

ABSTRACT

BANCROFT, ERIC DANIEL. Products of Triangular Idempotents and Units. (Under the direction of Mohan Putcha.)

We explore when elements of a monoid can be written as a product of units and idempotents. Specifically, we are interested in whether or not triangular elements can be decomposed in this manner. We first look at decompositions of nilpotent and partial permutation matrices in $T_n(k) = \{n \times n \text{ upper triangular matrices over the field } k\}$ into a product of idempotents in $T_n(k)$. It follows that every element of $T_n(k)$ can be expressed as a product of idempotents and a unit in $T_n(k)$. We then find necessary and sufficient conditions for a matrix in $T_n(k)$ to be decomposed into product of idempotents. The proof of this theorem gives us an algorithm for performing such a decomposition when it exists.

An implementation of this algorithm in Maple is given, along with a bound on the number of idempotents needed in the implementation.

Finally, we look at triangular elements of a ring with respect to orthogonal idempotents and necessary and sufficient conditions for them to be written as a product of units and idempotents.

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Products of Triangular Idempotents and Units

by
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DEDICATION

To my wife, Erin.

BIOGRAPHY

Eric Daniel Bancroft was born in Massachusetts in the early 1980's to Daniel and Cheryl Bancroft and grew up in central Florida. He was home-schooled from second grade until he graduated from high school. After high school Eric attended Seminole Community College, where he initially pursued an interest in automotive service. After a year and a half, he realized that working on cars was not the best career choice for him and turned his attention to mathematics. Eric finished his associate's degree at SCC and then transferred to the University of Central Florida to complete his bachelor's degree in mathematics. In 2005, Eric moved to Raleigh for graduate school in mathematics at North Carolina State University. He received his master's degree in 2007 and completed his Ph.D. in 2011. Along the way he met the love of his life, and married her in 2008. After completing their Ph.D.'s, Eric and his wife will be moving to northwest Pennsylvania to start a family and pursue careers in academia.

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Chapter 1

Motivation and Background

1.1 Semigroups

Definition 1.1.1. A *semigroup* is a non-empty set S with an associative binary operation [2].

Definition 1.1.2. $e \in S$ is called an *idempotent* if $e^2 = e$. We denote the set of idempotents of S by $E(S)$.

Definition 1.1.3. A semigroup M is a *monoid* if there is an identity element $1 \in M$.

Definition 1.1.4. Given a semigroup S , a subset $T \subseteq S$ which is also a semigroup is called a *subsemigroup*. If $T \subseteq S$ is also a group then it is called a *subgroup*.

Definition 1.1.5. An invertible element of a monoid M is called a *unit*. The group of units of M is denoted by G .

Theorem 1.1.6. If e is an idempotent and b is a unit in M , then $e' = beb^{-1}$ is also an idempotent:

$$(e')^2 = beb^{-1}beb^{-1} = beeb^{-1} = beb^{-1} = e'$$

Corollary 1.1.7. Suppose e_1, e_2, \dots, e_n are idempotents and b is a unit in M . Let $f_i =$

$be_i b^{-1}$, $g_i = b^{-1}e_i b$. Then f_i and g_i are idempotents and

$$e_1 \dots e_k b e_{k+1} \dots e_n = b g_1 \dots g_k e_{k+1} \dots e_n$$

and

$$e_1 \dots e_k b e_{k+1} \dots e_n = e_1 \dots e_k f_{k+1} \dots f_n b^{-1}$$

If $1 \notin S$, then we can simply add an identity element to form a monoid:

$$S^1 = \{1\} \cup S.$$

Example 1.1.8. Some natural examples of semigroups are:

1. The semigroup \mathcal{T}_n of all maps from $\{1, \dots, n\}$ to itself.
2. The multiplicative semigroup $M_n(k)$ of all $n \times n$ matrices over a field \mathbb{F} .
3. If k is an algebraically closed field and $G \subseteq Gl_n(k)$ is a reductive group, then the Zariski closure $M = \overline{G}$ of G in $M_n(k)$ is called a *reductive monoid*.

Definition 1.1.9. S is said to be *strongly π -regular* (written $s\pi r$) if for each $a \in S$, there exists a positive integer i such that a^i lies in a subgroup of S .

Definition 1.1.10. An element $s \in S$ is said to be *regular* if there exists $x \in S$ such that $a = axa$ and $x = xax$. S is called a *regular semigroup* if every element of S is regular.

All of the semigroups in Example 1.1.8 are regular semigroups.

Definition 1.1.11. For $a, b \in S$, $a|b$ (*a divides b*) if $b = xay$ for some $x, y \in S^1$.

We now turn our attention to inverse semigroups.

Definition 1.1.12. S is said to be an *inverse semigroup* if for all $a \in S$ there exists a unique $a^{-1} \in S$ such that $a^{-1}aa^{-1} = a^{-1}$ and $aa^{-1}a = a$.

From [2] we have the following theorem:

Theorem 1.1.13. *The following are three conditions on a semigroup S are equivalent:*

1. S is an inverse semigroup.
2. S is regular, and any two idempotents elements of S commute with each other.
3. Every principal right ideal and every principal left ideal of S has a unique idempotent generator.

Definition 1.1.14. Let \mathcal{J}_X denote the set of all one-to-one partial transformations on a set X . \mathcal{J}_X is an inverse semigroup, called the *symmetric inverse semigroup* (see [2], [15]).

Example 1.1.15. Within $M_n(k)$, the *rook monoid* R consisting of all partial permutation matrices is the symmetric inverse semigroup.

Suppose a monoid M with unit group G is generated by G and $E(M)$. Then $M = G\langle E(M) \rangle = \langle E(M) \rangle G$ (by Corollary 1.1.7) and G acts on $E(M)$ as ${}^g a = gag^{-1}$, where $g \in G, a \in \langle E(M) \rangle$. Hence $E(M) \times G$ becomes a semidirect product of $E(M)$ and G :

$$(a, g)(a', g') = (a^g a', gg')$$

and M is a homomorphic image of this semidirect product via the map $(a, g) \rightarrow ag$.

1.1.1 Connections to Geometry

Definition 1.1.16. G is an *algebraic group* if

1. G is a group,
2. G is an affine variety,
3. and the two maps $\mu : G \times G \rightarrow G$ and $\iota : G \rightarrow G$, given by $\mu(x, y) = xy$ and $\iota(x) = x^{-1}$, are morphisms of varieties.

The unique irreducible component of G containing 1 is denoted by G^c . G is *connected* if $G^c = G$

Definition 1.1.17. Let G be a connected group. $a \in G$ is *unipotent* if the only eigenvalue of a is 1. G is *unipotent* if every element of G unipotent.

Definition 1.1.18. Let G be a connected group. G is *reductive* if it has no non-trivial normal unipotent subgroups.

The idempotents of a semigroup often represent a connection to geometry. For a maximal torus T of G , $E(\overline{T})$ is isomorphic to the face lattice of a polytope (see [12]), with partial order $e \leq f$ when $e = ef = fe$ for $e, f \in E(\overline{T})$. For a reductive monoid M , $E(M)$ is closely related to the Tits building of G by [14].

Definition 1.1.19. Let S be a semigroup such that $S = \bigsqcup_{\alpha \in \Omega} S_\alpha$ with S_α 's being subsemigroups of S and for all $\alpha, \beta \in \Omega$, $S_\alpha S_\beta \cup S_\beta S_\alpha \subseteq S_\gamma$ for some $\gamma \in \Omega$. Then S has the structure of a *semilattice [union]* of semigroups S_α ($\alpha \in \Omega$).

Example 1.1.20. \overline{T} (the closure of a torus) is a semilattice of $e_i T$'s, where $e_i \in E(\overline{T})$.

Definition 1.1.21. Let S be a semigroup. $a \in S$ is said to be *completely regular* if a lies in some subgroup of S . S is completely regular if every element of S is completely regular. Equivalently, S is completely regular if it is a union of its subgroups.

Definition 1.1.22. A *completely simple semigroup* S is a $\pi\tau$ -semigroup with no ideals other than S .

Theorem 1.1.23. *A semigroup S is completely regular if and only if it is a semilattice of completely simple semigroups. (This concept is due to Clifford in [1].)*

Definition 1.1.24. A semigroup S is said to be *archimedean* if $a|b^i$ for some positive integer i and all $a, b \in S$.

Tamura and Kimura in [20] show that any commutative semigroup is a semilattice of archimedean semigroups S_α , $\alpha \in \Omega$.

General semilattices of archimedean semigroups were characterized by Putcha in [12]:

Theorem 1.1.25. *A semigroup S is a semilattice of archimedean semigroups if and only if for all $a, b \in S_\alpha$ there exists a positive integer i such that*

$$a|b \implies a^2|b^i.$$

It is shown in [13] that for a connected solvable subgroup $B \subseteq GL_n(k)$ the closure \overline{B} in $M_n(k)$ is a semilattice of archimedean semigroups. In particular, the monoid $T_n(k)$ of all upper triangular matrices over a field k is a semilattice of achimedean semigroups.

1.2 Question

How do we study idempotents? One way to study idempotents is to study the semigroup $\langle E(S) \rangle$ generated by $E(S)$. In [5], we see that the singular matrices of $M_n(k)$ are the product of idempotents. An analogous result was proved by J. Howie in [6]. The semigroup generated by idempotents is a reductive monoid and is studied in [18].

However, singular matrices in $T_n(k)$ are not necessarily products of idempotents in $T_n(k)$. So a natural question is whether $T_n(k)$ is generated by idempotent matrices and units in $T_n(k)$. For $T_n(\mathbb{F}_q)$, where \mathbb{F}_q is a finite field, this was shown by Putcha in [17]. This is proved in the more general context of triangular monoids within a *finite* reductive monoid using monoid Hecke algebras, not available in the infinite case. Our first question then is whether or not we can extend these results to an infinite field.

1.3 The Renner Decomposition

L.E. Renner in [19] develops a decomposition for reductive monoids which is analogous to the Bruhat decomposition for reductive groups. Renner's decomposition is an important tool for studying $\langle E(S) \rangle$.

Given a reductive group G , a maximal torus $T \subseteq G$, a Borel subgroup $B \subseteq G$ (i.e., a

maximal, connected, solvable subgroup) such that $T \subseteq B$, and Weyl group $W = N_G(T)/T$, then the Bruhat decomposition of G is

$$G = \bigsqcup_{w \in W} BwB$$

Example 1.3.1. In $GL_n(k)$, the Bruhat decomposition says that given $G = GL_n(k)$, the Borel subgroup B of upper triangular invertible matrices, and the Weyl group W of permutation matrices, then

$$G = \bigsqcup_{w \in W} BwB$$

So, we can express any element of G as the product of an upper triangular unit, a permutation matrix, and an upper triangular unit.

Within a reductive monoid, $M = \overline{G}$, the Renner monoid $R = \overline{N_G(T)}/T$ is an inverse monoid. Renner's decomposition extends to the Bruhat decomposition of G to M as

$$M = \bigsqcup_{\sigma \in R} B\sigma B$$

Example 1.3.2. In $M_n(k)$, Renner's decomposition allows us to take $M = M_n(k)$, $R = \{\text{partial permutation matrices}\}$, the Borel subgroup B of upper triangular invertible matrices, and then decompose M as

$$M = \bigsqcup_{r \in R} BrB$$

This allows us to express elements of M as the product of an upper triangular unit, a partial permutation matrix, and an upper triangular unit.

Example 1.3.3. If we restrict to upper triangular matrices, Renner's decomposition says that given $\overline{B} = T_n(k)$ and $R^+ = \{\text{upper triangular partial permutation matrices}\}$, then

$$\overline{B} = \bigsqcup_{r \in R^+} BrB$$

So, every upper triangular matrix can be written as the product of a unit, a partial permutation matrix, and a unit in $T_n(k)$.

We will later use this to show that every upper triangular matrix can be written as the product of a unit and idempotents in $T_n(k)$.

1.4 Notation

Trying to express how the nilpotent matrices could be written as a product of idempotents in general was rather tedious, and similar issues came up when looking at the partial permutation matrices. Putcha suggested a new notation to simplify the process. We need the following definition before we can define the notation:

Definition 1.4.1. Let $[n] = \{1, 2, \dots, n\}$. Then define an equivalence relation \sim with equivalence classes

$$X_1 = \{i_1^1 < \dots < i_{n_1}^1\}, X_2 = \{i_1^2 < \dots < i_{n_2}^2\}, \dots, X_t = \{i_1^t < \dots < i_{n_t}^t\},$$

where $\bigsqcup_{j=1}^t X_j = [n]$ (disjoint union), so $X_k \cap X_l = \emptyset$ for $k \neq l$, and $n_1 + n_2 + \dots + n_t = n$.

Note: For an element i_p^r , the upper index r indicates the equivalence class, and the lower index p indicates the index within the equivalence class. When using this notation to index the elements of a matrix A , we will denote the entry $(A)_{i_p^r, i_q^r}$ by $a_{p,q}^r$.

Example 1.4.2. For example, the $n_1 \times n_1$ matrix indexed by X_1 would be denoted as

$$A_{X_1} = \begin{bmatrix} a_{i_1^1, i_1^1} & a_{i_1^1, i_2^1} & \dots & a_{i_1^1, i_{n_1}^1} \\ a_{i_2^1, i_1^1} & a_{i_2^1, i_2^1} & \dots & a_{i_2^1, i_{n_1}^1} \\ \vdots & & \ddots & \vdots \\ a_{i_{n_1}^1, i_1^1} & a_{i_{n_1}^1, i_2^1} & \dots & a_{i_{n_1}^1, i_{n_1}^1} \end{bmatrix}$$

Definition 1.4.3 (The “window sum”). Let $A_{X_1}, A_{X_2}, \dots, A_{X_t}$ be $n_1 \times n_1, n_2 \times n_2, \dots, n_t \times n_t$ matrices, respectively, whose entries are indexed by elements of X_1, X_2, \dots, X_t . Then we can define

$$A = (A_{X_1} \boxplus A_{X_2} \boxplus \dots \boxplus A_{X_t})_{\sim},$$

where

$$(A)_{kl} = \begin{cases} (A_{X_r})_{kl}, & \text{if } k = i_p^r, l = i_q^r, 1 \leq r \leq t \\ 0, & \text{if } k = i_p^r, l = i_q^s, r \neq s \end{cases}$$

or

$$(A)_{i_p^r, i_q^s} = \begin{cases} (A_{X_r})_{pq}, & \text{if } r = s \\ 0, & \text{if } r \neq s \end{cases}$$

Example 1.4.4. Let

$$A_{X_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix}_{\{1,2,4\}}$$

and

$$A_{X_2} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}_{\{3,5\}}$$

Find:

$$A = A_{X_1} \boxplus A_{X_2}.$$

First we fill in the entries from the first matrix...

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix}$$

..then we fill in the entries from the second matrix...

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ & 1 & 2 \\ 3 & 1 & 0 \\ & 0 & 1 \end{bmatrix}$$

...and finally we fill in the rest of the matrix with zeros:

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 1.4.5. *If $A_{X_1}, A_{X_2}, \dots, A_{X_t}$ are upper-triangular (so $a_{i_p, q}^r = 0$ if $p > q$), then so is $A = (A_{X_1} \boxplus A_{X_2} \boxplus \dots \boxplus A_{X_t})_{\sim}$.*

Proof. Let $(A)_{kl} = a_{kl}$ be such that $k > l$. We want to show that $a_{kl} = 0$: If $k = i_p^r, l = i_q^s$, where $r \neq s$, then $a_{kl} = 0$ by definition. If $k = i_p^r, l = i_q^r$, then $k > l \Rightarrow i_p^r > i_q^r \Rightarrow p > q \Rightarrow a_{i_p, q}^r = a_{kl} = 0$. Thus, A is upper-triangular. **Q.E.D.**

Example 1.4.6. \boxplus -ing the upper triangular matrices

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{\{1,2,4,6\}} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_{\{3,5\}}$$

yields the upper triangular matrix

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{\{1,2,4,6\}} \boxplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_{\{3,5\}} = \begin{bmatrix} 1 & & 3 & 4 \\ 0 & 1 & & 2 \\ & & 1 & 1 \\ 0 & & 0 & 1 \\ 0 & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 1.4.7. Let $A = (A_{X_1} \boxplus A_{X_2} \boxplus \dots \boxplus A_{X_t})_{\sim}$ and $B = (B_{X_1} \boxplus B_{X_2} \boxplus \dots \boxplus B_{X_t})_{\sim}$.

Then

$$AB = (A_{X_1}B_{X_1} \boxplus A_{X_2}B_{X_2} \boxplus \dots \boxplus A_{X_t}B_{X_t})_{\sim}.$$

Proof. The kl^{th} entry of the left-hand side of the equation is going to be the dot product of the k^{th} row of A with the l^{th} column of B , i.e.,

$$\begin{pmatrix} a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix} \begin{pmatrix} b_{1l} \\ b_{2l} \\ \vdots \\ b_{nl} \end{pmatrix},$$

where $k = i_p^r, l = i_q^s$ for some r, s , and $a_{ku} = 0$ if $u \notin X_r, b_{vl} = 0$ if $v \notin X_s$ (by definition of $(\cdot)_{\sim}$ above). So then

$$\begin{pmatrix} a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix} \begin{pmatrix} b_{1l} \\ b_{2l} \\ \vdots \\ b_{nl} \end{pmatrix} = \sum_{j=1}^n a_{kj} b_{jl} = 0$$

unless $\{k, j, l\} \subseteq X_r$ for some r (i.e., $r = s$), in which case we would have

$$\sum_{j=1}^n a_{kj} b_{jl} = \sum_{j=1}^{n_r} a_{i_p^r, j} b_{i_q^r, j} \left(= \sum_{j=1}^{n_r} a_{k i_j^r} b_{i_j^r l} \right).$$

The kl^{th} entry of the right-hand side is going to be 0 if $k = i_p^r, l = i_q^r, r \neq s$ (again from

the definition of $(\)_{\sim}$. If $r = s$, then it comes from the $kl^{th} = i_{p,q}^r$ entry of $A_{X_r}B_{X_r}$, which is the k^{th} row of A_{X_r} dotted with the l^{th} column of B_{X_r} , i.e.,

$$\begin{pmatrix} a_{i_{k,1}^r} & a_{i_{k,2}^r} & \dots & a_{i_{k,n_r}^r} \end{pmatrix} \begin{pmatrix} b_{i_{1,l}^r} \\ b_{i_{2,l}^r} \\ \vdots \\ b_{i_{n_r,l}^r} \end{pmatrix} = \sum_{j=1}^{n_r} a_{i_{k,j}^r} b_{j,l}^r$$

But this is precisely what we had on the left-hand side.

Therefore

$$AB = (A_{X_1}B_{X_1} \boxplus A_{X_2}B_{X_2} \boxplus \dots \boxplus A_{X_t}B_{X_t})_{\sim}.$$

Q.E.D.

Remark 1.4.8. In the preceding theorem, the equivalence classes X_1, X_2, \dots, X_t are the same in both A and B . If this is not the case, then we cannot use this theorem to compute AB .

Example 1.4.9. Consider the two 5×5 matrices

$$\begin{bmatrix} 3 & 0 & -1 & 0 & 3 \\ 0 & 1 & 0 & -2 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 3 \\ 2 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}_{\{1,3,5\}} \boxplus \begin{bmatrix} 1 & -2 \\ -3 & 0 \end{bmatrix}_{\{2,4\}}$$

and

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -2 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -2 & 0 & -1 \\ 1 & 3 & -2 \end{bmatrix}_{\{1,3,5\}} \boxplus \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}_{\{2,4\}}$$

If we multiply in the usual way, we get:

$$\begin{bmatrix} -1 & 0 & 9 & 0 & -5 \\ 0 & -3 & 0 & -3 & 0 \\ -6 & 0 & 0 & 0 & -1 \\ 0 & -3 & 0 & 3 & 0 \\ 3 & 0 & -3 & 0 & 4 \end{bmatrix}$$

If we multiply before \boxplus -ing, then we get:

$$\begin{bmatrix} -1 & 9 & -5 \\ -6 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix}_{\{1,3,5\}} \boxplus \begin{bmatrix} -3 & -3 \\ -3 & 3 \end{bmatrix}_{\{2,4\}} = \begin{bmatrix} -1 & 0 & 9 & 0 & -5 \\ 0 & -3 & 0 & -3 & 0 \\ -6 & 0 & 0 & 0 & -1 \\ 0 & -3 & 0 & 3 & 0 \\ 3 & 0 & -3 & 0 & 4 \end{bmatrix}$$

Corollary 1.4.10. *If $A_{X_1}, A_{X_2}, \dots, A_{X_t}$ are idempotent, then $(A_{X_1} \boxplus A_{X_2} \boxplus \dots \boxplus A_{X_t})_{\sim}$ is also idempotent.*

Proof.

$$\begin{aligned} (A_{X_1} \boxplus A_{X_2} \boxplus \dots \boxplus A_{X_t})_{\sim}^2 &= (A_{X_1} \boxplus A_{X_2} \boxplus \dots \boxplus A_{X_t})_{\sim} (A_{X_1} \boxplus A_{X_2} \boxplus \dots \boxplus A_{X_t})_{\sim} \\ &= (A_{X_1}^2 \boxplus A_{X_2}^2 \boxplus \dots \boxplus A_{X_t}^2)_{\sim} \quad (\text{by the previous theorem}) \\ &= (A_{X_1} \boxplus A_{X_2} \boxplus \dots \boxplus A_{X_t})_{\sim} \end{aligned}$$

Q.E.D.

When the dimension of the matrices with which we are working is clear from the context, we may omit any identity matrices from the \boxplus -sum in order to simplify the notation:

$$A_{X_1} \boxplus \dots \boxplus A_{X_i} \boxplus I_{X_{i+1}} = A_{X_1} \boxplus \dots \boxplus A_{X_i}$$

We may also write the equivalence classes underneath the matrices to make the notation more

compact.

Example 1.4.11.

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}_{\{4,9\}} \boxplus_{\mathbb{I}_{[9] \setminus \{4,9\}}} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}_{\{4,9\}}$$

Chapter 2

Nilpotent and Partial Permutation Matrices

2.1 Nilpotent Matrices in $T_n(k)$

Having developed some background and notation, we now turn to the problem of decomposing a nilpotent upper triangular matrix into a product of upper triangular idempotent matrices.

We begin with two examples:

Example 2.1.1. The matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ can be written as a product of idempotents in

$T_4(\mathbb{R})$ as follows:

$$\begin{aligned}
& \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
& = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

This example begins to develop the method which we will use in general to compose an upper triangular nilpotent matrix into a product of idempotents. Reading the product from right to left and viewing each multiplication as acting on the rows of the first matrix, we begin with the idempotent matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and multiply it on the left by the idempotent matrix which moves the second row in the original matrix up into the first row (leaving zeros in the second row):

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The next multiplication moves the original third row to the second row and leaves zeros in the third row:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The final multiplication moves the original fourth row to the third row, leaving zeros in the fourth row and giving us the desired matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2.1.2. The nilpotent matrix

$$A = \begin{bmatrix} 0 & 3 & 2 & -5 & 8 \\ 0 & 0 & -1 & 2 & 6 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

can be written as the following product of idempotents in $T_5(\mathbb{R})$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

or, using the \boxplus notation, as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{[5]} \cdot \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{[4]} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}_{[3]} \cdot \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}_{[2]} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{[1]}$$

This example illustrates the general procedure for decomposing a nilpotent matrix into a product of idempotents. We begin with the matrix obtained by taking an identity matrix and

changing the $(1, 1)$ entry to 0 (so the first column of this matrix is the first column of A). To form the second column of A , we multiply on the left by the matrix which adds 3 times the second row to the first row (since the only non-zero element in the second row was the 1 in the $(2, 2)$ position, only the second column is changed by this action) and multiplies the second row by zero:

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We then multiply the result by the matrix which adds 2 times the third row and -1 times the third row to the first and second rows, respectively, and then multiplies the third row by zero. Because the only non-zero element of the third row was the 1 in the $(2, 2)$ position, this creates the second column of A without changing any other column:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The fourth and fifth rows are created in a similar manner:

$$\begin{bmatrix} 1 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 & -5 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 & 2 & -5 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 & -5 & 8 \\ 0 & 0 & -1 & 2 & 6 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The method we used in the preceding example is generalized in the proof of the following theorem:

Theorem 2.1.3. *Every nilpotent matrix $N \in T_n(k)$ can be written as a product of upper triangular idempotents.*

Proof. We start with the idempotent $n \times n$ matrix with 1's on diagonals two through n and zeros elsewhere:

$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ [1] \end{bmatrix}$$

We then multiply on the left by the idempotent matrix

$$\begin{bmatrix} 1 & a_{12} & 0 & \dots & 0 \\ 0 & 0 & & 0 & \vdots \\ \vdots & & 1 & & \\ & 0 & & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} \\ 0 & 0 \\ [2] \end{bmatrix}$$

to create the second column. Next we multiply by the idempotent matrix

$$\begin{bmatrix} 1 & 0 & a_{13} & 0 & \dots & 0 \\ 0 & 1 & a_{23} & 0 & & \vdots \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ \vdots & & & & \ddots & 0 \\ 0 & \dots & & 0 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}_{[3]}$$

to create the third column. We continue in this manner to create the remaining columns and complete our matrix. This yields the following decomposition into a product of upper triangular idempotents:

$$\begin{bmatrix} 0 & a_{1,2} & \dots & a_{1,n} \\ \vdots & \ddots & \ddots & \vdots \\ & & 0 & a_{n-1,n} \\ 0 & \dots & 0 & 0 \end{bmatrix} = \left(\prod_{j=0}^{n-2} \begin{bmatrix} 1 & 0 & \dots & 0 & a_{1,n-j} \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \\ & & & 1 & a_{n-j-1,n-j} \\ 0 & \dots & & 0 & 0 \end{bmatrix}_{[n-j]} \right) \cdot \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}_{[1]}$$

Q.E.D.

2.2 Partial Permutation Matrices in $T_n(k)$

Definition 2.2.1. A *partial permutation matrix* is a permutation matrix with some of the non-zero entries changed to zero.

Corollary 2.2.2. *Every partial permutation matrix in $T_n(k)$ can be written as a product of upper triangular idempotents.*

Proof. Given a partial permutation matrix $P \in T_n(k)$ we can decompose P into its nilpotent and fixed parts, writing it as $N_1 \boxplus \dots \boxplus N_m \boxplus I$ where each $N_i \in T_n(k)$ is nilpotent and I is

an identity matrix. By Theorem 2.1.3, each N_i can be written as a product of idempotents in $T_n(k)$, and thus P can be expressed as a product of upper triangular idempotents. **Q.E.D.**

Example 2.2.3. Consider the partial permutation matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the \boxplus -notation, we can write this matrix as

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{\{1,3\}} \boxplus \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{\{2,5,6\}} \boxplus I_{\{4\}}$$

where the first two matrices are nilpotent. Using Theorem 2.1.3 we can now write:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{\{1,3\}} \boxplus \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{\{2,5,6\}} \boxplus I_{\{4\}} = \\ \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_{\{1,3\}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \boxplus \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{\{2,5,6\}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \boxplus I_{\{4\}} \quad (1)$$

where the matrices on the right hand side are all idempotent.

To split this into a product of idempotent matrices, we must have the same number of matrices corresponding to each equivalence class. Since the second term in the \boxplus -sum has the greatest number of matrices (three), we need to write the other two terms as a product of three matrices. This can be accomplished in a number of ways. One way is to multiply by identity

matrices as needed:

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)_{\{1,3\}} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \mathbf{I} \right)_{\{1,3\}}$$

(the identity matrix may go in front or in-between as well) and

$$\mathbf{I}_{\{4\}} = (\mathbf{I} \cdot \mathbf{I} \cdot \mathbf{I})_{\{4\}}$$

We may also obtain the other matrices needed by inserting additional copies of any idempotent matrices. In this example, we can write

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right)_{\{1,3\}} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)_{\{1,3\}}$$

or

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 \right)_{\{1,3\}} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^2 \right)_{\{1,3\}}$$

So, one of several possible ways to write (1) would be

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)_{\{1,3\}} \boxplus \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)_{\{2,5,6\}} \boxplus (\mathbf{I} \cdot \mathbf{I} \cdot \mathbf{I})_{\{4\}}$$

Using Theorem 1.4.7 we then write as three separate idempotent matrices:

$$\begin{aligned}
& \left(\begin{array}{c} \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right]_{\{1,3\}} \boxplus \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right]_{\{2,5,6\}} \boxplus I_{\{4\}} \end{array} \right) \cdot \left(\begin{array}{c} \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right]_{\{1,3\}} \boxplus \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]_{\{2,5,6\}} \boxplus I_{\{4\}} \end{array} \right) \cdot \left(\begin{array}{c} \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right]_{\{1,3\}} \boxplus \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]_{\{2,5,6\}} \boxplus I_{\{4\}} \end{array} \right) \\
& = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Corollary 2.2.4. *Every matrix in $T_n(k)$ can be written as a product of upper triangular idempotents and a unit.*

Proof. Given $A \in T_n(k)$, by Renner's decomposition we can write A as a product of units and partial permutation matrices in $T_n(k)$. By Corollary 2.2.2, all of the partial permutation matrices can be written as a product of idempotents in $T_n(k)$. Thus we can write A as the product of units and idempotents. Finally, we use Corollary 1.1.7 to move the units to one side of the product and multiply them together to obtain a single unit. Q.E.D.

i.e., If we have consecutive 1's on the diagonal, then they must be part of an identity matrix block; other elements above the diagonal can be whatever we like.

Lemma 3.1.2. *All idempotents in $T_n(k)$ fit the form of the theorem.*

Proof. (By contradiction.) Suppose there exists an idempotent that does not fit the form of the theorem. This can happen in one of two ways:

1. The matrix has an element other than 0 or 1 on the diagonal, i.e.,

$$A = \begin{bmatrix} * & * & * \\ 0 & a & * \\ 0 & 0 & * \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} * & * & * \\ 0 & a^2 & * \\ 0 & 0 & * \end{bmatrix} \neq A$$

which is a contradiction. Thus, A must have only 0 or 1 on the diagonal.

2. There are consecutive 1's on the diagonal that are not part of an identity block, i.e.,

$$A = \begin{bmatrix} * & * & * \\ 0 & B & * \\ 0 & 0 & * \end{bmatrix}$$

where $B = I + U$ and $U \neq 0$ is strictly upper triangular. Then

$$A^2 = \left[\begin{array}{c|c|c} * & * & * \\ \hline 0 & B^2 & * \\ \hline 0 & 0 & * \end{array} \right]$$

so if B were idempotent then $B^2 = B \Rightarrow (I + U)^2 = I + U \Rightarrow I^2 + 2U + U^2 = I + U \Rightarrow U + U^2 = 0 \Rightarrow U(U + I) = 0 \Rightarrow U = 0$ or I , which is a contradiction since U is non-zero and strictly upper triangular. Thus any consecutive 1s on the diagonal must be part of an identity block.

Thus all idempotents in $T_n(k)$ must fit the form of the theorem.

Q.E.D.

Lemma 3.1.3. *The set of matrices that fit the form of the theorem is closed under matrix multiplication.*

Proof. Let A, B be arbitrary matrices of this form, and suppose we multiply them together in the order AB . We view both as block matrices, partitioning into blocks in such a way that the diagonal blocks of A are either 1×1 0-matrices or identity matrices containing adjacent/consecutive 1's, and made as large as possible. B is then partitioned into blocks based on the divisions of A , so that the block containing b_{ij} has the same dimensions and position as the block containing a_{ij} . Once we have partitioned B into blocks, the blocks on the diagonal of B

will also satisfy the terms of the conjecture. Now when we multiply the two together, we get

$$AB = \begin{bmatrix} 0 & * & * & * & * \\ 0 & I & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} B_{11} & * & * & * & * \\ 0 & B_{22} & * & * & * \\ 0 & 0 & B_{33} & * & * \\ 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 0 & B_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & * & * & * & * \\ 0 & B_{22} & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 0 & B_{nn} \end{bmatrix}$$

So, on the diagonal we either have 0's or block matrices that fit the form of the theorem. Thus, AB fits the form, so the set is closed under multiplication. Q.E.D.

Example 3.1.4. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & 6 & 9 & 9 & 0 \\ 0 & 1 & 3 & 4 & 7 & 0 & 8 & 1 \\ 0 & 0 & 0 & 5 & 8 & 1 & 7 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 & 9 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that both of these matrices fit the form of the theorem. We divide both into blocks as described above:

$$A = \begin{array}{c} \left[\begin{array}{ccc|cc|cc|c} 1 & 0 & 2 & 3 & 6 & 9 & 9 & 0 \\ 0 & 1 & 3 & 4 & 7 & 0 & 8 & 1 \\ \hline 0 & 0 & 0 & 5 & 8 & 1 & 7 & 2 \\ \hline 0 & 0 & 0 & 1 & 0 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 5 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \end{array}$$

$$B = \begin{array}{c} \left[\begin{array}{ccc|cc|cc|c} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 4 & 5 & 6 & 7 \\ \hline 0 & 0 & 0 & 1 & 8 & 9 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

Multiplying together we get

$$AB = \left[\begin{array}{cc|cc|cc|cc} 0 & 1 & 4 & 6 & 36 & 54 & 90 & 165 \\ 0 & 0 & 11 & 13 & 44 & 66 & 49 & 112 \\ \hline 0 & 0 & 0 & 5 & 40 & 61 & 36 & 92 \\ \hline 0 & 0 & 0 & 1 & 8 & 9 & 16 & 55 \\ 0 & 0 & 0 & 0 & 0 & 2 & 23 & 57 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 4 & 28 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

which also fits the form of the theorem.

Before proving the theorem, we introduce the following notation:

Definition 3.1.5. We use an arrow to denote a product written out right to left:

$$\Leftarrow \prod_{i=1}^n a_i = a_n \cdot a_{n-1} \cdot \dots \cdot a_2 \cdot a_1$$

Definition 3.1.6. For an integer $m \geq 1$ we define

$$\sigma(m) = \sum_{i=1}^m n_i$$

Proof of Theorem 3.1.1. If: This follows from Lemmas 3.1.2 and 3.1.3.

Only if: We show that a matrix that fits the form of the conjecture can be written as a product of idempotents in by using the following construction. All multiplication is left multiplication.

1. If the matrix starts with an identity block ($n_1 = 0$), then proceed to the next step.

If there are zeros on the diagonal ($n_1 > 0$), then we start with the idempotent matrix

formed by an identity matrix whose first row has been replaced with the first row of A :

$$\begin{bmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & 0 & 1 \end{bmatrix}_{[n]}$$

To obtain the second row of A , we then multiply on the left by the idempotent matrix formed by taking an identity matrix and replacing its second row with the second row of A :

$$\begin{bmatrix} 0 & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & 0 & 1 \end{bmatrix}_{[n]\setminus[1]} \cdot \begin{bmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & 0 & 1 \end{bmatrix}_{[n]}$$

This left-multiplication leaves the all but the second row of the first matrix fixed. In the second row, we have

$$\begin{aligned} & a_{2,3} \cdot [0 \ 0 \ 1 \ 0 \ \dots \ 0] + a_{2,4} \cdot [0 \ 0 \ 0 \ 1 \ 0 \ \dots \ 0] + \dots + a_{2,n} \cdot [0 \ \dots \ 0 \ 1] = \\ & = [\text{the second row of } A] \end{aligned}$$

Thus we now have a matrix whose first two rows are the same as A :

$$\begin{bmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & \cdots & a_{1,n} \\ 0 & 0 & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & \vdots \\ & & 0 & & 1 & 0 \\ 0 & & & & & 1 \end{bmatrix}$$

$[n]$

We repeating this process for the remaining $n_1 - 2$ rows, which yields the following product:

$$\Leftarrow \prod_{i=1}^{n_1} \begin{bmatrix} 0 & a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & 0 & 1 \end{bmatrix}$$

$[n] \setminus [i-1]$

The result is a matrix whose first n_1 rows are the first n_1 rows of A , with identity rows everywhere else.

2. To form the rows corresponding to the first identity block, we first create the $n_1 + 1$ through n_2 entries of the $(\sigma(2) + 1)^{st}$ column (the part of the column immediately to the right of the first identity block) by multiplying the result of the previous steps by the

idempotent matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & a_{n_1+1, \sigma(2)+1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & \vdots \\ & & & 1 & a_{\sigma(2), \sigma(2)+1} \\ 0 & & & 0 & 0 \end{bmatrix}_{\{n_1+1, \dots, \sigma(2)+1\}}$$

This multiplication leaves all but rows $n_1 + 1$ through $n_2 + 1$ fixed, adds $a_{i, \sigma(2)+1} \cdot e_{\sigma(2)+1}$ to row $n_1 + 1 \leq i \leq \sigma(2)$, and multiplies the $(\sigma(2) + 1)^{st}$ row by zero (we will use this zero row as a ‘work space’).

To form the remaining parts of rows $n_1 + 1$ through $\sigma(2)$, we start by multiplying by the idempotent matrix

$$\begin{bmatrix} 0 & a_{n_1+1, \sigma(2)+2} & \cdots & \cdots & a_{n_1+1, n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & 0 & 1 \end{bmatrix}_{[n] \setminus \{\sigma(2)\}}$$

This forms the remainder of row $n_1 + 1$ in the zero row left after the previous step.

We then move this part of row $n_1 + 1$ into place by multiplying by the idempotent matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_{\{n_1+1, \sigma(2)+1\}}$$

This matrix leaves all but rows $\sigma(2) + 1$ and $n_1 + 1$ fixed, adding row $\sigma(2) + 1$ to row $n_1 + 1$ to complete row $n_1 + 1$, and then zeros out row $\sigma(2) + 1$ (resetting our ‘work space’ for the next repetition).

We repeat this process for rows $n_1 + 2$ through $\sigma(2)$. This yields the following product:

$$\left\langle \prod_{i=n_1+1}^{\sigma(2)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle_{\{i, \sigma(2)+1\}} \begin{bmatrix} 0 & a_{i, \sigma(2)+2} & \cdots & \cdots & a_{i, n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ & & & 0 & 1 \end{bmatrix}_{[n] \setminus [\sigma(2)]} \begin{bmatrix} 1 & 0 & \cdots & 0 & a_{n_1+1, \sigma(2)+1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & \vdots \\ & & & 1 & a_{\sigma(2), \sigma(2)+1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{\{n_1+1, \dots, \sigma(2)+1\}}$$

The first $\sigma(2)$ rows of the resulting matrix are equal to the corresponding rows of A , the $(\sigma(2) + 1)^{st}$ row is zero, and remaining rows have ones on the diagonals and zeros everywhere else.

3. We continue this process, alternately using the process in step 1 for rows with zeros on the diagonal, and then the process in step 2 for rows corresponding to identity blocks.
4. When we get to $n_{(m)}$, the algorithm ends in one of two ways, depending on whether we end with zeros on the diagonal or an identity block.

If we end with zeros on the diagonal, then we still proceed as outlined in 2:

$$\left\langle \prod_{i=n_{(m-1)}+1}^n \begin{bmatrix} 0 & a_{i, i+1} & a_{i, i+2} & \cdots & a_{i, n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & 0 & 1 \end{bmatrix} \right\rangle_{[n] \setminus [i-1]}$$

where the last matrix we multiply by is

$$\begin{bmatrix} 0 \\ [n] \setminus [n-1] \end{bmatrix}$$

If we end with an identity block on the diagonal, then we stop after forming the previous n_{m-1} rows:

$$\begin{matrix} n_{(m-1)} \\ \Leftarrow \prod_{i=n_{(m-2)}+1} \end{matrix} \begin{bmatrix} 0 & a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & 0 & 1 \end{bmatrix} \begin{matrix} [n] \setminus [i-1] \end{matrix}$$

Thus, any matrix that fits the form of the theorem can be written as the following product of idempotents (**reading from right to left**):

$$\dots \cdot \begin{bmatrix} \begin{matrix} \sigma(1) \\ \Leftarrow \prod_{i=1} \end{matrix} \begin{bmatrix} 0 & a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & 0 & 1 \end{bmatrix} \\ [n] \setminus [i-1] \end{bmatrix}$$

$$\dots \left[\begin{array}{c} \left[\begin{array}{c} \left[\begin{array}{c} 1 & 1 \\ 0 & 0 \end{array} \right] \\ \left\{ i, \sigma(2)+1 \right\} \end{array} \right] \\ \leftarrow \prod_{i=\sigma(1)+1}^{\sigma(2)} \end{array} \right] \left[\begin{array}{c} \left[\begin{array}{cccc} 0 & a_{i, \sigma(2)+2} & \cdots & \cdots & a_{i, n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \end{array} \right] \\ [n] \setminus [\sigma(2)] \end{array} \right] \left[\begin{array}{c} \left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & a_{\sigma(1)+1, \sigma(2)+1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & \vdots \\ & & & 1 & a_{\sigma(2), \sigma(2)+1} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ \{ \sigma(1)+1, \dots, \sigma(2)+1 \} \end{array} \right] \end{array} \right]$$

$$\dots \left[\begin{array}{c} \left[\begin{array}{c} \left[\begin{array}{c} 0 & a_{i, i+1} & a_{i, i+2} & \cdots & a_{i, n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \end{array} \right] \\ [n] \setminus [i-1] \end{array} \right] \\ \leftarrow \prod_{i=\sigma(2)+1}^{\sigma(3)} \end{array} \right]$$

$$\dots \left[\begin{array}{c} \left[\begin{array}{c} \left[\begin{array}{c} 1 & 1 \\ 0 & 0 \end{array} \right] \\ \left\{ i, \sigma(4)+1 \right\} \end{array} \right] \\ \leftarrow \prod_{i=\sigma(3)+1}^{\sigma(4)} \end{array} \right] \left[\begin{array}{c} \left[\begin{array}{cccc} 0 & a_{i, \sigma(4)+2} & \cdots & \cdots & a_{i, n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \end{array} \right] \\ [n] \setminus [\sigma(3)] \end{array} \right] \left[\begin{array}{c} \left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & a_{\sigma(3)+1, \sigma(4)+1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & \vdots \\ & & & 1 & a_{\sigma(4), \sigma(4)+1} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ \{ \sigma(3)+1, \dots, n(4)+1 \} \end{array} \right] \end{array} \right]$$

$$\begin{aligned}
& \dots \left[\begin{array}{c} \left[\begin{array}{ccccc} 0 & a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & & 0 & 1 \end{array} \right] \\ \leftarrow \prod_{i=\sigma(j)+1}^{\sigma(j+1)} \\ [n] \setminus [i-1] \end{array} \right] \\
& \dots \left[\begin{array}{c} \left[\begin{array}{ccccc} 0 & a_{i,\sigma(j+2)+2} & \cdots & \cdots & a_{i,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & 0 & 1 \end{array} \right] \\ \leftarrow \prod_{i=\sigma(j+1)+1}^{\sigma(j+2)} \\ \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right] \\ \{i,\sigma(j+2)+1\} \\ [n] \setminus [\sigma(j+2)] \end{array} \right] \left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & a_{\sigma(j+1)+1,\sigma(j+2)+1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & \vdots \\ & & & 1 & a_{\sigma(j+2),\sigma(j+2)+1} \\ 0 & 0 & & 0 & 0 \end{array} \right] \\ \{\sigma(j+1)+1, \dots, n_{(j+2)+1}\} \end{array} \right] \\
& \left[\begin{array}{c} \left[\begin{array}{ccccc} 0 & a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & & 0 & 1 \end{array} \right] \\ \leftarrow \prod_{i=\sigma(m-1)+1}^n \\ [n] \setminus [i-1] \end{array} \right] \text{ or } \left[\begin{array}{c} \left[\begin{array}{ccccc} 0 & a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,n} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & & & 0 & 1 \end{array} \right] \\ \leftarrow \prod_{i=\sigma(m-2)+1}^{\sigma(m-1)} \\ [n] \setminus [i-1] \end{array} \right] \\
& \underbrace{\hspace{15em}}_{\text{if ends with consecutive zeros on diagonal}} \quad \underbrace{\hspace{15em}}_{\text{if ends with an identity block on diagonal}}
\end{aligned}$$

Q.E.D.

3.2 Examples

Example 3.2.1. We can write the matrix

$$\left[\begin{array}{cc|cccccc} 1 & 0 & 2 & 3 & 6 & 9 & 9 & 0 \\ 0 & 1 & 3 & 4 & 7 & 0 & 8 & 1 \\ \hline 0 & 0 & 0 & 5 & 8 & 1 & 7 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

as a product of idempotents as follows:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 6 & 9 & 9 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\dots \text{ form rest of first row, } \dots} \underbrace{\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Start with elts next to 1st identity block, } \dots} = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 6 & 9 & 9 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Form 6th row.}} \begin{bmatrix} 1 & 0 & 2 & 3 & 6 & 9 & 9 & 0 \\ 0 & 1 & 3 & 4 & 7 & 0 & 8 & 1 \\ 0 & 0 & 0 & 5 & 8 & 1 & 7 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 & 6 & 9 & 9 & 0 \\ 0 & 1 & 3 & 4 & 7 & 0 & 8 & 1 \\ 0 & 0 & 0 & 5 & 8 & 1 & 7 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Form 7th row (completes matrix).}} \begin{bmatrix} 1 & 0 & 2 & 3 & 6 & 9 & 9 & 0 \\ 0 & 1 & 3 & 4 & 7 & 0 & 8 & 1 \\ 0 & 0 & 0 & 5 & 8 & 1 & 7 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 & 6 & 9 & 9 & 0 \\ 0 & 1 & 3 & 4 & 7 & 0 & 8 & 1 \\ 0 & 0 & 0 & 5 & 8 & 1 & 7 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3.2.2. We can write the matrix

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 4 & 8 & 3 & 8 \\ 0 & 1 & 0 & 2 & 5 & 9 & 4 & 9 \\ 0 & 0 & 1 & 3 & 6 & 0 & 5 & 0 \\ \hline 0 & 0 & 0 & 0 & 7 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

as a product of idempotents as follows:

$$\underbrace{\begin{bmatrix} 0 & 4 & 8 & 3 & 8 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{[8]\setminus[3]} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{[4]} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 8 & 3 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

... form rest of 3rd row... Start with elts next to 1st identity block...

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\{1,4\}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 8 & 3 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 4 & 8 & 3 & 8 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

...and move into place.

$$\underbrace{\begin{bmatrix} 0 & 5 & 9 & 4 & 9 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{[8]\setminus[3]} \begin{bmatrix} 1 & 0 & 0 & 1 & 4 & 8 & 3 & 8 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 4 & 8 & 3 & 8 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 9 & 4 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Form rest of 2nd row...

Chapter 4

The Algorithm in Maple

4.1 Introduction

Given a matrix that fits the form of Theorem 3.1.1, writing out the decomposition into the product of idempotents as outlined in the proof is a tedious task. C. Mangum and I wanted to create a package for Maple that would check whether a matrix was of the form of the theorem and, if so, compute the decomposition for us (see [11]).

In this chapter we begin with descriptions of the Maple procedures used to implement the algorithm; examples using each procedure are included along with the code. We then find an upper bound on the number of idempotents necessary to perform the decomposition of a given $n \times n$ matrix.

4.2 WindowSum

The input in the “WindowSum” procedure is a user-determined number of square matrices and their corresponding index equivalence classes. The procedure outputs the “window sum” of these matrices. It begins with error-checking, ensuring that the user inputs the matrices and index equivalence classes in the correct order and also checks to whether the procedure is in fact needed (guarding against the user inputting nothing or only one matrix). Assuming arguments

are given properly, the procedure finds the size of the output matrix, and then reads off its entries from the arguments input by the user to create the window sum.

Code

```
WindowSum := proc()

#-----
# Input: Square matrices A[1] to A[k]
#       Matrix indices n[1] to n[k] (sets) respectively for each A[i]
#       (All of the matrices are given first, then the corresponding index lists)
#
# Output: M, the "window sum" of the k matrices
#-----

local h, i, j, M, m, n;

m:=0;
n:=0;

if nargs=0 then          #If no arguments given, alert the user.
  lprint("Error: No arguments");
  return;                #Return nothing
elif not type(nargs/(2),integer) then          #If number of args is odd, not all
matrices are paired with an index list
  lprint("Error: Matrices and index lists not paired");
  return;                #Return nothing
elif nargs=2 then          #If number of args is 2, there is 1 matrix with its
```

```

index list, so no sum needed.

    lprint("Only given one matrix; no sum needed");
    return args[1];          #In that case, return the matrix that was entered.
else          #Check to see if inputs are in the correct order (Matrices first then
eqv classes).
    for h from 1 to nargs do
        if h<=nargs/2 and not type(args[h], Matrix) then
            lprint("Error: Element in first half of args not of data type Matrix");
            return;          #Return nothing
        elif h>nargs/2 and not type(args[h], set) then
            lprint("Error: Element in second half of args not of data type set");
            return;          #Return nothing
        end if:
    end do:
end if:

#Otherwise, the window sum is necessary, so ...

for h from nargs/(2)+1 to nargs do          #For each index list entered by the
user...
    n:=max(args[h]);          #Find dimension of M
    if n>m then
        m:=n;
    end if:
end do:

M:=Array(1..m,1..m,[[0],[0]]);          #Filling M with zeroes

```

```

for h from 1 to (nargs)/(2) do          #For each of the matrices entered by the
user...
    for i from 1 to nops(args[(nargs)/(2)+h]) do
        for j from 1 to nops(args[(nargs)/(2)+h]) do
            M[args[(nargs)/(2)+h][i],args[(nargs)/(2)+h][j]]:= args[h][i,j];
#Fill M with corresponding entries from the matrices entered by the user.
        end do:
    end do:
end do:

return M;

end proc:

```

Examples

```

1. [ >A := Matrix([[5, 6], [7, 8]]);
    B := Matrix([[9, 10], [11, 12]]);
    C := Matrix([[13, 14], [15, 16]]);
    n := {1, 4};
    p := {2, 5};
    q := {3, 6};

```

$$A := \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$B := \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix}$$

$$C := \begin{bmatrix} 13 & 14 \\ 15 & 16 \end{bmatrix}$$

$$n := \{1, 4\}$$

$$p := \{2, 5\}$$

$$q := \{3, 6\}$$

```
[ > WindowSum(A, B, C, n, p, q);
```

$$\begin{bmatrix} 5 & 0 & 0 & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 10 & 0 \\ 0 & 0 & 13 & 0 & 0 & 14 \\ 7 & 0 & 0 & 8 & 0 & 0 \\ 0 & 11 & 0 & 0 & 12 & 0 \\ 0 & 0 & 15 & 0 & 0 & 16 \end{bmatrix}$$

4.3 IsIdempotent

The IsIdempotent procedure takes a matrix as input, tests to see whether it is idempotent, and then returns “true” or “false”.

Code

```
IsIdempotent := proc(A)

#-----
# Input: The matrix A.
# Output: Tests whether A is idempotent, returns true or false.
#-----

if IsMatrixShape(A^2-A, zero) then
    return true;
else
    return false;
end if:

end proc:
```

Examples

1. $\left[\begin{array}{l} >A := \text{Matrix}([[1 , 2 , 2], [0 , 1 , 3], [0 , 0 , 0]]); \\ \text{IsIdempotent}(A); \end{array} \right.$

$$A := \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

false

2. $\left[\begin{array}{l} >A := \text{Matrix}([[1 , 0 , 2], [0 , 1 , 3], [0 , 0 , 0]]); \\ \text{IsIdempotent}(A); \end{array} \right.$

$$A := \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

true

4.4 ZeroIdentitySize

The ZeroIdentitySize procedure searches along the diagonal the input matrix for consecutive entries which are equal. Although Theorem 3.1.1 only applies to matrices with zeros or ones on the diagonal, checking that that condition is satisfied is done with the FitsTheorem procedure. ZeroIdentitySize adds the starting position of each set of consecutive diagonal entries to the list ZeroIdentitySize[1], and cardinality of the set to the list ZeroIdentitySize[2]. In order to make referencing the entries in these lists easier later on, we make two assumptions:

1. That the matrix starts with zeros on the diagonal. If this is not the case, then the first starting position and cardinality are both 0.
2. That the cardinality of ZeroIdentitySize[1] is even. Given the first assumption, this means that the diagonal ends with ones. If this is not the case, then the last starting position is one more than the dimension of the matrix and the last cardinality is 0.

Code

```
ZeroIdentitySize := proc(A)
```

```
#-----
```

```
# Input: The matrix A.
```

```

# Output:
#     n: The list of the sizes of each set of consecutive
# zeros or identity block (n[1], n[2], ..., n[m]).
# Odd subscripts correspond to zeros (n[1]=0 if
# matrix starts with identity block), even subscripts
# correspond to identity blocks. nops(n) always even.
#     sigma: The list of starting positions of each set of
# consecutive zeros or identity block (if n[1]=0, then
# sigma[1] also =0).
#-----

local m1,i,i1,i2,j1,j2,n, sigma;

m1 := LinearAlgebra[RowDimension](A);
j1:=1;      #j1 = where we are on the diagonal
j2:=0;      #j2 = how many elements we have in list n
n:=[];      #Start off with an empty list (fixes recursion error with nops(n)
when starting with zeros on diag)

sigma:=[];

if LinearAlgebra[RowDimension](A)<>LinearAlgebra[ColumnDimension](A) then
    return print("Error: Matrix is not square.");
end if:

if A[1,1]<>0 then      #Checks to see if we start with zeros or not.
    n:=[];            #n is going to be the list of lengths of consecutive zeros/dim

```

```

of id blocks

    j2:=nops(n);          #j2 = how many elements currently we have in list n
    sigma:=[0,1];        # If we start with 1s, then say that 0s start in the [non-
                          existant] 0th place on the diagonal, and then 1s start on the 1st.

else

    sigma:=[1];          #Otherwise, simply have 1s starting on 1st place on the di-
                          agonal.
end if:

for i from 1 to m1-1 do      #This loops counts all but the last eltement of
n.
    if A[i,i]<>A[i+1, i+1] then      #If elt on diag is <> next elt on diag,
then...
        n:=[op(n), i-sum(n[j],j=1..j2)];
        j2:=nops(n);          #j2 = how many elements we currently have in list n
        sigma:=[op(sigma), i+1];      #Record where the next set of zeros or iden-
tity block starts.
    end if:
end do:

n:=[op(n), m1-sum(n[j],j=1..j2)];

if A[m1,m1]<>1 then      #Always end with the size of an identity block (0 if
dosn't exist).
    n:=[op(n), 0];
    sigma:=[op(sigma), m1+1];

```

end if:

return n,sigma;

end proc:

Examples

1. $\left[\begin{array}{l} > A := \\ \text{Matrix}([[1,9, 2, 3, 4], \\ [0, 0, 5, 6, 7], \\ [0, 0, 1, 0, 2], \\ [0, 0, 0, 1, 3], \\ [0, 0, 0, 0, 0]]); \\ \text{ZeroIdentitySize}(A); \end{array} \right.$

$$A := \begin{bmatrix} 1 & 9 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[0, 1, 1, 2, 1, 0], [0, 1, 2, 3, 5, 6]

2. $\left[\begin{array}{l} > A := \\ \text{Matrix}([[0,9, 2, 3, 4], \\ [0, 1, 0, 6, 7], \\ [0, 0, 1, 0, 2], \\ [0, 0, 0, 0, 3], \\ [0, 0, 0, 0, 1]]); \\ \text{ZeroIdentitySize}(A); \end{array} \right.$

$$A := \begin{bmatrix} 0 & 9 & 2 & 3 & 4 \\ 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[1, 2, 1, 1], [1, 2, 4, 5]

4.5 FitsTheorem

The FitsTheorem checks to see if a given matrix fits the form of Theorem 3.1.1. In order for this procedure to return true, the following three conditions must be satisfied:

1. The matrix must be upper triangular.
2. The matrix may only have zeros or ones on the diagonal.
3. Any consecutive ones must form an identity matrix block on the diagonal.

Any conditions not satisfied will result in an error message being printed. If the matrix fails to satisfy multiple conditions, the error message for each is printed.

Code

```
FitsTheorem := proc(A)

#-----
# Input:
#     A: A matrix.
#     sigma: The result of ZeroIdentitySize
#
```

```

# Output:
#     If it fits the form of the theorem,
# then it outputs true.
#     If not, then it prints an
# error message and returns false.
#-----

local i,j,k,sigma,NextSigma,errors;

sigma:=ZeroIdentitySize(A)[2];
if IsMatrixShape(A,triangular[upper])=false then      #Check to see if the ma-
trix is upper triangular (and square).
    lprint("Error: Matrix is not upper triangular.");
    errors:=true;
end if:

for i from 1 to LinearAlgebra[RowDimension](A) do      #Check to see if there is
anything other than 0 or 1 on the diagonal.
    if A[i,i]<>0 and A[i,i]<>1 then
        lprint("Error: Number other than 0 or 1 on diagonal.");
        errors:=true;
    end if:
end do:

for i from 1 to nops(sigma)/2 do      #These loops check to see if 1s are part
of an identity block.

```

```

    if 2*i+1 < nops(sigma) then          #Avoids the error that occurs if diago-
nal ends with an identity block (still need this).
        NextSigma:=sigma[2*i+1]-1
    else
        NextSigma:=LinearAlgebra[RowDimension](A)
    end if:

    for j from sigma[2*i] to NextSigma-1 do
        for k from j+1 to NextSigma do
            if A[j,k]<>0 then
                lprint("Error: Consecutive ones do not form an identity block.");
                errors:=true;
            end if:
        end do:
    end do:

end do:

if errors=true then
    return false;
else
    return true;
end if:

end proc:

```

Examples

1. $\left[\begin{array}{l} > A := \text{Matrix}([[1 , 0 , 2], [0 , 1 , 3], [0 , 0 , 0]]); \\ \text{FitsTheorem}(A); \end{array} \right.$

$$A := \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

true

2. $\left[\begin{array}{l} > A := \text{Matrix}([[1 , 0 , 2], [0 , 2 , 3], [0 , 0 , 0]]); \\ \text{FitsTheorem}(A); \end{array} \right.$

$$A := \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

“Error: Number other than 0 or 1 on diagonal”

3. $\left[\begin{array}{l} > A := \text{Matrix}([[1 , 0 , 2], [0 , 1 , 3], [4 , 0 , 0]]); \\ \text{FitsTheorem}(A); \end{array} \right.$

$$A := \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 4 & 0 & 0 \end{bmatrix}$$

“Error: Matrix is not upper triangular.”

4. $\left[\begin{array}{l} > A := \text{Matrix}([[1 , 1 , 2], [0 , 1 , 3], [0 , 0 , 0]]); \\ \text{FitsTheorem}(A); \end{array} \right.$

$$A := \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

“Error: Consecutive ones do not form an identity block.”

4.6 MakeIdempotents

The “MakeIdempotents” procedure utilizes the previous procedures and applies the algorithm described in the proof of Theorem 3.1.1 to decompose the input matrix into a product of upper triangular idempotents.

This procedure first calls the IsIdempotent procedure to see whether the input matrix is already idempotent (in which case the rest of the algorithm is not needed) and, if not, then uses FitsTheorem to test whether the input matrix that can be written as a product of upper triangular idempotents. If a decomposition into upper triangular idempotents is possible, then the algorithm is applied.

The procedure works with zero and identity blocks in pairs (that is, the procedure forms the idempotents related to the first zero and identity blocks together, then related to the second pair of blocks of each type, and so on) rather than separately.

There is a special case that deserves mention: if the input matrix ends with a 1×1 zero block on the diagonal, then some of the steps in the algorithm are unnecessary and yield matrices which, while they do not make the product incorrect, are not needed in the decomposition. A check for this case is included to remove these extraneous matrices from the output.

Code

```
MakeIdempotents:=proc(A)

#-----
# Input: A, a matrix of the form in the theorem
# Output: Idempotents whose product is A,
#       E[m]: the ith idempotent in the product (E[1] is rightmost)
#-----

local i,j,j1,m1,E,sigma;
```

```

m1 := LinearAlgebra[RowDimension](A);      #Size of A
j1 := 1;      #j1 = where we are on the diagonal
E:=[];      #List of idempotents
sigma:=ZeroIdentitySize(A)[2];      #List of starting points of each new block
(whether 0 or identity block)

if IsIdempotent(A)=true then      #If A is already idempotent, then...
    lprint("Matrix is idempotent.");
    return A;
end if:

if FitsTheorem(A)=false then
    return lprint("Matrix cannot be written as a product of upper triangular idem-
potents.");
end if:
#If A is not idempotent, then do algorithm

for j from 1 to nops(ZeroIdentitySize(A)[1])-1 do

    if type(j,even) then      #If we are at an even entry of n (i.e. an iden-
tity block), then...
        E:=[LinearAlgebra[RowOperation]...
            ... (LinearAlgebra[IdentityMatrix](m1),1,1),op(E)];
        for i from sigma[j] to sigma[j+1]-1 do
            E[1][i,sigma[j+1]]:=A[i,sigma[j+1]];      #Elements next to identity

```

```

block
  end do:

  E[1][sigma[j+1],sigma[j+1]]:=0;

  if j=nops(ZeroIdentitySize(A)[1])-2 and ZeroIdentitySize(A)[1][j+1]=1 and ZeroIdentitySize(A)[1][j+2]=0 then    #If...
    E:=[LinearAlgebra[RowOperation]...
      (LinearAlgebra[IdentityMatrix](m1),1,1),op(E)];
    E[1][sigma[j+1],sigma[j+1]]:=0;
    return op(E);

  else
    for i from sigma[j] to sigma[j+1]-1 do          #For each row in identity
block, do...
      E:=[LinearAlgebra[RowOperation]...
        (LinearAlgebra[IdentityMatrix](m1),1,1),op(E)];
      E[1][sigma[j+1]]:=A[j1];          #Filling "workspace" row
      E[1][sigma[j+1],sigma[j+1]], E[1][sigma[j+1],j1]:=0,0;
      E:=[LinearAlgebra[RowOperation]...
        (LinearAlgebra[IdentityMatrix](m1),1,1),op(E)];
      E[1][j1,sigma[j+1]]:=1;
      E[1][sigma[j+1],sigma[j+1]]:=0;    #Move into place
      j1:=j1+1;
    end do:

  end if:

```

```

end if:

#Finishing identity block

if type(j,odd) and ZeroIdentitySize(A)[1][j]<>0 then      #If we are at an EX-
ISTING odd entry of n (i.e. a zero block), then...
    for i from sigma[j] to sigma[j+1]-1 do      #For each row in the zero block,
do...
        E:=[LinearAlgebra[RowOperation]...
            (LinearAlgebra[IdentityMatrix](m1),1,1),op(E)];
        E[1][i]:=A[i];      #ith row of E[1] is ith row of A
        j1:=j1+1;
    end do:      #Finishing zero block
end if:

end do:

return op(E);

end proc:

end module;

```

Examples

```
1. > A:=  
   Matrix([[ 1 , -3 , 2 , 3 , 4 ],  
          [ 0 , 0 , 5 , 6 , 7 ],  
          [ 0 , 0 , 1 , 0 , 0 ],  
          [ 0 , 0 , 0 , 1 , 0 ],  
          [ 0 , 2 , 0 , 0 , 1 ]]);  
   MakeIdempotents(A);
```

$$A := \begin{bmatrix} 1 & -3 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

“Error: Matrix is not upper triangular.”

“Matrix cannot be written as a product of upper triangular idempotents.”

```
2. > A:=  
   Matrix([[ 1 , -3 , 2 , 3 , 4 ],  
          [ 0 , 0 , 5 , 6 , 7 ],  
          [ 0 , 0 , 1 , 0 , 0 ],  
          [ 0 , 0 , 0 , 1 , 0 ],  
          [ 0 , 0 , 0 , 0 , 1 ]]);  
   MakeIdempotents(A);
```

$$A := \begin{bmatrix} 1 & -3 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -3 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3. $\left[\begin{array}{l} > A:= \\ \text{Matrix}([[0, 0], \\ [0, 1]]); \\ \text{MakeIdempotents}(A); \end{array} \right.$

$$A := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

“Matrix is idempotent.”

4. $\left[\begin{array}{l} > A:= \\ \text{Matrix}([[1, 2, 3, 4], \\ [0, 0, 6, 5], \\ [0, 0, 1, 2], \\ [0, 0, 0, 0]], \\ \text{MakeIdempotents}(A); \end{array} \right.$

$$A := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. $\left[\begin{array}{l} >A := \text{Matrix}([0]); \\ \text{MakeIdempotents}(A); \end{array} \right.$

$$A := \begin{bmatrix} 0 \end{bmatrix}$$

“Matrix is idempotent”

$$\begin{bmatrix} 0 \end{bmatrix}$$

6. $\left[\begin{array}{l} > A:= \\ \text{Matrix}([[1 , 0 , 0 , 0 , 5 , -2 , -9 , 12 , 15 , 100], \\ [0 , 1 , 0 , 0 , 18 , 20 , 0 , 19 , -5/4 , 8], \\ [0 , 0 , 1 , 0 , \text{exp}(1) , -15 , 3 , 4 , 6 , -9], \\ [0 , 0 , 0 , 1 , 9 , -8 , 7 , -6 , 5 , -4], \\ [0 , 0 , 0 , 0 , 0 , 28 , \text{Pi} , -53/4 , 2 , 3], \\ [0 , 0 , 0 , 0 , 0 , 0 , 4 , 5 , 6 , 0], \\ [0 , 0 , 0 , 0 , 0 , 0 , 1 , 0 , 1 , 0], \\ [0 , 0 , 0 , 0 , 0 , 0 , 0 , 1 , 48 , 9], \\ [0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 999], \\ [0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 1]]); \\ \text{MakeIdempotents}(A); \end{array} \right.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 18 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & e^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4.7 Bounding the Algorithm

Given the MakeIdempotents algorithm, we want to find the least upper bound for the number of idempotent matrices needed to perform the decomposition. To do so, we sum the number needed for each block (that is, a zero or identity block on the diagonal of the input matrix) for an arbitrary input matrix. We consider two broad cases in this sum: if the matrix ends with a zero block on the diagonal, or an identity block. To be consistent with the way in which we wrote the code and for computational simplicity, we assume that the first block of the matrix is always a zero block (if in fact the matrix starts with an identity block, then the first zero block is defined to have size 0). Thus, letting m be the number of blocks in the matrix, we have two cases: m odd, and m even. (Note: in the following summations, n_j is the number of rows in the j th block.)

For m odd, the number of idempotents output by the algorithm is

$$\left(\sum_{i=1}^{\frac{m-1}{2}} (n_{2i-1} + 2n_{2i} + 1) \right) + n_m$$

n_{2i-1} is the number of idempotents for each zero block in the matrix (one idempotent is needed for each row of a zero block), and $2n_{2i} + 1$ is the number of idempotents for each identity block (one idempotent is needed for the elements in the column next to the identity block, and two

are needed for each row: filling the “workspace” row, and then moving that row into place as per the algorithm). The bound $\frac{m-1}{2}$ causes the summation to count all but the very last block in the inputted matrix; this block contributes n_m idempotents to the sum.

For m even, we have the following sum:

$$\left(\sum_{i=1}^{\frac{m}{2}-1} (n_{2i-1} + 2n_{2i} + 1) \right) + n_{m-1}$$

The new bound on the summation $\frac{m}{2} - 1$ counts all but the last two blocks of the matrix. The second to the last block contributes n_{m-1} idempotents to the sum. Since the last block is an identity matrix and there can be no column to the right of that last block, that block does not contribute any additional idempotents to the sum.

With these two sums, we can determine an upper bound for how many idempotents will be output. We group terms in each sum in such a way as to write the sum in terms of n , the dimension of the input matrix.

For the first case (m odd), we rearrange as follows:

$$\sum_{i=1}^{\frac{m-1}{2}} (n_{2i-1} + n_{2i}) + n_m + \sum_{i=1}^{\frac{m-1}{2}} (n_{2i} + 1) = n + (n_2 + 1 + n_4 + 1 + \dots + n_{m-3} + 1 + n_{m-1} + 1)$$

since $n = n_1 + n_2 + n_3 + \dots + n_{m-1} + n_m$. At this point, we need a few more cases to determine how the quantity in parentheses compares to n :

If $n_1 \neq 0$ (that is, if the matrix starts with a zero block on the diagonal), then this last quantity is strictly less than n because all other terms are less than or equal to their counterparts in the expression for n . If $n_1 = 0$, we then consider n_m . If $n_m > 1$, then the quantity in parentheses is strictly less than n (since the last term in parentheses is a 1). The only remaining case is when $n_1 = 0$ and $n_m = 1$. In that case, the summation alone says that the quantity in parentheses is less than or equal to n , and hence the overall sum is $\leq 2n$. However, $n_1 = 0$ and $n_m = 1$ puts us in the special case where the matrix ends with a 1×1 zero block

on the diagonal. In that case, the algorithm outputs unnecessary matrices (the “workspace” and “move into place” matrices are not needed when there is only one column to the right of the last identity block), so even though the sum is possibly equal to $2n$, the upper bound is *strictly* less than $2n$. Thus, when m is odd, in each case the upper bound for the number of idempotents output by the algorithm is $2n - 1$.

A similar computation is performed for the m even case. We rearrange the sum as:

$$\begin{aligned}
& \sum_{i=1}^{\frac{m}{2}-1} (n_{2i-1} + 2n_{2i} + 1) + n_{m-1} \\
&= \sum_{i=1}^{\frac{m}{2}-1} (n_{2i-1} + n_{2i} + n_{m-1}) + \sum_{i=1}^{\frac{m}{2}-1} (n_{2i} + 1) \\
&= (n_1 + n_2 + n_3 + \dots + n_{m-2} + n_{m-1}) + (n_2 + 1 + n_4 + 1 + \dots + n_{m-4} + 1 + n_{m-2} + 1)
\end{aligned}$$

Since $n_m > 0$, the quantity in the first set of parentheses is strictly less than n (i.e., $\leq n - 1$). The quantity in the second set of parentheses is $\leq n - 1$ for the same reason. Thus the sum is $\leq 2n - 2$, and so our upper bound when m is even is $2n - 2$.

Hence, considering all cases and values of n , the upper bound for the number of idempotents output by the algorithm is $2n - 1$.

For $n \leq 10$, C. Mangum and I have constructed examples to show that the bounds derived here are attained in certain cases; thus, they are in fact the *least* upper bounds. We conjecture that they are the least upper bounds for all n , but were not able to construct general examples to show that this bound can be attained for any n . The following examples show the upper bounds being attained when $n = 6$, for both m odd and m even:

Example 4.7.1. This example shows the bound $2n - 1$ being attained when m is odd:

```

> A:=
Matrix([[ 1 , 2 , 5 , 0 , 0 , 0 ],
[ 0 , 0 , 5 , 0 , 8 , 0 ],
[ 0 , 0 , 1 , 0 , 6 , 4 ],
[ 0 , 0 , 0 , 1 , 7 , 9 ],
[ 0 , 0 , 0 , 0 , 0 , 8 ],
[ 0 , 0 , 0 , 0 , 0 , 0 ]]);
MakeIdempotents(A);

```

$$A := \begin{bmatrix} 1 & 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 8 & 0 \\ 0 & 0 & 1 & 0 & 6 & 4 \\ 0 & 0 & 0 & 1 & 7 & 9 \\ 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chapter 5

Rings

5.1 Definitions

Definition 5.1.1. Let R be a ring, and $\Gamma = \{e_1, \dots, e_n\}$ be *orthogonal idempotents* of R :

1. $e_i^2 = e_i$,
2. $e_i e_j = 0$ for $i \neq j$,
3. $e_1 + \dots + e_n = 1$.

Definition 5.1.2. $a \in R$ is said to be *triangular* if $e_i a e_j = 0$ for $i > j$.

$a \in R$ is said to be *strictly triangular* if $e_i a e_j = 0$ for $i \geq j$.

Definition 5.1.3. $e_i R e_i = \{a \in R \mid a e_i = a = e_i a\}$

Definition 5.1.4. $T_\Gamma(R) = \{a \in R \mid e_i a e_j = 0, i > j\}$

Theorem 5.1.5. $T_\Gamma(R)$ is a ring, and hence a monoid with respect to multiplication.

5.2 Results for Rings

Theorem 5.2.1. The multiplicative monoid $T_\Gamma(R)$ is generated by idempotents and units (in $T_\Gamma(R)$) if and only if each multiplicative monoid $e_i R e_i$ is generated by idempotents and units

(in e_iRe_i).

Proof. If: Suppose $T_\Gamma(R)$ is generated by idempotents and units. There is a homomorphism $T_\Gamma(R) \rightarrow e_iRe_i$ given by $a \mapsto e_iae_i$, and thus e_iRe_i is also generated by idempotents and units.

Only if: Let $n = 2$, $\Gamma = \{e_1, e_2\}$ where e_1, e_2 are orthogonal idempotents, and $a \in T_\Gamma(R)$. Then

$$a = 1 \cdot a \cdot 1 = (e_1 + e_2)a(e_1 + e_2) = e_1ae_1 + e_1ae_2 + e_2ae_1 + e_2ae_2 = a_1 + a_{12} + 0 + a_2,$$

where $a_1 = e_1ae_1$, $a_{12} = e_1ae_2$, and $a_2 = e_2ae_2$.

Since each e_iRe_i is generated by idempotents and units, we can write

$$a_1 = u_1f_1 \dots f_l$$

and

$$a_2 = u_2h_1 \dots h_k$$

where u_1 is a unit in e_1Re_1 , $f_1 \dots f_l$ are idempotents e_1Re_1 , u_2 is a unit in e_2Re_2 , and $h_1 \dots h_k$ are idempotents e_2Re_2 .

$b = 1 + a_{12}$ is a unit in R with inverse $1 - a_{12}$:

$$(1 + a_{12})(1 - a_{12}) = 1 - a_{12} + a_{12} - a_{12}a_{12} = 1 - 0 = 1 = (1 - a_{12})(1 + a_{12})$$

Now, $u = u_1 + u_2$ is an unit in R with $u^{-1} = u_1^{-1} + u_2^{-1}$:

$$(u_1 + u_2)(u_1^{-1} + u_2^{-1}) = u_1u_1^{-1} + u_1u_2^{-1} + u_2u_1^{-1} + u_2u_2^{-1} = e_1 + 0 + 0 + e_2 = 1$$

Multiplying a by u^{-1} , we have

$$\begin{aligned}
u^{-1}a &= u^{-1}(a_1 + a_{12} + a_2) \\
&= (u_1^{-1} + u_2^{-1})(a_1 + a_{12} + a_2) \\
&= (u_1^{-1} + u_2^{-1})a_1 + (u_1^{-1} + u_2^{-1})a_{12} + (u_1^{-1} + u_2^{-1})a_2 \\
&= u_1^{-1}a_1 + u^{-1}a_{12} + u_2^{-1}a_2 \\
&= f_1 \dots f_l + u^{-1}a_{12} + h_1 \dots h_k
\end{aligned}$$

Also, $v = 1 + u^{-1}a_{12}$ is a unit in R with inverse $1 - u^{-1}a_{12}$:

$$\begin{aligned}
(1 + u^{-1}a_{12})(1 - u^{-1}a_{12}) &= 1 - u^{-1}a_{12} + u^{-1}a_{12} - u^{-1}a_{12}u^{-1}a_{12} \\
&= 1 - u^{-1}a_{12}u^{-1}a_{12} \\
&= 1 - u^{-1}a_{12}(u_1^{-1} + u_2^{-1})a_{12} \\
&= 1 - u^{-1}a_{12}(u_1^{-1}a_{12} + u_2^{-1}a_{12}) \\
&= 1 - u^{-1}a_{12}u_1^{-1}a_{12} \\
&= 1
\end{aligned}$$

So without loss of generality, we may assume $a_1 = f_1 \dots f_l$ and $a_2 = h_1 \dots h_k$.

We claim that

$$h'_i = e_1 + h_i$$

and

$$f'_j = f_j + e_2$$

are idempotents in R :

$$\begin{aligned}
(h'_i)^2 &= (e_1 + h_i)^2 \\
&= (e_1 + h_i)(e_1 + h_i) \\
&= e_1^2 + e_1 h_i + h_i e_1 + h_i^2 \\
&= e_1 + h_i = h'_i
\end{aligned}$$

$$\begin{aligned}
(f'_j)^2 &= (f_j + e_2)^2 \\
&= (f_j + e_2)(f_j + e_2) \\
&= f_j^2 + f_j e_2 + e_2 f_j + e_2^2 \\
&= f_j + e_2 = f'_j
\end{aligned}$$

Now

$$a = h'_1 \dots h'_k v f'_1 \dots f'_l :$$

We first show (by induction) that $h'_1 \dots h'_k = e_1 + h_1 \dots h_k$:

$$\begin{aligned}
h'_1 h'_2 &= (e_1 + h_1)(e_1 + h_2) \\
&= e_1^2 + e_1 h_2 + h_1 e_1 + h_1 h_2 \\
&= e_1 + 0 + 0 + h_1 h_2 \\
&= e_1 + h_1 h_2
\end{aligned}$$

Now suppose $h'_1 \dots h'_{k-1} = e_1 + h_1 \dots h_{k-1}$, and then

$$\begin{aligned}
(h'_1 \dots h'_{k-1})h'_k &= (e_1 + h_1 \dots h_{k-1})(e_1 + h_k) \\
&= e_1^2 + e_1 h_k + h_1 \dots h_{k-1} e_1 + h_1 \dots h_{k-1} h_k \\
&= e_1 - h_1 \dots h_k \\
&= e_1 + h_1 \dots h_k
\end{aligned}$$

Similarly, $f'_1 \dots f'_l = f_1 \dots f_l + e_2$

Then

$$\begin{aligned}
h'_1 \dots h'_k v f'_1 \dots f'_l &= (e_1 + h_1 \dots h_{k-1})(1 + u^{-1} a_{12})(f_1 \dots f_l + e_2) \\
&= (e_1 + h_1 \dots h_{k-1})(f_1 \dots f_l + e_2 + u^{-1} a_{12} f_1 \dots f_l + u^{-1} a_{12} e_2) \\
&= (e_1 + h_1 \dots h_{k-1})(f_1 \dots f_l + e_2 + 0 + u^{-1} a_{12}) \\
&= e_1 f_1 \dots f_l + e_1 e_2 + e_1 u^{-1} a_{12} + h_1 \dots h_{k-1} f_1 \dots f_l \\
&\quad + h_1 \dots h_{k-1} e_2 + h_1 \dots h_{k-1} u^{-1} a_{12} \\
&= f_1 \dots f_l + 0 + u^{-1} a_{12} + 0 + h_1 \dots h_{k-1} + 0 + 0 \\
&= f_1 \dots f_l + u^{-1} a_{12} + h_1 \dots h_k \\
&= u^{-1} a
\end{aligned}$$

So $u^{-1}a$ can be written as the product of a unit and idempotents, and thus a can be written as the product of a unit and idempotents.

Now let e_1, e_2, \dots, e_m be orthogonal idempotents, $\Gamma_{m-1} = \{e_1, e_2, \dots, e_{m-1}\}$, $\Gamma_m = \{e_1, e_2, \dots, e_m\}$, and assume the result holds for $T_{\Gamma_{m-1}}(R)$. Let $\tilde{e}_1 = e_1 + e_2 + \dots + e_{m-1}$, $\tilde{e}_2 = e_m$, and $a \in T_{\Gamma_m}(R)$. Then

$$a = 1 \cdot a \cdot 1 = (\tilde{e}_1 + \tilde{e}_2)a(\tilde{e}_1 + \tilde{e}_2) = \tilde{e}_1 a \tilde{e}_1 + \tilde{e}_1 a \tilde{e}_2 + \tilde{e}_2 a \tilde{e}_1 + \tilde{e}_2 a \tilde{e}_2 = \tilde{e}_1 a \tilde{e}_1 + \tilde{e}_1 a \tilde{e}_2 + \tilde{e}_2 a \tilde{e}_2,$$

By induction $\tilde{e}_1 a \tilde{e}_1 \in T_{\Gamma_{m-1}}(R)$ can be written as the product of a unit and idempotents, and then using the $n = 2$ case we can write $a \in T_{\Gamma_m}(R)$ as the product of a unit and idempotents.

Q.E.D.

Corollary 5.2.2. *Every block upper triangular matrix can be written as a product of block upper triangular units and idempotents.*

Proof. In the matrix case we have the orthogonal idempotents

$$e_1 = \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 \\ & I_{r_2} \\ 0 & 0 \end{bmatrix}, \dots, e_m = \begin{bmatrix} 0 & 0 \\ & & & \\ & & & I_{r_m} \end{bmatrix}$$

Then $T_{\Gamma}(M_n(k)) = \{\text{Block upper triangular matrices in } M_n(k)\}$ and $e_i M_n(k) e_i \cong e_i R e_i$.

$e_i M_n(k) e_i$ is a product of units and idempotents by the Renner's decomposition and Corollary 2.2.2, and thus the result follows from Theorem 5.2.1.

Q.E.D.

Example 5.2.3. We can write the matrix

$$\left[\begin{array}{cc|ccc} 1 & -1 & 1 & 2 & 3 \\ -1 & 1 & 4 & 5 & 6 \\ \hline 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & 5 & -3 \end{array} \right]$$

as a product of units and idempotents:

In this example we have

$$e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Also observe that we can write the blocks on the diagonal as products of idempotents as

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -\frac{10}{9} & -\frac{4}{9} \\ 5 & -\frac{16}{9} & -\frac{10}{9} \\ 1 & -\frac{5}{9} & \frac{7}{9} \end{bmatrix} \begin{bmatrix} 1 & \frac{25}{9} & -\frac{17}{9} \\ 0 & \frac{34}{9} & -\frac{17}{9} \\ 0 & \frac{50}{9} & -\frac{25}{9} \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 3 \\ -1 & 1 & 4 & 5 & 6 \\ \hline 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & -\frac{10}{9} & -\frac{4}{9} \\ 0 & 0 & 5 & -\frac{16}{9} & -\frac{10}{9} \\ 0 & 0 & 1 & -\frac{5}{9} & \frac{7}{9} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & \frac{25}{9} & -\frac{17}{9} \\ 0 & 0 & 0 & \frac{34}{9} & -\frac{17}{9} \\ 0 & 0 & 0 & \frac{50}{9} & -\frac{25}{9} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 5 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can also use Corollary 1.1.7 to move the unit to either side of the product and have

$$\begin{aligned}
 \begin{bmatrix} 1 & -1 & 1 & 2 & 3 \\ -1 & 1 & 4 & 5 & 6 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & 5 & -3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 5 & 6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 4 & -\frac{4}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -15 & \frac{25}{3} & \frac{10}{3} \\ 0 & 1 & -39 & \frac{65}{3} & \frac{26}{3} \\ 0 & 0 & 3 & -\frac{10}{9} & -\frac{4}{9} \\ 0 & 0 & 5 & -\frac{16}{9} & -\frac{10}{9} \\ 0 & 0 & 1 & -\frac{5}{9} & \frac{7}{9} \end{bmatrix} \\
 & \begin{bmatrix} 1 & 0 & 0 & -25 & 17 \\ 0 & 1 & 0 & -\frac{175}{3} & \frac{119}{3} \\ 0 & 0 & 1 & \frac{25}{9} & -\frac{17}{9} \\ 0 & 0 & 0 & \frac{34}{9} & -\frac{17}{9} \\ 0 & 0 & 0 & \frac{50}{9} & -\frac{25}{9} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

or

$$\begin{aligned}
 \begin{bmatrix} 1 & -1 & 1 & 2 & 3 \\ -1 & 1 & 4 & 5 & 6 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & 5 & -3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -\frac{10}{9} & -\frac{4}{9} \\ 0 & 0 & 5 & -\frac{16}{9} & -\frac{10}{9} \\ 0 & 0 & 1 & -\frac{5}{9} & \frac{7}{9} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{25}{9} & -\frac{17}{9} \\ 0 & 0 & 0 & \frac{34}{9} & -\frac{17}{9} \\ 0 & 0 & 0 & \frac{50}{9} & -\frac{25}{9} \end{bmatrix} \\
 & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 5 & 7 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 & 5 & 6 \\ 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 5 & 6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

Example 5.2.4. We can write

$$\left[\begin{array}{cc|cccc} 1 & -1 & 1 & 2 & 3 & 7 & 8 \\ -1 & 1 & 4 & 5 & 6 & 9 & -9 \\ \hline 0 & 0 & 2 & 2 & -2 & -8 & -7 \\ 0 & 0 & 5 & 1 & -3 & -6 & -5 \\ 0 & 0 & 1 & 5 & -3 & -4 & -3 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 4 & -2 \end{array} \right]$$

as a product of units and idempotents:

Observe that

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}$$

can be written as products of idempotents as in example 5.2.3.

Using Corollary 1.1.7 to move the units to one side (say, the left), we have

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 3 & 7 & 8 \\ -1 & 1 & 4 & 5 & 6 & 9 & -9 \\ 0 & 0 & 2 & 2 & -2 & -8 & -7 \\ 0 & 0 & 5 & 1 & -3 & -6 & -5 \\ 0 & 0 & 1 & 5 & -3 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 7 & 8 \\ 0 & 1 & 4 & 5 & 6 & 9 & -9 \\ 0 & 0 & 1 & 0 & 0 & -8 & -7 \\ 0 & 0 & 0 & 1 & 0 & -6 & -5 \\ 0 & 0 & 0 & 0 & 1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -68 & 34 \\ 0 & 1 & 0 & 0 & 0 & -124 & 62 \\ 0 & 0 & 1 & 0 & 0 & 14 & -7 \\ 0 & 0 & 0 & 1 & 0 & 10 & -5 \\ 0 & 0 & 0 & 0 & 1 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 73 \\ 0 & 1 & 0 & 0 & 0 & 0 & 157 \\ 0 & 0 & 1 & 0 & 0 & 0 & -15 \\ 0 & 0 & 0 & 1 & 0 & 0 & -11 \\ 0 & 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 39 & 0 \\ 0 & 1 & 0 & 0 & 0 & 95 & 0 \\ 0 & 0 & 1 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 107 & 0 \\ 0 & 1 & 0 & 0 & 0 & 219 & 0 \\ 0 & 0 & 1 & 0 & 0 & -22 & 0 \\ 0 & 0 & 0 & 1 & 0 & -16 & 0 \\ 0 & 0 & 0 & 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 4 & -\frac{4}{3} & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -15 & \frac{25}{3} & \frac{10}{3} & 0 & 0 \\ 0 & 1 & -39 & \frac{65}{3} & \frac{26}{3} & 0 & 0 \\ 0 & 0 & 3 & -\frac{10}{9} & -\frac{4}{9} & 0 & 0 \\ 0 & 0 & 5 & -\frac{16}{9} & -\frac{10}{9} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{9} & \frac{7}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -25 & 17 & 0 & 0 \\ 0 & 1 & 0 & -\frac{175}{3} & \frac{119}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{25}{9} & -\frac{17}{9} & 0 & 0 \\ 0 & 0 & 0 & \frac{34}{9} & -\frac{17}{9} & 0 & 0 \\ 0 & 0 & 0 & \frac{50}{9} & -\frac{25}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Chapter 6

Future Directions

In the future, I would like to continue to expand on the ideas developed in this dissertation. Some specific problems which I would like to answer are:

Problem 1: When is a block upper triangular matrix expressible as a product of block upper triangular idempotent matrices?

Once proved, the following conjecture will provide a partial answer to this question:

Conjecture 6.0.5. *Nilpotent block upper triangular matrices in $M_n(k)$ can be written as a product of block upper triangular idempotents.*

Example 6.0.6. The nilpotent block upper triangular matrix

$$N = \left[\begin{array}{ccc|cc} 2 & 2 & -2 & 1 & 2 \\ 5 & 1 & -3 & 3 & 4 \\ 1 & 5 & -3 & 5 & 6 \\ \hline 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 2 & 2 \end{array} \right] \in M_n(\mathbb{R})$$

can be written as a product of block upper triangular idempotents as follows:

$$N = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -\frac{10309}{648} & 0 \\ 0 & 1 & 0 & -\frac{793}{108} & 0 \\ 0 & 0 & 1 & -\frac{55}{18} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array} \right] \left[\begin{array}{ccc|cc} 1 & 0 & 0 & \frac{637}{81} & -\frac{637}{324} \\ 0 & 1 & 0 & \frac{98}{27} & -\frac{49}{54} \\ 0 & 0 & 1 & \frac{10}{9} & -\frac{5}{18} \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array} \right] \\
\left[\begin{array}{ccc|cc} \frac{389}{9} & -\frac{1615}{18} & -\frac{1235}{324} & \frac{1235}{243} & \frac{6175}{1944} \\ 20 & -\frac{83}{2} & -\frac{65}{36} & \frac{65}{27} & -\frac{325}{216} \\ 8 & -17 & \frac{5}{18} & \frac{26}{27} & -\frac{65}{108} \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc|cc} -\frac{109}{9} & \frac{295}{9} & -\frac{767}{81} & -\frac{118}{243} & -\frac{649}{486} \\ -6 & 16 & -\frac{13}{3} & -\frac{2}{9} & -\frac{11}{18} \\ -4 & 10 & -\frac{17}{9} & -\frac{4}{27} & -\frac{11}{27} \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \\
\left[\begin{array}{ccc|cc} 1 & \frac{169}{12} & -\frac{845}{24} & \frac{2197}{216} & -\frac{3887}{1296} \\ 0 & \frac{15}{2} & -\frac{65}{4} & \frac{169}{36} & -\frac{299}{216} \\ 0 & 3 & -\frac{13}{2} & \frac{13}{6} & -\frac{23}{36} \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

Problem 2: Theorem 5.2.1 gives necessary and sufficient conditions for an element of $T_\Gamma(R)$ to be expressible as a product of idempotents and units in $T_\Gamma(R)$. I would like to determine when an element of $T_\Gamma(R)$ is expressible strictly as a product of idempotents in $T_\Gamma(R)$.

Problem 3: Another way in which I would like to expand on the work done in Chapter 5 is by determining $\langle E(R) \rangle$, where R is a ring.

Problem 4: In Chapter 3 we determined $\langle E(\overline{B}) \rangle$, where $B \subseteq Gl_n(k)$ is the Borel subgroup of upper triangular invertible matrices and $\overline{B} = T_n(k)$. I would like to generalize this and determine when, in a reductive monoid M with a Borel subgroup B , an element of \overline{B} is a product of idempotents in \overline{B} .

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