete series). The next step was to study the representation theory of the general semisimple symmetric spaces. For these most of the work was done in the late 70's and 80's by a number of mathematicians, including Faraut, Flensted-Jensen, Oshima, Sekiguchi, Matsuki, Brylinski, Delorme, Schlichtkrul and van der Ban (see [HC84, FJ80, ŌS80, ŌM84, BD92, vdBS97, Del98])). The Plancherel formula for real reductive symmetric spaces was recently completed by Delorme (see [Del98]).

More recently a number of people have started to study the representation theory and Plancherel formulas of symmetric k-varieties over the $\mathfrak{p}$-adic numbers. This includes recent work of Jacquet, Lai and Rallis [JLR93] on a trace formula for symmetric k-varieties, work of Rader and Rallis [RR96] on spherical characters for $\mathfrak{p}$-adic symmetric k-varieties, a number of results for rank one symmetric k-varieties by Bosman [Bos92] and some results about the space of H_k-distribution vectors. This includes sharp estimates for the multiplicities in the Plancherel decomposition of $L^2(G_k/H_k)$ (see [HH99]).

The representation theory of reductive symmetric spaces is also of interest over other base fields besides real numbers and local fields. For example reductive symmetric spaces over the complex numbers play an essential role in the study of Harish Chandra modules (see for example [BB81] and [Vog83, Vog82])). Another interesting case are the reductive symmetric spaces defined over a finite field. These have been studied by Lusztig and his students (see for example [Lus90] and [Gro92]).

1.1.5 Fixed point groups and Symmetric k-Varieties

For $k = \mathbb{R}$ the Plancherel formula was first determined in the case of the Riemannian symmetric spaces (i.e. the case that O is compact). The main reason for this is that in this case the structure of the corresponding reductive symmetric space is simpler, then in the general case. For example all elements of X are semisimple and the left regular representation decomposes multiplicatively free.

For $k = \mathbb{Q}_p$, one gets a generalization of the real Riemannian symmetric space.

The fixed-point group $H = G^\delta$ for an involution δ over G is defined by

$$G^\delta = \{x \in G \mid \delta(x) = x\}.$$

The fixed point group determines much of the structure of the corresponding symmetric k-variety

$$X := \{g\delta(g)^{-1} \mid g \in G\}.$$

It is easy to see that $X \simeq G/G^\delta$. Moreover if G^δ is compact, then from [HW93] it follows that X consists of semisimple elements.

Proposition 1 ([HW93, Proposition 10.8]). *Let G be a connected reductive algebraic k-group with $\operatorname{char}(k) = 0$ and $X = \{g\delta(g)^{-1} | g \in G\}$. Suppose that $H \cap [G, G]$ is anisotropic over k. Then X_k consists of semi-simple elements.*

We note that for $k = \mathbb{R}$ or $\mathbb{Q}_p$ all k-anisotropic subgroups are compact. There are many other similarities between the structure of the real and p-adic reductive symmetric spaces. For example in both cases the left regular representation decomposes multiplicatively free (see [HH99, Corollary 8.3]). So for $k = \mathbb{Q}_p$ this is also the natural first case to study the Plancherel decomposition. In view of these results it is important to determine for $k = \mathbb{Q}_p$ which involutions have a compact fixed point group. We will determine this for each of the involutions we study in this thesis.

1.1.6 Notations

Throughout this thesis $\bar{G}$ will be a reductive algebraic group defined over a field k of characteristic not 2. We will mainly consider the case that k is an algebraically closed field, the real numbers, a finite field or the p-adic numbers. We will write $\bar{k}$ for the algebraic closure of k and $k_1 \subset \bar{k}$ denotes an extension field of k. We will write G for the set of k-rational points of $\bar{G}$. Following Borel [Bor91,] the group G is called *k-split* if $\bar{G}$ contains a maximal torus T, which is k-split as well. We note that every group $\bar{G}$ contains a k-split k-form G, which is unique up to isomorphy.

For $A \in \mathrm{GL}(n, k)$ let $\mathrm{Int}(A) = \mathcal{I}_A$ denote the inner automorphism defined by $\mathcal{I}_A(X) = A^{-1}XA$, $X \in \mathrm{GL}(n, k)$. Let $\mathrm{Aut}(G)$ denote the set of automorphisms of G, $\mathrm{Int}_k(G) = \{\mathrm{Int}(x) \mid x \in G\}$ the set of inner automorphisms of G and $\mathrm{Int}(G) = \{\mathrm{Int}(x) \mid x \in \bar{G} \text{ and } \mathrm{Int}(x)(G) \subset G\}$. An automorphism ϕ of G is called of *inner type* if $\phi = \mathcal{I}_A$ for some $A \in \mathrm{GL}(n, \bar{k})$. Otherwise ϕ is called of *outer type*.

In this thesis we will mainly consider the cases that $\bar{G}$ is one of $\mathrm{SL}(2, \bar{k})$, $\mathrm{SL}(n, \bar{k})$ or $\mathrm{SO}(n, \bar{k})$ and G is the unique k-split k-form of $\bar{G}$. In the case that $\bar{G} = \mathrm{SL}(n, \bar{k})$, then $G = \mathrm{SL}(n, k)$. In the case that $\bar{G} = \mathrm{SO}(n, \bar{k})$ the k-split k-form comes from a a bilinear form which is maximally isotropic.

Some more notation

We will write $I_{s,t}$ for the $n \times n$-matrix

$$I_{s,t} = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}$$

where $s + t = n$, for $p \in \bar{k}$ we will write $L_{n,p}$ for the $2n \times 2n$-matrix

$$L_{n,p} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ p & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & p & 0 \end{pmatrix}.$$

For $p = -1$ we will also write J_n for the $2n \times 2n$-matrix $L_{n,-1}$. Finally we will write $K_{n,x,y,z}$ for the $n \times n$-matrix

$$K_{n,x,y,z} = \begin{pmatrix} I_{n-3} & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{pmatrix}$$

where $x, y, z \in k$. We will also write $M_{n,x}$ for $K_{n,1,1,x}$ and $N_{n,x,y}$ for $K_{n,1,x,y}$.

1.2 Isomorphy classes of involutions

To study these symmetric k-varieties one needs first a classification of the related involutions up to isomorphy. Before we define what we mean with isomorphy of involutions we need a bit more notation.

For $A \in \mathrm{GL}(n,k)$ let $\mathrm{Int}(A) = \mathcal{I}_A$ denote the inner automorphism defined by $\mathcal{I}_A(X) = A^{-1}XA$, $X \in \mathrm{GL}(n,k)$. Let $\mathrm{Aut}(G)$ denote the set of automorphisms of G, $\mathrm{Int}_k(G) = \{\mathrm{Int}(x) \mid x \in G\}$ the set of inner automorphisms of G and $\mathrm{Int}(G) = \{\mathrm{Int}(x) \mid x \in \bar{G} \text{ and } \mathrm{Int}(x)(G) \subset G\}$. An automorphism ϕ of G is called of *inner type* if $\phi = \mathcal{I}_A$ for some $A \in \mathrm{GL}(n,\bar{k})$. Otherwise ϕ is called of *outer type*. Note that for $G = \mathrm{SL}(n,k)$ one can consider conjugation by elements of $\mathrm{GL}(n,\bar{k})$ instead of conjugation by elements of $\mathrm{SL}(n,\bar{k})$.

Definition 2. $\theta, \phi \in \mathrm{Aut}(G)$ are said to be k_1-conjugate or k_1-isomorphic if and only if there is a $\chi \in \mathrm{Int}(G_1)$, such that $\chi^{-1}\theta\chi = \phi$. In the case that $k = k_1$ we will also say that they are conjugate or isomorphic.

A characterization of the isomorphy classes of the involutions was given in [Hel00] essentially using the following 3 invariants:

(1) classification of admissible (Γ, θ)-indices.

(2) classification of the G_k-isomorphy classes of k-involutions of the k-anisotropic kernel of G.

(3) classification of the G_k-isomorphy classes of k-inner elements of G.

For more details, see [Hel00]. The admissible (Γ, θ)-indices determine most of the fine structure of the symmetric k-varieties and a classification of these was included in [Hel00] as well. For k algebraically closed or k the real numbers the full classification can be found in [Hel88]. For other fields a classification of the remaining two invariants is still lacking. In particular the case of symmetric k-varieties over the $\mathfrak{p}$-adic numbers is of interest. We note that the above characterization only holds for k a perfect field.

To classify the remaining two invariants we start in this thesis with a full classification of the cases that $\bar{G} = \mathrm{SL}(n, \bar{k})$, resp. $\bar{G} = \mathrm{SO}(2n+1, \bar{k})$ and G is the corresponding k-split k-form of $\bar{G}$. We will not use the above characterization from [Hel00] and give a direct classification of the isomorphy classes.

1.2.1 k-inner elements

Assume G is k-split and let T be a maximal k-split torus of $\bar{G}$. Since G is k-split T is a maximal torus of $\bar{G}$ as well. Moreover from [Hel00, Theorem 8.33] it follows that we have the following characterization of the isomorphy classes:

Theorem 2 ([Hel00, Theorem 8.33]). *Any k-involution of G is isomorphic to one of the form $\sigma\,\mathrm{Int}(a)$, where σ is a representative of a $\bar{G}$-isomorphy class of k-involutions, A a maximal (σ, k)-split torus and $a \in A$.*

The set of *set of k-inner elements of* A is defined as the set of those $a \in A$ such that $\sigma\,\mathrm{Int}(a)$ is a k-involution of G by $I_k(A)$. We recall that from [Hel00, Lemma 9.7] it follows that one can find a set of representatives for the isomorphy classes of the involutions $\sigma\,\mathrm{Int}(A)$ in the set $I_k(A)/A_k^2$. Here A_k is the set a k-regular elements of A and $A_k^2 = \{a^2 \mid a \in A_k\}$. Note that the set $A_k/A_k^2 = \simeq (k^*/(k^*)^2)^n$. We rewrote the representatives for the isomorphy classes in the form $\sigma\,\mathrm{Int}(a)$ with σ one of the representatives from the algebraically closed case and in particular find a set of k-inner elements of A representing these isomorphy classes.

1.3 Summary of results in this thesis

Case by case, we classify the involutions, and determine the compactness of the fixed point group of each isomorphy class. We study the cases of $\mathrm{SL}(2,k)$, $\mathrm{SL}(n,k)$, and $\mathrm{SO}(2n+1,k)$, where the field k is one of algebraically closed, real, finite and p-adic.

1.3.1 $\mathrm{SL}(2,k)$

First we study the simple and fundamental case of $\mathrm{SL}(2,k)$. For $\mathrm{SL}(2,k)$, all involutions are inner automorphisms. And the representatives of different classes up to isomorphy can be chosen as $\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$, where q is a representative of k^*/k^{*2}. The fixed point groups of these involutions are $H^q = \left\{ \begin{pmatrix} x & y \\ ay & x \end{pmatrix} \,\middle|\, x^2 - qy^2 = 1 \right\}$. For $k = \mathbb{R}$ the real numbers only the group $H^{-1} = H^\theta$ is compact. Here θ is the involution defined by $\theta(g) = {}^Tg^{-1}$. For $k = \mathbb{Q}_p$ the p-adic numbers, all the fixed point group are compact except for the class corresponding to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

1.3.2 $\mathrm{SL}(n,k)$ $(n \geq 3)$

The results for $\mathrm{SL}(n,k)$ and $\mathrm{GL}(n,k)$ are very similar, the result of $\mathrm{SL}(n,k)$ can be extended to $\mathrm{GL}(n,k)$. Note that for $G = \mathrm{SL}(n,k)$ one can consider conjugation by elements of $\mathrm{GL}(n,\bar{k})$ instead of conjugation by elements of $\mathrm{SL}(n,\bar{k})$. Recall that if k is algebraically closed and $n \geq 2$, then $\|\operatorname{Aut}(G)/\operatorname{Int}(G)\| = 2$. The involution θ defined by $\theta(A) = ({}^TA)^{-1}$, $A \in \mathrm{GL}(n,k)$, is of outer type. So any automorphism (thus involution as well) can be written as $\mathfrak{I}_A$ or $\theta\mathfrak{I}_A$. We have the classifications for individual fields as following.

$k = \bar{k}$: algebraically closed

(1) If n is odd, there are $\frac{n+1}{2}$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_A$ with A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$ and θ.

(2) If n is even, there are $\frac{n}{2} + 2$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_A$ with A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$, θ and $\theta\mathfrak{I}_{J_n}$.

$k = \mathbb{R}$: the real numbers

(1) If n is odd, there are n isomorphy classes of involutions. Representatives are θ, $\mathfrak{I}_A$ and $\theta\mathfrak{I}_A$ with A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$.

(2) If n is even, there are $n+3$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_{J_n}$, θ, $\theta\mathfrak{I}_{J_n}$, $\mathfrak{I}_A$ and $\theta\mathfrak{I}_A$ with A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$.

$k = \mathbb{F}_p$: finite field, $p \neq 2$

Let N_p be a non trivial representative of $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$.

(1) If n is odd, there are $\frac{n-1}{2} + 2$ isomorphy classes of involutions. Representatives are θ, $\mathfrak{I}_A$ and $\theta\mathfrak{I}_B$ where A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$ and B is M_{n,N_p}.

(2) If n is even, there are $\frac{n}{2} + 4$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_A$, $\mathfrak{I}_B$, θ, $\theta\mathfrak{I}_{J_n}$ and $\theta\mathfrak{I}_C$ with A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$, B is L_{n,N_p}, and C is M_{n,N_p}.

$k = \mathbb{Q}_p$: the p-adic numbers

If $p \neq 2$, then we take $1, p, N_p, pN_p$ as representatives of $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ and if $p = 2$, then we take $\{1, -1, 2, -2, -3, 3, 6, -6\}$ as representatives.

(1) If n is even, then there are $\frac{n}{2} + 9$ isomorphy classes of involutions for $p \neq 2$, $\frac{n}{2} + 17$ for $p = 2$. Representatives are

(a) $p \neq 2$: $\mathfrak{I}_A$, $\mathfrak{I}_B$, θ, $\theta\mathfrak{I}_{J_n}$ and $\theta\mathfrak{I}_C$ and $\theta\mathfrak{I}_D$. Here A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$, B is $L_{n,x}$ with $x = N_p,\ p$, or pN_p and C is $M_{n,x}$ with $x = N_p,\ p$, or pN_p. For the matrix D we have the following cases:

$$D = \begin{cases} K_{n,p,N_p,pN_p} & \text{if } -1 \in \mathbb{Q}_p^2 \\ N_{n,p,p} & \text{if } -1 \notin \mathbb{Q}_p^2 \text{ and } n = 4k \\ K_{n,p,p,N_p} & \text{if } -1 \notin \mathbb{Q}_p^2 \text{ and } n = 4k+2 \end{cases}$$

(b) $p = 2$: The same as $p \neq 2$, but x in B and C are chosen from 2, 3, 6, -1, -2, -3, -6, and D is $I_{n-2,2}$ if $n = 4k$ and $K_{n,2,3,-6}$ if $n = 4k+2$.

(2) If $n = 4k+1$, there are $\frac{n-1}{2} + 2$ isomorphy classes of involutions if $-1 \in \mathbb{Q}_p^2$, otherwise $\frac{n-1}{2} + 1$. Representatives are $\mathfrak{I}_A$, θ, and possibly $\mathfrak{I}_D$ if $-1 \in \mathbb{Q}_p^2$, where A is one of the following: $I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n-1}{2}$ and D is K_{n,p,N_p,pN_p}.

(3) If $n = 4k+3$, there are $\frac{n-1}{2} + 2$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_A$, θ and $\theta\mathfrak{I}_D$, where A is one of the following: $I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n-1}{2}$ and D is

$$D = \begin{cases} K_{n,p,N_p,pN_p} & \text{if } -1 \in \mathbb{Q}_p^2 \\ N_{n,p,p} & \text{if } -1 \notin \mathbb{Q}_p^2 \\ I_{n-2,2} & \text{if } p = 2 \end{cases}$$

Compact fixed point groups

For the classification, we determine which fixed point group are compact. For $k = \mathbb{R}$, we proved that the only compact fixed point group is for the involution θ. For $\mathbb{Q}_p$ with $p \neq 2$, the involutions with compact fixed point groups are

(1) $\operatorname{rank}(G) = n = 3$: $\theta\mathfrak{I}_A$ and $\theta\mathfrak{I}_B$, where A is $M_{3,p}$ and B is M_{3,pN_p}.

(2) $\operatorname{rank}(G) = n = 4$: $\theta\mathfrak{I}_A$ if $-1 \in \mathbb{Q}_p^2$, where A is $N_{4,p,p}$.

(3) $\operatorname{rank}(G) = n > 4$. None.

For $\mathbb{Q}_2$ the involutions with compact fixed point groups are

(1) $\operatorname{rank}(G) = n = 3$: θ, $\theta\mathfrak{I}_{M_{3,2}}$, $\theta\mathfrak{I}_{M_{3,-3}}$and $\theta\mathfrak{I}_{M_{3,-6}}$.

(2) $\operatorname{rank}(G) = n = 4$: θ.

(3) $\operatorname{rank}(G) = n > 4$. None.

***k*-inner elements.**

We translate those explicit classifications to fit the characterization given in [Hel00] and introduce the (Γ, σ)-indices, by which we could verify our results in different point of view.

1.3.3 $\mathrm{SO}(2n+1, k)$

Classification for individual fields

For $k = \mathbb{C}$, $\mathbb{R}$ and $\mathbb{F}_p$ we have the following classification of the isomorphy classes of involutions of $\mathrm{SO}(2n+1, k)$:

(1) $k = \mathbb{C}$, the complex numbers. There is only one $\mathrm{SO}(2n+1, k)$-conjugacy class for each $\mathrm{GL}(2n+1, k)$-conjugacy class.

(2) $k = \mathbb{R}$, the real numbers. There is only one $\mathrm{SO}(2n+1, k)$-conjugacy class for each $\mathrm{GL}(2n+1, k)$-conjugacy class.

(3) $k = \mathbb{F}_p$, $(p \neq 2)$ a finite field. There are two $\mathrm{SO}(2n+1, k)$-conjugacy classes for each $\mathrm{GL}(2n+1, k)$-conjugacy class. Representatives are I_A with $A = X^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}X$ and $X = \mathrm{Id}$ or $X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & a & b & 0 \\ \vdots & -b & a & \vdots \\ 0 & 0 & \dots & I_{t-1} \end{pmatrix}$, where $a^2 + b^2 \notin k^2$.

(4) $k = \mathbb{Q}_p$, $p \neq 2$ the p-adic numbers. There are in general more subclasses for each $\mathrm{GL}(2n+1, \mathbb{Q}_p)$ class. We have the following possible representatives for each $\mathrm{GL}(2n+1, k)$ class provided t is big enough. Where $c_1 + 1$, $c_2 + 1$ and $c_3 + 1$ are in the p, N_p and pN_p subsets respectively of $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$.

(a) $$A^{(3)} = \begin{pmatrix} I_{s-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & -I_{t-3} & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 & -c_1 & 0 & 0 \\ 0 & 0 & d_2 & 0 & 0 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & d_3 & 0 & 0 & 0 & -c_3 \end{pmatrix}.$$

(b) $$A_1^{(1)} = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & c_1 & 0 & d_1 \\ 0 & 0 & -I_{t-1} & 0 \\ 0 & d_1 & 0 & -c_1 \end{pmatrix}.$$

(c) $$A_{2,3}^{(2)} = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & d_2 & 0 \\ 0 & 0 & c_3 & 0 & 0 & d_3 \\ 0 & 0 & 0 & -I_{t-2} & 0 & 0 \\ 0 & d_2 & 0 & 0 & -c_2 & 0 \\ 0 & 0 & d_3 & 0 & 0 & -c_3 \end{pmatrix}.$$

(d) $$A_2^{(1)} = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & c_2 & 0 & d_2 \\ 0 & 0 & -I_{t-1} & 0 \\ 0 & d_2 & 0 & -c_2 \end{pmatrix}.$$

(e) $$A_{1,3}^{(2)} = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & d_1 & 0 \\ 0 & 0 & c_3 & 0 & 0 & d_3 \\ 0 & 0 & 0 & -I_{t-2} & 0 & 0 \\ 0 & d_1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & d_3 & 0 & 0 & -c_3 \end{pmatrix}.$$

(f) $$A_3^{(1)} = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & c_3 & 0 & d_3 \\ 0 & 0 & -I_{t-1} & 0 \\ 0 & d_3 & 0 & -c_3 \end{pmatrix}.$$

(g) $$A_{1,2}^{(2)} = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & d_1 & 0 \\ 0 & 0 & c_2 & 0 & 0 & d_2 \\ 0 & 0 & 0 & -I_{t-2} & 0 & 0 \\ 0 & d_1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & d_2 & 0 & 0 & -c_2 \end{pmatrix}.$$

Fixed point groups

For algebraically closed fields and the real numbers there is a unique isomorphy class of involutions represented by the involution $\mathcal{I}_{I_{s,t}}$, which has as fixed point group

$$\left\{ \begin{pmatrix} N_s & 0 \\ 0 & N_t \end{pmatrix} \in \mathrm{SO}(2n+1,k) \;\middle|\; N_s \in \mathrm{SO}(s,k) \text{ and } N_t \in \mathrm{SO}(t,k) \right\},$$

which is isomorphic to $\mathrm{SO}(s,k) \times \mathrm{SO}(t,k)$. This group is never compact for algebraically closed fields and compact for the field of the real numbers.

The only compact fixed point groups for $\mathrm{SO}(2n+1,\mathbb{Q}_p)$ are in $\mathrm{SO}(3,\mathbb{Q}_p)$ with $t=1$ or $t=0$. All possibilities are for the following values of X^TX.

(1) $-1 \in \mathbb{Q}_p^2$.

(a) in the $\mathrm{GL}(3,k)$-conjugate class of $I_{2,1}$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & N_p & 0 \\ 0 & 0 & N_p \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & pN_p & 0 \\ 0 & 0 & pN_p \end{pmatrix}$.

(b) in the $\mathrm{GL}(3,k)$-conjugate class of $I_{3,0}$: None.

(2) $-1 \notin \mathbb{Q}_p^2$.

(a) in the $\mathrm{GL}(3,k)$-conjugate class of $I_{2,1}$: I, $\begin{pmatrix} N_p & 0 & 0 \\ 0 & pN_p & 0 \\ 0 & 0 & p \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & N_p & 0 \\ 0 & 0 & N_p \end{pmatrix}$, $\begin{pmatrix} p & 0 & 0 \\ 0 & N_p & 0 \\ 0 & 0 & pN_p \end{pmatrix}$.

(b) in the $\mathrm{GL}(3,k)$-conjugate class of $I_{3,0}$: I.

(Γ, σ)-indices

In this section, we only consider the situation $-1 \in \mathbb{Q}_p^2$. Since the group is k-split the (Γ,σ)-indices are exactly the σ-indices of the case that $k = \bar{k}$ is algebraically closed, only with an additional label Γ under all the black nodes in the σ-index. The latter were classified [Hel88, Table II]. We recall that in the case $k = \bar{k}$ there is a bijective correspondence between the isomorphy classes of k-involutions and the congruence classes of σ-indices (see [Hel88, Theorem 3.11]). For notations on the (Γ, σ)-indices we refer to [Hel00, Section 5]. And more details are provided in later chapters. The involutions of $\bar{G} = \mathrm{SO}(2n+1, \bar{k})$ corresponding to these (Γ, σ)-indices are the following: $\mathcal{I}_A$ with A is one of $I_{s,t}$, $s = 1, 2, \ldots, t$, $s \geq t$ with

$$I_{s,t} = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}.$$

In some of the above cases the restricted root system for the related symmetric k-variety changes its type for different values of its rank s. Therefor in some cases we have more then one type of (Γ, σ)-index related to involutions of similar type.

k-inner elements for $\mathrm{SO}(2n+1, k)$

In the section, we restrict the field $\mathbb{Q}_p$ to be the ones in which -1 is a square. We first determine the maximal (σ, k)-split tori. Let T be a maximal k-split torus of $\bar{G}$. Since G is k-split T is a maximal torus of $\bar{G}$ as well. Our main results and classification are as following: First we prove that $T = \begin{pmatrix} a_1 & b_1 & \ldots & 0 & 0 & 0 \\ -b_1 & a_1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n & b_n & 0 \\ 0 & 0 & \ldots & -b_n & a_n & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}$ is a maximal torus for $\mathrm{SO}(2n+1, k)$, where $a_i^2 + b_i^2 = 1$, for $i = 1, 2, \ldots, n$.

Furthermore we prove that the maximal (σ, k)-split tori for $I_{s,t}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$ can be chosen as:

$$A_{s,n} = \begin{pmatrix} a_1 & \ldots & 0 & \ldots & 0 & \ldots & b_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & a_s & \ldots & b_s & \ldots & 0 \\ \vdots & \vdots & \vdots & I_{n-2s} & \vdots & \vdots & \vdots \\ 0 & \ldots & -b_s & \ldots & a_s & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_1 & \ldots & 0 & \ldots & 0 & \ldots & a_1 \end{pmatrix},$$

the dimension of the maximal (σ, k)-split torus is of course s; Next, we show that the matrices $A = \begin{pmatrix} -a & b \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} -c & d \\ d & c \end{pmatrix}$ are conjugate over $\mathrm{SO}(2, k)$ iff $a + 1 =$

$e^2(c+1)$ for some $e \in k$, where $a^2+b^2 = c^2+d^2 = 1$ and $a \neq -1$, $c \neq -1$. Thus for the real numbers the matrices $A = \begin{pmatrix} -a & b \\ b & a \end{pmatrix}$ are all conjugate over $\mathrm{SO}(2n+1,k)$, while for the field $\mathbb{Q}_p$, the matrices $A = \begin{pmatrix} -a & b \\ b & a \end{pmatrix}$ have four (respective eight) different isomorphy classes for $p \neq 2$ (respectively $p = 2$), where $a^2+b^2 = 1$. Finally we prove that all the k-inner elements (thus the involutions) for $\mathrm{SO}(2n+1,k)$ are conjugate to one of the following over $\mathrm{SO}(2n+1,k)$:

$$\begin{pmatrix} -a_1 & b_1 & \dots & 0 & 0 & 0 \\ b_1 & a_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -a_t & b_t & 0 \\ 0 & 0 & \dots & b_t & a_t & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}$$

Where $a_i^2 + b_i^2 = 1$ and $a_i \neq -1$ for $i = 1, 2, \dots, t$. For k the real numbers, each of these forms are conjugate over $\mathrm{SO}(2,\mathbb{R})$ as long as long t is fixed. And for p-adic, we prove that the $n \times n$ matrices

$$\begin{pmatrix} -a_1 & b_1 & \dots & 0 & 0 & 0 \\ b_1 & a_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -a_t & b_t & 0 \\ 0 & 0 & \dots & b_t & a_t & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix} \text{ and } \begin{pmatrix} -a_1' & b_1' & \dots & 0 & 0 & 0 \\ b_1' & a_1' & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -a_t' & b_t' & 0 \\ 0 & 0 & \dots & b_t' & a_t' & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}$$

are conjugate over $\mathrm{SO}(2n+1,k)$ iff the matrices

$$\begin{pmatrix} (a_1+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a_t+1) \end{pmatrix} \text{ and } \begin{pmatrix} (a_1'+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a_t'+1) \end{pmatrix}$$

are congruent. Therefore there are at most eight (resp. sixteen) isomorphy classes over $\mathrm{SO}(2n+1,\mathbb{Q}_p)$ for $p \neq 2$ (resp. $p = 2$).

Chapter 2

SL$(2, k)$ case

2.1 Involutions and their invariants

In this chapter we give a simple characterization of the isomorphy classes of involutions of SL$(2, k)$ with k any field of characteristic not 2. We also classify the isomorphy classes of involutions for k algebraically closed, the real numbers, the $\mathfrak{p}$-adic numbers and finite fields. We determine in which cases the corresponding fixed point group H is k-anisotropic. In those cases the corresponding symmetric k-variety consists of semisimple elements.

Let G be a connected reductive algebraic group defined over a field k of characteristic not 2, θ an involution of G defined over k, H a k-open subgroup of the fixed point group of θ and G_k (resp. H_k) the set of k-rational points of G (resp. H). The variety G_k/H_k is called a symmetric k-variety. These varieties occur in many problems in representation theory, geometry and singularity theory. To study these symmetric k-varieties one needs first a classification of the related k-involutions. A characterization of the isomorphy classes of the k-involutions was given in [Hel00] essentially using the following 3 invariants:

(1) classification of admissible (Γ, θ)-indices.

(2) classification of the G_k-isomorphy classes of k-involutions of the k-anisotropic kernel of G.

(3) classification of the G_k-isomorphy classes of k-inner elements of G.

For more details, see [Hel00]. The admissible (Γ, θ)-indices determine most of the fine structure of the symmetric k-varieties and a classification of these was included in [Hel00] as well. For k algebraically closed or k the real numbers the full classification can be found in [Hel88]. For other fields a classification of the remaining two invariants is still lacking. In particular the case of symmetric k-varieties over the $\mathfrak{p}$-adic numbers is of interest. We note that the above characterization only holds for k a perfect field.

To classify the remaining two invariants we start in this chapter with a full classification of the case that $G = \mathrm{SL}(2, k)$. This case will be fundamental in the analysis of the general case. We will first give a simple characterization of the isomorphy classes of k-involutions, which does not depend on any of the results in [Hel00]. Also these results hold for any field of characteristic not 2, not only perfect fields. Next we classify the possible isomorphy classes for k algebraically closed, the real numbers, the $\mathfrak{p}$-adic numbers and finite fields. Finally we determine the fixed point groups and determine which are k-anisotropic. For k the $\mathfrak{p}$-adic numbers the symmetric k-varieties G_k/H_k with H_k k-anisotropic have a similar structure as the Riemannian symmetric spaces and therefor these cases are of particular interest for studying their representations. The results in this paper will play a fundamental role in the classification of the isomorphy classes of involutions of $\mathrm{SL}(n, k)$. This will be discussed in the following chapters.

2.1.1 Characterization of the isomorphy classes of involutions

Our basic reference for reductive groups will be the papers of Borel and Tits [BT65, BT72] and also the books of Borel [Bor91], Humphreys [Hum75] and Springer [Spr81]. We shall follow their notations and terminology. All algebraic groups and algebraic varieties are taken over an arbitrary field k (of characteristic $\neq 2$) and all algebraic groups considered are linear algebraic groups.

2.1.2 Automorphisms of G

For $A \in \bar{G}$ let $\mathrm{Int}(A) = \mathcal{I}_A$ denote the inner automorphism defined by $\mathcal{I}_A(X) = A^{-1}XA$, $X \in \bar{G}$. Let $\mathrm{Aut}(G)$ denote the set of automorphisms of G and $\mathrm{Int}(G) = \{\mathrm{Int}(x) \mid x \in G\}$ the set of inner automorphisms of G. In this subsection we characterize the group $\mathrm{Aut}(G)$. First we recall the following:

Definition 3. $\theta, \phi \in \mathrm{Aut}(\bar{G})$ are said to be k-conjugate if and only if there is a $\chi \in \mathrm{Int}(G)$, such that $\chi^{-1}\theta\chi = \phi$.

We recall the following results, which can be found in [Bor91]:

Lemma 1. *If k is an algebraically closed field, then we have* $\mathrm{Aut}(G) = \mathrm{Int}(G)$.

Remark 1. For $\theta \in \mathrm{Aut}(G)$ and k is not algebraically closed, there always exists an extension field k_1 of k and $\tau \in \mathrm{Int}(G_1)$ such that $\tau|G = \theta$, i.e. there exists a 2×2-matrix $A \in G_1$, such that $\theta = \tau|G = \mathcal{I}_A|G$.

Lemma 2. *Suppose $A \in \mathrm{GL}(2,k_1)$. If $\mathcal{I}_A|G = \mathrm{Id}$, then $A = p\,\mathrm{Id}$ for some $p \in k_1$.*

Proof. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in k_1$. Since $\mathcal{I}_A|G = \mathrm{Id}$, we have for all $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in G$, $\mathcal{I}_A(X) = AXA^{-1} = X$, i.e. $AX = XA$. So

$$\begin{pmatrix} ax_1 + bx_3 & ax_2 + bx_4 \\ cx_1 + dx_3 & cx_2 + dx_4 \end{pmatrix} = \begin{pmatrix} ax_1 + cx_2 & bx_1 + dx_2 \\ ax_3 + cx_4 & bx_3 + dx_4 \end{pmatrix}.$$

Since X is arbitrary, we have $a = d$ and $b = c = 0$, i.e. $A = a\,\mathrm{Id}$. □

This result enables us to determine when an element of $\mathrm{Int}(G_1)$ keeps G invariant:

Lemma 3. *$\mathcal{I}_A \in \mathrm{Int}(G_1)$ keeps G invariant if and only if $A = pB$ for some $p \in k_1$, $B \in \mathrm{GL}(2,k)$.*

Proof. Write $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $a_i \in k_1$, $i = 1,2,3,4$. So $A^{-1} = (\det A)^{-1} \begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}$. For all $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in G$, we have $\mathcal{I}_A(X) = A^{-1}XA =$

$$(\det A)^{-1} \begin{pmatrix} a_1a_4x_1 + a_3a_4x_2 - a_1a_2x_3 - a_2a_3x_4 & a_2a_4x_1 + a_4^2x_2 - a_2^2x_3 - a_2a_4x_4 \\ -a_1a_3x_1 - a_3^2x_2 + a_1^2x_3 + a_3a_4x_4 & -a_2a_3x_1 - a_3a_4x_2 + a_1a_2x_3 + a_1a_4x_4 \end{pmatrix}.$$

Since X is arbitrary we have

$$\mathcal{I}_A(X) \in G \Longleftrightarrow \frac{a_ia_j}{(a_1a_4 - a_2a_3)} \in k,\ \forall i,j = 1,2,3,4 \Longleftrightarrow \frac{a_i}{a_j} \in k, \forall i,j = 1,2,3,4,$$

provided $a_j \neq 0$, i.e. $A = pB$ for some $p \in k_1$, $B \in \mathrm{GL}(2,k)$. □

Remark 2. Since for all $p \in k_1$ and $B \in \mathrm{GL}(2,k_1)$, we always have $\mathcal{I}_{pB} = \mathcal{I}_B$. It follows that every automorphism over $\mathrm{SL}(2,k)$ can be written as the restriction to $\mathrm{SL}(2,k)$ of a inner automorphism over $\mathrm{GL}(2,k)$.

2.2 Involutions on $G = \mathrm{SL}(2,k)$

Now let's turn to involutions on $G = \mathrm{SL}(2,k)$. Suppose $\theta \in \mathrm{Aut}(G)$ is an involution, i.e. $\theta^2 = \mathrm{Id}$. By Lemma 1, there is a field $k_1 \supset k$ and an $A \in G_1 = \mathrm{SL}(2,k_1)$ such that $\theta = \mathcal{I}_A|G$.

Lemma 4. *Suppose $k \subset k_1$, $\theta \in \mathrm{Aut}(G)$ is an involution, then there is a matrix $A \in \mathrm{GL}(2,k)$, such that $\theta = \mathfrak{I}_A|G$ and A is conjugate to $\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$, for some $p, q \in k$.*

Proof. Since θ is an involution, we know there is a matrix $A \in \mathrm{GL}(2,k)$ such that $\mathfrak{I}_A^2 = \theta^2 = \mathrm{Id}$, i.e. $\mathfrak{I}_{A^2} = \mathrm{Id}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so $A^2 = \begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & d^2+bc \end{pmatrix}$ By Lemma 2, we have $A^2 = p\,\mathrm{Id}$, for some $p \in k$, i.e. $a^2+bc = d^2+bc$ and $(a+d)c = (a+d)b = 0$.

(1) If $a+d=0$, then we have two subcases:

(a) If we also have $b=c=0$, then we get $A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$.

Let $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, then $P^{-1}AP = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$.

(b) If $a+d=0$ and both b and c are not equal to 0, then without loss of generality, we may assume that $b \neq 0$. Let $P = \begin{pmatrix} 1 & 0 \\ -a/b & 1 \end{pmatrix}$, then we have $P^{-1}AP = \begin{pmatrix} 0 & b \\ a^2/b+c & 0 \end{pmatrix}$.

(2) $a+d \neq 0 \Rightarrow a = d \neq 0, b = c = 0$ i.e. $A = a\,\mathrm{Id}, a \in k$, then θ is the identity, which is not a involution.

□

Since by remark 2, we can multiply A by any constant without changing the involution, we get:

Corollary 1. *All the classes of involutions over G can be represented by the matrices $\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \in \mathrm{GL}(2,k)$.*

Lemma 5. *Let $A = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \in G$. Then $\mathfrak{I}_A$ is conjugate to $\mathfrak{I}_B$ if and only if there is a matrix $X \in G$ and a constant $c \in k$, such that $X^{-1}AX = cB$.*

Proof. The result follows from the following equivalent statements:

- $\mathfrak{I}_A$ is conjugate to $\mathfrak{I}_B$.
- there is a matrix $X \in \mathrm{GL}(2,k)$, such that $\mathfrak{I}_{X^{-1}}\mathfrak{I}_A\mathfrak{I}_X = \mathfrak{I}_B$.
- $\mathfrak{I}_{X^{-1}AXB^{-1}} = \mathrm{Id}$.

- there is $c \in k$, such that $X^{-1}AXB^{-1} = c\,\mathrm{Id}$ (Lemma 2, $k_1 = k$).
- $X^{-1}AX = cB$.

$\square$

Theorem 3. *Suppose $\theta, \phi \in \mathrm{Aut}(G)$ involutions of which the corresponding matrices in G are $A = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$. Then θ is conjugate to ϕ if and only if q/p is a square in k^*.*

Proof. ($\Rightarrow$) By the result of Lemma 5, and by taking determinants on both sides, we have $-p = -c^2 q$, i.e. $c^2{=}p/q$.
($\Leftarrow$) Suppose $c^2{=}q/p$, $c \in k$, $X = \begin{pmatrix} 0 & 1 \\ pc & 0 \end{pmatrix}$, then $X^{-1}BX = cA$. By Lemma 5, we're done. $\square$

For a field k let k^* denote the product group of all the nonzero elements and $(k^*)^2 = \{a^2 \mid a \in k^*\}$. Then $(k^*)^2$ is a normal subgroup of k^*.

Corollary 2. *The number of isomorphy classes of involutions over G equals the order of $k^*/(k^*)^2$.*

Remark 3. For k a perfect field the results in this section could also have been derived from the characterization of the k-involutions in [Hel00]. The diagonal subgroup T is a maximal k-split torus, hence there is no k-anisotropic kernel. Since the root system $\Phi(T)$ of T with respect to G is of type A_1, there is only one nontrivial involution of $\Phi(T)$. So in this case one only needs to check the third invariant mentioned in the introduction. Let $\theta \in \mathrm{Aut}(G)$ be the involution defined by $\theta(g) = {}^T g^{-1}$, $g \in G$. Note that $\theta = \mathfrak{I}_A$ with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then any other involution of G is isomorphic to $\theta\,\mathrm{Int}(x)$ for some $x \in T$. It suffices to check then the isomorphy classes of these involutions. It is easy to check that for the choice of x one can restrict to the elements $x = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with a a representative of $k^*/(k^*)^2$. Then finally one has to check that none of these involutions are isomorphic to each other.

2.3 Classification of the isomorphy classes of involutions

In this section we give a classification of the isomorphy classes of involutions for k algebraically closed, the real numbers, finite fields and the $\mathfrak{p}$-adic numbers.

Let $\theta = \mathfrak{I}_A$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\tau = \mathfrak{I}_B$, where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(1) **Algebraically closed fields.** In this case $|k^*/(k^*)^2| = 1$, so there is only one involution θ, where $\theta(g) = \mathcal{I}_A(g) = {}^T g^{-1}$.

(2) **Real numbers $\mathbb{R}$.** In this case $\mathbb{R}^*/(\mathbb{R}^*)^2 \simeq \mathbb{Z}_2$, so there are two involutions θ and τ.

(3) **Rational numbers $\mathbb{Q}$.** In this case $|k^*/(k^*)^2| = \infty$, so there are infinitely many involutions.

(4) **Finite fields ($\mathbb{F}_p$, $p \neq 2$).** In this case $\mathbb{F}_p^*/\mathbb{F}_p^{*2} \simeq \mathbb{Z}_2$, so there are only two involutions. This can be seen as follows. Let $\phi : \mathbb{F}_p \to \mathbb{F}_p$ be $\phi(x) = x^2$, then $\phi(\mathbb{F}_p^*) = \mathbb{F}_p^{*2}$ is a normal subgroup of $\mathbb{F}_p^*$ and $|\mathbb{F}_p^*/\mathbb{F}_p^{*2}| = |\operatorname{Ker}(\phi)| = 2$. Let $1, 2, \ldots, p-1$ be the the representative of $\mathbb{F}_p^*$, N_p be the "smallest number" in $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$, so 1 and N_p are the representatives of $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$, hence the corresponding involutions are also clear now.

(5) **$\mathfrak{p}$-adic numbers $\mathbb{Q}_p$.** If $p \neq 2$, there are four involutions and $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. This can be seen as follows. Recall that $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ relative to the $\mathfrak{p}$-adic norm, and explicitly

$$\mathbb{Q}_p = \left\{ \sum_{i=-n}^{\infty} a_i p^i \;\middle|\; a_i = 0, 1, \ldots, p-1, \text{ with } a_n \neq 0, n \in \mathbb{Z} \right\}.$$

The norm of $x = \sum_{i=-n}^{\infty} a_i p^i$ with $a_n \neq 0,\ a_i = 0, 1, \ldots, p-1$ is defined as p^{-n}, i.e. $|x|_p = p^{-n}$. With this norm, the distance of any two points is one of the countable choices $p^{-n}, n \in \mathbb{Z}$. Now let's turn to the squares of $\mathfrak{p}$-adic numbers. First of all, p can't be a square. Otherwise if we have $x \in \mathbb{Q}_p$ such that $x^2 = p$, then x has to begin with positive power of p, which makes the square of x begin with p^2 at least and disagree with p. Note that the squares of the $\mathfrak{p}$-adic numbers must have its leading coefficient in $\mathbb{F}_p^{*2}$, so N_p (same notation as (4)) couldn't be a square. $N_p p^{-1}$ is not a square either by the same argument. That means p and N_p are in different cosets and $pN_p = p^{-1}N_p p^2$ isn't a square either. It's easy to check 1, p, N_p are pN_p are in different cosets. And actually they are representatives for $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, i.e. the matrices corresponding to different involutions for $\mathbb{Q}_p (p \neq 2)$ are

$$\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, a = 1, p, N_p \textit{ or } pN_p. \tag{2.1}$$

Interestingly $\mathbb{Q}_2$ has eight involutions. The corresponding matrices are

$$\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, a \in \{1,-1,2,-2,-3,3,6,-6\} \tag{2.2}$$

The details about cosets for $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$ can also be found in Mahler's book [Mah81].

2.4 Fixed point groups and symmetric k-varieties

The fixed-point group H^δ for an involution δ over G is defined by

$$H^\delta = \{x \in G \mid \delta(x) = x\}.$$

The fixed point group determines a lot of the structure of the corresponding symmetric k-variety $X := \{g\delta(g)^{-1} \mid g \in G\}$. It is easy to see that $X \simeq G/H^\delta$. Moreover if H^δ is compact then from [HW93] it follows that X consists of semisimple elements:

Proposition 2 ([HW93, Proposition 10.8]). *Let G be a connected reductive algebraic k-group with* $\mathrm{char}(k) = 0$ *and $X = \{g\theta(g)^{-1} | g \in G\}$. Suppose that $H \cap [G,G]$ is anisotropic over k. Then X_k consists of semi-simple elements.*

In view of this result it is important to determine which involutions have an k-anisotropic fixed point group. For $k = \mathbb{R}$ or $\mathbb{Q}_p$ all k-anisotropic subgroups are compact. In this section we determine the fixed point groups of the involutions in the previous section. Our main focus will be given to the $\mathfrak{p}$-adic fields $\mathbb{Q}_p$. From [HW93, Proposition 1.2] we know that two involutions are conjugate if and only if their corresponding fixed-point groups are conjugate. Combined with Corollary 1, it follows that it suffices to determine the fixed-point group of the involutions $\delta = \mathcal{I}_A$ with $A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$ and a representative of k^*/k^{*2}. We will also write H^a for H^δ.

$$H^a = \left\{ \{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \;\middle|\; \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, xw - yz = 1 \right\}.$$

That is

$$H^a = \left\{ \begin{pmatrix} x & y \\ ay & x \end{pmatrix} \;\middle|\; x^2 - ay^2 = 1 \right\}.$$

For k the complex numbers we have only one involution and for k the real numbers there are two. Corresponding to the involutions θ and τ, the two fixed-point groups are

$$H^\theta = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and

$$H^\tau = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

They are conjugate over the complex numbers but not over the real numbers, since over the real numbers H^θ is compact while H^τ is not compact.

For $\mathbb{Q}_p$, choose a from (2.1) or (2.2), then the fixed-point groups corresponding to the representatives of involutions are

$$\left\{ \begin{pmatrix} x & y \\ ay & x \end{pmatrix} \;\middle|\; x^2 - ay^2 = 1 \right\}$$

Consider the curves

$$\mathbf{O}_a = \{(x, y) \mid x^2 - ay^2 = 1, x, y \in \mathbb{Q}_p\}$$

Where a is chosen from (2.1) or (2.2). Before we go any further, we recall some useful Lemmas:

Lemma 6. *Let $x \in \mathbb{Q}_p (p \neq 2)$. If $|x|_p \leq p^{-1}$, then $1 + x \in \mathbb{Q}_p^{*2}$. If $x \in \mathbb{Q}_2$ and $|x|_p \leq \frac{1}{8}$, then $1 + x \in \mathbb{Q}_2^{*2}$.*

Lemma 7. *For $p \neq 2$, $x^2 - ay^2 \leq 1$ implies $\max(|x|_p, |y|_p) \leq 1$. Where $a = p$, N_p or pN_p and $x, y \in \mathbb{Q}_p$.*

Lemma 8. *Suppose $x, y \in \mathbb{Q}_2$, $a = -1, -2, -6, 2, 3$ or 6, then $x^2 - ay^2 \leq 1$ implies $\max(|x|_p, |y|_p) \leq 1$; $x^2 + 3y^2 \leq 1$ implies $\max(|x|_p, |y|_p) \leq 2$.*

Similar results can be found in [Mah81], we omit the proofs here.

Lemma 9. *A bounded sequence in $\mathbb{Q}_p$ has a limit.*

Proof. Without loss of generality, suppose $\{x_n \in \mathbb{Q}_p \mid |x_n|_p \leq 1, n = 1, 2, \ldots\}$ is a bounded sequence. Name the sequence Φ^0 and write every $x_n^0 = x_n$ as the standard form

$$x_n^0 = \sum_{i=0}^{\infty} a_{ni}^0 p^i, a_{ni} = 0, 1, \ldots, p-1, \text{ and } n = 0, 1, 2, \ldots$$

Since a_{n0}^0 has only p choices, there must be a infinite subsequence $\{x_{n_k}^0\}$ with the same value for its first coefficients, i.e. all $a_{n_k 0}^0$'s are all the same, say b_0. Name the subsequence Φ^1 and write every x_n^1 as

$$x_n^1 = b_0 + \sum_{i=1}^{\infty} a_{ni}^1 p^i, a_{ni} = 0, 1, \ldots, p-1, \text{ and } n = 0, 1, 2, \ldots$$

To choose b_1 and b_i, perform the same choice on Φ^1 and $a^1_{n_1}$ and so on. By construction, the sequence has limit

$$x = \sum_{i=0}^{\infty} b_i p^i$$

□

Theorem 4. *(i) The orbits of the curves $\mathbf{O}_a$, $a \neq 1$ are all closed, bounded and (sequentially) compact.*

(ii) The orbit of the curve $\mathbf{O}_1$ is closed, but neither compact nor bounded.

Proof. (i). Take the norm of $\mathbb{Q}_p \times \mathbb{Q}_p$ to be the maximum of that of its components, i.e. for $z = (x, y) \in \mathbb{Q}_p \times \mathbb{Q}_p$, the norm of z is defined by $\|z\|_p = \max(|x|_p, |y|_p)$. First let's consider the functional $f(x, y)$ defined by:

$$(x, y) \in \mathbb{Q}_p \times \mathbb{Q}_p \to f(x, y) = x^2 - ay^2 \in \mathbb{Q}_p.$$

Since $|x^2 - ay^2|_p \leq |x|_p^2 + |a|_p|y|_p^2$, $f(x, y)$ is continuous. The orbit, which is the preimage of the closed set $\{1\}$ is also closed. Boundedness is immediate from Lemma 7 and 8. For $\mathbb{Q}_p$, compactness and sequential compactness are equivalent. For a infinite sequence in one of the orbits, it has a limit by Lemma 9. Since it's closed, the limit lies on the orbit. That proves the compactness and thus the first part of the theorem as well.

(ii). The same argument proves that $\mathbf{O}_1$ is closed. Now assume $x = p^{-n}\overline{x}$ and $y = p^{-n}$, we prove for some choices of $\overline{x}$, (x, y) lies on the orbit. Actually

$$x^2 - y^2 = p^{-2n}\overline{x}^2 - p^{-2n} = 1.$$

But then $\overline{x}^2 = 1 + p^{2n}$. By Lemma 6, there is a solution for $\overline{x}$ for $n \geq 2$. And $|y|_p = p^n$ converges to infinity as n goes to infinity. So $\mathbf{O}_1$ is unbounded and thus noncompact. □

2.4.1 k-anisotropic fixed point groups

To determine for general fields k if the fixed point groups H^a are k-anisotropic or k-split we use the following result:

Lemma 10. *The matrix $\left(\begin{smallmatrix} x & y \\ ay & x \end{smallmatrix}\right)$ is k-split (i.e. diagonizable over k) if $a \in (k^*)^2$ and k-anisotropic if $a \notin (k^*)^2$.*

Proof. The minimum polynomial of $\left(\begin{smallmatrix} x & y \\ ax & y \end{smallmatrix}\right)$ is $\lambda^2 - 2x\lambda + x^2 - ay^2$. This polynomial factors over k if and only if $a \in (k^*)^2$. □

Corollary 3. *The fixed point group H^a is k-split if $a \in (k^*)^2$ and k-anisotropic if $a \notin (k^*)^2$.*

It follows from this result that for $k = \mathbb{Q}_p$, when we restrict ourselves to the representatives of the isomorphy classes, the group H^a is k-anisotropic if and only if $a \neq 1$ as we saw earlier in this section.

2.4.2 SO$(2, k)$

For $a = -1$ the group H^a is the special orthogonal group $\mathrm{SO}(2,k) = \{\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right) \mid x^2 + y^2 = 1\}$. As we saw above this group is k-anisotropic for $k = \mathbb{R}$. In the following we determine when this group is k-split or k-anisotropic for $k = \mathbb{F}_q$ and $\mathbb{Q}_p$. For this we recall the following well known result for $k = \mathbb{F}_q$:

Lemma 11. *-1 is a square in $\mathbb{F}_q$ if $q \equiv 1(\mod 4)$ and -1 is not a square in $\mathbb{F}_q$ if $q \equiv 3(\mod 4)$.*

Proof. The group $\mathbb{F}_q^*$ is a cyclic group of order $q - 1$ and -1 is the only element of order 2. So -1 is a square if and only if the order of $\mathbb{F}_q^*$ is a multiple of 4. $\square$

It follows from the above results that for $k = \mathbb{F}_q$ the group SO$(2, k)$ is k-split if and only if $q \equiv 1(\mod 4)$. A similar results holds in the case that $k = \mathbb{Q}_p$.

Chapter 3

$\mathbf{SL}(\boldsymbol{n}, \boldsymbol{k}),\ (\boldsymbol{n} \geq 3)$

3.1 Preliminaries

Our basic reference for reductive groups will be the papers of Borel and Tits [BT65, ?] and also the books of Borel [Bor91], Humphreys [Hum75] and Springer [Spr81]. We shall follow their notations and terminology. All algebraic groups and algebraic varieties are taken over an arbitrary field k (of characteristic $\neq 2$) and all algebraic groups considered are linear algebraic groups.

We'll use the following notations for this chapter: $G = \mathrm{SL}(n, k)$, $G_1 = \mathrm{SL}(n, k_1)$, $\bar{G} = \mathrm{SL}(n, \bar{k})$, where field $\bar{k}$ is the algebraic closure of k and k_1 is an extension field of k. Let k^* denote the product group of all the nonzero elements and $(k^*)^2 = \{a^2 \mid a \in k^*\}$. $\mathcal{I}$ is the identity automorphism.

3.2 Automorphisms of G

For $A \in \mathrm{GL}(n, k)$ let $\mathrm{Int}(A) = \mathcal{I}_A$ denote the inner automorphism defined by $\mathcal{I}_A(X) = A^{-1}XA$, $X \in \mathrm{GL}(n, k)$. Let $\mathrm{Aut}(G)$ denote the set of automorphisms of G, $\mathrm{Int}_k(G) = \{\mathrm{Int}(x) \mid x \in G\}$ the set of inner automorphisms of G and $\mathrm{Int}(G) = \{\mathrm{Int}(x) \mid x \in \bar{G} \text{ and } \mathrm{Int}(x)(G) \subset G\}$. An automorphism ϕ of G is called of *inner type* or inner automorphism if $\phi = \mathcal{I}_A|G$ for some $A \in \mathrm{GL}(n, \bar{k})$. Otherwise ϕ is called of *outer type*, or outer automorphism.

Definition 4. $\theta, \phi \in \mathrm{Aut}(G)$ are said to be k_1-conjugate or k_1-isomorphic if and only if there is a $\chi \in \mathrm{Int}(G_1)$, such that $\chi^{-1}\theta\chi = \phi$. In the case that $k = k_1$ we will also say that they are conjugate or isomorphic.

Lemma 12. *If k is algebraically closed (i.e. $k = \bar{k}$) and $n \geq 2$, then* $\|\mathrm{Aut}(G)/\mathrm{Int}(G)\| = 2$.

3.3 Inner automorphisms

In this section we will first consider the case of involutions of inner type. By definition, for any automorphism θ of inner type, there exist a $n \times n$-matrix $A \in \mathrm{GL}(n, \bar{k})$, such that $\theta = \mathfrak{I}_A|G$.

Lemma 13. *Let $A \in \mathrm{GL}(n, \bar{k})$. If $\mathfrak{I}_A|G = \mathfrak{I}$, then $A = pI$ for some $p \in \bar{k}$, i.e. $\mathfrak{I}_A = \mathfrak{I}$ over* $\mathrm{GL}(\bar{V})$.

Proof. Since $\mathfrak{I}_A|G = \mathfrak{I}$, we have for all $X \in \mathrm{SL}(n, k)$, $\mathfrak{I}_A(X) = A^{-1}XA = X$, i.e. $AX = XA$. Since X is arbitrary it follows that $A = pI$ for some $p \in \bar{k}$. Furthermore $\mathfrak{I}_A = \mathfrak{I}_{pI} = \mathfrak{I}_I = \mathfrak{I}$. Let $A = (a_{ij})_{2n+1\times 2n+1}$ and for $s = 1, 2, \ldots, n-1$ let $X_s = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-s-1} \end{pmatrix}$. We have $X_s \in \mathrm{SL}(n, k)$, and $AX_s = X_sA$. That forces $a_{ij} = 0$ if $i \neq j$ and $a_{11} = a_{22} = \cdots = a_{n,n}$. $\square$

Lemma 14. *For any inner automorphism $\phi \in \mathrm{Int}(G)$, suppose $A \in \mathrm{GL}(n, \bar{k})$. Then $\phi = \mathfrak{I}_A \in \mathrm{Int}(\bar{G})$ keeps G invariant if and only if $A = pB$, for some $p \in \bar{k}$ and $B \in \mathrm{GL}(n, k)$. In other words, there is a matrix $B \in \mathrm{GL}(V)$ such that $\phi = \mathfrak{I}_B|G$.*

Proof. To prove ($\Longleftarrow$) is obvious. We'll concentrate on the other way.

($\Longrightarrow$) Assume

$$A = (a_{ij})_{n\times n} \in \mathrm{GL}(\bar{V})$$

and

$$X = (x_{ij})_{n\times n} \in G$$

where $a_{ij} \in \bar{k}$, $x_{ij} \in k$. We have $A^{-1} = \det(A)^{-1}(A_{ij})_{n\times n}$, where A_{ij} is the ij-th minor of A. Then

$$\mathfrak{I}_A(X) = A^{-1}XA = \det(A)^{-1}\left(\sum_{k=1}^{n}\sum_{l=1}^{n} A_{ik}x_{kl}a_{lj}\right)_{n\times n}.$$

We'll prove $\det(A)^{-1}A_{ij}a_{kl} \in k$. Without loss of generality, we prove the case for $\det(A)^{-1}A_{i1}a_{1l}$. Let $X = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \frac{1}{\delta} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$, and $Y = (y_{il}) = \mathfrak{I}_A(X)$, we have

$$y_{il} = \det(A)^{-1}\left(\delta A_{i1}a_{1l} + \frac{1}{\delta}A_{i2}a_{2l} + A_{i3}a_{3l} + \cdots + A_{in}a_{nl}\right) \in k,$$

for all $i, l = 1, 2, \ldots n$. Since $0 \neq \delta \in k$ is arbitrary, $\det(A)^{-1} A_{i1} a_{1l} \in k$ and in general $\det(A)^{-1} A_{ij} a_{kl} \in k$. Hence $a_{ij}/a_{kl} \in k$, for all $i, j, k, l = 1, 2, \ldots n$, provided $a_{kl} \neq 0$, i.e. $A = pB$, for some $p \in \bar{k}$ and $B \in \mathrm{GL}(n,k)$. And $\phi = \mathcal{I}_A = \mathcal{I}_{pB} = \mathcal{I}_B$, where $B \in \mathrm{GL}(V)$ $\square$

Lemma 15. *Suppose $A \in \mathrm{GL}(n,k)$ with $A^2 = pI$. Then*

(1) If $p = c^2 \in k^{*2}$ then A is conjugate to $cI_{i,n-i}$ for some $i = 0, 1, \ldots, n$.

(2) If p is not in k^{*2}, then n is even and A is conjugate to $L_{\frac{n}{2},p}$.

Proof. If there is a $c \in k$, such that $p = c^2$, then the characteristic polynomial of A is $(x-c)^i (x+c)^{n-i}$, and the minimal polynomial is a factor of $(x+c)(x-c)$. So A is conjugate to $cI_{i,n-i}$ for some $i = 0, 1, \ldots, n$.

If p is not in k^{*2}, then the minimal polynomial is $(x^2 - p)$, which does not factor over k, therefore the characteristic polynomial is a power of the minimal polynomial. Hence n, which is the degree of the characteristic polynomial, is even. Furthermore, A is conjugate to $L_{\frac{n}{2},p}$ since they have the same minimal and characteristic polynomials. $\square$

Lemma 16. *Suppose $\theta \in \mathrm{Aut}(G)$ is an involution of inner type. Then there is a matrix $A \in \mathrm{GL}(n,k)$, such that $\theta = \mathcal{I}_A$, where matrix A is conjugate to $cI_{i,n-i}$ or $L_{\frac{n}{2},p}$. Where $i \in \{0, 1, \ldots, n\}$, $c \in k^*$ and $p \in \bar{k}^* \setminus \bar{k}^{*2}$.*

Proof. By Lemma 13 we know that there is a matrix $A \in \mathrm{GL}(n,k)$, such that $\theta = \mathcal{I}_A$. Since θ is an involution we have $\theta^2 = \mathcal{I}_{A^2} = \mathcal{I}$. By Lemma 14 we have $A^2 = pI$ for some $p \in \bar{k}$. It follows that it is conjugate to one of the forms in the previous Lemma. $\square$

We established above that for any inner involution θ of G, there exists a matrix $A \in \mathrm{GL}(n,k)$, such that $\theta = \mathcal{I}_A$, and A is conjugate to $cI_{i,n-i}$ for some $i = 0, 1, \ldots, n$ and $c \in k^*$ or conjugate to $L_{\frac{n}{2},p}$. To determine the isomorphy classes of involutions of inner type we still need to determine which of these matrices lead to conjugate involutions. We consider this question in the following.

Lemma 17. *The matrices $I_{i,n-i}$ and $cI_{j,n-j}$ are conjugate for some $c \in k$ if and only if c is one of the following:*

(1) $c = 1$ and $i = j$.

(2) $c = -1$ and $i + j = n$.

Proof. Since the eigenvalues of both $I_{i,n-i}$ and $cI_{j,n-j}$ have to be exactly the same, that forces c to be 1 or -1. If $c = 1$, $I_{i,n-i}$ and $I_{j,n-j}$ are conjugate, therefore $i = j$. If $c = -1$, $I_{i,n-i}$ and $-I_{j,n-j} = I_{n-j,j}$ are conjugate, therefore $i = n-j$, i.e. $i+j = n$. $\square$

Lemma 18. *Let $p, q \in \bar{k}^* \setminus \bar{k}^{*2}$. The matrix $L_{\frac{n}{2},p}$ is conjugate to $cL_{\frac{n}{2},q}$ for some $c \in k$ if and only if $\frac{p}{q} \in k^{*2}$.*

Proof. The minimal polynomial of $L_{\frac{n}{2},p}$ is $(x^2 - p)$ and that of $cL_{\frac{n}{2},q}$ is $(x^2 - c^2 q)$. The characteristic polynomial of $L_{\frac{n}{2},p}$ is $(x^2 - p)^n$ and that of $cL_{\frac{n}{2},q}$ is $(x^2 - c^2 q)^n$, so $L_{\frac{n}{2},p}$ and $cL_{\frac{n}{2},q}$ are conjugate if and only $p = c^2 q$, what forces $\frac{p}{q} = c^2 \in k^{*2}$. ☐

Lemma 19. *The inner automorphisms $\theta = \mathcal{I}_A$ and $\phi = \mathcal{I}_B$ are conjugate iff the matrix A is conjugate to cB for some $c \in \bar{k}$.*

Proof. The result follows from the following equivalent statements:

- $\theta = \mathcal{I}_A$ is conjugate to $\phi = \mathcal{I}_B$.
- there is a matrix $X \in \mathrm{GL}(n, k)$, such that $\mathcal{I}_{X^{-1}} \mathcal{I}_A \mathcal{I}_X = \mathcal{I}_B$.
- $\mathcal{I}_{X^{-1}AXB^{-1}} = \mathrm{Id}$.
- there is $c \in k$, such that $X^{-1}AXB^{-1} = c\,\mathrm{Id}$ for some $c \in \bar{k}$ (see Lemma 13).
- $X^{-1}AX = cB$ for some $c \in \bar{k}$.
- A is conjugate to cB for some $c \in \bar{k}$.

☐

Theorem 5. *Suppose the involution $\theta \in \mathrm{Aut}(G)$ is of inner type. Then up to isomorphy θ is one of the following:*

(1) $\mathcal{I}_A|G$, where $A = I_{i,n-i} \in \mathrm{GL}(V)$ where $i \in \{1, 2, \ldots, [\frac{n}{2}]\}$.

(2) $\mathcal{I}_A|G$, where $A = L_{\frac{n}{2},p} \in \mathrm{GL}(V)$ where $p \in k^* \setminus k^{*2}$, $p \not\equiv 1 \mod k^{*2}$.

Note that (2) can only occur when n is even.

Proof. By Lemma 16, the matrix $A \in \mathrm{GL}(V)$ s.t. $\theta = \mathcal{I}_A$ is conjugate to $cI_{i,n-i}$ for some $c \in k^*$ and $i = 1, 2, \ldots, n$ or $L_{\frac{n}{2},p}$ for some $p \in k^* \setminus k^{*2}$. If A is of the form $I_{i,n-i}$, then A is conjugate to $(-1)I_{n-i,i}$. This limits i to $1, 2, \ldots, [\frac{n}{2}]$, each of which represents a unique isomorphy class. The reason for this is Lemma 19.

For (2)Since $p \in k^* \setminus k^{*2}$, by Lemma 18, we can choose a representative q from $k^* \setminus k^{*2}$, $q \not\equiv 1 \mod k^{*2}$. Since $\mathcal{I}_A = \mathcal{I}_{cA}$, we can factor out the constant.

The above two classes are not conjugate to each other since those in (1) are split and those in (2) are not, thus it's impossible for them to have same eigenvalues. ☐

Some more notation

For a field k let k^* denote the product group of all the nonzero elements and $(k^*)^2 = \{a^2 \mid a \in k^*\}$. Then $(k^*)^2$ is a normal subgroup of k^*.

Recall from [HW02, Section 2.1] that $k^*/(k^*)^2 \simeq \{1\}$ if $k = \bar{k}$, $k^*/(k^*)^2 \simeq \mathbb{Z}_2$ if $k = \mathbb{R}$ or $\mathbb{F}_p$ and $k^*/(k^*)^2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ if $\mathbb{Q}_p$, $p \neq 2$. We will denote the nontrivial representative of $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$ by N_p. Then 1, p, N_p are pN_p are representatives of $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$. In the case of $k = \mathbb{Q}_2$ a set of representatives of $k^*/(k^*)^2$ are $\{1, -1, 2, -2, -3, 3, 6, -6\}$.

Corollary 4. *The number of involutions of inner type of G up to isomorphy is equal to $\| k^*/(k^*)^2 \| + \frac{n}{2} - 1$ if n is even and $\frac{n-1}{2}$ if n is odd.*

For $k = \bar{k}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{F}_p$ and $\mathbb{Q}_p$ we summarize the number of isomorphy classes of involutions of inner type in the following.

(1) $k = \bar{k}$ algebraically closed. There are $[\frac{n}{2}]$ isomorphy classes of involutions of inner type.

(2) $k = \mathbb{R}$ the real numbers. There are $\frac{n}{2} + 1$ isomorphy classes of involutions of inner type for n is even and $\frac{n-1}{2}$ for n is odd.

(3) $k = \mathbb{Q}$ the rational numbers. There are infinite many isomorphy classes of involutions of inner type.

(4) $k = \mathbb{F}_p$ finite field $(p \neq 2)$. There are $\frac{n}{2} + 1$ isomorphy classes of involutions of inner type for n is even and $\frac{n-1}{2}$ for n is odd.

(5) $k = \mathbb{Q}_p$ the p-adic numbers. For $p \neq 2$, there are $\frac{n}{2} + 3$ isomorphy classes of involutions of inner type for n is even and $\frac{n-1}{2}$ for n is odd. For $p = 2$ there are $\frac{n}{2} + 7$ isomorphy classes of involutions of inner type for n is even and $\frac{n-1}{2}$ for n is odd.

3.4 Involutions of outer type

By Lemma 12, any outer automorphism can be written as $\theta \mathfrak{I}_A$, where θ is a fixed outer automorphism. We choose the fixed outer automorphism to be the map θ defined by $\theta(x) = {}^T x^{-1}$, $x \in G$. This is known to be an outer automorphism for $n > 2$. In the remainder of this chapter, let $Z(G)$ denote the center of G.

Lemma 20. *Let θ be defined by $\theta(x) = {}^T x^{-1}$, $x \in G$. Then we have the following results:*

(1) $\theta \mathfrak{I}_A$ is involution if and only if $\theta(A)A \in Z(G)$.

(2) $\theta(A)A \in Z(G)$ if and only if ${}^T A = A$ or ${}^T A = -A$. The later case only occurs when n is even.

Proof. $\theta \mathfrak{I}_A$ is involution, that is $\theta \mathfrak{I}_A \theta \mathfrak{I}_A(X) = X$ for any $X \in G$, i.e. $\theta(A)^{-1} A^{-1} \theta(X) A \theta(A) = \theta(X)$, i.e. $\theta(X) A \theta(A) = A \theta(A) \theta(X)$, therefore $X \theta(A) A = \theta(A) A X$. The above also works reversely. This proves (1).

For our choice of θ, we have $\theta(A)A \in Z(G)$ if and only if ${}^T A = zA$ for some $z \in Z(G)$. We note that $Z(\mathrm{SL}(n, \bar{k})) \simeq \mathbb{Z}_n$. So ${}^T A = tA$, where $t \in k$ and $t^n = 1$. Furthermore $A = {}^T({}^T A) = t^2 A$, what forces $t = 1$ or $t = -1$. This proves (2). ∎

Recall that two matrices A and B are called *congruent* if there exists a matrix X such that $({}^TX)AX = B$.

Lemma 21. *The outer automorphisms $\theta\mathfrak{I}_A$ and $\theta\mathfrak{I}_B$ are conjugate if and only if the matrix A is congruent to pB for some $p \in \bar{k}$.*

Proof. Since $\mathfrak{I}_{C^{-1}}\theta\mathfrak{I}_A\mathfrak{I}_C = \theta\mathfrak{I}_{{}^TCAC}$, it follows that $\mathfrak{I}_{C^{-1}}\theta\mathfrak{I}_A\mathfrak{I}_C = \theta\mathfrak{I}_B$ if and only if $\theta\mathfrak{I}_{{}^TCAC} = \theta\mathfrak{I}_B$, i.e. $\mathfrak{I}_{{}^TCAC} = \mathfrak{I}_B$. Then Lemma 13 implies that A is congruent to pB for some $p \in \bar{k}$. $\square$

Essentially congruence means that one applies the same operation on the columns of a matrix as on the rows. Since all operations on a matrix are the combinations of three fundamental ones, namely multiplying a row (column) with a constant, exchanging two rows (columns) and adding a multiple of a row (column) to another. Corresponding to these operation, there are three fundamental matrices.

Lemma 22. *(1) Symmetric matrices are congruent to diagonal matrices and skew symmetric matrices of even rank are congruent to J_n.*

(2) If $b_1, \dots b_n \in k^*$, then $A = \operatorname{diag}(a_1, \dots, a_n)$ is congruent to $B = \operatorname{diag}(b_1^2 a_1, \dots, b_n^2 a_n)$.

Proof. (1) Assume first that A is a symmetric $n \times n$-matrix. We prove the result by induction on n.

If $n = 1$, the result is obvious. Suppose that for $n = \pi$ all symmetric matrices are congruent to a diagonal matrix. For $n = \pi + 1$, let $A = \begin{pmatrix} a_1 & a_{11} & a_{12} & \dots & a_{1\pi} \\ a_{11} & a_2 & a_{22} & \dots & a_{2\pi} \\ a_{12} & a_{22} & a_3 & \dots & a_{3\pi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1\pi} & a_{2\pi} & a_{3\pi} & \dots & a_{\pi+1} \end{pmatrix}$.

Assume first that $a_1 \neq 0$ and let $P = \begin{pmatrix} 1 & -\frac{a_{11}}{a_1} & \dots & -\frac{a_{1\pi}}{a_1} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$. Then ${}^TPAP$ is of the form $\begin{pmatrix} a_1 & 0 \\ 0 & A_\pi \end{pmatrix}$ with A_π a symmetric matrix of dimension π. Using the induction hypothesis, the result follows. If $a_{11} = 0$, then a_{1i} is not equal to zero for all i, e.g. without loss of generality we may assume $a_{11} \neq 0$. Let $Q = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{\pi-1} \end{pmatrix}$. Then ${}^TQAQ$ is a symmetric matrix with the first entry nonzero. A similar matrix moves $a_{1,i}$ to the first entry, when it is nonzero.

For skew symmetric matrices, its size has to be even to ensure its non singularity. Assume its size is $2n$. We prove the result by induction on n. If $n = 1$, then $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$. Let $P = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$, then ${}^TPAP = J_1$.

Suppose that all skew symmetric matrices of size $2n = 2\pi$ are congruent to J_π. For $n = \pi + 1$, assume $A = \begin{pmatrix} 0 & a_{1,2} & \dots & a_{1,2\pi} \\ -a_{1,2} & 0 & \dots & a_{2,2\pi} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1,2\pi} & -a_{2,2\pi} & \dots & 0 \end{pmatrix}$. Assume first that $a_{1,2} \neq 0$ and let $P = \begin{pmatrix} \frac{1}{a_{1,2}} & 0 & \frac{a_{2,3}}{a_{1,2}} & \dots & \frac{a_{2,2\pi}}{a_{1,2}} \\ 0 & \frac{1}{a_{1,2}} & -\frac{a_{1,3}}{a_{1,2}} & \dots & \frac{a_{1,2\pi}}{a_{1,2}} \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$. Then ${}^TPAP$ is the form $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A_{2\pi} \end{pmatrix}$, where $A_{2\pi}$ is a skew symmetric matrix of size 2π and by the induction hypothesis, we're done.

If $a_{1,2} = 0$, then using a similar matrix as in the case of symmetric matrices, we can exchange $a_{1,2}$ with one of the non zero entries in the first row and then the result follows using the above argument.

(2) Let $P = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_n \end{pmatrix}$. Then ${}^TPAP = B$. □

It follows from the above result that for the entries of the diagonal matrix one can mod out squares without changing it's congruence class. Therefore, for the congruence classes, it is enough to consider the diagonal matrices with entries chosen from the representatives of $k^*/(k^*)^2$. For $k = \bar{k}$, $\mathbb{R}$, $\mathbb{F}_p$ and $\mathbb{Q}_p$ a set of representatives is given in [HW02, Section 2.1] (see also 3.3). Combined with the above results this yields the following description of the isomorphy classes in the case that $k = \bar{k}$, $\mathbb{R}$ or $\mathbb{F}_p$:

Theorem 6. *Let $\theta\Im_A$ be an involution of outer type.*

(1) If $k = \bar{k}$ algebraically closed, then there is only one isomorphy class of involutions of outer type if n is odd and two if n is even.

(2) If $k = \mathbb{R}$ the real numbers, then there are $\frac{n}{2} + 1$ isomorphy classes of involutions of outer type if n is even and $\frac{n+1}{2}$ if n is odd. Representatives are the involutions $\theta\Im_A$ with $A = I_{n-i,i}$ for $i \leq n - i$ and J_n.

(3) *If $k = \mathbb{F}_p$ a finite field with $p \neq 2$, then there are two isomorphy classes of involutions of outer type if n is odd, three if n is even. Representatives are the involutions $\theta\mathfrak{I}_A$ with $A = I$, M_{n,N_p} and J_n.*

Proof. (1) and (2) follow from Lemma 21 and 22. So it suffices to prove (3).

Let $\sigma = \theta\mathfrak{I}_A$ be an involution. By Lemma 22 we may assume that A is diagonal or equal to J_n. In the latter case we are done, so assume A is diagonal. It's easy to see that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is congruent to $\begin{pmatrix} ab/(b+ak^2) & 0 \\ 0 & b+ak^2 \end{pmatrix}$ with $b + ak^2 \neq 0$. Since any elements of $\mathbb{F}_p$ can be written as the sum of two squares, it follows that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & N_p \end{pmatrix}$ depending on whether a and b are from the same coset of $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$. Here N_p is a representative of the coset of all the non-square elements. By successively going down the diagonal of A, applying the above congruence whenever an element is not equal to 1 it follows that A is congruent to I or M_{n,N_p}. This proves the result. □

Next we consider the case that $k = \mathbb{Q}_p$ the p-adic numbers. We still have that A is congruent to a diagonal matrix or J_n. Assume that A is diagonal. Since we can mod out any squares of $\mathbb{Q}_p$, the entries on the diagonal can be limited to the representatives of $\mathbb{Q}_p^{*2}/\mathbb{Q}_p^*$, i.e. 1, p, N_p and pN_p. For details about notations and a discussion about $\mathbb{Q}_p^{*2}/\mathbb{Q}_p^*$, see [HW02, Section 2.1] or 3.3.

Let D_i be the i by i determinant in the upper left hand corner of a symmetric matrix A. For α and β non-zero p-adic numbers we define the symbol

$$(\alpha, \beta)_p = +1 \text{ or } -1$$

according to whether

$$\alpha x^2 + \beta y^2 = 1$$

has or has no solution in $\mathbb{Q}_p$. Furthermore let $c_p(A)$ denote the Hasse symbol of A, which is defined as

$$c_p(A) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p.$$

Lemma 23. *For $\alpha \in \mathbb{F}_p$ and $n \times n$ matrix A we have*

$$c_p(\alpha A) = \begin{cases} (\alpha, -1)_p^{n/2} (\alpha, D_n)_p c_p(A) & \text{if } n \text{ is even} \\ (\alpha, -1)_p^{(n+1)/2} c_p(A) & \text{if } n \text{ is odd} \end{cases}$$

Proof. $(-1, -\alpha^n D_n)_p = (-1, x)_p^n (-1, -D_n)_p$, and

$$\begin{aligned}(\alpha^i D_i, -\alpha^{i+1} D_{i+1})_p &= (\alpha^i, \alpha^{i+1})_p (\alpha^i, -D_{i+1})_p (D_i, \alpha^{i+1})_p (D_i, -D_{i+1}) \\ &= (\alpha^i, -D_{i+1})_p (D_i, \alpha^{i+1})_p (D_i, -D_{i+1}) \\ &= \begin{cases} (\alpha, D_i)_p (D_i, -D_{i+1})_p & \text{if } n \text{ is even} \\ (\alpha, -D_{i+1})_p (D_i, -D_{i+1})_p & \text{if } n \text{ is odd} \end{cases}\end{aligned}$$

But then

$$\begin{aligned}c_p(\alpha A) &= (\alpha, -1)^n (\alpha, -D_2)(\alpha, D_2)(\alpha, -D_4)(\alpha, D_4) \ldots C_p(A) \\ &= \begin{cases} (\alpha, -1)_p^{n/2} (\alpha, D_n)_p c_p(A) & \text{if } n \text{ is even} \\ (\alpha, -1)_p^{(n+1)/2} c_p(A) & \text{if } n \text{ is odd} \end{cases}\end{aligned}$$

□

Theorem 7. *Two symmetric matrices A and B are congruent if and only if*

$$\det A = \tau^2 \det B \ \text{ and } \ c_p(A) = c_p(B).$$

where $\tau \in \mathbb{Q}_p^*$ and $c_p(X)$ is the Hasse symbol of X.

Proof. For a proof of this result we refer to [Jon55, Theorem ??]. □

Modulo a square the determinant of a p-adic matrix is equal to $1, p, N_p, pN_p$ if $p \neq 2$ and equal to $(1, 2, 3, 6, -1, -2, -3, -6)$ if $p = 2$, see [HW02, Section 2.1] or 3.3. Moreover for the Hasse symbol we only have two choices 1 or -1. So, up to isomorphy there are at most eight isomorphy classes of involutions of outer type of the form $\theta \operatorname{Int}(A)$ with A symmetric if $p \neq 2$ and sixteen if $p = 2$. They are represented by the dual value (δ, c), where δ is a representative for $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, and c is the Hasse symbol, which is equal to 1 or -1. What remains is to determine whether any of these dual values give isomorphic involutions.

Theorem 8. *Let $G = \mathrm{SL}(n,k)$ or $\mathrm{GL}(n,k)$ with $k = \mathbb{Q}_p$ the p-adic numbers and $\theta\mathfrak{I}_A$ an involution with A symmetric. Then*

(1) If n is even, there are five isomorphy classes of involutions of outer type for $p \neq 2$ and nine for $p = 2$.

(2) If $n = 4k+1$, there are two isomorphy classes of involutions of outer type if $-1 \in \mathbb{Q}_p$ is a square, one otherwise.

(3) If $n = 4k+3$, there are two isomorphy classes of involutions of outer type in $\mathrm{SL}(n, \mathbb{Q}_p)$.

Remark 4. $-1 \notin \mathbb{Q}_2^2$, and (2) and (3) hold both in the case that $p \neq 2$ and $p = 2$.

Proof. (1). By Lemma 21 it suffices to consider when a matrix is congruent to a multiple of another matrix. In particular we can multiply a matrix with a constant without changing the isomorphy class of the corresponding involution. Since n is even we have $\det(\alpha A) = \det A$ modulo a square. So in this case each symmetric matrix with a different determinant modulo a square represents a different isomorphy class of involutions of type $\theta \mathfrak{I}_A$.

First consider the case that $n = 4k$. Then $c_p(\alpha A) = (\alpha, D_n)_p c_p(A)$. If $\det A = 1$, then $c_p(\alpha A) = c_p(A)$, so the two outer automorphisms corresponding to the dual values $(1,1)$ and $(1,-1)$ are not conjugate since multiplication by any α can never make the Hasse symbols equal. In the case that $\det A$ is not 1, we have only one outer automorphism for each value of $\delta \in k^*/(k^*)^2$ not equal to 1, since we can multiply with a constant $\alpha \in k^*$ to make the Hasse symbol change sign. That means we can make the dual value of αA equal to $(\delta, 1)$ as well as $(\delta, -1)$, so we take $(\delta, 1)$ as our representative.

In the case that $n = 4k+2$, then

$$\begin{aligned} c_p(\alpha A) &= (\alpha, -1)_p (\alpha, D_n)_p c_p(A) \\ &= (\alpha, -D_n)_p c_p(A) \end{aligned}$$

So for $\det A = -1$ there are two outer automorphisms with dual values $(-1,1)$ and $(-1,-1)$, and one for each of the others dual values $(\delta, 1)$ with $\delta \in k^*/(k^*)^2$, $\delta \neq -1$. Note that $\det A = -1$ modulo a square equals 1 or N_p depending on p. In both cases we get five outer automorphisms for $p \neq 2$ and nine for $p = 2$.

(2) and (3). In the case that n is odd we can again multiply the matrix A with a constant α without changing the isomorphy class of the corresponding involution, but in this case $\det(\alpha A) = \alpha \det A$ modulo a square. So we can take representatives whose corresponding matrix have determinant 1.

For $n = 4k+1$ we have $c_p(\alpha A) = (\alpha, -1)_p c_p(A)$. Therefore if -1 is a square, no matter what your choice of α is, the Hasse symbol remains the same and there are two involutions of outer type with dual values $(1,1)$ and $(1,-1)$. If -1 is a not square there is only one isomorphy class of outer automorphism since we can multiply with a constant $\alpha \in k^*$ to make the Hasse symbol change sign.

For $n = 4k+3$ we have $c_p(\alpha A) = c_p(A)$. So there are two outer automorphisms since multiplication with any constant $\alpha \in k^*$ can never change the Hasse symbol, i.e. the involutions with dual value $(1,1)$ and $(1,-1)$ are never conjugate. □

Remark 5. Let $D_1 = (1,1)$, $D_2 = (1,-1)$, $D_3 = (p,1)$, $D_4 = (p,-1)$, $D_5 = (N_p, 1)$, $D_6 = (N_p, -1)$, $D_7 = (pN_p, 1)$, $D_8 = (pN_p, -1)$ be the dual values in $\mathbb{Q}_p$, $(p \neq 2)$.

And let $d_1 = (1,1)$, $d_2 = (1,-1)$, $d_3 = (-1,1)$, $d_4 = (-1,-1)$, $d_5 = (2,1)$, $d_6 = (2,-1)$, $d_7 = (-2,1)$, $d_8 = (-2,-1)$, $d_9 = (3,1)$, $d_{10} = (3,-1)$, $d_{11} = (-3,1)$, $d_{12} = (-3,-1)$ $d_{13} = (6,1)$, $d_{14} = (6,-1)$, $d_{15} = (-6,1)$, $d_{16} = (-6,-1)$ be the dual values in $\mathbb{Q}_2$. Table 1 gives the dual values of the representatives of the isomorphy classes of the involutions $\theta\mathfrak{I}_A$ with A symmetric.

dual value	$-1 \in \mathbb{Q}_p^2$	$-1 \notin \mathbb{Q}_p^2$	$p = 2$
$n = 4k$	D_1, D_2, D_3, D_5, D_7	D_1, D_2, D_3, D_5, D_7	$d_1, d_2, d_3, d_5, d_7, d_9, d_{11}, d_{13}, d_{15}$
$n = 4k+2$	D_1, D_2, D_3, D_5, D_7	D_1, D_3, D_5, D_6, D_7	$d_1, d_3, d_4, d_5, d_7, d_9, d_{11}, d_{13}, d_{15}$
$n = 4k+1$	D_1, D_2	D_1	d_1
$n = 4k+3$	D_1, D_2	D_1, D_2	d_1, d_2

Table 3.1: Corresponding dual values (up to isomorphy)

3.5 Summary of the classification on SL(n, k)

As in the previous section we let θ be the involution defined by $\theta(x) = {}^T x^{-1}$, $x \in G$. In this section we summarize the classification of involutions in the case that the field k is algebraically closed, the real numbers, the p-adic numbers or a finite field $\mathbb{F}_p$ and we give representatives for each of the isomorphy classes.

$k = \bar{k}$: algebraically closed

(1) If n is odd, there are $\frac{n+1}{2}$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_A$ with A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$ and θ.

(2) If n is even, there are $\frac{n}{2} + 2$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_A$ with A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$, θ and $\theta\mathfrak{I}_{J_n}$.

$k = \mathbb{R}$: the real numbers

(1) If n is odd, there are n isomorphy classes of involutions. Representatives are θ, $\mathfrak{I}_A$ and $\theta\mathfrak{I}_A$ with A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$.

(2) If n is even, there are $n + 3$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_{J_n}$, θ, $\theta\mathfrak{I}_{J_n}$, $\mathfrak{I}_A$ and $\theta\mathfrak{I}_A$ with A is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$.

$k = \mathbb{F}_p$: finite field, $p \neq 2$

Let N_p be a non trivial representative of $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$.

(1) If n is odd, there are $\frac{n-1}{2}+2$ isomorphy classes of involutions. Representatives are θ, $\mathfrak{I}_A$ and $\theta\mathfrak{I}_B$ where A is one of the following $I_{n-i,i}$, $i = 1,2,\dots,\lceil\frac{n-1}{2}\rceil$ and B is M_{n,N_p}.

(2) If n is even, there are $\frac{n}{2}+4$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_A$, $\mathfrak{I}_B$, θ, $\theta\mathfrak{I}_{J_n}$ and $\theta\mathfrak{I}_C$ with A is one of the following $I_{n-i,i}$, $i = 1,2,\dots,\frac{n}{2}$, B is L_{n,N_p}, and C is M_{n,N_p}.

$k = \mathbb{Q}_p$: the p-adic numbers

If $p \neq 2$, then we take $1, p, N_p, pN_p$ as representatives of $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ and if $p = 2$, then we take $\{1,-1,2,-2,-3,3,6,-6\}$ as representatives.

(1) If n is even, then there are $\frac{n}{2}+9$ isomorphy classes of involutions for $p \neq 2$, $\frac{n}{2}+17$ for $p = 2$. Representatives are

(a) $p \neq 2$: $\mathfrak{I}_A$, $\mathfrak{I}_B$, θ, $\theta\mathfrak{I}_{J_n}$ and $\theta\mathfrak{I}_C$ and $\theta\mathfrak{I}_D$. Here A is one of the following $I_{n-i,i}$, $i = 1,2,\dots,\frac{n}{2}$, B is $L_{n,x}$ with $x = N_p,\ p$, or pN_p and C is $M_{n,x}$ with $x = N_p,\ p$, or pN_p. For the matrix D we have the following cases:

$$D = \begin{cases} K_{n,p,N_p,pN_p} & \text{if } -1 \in \mathbb{Q}_p^2 \\ N_{n,p,p} & \text{if } -1 \notin \mathbb{Q}_p^2 \text{ and } n = 4k \\ K_{n,p,p,N_p} & \text{if } -1 \notin \mathbb{Q}_p^2 \text{ and } n = 4k+2 \end{cases}$$

(b) $p = 2$: The same as $p \neq 2$, but x in B and C are chosen from 2, 3, 6, -1, -2, -3, -6, and D is $I_{n-2,2}$ if $n = 4k$ and $K_{n,2,3,-6}$ if $n = 4k+2$.

(2) If $n = 4k+1$, there are $\frac{n-1}{2}+2$ isomorphy classes of involutions if $-1 \in \mathbb{Q}_p^2$, otherwise $\frac{n-1}{2}+1$. Representatives are $\mathfrak{I}_A$, θ, and possibly $\mathfrak{I}_D$ if $-1 \in \mathbb{Q}_p^2$, where A is one of the following: $I_{n-i,i}$, $i = 1,2,\dots,\frac{n-1}{2}$ and D is K_{n,p,N_p,pN_p}.

(3) If $n = 4k+3$, there are $\frac{n-1}{2}+2$ isomorphy classes of involutions. Representatives are $\mathfrak{I}_A$, θ and $\theta\mathfrak{I}_D$, where A is one of the following: $I_{n-i,i}$, $i = 1,2,\dots,\frac{n-1}{2}$ and D is

$$D = \begin{cases} K_{n,p,N_p,pN_p} & \text{if } -1 \in \mathbb{Q}_p^2 \\ N_{n,p,p} & \text{if } -1 \notin \mathbb{Q}_p^2 \\ I_{n-2,2} & \text{if } p = 2 \end{cases}$$

Note that $-1 \notin \mathbb{Q}_2$, and (2) and (3) hold for $p \neq 2$ and $p = 2$.

3.6 Fixed Point Groups and Symmetric k-Varieties

The fixed-point group $H = G^\delta$ for an involution δ over G is defined by

$$G^\delta = \{x \in G \mid \delta(x) = x\}.$$

The fixed point group determines a lot of the structure of the corresponding symmetric k-variety $X := \{g\delta(g)^{-1} \mid g \in G\}$. It is easy to see that $X \simeq G/G^\delta$. Moreover if G^δ is compact, then from [HW93] it follows that X consists of semisimple elements:

Proposition 3 ([HW93, Proposition 10. 8]). *Let G be a connected reductive algebraic k-group with* $\operatorname{char}(k) = 0$ *and* $X = \{g\delta(g)^{-1} | g \in G\}$. *Suppose that* $H \cap [G, G]$ *is anisotropic over k. Then X_k consists of semi-simple elements.*

In view of this result it is important to determine which involutions have an k-anisotropic fixed point group. For $k = \mathbb{R}$ or $\mathbb{Q}_p$ all k-anisotropic subgroups are compact. Our main attention will be given to $\mathbb{R}$ and $\mathbb{Q}_p$. In this section, we'll study the compactness of the fixed point groups.

Lemma 24. *For the matrix $A = I_{n-i,i}$, the fixed point group $G^{\mathfrak{I}_A}$ consists of the matrices* $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ *where* $X \in \mathrm{GL}(n-i, k)$, $Y \in \mathrm{GL}(i, k)$ *and* $\det X . \det Y = 1$ *and the group $G^{\mathfrak{I}_A}$ is noncompact.*

Proof. For $A = I_{n-i,i}$ and $\mathfrak{I}_A(X) = X$ write $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ with X_1 a $n - i \times n - i$ block and X_4 a $i \times i$ block. Then

$$\mathfrak{I}_A(X) = \begin{pmatrix} I_{n-i} & 0 \\ 0 & -I_i \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} I_{n-i} & 0 \\ 0 & -I_i \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

i.e. $\begin{pmatrix} X_1 & -X_2 \\ -X_3 & X_4 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$.

So $X_2 = 0$ and $X_3 = 0$. Since $X \in \mathrm{SL}(n, k)$ and $\det X = \det X_1 . \det X_4$ the result follows. $\square$

Lemma 25. *For the matrix $B = L_{n,p}$, the noncompact fixed point group $G^{\mathfrak{I}_B}$ consists of the elements*

$$\begin{pmatrix} X_{11} & X_{12} & \dots & X_{1m} \\ X_{21} & X_{22} & \dots & X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \dots & X_{mm} \end{pmatrix},$$

where $m = \frac{n}{2}$, and $X_{ij} = \begin{pmatrix} x_{ij} & y_{ij} \\ py_{ij} & x_{ij} \end{pmatrix}$.

Proof. Let $X = \begin{pmatrix} X_{11} & X_{12} & \dots & X_{1m} \\ X_{21} & X_{22} & \dots & X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \dots & X_{mm} \end{pmatrix}$ and assume $I_B(X) = X$. Then

$$-\begin{pmatrix} J_2 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_2 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} & \dots & X_{1m} \\ X_{21} & X_{22} & \dots & X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \dots & X_{mm} \end{pmatrix}\begin{pmatrix} J_2 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_2 \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} & \dots & X_{1m} \\ X_{21} & X_{22} & \dots & X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \dots & X_{mm} \end{pmatrix}$$

That is $-J_2X_{ij}J_2 = X_{ij}$ for all $i,j = 1,\dots,n$. Let $X_{ij} = \begin{pmatrix} x_{ij} & y_{ij} \\ z_{ij} & w_{ij} \end{pmatrix}$. Then $-J_2X_{ij}J_2 = X_{ij}$ implies that $w_{ij} = x_{ij}$ and $z_{ij} = py_{ij}$, what proves the result. $\square$

Lemma 26. *$x_1^2 + x_2^2 + x_3^2 + ax_4^2 = 1$ has only bounded solutions in $\mathbb{Q}_2$ if and only if $a \in \mathbb{Q}_2^2$.*

Proof. Assume first that $a = 1$, i.e. $x_1^2+x_2^2+x_3^2+x_4^2 = 1$ and write $x_i = 2^{s_i} + \sum_{j=s_i+1}^{\infty} \delta_{ij}2^j$.

Then $x_i^2 = 2^{2s_i} + \sum_{j=2s_i+3}^{\infty} \pi_{ij}2^j$. Without loss of generality we may first assume that $s_1 \leq s_2 \leq s_3 \leq s_4$. If $s_1 \neq s_2$, then

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2^{2s_1} + \sum_{j=2s_i+2}^{\infty} \phi_{ij}2^j.$$

To make this equal to 1, we must have $s_1 = 0$ and x_1, x_2, x_3 and x_4 are units. If $s_1 = s_2$ and $s_2 \neq s_3$, then

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2^{2s_1+1} + \sum_{j=2s_i+2}^{\infty} \phi_{ij}2^j$$

which cannot equal 1. If $s_1 = s_2 = s_3$ and $s_3 \neq s_4$, then $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2^{2s_1} + 2^{2s_1+1} + \sum_{j=2s_i+2}^{\infty} \phi_{ij}2^j$, which is not going to equal to 1. If $s_1 = s_2 = s_3 = s_4$ then

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2^{2s_1+2} + \sum_{j=2s_i+3}^{\infty} \phi_{ij}2^j,$$

To make this equal to 1, we need to have $s_1 = -1$, which forces the norms $|x_i| = 2$. So $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ forces $|x_i| \leq 2$. Thus all solutions are in the ball with radius 2. On the other hand if $a \notin \mathbb{Q}_2^2$ we can take a to be one of the representatives 2, 3, 6, -1, -2, -3 or -6 of $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, which can all be written as $\delta_0 + 2\delta_1 + 4\delta_2 + \sum_{s=3}^{\infty} \phi_s 2^s$. Since δ_i and ϕ_s are 0 or 1 we get

$$ax_4 = \delta_0 2^{2s_4} + \delta_1 22s_4 + 1 + \delta_2 22s_4 + 2 + \sum_{k=2s_4+3}^{\infty} \pi_s 2^s.$$

No matter what δ_i is, we can carefully choose x_1, x_2 and x_3 to make the coefficients all zero for those whose powers are less than $2s_4 + 3$. For the coefficients whose power are larger or equal to $2k_4 + 3$, we can choose x_1, x_2 and x_3 as we desire. In particular we can make the sum equal to 1. Since $s_4 \leq -2$ is an arbitrary integer we can get as large a solution as we desire. □

Lemma 27. *The fixed point group of* θ *is the group* $\mathrm{SO}(n, k) = \{A \in G \mid {}^T AA = \mathrm{Id}\}$. *For* $k = \mathbb{R}$, *the group* $\mathrm{SO}(n, k)$ *is compact, for* $k = \mathbb{Q}_p$ *the p-adic numbers* ($p \neq 2$), *it's not. For* $\mathbb{Q}_2$, *if the rank of* G *is* 3 *or* 4, *it's compact, noncompact if the rank of* G *is larger than or equal to* 5.

Proof. For $k = \mathbb{R}$ and $\mathbb{Q}_p$ compactness means closed and bounded. It's easy to see that the fixed point group is closed. For $k = \mathbb{R}$, since the norm $\|A\| = n$, it follows that it is bounded.

For $\mathbb{Q}_2$, consider the case of $\mathrm{rank}(G) = 4$ first. If $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, then by Lemma 26 $|x_i| \leq 2$. Therefore for $\mathrm{rank}(G) = 4$ (thus 3 as well), the fixed point group is compact.

For rank of G of 5 or bigger, let $x_5 = x_4$, we have $x_1^2 + x_2^2 + x_3^2 + 2x_4^2 = 1$, also by Lemma 26 we know we can choose x_i as big as we want, so it's noncompact.

For $\mathbb{Q}_p$, consider first the case $n = 3$. The matrices $A_3 = \begin{pmatrix} a & b & c \\ \frac{ac}{b-1} & c & 1 + \frac{c^2}{b-1} \\ b + \frac{c^2}{b-1} & -a & -\frac{ac}{b-1} \end{pmatrix}$ are in the fixed point group as long as $a^2 + b^2 + c^2 = 1$. We know when $n \geq 3$, we can choose a, b, c as large as we want, hence the norms of the matrices are not all finite, therefore the set of these matrices is not bounded. For G of higher rank the matrices $\begin{pmatrix} A_3 & 0 \\ 0 & I_{n-3} \end{pmatrix}$ are in the fixed point group. □

Lemma 28. *The fixed point group of* $\theta \Im_{J_n}$ *is noncompact.*

Proof. The fixed point group is $H = \{A \mid \theta(A) = \mathfrak{I}_{J_n}(A)\}$. Clearly

$$\begin{pmatrix} r_1 & 0 & \dots & 0 & 0 \\ 0 & r_1^{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & r_m & 0 \\ 0 & 0 & \dots & 0 & r_m^{-1} \end{pmatrix}$$

is in H, which is not bounded. $\square$

Lemma 29. *Let $k = \mathbb{R}$ or $\mathbb{Q}_2$ and $A = I_{n-i,i}$. Then the fixed point group of $\theta\mathfrak{I}_A$ is noncompact.*

Proof. The matrix

$$\begin{pmatrix} I_{n-i-1} & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ 0 & 0 & 0 & I_{i-1} \end{pmatrix}$$

is in the fixed point group as long as $a^2 - b^2 = 1$. $\square$

So far we proved that the only compact fixed point group is that of the involution θ for $k = \mathbb{R}$ and $k = \mathbb{Q}_2$ and $n = 3$ or 4.

Lemma 30. *Let $k = \mathbb{Q}_p$, $(p \neq 2$ and $p = 2)$ and $C = M_{n,x}$. The fixed point group of $\theta\mathfrak{I}_C$ is noncompact if the* rank$(G) = n \geq 4$.

Proof. For $p \neq 2$ The matrices $\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$, where M is $n-1 \times n-1$ matrix such that ${}^{T}MM = \text{Id}$, is in the fixed point group. By Lemma 27, if $n \geq 4$, it's unbounded, therefore the fixed point group is unbounded, thus noncompact. For $p = 2$, by Lemma 26, for rank of G bigger than or equal to 4, it's noncompact. $\square$

For the matrices form C, we still need to consider the situation of rank$(G) = n = 3$.

Lemma 31. *Let $k = \mathbb{Q}_p$ and $C = M_{3,x}$. The fixed point group of $\theta\mathfrak{I}_C$ is noncompact if $-1 \in \mathbb{Q}_p^2$.*

Proof. The matrices $\begin{pmatrix} a & b & 0 \\ -b & \frac{1-b^2}{a} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are in the fixed point group as long as $a^2+b^2 = 1$. While $-1 \in \mathbb{Q}_p^2$, the norm of a and b can be chosen as big as we want, hence the fixed point group of $\theta\mathfrak{I}_C$ is noncompact. $\square$

Lemma 32. *Let $k = \mathbb{Q}_p$ and $C = M_{3,N_p}$. The fixed point group of $\theta \mathfrak{I}_C$ is noncompact.*

Proof. The matrix $A_3 = \begin{pmatrix} a & b & c \\ \frac{x^2c-a^2b}{1-b^2} & a & \frac{ac}{b-1} \\ \frac{acx}{b-1} & xc & \frac{xc^2}{1-b}-1 \end{pmatrix}$ is in the fixed point group as long as $a^2+b^2+xc^2=1$ has a solution. For $x = N_p$, we have infinite solutions, and we can choose the norms of the roots to be as large as we want. $\square$

Lemma 33. *Let $k = \mathbb{Q}_p$ and $C = M_{3,x}$, where x is p or pN_p. The fixed point group of $\theta \mathfrak{I}_C$ is compact if $-1 \notin \mathbb{Q}_p^2$.*

Proof. The fixed point group of $\theta \mathfrak{I}_C$ is the set of the matrices $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where

$$\begin{pmatrix} a_{11} & a_{12} & xa_{13} \\ a_{21} & a_{22} & xa_{23} \\ \frac{1}{x}a_{31} & \frac{1}{x}a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{11} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = I.$$

For x is N_p or pN_p and $-1 \notin \mathbb{Q}_p^2$, all the solutions for $a_{i1}^2 + a_{i2}^2 + xa_{i3}^2 = 1$ or x are unit (norm less or equal 1), therefore the fixed point group is bounded, hence compact. $\square$

Lemma 34. *Let $k = \mathbb{Q}_2$ and $C = M_{3,x}$, where $x \in \{2, 3, 6, -1, -2, -3, -6\}$. The fixed point group of $\theta \mathfrak{I}_C$ is compact if $x = 2$, -3 or -6, noncompact otherwise.*

Proof. Whether the fixed point group is bounded or not depends on whether the equation $a_1^2+a_2^2+xa_3^2 = 1$ has only bounded solutions or not, as follows from Lemma 32 and 33. Let $a_i = 2^{k_i} + \sum_{j=k_i+1}^{\infty} \delta_{ij} 2^j$. Without loss of generality we may assume $k_1 \leq k_2$. If $k_1 \geq k_3 + 1$, with $x \in \{-1, -2, -3, -6, 2, 3, 6\}$, then $a_1^2 + a_2^2 + xa_3^2 = x2^{2k_3} + x + \sum_{j=2k_3+2}^{\infty} \delta_{ij} 2^j$ is either not equal to 1 or $k_3 = 0$. This forces $|a_3| = 1$, thus $\max(|a_1|, |a_2|) \leq \frac{1}{2}$. So we can possibly only get a noncompact fixed group when $k_3 \geq k_1$. Therefore we can write $a_2^2 = 2^{2k_1} \sum_{j=0}^{\infty} \delta_j 2^j$, and $xa_3^2 = 2^{2k_1} \sum_{j=0}^{\infty} \pi 2^j$. We also know that $a_1^2 = 2^{2k_1} + 2^{2k_1} \sum_{j=3}^{\infty} \phi_j 2^j$ If we want $k_1 \leq -1$ and $a_1^2 + a_2^2 + xa_3^2 = 1$, then we must have $1 + \delta_0 + \pi_0 = 2$, $\delta_1 + \pi_1 = 1$, $\delta_2 + \pi_2 = 1$. Furthermore, we know that if $\delta_0 = 1$, then $\delta_1 = \delta_2 = 0$ and if $\delta_0 = 0$, then $\delta_1 = 0$ since $a_2^2 = 2^{2k_1} \sum_{j=0}^{\infty} \delta_j 2^j$. So if

$\delta_0 = 1$, we have $\phi_0 = 0$, and $\phi_1 = \phi_2 = 1$, so $x = 6$ or -2. If $\delta_0 = 0$, we have $\phi_0 = 1$ and $\phi_1 = 1$, so $x = 3$, or -1. From the above discussion we know that for x is -1, -2, 3 and 6, we can choose a_{ij} to be as big as possible, thus the fixed point group is noncompact while for x is 2, -3 or -6 the a_{ij}'s can only be chosen from the unit ball. $\square$

Lemma 35. *For the matrix* $D = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$, *the fixed point group of* $\theta \mathfrak{I}_D$ *are the matrices* $\begin{pmatrix} a & b & c \\ \frac{\delta ybi - zacf}{xa^2+yb^2} & -\frac{\delta xai + zbcf}{xa^2+yb^2} & f \\ -\frac{z(aci+\delta bf)}{xa^2+yb^2} & \frac{z(\delta xaf - ybci)}{y(xa^2+yb^2)} & i \end{pmatrix}$ *where* δ *is* 1 *or* -1 *and* a, b, c, f *and* i *satisfy the following equations:*

$$a^2 + \frac{y}{x}b^2 + \frac{z}{x}c^2 = 1$$

and

$$\frac{z}{x}c^2 + \frac{z}{y}f^2 + i^2 = 1.$$

Proof. Let $X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be in the fixed point group of $\theta \mathfrak{I}_D$. Then $\theta(X) = \mathfrak{I}_D(X)$, that is $D^{-1}XD^TX = I$, i.e.

$$\begin{pmatrix} a & \frac{y}{x}b & \frac{z}{x}c \\ \frac{x}{y}d & e & \frac{z}{y}f \\ \frac{x}{z}g & \frac{y}{z}h & i \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = I.$$

From $\frac{x}{y}ad + be + \frac{z}{y}fc = 0$, it follows that $e = -\frac{(xad+zfc)}{yb}$. Therefore

$$\frac{x}{y}d^2 + e^2 + \frac{z}{y}f^2 = 1$$

$$\Longrightarrow \frac{x}{y}d^2 + \left(\frac{xa}{yb}\right)^2 d^2 + \left(\frac{2xzacf}{yb}\right) d + \frac{z^2f^2c^2}{y^2b^2} + \frac{z}{y}f^2 - 1 = 0$$

$$\Longrightarrow \left(xyb^2 + x^2a^2\right) d^2 + (2xzacf)\, d + z^2c^2f^2 + yzb^2f^2 - y^2b^2 = 0$$

Solve

$$d = \frac{\delta ybi - zacf}{xa^2 + yb^2}, \text{ where } \delta \text{ is 1 or } -1.$$

Then

$$e = -\frac{xad + zcf}{yb} = -\frac{\delta xai + zbcf}{xa^2 + yb^2}$$

Furthermore we have

$$ag + \frac{y}{x}bh = \frac{z}{x}ci = 0$$

and

$$\frac{x}{y}d + eh + \frac{z}{y}fi = 0$$

So

$$g = \frac{z\,(ce - bf)\,i}{y\,(bd - ae)}$$

$$h = \frac{z\,(af - cd)\,i}{y\,(bd - ae)}$$

And if we plug in d and e into $db - ae$, we have $db - ae = \delta i$, So

$$g = \delta z\,(ce - bf) = -\frac{z\,(aci + \delta bf)}{xa^2 + yb^2}$$

and

$$h = \frac{\delta z}{y}\,(af - cd) = \frac{z\,(\delta xaf - ybci)}{y\,(xa^2 + yb^2)}$$

$\square$

Lemma 36. *For matrix $D = K_{n,p,N_p,pN_p}$, and $-1 \in \mathbb{Q}_p^2$ the fixed point group of $\theta\Im_D$ is noncompact.*

Proof. matrix $\begin{pmatrix} I_{n-3} & 0 \\ 0 & M_3 \end{pmatrix}$ is in the fixed point group if and only if M_3 is the form of Lemma 35. For $-1 \in \mathbb{Q}_p^2$, we can choose a, b and c as large as we want to make the equations in Lemma 34 satisfied. $\square$

Lemma 37. *Assume $-1 \notin \mathbb{Q}_p^2$. For the matrix $D = N_{n,p,p}$ with $n \geq 5$, or $D = K_{n,p,p,N_p}$ with $n \geq 6$, the fixed point group of $\theta\Im_D$ is noncompact.*

Proof. The result follows with a simular arguments as in Lemma 30, using Lemma 27. $\square$

This proves that for each choice of the matrices D in section 3(5)(a)(i) the fixed point groups are noncompact except in the case that the rank of G is 4 and $-1 \notin \mathbb{Q}_p^2$, which is compact as follows from the following Lemma.

Lemma 38. *For the matrix $D = N_{4,p,p}$, and $-1 \notin \mathbb{Q}_p^2$, the fixed point group of $\theta\Im_D$ is compact.*

Proof. The fixed point group is the set of all the matrices $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$, where

$$\begin{pmatrix} a_{11} & a_{12} & pa_{13} & pa_{14} \\ a_{21} & a_{22} & pa_{23} & pa_{24} \\ \frac{1}{p}a_{31} & \frac{1}{p}a_{32} & a_{33} & a_{34} \\ \frac{1}{p}a_{41} & \frac{1}{p}a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = I.$$

Since $-1 \notin \mathbb{Q}_p$, x^2+y^2 begins with even power of p if we write it in the standard form. So consider the diagonal element in the above matrix identity. $a_{11}^2+a_{12}^2+pa_{13}^2+pa_{14}^2 = 1$ can only occur inside the unit ball (with norm less or equal 1). A similar arguments hold for the other rows. □

Summary of the compact fixed groups

For $\mathbb{R}$, we proved that the only compact fixed point group is for the involution θ, and for $\mathbb{Q}_p$ with $p \neq 2$, the involutions with compact fixed point groups are

(1) rank$(G) = n = 3$: $\theta\mathfrak{I}_A$ and $\theta\mathfrak{I}_B$, where A is $M_{3,p}$ and B is M_{3,pN_p}.

(2) rank$(G) = n = 4$: $\theta\mathfrak{I}_A$ if $-1 \in \mathbb{Q}_p^2$, where A is $N_{4,p,p}$.

(3) rank$(G) = n > 4$. None.

Finally for $\mathbb{Q}_2$ the involutions with compact fixed point groups are

(1) rank$(G) = n = 3$: θ, $\theta\mathfrak{I}_{M_{3,2}}$, $\theta\mathfrak{I}_{M_{3,-3}}$and $\theta\mathfrak{I}_{M_{3,-6}}$.

(2) rank$(G) = n = 4$: θ.

(3) rank$(G) = n > 4$. None.

3.7 Involutions and k-inner elements

Let G be a connected reductive algebraic group defined over a field k of characteristic not 2, σ an involution of G defined over k, H a k-open subgroup of the fixed point group of σ and G_k (resp. H_k) the set of k-rational points of G (resp. H). The variety G_k/H_k is called a symmetric k-variety. To study these symmetric k-varieties one needs first a classification of the related k-involutions. In [Hel00] the isomorphy classes of k-involutions were characterized by essentially using the following 3 invariants:

(1) classification of admissible (Γ, σ)-indices.

(2) classification of the G_k-isomorphy classes of k-involutions of the k-anisotropic kernel of G.

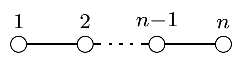
(3) classification of the G_k-isomorphy classes of k-inner elements of G.

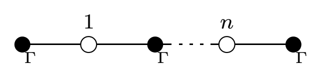
For more details, see [Hel00]. The admissible (Γ, σ)-indices determine most of the fine structure of the symmetric k-varieties and a classification of these was included in [Hel00] as well. To complete the classification it remains to classify the second and third invariant. As was shown in [Hel00] a classification of the k-inner elements depend on the base field k and for general G a classification of this second and third invariant can be quite complicated. For $k = \mathbb{R}$ the k-inner elements were classified in [Hel88] by using signatures as an invariant. For other fields additional invariants are needed. To get a good idea of the kind of invariant that might be needed we study the case that G is k-split first. In this case there is a maximal torus T which is k-split and hence there is no k-anisotropic kernel. So in this case we only need to classify the third invariant: the G_k-isomorphy classes of k-inner elements. In this chapter we study the case that $G_k = \mathrm{SL}(n, k)$, which is k-split. In the previous sections we gave a characterization of the isomorphy classes of k-involutions and classified them for k algebraically closed, the real numbers, the p-adic numbers or a finite field $\mathbb{F}_p$. This classification was independent of the characterization in [Hel00]. To be able to use these results as an indication of how to proceed with a general classification of the k-inner elements we need to translate the results in this chapter to fit the invariants/ characterization given in [Hel00]. We discuss this in this section.

3.8 (Γ, σ)-indices

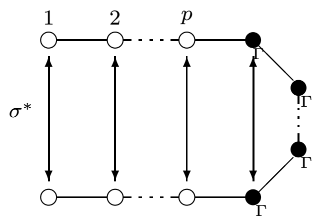
3.8.1 (σ, k)-split tori

Since the group is k-split the (Γ, σ)-indices are exactly the σ-indices of the case that $k = \bar{k}$ is algebraically closed, only with an additional label Γ under all the black nodes in the σ-index. The latter were classified [Hel88, Table II]. We recall that in the case $k = \bar{k}$ there is a bijective correspondence between the isomorphy classes of k-involutions and the congruence classes of σ-indices (see [Hel88, Theorem 3.11]). So the (Γ, σ)-indices for $G_k = \mathrm{SL}(n, k)$ are:

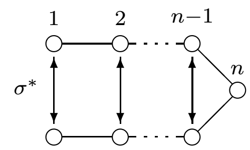
$A^{(1)}_{n,n}(I)$: 

$A^{(1)}_{2n+1,2n+1}(II)$: 

$A^{(1)}_{n,n}(III^p_a)$:



$A^{(1)}_{2n-1,2n-1}(III_b)$:



For notations on the (Γ,σ)-indices we refer to [Hel00, Section 5]. The involutions of $\bar{G} = \mathrm{SL}(n,\bar{k})$ corresponding to these (Γ,σ)-indices are respectively θ, $\theta\mathfrak{I}_{J_n}$ and $\mathfrak{I}_A$ with A is one of $I_{n-i,i}$, $i = 1,2,\ldots,\lceil\frac{n-1}{2}\rceil$. The latter two (Γ,σ)-indices are both related to the involutions of inner type, but since the restricted root system for the related symmetric k-variety is of a different type both (Γ,σ)-indices occur in the list of (Γ,σ)-indices.

3.8.2 k-inner elements

Let T be a maximal k-split torus of $\bar{G}$. Since G is k-split T is a maximal torus of $\bar{G}$ as well. Since G is k-split it follows from [Hel00, Theorem 8.33] that we have the following characterization of the isomorphy classes:

Theorem 9 ([Hel00, Theorem 8.33]). *Any k-involution of G is isomorphic to one of the form $\sigma\,\mathrm{Int}(a)$, where σ is a representative of a $\bar{G}$-isomorphy class of k-involutions, A a maximal (σ,k)-split torus and $a \in A$.*

The set of *set of k-inner elements of A* is defined as the set of those $a \in A$ such that $\sigma\,\mathrm{Int}(a)$ is a k-involution of G by $I_k(A)$. We recall that from [Hel00, Lemma 9.7] it follows that one can find a set of representatives for the isomorphy classes of the involutions $\sigma\,\mathrm{Int}(A)$ in the set $I_k(A)/A_k^2$. Here A_k is the set a k-regular elements of A and $A_k^2 = \{a^2 \mid a \in A_k\}$. Note that the set $A_k/A_k^2 =\simeq (k^*/(k^*)^2)^n$.

In the remainder of this section we will rewrite the representatives for the isomorphy classes in the form $\sigma\,\mathrm{Int}(a)$ with σ one of the representatives from the algebraically closed case and in particular find a set of k-inner elements of A representing these isomorphy classes. This will lead to a set of invariants classifying these elements in the cases that $G = \mathrm{SL}(n,k)$.

Computing the maximal (σ,k)-split tori

For each of the different types of involutions over the algebraically closed field $\bar{k}$ we will compute in the following first the maximal (σ,k)-split torus A and after that the k-inner elements representing the different isomorphy classes of involutions. In the following let T be the maximal k-split torus consisting of all the diagonal matrices.

(1) If $\sigma = \theta$, then the maximal (σ, k)-split torus is $S_1 = T_\sigma^- = \{t \in T \mid \sigma(t) = t^{-1}\} = T$.

(2) If $\sigma = \theta \mathfrak{I}_{J_n}$, then let $T' = X^{-1}TX$ with $X \in \mathrm{SL}(n, k)$. We need to choose X such that $S_2 = T_\sigma^- = \{X^{-1}tX \mid t \in T, \sigma(X^{-1}tX) = (X^{-1}tX)^{-1}\}^0$ has maximal dimension. Note that

$$\begin{aligned}\sigma(X^{-1}tX) = (X^{-1}tX)^{-1} &\Rightarrow \theta\mathfrak{I}_{J_n}(X^{-1}tX) = X^{-1}t^{-1}X\\ &\Rightarrow J_n^{-1}(X^{-1}tX)J_n = \theta(X^{-1}t^{-1}X) = {}^TXt^TX^{-1}\\ &\Rightarrow XJ_n{}^TXt = tXJ_n{}^TX.\end{aligned}$$

For $X = I$, the dimension of A_2 is maximal and equal to $\frac{n}{2}$. In particular the maximal (σ, k)-split torus is:

$$A_2 = \{\mathrm{diag}(a_1, a_2, \ldots, a_n) \mid a_1 = a_2, a_3 = a_4, \ldots a_{n-1} = a_n\}.$$

(3) If $\sigma = \mathfrak{I}_A$ with A one of $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$, then let $T' = X^{-1}TX$ with $X \in \mathrm{SL}(n, k)$. We need to choose X such that $S_{n-i,i} = T_\sigma^- = \{X^{-1}tX \mid t \in T, \sigma(X^{-1}tX) = (X^{-1}tX)^{-1}\}^0$ has maximal dimension.

For the maximal (σ, k)-split torus and their dimensions, we have

Lemma 39. *The maximal (σ, k)-split torus for $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$ can be chosen as:*

$$A_{n-i,i} = \{X^{-1}\,\mathrm{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1)X\},$$

where X satisfies $XAX^{-1} = \begin{pmatrix} J & 0 \\ 0 & I_{n-2i} \end{pmatrix}$. The dimension of the maximal (σ, k)-split torus is of course i.

Proof. Note that

$$\begin{aligned}\sigma(X^{-1}tX) = (X^{-1}tX)^{-1} &\Rightarrow \mathfrak{I}_A(X^{-1}tX) = X^{-1}t^{-1}X\\ &\Rightarrow A^{-1}(X^{-1}tX)A = X^{-1}t^{-1}X\\ &\Rightarrow tXAX^{-1} = XAX^{-1}t^{-1}.\end{aligned}$$

Since t is conjugate to t^{-1}, then highest possible dimension can only be less or equal to $\frac{n}{2}$. Furthermore, if $t = \mathrm{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1)$, and $tY = Yt^{-1}$, then we have $Y = \begin{pmatrix} J & 0 \\ 0 & Y_{n-2i} \end{pmatrix}$, therefore, if the (σ, k)-split tori has dimension of i, the corresponding $I_{n-j,j}$ has to be conjugate to $\begin{pmatrix} J & 0 \\ 0 & Y_{n-2i} \end{pmatrix}$. Hence the (σ, k)-split tori has dimension of i iff the corresponding $I_{n-j,j}$ s.t. $j \geq i$, i.e. the maximal (σ, k)-split tori is dimension j for $I_{n-j,j}$. $\square$

Computing the k-inner elements representing the isomorphy classes

From [Hel00, Theorem 8.33] we know now that any k-involution is conjugate to one of the following:

(1) $\theta\,\mathrm{Int}(a),\ a \in S_1 = T,,$

(2) $\theta\mathfrak{I}_{J_n}\,\mathrm{Int}(a),\ a \in S_2,$

(3) $\mathfrak{I}_A\,\mathrm{Int}(a), A = I_{n-i,i}, a \in S_{n-i,i}.$

Next we will compute the k-inner elements corresponding to the representatives of the isomorphy classes of involutions in Section 3.5. Note that for $k = \mathbb{R}$ a classification of these k-inner elements can also be found in [Hel88, Table II and IV], where they are called quadratic elements.

(1) $\boldsymbol{k} = \mathbb{R}$: the real numbers (see Section 3.5)

  (a) θ is in case (1) with $a = I$.

  (b) $\mathfrak{I}_A$ is in case (3) with $a = I$.

  (c) $\mathfrak{I}_{J_n}$ is in case (3) with $a = X^{-1}tX \in A_{\frac{n}{2},\frac{n}{2}}$, where $t = \mathrm{diag}(i, \ldots, i, -i, \ldots, -i)$.

  (d) $\theta\mathfrak{I}_{J_n}$ is in case (2) with $a = I$.

  (e) $\theta\mathfrak{I}_A$ is in the case (1) with $a = A$.

(2) $\boldsymbol{k} = \mathbb{Q}_{\boldsymbol{p}}$: the p-adic numbers (see Section 3.5)

  (a) θ is in case (1) with $a = I$.

  (b) $\mathfrak{I}_A$ is in case (3) with $a = I$.

  (c) $\theta\mathfrak{I}_A$ is in case (1) with $a = A$.

  (d) $\mathfrak{I}_B$ is in case (3) with $a = X^{-1}tX \in A_{\frac{n}{2},\frac{n}{2}}$, where $t = (\sqrt{x})^{-1}\,\mathrm{diag}(x, \ldots, x, 1 \ldots, 1)$, and x is p, N_p, pN_p for $p \neq 2$ and $-1, -2, -3, -6, 2, 3, 6$ for $\mathbb{Q}_2$.

  (e) $\theta\mathfrak{I}_{J_n}$ is in case (2) with $a = I$.

Chapter 4

Involutions of SO($2n+1, k$)

4.1 Preliminaries

For this chapter, let k be a field and k_1 is an extension field of k, $\bar{k}$ the algebraic closure of k, $G = \mathrm{SO}(2n+1, k)$, $G_1 = \mathrm{SO}(2n+1, k_1)$ and $\bar{G} = \mathrm{SO}(2n+1, \bar{k})$. We assume that the characteristic of k, k_1 and $\bar{k}$ is not equal to 2.

Definition 5. $\theta, \phi \in \mathrm{Aut}(G_1)$ are said to be k-conjugate if and only if there is a $\chi \in \mathrm{Int}(G)$, such that $\chi^{-1}\theta\chi = \phi$.

Similar as in the case of $\mathrm{SL}(2, \bar{k})$ in general we have the following result, which can be found in [Bor91]:

Lemma 40. *If k is an algebraically closed field, then we have* $\mathrm{Aut}(G) = \mathrm{Int}(G)$.

4.2 Isomorphy classes of Involutions of SO($2n+1, k$)

From Lemma 40 it follows that for any $\theta \in \mathrm{Aut}(G)$, there is a $(2n+1) \times (2n+1)$ matrix $A \in \mathrm{GL}(2n+1, \bar{k})$, such that $\theta = \mathfrak{I}_A|_G$. So any automorphism θ is an inner automorphism. We will also call this an innermorphism. For inner automorphisms, we have the following:

Lemma 41. *If $\mathfrak{I}_A|_G = \mathfrak{I}$ for some $A \in \mathrm{GL}(2n+1, \bar{k})$, then $A = pI$ for some $p \in \bar{k}$.*

Proof. Since $\mathfrak{I}_A|_G = \mathfrak{I}$, we have for all $X \in G$, $\mathfrak{I}_A(X) = A^{-1}XA = X$ i.e. $AX = XA$. Let $A = (a_{ij})_{2n+1 \times 2n+1}$ and $X_{s,t} = \begin{pmatrix} I_{s-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & I_{t-s-1} & \vdots & \vdots \\ 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & I_{2n+1-t} \end{pmatrix}$, where

$s < t \leq 2n+1$. We have $X_{s,t} \in \mathrm{SO}(2n+1,k)$, and $AX_{s,t} = X_{s,t}A$. From $AX_{s_1,t_1}X_{s_2,t_2}\dots X_{s_r,t_r} = X_{s_1,t_1}X_{s_2,t_2}\dots X_{s_r,t_r}A$ with $s_i < t_i \leq 2n+1$, $i = 1,\dots r$, it follows that $a_{ij} = 0$ if $i \neq j$ and $a_{11} = a_{22} = \cdots = a_{2n+1,2n+1}$. That is $A = p\,\mathrm{Id}$ for some $p \in \bar{k}$. $\square$

Next we characterize the inner automorphisms $\mathcal{I}_A$ with $A \in \mathrm{GL}(2n+1,\bar{k})$ which keep $\bar{G}$ and G invariant. We have the following result.

Lemma 42. *Suppose* $A \in \mathrm{GL}(2n+1,\bar{k})$*,* $G = \mathrm{SO}(2n+1,k)$ *and* $\bar{G} = \mathrm{SO}(2n+1,\bar{k})$*.*

(a) The inner automorphism $\mathcal{I}_A$ *keeps* $\bar{G}$ *invariant if and only if* $A = pM$*, for some* $p \in \bar{k}$ *and* $M \in \bar{G}$*.*

(b) If $A \in \bar{G}$*, then* $\mathcal{I}_A$ *keeps* G *invariant if and only if* $A = pM$*, for some* $p \in \bar{k}$ *and* $M \in G$*.*

In particular any inner automorphism $\sigma \in \mathrm{Int}(G)$ *can be written as* $\sigma = \mathcal{I}_M$*, where* $M \in \mathrm{SO}(2n+1,k)$*.*

Proof. (a). The proof of ($\Longleftarrow$) is obvious.

($\Longrightarrow$) Let $X \in \bar{G}$. Then $B = \mathcal{I}_A(X) = A^{-1}XA \in \bar{G}$. From ${}^TB^{-1} = B$ it follows that $A^TAX(A^TA)^{-1} = X$. But then by Lemma 41 we have $A^TA = q\,\mathrm{Id}$ for some $q \in \bar{k}^*$. Let $p \in \bar{k}^*$ such that $p^{2(2n+1)} = q^{-1}$. Then $A_1 = pA$ satisfies $A_1{}^TA_1 = \mathrm{Id}$, so $A_1 \in \mathrm{O}(2n+1,\bar{k})$. If $\det(A_1) = 1$ we are done and if $\det(A_1) = -1$, then since $2n+1$ is odd the matrix $-A_1 \in \mathrm{SO}(2n+1,\bar{k})$.

(b). The proof of ($\Longleftarrow$) is obvious.

($\Longrightarrow$) Since $A \in \bar{G}$ we have that $A^{-1} = {}^TA$. Let $X = (x_{ij})_{2n+1\times 2n+1} \in \mathrm{SO}(2n+1,k)$ with $x_{ij} \in k$ and assume $A = (a_{ij})_{2n+1\times 2n+1} \in \mathrm{SO}(2n+1,\bar{k})$ with $a_{ij} \in \bar{k}$. Then

$$\mathcal{I}_A(X) = A^{-1}XA = \left(\sum_{s=1}^{2n+1}\sum_{l=1}^{2n+1} a_{si}x_{sl}a_{lj}\right)_{(2n+1)\times(2n+1)}$$

Let $e_{s,t}$ be the matrix $(x_{i,j})$ where $x_{i,j} = 0$ except $x_{s,t} = 1$.

(1) Let $X_{s,t} = I - 2e_{s,s} - 2e_{t,t}$, then $X \in G$ for $s \neq t$. Since $\mathcal{I}_A(I)$, and $\mathcal{I}_A(X_{st}) \in G$, then $\mathcal{I}_A(X_{st}) - \mathcal{I}_A(I) \in k$, i.e. $\mathcal{I}_A(e_{ss}) + \mathcal{I}_A(e_{tt}) \in k$. i.e. $a_{si}a_{sj} + a_{ti}a_{tj} \in G$ for all $i,j,s \neq t$. Next we prove $a_{si}a_{sj} \in G$ for all i,j,s. Without loss of generality it suffices to prove that $a_{1i}a_{1j} \in G$. Since $a_{1i}a_{1j} + \cdots + a_{2n+1,i}a_{2n+1,j} \in G$ (in fact it must be 1 or 0 since $A^TA = I$), we have

$$a_{1i}a_{1j} = \frac{1}{2}(2(a_{1i}a_{1j} + \cdots + a_{2n+1,i}a_{2n+1,j}) - (a_{2i}a_{2j} + a_{3i}a_{3j}) - \\ - \cdots - (a_{2n+1,i}a_{2n+1,j} + a_{2i}a_{2j})) \in G.$$

(2) We prove next that $a_{si}a_{tj} \in G$ for $s \neq t$. Without loss of generality it suffices to prove that $a_{1i}a_{2j} \in G$. Let $X_1 = \begin{pmatrix} & 0 & 1 & 0 \\ & -1 & 0 & 0 \\ 0 & 0 & I_{2n-1} & \end{pmatrix}$, and $X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & I_{2n-2} \end{pmatrix}$, Since $\mathcal{I}_A(X_1) \in G$ and $\mathcal{I}_A(X_2) \in G$, i.e. $-a_{2i}a_{1j} + a_{1i}a_{2j} + a_{3i}a_{3j} + a_{4,i}a_{4j} + \cdots + a_{2n+1,i}a_{2n+1,j} \in G$, and $a_{2i}a_{1j} + a_{1i}a_{2j} - a_{3i}a_{3j} + a_{4,i}a_{4j} + \cdots + a_{2n+1,i}a_{2n+1,j} \in G$. By (1) $a_{si}a_{sj} \in G$, therefore we have $-a_{2i}a_{1j} + a_{1i}a_{2j} \in G$ and $a_{2i}a_{1j} + a_{1i}a_{2j} \in G$. Hence $a_{1i}a_{2j} \in G$.

By (1) and (2), $a_{is}a_{tj} \in k$ for all i, j, s, t. Therefore $A = pB$, where $p = a_{st}^{-1} \neq 0$ and $B \in \mathrm{GL}(n,k)$ for some s, t.

Furthermore, $I_A = I_B$ and $I_B(X) \in \mathrm{SO}(2n+1,k)$, i.e. $I_B(X)^T I_B(X) = I$, i.e. $XB^TB = B^TBX$, so $B^TB = qI$ for some $q \in k$. Taking determinants on both sides, we get $(\det B)^2 = q^{2n+1}$, so $q = r^{-2}$ for some $r \in k$. Let $M = rB$, we have $M^TM = I$ and $\mathcal{I}_A = \mathcal{I}_M$. If $M \notin G$, i.e. $\det M = -1$, we choose instead $-M \in G$ and $\mathcal{I}_A = \mathcal{I}_M$. $\square$

Now let's turn our attention to involutions of $G = \mathrm{SO}(2n+1,k)$. Since $\theta \in \mathrm{Aut}(G)$ is always an inner automorphism, furthermore by Lemma 42, there is a matrix $A \in \mathrm{SO}(2n+1,k)$ such that $\theta = \mathcal{I}_A$. And if it's involution, i.e. $\theta^2 = I$. Then we have the following

Lemma 43. *Suppose θ is an involution of $G = \mathrm{SO}(2n+1,k)$. Then there is a matrix $A \in \mathrm{SO}(2n+1,k)$, such that $\mathcal{I}_A = \theta$ and $A = X^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} X$ for some $X \in \mathrm{GL}(n,k)$ with X^TX a diagonal matrix with $s+t = 2n+1$ and s is odd.*

Proof. By Lemma 42 we know there is a matrix $A \in G$, such that $\theta = \mathcal{I}_A$. Since θ is an involution we have $\theta^2 = \mathcal{I}_A^2 = \mathcal{I}$. By Lemma 41, $A^2 = \delta I$. Since $A \in \mathrm{SO}(2n+1,k)$ it follows that $\delta = 1$ and $A^2 = I$. So A is semisimple and the eigenvalues of A are 1 and/or -1. Then there is a matrix $Y \in \mathrm{GL}(n,k)$ such that $A = Y^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Y$. Since $A \in G$ and the determinant of A is 1, s must be odd. Since $A = {}^TA$, we get

$$Y^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Y = {}^TY \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} {}^TY^{-1},$$

i.e.

$$\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Y^TY = Y^TY \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}.$$

Therefore

$$Y^TY = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix},$$

where ${}^TY_1 = Y_1$ and ${}^TY_2 = Y_2$. Since symmetric matrices are congruent to diagonal matrices (ref to Lemma 22), there is a matrix $N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$ such that NY^TY^TN is diagonal. Let $X = NY$. Then

$$X^TX = \operatorname{diag}(a_1, \dots, a_{2n+1}),$$

and

$$X^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} X = (NY)^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} NY = Y^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Y = A.$$

□

It follows from the above result that for any involution θ of G, we have a matrix $A \in G$, such that $\theta = \mathcal{I}_A$, and A satisfies the conditions in Lemma 43. Since $\mathcal{I}_A = \mathcal{I}_{-A}$, instead of restrict s to be odd, we assume $s > t$ to iterate all situations. For such a choice, we must pay attention we don't require $A \in \mathrm{SO}(2n+1,k)$ anymore. All we need is $A^TA = I$.

Now we need to figure out which of these matrices A give involutions which are isomorphic over $\mathrm{SO}(2n+1,k)$. Since $\mathrm{SO}(2n+1,k)$ is a subset of $\mathrm{GL}(2n+1,k)$, whenever two matrices are conjugate over $\mathrm{SO}(2n+1,k)$, they are also conjugate over $\mathrm{GL}(2n+1,k)$, so we only need to find out how many conjugate $\mathrm{SO}(2n+1,k)$-subclasses there are within each $\mathrm{GL}(2n+1,k)$-conjugate class. For fixed $s > t$ $(s+t = 2n+1)$ all the matrices, which are $\mathrm{GL}(2n+1,k)$-conjugate to $\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}$ can be written as

$$\left\{ X^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} X \;\middle|\; X^TX = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{2n+1} \end{pmatrix} \right\} \tag{4.1}$$

In the following for the field of $\mathbb{Q}_p$, we let $c_p(x_1, \dots, x_j)$ denote the Hasse symbol of the diagonal matrix

$$\begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_j \end{pmatrix}.$$

Lemma 44. *Suppose θ and ϕ are two involutions of $\mathrm{SO}(2n+1,k)$ in the same $\mathrm{GL}(2n+1,k)$-conjugacy class, and let A and B be two matrices satisfying the conditions in Lemma 43, such that $I_A=\theta$ and $I_B=\phi$. Remember $I_{s,t}=\begin{pmatrix} I_s & 0\\ 0 & -I_t\end{pmatrix}$ Write $A=X^{-1}I_{s,t}X$, $B=Y^{-1}I_{s,t}Y$ where $X^TX=\begin{pmatrix} a_1 & 0 & \dots & 0\\ 0 & a_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_{2n+1}\end{pmatrix}$ and $Y^TY=\begin{pmatrix} b_1 & 0 & \dots & 0\\ 0 & b_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & b_{2n+1}\end{pmatrix}$. Then the following are equivalent*

(1) θ is conjugate to ϕ (over $\mathrm{SO}(2n+1,k)$);

(2) A is conjugate to B (over $\mathrm{SO}(2n+1,k)$);

(3) $\begin{pmatrix} a_1 & 0 & \dots & 0\\ 0 & a_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_s\end{pmatrix}$ is congruent to $\begin{pmatrix} b_1 & 0 & \dots & 0\\ 0 & b_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & b_s\end{pmatrix}$ and $\begin{pmatrix} a_{s+1} & 0 & \dots & 0\\ 0 & a_{s+2} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_{2n+1}\end{pmatrix}$ is congruent to $\begin{pmatrix} b_{s+1} & 0 & \dots & 0\\ 0 & b_{s+2} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & b_{2n+1}\end{pmatrix}$;

(4) For k the p-adic numbers: $a_1a_2\dots a_i=\tau^2b_1b_2\dots b_i$, $c_p(a_1,\dots,a_s)=c_p(b_1,\dots,b_s)$ and $c_p(a_{s+1},\dots,a_{2n+1})=c_p(b_{s+1},\dots,b_{2n+1})$.

Proof. (2) $\Longleftrightarrow$ (1) is obvious thanks to Lemma 42 and 43.

(2) $\Longrightarrow$ (3): Assume there is a $N\in G$, such that $N^{-1}AN=B$. Then

$$N^{-1}X^{-1}I_{s,t}XN=Y^{-1}I_{s,t}Y,$$

i.e.

$$I_{s,t}XNY^{-1}=XNY^{-1}I_{s,t}.$$

Therefore $XNY^{-1}=\begin{pmatrix} N_1 & 0\\ 0 & N_2\end{pmatrix}$, where N_1 is an $s\times s$-matrix and N_2 is an $t\times t$-matrix, i.e. $XN = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} Y$. But then

$$X^T X = X N^T N^T X = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} Y^T Y \begin{pmatrix} {}^T N_1 & 0 \\ 0 & {}^T N_2 \end{pmatrix},$$

i.e.

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{2n+1} \end{pmatrix} = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{2n+1} \end{pmatrix} \begin{pmatrix} {}^T N_1 & 0 \\ 0 & {}^T N_2 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_s \end{pmatrix} = N_1 \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_s \end{pmatrix} {}^T N_1$$

and

$$\begin{pmatrix} a_{s+1} & 0 & \dots & 0 \\ 0 & a_{s+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{2n+1} \end{pmatrix} = N_2 \begin{pmatrix} b_{s+1} & 0 & \dots & 0 \\ 0 & b_{s+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{2n+1} \end{pmatrix} {}^T N_2.$$

(3) $\Longrightarrow$ (2): Assume there is a $s \times s$-matrix N_1 and a $t \times t$-matrix N_2 such that

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_s \end{pmatrix} = N_1 \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_s \end{pmatrix} {}^T N_1$$

and

$$\begin{pmatrix} a_{s+1} & 0 & \dots & 0 \\ 0 & a_{s+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{2n+1} \end{pmatrix} = N_2 \begin{pmatrix} b_{s+1} & 0 & \dots & 0 \\ 0 & b_{s+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{2n+1} \end{pmatrix} {}^T N_2.$$

Let $N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$ and $M = X^{-1}NY$. Then

$$\begin{aligned} M^{-1}AM &= Y^{-1}N^{-1}XX^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}XX^{-1}NY \\ &= Y^{-1}N^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}NY \\ &= Y^{-1}\begin{pmatrix} N_1^{-1}I_sN_1 & 0 \\ 0 & -N_2^{-1}I_tN_2 \end{pmatrix}Y \\ &= Y^{-1}\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}Y \\ &= B. \end{aligned}$$

Furthermore, $M^TM = X^{-1}(NY^TY^TN)^TX^{-1} = X^{-1}(X^TX)^TX^{-1} = I$. If $M \notin G$, i.e. $\det M = -1$, choose $-M$ and all of the above still hold.

The equivalence of (4) and (3) is immediate from Theorem 7: $\square$

So for k the p-adic numbers, by (4) of Lemma 44, the triple values (δ, t_1, t_2) determine the $\mathrm{SO}(2n+1,k)$-conjugacy classes, where $\delta = a_1a_2\ldots a_i$ is the representative in $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ of the determinant of the upper-left part of the diagonal matrix in (4.1), t_1, resp. t_2 is the Hasse symbol of the upper-left resp. bottom-right part of the diagonal matrix in (4.1).

Lemma 45. $c_p(a_1,\ldots,a_{2n+1}) = c_p(a_1,\ldots,a_s)c_p(a_{s+1},\ldots,a_{2n+1})(-1,-1)_p(\delta_1,\delta_2)_p$, *where* $\delta_1 = a_1\ldots a_s$, $\delta_2 = a_{s+1}\ldots a_{2n+1}$.

Proof. In general $c_p(x_1,x_2,\ldots,x_m) = (-1,-D_n)_p\prod_{i=1}^{m-1}(D_i,-D_{i+1})_p$, where $D_i = x_1x_2\ldots x_i$. So

$$c_p(x_1,x_2,\ldots,x_m) = (-1,-1)_p\prod_{i=1}^{m}(-1,x_i)\prod_{j=1}^{m-1}\prod_{i=j+1}^{m}(x_j,x_i)_p.$$

Therefore

$$c_p(a_1,\dots,a_s)c_p(a_{s+1},\dots,a_{2n+1})(-1,-1)_p(\delta_1,\delta_2)_p =$$

$$\begin{aligned}
=&(-1,-1)_p\prod_{i=1}^{s}(-1,a_i)\prod_{j=1}^{s-1}\prod_{i=j+1}^{s}(a_j,a_i)_p(-1,-1)_p\prod_{i=s+1}^{2n+1}(-1,a_i).\\
&\prod_{j=1}^{2n-s}\prod_{i=j+1}^{2n+1-s}(a_{s+j},a_{s+i})_p(-1,-1)_p(\delta_1,\delta_2)_p\\
=&(-1,-1)_p\prod_{i=1}^{2n+1}(-1,a_i)\prod_{j=1}^{s-1}\prod_{i=j+1}^{s}(a_j,a_i)_p\prod_{j=s+1}^{2n}\prod_{i=j+1}^{2n+1}(a_j,a_{s+i})_p\prod_{i=1}^{s}\prod_{j=s+1}^{2n+1}(a_i,a_j)_p\\
=&(-1,-1)_p\prod_{i=1}^{2n+1}(-1,a_i)\prod_{j=1}^{2n}\prod_{i=j+1}^{2n+1}(a_j,a_i)_p\\
=&c_p(a_1,\dots,a_{2n+1}).
\end{aligned}$$

□

Theorem 10. *For $k=\mathbb{C}$, $\mathbb{R}$ and $\mathbb{F}_p$ we have the following classification of the isomorphy classes of involutions of* $\mathrm{SO}(2n+1,k)$*:*

(1) $k=\mathbb{C}$, the complex numbers. There is only one $\mathrm{SO}(2n+1,k)$-conjugacy class for each $\mathrm{GL}(2n+1,k)$-conjugacy class.

(2) $k=\mathbb{R}$, the real numbers. There is only one $\mathrm{SO}(2n+1,k)$-conjugacy class for each $\mathrm{GL}(2n+1,k)$-conjugacy class.

(3) $k=\mathbb{F}_p$, $(p\neq 2)$ a finite field. There are two $\mathrm{SO}(2n+1,k)$-conjugacy classes for each $\mathrm{GL}(2n+1,k)$-conjugacy class. Representatives are I_A with $A=X^{-1}\begin{pmatrix}I_s & 0\\ 0 & -I_t\end{pmatrix}X$ and $X=\mathrm{Id}$ or $X=\begin{pmatrix}I_{s-1} & 0 & 0 & 0\\ 0 & a & b & 0\\ \vdots & -b & a & \vdots\\ 0 & 0 & \dots & I_{t-1}\end{pmatrix}$, with $a^2+b^2\notin k^2$.

Proof. As in Lemma 44 suppose $\theta=\mathcal{I}_A$ with $A=X^{-1}\begin{pmatrix}I_s & 0\\ 0 & -I_t\end{pmatrix}X$ and $X^TX=\begin{pmatrix}N_1 & 0\\ 0 & N_2\end{pmatrix}$, where $N_1=\begin{pmatrix}a_1 & 0 & \dots & 0\\ 0 & a_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_s\end{pmatrix}$, $N_2=\begin{pmatrix}a_{s+1} & 0 & \dots & 0\\ 0 & a_{s+2} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_{2n+1}\end{pmatrix}$.

(1) Since for k algebraically closed all diagonal matrices are congruent to the identity matrix the result in this case is immediate from Lemma 44(3).

(2) For $k=\mathbb{R}$ all the diagonal entries $a_1, a_2, \ldots, a_{2n+1}$ of X^TX are positive since X^TX is positive-definite. So X^TX is congruent with Id and the result follows from 44(3).

(3) By Theorem 6 we may assume that N_1 is congruent to Id or M_{s,N_p} and similarly that N_2 is congruent to Id or M_{t,N_p}. Since $\det(X^TX)=1$ modulo a square we get that $\det(N_1)=\det(N_2)$ modulo a square. So either both N_1 and N_2 are congruent to Id or N_1 is congruent to M_{s,N_p} and N_2 is congruent to M_{t,N_p}. In the first case we have $X=\mathrm{Id}$ and in the second case we can take $X=\begin{pmatrix} I_{s-1} & 0 & 0 & 0\\ 0 & a & b & 0\\ \vdots & -b & a & \vdots \\ 0 & 0 & \ldots & I_{t-1}\end{pmatrix}$, where $a^2+b^2\notin k^2$. □

4.2.1 Classification for $k=\mathbb{Q}_p$: the p-adic numbers

Theorem 11. *Suppose $\theta=\mathfrak{I}_A$ is an involution of $\mathrm{SO}(2n+1,k)$. Assume $X\in \mathrm{GL}(2n+1,k)$ such that $A=X^{-1}I_{s,t}X$, with $X^TX=\begin{pmatrix} N_1 & 0\\ 0 & N_2\end{pmatrix}$, where*

$$N_1=\begin{pmatrix} a_1 & 0 & \ldots & 0\\ 0 & a_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & a_s\end{pmatrix} \quad \textit{and} \quad N_2=\begin{pmatrix} a_{s+1} & 0 & \ldots & 0\\ 0 & a_{s+2} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & a_{2n+1}\end{pmatrix}.$$

Let $\delta=a_1a_2\ldots a_s$, $\delta'=a_{s+1}a_{s+2}\ldots a_{2n+1}$ be the representative in $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ for the determinants of N_1 and N_2 respectively and let t_1, t_2 be the Hasse symbol of N_1 and N_2 respectively. For each $\mathrm{GL}(2n+1,k)$-conjugacy class of an involution $\theta=\mathfrak{I}_a$ there are at most 8 $\mathrm{SO}(2n+1,k)$-conjugacy classes of θ corresponding to the triple values $(\delta, t_2(\delta,\delta)_p, t_2)$.

Proof. By (4) of Lemma 44, the triple value (δ,t_1,t_2) determines the $\mathrm{SO}(2n+1,k)$-conjugacy classes. Since $X\mathfrak{I}^TX=X^TX=\begin{pmatrix} a_1 & 0 & \ldots & 0\\ 0 & a_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & a_{2n+1}\end{pmatrix}$, i.e. $\begin{pmatrix} a_1 & 0 & \ldots & 0\\ 0 & a_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & a_{2n+1}\end{pmatrix}$ is congruent to I, $c_p(a_1,a_2,\ldots,a_{2n+1})=(-1,-1)_p$, and $\delta=\tau^2\delta'$. By Lemma 44 $(-1,-1)_p=t_1t_2(-1,-1)_p(\delta,\delta')_p$ i.e. $t_1=t_2(\delta,\delta)_p$. □

Remark 6. The triple value only has two independent unknowns.

Next we will determine which of the triple values $(\delta, t_2(\delta,\delta)_p, t_2)$ in Theorem 11 give different $\mathrm{SO}(2n+1,k)$-conjugacy classes and give representatives for each of these conjugacy classes.

Corollary 5. *Let* $L_2 = \begin{pmatrix} p & 0 \\ 0 & N_p \end{pmatrix}$, $M_2 = \begin{pmatrix} N_p & 0 \\ 0 & pN_p \end{pmatrix}$, $N_2 \begin{pmatrix} p & 0 \\ 0 & pN_p \end{pmatrix}$ *and* $M_3 = \begin{pmatrix} p & 0 & 0 \\ 0 & N_p & 0 \\ 0 & 0 & pN_p \end{pmatrix}$. *Then all the conjugacy classes fill into one of the* $\mathrm{SO}(2n+1,k)$-*conjugacy subclasses in Table 4.1 and 4.2 where* $s+t=2n+1$ *and* $s>t$.

4.2.2 The matrix representatives A such that $\theta = \mathfrak{I}_A$

From the Tables 4.1 and 4.2, we know the representatives of X^TX. In this subsection we'll choose a representative X for each value of X^TX and compute A.

Assume $a_1^2+b_1^2=p$, $a_2^2+b_2^2=N_p$,$a_3^2+b_3^2=pN_p$ and for $i=1$, 2, or 3 $c_i=\frac{a_i^2-b_i^2}{a_i^2+b_i^2}$, $d_i=\frac{2a_ib_i}{a_i^2+b_i^2}$. (Note that $c_i^2+d_i^2=1$). In the following we list for each type of X^TX a representative X and the corresponding matrix A. The notation in the following follows the pattern: the sup index in parenthesis indicates how many independent variables there are, and the sub indexes indicating which ones of the c_i's. For example $A^{(2)}_{1,2}$ means there are 2 of c_1, c_2, c_3 occur in the matrix and these two are c_1 and c_2. And we brief $A^{(3)}_{1,2,3}$ as $A^{(3)}$ since for sup index is three ,there is only one choice.
(1) $-1 \in \mathbb{Q}_p^2$.

(a) $X^TX = \begin{pmatrix} I_{s-3} & 0 & 0 & 0 \\ 0 & M_3 & 0 & 0 \\ 0 & 0 & I_{t-3} & 0 \\ 0 & 0 & 0 & M_3 \end{pmatrix}$, we choose

$$X = \begin{pmatrix} I_{s-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & b_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & 0 & I_{t-3} & 0 & 0 & 0 \\ 0 & -b_1 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & -b_2 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & -b_3 & 0 & 0 & 0 & a_3 \end{pmatrix},$$

hence

$$A^{(3)} = X^{-1} I_{s,t} X = \begin{pmatrix} I_{s-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & -I_{t-3} & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 & -c_1 & 0 & 0 \\ 0 & 0 & d_2 & 0 & 0 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & d_3 & 0 & 0 & 0 & -c_3 \end{pmatrix}.$$

(b) $X^T X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$, we choose $X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & a_1 & 0 & b_1 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & -b_1 & 0 & a_1 \end{pmatrix}$,

hence $A_1^{(1)} = X^{-1} I_{s,t} X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & c_1 & 0 & d_1 \\ 0 & 0 & -I_{t-1} & 0 \\ 0 & d_1 & 0 & -c_1 \end{pmatrix}$.

(c) $X^T X = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & I_{t-2} & 0 \\ 0 & 0 & 0 & M_2 \end{pmatrix}$, we choose $X = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 & 0 & 0 & \\ 0 & a_2 & 0 & 0 & b_2 & 0 & \\ 0 & 0 & a_3 & 0 & 0 & b_3 & \\ 0 & 0 & 0 & I_{t-2} & 0 & 0 & \\ 0 & -b_2 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & -b_3 & 0 & 0 & 0 & a_3 \end{pmatrix}$,

hence $A_{2,3}^{(2)} = X^{-1} I_{s,t} X = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & d_2 & 0 \\ 0 & 0 & c_3 & 0 & 0 & d_3 \\ 0 & 0 & 0 & -I_{t-2} & 0 & 0 \\ 0 & d_2 & 0 & 0 & -c_2 & 0 \\ 0 & 0 & d_3 & 0 & 0 & -c_3 \end{pmatrix}$.

(d) $X^T X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & N_p & 0 & 0 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & 0 & 0 & N_p \end{pmatrix}$, we choose $X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & a_2 & 0 & b_2 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & -b_2 & 0 & a_2 \end{pmatrix}$, hence

$A_2^{(1)} = X^{-1} I_{s,t} X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & c_2 & 0 & d_2 \\ 0 & 0 & -I_{t-1} & 0 \\ 0 & d_2 & 0 & -c_2 \end{pmatrix}$.

(e) $X^TX = \begin{pmatrix} I_{i-2} & 0 & 0 & 0 \\ 0 & N_2 & 0 & 0 \\ 0 & 0 & I_{2n+1-i} & 0 \\ 0 & 0 & 0 & N_2 \end{pmatrix}$, we choose $X = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & b_1 & 0 \\ 0 & 0 & a_3 & 0 & 0 & b_3 \\ 0 & 0 & 0 & I_{t-2} & 0 & 0 \\ 0 & -b_1 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & -b_3 & 0 & 0 & 0 & a_3 \end{pmatrix}$,

hence $A^{(2)}_{1,3} = X^{-1}I_{s,t}X = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & d_1 & 0 \\ 0 & 0 & c_3 & 0 & 0 & d_3 \\ 0 & 0 & 0 & -I_{t-2} & 0 & 0 \\ 0 & d_1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & d_3 & 0 & 0 & -c_3 \end{pmatrix}$.

(f) $X^TX = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & pN_p & 0 & 0 \\ 0 & 0 & I_{2n-i} & 0 \\ 0 & 0 & 0 & pN_p \end{pmatrix}$, we choose $X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & a_3 & 0 & b_3 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & -b_3 & 0 & a_3 \end{pmatrix}$,

hence $A^{(1)}_{3} = X^{-1}I_{s,t}X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & c_3 & 0 & d_3 \\ 0 & 0 & -I_{t-1} & 0 \\ 0 & d_3 & 0 & -c_3 \end{pmatrix}$.

(g) $X^TX = \begin{pmatrix} I_{i-2} & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & I_{2n+1-i} & 0 \\ 0 & 0 & 0 & L_2 \end{pmatrix}$, we choose

$$X = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & b_1 & 0 \\ 0 & 0 & a_3 & 0 & 0 & b_3 \\ 0 & 0 & 0 & I_{t-2} & 0 & 0 \\ 0 & -b_1 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & -b_3 & 0 & 0 & 0 & a_3 \end{pmatrix},$$

hence $A^{(2)}_{1,2} = X^{-1}I_{s,t}X = \begin{pmatrix} I_{s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & d_1 & 0 \\ 0 & 0 & c_2 & 0 & 0 & d_2 \\ 0 & 0 & 0 & -I_{t-2} & 0 & 0 \\ 0 & d_1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & d_2 & 0 & 0 & -c_2 \end{pmatrix}$.

(2) $-1 \notin (\mathbb{Q}_p^*)^2$. For those cases that also appear in case $-1 \in (\mathbb{Q}_p^*)^2$, the same matrices still hold, for the rest, we decide to leave it open since it requires to compute $X^TX = M_3$, which does not have a nice representative.

4.3 Fixed Point Groups

Lemma 46. *Suppose* $\theta = \mathcal{I}_A$ *with* $A = X^{-1}I_{s,t}X$, $X^TX = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{2n+1} \end{pmatrix}$.

Let

$$M_s = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_s \end{pmatrix} \quad \text{and} \quad M_t = \begin{pmatrix} a_{s+1} & 0 & \dots & 0 \\ 0 & a_{s+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{2n+1} \end{pmatrix}.$$

Then the fixed point group is

$$G^\theta = \left\{ X^{-1} \begin{pmatrix} N_s & 0 \\ 0 & N_t \end{pmatrix} X \;\middle|\; N_s M_s{}^T N_s = M_s,\ N_t M_t{}^T N_t = M_t \right\}.$$

Proof. Suppose $Y \in \mathrm{SO}(2n+1,k)$ such that $\theta(Y) = Y$, i.e. $X^{-1}I_{s,t}XYX^{-1}I_{s,t}X = Y$. So we have $I_{s,t}XYX^{-1} = XYX^{-1}I_{s,t}$. So $XYX^{-1} = \begin{pmatrix} N_s & 0 \\ 0 & N_t \end{pmatrix}$, i.e. $Y = X^{-1}\begin{pmatrix} N_s & 0 \\ 0 & N_t \end{pmatrix} X$. Since $Y \in \mathrm{SO}(2n+1,k)$, we have $X^{-1}\begin{pmatrix} N_s & 0 \\ 0 & N_t \end{pmatrix} X^TX \begin{pmatrix} {}^TN_s & 0 \\ 0 & {}^TN_t \end{pmatrix} {}^TX^{-1} = \mathcal{I}$, i.e.

$$\begin{pmatrix} N_s & 0 \\ 0 & N_t \end{pmatrix} X^TX \begin{pmatrix} {}^TN_s & 0 \\ 0 & {}^TN_t \end{pmatrix} = X^TX.$$

i.e. $N_s M_s{}^T N_s = M_s$, $N_t M_t{}^T N_t = M_t$. ☐

It follows from this result that in order to determine whether the fixed point group is compact comes down to determining whether the two parts N_1 and N_2 are compact, where

$$N_1 \begin{pmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_s \end{pmatrix} {}^TN_1 = \begin{pmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_s \end{pmatrix} \quad \text{and}$$

$$N_2 \begin{pmatrix} a_{s+1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{2n+1} \end{pmatrix} {}^TN_2 = \begin{pmatrix} a_{s+1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{2n+1} \end{pmatrix}.$$

The choices of the a_i's can be limited to those in Corollary 5. In particular for $k = \mathbb{Q}_p$, with m equal to s or t, M_s, M_t are one of the following:

(1) $\begin{pmatrix} I_{m-1} & 0 \\ 0 & a \end{pmatrix}$, with $a = p,\ N_p$ *or* pN_p.

(2) $\begin{pmatrix} I_{m-2} & 0 \\ 0 & A_2 \end{pmatrix}$, with $A_2 = L_2,\ M_2$ *or* N_2.

(3) $\begin{pmatrix} I_{m-3} & 0 \\ 0 & M_3 \end{pmatrix}$.

.

4.3.1 k is algebraically closed or $k = \mathbb{R}$ the real numbers

The following result gives the fixed point group for the case that $A = I_{s,t}$, which holds for any field.

Lemma 47. *For the matrix $A = I_{s,t}$, $s+t = 2n+1$, and $s > t$, the fixed point group $G^{\mathfrak{I}_A}$ is the matrix $\begin{pmatrix} N_s & 0 \\ 0 & N_t \end{pmatrix} \in \mathrm{SO}(2n+1,k)$, where $N_s \in \mathrm{SO}(s,k)$ and $N_t \in \mathrm{SO}(t,k)$.*

Proof. Choose X to be I, the identity matrix. The result follows immediately from Lemma 46. □

Remark 7. For k is algebraically closed the groups $\mathrm{SO}(s,k)$ and $\mathrm{SO}(t,k)$ are unbounded and therefor $G^{\mathfrak{I}_A}$ is non-compact. For $k = \mathbb{R}$ the group $\mathrm{SO}(2n+1,\mathbb{R})$ itself is compact and any closed subgroup is compact as well. So in this case $G^{\mathfrak{I}_A}$ is compact.

For k the complex numbers (or an algebraically closed field in general) and k the real numbers, there is only one conjugacy class of involutions and the the fixed point group is the one specified by Lemma 47. And it's compact for the real numbers and non-compact for the complex numbers (or algebraically closed fields in general).

4.3.2 k is $\mathbb{F}_p$, a finite field

Besides the involution $\mathfrak{I}_A$ where $A = I_{s,t}$, there is another isomorphy class of involutions represented by $\mathfrak{I}_B$ with $B = X^{-1} I_{s,t} X$, where

$$X = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & a & b & 0 \\ \vdots & -b & a & \vdots \\ 0 & 0 & \dots & -I_{t-1} \end{pmatrix} \text{ with } a^2+b^2 \notin k^2. \text{ That is } B = \begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & c & d & 0 \\ \vdots & d & -c & \vdots \\ 0 & 0 & \dots & I_{t-1} \end{pmatrix},$$

where $c^2 + d^2 = 1$. $c = \frac{a^2-b^2}{a^2+b^2}$, $d = \frac{2ab}{a^2+b^2}$, with $a^2 + b^2 \notin k^2$.

4.3.3 $\boldsymbol{k}$ is $\mathbb{Q}_p$, the $\boldsymbol{p}$-adic numbers

Lemma 48. *Assume* $x^2 + aby^2 = 1$ *and let* $C = \begin{pmatrix} x & ay \\ by & -x \end{pmatrix}$. *Then*

$$C \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} {}^T C = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where C *is compact if and only if* $-ab \in \mathbb{Q}_p^2$.

Proof. The first part follows immediately from a direct matrix computation.

The compactness is equivalent to closed and bounded. The set of matrices is closed, and bounded iff $x^2 + aby^2 = 1$ has only bounded solution. That is, $-ab \in \mathbb{Q}_p^2$. □

Lemma 49. *For* $D = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$, *the matrices* N *such that* $ND^TN = D$ *are the following non-compact matrices* $\begin{pmatrix} a & b & c \\ \frac{\delta ybi - zacf}{xa^2+yb^2} & -\frac{\delta xai + zbcf}{xa^2+yb^2} & f \\ -\frac{z(aci+\delta bf)}{xa^2+yb^2} & \frac{z(\delta xaf - ybci)}{y(xa^2+yb^2)} & i \end{pmatrix}$, *where* δ *is* 1 *or* -1 *and* a, b, c, f *and* i *satisfy the following equations:*

$$a^2 + \frac{y}{x}b^2 + \frac{z}{x}c^2 = 1$$

and

$$\frac{z}{x}c^2 + \frac{z}{y}f^2 + i^2 = 1.$$

Proof. Let θ be the automorphism such that $\theta(X) = {}^TX^{-1}$. Then N is the fixed point group of $\theta \mathfrak{I}_D$ and the result follows from Lemma 35. □

The above two Lemmas give us the tools to reduce the possible compact situations to a limited number. Let's take a look. By Lemma 49, the only possible compact N would have to come from those whose sizes are less than 3 since the x, y, z have no restrictions in the Lemma. So N could be $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or a 1×1-matrix. Furthermore for the 2×2-matrices Lemma 48 allows us only those with $-ab \notin \mathbb{Q}_p^2$. Namely

(1) $-1 \in \mathbb{Q}_p^2$. $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & N_p \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & pN_p \end{pmatrix}$, L_2, M_2, N_2.

(2) $-1 \notin \mathbb{Q}_p^2$. $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\0 & p\end{pmatrix}$, $\begin{pmatrix}1 & 0\\0 & pN_p\end{pmatrix}$, $\begin{pmatrix}p & 0\\0 & p\end{pmatrix}$, $\begin{pmatrix}N_p & 0\\0 & N_p\end{pmatrix}$, $\begin{pmatrix}pN_p & 0\\0 & pN_p\end{pmatrix}$, L_2, M_2.

Theorem 12. *The only compact fixed point groups for* $\mathrm{SO}(2n+1,\mathbb{Q}_p)$ *are in* $\mathrm{SO}(3,\mathbb{Q}_p)$ *with* $t=1$ *or* $t=0$. *All possibilities are for the following values of* X^TX.

(1) $-1 \in \mathbb{Q}_p^2$.

(a) in the $\mathrm{GL}(3,k)$*-conjugate class of* $I_{2,1}$*:* $\begin{pmatrix}1 & 0 & 0\\0 & p & 0\\0 & 0 & p\end{pmatrix}$, $\begin{pmatrix}1 & 0 & 0\\0 & N_p & 0\\0 & 0 & N_p\end{pmatrix}$, $\begin{pmatrix}1 & 0 & 0\\0 & pN_p & 0\\0 & 0 & pN_p\end{pmatrix}$.

(b) in the $\mathrm{GL}(3,k)$*-conjugate class of* $I_{3,0}$*: None.*

(2) $-1 \notin \mathbb{Q}_p^2$.

(a) in the $\mathrm{GL}(3,k)$*-conjugate class of* $I_{2,1}$*:* I, $\begin{pmatrix}N_p & 0 & 0\\0 & pN_p & 0\\0 & 0 & p\end{pmatrix}$, $\begin{pmatrix}1 & 0 & 0\\0 & N_p & 0\\0 & 0 & N_p\end{pmatrix}$, $\begin{pmatrix}p & 0 & 0\\0 & N_p & 0\\0 & 0 & pN_p\end{pmatrix}$.

(b) in the $\mathrm{GL}(3,k)$*-conjugate class of* $I_{3,0}$*:* I.

4.4 k-split k-form of $\mathrm{SO}(n,\bar{k})$

Contrary to the case of $\mathrm{SL}(n,k)$ the group $\mathrm{SO}(n,k)$ is usually not k-split and actually the opposite is the case.

For example if $k=\mathbb{R}$ the group $\mathrm{SO}(n,\mathbb{R})$ is k-anisotropic, i.e. compact, so the opposite of k-split. For $k=\mathbb{F}_p$ and $k=\mathbb{Q}_p$ the group $\mathrm{SO}(n,k)$ is not k-anisotropic, but not always k-split either. In order to get the k-split k-form of $\mathrm{SO}(n,\bar{k})$, then instead of the standard symmetric bilinear form $B(x,y)={}^Txy$ one has to take a different symmetric bilinear form $B_1(x,y)={}^TxMy$ and consider the group $\mathrm{SO}(n,k,B_1):=\{A\in\mathrm{GL}(n,k)\mid B_1(A(x),A(y))=(x,y)\text{ for all }x,y\in k^n\}$, which is also a k-form of $\mathrm{SO}(n,\bar{k})$. Naturally the symmetric bilinear forms B and B_1 are congruent over $\bar{k}$ and consequently the corresponding groups $\mathrm{SO}(n,k)=\mathrm{SO}(n,k,B)$ and $\mathrm{SO}(n,k,B_1)$ are isomorphic. In the next subsection we will discuss the k-split k-form of $\mathrm{SO}(n,\bar{k})$ and show that for $k=\mathbb{F}_p$ and $k=\mathbb{Q}_p$ the group $\mathrm{SO}(n,k)$ is k-split if n is odd, but is not always k-split if n is even.

A consequence of this all is that for an arbitrary field k we cannot always use the characterization of the involutions using the k-inner elements, like we did in the case of $\mathrm{SL}(n,k)$, since this holds only for k-split k-forms.

For other k-forms the result could still hold, but there might be fewer isomorphy classes due to the fact that we need to consider isomorphy under $(H.Z_G(A))_k$ instead of H_k. Here $Z_G(A)$ is the centralizer of the maximal (θ,k)-split torus A and X_k denotes the set of k-rational points of a variety X, defined over k. We will discuss this further in this section.

In general we can get a k-split k-form of $\mathrm{SO}(n,\bar{k})$ as follows. Let $B_1(x,y) = {}^T x M_1 y$ be the symmetric bilinear form with M_1 the $n\times n$-matrix

$$M_1 = \begin{pmatrix} I_{n-l} & 0 \\ 0 & -I_l \end{pmatrix}$$

and $l = \lfloor \frac{n}{2} \rfloor$. Let

$$T_1 = \left\{ \begin{pmatrix} a_1 & 0 & \dots & 0 & b_1 & 0 & \dots & 0 & 0 \\ 0 & a_2 & \dots & 0 & 0 & b_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_n & 0 & 0 & \dots & b_n & 0 \\ b_1 & 0 & \dots & 0 & a_1 & 0 & \dots & 0 & 0 \\ 0 & b_2 & \dots & 0 & 0 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_n & 0 & 0 & \dots & a_n & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \;\middle|\; a_i^2 - b_i^2 = 1, \text{ for } i = 1,2,\dots,n \right\}$$

and

$$T_2 = \left\{ \begin{pmatrix} a_1 & 0 & \dots & 0 & b_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n & 0 & 0 & \dots & b_n \\ b_1 & 0 & \dots & 0 & a_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 & 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n & 0 & 0 & \dots & a_n \end{pmatrix} \;\middle|\; a_i^2 - b_i^2 = 1, \text{ for } i = 1,2,\dots,n \right\}.$$

Then we have the following:

Proposition 4. *T_1 is a maximal torus of $\mathrm{SO}(2n+1,k,B_1)$, which is k-split and T_2 is a maximal torus of $\mathrm{SO}(2n,k,B_1)$, which is k-split. So in particular the group $\mathrm{SO}(n,k,B_1)$ is k-split.*

Proof. From Lemma 10 it follows that T_1 and T_2 consist of k-split semisimple elements. Since the dimension of T_1 and T_2 is equal to n, which is equal to the rank of $\mathrm{SO}(2n+1,k,B_1)$ and $\mathrm{SO}(2n,k,B_1)$ the result follows. □

Remark 8. If B_1 and B_2 are symmetric bilinear forms for which the corresponding matrices M_1 and M_2 are congruent, then the groups $\mathrm{SO}(n,k,B_1)$ and $\mathrm{SO}(n,k,B_2)$ are k-isomorphic. In particular if $X \in \mathrm{GL}(n,k)$ such that ${}^TXM_1X = M_2$, then $X^{-1}\,\mathrm{SO}(n,k,B_1)X = \mathrm{SO}(n,k,B_2)$.

Corollary 6. *If* $-1 \in (k^*)^2$*, then* $\mathrm{SO}(n,k,B_1)$ *is isomorphic to* $\mathrm{SO}(n,k)$ *and in particular* $\mathrm{SO}(n,k)$ *is* k*-split.*

Proof. Since M_1 is congruent to Id the corresponding groups are isomorphic as well. □

Remark 9. For $k = \mathbb{F}_p$ and $\mathbb{Q}_p$ we have by 2.4.2 that $-1 \in (k^*)^2$ if and only if $p \equiv 1($ mod $4)$.

For many fields one can reduce the number of duplicate diagonal entries:

Lemma 50. *Let* $\epsilon \in k$ *such that* ϵ *can be written as a sum of 2 squares. Then* $M_1 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ *is congruent to* Id.

Proof. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,k)$. Then

$$ {}^TXM_1X = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \epsilon(a^2+c^2) & \epsilon(ab+cd) \\ \epsilon(ab+cd) & \epsilon(b^2+d^2) \end{pmatrix}. $$

Taking $a = d$, $b = -c$ and $(a^2+c^2) = \epsilon^{-1}$ the result follows. □

Both $k = \mathbb{F}_p$ and $\mathbb{Q}_p$ satisfy the property that any element of k^* can be written as a sum of 2 squares, so in these cases $M_1 = \begin{pmatrix} I_{n-l} & 0 \\ 0 & -I_l \end{pmatrix}$ is congruent to $\begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$ or Id. For $k = \mathbb{R}$ the element -1 is clearly not the sum of two squares and in this case we cannot eliminate any -1's from the diagonal.

We can show now that the group $\mathrm{SO}(2n+1,k)$ for $k = \mathbb{F}_p$ and $\mathbb{Q}_p$ is k-split.

Proposition 5. *For* $k = \mathbb{F}_p$ *and* $\mathbb{Q}_p$ *the group* $\mathrm{SO}(2n+1,k)$ *is* k*-split.*

Proof. Let $B_1(x,y) = {}^TxM_1y$ be the symmetric bilinear form with $M_1 = \begin{pmatrix} I_{2n+1-l} & 0 \\ 0 & -I_l \end{pmatrix}$ and $l = \lfloor \frac{n}{2} \rfloor$.

By Proposition 4 the group $\mathrm{SO}(2n+1,k,B_1)$ is k-split and as we saw above for $k=\mathbb{F}_p$ and $\mathbb{Q}_p$ the matrix M_1 is congruent to $\begin{pmatrix} I_{2n} & 0 \\ 0 & -1 \end{pmatrix}$ or Id. If M_1 is congruent to $\begin{pmatrix} I_{2n} & 0 \\ 0 & -1 \end{pmatrix}$, then by Lemma 50 $\begin{pmatrix} I_{2n} & 0 \\ 0 & -1 \end{pmatrix}$ is congruent to $-\mathrm{Id}$. Since the groups for $M_1 = \mathrm{Id}$ and $M_1 = -\mathrm{Id}$ are the same, the result follows. □

Remark 10. The same proof does not work for $\mathrm{SO}(2n,k,B_1)$ and consequently the group $\mathrm{SO}(2n,k)$ does not need to be k-split for $k=\mathbb{F}_p$ and $k=\mathbb{Q}_p$.

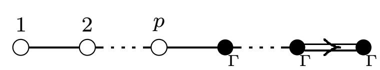
4.5 *k*-inner elements

For $k=\mathbb{R}$ the group $\mathrm{SO}(2n+1,k)$ is k-anisotropic, so there are no (θ,k)-split tori, so consequently also no k-inner elements. By [Hel88, Lemma 10.3] the isomorphy classes of involutions of k-anisotropic real groups correspond to those of the corresponding complex group. Moreover for k algebraically closed or the real numbers there is only one isomorphy class of involutions $\mathcal{I}_A$ with $A = I_{s,t}$ for each value of s. So also for this reason there are no k-inner elements in these two cases. In this section we will determine the k-inner elements for $k=\mathbb{Q}_p$, but first we list the admissible (Γ,σ)-indices for $G=\mathrm{SO}(2n+1,k)$.

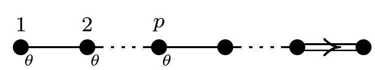
4.5.1 (Γ,σ)-indices

The (Γ,σ)-indices for the involutions of $G=\mathrm{SO}(2n+1,k)$ for those fields for which we classified the involutions in the previous sections are the following:

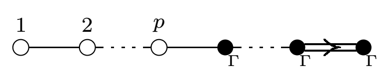
$k=\bar{k}$: $B_{2n+1,2n+1}(I^p), 1 \le p \le n$:



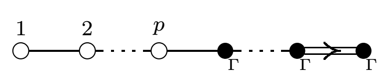
$k=\mathbb{R}$: $B_{2n+1,0}(I^p), 1 \le p \le n$:



$k=\mathbb{F}_p$: $B_{2n+1,2n+1}(I^p), 1 \le p \le n$:



$k=\mathbb{Q}_p$: $B_{2n+1,2n+1}(I^p), 1 \le p \le n-1$:



In the case that $k=\mathbb{F}_p$ and $k=\mathbb{Q}_p$ the group $\mathrm{SO}(2n+1,k)$ is k-split and we can classify the k-inner elements, similarly as in the case of involutions of $\mathrm{SL}(n,k)$.

4.6 Classification of k-inner elements for $\mathrm{SO}(2n+1,\mathbb{Q}_p)$

4.6.1 The case $-1 \in \mathbb{Q}_p^2$

As we saw in section 2.4.2 the group $\mathrm{SO}(2,k)$ for $k = \mathbb{F}_p$ and $\mathbb{Q}_p$ is k-split if $-1 \in (k^*)^2$ and k-anisotropic if $-1 \notin (k^*)^2$. As we saw from Lemma 10 the matrix $\left(\begin{smallmatrix} a & b \\ \lambda b & a \end{smallmatrix}\right)$ is k-split (i.e. diagonizable over k) iff $\lambda \in (k^*)^2$. We have the following Lemma to determine the maximal k-split and maximal (θ,k)-split tori.

Lemma 51.

$$T = \left\{ \begin{pmatrix} a_1 & b_1 & \dots & 0 & 0 & 0 \\ -b_1 & a_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n & b_n & 0 \\ 0 & 0 & \dots & -b_n & a_n & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \middle|\ a_i^2 + b_i^2 = 1,\ \mathit{for}\, i = 1,2,\dots,n \right\}$$

is a maximal k-split torus of $\mathrm{SO}(2n+1,k)$ *if* $-1 \in {k^*}^2$.

Proof. From Lemma 11 it follows that T consist of k-split semisimple elements. Since the dimension of T is equal to n, which is equal to the rank of $\mathrm{SO}(2n+1,k)$ it follows that T is a maximal torus as well. $\square$

We know that the involutions of $\bar{G}$ are $\mathcal{I}_A$ where $A = I_{s,t} = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}$, $s+t = 2n+1$ and $s > t$. The following Lemma will help us to determine the conjugacy classes of involutions of G.

Lemma 52. *For $\sigma = I_A$, where $A = I_{s,t}$, with $t = 1,2,\dots,n$, $s+t = 2n+1$, the maximal (σ,k)-split torus can be chosen as:*

$$A_{s,t} = \left\{ \begin{pmatrix} a_1 & \dots & 0 & \dots & 0 & \dots & b_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & a_t & \dots & b_t & \dots & 0 \\ \vdots & \vdots & \vdots & I_{2n+1-2t} & \vdots & \vdots & \vdots \\ 0 & \dots & -b_t & \dots & a_t & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_1 & \dots & 0 & \dots & 0 & \dots & a_1 \end{pmatrix} \middle|\ a_i^2 + b_i^2 = 1,\ \mathit{for}\ i = 1,\dots,t. \right\}.$$

The dimension of the maximal (σ,k)-split torus is t.

Proof. It's obvious that $A_{s,t}$ is a (σ,k)-split torus for $\sigma = I_A$ with $A = I_{s,t}$. We prove next that the maximal dimension of a (σ,k)-split torus for $\sigma = I_A$ with $A = I_{s,t}$ is t. Let $T_1 \in \mathrm{SO}(2n+1,\bar{k})$ be a maximal torus, then there is $X \in \mathrm{SO}(2n+1,\bar{k})$ s.t. $T = X^{-1}T_1X$, where T_1 is maxiaml torus of diagonal matrices. For a maximal (σ,k)-split torus $T = X^{-1}T_1X$, we have for $\tau \in T_1$,

$$\begin{aligned}\sigma(X^{-1}\tau X) = (X^{-1}\tau X)^{-1} &\Rightarrow \mathcal{I}_A(X^{-1}\tau X) = X^{-1}\tau^{-1}X\\ &\Rightarrow A^{-1}(X^{-1}\tau X)A = X^{-1}\tau^{-1}X\\ &\Rightarrow \tau XAX^{-1} = XAX^{-1}\tau^{-1}.\end{aligned}$$

Since τ is conjugate to τ^{-1}, the highest possible dimension can only be less or equal to n. Furthermore, if $\tau = \mathrm{diag}(a_1,\dots,a_i,a_i^{-1},\dots,a_1^{-1},1,\dots,1)$, and $\tau Y = Y\tau^{-1}$, then we have $Y = \begin{pmatrix} J & 0 \\ 0 & Y_{2n+1-2i}\end{pmatrix}$, therefore, if the (σ,k)-split tori has dimension i, the corresponding $I_{2n+1-j,j}$ has to be conjugate to $\begin{pmatrix} J & 0 \\ 0 & Y_{2n+1-2i}\end{pmatrix}$. Hence the (σ,k)-split tori have dimension i if and only if the corresponding $I_{2n+1-j,j}$ such that $j \geq i$, i.e. the maximal (σ,k)-split tori is of dimension t for $I_{2n+1-t,t} = I_{s,t}$. □

Lemma 53. *The matrices* $A = \begin{pmatrix} -a & b \\ b & a\end{pmatrix}$ *and* $B = \begin{pmatrix} -c & d \\ d & c\end{pmatrix}$ *are conjugate over* $\mathrm{SO}(2,k)$ *if and only if* $a+1 = e^2(c+1)$ *for some* $e \in k$*, where* $a^2+b^2 = c^2+d^2 = 1$ *and* $a \neq -1$*,* $c \neq -1$*.*

Proof. Let $X = \begin{pmatrix} b & a+1 \\ a+1 & -b\end{pmatrix}$ and $Y = \begin{pmatrix} d & c+1 \\ c+1 & -d\end{pmatrix}$, $I_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$.

($\Rightarrow$) $A = X^{-1}I_{1,1}X$ and $B = Y^{-1}I_{1,1}Y$,

Since A is conjugate to B over $\mathrm{SO}(2,k)$ there is a matrix $Z \in \mathrm{SO}(2,k)$, such that $Z^{-1}AZ = B$, i.e. $Z^{-1}X^{-1}I_{1,1}XZ = Y^{-1}I_{1,1}Y$, therefore $I_{1,1}XZY^{-1} = XZY^{-1}I_{1,1}$, so $XZY^{-1} = \begin{pmatrix} e & 0 \\ 0 & f\end{pmatrix}$, i.e. $XZ = \begin{pmatrix} e & 0 \\ 0 & f\end{pmatrix}Y$. Since $Z \in \mathrm{SO}(2,k)$, we have $X^TX = XZ^TZ^TX = \begin{pmatrix} e & 0 \\ 0 & f\end{pmatrix}Y^TY\begin{pmatrix} e & 0 \\ 0 & f\end{pmatrix}$, i.e. $\begin{pmatrix} 2(a+1) & 0 \\ 0 & 2(a+1)\end{pmatrix} = \begin{pmatrix} 2e^2(c+1) & 0 \\ 0 & 2f^2(c+1)\end{pmatrix}$, i.e. $a+1 = e^2(c+1)$.

($\Leftarrow$) If $a+1 = e^2(c+1)$, let $Z = eX^{-1}Y$, then $Z \in \mathrm{SO}(2,k)$ and $Z^{-1}AZ = B$. □

Remark 11. Since inner automorphism $\mathcal{I}_A = \mathcal{I}_{-A}$, at the time $a = -1$, we can consider $-A$ instead. And in the above proof, if we take $Z = eX^{-1}I_{1,1}Y$, we still have $Z^{-1}AZ = B$, $Z^TZ = I$ and $\det Z = -1$.

Corollary 7. *For the field* $\mathbb{Q}_p$*, the matrices* $A = \begin{pmatrix} -a & b \\ b & a\end{pmatrix}$ *have four (respective eight) different isomorphic classes for* $p \neq 2$ *(respective* $p = 2$*), where* $a^2+b^2 = 1$*.*

Let T be a maximal k-split torus of $\bar{G}$. Since G is k-split T is a maximal torus of $\bar{G}$ as well. Since G is k-split it follows from [Hel00, Theorem 8. 33] that we have the following characterization of the isomorphy classes:

Theorem 13 ([Hel00, Theorem 8. 33]). *Assume G is k-split. Then any k-involution of G is isomorphic to one of the form $\sigma\,\mathrm{Int}(a)$, where σ is a representative of a $\bar{G}$-isomorphy class of k-involutions, A is a maximal (σ,k)-split torus and $a \in A$.*

The set of *set of k-inner elements of A* is defined as the set of those $a \in A$ such that $\sigma\,\mathrm{Int}(a)$ is a k-involution of G by $I_k(A)$. We recall that from [Hel00, Lemma 9. 7] it follows that one can find a set of representatives for the isomorphy classes of the involutions $\sigma\,\mathrm{Int}(A)$ in the set $I_k(A)/A_k^2$. Here A_k is the set a k-regular elements of A and $A_k^2 = \{a^2 \mid a \in A_k\}$. Note that the set $A_k/A_k^2 = \simeq (k^*/(k^*)^2)^n$.

Lemma 54. *All the k-inner elements for $\mathrm{SO}(2n+1,k)$ are conjugate to $\mathfrak{I}_A$ over $\mathrm{SO}(2n+1,k)$ with A one of the following:*

$$\begin{pmatrix} -a_1 & b_1 & \dots & 0 & 0 & 0 \\ b_1 & a_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -a_t & b_t & 0 \\ 0 & 0 & \dots & b_t & a_t & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}$$

where $a_i^2 + b_i^2 = 1$ and $a_i \neq -1$ for $i = 1, 2, \dots, t$.

Proof. There is only one type of involutions over $\bar{k}$, $\mathfrak{I}_{I_{s,t}}$. Since the maximal (θ,k)-split torus is $A_{s,t}$ (Lemma 52), the k-inner elements are $\mathfrak{I}_{I_{s,t}}\mathfrak{I}_{A_{s,t}}$, i.e. the k-inner elements are $\mathfrak{I}_B$ with $B = I_{s,t}A_{s,t}$, which is

$$\begin{pmatrix} a_1 & \dots & 0 & \dots & 0 & \dots & b_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & a_t & \dots & b_t & \dots & 0 \\ \vdots & \vdots & \vdots & I_{2n+1-2t} & \vdots & \vdots & \vdots \\ 0 & \dots & b_t & \dots & -a_t & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1 & \dots & 0 & \dots & 0 & \dots & -a_1 \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} b_1 & a_1+1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\ a_1+1 & -b_1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & b_2 & a_2+1 & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & a_2+1 & -b_2 & \dots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & b_t & a_t+1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & a_t+1 & -b_t & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}$$

and

$$Y = \begin{pmatrix} a_1+1 & \dots & 0 & \dots & 0 & \dots & b_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & a_t+1 & \dots & b_t & \dots & 0 \\ \vdots & \vdots & \vdots & I_{2n+1-2t} & \vdots & \vdots & \vdots \\ 0 & \dots & -b_t & \dots & a_t+1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_1 & \dots & 0 & \dots & 0 & \dots & a_1+1 \end{pmatrix},$$

then

$$C = X^{-1}AX = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}$$

and $Y^{-1}BY = I_{s,t}$. Let $L_{i,j}$ be the fundamental matrix which exchanges the i'-th and j'-th row(column),

$$L = L_{2t,2n+1}L_{2t-2,2n}\dots L_{2,2n+1-(t-1)},$$

and $Z = YL^{-1}X^{-1}$ then we have $L^{-1}CL = I_{s,t}$ and

$$\begin{aligned} Z^{-1}BZ &= XLY^{-1}YI_{s,t}Y^{-1}YL^{-1}X^{-1} \\ &= XLI_{s,t}L^{-1}X^{-1} = XCX^{-1} = A \end{aligned}$$

is this form. Furthermore ${}^{T}ZZ = I$ and $\det Z = 1$. $\square$

Lemma 55. *Consider the* $n \times n$ *matrices*

$$A = \begin{pmatrix} -a_1 & b_1 & \dots & 0 & 0 & 0 \\ b_1 & a_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -a_t & b_t & 0 \\ 0 & 0 & \dots & b_t & a_t & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}$$

and

$$B = \begin{pmatrix} -a'_1 & b'_1 & \dots & 0 & 0 & 0 \\ b'_1 & a'_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -a'_t & b'_t & 0 \\ 0 & 0 & \dots & b'_t & a'_t & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}.$$

They are conjugate over $\mathrm{SO}(2n+1,\mathbb{Q}_p)$ *if and only if* $\begin{pmatrix} (a_1+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a_t+1) \end{pmatrix}$ *and* $\begin{pmatrix} (a'_1+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a'_t+1) \end{pmatrix}$ *are congruent.*

Proof. Suppose

$$A = \begin{pmatrix} -a_1 & b_1 & \dots & 0 & 0 & 0 \\ b_1 & a_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -a_t & b_t & 0 \\ 0 & 0 & \dots & b_t & a_t & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}$$

and

$$B = \begin{pmatrix} -a'_1 & b'_1 & \dots & 0 & 0 & 0 \\ b'_1 & a'_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -a'_t & b'_t & 0 \\ 0 & 0 & \dots & b'_t & a'_t & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}$$

are conjugate. Let

$$X = \begin{pmatrix} b_1 & 0 & \dots & 0 & 0 & \dots & 0 & a_1+1 & 0 \\ a_1+1 & 0 & \dots & 0 & 0 & \dots & 0 & -b_1 & 0 \\ 0 & b_2 & \dots & 0 & 0 & \dots & a_2+1 & 0 & 0 \\ 0 & a_2+1 & \dots & 0 & 0 & \dots & -b_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_t & a_t+1 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & a_t+1 & -b_t & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}$$

and

$$Y = \begin{pmatrix} b_1' & 0 & \dots & 0 & 0 & \dots & 0 & a_1'+1 & 0 \\ a_1'+1 & 0 & \dots & 0 & 0 & \dots & 0 & -b_1' & 0 \\ 0 & b_2' & \dots & 0 & 0 & \dots & a_2'+1 & 0 & 0 \\ 0 & a_2'+1 & \dots & 0 & 0 & \dots & -b_2' & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_t' & a_t'+1 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & a_t'+1 & -b_t' & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & I_{2n+1-2t} \end{pmatrix}.$$

Then $A = -XI_{t,s}X^{-1}$ and $B = -YI_{t,s}Y^{-1}$. Since A and B are conjugate over $\mathrm{SO}(2n+1,k)$, there is some $Z \in \mathrm{SO}(2n+1,k)$ such that $Z^{-1}AZ = B$, i.e. $Z^{-1}XI_{t,s}X^{-1}Z = YI_{t,s}Y^{-1}$. So $I_{t,s}X^{-1}ZY = X^{-1}ZYI_{t,s}$, thus $X^{-1}ZY = \begin{pmatrix} A_t & 0 \\ 0 & A_s \end{pmatrix}$, where A_s is $s \times s$ and A_t is $t \times t$. Therefore $ZY = X\begin{pmatrix} A_t & 0 \\ 0 & A_s \end{pmatrix}$. Since $Z \in \mathrm{SO}(2n+1,k)$, we have

$$^TYY = {}^TY^TZZY = \begin{pmatrix} ^TA_t & 0 \\ 0 & ^TA_s \end{pmatrix} {}^TXX \begin{pmatrix} A_t & 0 \\ 0 & A_s \end{pmatrix}.$$

i.e.

$$\begin{pmatrix} 2(a_1'+1) & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 2(a_t'+1) & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 2(a_t'+1) & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \ddots & 0 & 0 \\ 0 & \dots & 0 & 0 & \dots & 2(a_1'+1) & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & I_{2n+1-2t} \end{pmatrix} =$$

$$\begin{pmatrix} {}^TA_t & 0 \\ 0 & {}^TA_s \end{pmatrix} \begin{pmatrix} 2(a_1+1) & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 2(a_t+1) & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 2(a_t+1) & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \ddots & 0 & 0 \\ 0 & \dots & 0 & 0 & \dots & 2(a_1+1) & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & I_{2n+1-2t} \end{pmatrix} \begin{pmatrix} A_t & 0 \\ 0 & A_s \end{pmatrix},$$

therefore $\begin{pmatrix} (a'_1+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a'_t+1) \end{pmatrix}$ is congruent to $\begin{pmatrix} (a_1+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a_t+1) \end{pmatrix}$.

($\Leftarrow$) If $\begin{pmatrix} (a'_1+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a'_t+1) \end{pmatrix}$ is congruent to $\begin{pmatrix} (a_1+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a_t+1) \end{pmatrix}$, then $\begin{pmatrix} (a'_t+1) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & (a'_1+1) & 0 \\ 0 & \dots & 0 & I_{2n+1-t} \end{pmatrix}$ is also congruent to $\begin{pmatrix} (a_t+1) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & (a_1+1) & 0 \\ 0 & \dots & 0 & I_{2n+1-t} \end{pmatrix}$, therefore there is a $s\times s$ matrix M and a $t\times t$ matrix N, and $L = \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}$, such that ${}^TL^TXXL = {}^TYY$. Let $Z = XLY^{-1}$, therefore $Z^{-1}AZ = -YL^{-1}X^{-1}XI_{t,s}X^{-1}XLY^{-1} = -YL^{-1}I_{t,s}LY^{-1} = -YI_{t,s}Y^{-1} = B$. Furthermore we have ${}^TZZ = {}^TY^{-1T}L^TXXLY^{-1} = I$, if $\det Z = -1$ we take $-Z$ instead, since the size of the matrix is $2n+1$, therefore $\det -Z = 1$. □

So for the p-adic numbers, by Lemma 55, the dual values (δ, t) determine the $\mathrm{SO}(2n+1,k)$-conjugate classes, where $\delta = (a_1+1)(a_2+1)\dots(a_t+1)(\mod \mathbb{Q}_p^{*2})$, the representative in $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ and τ is the Hasse symbol of $\begin{pmatrix} (a_1+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a_t+1) \end{pmatrix}$. Since there are only 2 possibilities for τ and 4 for δ, there are at most eight possible involutions up to conjugation.

Corollary 8. *Let $K = \begin{pmatrix} -a & b \\ b & a \end{pmatrix}$, where $a \neq -1$, then K is divided as four subset: K_1, K_p, K_{N_p} and K_{pN_p} according to which coset of $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} (p \neq 2)$ $a+1$ is in. Then all the $\mathrm{SO}(2n+1,k)$-conjugacy subclasses are in table 4 for $-1 \in \mathbb{Q}_p^2$ if we assume $s > t$,* *Where D is a representative of* $\begin{pmatrix} (a_1+1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (a_t+1) \end{pmatrix}$, *and I_A is a representative of the conjugacy class of involutions. In particular we can take* $K_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Remark 12. Note that the classification is the a variation (up to conjugacy) of that of the previous section. And we verified our classification.

δ	$(\delta\delta)_p$	Triple value	representative of X^TX	$t \geq 3$	$t=2$	$t=1$	$t=0$
1	1	$(1,1,1)$	I	Y	Y	Y	Y
1	1	$(1,-1,-1)$	$\begin{pmatrix} I_{s-3} & 0 & 0 & 0 \\ 0 & M_3 & 0 & 0 \\ 0 & 0 & I_{t-3} & 0 \\ 0 & 0 & 0 & M_3 \end{pmatrix}$	Y	N	N	N
p	1	$(p,1,1)$	$\begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$	Y	Y	Y	N
p	1	$(p,-1,-1)$	$\begin{pmatrix} I_{s-2} & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & I_{t-2} & 0 \\ 0 & 0 & 0 & M_2 \end{pmatrix}$	Y	Y	N	N
N_p	1	$(N_p,1,1)$	$\begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & N_p & 0 & 0 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & 0 & 0 & N_p \end{pmatrix}$	Y	Y	Y	N
N_p	1	$(N_p,-1,-1)$	$\begin{pmatrix} I_{s-2} & 0 & 0 & 0 \\ 0 & N_2 & 0 & 0 \\ 0 & 0 & I_{t-2} & 0 \\ 0 & 0 & 0 & N_2 \end{pmatrix}$	Y	Y	N	N
pN_p	1	$(pN_p,1,1)$	$\begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & pN_p & 0 & 0 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & 0 & 0 & pN_p \end{pmatrix}$	Y	Y	Y	N
pN_p	1	$(pN_p,-1,-1)$	$\begin{pmatrix} I_{s-2} & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & I_{t-2} & 0 \\ 0 & 0 & 0 & L_2 \end{pmatrix}$	Y	Y	N	N
		In total		8	7	4	1

Table 4.1: $-1 \in \mathbb{Q}_p^2$

δ	$(\delta\delta)_p$	Triple value	representative of X^TX	$t \geq 3$	$t=2$	$t=1$	$t=0$
1	1	$(1,1,1)$	I	Y	Y	Y	Y
1	1	$(1,-1,-1)$	$\begin{pmatrix} I_{s-3} & 0 & 0 & 0 \\ 0 & M_3 & 0 & 0 \\ 0 & 0 & I_{t-3} & 0 \\ 0 & 0 & 0 & M_3 \end{pmatrix}$	Y	N	N	N
p	-1	$(p,1,-1)$	$\begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & I_{t-2} & 0 \\ 0 & 0 & 0 & M_2 \end{pmatrix}$	Y	Y	N	N
p	-1	$(p,-1,1)$	$\begin{pmatrix} I_{s-2} & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$	Y	Y	Y	N
N_p	1	$(N_p,1,1)$	$\begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & N_p & 0 & 0 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & 0 & 0 & N_p \end{pmatrix}$	Y	Y	Y	N
N_p	1	$(N_p,-1,-1)$	$\begin{pmatrix} I_{s-2} & 0 & 0 & 0 \\ 0 & N_2 & 0 & 0 \\ 0 & 0 & I_{t-2} & 0 \\ 0 & 0 & 0 & N_2 \end{pmatrix}$	Y	Y	N	N
pN_p	-1	$(pN_p,1,-1)$	$\begin{pmatrix} I_{s-1} & 0 & 0 & 0 \\ 0 & pN_p & 0 & 0 \\ 0 & 0 & I_{t-2} & 0 \\ 0 & 0 & 0 & L_2 \end{pmatrix}$	Y	Y	N	N
pN_p	-1	$(pN_p,-1,1)$	$\begin{pmatrix} I_{2-2} & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & I_{t-1} & 0 \\ 0 & 0 & 0 & pN_p \end{pmatrix}$	Y	Y	Y	N
		In total		8	7	4	1

Table 4.2: $-1 \notin \mathbb{Q}_p^2$

δ	τ	rep. for $-1 \in \mathbb{Q}_p^2$	rep. for $-1 \notin \mathbb{Q}_p^2$
1	1	I	1
1	-1	$K_{n,p,N_p,pN_p},$	$N_{n,p,p}$
p	1	$M_{n,p}$	$M_{n,p}$
p	-1	$N_{n,N_p,pN_p},$	$N_{n,N_p,pN_p},$
N_p	1	M_{n,N_p}	M_{n,N_p}
N_p	-1	$N_{n,p,pN_p},$	N_{n,p,p,N_p}
pN_p	1	M_{n,pN_p}	M_{n,pN_p}
pN_p	-1	$N_{n,p,N_p},$	N_{n,p,N_p}

Table 4.3: Representative of matrices satisfying δ and Hasse symbol value in $\mathbb{Q}_p$, $(p \neq 2)$

δ	1	1	2	2	3	3	6	6
Hasse symbol	1	-1	1	-1	1	-1	1	-1
matrix rep.	I	$N_{n,3,3}$	$M_{n,2}$	$N_{n,-1,-2}$	$M_{n,3}$	$N_{n,2,6}$	$M_{n,6}$	$N_{n,2,3}$
δ	-1	-1	-2	-2	-3	-3	-6	-6
Hasse symbol	1	-1	1	-1	1	-1	1	-1
matrix rep.	$M_{n,-1}$	$K_{n,2,3,-6}$	$M_{n,-2}$	$N_{n,3,-6}$	$M_{n,-3}$	$N_{n,2,-6}$	$M_{n,-6}$	$N_{n,2,-3}$

Table 4.4: Representative of matrices satisfying δ and Hasse symbol value in $\mathbb{Q}_2$

δ	τ	D	A	$t \geq 3$	$t=2$	$t=1$	$t=0$
1	1	I	$\begin{pmatrix} I_s & 0 \\ 0 & I_t \end{pmatrix}$	Y	Y	Y	Y
1	-1	K_{t,p,N_p,pN_p}	$\begin{pmatrix} I_{s-3} & 0 & 0 & 0 & 0 \\ 0 & K_p & 0 & 0 & 0 \\ 0 & 0 & K_{N_p} & 0 & 0 \\ 0 & 0 & 0 & K_{pN_p} & 0 \\ 0 & 0 & 0 & 0 & I_{t-3} \end{pmatrix}$	Y	N	N	N
p	1	$M_{t,p}$	$\begin{pmatrix} I_{s-1} & 0 & 0 \\ 0 & K_p & 0 \\ 0 & 0 & I_{t-1} \end{pmatrix}$	Y	Y	Y	N
p	-1	N_{t,N_p,pN_p}	$\begin{pmatrix} I_{s-2} & 0 & 0 & 0 \\ 0 & K_{N_p} & 0 & 0 \\ 0 & 0 & K_{pN_p} & 0 \\ 0 & 0 & 0 & I_{t-2} \end{pmatrix}$	Y	Y	N	N
N_p	1	M_{tN_p}	$\begin{pmatrix} I_{s-1} & 0 & 0 \\ 0 & K_{N_p} & 0 \\ 0 & 0 & I_{t-1} \end{pmatrix}$	Y	Y	Y	N
N_p	-1	N_{t,p,pN_p}	$\begin{pmatrix} I_{s-2} & 0 & 0 & 0 \\ 0 & K_p & 0 & 0 \\ 0 & 0 & K_{pN_p} & 0 \\ 0 & 0 & 0 & I_{t-2} \end{pmatrix}$	Y	Y	N	N
pN_p	1	M_{t,pN_p}	$\begin{pmatrix} I_{s-1} & 0 & 0 \\ 0 & K_{pN_p} & 0 \\ 0 & 0 & I_{t-1} \end{pmatrix}$	Y	Y	Y	N
pN_p	-1	N_{t,p,N_p}	$\begin{pmatrix} I_{s-2} & 0 & 0 & 0 \\ 0 & K_p & 0 & 0 \\ 0 & 0 & K_{N_p} & 0 \\ 0 & 0 & 0 & I_{t-2} \end{pmatrix}$	Y	Y	N	N
		In total		8	7	4	1

Table 4.5: Representatives of involutions up to conjugacy for SO$(2n+1,k)$, $(-1 \in \mathbb{Q}_p^2)$

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